

Department of Mathematics



Numerical Analysis of Random Periodicity of Stochastic Differential Equations

by

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Abstract

In this thesis, we discuss the numerical approximation of random periodic solutions (r.p.s.) of stochastic differential equations (SDEs) with multiplicative noise. We prove the existence of the random periodic solution as the limit of the pull-back flow when the starting time tends to $-\infty$ along the multiple integrals of the period. As the random periodic solution is not explicitly constructible, it is useful to study the numerical approximation. We discretise the SDE using the Euler-Maruyama scheme and modified Milstein scheme. Subsequently we obtain the existence of the random periodic solution as the limit of the pull-back of the discretised SDE. We prove that the latter is an approximated random periodic solution with an error to the exact one at the rate of $\sqrt{\Delta t}$ in the mean-square sense in Euler-Maruyama method and Δt in the modified Milstein method. We obtain the weak convergence result in infinite horizon for the approximation of the average periodic measure.

Keywords: random periodic solution, periodic measure, Euler-Maruyama method, modified Milstein method, infinite horizon, rate of convergence, pull-back, weak convergence.

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Chapter 1

Introduction

Periodicity plays a very important role in the study of many different areas in science. There are many periodic phenomena in our real life. Considering the sunrise and sunset each day, we notice this periodic behaviour is driven by a dynamical system in the celestial mechanics. Even now, it is still very hard to describe the process of evolution in the formation of the solar system. But we also benefit from the periodicity which is predictable in some sense. The long time behaviour of the universe inspires people to investigate the long time limits of relevant dynamical systems. In the deterministic dynamical system theory, fixed points or periodic solutions capture the intuitive idea of a stationary state or an equilibrium of a dynamical system. Mathematicians have made enormous progress in the study of deterministic systems.

However, many systems in our real life are influenced by some noise factors from internal or external sources. For instance, when we consider the maximum daily temperature in any particular region, it certainly has periodic nature driven by the divine clock due to the revolution of the earth around the sun. But the randomness may come from the uncertainty of the reaction in sun, which influence the heat delivery to the earth. On the other hand, the change of the climate on the earth also provides chaotic disturbance onto the underlying dynamical system. Also the price of wheat in financial market shows the combination of periodicity and randomness. The intrinsic seasonality of wheat growth suggests the periodicity of the price should follow the change of seasons. However, the strike price is always being influenced by not only the real time supplies and demands, but also trading reactions in the international market. Hence the importance of studying stochastic

dynamical systems can hardly be overestimated.

The idea to regard stochastic differential equations (SDEs) as random dynamical systems can be traced back to late 1970's and early 1980's with a number of seminal works by Elworthy, Meyer, Baxendale, Bismut, Ikeda, Watanabe, Kunita and others ([1], [4], [5], [11], [25], [31], [33] etc). Later this was further developed to include stochastic partial differential equations (SPDEs) by Flandoli [19], Garrido-Atienza, Lu and Schmalfuss [20], Mohammed, Zhang and Zhao[38].

The concept of stationary solutions of stochastic dynamical systems has been known for some time and is a stochastic counterpart of the notion of fixed points in the theory of dynamical systems. There are many works studying their existence for SDEs and SPDEs, such as Caraballo, Kloeden and Schmalfuss [8], Khanin, Mazel and Sinai [28], Schmalfuss [41], Sinai [42], Zhao and Zheng [54] etc. An ergodic theory of random dynamical systems has been built under the stationary regime, in which stationary solutions and stationary measures, which are "equivalent", are fundamental objects.

Periodic solution has been a central concept in the theory of dynamical systems since Poincaré's pioneering work [40]. As the random counterpart of periodic solution, the concept of random periodic solutions (RPS) began to be addressed recently for a C^1 -cocycle in the work of Zhao and Zheng[55]. Later the definition of random periodic solutions and their existence for semi-flows generated by non-autonomous SDEs with additive noise were given by Feng, Zhao and Zhou[12], and it was developed to include SPDEs by the work of Feng and Zhao[13].

Denote by $\Delta := \{(t, s) \in \mathbb{R}^2, s \leq t\}$. Let \mathbb{X} be a separable Banach space. Denote by $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}})$ a metric dynamical system and $\theta_s : \Omega \rightarrow \Omega$ is assumed to be measurably invertible for all $s \in \mathbb{R}$. Consider a stochastic periodic semi-flow $u : \Delta \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ of period τ , which satisfies the semi-flow relation.

$$u(t, r, \omega) = u(t, s, \omega) \circ u(s, r, \omega), \quad (1.0.1)$$

for all $r \leq s \leq t$, $r, s, t \in \mathbb{R}$ and almost every $\omega \in \Omega$.

Definition 1.0.1. ([17]) *We call u a τ -periodic stochastic semi-flow if it satisfies an additional periodicity property: there exists a constant $\tau > 0$ such that*

$$u(t + \tau, s + \tau, \omega) = u(t, s, \theta_\tau \omega), \quad (1.0.2)$$

for any $t \geq s$ and almost every $\omega \in \Omega$.

Remark 1.0.2. ([17]) (i) The periodicity assumption (1.0.2) is very natural. It can be verified that solutions for SDEs or SPDEs with time periodic coefficients satisfy (1.0.2) by the same argument as verifying the cocycle property for autonomous stochastic systems. In the cocycle case, (1.0.2) holds for all $\tau > 0$ i.e.

$$u(t, s, \omega) = u(t - s, 0, \theta_s \omega)$$

for any $t \geq s$ and almost every $\omega \in \Omega$.

(ii) The periodicity assumption (1.0.2) plays a crucial role to enable us to lift the semi-flow u to a cocycle on the cylinder $[0, \tau) \times \mathbb{X}$.

The lift case provides the possibility to investigate the exponential contraction of partial derivatives in the analysis of weak approximation. SDEs and SPDEs with time-dependent coefficients which are periodic in time generate periodic semiflows satisfying (1.0.1) and (1.0.2) ([12]-[14]). The following definition of random periodic paths (solutions) for stochastic semi-flow was given by Feng, Zhao and Zhou.

Definition 1.0.3. ([12],[13]) A random periodic path of period τ of the semi-flow $u : \Delta \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ is an \mathcal{F} -measurable map $Y : \mathbb{R} \times \Omega \rightarrow \mathbb{X}$ such that

$$u(t, s, \omega)Y(s, \omega) = Y(t, \omega), \quad Y(s + \tau, \omega) = Y(s, \theta_\tau \omega),$$

for any $(t, s) \in \Delta$ and almost every $\omega \in \Omega$.

It has been proved that random periodic solutions exist for many SDEs and SPDEs ([12]-[14]). Recently, “equivalence” of random periodic paths and periodic measures has been proved in [17] and some results of the ergodicity of periodic measures have been obtained. These results are proved in cocycle case and semi-flow case. To consider the semi-flow case, lifts on the semi-flow and periodic measure played a critical role.

Note that many phenomena in the real world have both periodic and random nature, e.g. daily temperature, energy consumption, airline passenger volumes, CO_2 concentration etc. The concept and its study are relevant to modelling random periodicity in the real world.

In literature, there have been a number of recent works such as [9] on random attractors of the stochastic TJ model in climate dynamics; [3] on stochastic lattice

systems; [10] on stochastic resonance; [14] for SDEs with multiplicative linear noise; and [51] on bifurcations of stochastic reaction diffusion equations. All these results are theoretical on the existence of random periodic paths.

In deterministic cases of dynamical system, numerical schemes were widely applied on the solutions of ordinary differential equations. The stability and efficiency of these schemes were well studied in various brilliant works, including but not limited to Butcher [7], Stuart and Humphries [44], Stoer and Bulirsch [43].

In general, neither stationary solutions nor random periodic solutions can be constructed explicitly, so numerical approximation is another indispensable tool to study stochastic dynamics, especially to physically relevant problems. It is worth mentioning here that this is a numerical approximation of an infinite time horizon problem. There are numerous works on numerical analysis of SDEs on a finite horizon, and a number of excellent monographs (Kloeden and Platen [29], Milstein [36]). However, there are only a few works on infinite horizon problems. A numerical analysis of approximation to the stationary solutions and invariant measures of SDEs through discretising the pull-back, was given by Mattingly, Stuart and Higham [32], Talay [45], Talay and Rubaro [46], Tocino and Ardanuy [47], Yevik and Zhao[52]. Numerical approximations to stable zero solutions of SDEs were given by Higham, Mao and Stuart [23], Kloeden and Platen[29].

Numerical analysis for random periodic solutions was not considered in previous work. The infinite horizon stochastic integral equation (IHSIE) method can deal with anticipated cases ([12]-[14]). But it is still not clear how to numerically approximate two-sided IHSIE and anticipate random periodic solutions. The pull-back method used in this thesis is a popular way to study random attractors. Here we use this to deal with stable adapted random periodic solutions of dissipative systems for the first time. The pull-back method has some advantages. First, stability can be obtained immediately. Secondly, it can deal with some dissipative equations that cannot be dealt with by the IHSIE, especially the current IHSIE technique requires equations to have multiplicative linear noise or additive noise and f being bounded. Thirdly in this thesis, we study numerical approximations of random periodic solutions of dissipative SDEs and with the pull-back idea, a random periodic solution of the discretised system can be obtained as well.

The schemes in this thesis can be used to numerically compute random periodic

solutions and periodic measures for many concrete stochastic differential equations arising in various real world problems. This thesis provides rigorous theoretical error analysis to these schemes.

The structure of the thesis is as follows: in Chapter 3, We will first study the Euler-Maruyama numerical scheme in infinite horizon and obtain an approximating random periodic solution (r.p.s.) \widehat{X}_r^* . We will prove that the latter converges to the exact r.p.s. in $L^2(\Omega)$ at the rate of $\sqrt{\Delta t}$ when the time mesh Δt tends to zero. This result will be numerically verified. Despite its lower order of the approximation only at the rate of $\sqrt{\Delta t}$, the advantage of this scheme is its simplicity, and it is relatively easy to implement in actual computations. It works well for the SDE we consider in this thesis.

We also consider more advanced numerical schemes, e.g. Milstein scheme ([27], [29], [35], [36], [37], [47]), for high order convergence. We improve the rate of approximation from $\sqrt{\Delta t}$ in Euler-Maruyama scheme to Δt .

We will also do some numerical simulations to sample paths of the r.p.s. (Fig. 3.1). However, simulation of one pathwise trajectory is not a reliable way to tell whether or not it is random periodic though it looks very much like to be. Here we provide two reliable methods for this from numerical simulations. One method is to simulate $\{X_t^*(\omega), t \in \mathbb{R}\}$ and $\{X_t^*(\theta_{-\tau}\omega), t \in \mathbb{R}\}$ for the same ω . These two trajectories should be repeating each other, but with a shift of one period of time. See Fig. 3.1 as an example. The other way is to simulate $\{X_t^*(\theta_{-t}\omega), t \in \mathbb{R}\}$, which is periodic if and only if $X_t^*(\omega)$ is random periodic. As an example, see Fig. 3.2. These two approaches would apply to any other stochastic differential equations should they have a random periodic solution.

It was known from the recent work [17] that the law of the random periodic solution is the periodic measure of the corresponding Markov semigroup. Thus we will consider the convergence of transition probabilities generated by (2.0.1) and its numerical scheme along the integral multiples of period to the periodic measure and discretised periodic measure respectively and error estimate of the two periodic measures in the weak topology. Under the Lyapunov-Floquet transformation, our model can be extended to consider more general problems.

The strong approximations of the random periodic solution give us good understanding of the random periodicity. One would be also interested in an approx-

imation in the weak topology. This allows us to consider multi-dimensional noise numerically with greater efficiency and more general SDEs without demanding a strong dissipative condition as long as the system is non-degenerate. Note the approximation of random periodicity is not a classical finite initial value, but infinite horizon problem. For this reason, the exponential decay of the partial derivatives over long time, which plays a key role in this analysis, is obtained. In Chapter 4, a result on the average of the periodic measure is given, which provides a way to approximate periodic measures numerically.

Chapter 2

Assumptions, backgrounds and preliminary results

In this and the chapter of strong approximation, we study stochastic differential equations, which possess random periodic solutions and approximate them by Euler-Maruyama and modified Milstein schemes. Consider the following m -dimensional SDE

$$\begin{cases} dX_t^{t_0} &= [AX_t^{t_0} + f(t, X_t^{t_0})]dt + g(t, X_t^{t_0})dW_t, \\ X_{t_0}^{t_0} &= \xi \end{cases}, \quad (2.0.1)$$

where $f : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^m, g : \mathbb{R} \times \mathbb{R}^m \rightarrow \mathbb{R}^{m \times d}$, A is a symmetric and negative-definite $m \times m$ matrix, W_t is a two-sided Wiener process in \mathbb{R}^d on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The filtration is defined as follows

$$\mathcal{F}_s^t = \sigma\{W_u - W_v : s \leq v \leq u \leq t\}, \quad \mathcal{F}^t = \mathcal{F}_{-\infty}^t = \bigvee_{s \leq t} \mathcal{F}_s^t,$$

the random variable ξ is \mathcal{F}^{t_0} -measurable. We assume that the functions f and g are τ -periodic in time. By the variation of constant formula, the solution of (2.0.1) is given

$$X_t^{t_0}(\xi) = e^{A(t-t_0)}\xi + e^{At} \int_{t_0}^t e^{-As} f(s, X_s^{t_0}) ds + e^{At} \int_{t_0}^t e^{-As} g(s, X_s^{t_0}) dW_s. \quad (2.0.2)$$

Denote the standard P -preserving ergodic Wiener shift by $\theta : \mathbb{R} \times \Omega \rightarrow \Omega$,

$$\theta_t(\omega)(s) := W(t+s) - W(t), \quad t, s \in \mathbb{R}.$$

The solution X of the non-autonomous SDE does not satisfy the cocycle property, but $u(t, t_0) : \Omega \times R^m \rightarrow R^m$ given by

$$u(t, t_0)\xi = X_t^{t_0}(\xi)$$

satisfies the semi-flow property (1.0.1) and periodicity (1.0.2).

Denote by $X_r^{-k\tau}(\xi, \omega)$ the solution starting from time $-k\tau$. Then we have for any $k > 0$, $r > -k\tau$ and $\mathcal{F}^{-k\tau}$ -measurable random variable ξ ,

$$X_r^{-k\tau} = e^{A(r+k\tau)}\xi + e^{Ar} \int_{-k\tau}^r e^{-As} f(s, X_s^{-k\tau}) ds + e^{Ar} \int_{-k\tau}^r e^{-As} g(s, X_s^{-k\tau}) dW_s. \quad (2.0.3)$$

We will show that when $k \rightarrow \infty$, the pull-back $X_r^{-k\tau}(\xi)$ has a limit X_r^* in $L^2(\Omega)$ and X_r^* is the random periodic solution of SDE (2.0.1). It satisfies the infinite horizon stochastic integral equation (IHSIE)

$$X_r^* = \int_{-\infty}^r e^{A(r-s)} f(s, X_s^*) ds + \int_{-\infty}^r e^{A(r-s)} g(s, X_s^*) dW_s.$$

We separate the linear term AX from the nonlinear term in (2.0.1) to enable us to represent the random periodic solution by IHSIE ([12], [14]). This is helpful to formulate the scheme for SPDEs for which random periodic solutions were considered in [13].

We fix some notation. Let $p \geq 1$ and denote the L^p -norm of a random variable ξ by

$$\|\xi\|_p = (\mathbb{E} |\xi|^p)^{\frac{1}{p}},$$

and the Frobenius norm of any $d_1 \times d_2$ matrix B by

$$|B| = \left(\sum_{i=1}^{d_1} \sum_{j=1}^{d_2} B_{ij}^2 \right)^{\frac{1}{2}}.$$

2.1 Conditions for the SDE

We assume the following conditions for our model.

Condition (A). *The eigenvalues of the symmetric matrix A , which we denote by $\{\lambda_j\}_{j=1,2,\dots,m}$, satisfy $0 > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$.*

Condition (1). Assume there exists a constant $\tau > 0$ such that for any $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $f(t + \tau, x) = f(t, x)$, $g(t + \tau, x) = g(t, x)$, and there exist constant $C_0, \beta_1, \beta_2 > 0$ with $\beta_1 + \frac{\beta_2^2}{2} < |\lambda_1|$ such that for any $s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^m$,

$$\begin{aligned} |f(s, x) - f(t, y)| &\leq C_0 |s - t|^{1/2} + \beta_1 |x - y|, \\ |g(s, x) - g(t, y)| &\leq C_0 |s - t|^{1/2} + \beta_2 |x - y|. \end{aligned}$$

Condition (2). There exists a constant $K^* > 0$ such that the initial condition ξ satisfies

$$\|\xi\|_2 \leq K^*.$$

From Condition (1) it follows that for any $x \in \mathbb{R}^m$, the linear growth condition also holds:

$$|f(t, x)| \leq \beta_1 |x| + C_1, \quad |g(t, x)| \leq \beta_2 |x| + C_2,$$

where the constants C_1, C_2 are strictly positive. It is easy to see that there exists a constant α such that

$$\beta_1 + \frac{\beta_2^2}{2} < \alpha < |\lambda_1|.$$

In the following two chapters, we always assume that α satisfies this condition in all the following proofs. Set $\rho := |\lambda_m|$, where λ_m is the eigenvalue with largest module. For the SDE case, this quantity is certainly finite and for simplicity, we choose numerical schemes to treat the linear part explicitly, which simplify the proof of the pull-back convergence to the random periodic solutions for the discretised systems. However, in a case of SPDEs, this technical assumption is no longer true, but can be removed by employing exponential Euler-Maruyama method and Milstein scheme ([2], [26]). This will be studied in future work.

2.2 Existence and uniqueness of random periodic solution

First recall the following lemmas proved in [52], which will be needed later.

Lemma 2.2.1. *Assume that the matrix A is symmetric and satisfies Condition (A). Then for any $\Delta t > 0$, the matrix*

$$e^{A\Delta t} - \sum_{i=0}^p \frac{1}{i!} (A\Delta t)^i$$

is positive-definite for odd $p \in \mathbb{N}$ and negative-definite for even $p \in \mathbb{N}$ and $p = 0$.

Lemma 2.2.2. *Assume that the matrix A is symmetric and satisfies Condition (A), and let ρ be as above. Then, for $0 < \Delta t \leq \frac{1}{\rho}$, the matrix*

$$e^{A\Delta t j} - (I + A\Delta t)^j$$

is positive-definite for any $j \in \mathbb{N}$.

The proofs of the above two lemmas are specified in appendix. Now we first consider the boundedness of the solution in $L^2(\Omega)$.

Lemma 2.2.3. *Assume Conditions (A), (1) and (2). Then there exists a constant $C > 0$ such that for any $k \in \mathbb{N}$, $r \geq -k\tau$, we have $\mathbb{E} |X_r^{-k\tau}|^2 \leq C$.*

Proof. First, using Itô's formula to $e^{2\alpha r} |X_r^{-k\tau}|^2$, we have

$$\begin{aligned} e^{2\alpha r} |X_r^{-k\tau}|^2 &= e^{-2\alpha k\tau} \|\xi\|^2 + 2\alpha \int_{-k\tau}^r e^{2\alpha s} |X_s^{-k\tau}|^2 ds + 2 \int_{-k\tau}^r e^{2\alpha s} (X_s^{-k\tau})^T A X_s^{-k\tau} ds \\ &\quad + 2 \int_{-k\tau}^r e^{2\alpha s} (X_s^{-k\tau})^T f(s, X_s^{-k\tau}) ds + \int_{-k\tau}^r e^{2\alpha s} |g(s, X_s^{-k\tau})|^2 ds \\ &\quad + 2 \int_{-k\tau}^r e^{2\alpha s} (X_s^{-k\tau})^T g(s, X_s^{-k\tau}) dW_s \end{aligned} \quad (2.2.1)$$

Firstly note the sum of the second and third terms of the right-hand side is non-positive as the matrix $(\alpha I + A)$ is non-positive-definite. Take the expectation of both sides of (2.2.1), apply the above inequality and use linear growth conditions to obtain

$$\begin{aligned} &e^{2\alpha r} \mathbb{E} |X_r^{-k\tau}|^2 \\ &\leq e^{-2\alpha k\tau} \|\xi\|_2^2 + 2 \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} \left[(X_s^{-k\tau})^T f(s, X_s^{-k\tau}) \right] ds \\ &\quad + \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} |g(s, X_s^{-k\tau})|^2 ds \end{aligned}$$

$$\begin{aligned}
 &\leq e^{-2\alpha k\tau} \|\xi\|_2^2 + (2\beta_1 + \beta_2^2) \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} |X_s^{-k\tau}|^2 ds \\
 &\quad + 2(C_1 + \beta_2 C_2) \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} |X_s^{-k\tau}| ds + (2\alpha)^{-1} C_2^2 (e^{2\alpha r} - e^{-2\alpha k\tau}). \quad (2.2.2)
 \end{aligned}$$

Also, there exists $\varepsilon > 0$, such that

$$\left(\beta_1 + \frac{\beta_2^2}{2} \right) (1 + \varepsilon) < \alpha < |\lambda_1|.$$

By Young's inequality

$$2(C_1 + \beta_2 C_2) |X_s^{-k\tau}| \leq \frac{(C_1 + \beta_2 C_2)^2}{\varepsilon(2\beta_1 + \beta_2^2)} + \varepsilon(2\beta_1 + \beta_2^2) |X_s^{-k\tau}|^2.$$

Then we have

$$e^{2\alpha r} \mathbb{E} |X_r^{-k\tau}|^2 \leq K_1 + K_2 e^{2\alpha r} + K_3 \int_{-k\tau}^r e^{2\alpha s} \|X_s^{-k\tau}\|_2^2 ds,$$

where

$$\begin{aligned}
 K_1 &= e^{-2\alpha k\tau} \|\xi\|_2^2 - \left(\frac{C_2^2}{2\alpha} + \frac{(C_1 + \beta_2 C_2)^2}{2\alpha\varepsilon(2\beta_1 + \beta_2^2)} \right) e^{-2\alpha k\tau}, \\
 K_2 &= \frac{C_2^2}{2\alpha} + \frac{(C_1 + \beta_2 C_2)^2}{2\alpha\varepsilon(2\beta_1 + \beta_2^2)}, \\
 K_3 &= (2\beta_1 + \beta_2^2)(1 + \varepsilon) < 2\alpha.
 \end{aligned}$$

Now applying Gronwall's inequality, we have

$$\begin{aligned}
 &e^{2\alpha r} \mathbb{E} |X_r^{-k\tau}|^2 \\
 &\leq K_1 + K_2 e^{2\alpha r} + \int_{-k\tau}^r (K_1 + K_2 e^{2\alpha s}) K_3 e^{\int_s^r K_3 dr} ds \\
 &= K_1 e^{K_3(r+k\tau)} + K_2 e^{2\alpha r} + \frac{K_2 K_3}{2\alpha - K_3} (e^{2\alpha r} - e^{K_3 r + (K_3 - 2\alpha)k\tau}) \\
 &\leq (K_1 e^{2\alpha k\tau} + K_2) e^{2\alpha r} + \frac{K_2 K_3}{2\alpha - K_3} e^{2\alpha r}.
 \end{aligned}$$

Here we notice that $K_1 e^{2\alpha k\tau} + K_2 = \|\xi\|_2^2$. Therefore, by Condition (2)

$$\mathbb{E} |X_r^{-k\tau}|^2 \leq \|\xi\|_2^2 + \frac{2\alpha K_2}{2\alpha - K_3} \leq K^* + \frac{2\alpha K_2}{2\alpha - K_3}. \quad \square$$

In the next lemma, we will also obtain a bound on the norm $\|X_{t_1}^{-k\tau} - X_{t_2}^{-k\tau}\|_2$ for any fixed time t_1, t_2 . This will be essential for us to estimate the error of the numerical approximation in Section 3.1.2.

Lemma 2.2.4. *Assume Conditions (A), (1) and (2). Then there exist constants $C_3 > 0$, $C_4 > 0$, such that for any positive $k \in \mathbb{N}$ and any $t_1, t_2 \geq 0, t_1 \geq t_2$, the solution of (2.0.1) satisfies*

$$\|X_{t_1}^{-k\tau} - X_{t_2}^{-k\tau}\|_2 \leq C_3(t_1 - t_2) + C_4\sqrt{t_1 - t_2}.$$

Proof. From (2.0.2), we see that

$$\begin{aligned} & \|X_{t_1}^{-k\tau} - X_{t_2}^{-k\tau}\|_2 \\ & \leq e^{2Ak\tau} \|\xi\|_2 |e^{At_1} - e^{At_2}| \\ & \quad + \left\| e^{At_1} \int_{-k\tau}^{t_1} e^{-As} g(s, X_s^{-k\tau}) dW_s - e^{At_2} \int_{-k\tau}^{t_2} e^{-As} g(s, X_s^{-k\tau}) dW_s \right\|_2 \\ & \quad + \left\| e^{At_1} \int_{-k\tau}^{t_1} e^{-As} f(s, X_s^{-k\tau}) ds - e^{At_2} \int_{-k\tau}^{t_2} e^{-As} f(s, X_s^{-k\tau}) ds \right\|_2. \end{aligned} \quad (2.2.3)$$

We evaluate each term on the right-hand side of (2.2.3). Now we consider the first term with Lemma 2.2.1,

$$\begin{aligned} & |e^{At_1} - e^{At_2}| \\ & = \sqrt{\text{Tr} \left(e^{2At_2} (I - e^{A(t_1-t_2)})^2 \right)} \\ & \leq \sqrt{\text{Tr} \left((I - e^{A(t_1-t_2)})^2 \right)} \\ & = \sqrt{\text{Tr} \left(A^2 (t_1 - t_2)^2 + (e^{A(t_1-t_2)} - I - A(t_1 - t_2)) (e^{A(t_1-t_2)} - I + A(t_1 - t_2)) \right)} \\ & \leq \sqrt{\text{Tr} \left(A^2 (t_1 - t_2)^2 \right)} \\ & = |A| (t_1 - t_2). \end{aligned}$$

Then we estimate the second term with the Minkowski inequality, Itô's isometry and the linear growth property

$$\begin{aligned} & \left\| e^{At_1} \int_{-k\tau}^{t_1} e^{-As} g(s, X_s^{-k\tau}) dW_s - e^{At_2} \int_{-k\tau}^{t_2} e^{-As} g(s, X_s^{-k\tau}) dW_s \right\|_2 \\ & \leq \left\| \int_{-k\tau}^{t_2} (e^{At_1} - e^{At_2}) e^{-As} g(s, X_s^{-k\tau}) dW_s \right\|_2 + \left\| \int_{t_2}^{t_1} e^{-A(s-t_1)} g(s, X_s^{-k\tau}) dW_s \right\|_2 \\ & \leq \sqrt{\int_{-k\tau}^{t_2} |(e^{At_1} - e^{At_2}) e^{-As}|^2 \mathbb{E} [\beta_2 (|X_s^{-k\tau}|) + C_2]^2 ds} \end{aligned}$$

$$\begin{aligned}
 & + \sqrt{\int_{t_2}^{t_1} |e^{-A(s-t_1)}|^2 \mathbb{E} [\beta_2 (|X_s^{-k\tau}|) + C_2]^2 ds} \\
 \leq & \sqrt{\int_{-k\tau}^{t_2} |(e^{At_1} - e^{At_2}) e^{-As}|^2 \left(2\beta_2^2 \mathbb{E} |X_s^{-k\tau}|^2 + 2C_2^2 \right) ds} \\
 & + \sqrt{\int_{t_2}^{t_1} |e^{-A(s-t_1)}|^2 \left(2\beta_2^2 \mathbb{E} |X_s^{-k\tau}|^2 + 2C_2^2 \right) ds} \\
 \leq & K_4 \sqrt{\int_{-k\tau}^{t_2} |(e^{At_1} - e^{At_2}) e^{-As}|^2 ds} + K_4 \sqrt{\int_{t_2}^{t_1} |e^{-A(s-t_1)}|^2 ds}.
 \end{aligned}$$

Here we take some constant K_4 because $\mathbb{E} |X_s^{-k\tau}|^2$ is bounded above according to Lemma 2.2.3. So we just need to consider the remaining part

$$\begin{aligned}
 & \int_{-k\tau}^{t_2} |(e^{At_1} - e^{At_2}) e^{-As}|^2 ds \\
 = & Tr \left(\int_{-k\tau}^{t_2} e^{-2A(s-t_2)} (I - e^{A(t_1-t_2)})^2 ds \right) \\
 = & Tr \left((-2A)^{-1} (I - e^{2A(k\tau+t_2)}) (I - e^{A(t_1-t_2)})^2 \right) \\
 \leq & Tr \left((-2A)^{-1} (-A(t_1 - t_2) - (e^{A(t_1-t_2)} - I - A(t_1 - t_2)))^2 \right) \\
 \leq & Tr(-A) \frac{(t_1 - t_2)^2}{2}.
 \end{aligned}$$

Using the similar method to get

$$\begin{aligned}
 & \int_{t_2}^{t_1} |e^{-A(s-t_1)}|^2 ds \\
 = & Tr \left(\int_{t_2}^{t_1} e^{-2A(s-t_1)} ds \right) \\
 = & Tr \left((-2A)^{-1} (I - e^{2A(t_1-t_2)}) \right) \\
 = & Tr \left((-2A)^{-1} (-2A(t_1 - t_2) - (e^{2A(t_1-t_2)} - I - 2A(t_1 - t_2))) \right) \\
 \leq & (t_1 - t_2).
 \end{aligned}$$

Therefore we have

$$\begin{aligned}
 & \left\| e^{At_1} \int_{-k\tau}^{t_1} e^{-As} g(s, X_s^{-k\tau}) dW_s - e^{At_2} \int_{-k\tau}^{t_2} e^{-As} g(s, X_s^{-k\tau}) dW_s \right\|_2 \\
 \leq & K_4 \sqrt{\frac{Tr(-A)}{2}} (t_1 - t_2) + K_4 \sqrt{t_1 - t_2}.
 \end{aligned}$$

Lastly, we consider the third term of (2.2.3) with Minkowski inequality

$$\begin{aligned}
 & \left\| e^{At_1} \int_{-k\tau}^{t_1} e^{-As} f(s, X_s^{-k\tau}) ds - e^{At_2} \int_{-k\tau}^{t_2} e^{-As} f(s, X_s^{-k\tau}) ds \right\|_2 \\
 & \leq \left\| \int_{-k\tau}^{t_2} (e^{At_1} - e^{At_2}) e^{-As} f(s, X_s^{-k\tau}) ds \right\|_2 + \left\| \int_{t_2}^{t_1} e^{-A(s-t_1)} f(s, X_s^{-k\tau}) ds \right\|_2 \\
 & \leq \int_{-k\tau}^{t_2} \|(e^{At_1} - e^{At_2}) e^{-As} f(s, X_s^{-k\tau})\|_2 ds + \int_{t_2}^{t_1} \|e^{-A(s-t_1)} f(s, X_s^{-k\tau})\|_2 ds \\
 & \leq \int_{-k\tau}^{t_2} |(e^{At_1} - e^{At_2}) e^{-As}| \|f(s, X_s^{-k\tau})\|_2 ds + \int_{t_2}^{t_1} |e^{-A(s-t_1)}| \|f(s, X_s^{-k\tau})\|_2 ds \\
 & \leq K_5 \left(\int_{-k\tau}^{t_2} |(e^{At_1} - e^{At_2}) e^{-As}| ds + \int_{t_2}^{t_1} |e^{-A(s-t_1)}| ds \right) \\
 & \leq 2K_5(t_1 - t_2),
 \end{aligned}$$

for a constant $K_5 > 0$. Combining the above estimates we obtain the lemma with the constants C_3, C_4 being independent of k and t_1, t_2 . \square

Now we continue to consider the difference of the solutions under various initial values. For simplicity, we here study two different initial values ξ and η .

Lemma 2.2.5. *Denote by $X_r^{-k\tau}$ and $Y_r^{-k\tau}$ two solutions of (2.0.1) with different initial values ξ and η respectively. Assume Conditions (A), (1) and Condition (2) for both initial values. Then*

$$\left\| X_r^{-k\tau} - Y_r^{-k\tau} \right\|_2 \leq e^{\left(\beta_1 + \frac{\beta_2^2}{2} - \alpha\right)(r+k\tau)} \|\xi - \eta\|_2.$$

Proof. According to (2.0.2) we have

$$\begin{aligned}
 & X_r^{-k\tau} - Y_r^{-k\tau} \\
 & = e^{A(r+k\tau)} (\xi - \eta) + e^{Ar} \int_{-k\tau}^r e^{-As} (f(s, X_s^{-k\tau}) - f(s, Y_s^{-k\tau})) ds \\
 & \quad + e^{Ar} \int_{-k\tau}^r e^{-As} (g(s, X_s^{-k\tau}) - g(s, Y_s^{-k\tau})) dW_s.
 \end{aligned}$$

For simplicity, denote $\zeta_r^{-k\tau} = X_r^{-k\tau} - Y_r^{-k\tau}$. Then according to the method used in Lemma 2.2.3, and the global Lipschitz condition, we have

$$e^{2\alpha r} \left\| \zeta_r^{-k\tau} \right\|_2^2$$

$$\begin{aligned}
&\leq e^{-2\alpha k\tau} \|\xi - \eta\|_2^2 + 2 \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} \left[(\zeta_s^{-k\tau})^T (f(s, X_s^{-k\tau}) - f(s, Y_s^{-k\tau})) \right] ds \\
&\quad + \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} |g(s, X_s^{-k\tau}) - g(s, Y_s^{-k\tau})|^2 ds. \\
&\leq e^{-2\alpha k\tau} \|\xi - \eta\|_2^2 + 2 \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} \left[\beta_1 |\zeta_s^{-k\tau}|^2 \right] ds + \int_{-k\tau}^r e^{2\alpha s} \mathbb{E} \left[\beta_2^2 |\zeta_s^{-k\tau}|^2 \right] ds \\
&\leq e^{-2\alpha k\tau} \|\xi - \eta\|_2^2 + (2\beta_1 + \beta_2^2) \int_{-k\tau}^r e^{2\alpha s} \|\zeta_s^{-k\tau}\|_2^2 ds.
\end{aligned}$$

Then applying the Gronwall inequality to have

$$e^{2\alpha r} \|\zeta_r^{-k\tau}\|_2^2 \leq e^{-2\alpha k\tau} \|\xi - \eta\|_2^2 e^{(2\beta_1 + \beta_2^2)(r+k\tau)}.$$

Therefore

$$\|X_r^{-k\tau} - Y_r^{-k\tau}\|_2 \leq e^{\left(\beta_1 + \frac{\beta_2^2}{2} - \alpha\right)(r+k\tau)} \|\xi - \eta\|_2.$$

□

Now we can prove the following theorem.

Theorem 2.2.6. *Assume Conditions (A), (1). Then there exists a unique random periodic solution $X^*(r, \cdot) \in L^2(\Omega)$, $r \geq 0$ such that for any fixed initial value ξ , the solution of (2.0.1) satisfies*

$$\lim_{k \rightarrow \infty} \|X_r^{-k\tau}(\xi) - X^*(r)\|_2 = 0.$$

Proof. Condition (2) implies that the initial value ξ belongs to $L^2(\Omega)$. According to Lemma 2.2.3, $X_r^{-k\tau}(\cdot)$ maps $L^2(\Omega)$ to itself. Now we use the semi-flow property to get that for any $r, k, p \geq 0$,

$$X_r^{-k\tau-p\tau}(\xi) = X_r^{-k\tau}(\omega) \circ X_{-k\tau}^{-(k+p)\tau}(\omega, \xi).$$

Thus we can apply Lemma 2.2.5 to have for any $\varepsilon > 0$ there exists $k^* > 0$ such that for any $k \geq k^*$,

$$\|X_r^{-k\tau}(\xi) - X_r^{-(k+p)\tau}(\xi)\|_2 < \varepsilon.$$

This means that there exists $N > 0$ such that for any $l, m \geq N$, we have

$$\|X_r^{-l\tau}(\xi) - X_r^{-m\tau}(\xi)\|_2 < \varepsilon,$$

i.e. $\{X_r^{-k\tau}(\xi)\}_{k \in \mathbb{N}}$ is a Cauchy sequence, so converges to some $X^*(r, \omega)$ in $L^2(\Omega)$, when $k \rightarrow \infty$.

If we set $u(t, r)(\xi) = X_t^r(\xi)$, then $u(t, r) : \Omega \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ defines a semi-flow of homeomorphism (Kunita [31]). By the continuity of $X_t^r(\omega) : L^2(\Omega, \mathbb{R}^m) \rightarrow L^2(\Omega, \mathbb{R}^m)$, $t \geq r$, then

$$u(t, r, \omega) (X_r^{-k\tau}(\xi, \omega)) \xrightarrow[L^2(\Omega)]{k \rightarrow \infty} u(t, r, \omega) \circ (X^*(r, \omega)).$$

But

$$u(t, r, \omega) (X_r^{-k\tau}(\xi, \omega)) = X_t^{-k\tau}(\xi, \omega) \xrightarrow[L^2(\Omega)]{k \rightarrow \infty} X^*(t, \omega).$$

So $u(t, r, \omega) (X^*(r, \omega)) = X^*(t, \omega)$, $\mathbb{P} - a.s.$

Taking some other initial value η satisfying Condition (2), we have

$$\|X_r^* - X_r^{-k\tau}(\eta)\|_2 \leq \|X_r^* - X_r^{-k\tau}(\xi)\|_2 + \|X_r^{-k\tau}(\xi) - X_r^{-k\tau}(\eta)\|_2.$$

Applying Lemma 2.2.5 again, we can make the right-hand side small enough when $k \rightarrow \infty$. Therefore the convergence is independent of the initial value.

Now we need to prove the random periodicity of the $X^*(r, \omega)$. Note by the continuity of f and g ,

$$\begin{aligned} & X_{r+\tau}^{-(k-1)\tau}(\xi) \\ &= e^{A(r+k\tau)}\xi + \int_{-(k-1)\tau}^{r+\tau} e^{A(r+\tau-s)} f(s, X_s^{-(k-1)\tau}(\xi)) ds \\ & \quad + \int_{-(k-1)\tau}^{r+\tau} e^{A(r+\tau-s)} g(s, X_s^{-(k-1)\tau}(\xi)) dW_s \\ &= e^{A(r+k\tau)}\xi + \int_{-k\tau}^r e^{A(r-s)} f(s+\tau, X_{s+\tau}^{-(k-1)\tau}(\xi)) ds \\ & \quad + \int_{-k\tau}^r e^{A(r-s)} g(s+\tau, X_{s+\tau}^{-(k-1)\tau}(\xi)) d\widetilde{W}_s \\ &= e^{A(r+k\tau)}\xi + \int_{-k\tau}^r e^{A(r-s)} [f(s, X_{s+\tau}^{-(k-1)\tau}(\xi)) ds + g(s, X_{s+\tau}^{-(k-1)\tau}(\xi)) d\widetilde{W}_s]. \end{aligned}$$

where $\widetilde{W}_s := (\theta_\tau \omega)(s) = W_{s+\tau} - W_\tau$. On the other hand,

$$\theta_\tau X_{-k\tau}^r(\xi) = e^{A(r+k\tau)}\theta_\tau \xi + \int_{-k\tau}^r e^{A(r-s)} [f(s, \theta_\tau X_s^{-k\tau}) ds + g(s, \theta_\tau X_s^{-k\tau}) d\widetilde{W}_s],$$

By pathwise uniqueness of the solution of (2.0.1), we have

$$X_r^{-k\tau}(\theta_\tau \omega, \xi(\theta_\tau \omega)) = \theta_\tau X_r^{-k\tau}(\xi) = X_{r+\tau}^{-(k-1)\tau}(\omega, \xi(\omega)). \quad (2.2.4)$$

From the proof of convergence we have

$$\begin{aligned} X_{r+\tau}^{-(k-1)\tau}(\omega, \xi) &\xrightarrow[L^2(\Omega)]{k \rightarrow \infty} X^*(r + \tau, \omega), \\ X_r^{-k\tau}(\theta_\tau \omega, \xi(\theta_\tau \omega)) &\xrightarrow[L^2(\Omega)]{k \rightarrow \infty} X^*(r, \theta_\tau \omega). \end{aligned}$$

Therefore

$$X^*(r + \tau, \omega) = X^*(r, \theta_\tau \omega), \quad \mathbb{P} - a.s.$$

□

2.3 Numerical approximations

To consider the numerical approximation for stochastic differential equations, there are some fundamental results including Kloeden and Platen [29], Milstein [36], Milstein and Tretyakov [37]. The Taylor expansions are key to numerical approximation of solutions of ODEs. As a generalization of deterministic Taylor expansions as well as Itô formula, the stochastic Taylor expansions allow various kinds of approximations of functionals of diffusion processes. Therefore we start by investigating stochastic Taylor expansions and their applications.

2.3.1 Stochastic Taylor expansions

The introduction of stochastic Taylor expansions follows the layout of Kloeden and Platen [29]. When considering the solution X_t of a 1-dimensional ordinary differential equation

$$\frac{d}{dt} X_t = f(X_t),$$

with initial value X_{t_0} , for $t \in [t_0, T]$, $0 \leq t_0 < T$, the equivalent integral equation form is

$$X_t = X_{t_0} + \int_{t_0}^t f(X_s) ds. \quad (2.3.1)$$

We will require the function $f : \mathbb{R} \rightarrow \mathbb{R}$ to be sufficiently smooth and to have a linear growth to justify the following constructions. The chain rule is applied on such function f ,

$$\frac{d}{dt} f(X_t) = \frac{\partial}{\partial x} f(X_t) \frac{d}{dt} X_t,$$

or in integral form

$$f(X_t) = f(X_{t_0}) + \int_{t_0}^t \frac{\partial}{\partial x} f(X_s) \frac{d}{ds} X_s ds. \quad (2.3.2)$$

Then the equation (2.3.1) becomes

$$\begin{aligned} X_t &= X_{t_0} + \int_{t_0}^t \left(f(X_{t_0}) + \int_{t_0}^s f(X_z) \frac{\partial}{\partial x} f(X_z) dz \right) ds \\ &= X_{t_0} + f(X_{t_0}) \int_{t_0}^t ds + \int_{t_0}^t \int_{t_0}^s f(X_z) \frac{\partial}{\partial x} f(X_z) dz ds \end{aligned}$$

This simplest Taylor expansion can be continued by applying (2.3.2) to $f(X_z) \frac{\partial}{\partial x} f(X_z)$ and so on. Denote by $L = f \frac{\partial}{\partial x}$. Then for a general $r + 1$ times continuously differentiable function $f : \mathbb{R} \rightarrow \mathbb{R}$, this method gives the classical Taylor formula in integral form:

$$X_t = X_{t_0} + \sum_{i=1}^r \frac{(t - t_0)^i}{i!} L^i f(X_{t_0}) + \int_{t_0}^t \dots \int_{t_0}^{s_2} L^{r+1} X_{s_1} ds_1 \dots ds_{r+1},$$

for $t \in [t_0, T]$ and $r = 1, 2, 3, \dots$. The Taylor formula is a very useful tool in theoretical and practical investigations, particular in numerical analysis. This expansion depends on the values of the function and some of its higher derivatives at the expansion point, weighted by corresponding multiple time integrals. The remainder term contains the next multiple time integral. With sufficiently smooth function in a neighbourhood of a given point, we have the approximation with the desired order of accuracy. There are many possibilities to extend the Taylor expansions, one important and direct way of extension is based on the iteration of application of Itô formula (analogue of chain rule in deterministic case), which is known as Itô-Taylor expansions.

Suppose X_t is the solution of the one-dimensional Itô stochastic differential equation in integral form

$$X_t = X_{t_0} + \int_{t_0}^t f(X_s) ds + \int_{t_0}^t g(X_s) dW_s,$$

for $t_0 \leq t \leq T$ and $f, g : \mathbb{R} \rightarrow \mathbb{R}$. With sufficient smoothness and linear bound of functions f and g , we apply the Itô formula to these functions:

$$X_t = X_{t_0} + \int_{t_0}^t \left(f(X_{t_0}) + \int_{t_0}^s L^0 f(X_z) dz + \int_{t_0}^s L^1 f(X_z) dW_z \right) ds$$

$$\begin{aligned}
& + \int_{t_0}^t \left(g(X_{t_0}) + \int_{t_0}^s L^0 g(X_z) dz + \int_{t_0}^s L^1 g(X_z) dW_z \right) dW_s \\
& = X_{t_0} + f(X_{t_0}) \int_{t_0}^t ds + g(X_{t_0}) \int_{t_0}^t dW_s + R,
\end{aligned} \tag{2.3.3}$$

where

$$\begin{aligned}
L^0 &= f \frac{\partial}{\partial x} + \frac{1}{2} g^2 \frac{\partial^2}{\partial x^2}, \\
L^1 &= g \frac{\partial}{\partial x},
\end{aligned}$$

and

$$\begin{aligned}
R &= \int_{t_0}^t \int_{t_0}^s L^0 f(X_z) dz ds + \int_{t_0}^t \int_{t_0}^s L^1 f(X_z) dW_z ds \\
&+ \int_{t_0}^t \int_{t_0}^s L^0 g(X_z) dz dW_s + \int_{t_0}^t \int_{t_0}^s L^1 g(X_z) dW_z dW_s.
\end{aligned}$$

This is the simplest non-trivial Itô-Taylor expansion. As in deterministic case, the process can be continued by applying the Itô formula to $L^1 g$ to obtain

$$\begin{aligned}
X_t &= X_{t_0} + f(X_{t_0}) \int_{t_0}^t ds + g(X_{t_0}) \int_{t_0}^t dW_s \\
&+ L^1 g(X_{t_0}) \int_{t_0}^t \int_{t_0}^s dW_z dW_s + \bar{R}
\end{aligned} \tag{2.3.4}$$

with the remainder

$$\begin{aligned}
\bar{R} &= \int_{t_0}^t \int_{t_0}^s L^0 f(X_z) dz ds + \int_{t_0}^t \int_{t_0}^s L^1 f(X_z) dW_z ds \\
&+ \int_{t_0}^t \int_{t_0}^s L^0 g(X_z) dz dW_s + \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^0 L^1 g(X_u) du dW_z dW_s \\
&+ \int_{t_0}^t \int_{t_0}^s \int_{t_0}^z L^1 L^1 g(X_u) dW_u dW_z dW_s
\end{aligned}$$

These two Itô-Taylor expansions give us the famous numerical schemes, Euler-Maruyama scheme and Milstein scheme, for the numerical approximation of stochastic differential equations. We quote the book of Kloeden and Platen [29] to present these two schemes.

2.3.2 Euler-Maruyama scheme

The Euler-Maruyama scheme represent the simplest Itô-Taylor expansion (2.3.3). For the one-dimensional stochastic differential equation

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t,$$

the equidistant Euler-Maruyama scheme divide $[0, T]$ into N intervals with length Δt , therefore we have $T = N\Delta t$. The iteration formula with initial condition $Y_0 = X_0$ is

$$Y_{n+1} = Y_n + f(n\Delta t, Y_n)\Delta t + g(n\Delta t, Y_n)\Delta W_n,$$

for $i = 0, \dots, N - 1$. Here $\Delta W_n = W_{(i+1)\Delta t} - W_{i\Delta t}$ is the Brownian motion increments. If we denote the numerical approximation by $Y_t^{\Delta t}$, then the order strong convergence is given in the following theorem:

Theorem 2.3.1. [29] *Suppose that*

$$\mathbb{E}(|X_0|^2) < \infty, \tag{2.3.5}$$

$$\mathbb{E} \left(|X_0 - Y_0^{\Delta t}|^2 \right)^{1/2} \leq K_1(\Delta t)^{1/2}, \tag{2.3.6}$$

$$|f(t, x) - f(t, y)| + |g(t, x) - g(t, y)| \leq K_2 |x - y|, \tag{2.3.7}$$

$$|f(t, x)| + |g(t, x)| \leq K_3(1 + |x|), \tag{2.3.8}$$

and

$$|f(s, x) - f(t, x)| + |g(s, x) - g(t, x)| \leq K_4(1 + |x|) |s - t|^{1/2} \tag{2.3.9}$$

for all $s, t \in [0, T]$ and $x, y \in \mathbb{R}$, where the constants K_1, \dots, K_4 do not depend on Δt . Then there exists a positive constant K_5 , independent of Δt , such that the Euler-Maruyama approximation $Y_t^{\Delta t}$ satisfies:

$$\mathbb{E} \left(|X_T - Y_T^{\Delta t}| \right) \leq K_5(\Delta t)^{1/2}.$$

The proof of this theorem firstly derives the boundedness of the second moment of the process X_t for any $0 \leq t \leq T$ as well as that of the approximation process $Y_t^{\Delta t}$, where $Y_t^{\Delta t}$ is constructed piecewise between the discretization points on time t . Then the error criterion appeals in the both side of inequality, which gives us the result with the Gronwall inequality.

Remark 2.3.2. *The constant K_5 in the previous theorem involves an exponential function of time T , which comes from the Gronwall inequality. When we consider the infinite horizon problem, it becomes a problem.*

2.3.3 Milstein scheme

For the order of accuracy, a more efficient method was developed originally by Milstein [35]. There are also many sources that can be found in Milstein [36], Milstein and Tretyakov [37], Kloeden and Platen [29]. The Milstein scheme represent the Itô-Taylor expansions (2.3.4). The construction for each steps are as follows:

$$Y_{i+1} = Y_i + f(i\Delta t, Y_i)\Delta t + g(i\Delta t, Y_i)\Delta W_i + \frac{1}{2}g(i\Delta t, Y_i)g'(i\Delta t, Y_i)((\Delta W_i)^2 - \Delta t),$$

for $i = 0, \dots, N - 1$ and $\Delta W_i = W_{(i+1)\Delta t} - W_{i\Delta t}$. It is well known that it is difficult to approximate the multi-dimensional Brownian motion increment. Therefore we mainly consider the diagonal noise. Also the higher order smoothness and Lipschitz continuity are required for Milstein scheme. Here we quote the result as follows:

Theorem 2.3.3. [29] *Suppose that*

$$\mathbb{E}(|X_0|^2) < \infty, \quad (2.3.10)$$

$$\mathbb{E}\left(|X_0 - Y_0^{\Delta t}|^2\right)^{1/2} \leq K_1(\Delta t)^{1/2}, \quad (2.3.11)$$

$$\begin{aligned} |f(t, x) - f(t, y)| &\leq K_2|x - y|, \\ |g(t, x) - g(t, y)| &\leq K_2|x - y|, \\ |L^1g(t, x) - L^1g(t, y)| &\leq K_2|x - y|, \end{aligned} \quad (2.3.12)$$

$$|f(t, x)| + |L^j f(t, x)| \leq K_3(1 + |x|),$$

$$\begin{aligned}
|g(t, x)| + |L^j g(t, x)| &\leq K_3(1 + |x|), \\
|L^j L^0 g(t, x)| &\leq K_3(1 + |x|),
\end{aligned} \tag{2.3.13}$$

and

$$\begin{aligned}
|f(s, x) - f(t, x)| &\leq K_4(1 + |x|) |s - t|^{1/2}, \\
|g(s, x) - g(t, x)| &\leq K_4(1 + |x|) |s - t|^{1/2}, \\
|L^1 g(s, x) - L^1 g(t, x)| &\leq K_4(1 + |x|) |s - t|^{1/2},
\end{aligned} \tag{2.3.14}$$

for all $s, t \in [0, T]$, and $x, y \in \mathbb{R}$, $j = 0, 1$, where the constants K_1, \dots, K_4 do not depend on Δt .

Then for the Milstein approximation $Y^{\Delta t}$, the estimate

$$\mathbb{E} |X_T - Y_T^{\Delta t}| \leq K_5 \Delta t$$

holds, where the constant K_5 does not depend on Δt .

The additional conditions compared with Euler-Maruyama scheme guaranteed the required order of local error and the boundedness of corresponding coefficients. Gronwall inequality is the main tool to accomplish the proof of the theorem.

Remark 2.3.4. *When considering the infinite horizon problem, we need to modify the scheme by borrowing terms from higher order scheme to satisfy the required order. Therefore the corresponding assumption on function f , like the Lipschitz continuity and linear growth of the terms $L^j f(t, x)$, $j = 0, 1$, would be necessary.*

Chapter 3

Strong Approximations

3.1 Euler-Maruyama scheme

3.1.1 Numerical approximation for random periodic solution

In this section, we will introduce the basic Euler-Maruyama method to approximate the solution on infinite horizon. Take $\Delta t = \tau/n$, which will be taken to be sufficiently small such that $\Delta t \leq \frac{1}{\rho}$, for some $n \in \mathbb{N}$, in the remaining part of the thesis. Let $N = kn$. The time domain from time $-k\tau$ to time 0 is divided into N intervals of length Δt such that $N\Delta t = k\tau$. The scheme starts from an $\mathcal{F}^{-k\tau}$ -measurable random variable ξ at a time $-k\tau$. At each of the points $i\Delta t$ we set the value $\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}$ with the iteration formula

$$\begin{aligned}\widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} &= \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + A\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\Delta t + f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\Delta t \\ &\quad + g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) (W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t}),\end{aligned}\tag{3.1.1}$$

where $i = 0, 1, 2, \dots$, and $\widehat{X}_{-k\tau+0\Delta t}^{-k\tau} = \xi$.

It is easy to see that for any $M \geq 0$,

$$\begin{aligned}\widehat{X}_{-k\tau+M\Delta t}^{-k\tau} &= (I + A\Delta t)^M \xi + \Delta t \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \\ &\quad + \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) (W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t}).\end{aligned}\tag{3.1.2}$$

Moreover, we can set up a discrete semi-flow given by recalling the standard P -preserving ergodic Wiener shift $\theta_t(\omega)(s) := W(t+s) - W(t)$, $t, s \in \mathbb{R}$,

$$\hat{u}_{i,j}(\xi) = \hat{X}_{i\Delta t}^{j\Delta t}(\xi), \quad i \geq j, \quad i, j \in \{-kn, -kn+1, \dots\}, \quad \hat{\theta} = \theta_{\Delta t}, \quad \hat{\theta}^n = \hat{\theta}\hat{\theta} \dots \hat{\theta}.$$

By the scheme (3.1.2), we have

$$\begin{aligned} \hat{u}_{j,l}(\xi) &= (I + A\Delta t)^{j-l}\xi + \Delta t \sum_{m=l}^{j-1} (I + A\Delta t)^{j-m-1} f(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) \\ &\quad + \sum_{m=l}^{j-1} (I + A\Delta t)^{j-m-1} g(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) (W_{(m+1)\Delta t} - W_{m\Delta t}). \end{aligned}$$

And then,

$$\begin{aligned} &\hat{u}_{i,j} \circ \hat{u}_{j,l}(\xi) \\ &= (I + A\Delta t)^{i-j} (I + A\Delta t)^{j-l}\xi + \Delta t (I + A\Delta t)^{i-j} \sum_{m=l}^{j-1} (I + A\Delta t)^{j-m-1} f(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) \\ &\quad + (I + A\Delta t)^{i-j} \sum_{m=l}^{j-1} (I + A\Delta t)^{j-m-1} g(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) (W_{(m+1)\Delta t} - W_{m\Delta t}) \\ &\quad + \Delta t \sum_{m=j}^{i-1} (I + A\Delta t)^{i-m-1} f(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) \\ &\quad + \sum_{m=j}^{i-1} (I + A\Delta t)^{i-m-1} g(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) (W_{(m+1)\Delta t} - W_{m\Delta t}). \\ &= (I + A\Delta t)^{i-l}\xi + \Delta t \sum_{m=l}^{i-1} (I + A\Delta t)^{i-m-1} f(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) \\ &\quad + \sum_{m=l}^{i-1} (I + A\Delta t)^{i-m-1} g(m\Delta t, \hat{X}_{m\Delta t}^{l\Delta t}) (W_{(m+1)\Delta t} - W_{m\Delta t}) \\ &= \hat{u}_{i,l}(\xi) \end{aligned}$$

Therefore we proved that u satisfies the semi-flow property

$$\hat{u}_{i,j}(\omega) \circ \hat{u}_{j,l}(\omega) = \hat{u}_{i,l}(\omega), \quad \text{for } i \geq j \geq l.$$

Now we consider

$$\hat{u}_{i+n,j+n}(\omega)$$

$$\begin{aligned}
&= (I + A\Delta t)^{i-j}\xi + \Delta t \sum_{m=j+n}^{i+n-1} (I + A\Delta t)^{i+n-m-1} f(m\Delta t, \widehat{X}_{m\Delta t}^{(j+n)\Delta t}) \\
&\quad + \sum_{m=j+n}^{i+n-1} (I + A\Delta t)^{i+n-m-1} g(m\Delta t, \widehat{X}_{m\Delta t}^{(j+n)\Delta t}) (W_{(m+1)\Delta t} - W_{m\Delta t}) \\
&= (I + A\Delta t)^{i-j}\xi + \Delta t \sum_{m=j}^{i-1} (I + A\Delta t)^{i-m-1} f(m\Delta t, \widehat{\theta}^n \widehat{X}_{m\Delta t}^{(j)\Delta t}) \\
&\quad + \sum_{m=j}^{i-1} (I + A\Delta t)^{i-m-1} g(m\Delta t, \widehat{\theta}^n \widehat{X}_{m\Delta t}^{(j)\Delta t}) (W_{(m+n+1)\Delta t} - W_{(m+n)\Delta t}) \\
&= \widehat{u}_{i,j}(\widehat{\theta}^n \omega).
\end{aligned}$$

That is the periodic property of u ,

$$\widehat{u}_{i+n,j+n}(\omega) = \widehat{u}_{i,j}(\widehat{\theta}^n \omega), \text{ for } i \geq j.$$

In order to prove the convergence of the discretized semi-flow to a random periodic solution, we first derive some similar estimates as in Lemma 2.2.3 and Lemma 2.2.5. Then a discrete analogue of Theorem 2.2.6 will give us the result.

Lemma 3.1.1. *Assume Conditions (A), (1) and (2). Then there exists a constant $\widehat{C} > 0$ such that for any natural numbers $k \geq 0$, $M \geq 0$, and sufficiently small Δt , the numerical solution $\widehat{X}_{-k\tau+M\Delta t}^{-k\tau}$ defined by (3.1.2) satisfies*

$$\mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \leq \widehat{C}.$$

Proof. We still choose α such that $\beta_1 + \frac{\beta^2}{2} < \alpha < |\lambda_1|$. Then for any $M \geq 0$,

$$\begin{aligned}
&(1 - \alpha\Delta t)^{-2M} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \\
&= |\xi|^2 + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(\frac{\left| \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \right). \quad (3.1.3)
\end{aligned}$$

This is not hard to verify by expanding the sum and noting cancellations. Notice that

$$\frac{\left| \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2$$

$$\begin{aligned}
&= \left(\left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} - I \right) + \frac{\Delta t}{1 - \alpha\Delta t} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \right. \\
&\quad \left. + \frac{(W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t})^T g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T}{1 - \alpha\Delta t} \right) \\
&\quad \times \left(\left(\frac{I + A\Delta t}{1 - \alpha\Delta t} + I \right) \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + \frac{\Delta t}{1 - \alpha\Delta t} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right. \\
&\quad \left. + \frac{g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) (W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t})}{1 - \alpha\Delta t} \right). \quad (3.1.4)
\end{aligned}$$

Note $\left(\frac{I+A\Delta t}{1-\alpha\Delta t} - I\right) \left(\frac{I+A\Delta t}{1-\alpha\Delta t} + I\right)$ is non-positive definite, where Δt satisfies $0 < \Delta t \leq \frac{1}{\rho}$ as defined before, and for each i , $f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})$ and $g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})$ are both independent of $(W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t})$. Take expectation on both sides of (3.1.3), consider (3.1.4), apply the linear growth property and Young's inequality to have

$$\begin{aligned}
&(1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \\
&\leq \|\xi\|_2^2 + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(\frac{\Delta t}{1 - \alpha\Delta t} \right)^2 \mathbb{E} \left| f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
&\quad + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
&\quad + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{2\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left[\left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T (I + A\Delta t) f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right] \\
&\leq \widehat{K}_1 + (1 - \alpha\Delta t)^{-2M} \widehat{K}_2 + \widehat{K}_3 \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \mathbb{E} \left| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2, \quad (3.1.5)
\end{aligned}$$

where,

$$\begin{aligned}
\widehat{K}_1 &= \|\xi\|_2^2, \\
\widehat{K}_2 &= \frac{C_1^2 (\Delta t)^2 + C_2^2 \Delta t}{2\alpha\Delta t - \alpha^2 (\Delta t)^2} + \frac{\Delta t}{2\alpha\Delta t - \alpha^2 (\Delta t)^2} \frac{(C_1 + \beta_2 C_2 + \Delta t C_1 (\beta_1 + |A|))^2}{\widehat{\varepsilon} (2\beta_1 + \beta_2^2 + \Delta t (\beta_1^2 + 2\beta_1 |A|))}, \\
\widehat{K}_3 &= \frac{\Delta t}{(1 - \alpha\Delta t)^2} (1 + \widehat{\varepsilon}) (2\beta_1 + \beta_2^2 + \Delta t (\beta_1^2 + 2\beta_1 |A|)).
\end{aligned}$$

Here Δt and $\widehat{\varepsilon}$ need to be chosen small enough such that

$$(1 + \widehat{\varepsilon}) (2\beta_1 + \beta_2^2 + \Delta t (\beta_1^2 + 2\beta_1 |A|)) + \alpha^2 \Delta t < 2\alpha.$$

This guarantees that

$$(1 - \alpha\Delta t)^2 (1 + \widehat{K}_3) < 1.$$

By the discrete Gronwall inequality,

$$\begin{aligned}
& (1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \\
& \leq \widehat{K}_1 + \widehat{K}_2 (1 - \alpha\Delta t)^{-2M} + \sum_{i=0}^{M-1} \left(\widehat{K}_1 + \widehat{K}_2 (1 - \alpha\Delta t)^{-2i} \right) \widehat{K}_3 \left(1 + \widehat{K}_3 \right)^{M-i-1} \\
& = \widehat{K}_1 + \widehat{K}_2 (1 - \alpha\Delta t)^{-2M} + \widehat{K}_1 \left(1 + \widehat{K}_3 \right)^M - \widehat{K}_1 \\
& \quad + \widehat{K}_2 \widehat{K}_3 \frac{\left(1 + \widehat{K}_3 \right)^M - (1 - \alpha\Delta t)^{-2M}}{\left(1 + \widehat{K}_3 \right) - (1 - \alpha\Delta t)^{-2}}.
\end{aligned}$$

It turns out that,

$$\begin{aligned}
\mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 & \leq \widehat{K}_2 + \widehat{K}_1 \left(\left(1 + \widehat{K}_3 \right) (1 - \alpha\Delta t)^2 \right)^M \\
& \quad + \frac{\widehat{K}_2 \widehat{K}_3 (1 - \alpha\Delta t)^2 \left(1 - \left(\left(1 + \widehat{K}_3 \right) (1 - \alpha\Delta t)^2 \right)^M \right)}{1 - \left(1 + \widehat{K}_3 \right) (1 - \alpha\Delta t)^2} \leq \widehat{C}.
\end{aligned}$$

Note the choice of the constant \widehat{C} is independent of k and the lemma holds for sufficiently small time-step Δt and constant $\widehat{\varepsilon}$. \square

The following lemma is a discrete analogue of Lemma 2.2.5.

Lemma 3.1.2. *Denote by $\widehat{X}_{-k\tau+M\Delta t}^{-k\tau}$ and $\widehat{Y}_{-k\tau+M\Delta t}^{-k\tau}$ solutions of the Euler-Maruyama scheme with initial values ξ and η respectively. Assume Conditions (A), (1) and Condition (2) for both initial values. Let $\Delta t = \tau/n$, $n \in \mathbb{Z}^+$, be sufficiently small such that $0 < \Delta t \leq \frac{1}{\rho}$. Then for any $\varepsilon > 0$, there exists an integer $M^* > 0$ such that for any $M \geq M^*$, we have*

$$\left\| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} - \widehat{Y}_{-k\tau+M\Delta t}^{-k\tau} \right\|_2 < \varepsilon.$$

Proof. According to scheme (3.1.2) we have

$$\begin{aligned}
& \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} - \widehat{Y}_{-k\tau+M\Delta t}^{-k\tau} \\
& = (I + A\Delta t)^M (\xi - \eta) + \Delta t \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} \widehat{F}_i \\
& \quad + \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} \widehat{G}_i (W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t}).
\end{aligned}$$

Here

$$\begin{aligned}\widehat{F}_i &= f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - f(i\Delta t, \widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}), \\ \widehat{G}_i &= g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - g(i\Delta t, \widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}).\end{aligned}$$

Denote

$$\widehat{\zeta}_i = \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} - \widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}.$$

Then by Condition (1), we have $|\widehat{F}_i| \leq \beta_1 |\widehat{\zeta}_i|$ and $|\widehat{G}_i| \leq \beta_2 |\widehat{\zeta}_i|$. According to the method used in Lemma 3.1.1, we get the following result similar to inequality (3.1.5)

$$\begin{aligned}& (1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{\zeta}_M \right|^2 \\ & \leq \|\xi - \eta\|_2^2 + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(\frac{\Delta t}{1 - \alpha\Delta t} \right)^2 \mathbb{E} \left| \widehat{F}_i \right|^2 \\ & \quad + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| \widehat{G}_i \right|^2 \\ & \quad + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{2\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left[\left(\widehat{\zeta}_i \right)^T (I + A\Delta t) \widehat{F}_i \right] \\ & \leq \|\xi - \eta\|_2^2 + \widehat{K}_4 \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \mathbb{E} \left| \widehat{\zeta}_i \right|^2,\end{aligned}$$

where

$$\widehat{K}_4 = \frac{\Delta t}{(1 - \alpha\Delta t)^2} (2\beta_1 + \beta_2^2 + \Delta t (\beta_1^2 + 2\beta_1 |A|)).$$

We choose Δt small enough such that

$$2\beta_1 + \beta_2^2 + \Delta t (\beta_1^2 + 2\beta_1 |A|) + \alpha^2 \Delta t < 2\alpha.$$

Then, we have

$$(1 - \alpha\Delta t)^2 (1 + \widehat{K}_4) < 1.$$

Again the discrete Gronwall inequality implies

$$(1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{\zeta}_M \right|^2 \leq \|\xi - \eta\|_2^2 \prod_{i=0}^{M-1} (1 + \widehat{K}_4) = \|\xi - \eta\|_2^2 (1 + \widehat{K}_4)^M.$$

Finally

$$\mathbb{E} \left| \widehat{\zeta}_M \right|^2 \leq \|\xi - \eta\|_2^2 \left((1 - \alpha\Delta t)^2 (1 + \widehat{K}_4) \right)^M < \varepsilon$$

with sufficiently large M . □

In the numerical scheme we split the process into two time intervals, $[-k\tau, 0)$ and $[0, r]$. Define

$$\widehat{X}_r^{-k\tau} := \widehat{X}(r, 0, \omega) \circ \widehat{X}_0^{-k\tau}, \quad (3.1.6)$$

where $\widehat{X}(r, 0, \omega)$, $r \geq 0$, is finite time Euler-Maruyama approximation of the solution of stochastic differential equation with time step size Δt , till $N'\Delta t \leq r$, where N' is the unique number such that $N'\Delta t \leq r$ and $(N' + 1)\Delta t > r$. If $N'\Delta t < r$, define

$$\begin{aligned} \widehat{X}(r, 0, \omega) &= \widehat{X}(N'\Delta t, 0, \omega) + f(N'\Delta t, \widehat{X}(N'\Delta t, 0, \omega))(r - N'\Delta t) \\ &\quad + g(N'\Delta t, \widehat{X}(N'\Delta t, 0, \omega))(W_r - W_{N'\Delta t}) \end{aligned} \quad (3.1.7)$$

Lemma 3.1.3. (*Continuity of the discrete semi-flow with respect to the initial value*)

Denote by \widetilde{X}_r^0 and \widetilde{Y}_r^0 the solution of the finite time Euler-Maruyama scheme with the initial values $\widetilde{\xi}$ and $\widetilde{\eta}$ at time 0. Assume Conditions (A), (1) and Condition (2) for both initial values. Let Δt be sufficiently small, $p \geq 1$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $\|\widetilde{\xi} - \widetilde{\eta}\|_p < \delta$, we have

$$\left\| \widetilde{X}_r^0(\omega, \widetilde{\xi}) - \widetilde{Y}_r^0(\omega, \widetilde{\eta}) \right\|_p < \varepsilon. \quad (3.1.8)$$

Proof. Note that $\widetilde{X}_{N'\Delta t}^0$ and $\widetilde{Y}_{N'\Delta t}^0$ satisfy analogues of (3.1.2), with initial value $\widetilde{\xi}$ and $\widetilde{\eta}$ at time 0 instead of $-k\tau$. Apply the Euler-Maruyama scheme on the finite time $r' = N'\Delta t$ to obtain

$$\begin{aligned} & \left| \widetilde{X}_{r'}^0(\omega, \widetilde{\xi}) - \widetilde{Y}_{r'}^0(\omega, \widetilde{\eta}) \right|^p \\ &= \left| (I + A\Delta t)^{N'} (\widetilde{\xi} - \widetilde{\eta}) + (\Delta t) \sum_{i=0}^{N'-1} (I + A\Delta t)^{N'-i-1} \left(f(i\Delta t, \widetilde{X}_{i\Delta t}^0) - f(i\Delta t, \widetilde{Y}_{i\Delta t}^0) \right) \right. \\ & \quad \left. + \sum_{i=0}^{N'-1} (I + A\Delta t)^{N'-i-1} \left(g(i\Delta t, \widetilde{X}_{i\Delta t}^0) - g(i\Delta t, \widetilde{Y}_{i\Delta t}^0) \right) (W_{(i+1)\Delta t} - W_{i\Delta t}) \right|^p \\ &\leq 3^{p-1} \left| (I + A\Delta t)^{pN'} \right| \left| \widetilde{\xi} - \widetilde{\eta} \right|^p + 3^{p-1} (\Delta t)^p \left| (I + A\Delta t)^{pN'} \right| \left| \sum_{i=0}^{N'-1} (I + A\Delta t)^{-i-1} \widetilde{F}_i \right|^p \\ & \quad + 3^{p-1} \left| (I + A\Delta t)^{pN'} \right| \left| \sum_{i=0}^{N'-1} (I + A\Delta t)^{-i-1} \widetilde{G}_i (W_{(i+1)\Delta t} - W_{i\Delta t}) \right|^p, \end{aligned} \quad (3.1.9)$$

where

$$\widetilde{F}_i := f(i\Delta t, \widetilde{X}_{i\Delta t}^0) - f(i\Delta t, \widetilde{Y}_{i\Delta t}^0),$$

$$\tilde{G}_i := g(i\Delta t, \tilde{X}_{i\Delta t}^0) - g(i\Delta t, \tilde{Y}_{i\Delta t}^0).$$

Denote $\tilde{\zeta}_i := \tilde{X}_{i\Delta t}^0 - \tilde{Y}_{i\Delta t}^0$. For convenience, we denote $C_p = 3^{p-1}$, $C_{p,N'} = 3^{p-1}N'^{p-1}$. Taking expectation on both sides of (3.1.9), and noting that the Lipschitz condition of function f and g , we have

$$\begin{aligned} & (1 - \alpha\Delta t)^{-pN'} \left\| \tilde{\zeta}_{N'} \right\|_p^p \\ & \leq C_p \left\| \tilde{\xi} - \tilde{\eta} \right\|_p^p + C_{p,N'} (\Delta t)^p \sum_{i=0}^{N'-1} (1 - \alpha\Delta t)^{-(i+1)p} \beta_1^p \left\| \tilde{\zeta}_i \right\|_p^p \\ & \quad + C_{p,N'} (\Delta t)^{p/2} \sum_{i=0}^{N'-1} (1 - \alpha\Delta t)^{-(i+1)p} \beta_2^p \left\| \tilde{\zeta}_i \right\|_p^p \\ & = C_p \left\| \tilde{\xi} - \tilde{\eta} \right\|_p^p + \tilde{K} \sum_{i=0}^{N'-1} (1 - \alpha\Delta t)^{-ip} \left\| \tilde{\zeta}_i \right\|_p^p, \end{aligned}$$

where

$$\tilde{K} = \frac{C_{p,N'} \left((\Delta t)^p \beta_1^p + (\Delta t)^{p/2} \beta_2^p \right)}{(1 - \alpha\Delta t)^p},$$

which is bounded for any $1 \leq p < +\infty$.

Then by the Gronwall inequality, we have

$$(1 - \alpha\Delta t)^{-pN'} \left\| \tilde{\zeta}_{N'} \right\|_p^p \leq C_p \left\| \tilde{\xi} - \tilde{\eta} \right\|_p^p \prod_{i=0}^{N'-1} (1 + \tilde{K}).$$

So

$$\left\| \tilde{\zeta}_{N'} \right\|_p^p \leq C_p \left\| \tilde{\xi} - \tilde{\eta} \right\|_p^p \left((1 + \tilde{K})(1 - \alpha\Delta t)^p \right)^{N'}.$$

Note

$$\begin{aligned} & (1 + \tilde{K})(1 - \alpha\Delta t)^p \\ & \leq (1 - \alpha\Delta t)^p + C_{p,N'} \left((\Delta t)^p \beta_1^p + (\Delta t)^{p/2} \beta_2^p \right) \\ & \leq 1 + C_{p,N'}. \end{aligned}$$

The result (3.1.8) at $r' = N'\Delta t$ follows by taking

$$\delta = \frac{\varepsilon}{C_p} (1 + C_{p,N'})^{-N'}.$$

Finally (3.1.8) at time r follows from (3.1.7) and the estimate at $r' = N'\Delta t$. \square

Theorem 3.1.4. *Assume that Condition (1) is satisfied and Δt is fixed and small enough. The time domain is divided as $\tau = n\Delta t$. Then there exists $\widehat{X}_r^* \in L^2(\Omega)$ such that for any fixed initial values ξ , the solution of the Euler-Maruyama scheme satisfies*

$$\lim_{k \rightarrow \infty} \left\| \widehat{X}_r^{-k\tau}(\xi) - \widehat{X}_r^* \right\|_2 = 0, \quad (3.1.10)$$

and \widehat{X}_r^* satisfies the random periodicity property.

Proof. Firstly we note that the proof of the convergence of the process $\widehat{X}_0^{-k\tau}$ can be made similarly as that of Theorem 2.2.6. According to Lemma 3.1.1 we know that for any M , we have $\widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \in L^2(\Omega)$. We use a similar construction of a Cauchy sequence as in Theorem 2.2.6. As we assume that $\tau = n\Delta t$ and $k\tau = kn\Delta t =: N\Delta t$, we have the following result by using semi-flow property, for any $m \geq 1$,

$$\widehat{X}_0^{-(k+m)\tau} = \widehat{X}_0^{-(N+mn)\Delta t} = \widehat{X}_0^{-N\Delta t} \circ \widehat{X}_{-N\Delta t}^{-(N+mn)\Delta t}.$$

It is a same process as $\widehat{X}_0^{-N\Delta t}$ with a different initial value. By Lemma 3.1.2 we have that for any $\varepsilon > 0$ there exists N^* such that for any $N \geq N^*$, $\Delta t > 0$, we have

$$\left\| \widehat{X}_0^{-k\tau} - \widehat{X}_0^{-(k+m)\tau} \right\|_2 = \left\| \widehat{X}_0^{-N\Delta t} - \widehat{X}_0^{-(N+mn)\Delta t} \right\|_2 < \varepsilon.$$

Then we construct the Cauchy sequence $\widehat{X}_i = \widehat{X}_0^{-i\tau}$, which converges to some \widehat{X}^* in $L^2(\Omega)$. We now use the same method to prove the convergence is independent of the initial point. Note for fixed Δt ,

$$\left\| \widehat{X}^* - \widehat{X}_0^{-k\tau}(\eta) \right\|_2 \leq \left\| \widehat{X}^* - \widehat{X}_0^{-k\tau}(\xi) \right\|_2 + \left\| \widehat{X}_0^{-k\tau}(\xi) - \widehat{X}_0^{-k\tau}(\eta) \right\|_2 \xrightarrow{N \rightarrow \infty} 0,$$

where $N \rightarrow \infty$ is equivalent to $k \rightarrow \infty$.

Define $\widehat{X}^*(r, \omega) := \widehat{X}(r, 0, \omega) \circ \widehat{X}^*$, $r \geq 0$. According to Lemma 3.1.3, we have

$$\widehat{X}_r^{-k\tau}(\omega) = \widehat{X}(r, 0, \omega) \circ \widehat{X}_0^{-k\tau}(\omega) \xrightarrow[k \rightarrow \infty]{L^2(\Omega)} \widehat{X}(r, 0, \omega) \circ \widehat{X}^*(\omega) = \widehat{X}^*(r, \omega),$$

so (3.1.10) holds. On the other hand, similar to the proof of (2.2.4), we obtain

$$\widehat{X}_{r+\tau}^\tau(\omega, \xi(\omega)) = \widehat{X}_r^0(\theta_\tau \omega, \xi(\theta_\tau \omega)) = \theta_\tau \widehat{X}_r^0(\omega, \xi(\omega)).$$

Therefore,

$$\widehat{X}_r^{-k\tau}(\theta_\tau \omega) = \widehat{X}(r, 0, \theta_\tau \omega) \circ \widehat{X}_0^{-k\tau}(\theta_\tau \omega) \xrightarrow[k \rightarrow \infty]{L^2(\Omega)} \widehat{X}(r, 0, \theta_\tau \omega) \circ \widehat{X}^*(\theta_\tau \omega) = \widehat{X}^*(r, \theta_\tau \omega).$$

But,

$$\widehat{X}_{r+\tau}^{-k\tau+\tau}(\omega) \xrightarrow[L^2(\Omega)]{k \rightarrow \infty} \widehat{X}^*(r+\tau, \omega), \text{ and } \widehat{X}_{r+\tau}^{-k\tau+\tau}(\omega) = \widehat{X}_r^{-k\tau}(\theta_\tau \omega), \mathbb{P} - a.s,$$

thus we have $\widehat{X}^*(r+\tau, \omega) = \widehat{X}^*(r, \theta_\tau \omega), \mathbb{P} - a.s.$ \square

Example 3.1.5. Consider a specific stochastic differential equation

$$dX_t^{t_0} = -\pi X_t^{t_0} dt + \sin(\pi t) dt + X_t^{t_0} dW_t. \quad (3.1.11)$$

According to Theorem 2.2.6, the SDE (3.1.11) has a random periodic solution. By Theorem 3.1.4, its Euler-Maruyama discretisation also has a random periodic path.

To see the “periodicity” numerically, we provided two methods. One approach is to simulate the processes

$$\widehat{X}_t^*(\omega) = \widehat{X}_t^{-6}(\omega, 0.5), -5 \leq t \leq 0,$$

and

$$\widehat{X}_t^*(\theta_{-2}\omega) = \widehat{X}_t^{-6}(\theta_{-2}\omega, 0.5), -5 \leq t \leq 2,$$

with the same ω and step size $\Delta t = 0.01$ (Fig. 3.1).

One can see that these two trajectories exactly repeat each with a time shift of one period (only comparing the graph of $\widehat{X}_t^*(\theta_{-2}\omega)$ for $-3 \leq t \leq 2$). The second method is the simulation of $\{\widehat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$ for the same realisation ω and step size as before (Fig. 3.2). One can easily see that Fig. 3.2 is a perfect periodic curve. This agrees with the fact that if $\widehat{X}_t^*(\omega)$ is a random periodic path iff $\widehat{X}_t^*(\theta_{-t}\omega)$ is periodic, i.e. $\widehat{X}_{t+\tau}^*(\theta_{-(t+\tau)}\omega) = \widehat{X}_t^*(\theta_{-t}\omega)$. Note in theory $\widehat{X}_t^* = \widehat{X}_t^{-\infty}$, but we take pull-back time -6 as this is already enough to generate a good convergence to the random periodic paths $\widehat{X}_t^*(\cdot)$ for $t \geq -5$ by the solution starting at -6 from 0.5 for both cases. The choice of the initial position does not affect random periodic paths, but the time to take for the convergence.

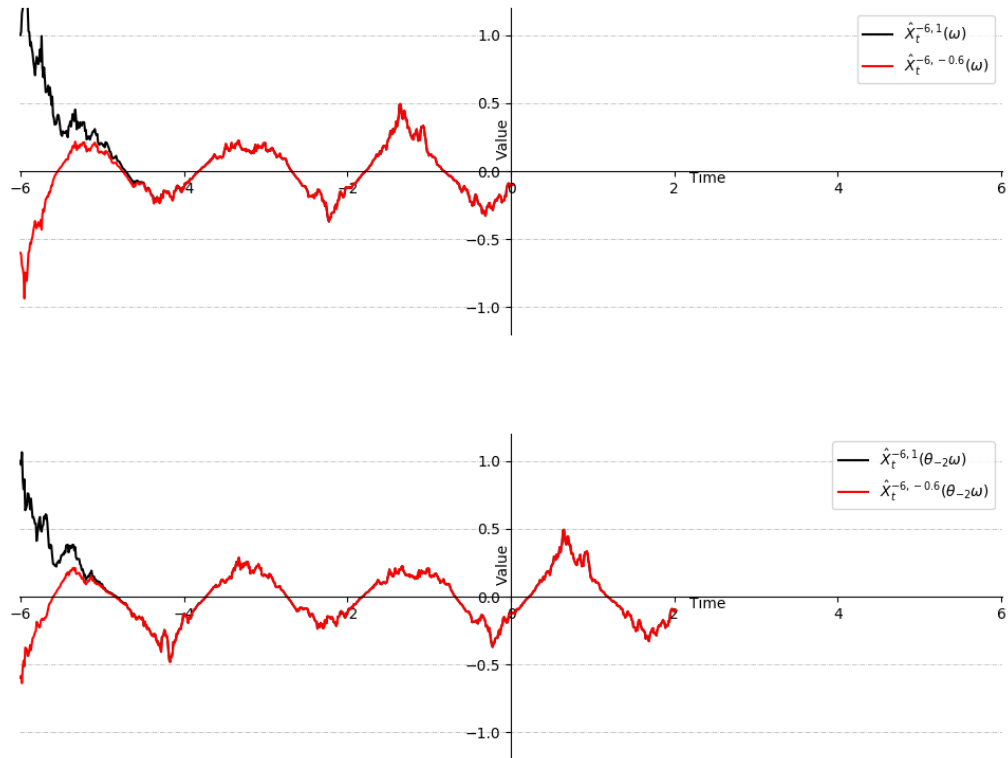


Figure 3.1: Simulations of the processes $\{\hat{X}_t^*(\omega), -5 \leq t \leq 0\}$ and $\{\hat{X}_t^*(\theta_{-2}\omega), -5 \leq t \leq 2\}$.

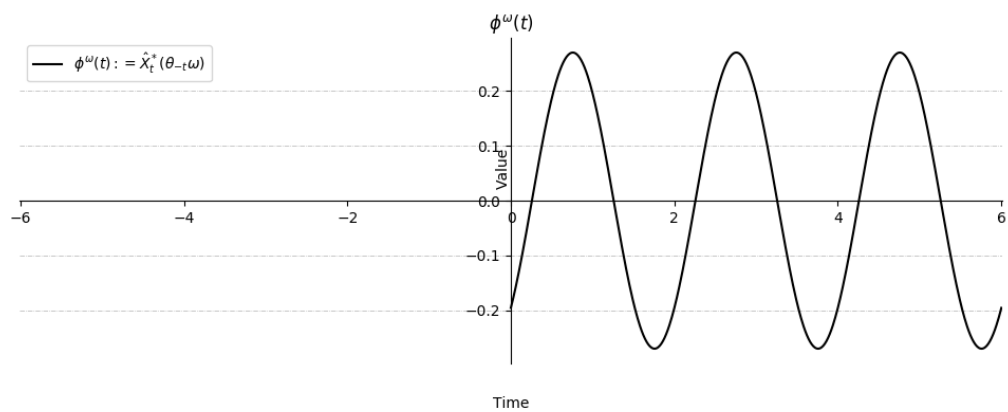


Figure 3.2: Simulation of the process $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

3.1.2 The error estimation

In the above sections, we proved the existence of random periodic solutions of SDE (2.0.1) and its discretisations as the limits of semi-flows when the starting times were pushed to $-\infty$. The next step is to estimate the error between these two limits. Now we need to consider the difference between the discrete approximate solution and the exact solution. The exact solution at time $-k\tau + M\Delta t$ is as follows

$$\begin{aligned} X_{-k\tau+M\Delta t}^{-k\tau}(\omega, \xi) = & e^{AM\Delta t}\xi + e^{A(M\Delta t-k\tau)} \int_{-k\tau}^{M\Delta t-k\tau} e^{-As} f(s, X_s^{-k\tau}) ds \\ & + e^{A(M\Delta t-k\tau)} \int_{-k\tau}^{M\Delta t-k\tau} e^{-As} g(s, X_s^{-k\tau}) dW_s. \end{aligned} \quad (3.1.12)$$

Lemma 3.1.6. *Assume Conditions (A), (1) and (2). Choose $\Delta t = \tau/n$ for some $n \in \mathbb{N}$ and $N = kn$. Then there exists a constant $K > 0$ such that for any sufficiently small fixed Δt and $N' \in \mathbb{N}$, , we have*

$$\limsup_{k \rightarrow \infty} \left\| X_{N'\Delta t}^{-k\tau} - \widehat{X}_{N'\Delta t}^{-k\tau} \right\|_2 \leq K\sqrt{\Delta t},$$

where $X_{N'\Delta t}^{-k\tau}$ and $\widehat{X}_{N'\Delta t}^{-k\tau}$ are the exact and the numerical solutions given by (3.1.12) and (3.1.2) respectively, K is independent of N' and Δt .

Proof. In the following proof, we always denote by \widehat{K}_i the constants derived from the underlining computation unless otherwise stated. For any $M \in \mathbb{N}$, we have

$$\begin{aligned} & X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \\ = & \left(e^{AM\Delta t} - (I + A\Delta t)^M \right) \xi + e^{A(M\Delta t-k\tau)} \int_{-k\tau}^{M\Delta t-k\tau} e^{-As} f(s, X_s^{-k\tau}) ds \\ & - \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \Delta t \\ & + e^{A(M\Delta t-k\tau)} \int_{-k\tau}^{M\Delta t-k\tau} e^{-As} g(s, X_s^{-k\tau}) dW_s \\ & - \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) (W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t}). \end{aligned}$$

Similar to the method of Lemma 3.1.1, firstly consider

$$(1 - \alpha\Delta t)^{-2M} \left| X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2$$

$$= \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(\frac{\left| X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \right). \quad (3.1.13)$$

For simplicity we denote

$$B_1 = \frac{1}{1 - \alpha\Delta t} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left(e^{-A(s+k\tau-(i+1)\Delta t)} f(s, X_s^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) ds,$$

$$B_2 = \frac{1}{1 - \alpha\Delta t} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left(e^{-A(s+k\tau-(i+1)\Delta t)} g(s, X_s^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) dW_s.$$

Therefore,

$$\begin{aligned} & X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \\ &= e^{A\Delta t} X_{-k\tau+i\Delta t}^{-k\tau} - (I + A\Delta t) \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + (1 - \alpha\Delta t) (B_1 + B_2). \end{aligned}$$

Now we consider

$$\begin{aligned} & \frac{\left| X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \\ &= \left((X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} - I \right) - (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} - I \right) + B_1^T + B_2^T \right) \\ & \quad \times \left(\left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} + I \right) X_{-k\tau+i\Delta t}^{-k\tau} - \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} + I \right) \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + B_1 + B_2 \right) \\ &= \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} - I \right) \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} + I \right) \\ & \quad \times \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) \\ & \quad + \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right)^2 \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) + B_1^T B_1 + B_2^T B_2 \\ & \quad + 2 \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} \right) \left(\frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right) \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) \\ & \quad + 2 \left((X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} \right) - (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} \right) \right) B_1 \\ & \quad + 2 \left((X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} \right) - (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} \right) \right) B_2 + 2B_1^T B_2. \end{aligned} \quad (3.1.14)$$

We note that the matrix $\left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} - I\right)\left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} + I\right)$ can be non-positive-definite when we choose the Δt small enough. Now we consider each term in (3.1.14). First,

$$\begin{aligned}
& \mathbb{E} \left[\left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right)^2 \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right] \\
& \leq \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \left| \frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right|^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \\
& = \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \left| \frac{\frac{1}{2}A^2(\Delta t)^2 + (e^{A\Delta t} - I - A\Delta t - \frac{1}{2}A^2(\Delta t)^2)}{1 - \alpha\Delta t} \right|^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \\
& \leq \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \left| \frac{\frac{1}{2}A^2(\Delta t)^2}{1 - \alpha\Delta t} \right|^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \\
& \leq \widehat{K}_5(\Delta t)^4.
\end{aligned}$$

Next,

$$\begin{aligned}
& \mathbb{E} [B_1^T B_1] = \mathbb{E} |B_1|^2 \\
& = \frac{1}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left(e^{-A(s+k\tau-(i+1)\Delta t)} f(s, X_s^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) ds \right|^2 \\
& \leq \frac{1}{(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| e^{-A(s+k\tau-(i+1)\Delta t)} f(s, X_s^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 ds \right)^2 \\
& \leq \frac{2(1 + \mu)}{\mu(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} |e^{-A(s+k\tau-(i+1)\Delta t)} - I| \left\| f(s, X_s^{-k\tau}) \right\|_2 ds \right)^2 \\
& \quad + \frac{2(1 + \mu)}{\mu(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(s, X_s^{-k\tau}) - f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 ds \right)^2 \\
& \quad + \frac{1 + \mu}{(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 ds \right)^2,
\end{aligned} \tag{3.1.15}$$

where μ is a small number from Young's inequality, which will be fixed later. By linear growth property of f and Lemma 2.2.3, we know that $\left\| f(s, X_s^{-k\tau}) \right\|_2$ is bounded. So for the first term in (3.1.15) we only need to estimate

$$\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} |e^{-A(s+k\tau-(i+1)\Delta t)} - I| ds \leq \frac{(\Delta t)^2}{2} \text{Tr}(-A).$$

By Condition (1) and Lemma 2.2.4, the second term in (3.1.15) becomes

$$\begin{aligned}
& \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(s, X_s^{-k\tau}) - f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 ds \\
& \leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left(\left\| f(s, X_s^{-k\tau}) - f(i\Delta t, X_s^{-k\tau}) \right\|_2 \right. \\
& \quad \left. + \left\| f(i\Delta t, X_s^{-k\tau}) - f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 \right) ds \\
& \leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} C_0 |s - i\Delta t + k\tau|^{1/2} ds + \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \beta_1 \left\| X_s^{-k\tau} - X_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 ds \\
& \leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} (C_0 + \beta_1 C_4) \sqrt{s - i\Delta t + k\tau} ds \\
& \leq \widehat{K}_6 (\Delta t)^{\frac{3}{2}}.
\end{aligned}$$

Applying the global Lipschitz condition, the third term of (3.1.15) becomes

$$\begin{aligned}
& \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 ds \\
& \leq \beta_1 \Delta t \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2.
\end{aligned}$$

We summarise the above inequalities to have

$$\mathbb{E} [B_1^T B_1] \leq \widehat{K}_7 (\Delta t)^3 + \frac{(1 + \mu)\beta_1^2 (\Delta t)^2}{(1 - \alpha\Delta t)^2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2. \quad (3.1.16)$$

This term is of the 3rd order of Δt and 2nd order of Δt with $\left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2$. Similar to the $\mathbb{E} [B_1^T B_1]$, the following term can be estimated as

$$\begin{aligned}
& \mathbb{E} [B_2^T B_2] = \mathbb{E} |B_2|^2 \\
& = \frac{1}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left(e^{-A(s+k\tau-(i+1)\Delta t)} g(s, X_s^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) dW_s \right|^2 \\
& = \frac{1}{(1 - \alpha\Delta t)^2} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| e^{-A(s+k\tau-(i+1)\Delta t)} g(s, X_s^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2^2 ds \\
& \leq \frac{2(1 + \mu)}{\mu(1 - \alpha\Delta t)^2} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left| e^{-A(s+k\tau-(i+1)\Delta t)} - I \right|^2 \left\| g(s, X_s^{-k\tau}) \right\|_2^2 ds \\
& \quad + \frac{2(1 + \mu)}{\mu(1 - \alpha\Delta t)^2} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| g(s, X_s^{-k\tau}) - g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2^2 ds \\
& \quad + \frac{1 + \mu}{(1 - \alpha\Delta t)^2} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2^2 ds, \quad (3.1.17)
\end{aligned}$$

where μ is a small number from Young's inequality, which will be fixed later. By the linear growth property of g and Lemma 2.2.3, we know that $\|g(s, X_s^{-k\tau})\|_2^2$ is bounded. So we only need to estimate

$$\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} |e^{-A(s+k\tau-(i+1)\Delta t)} - I|^2 ds \leq \frac{2}{3} (\Delta t)^3 Tr(A^2).$$

By Condition (1) and Lemma 2.2.4, the second term in (3.1.17) becomes

$$\begin{aligned} & \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \|g(s, X_s^{-k\tau}) - g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau})\|_2^2 ds \\ & \leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} 2(C_0^2 + \beta_2^2 C_4^2) |s - i\Delta t + k\tau| ds \leq \widehat{K}_8 (\Delta t)^2. \end{aligned}$$

The third term follows from the global Lipschitz condition

$$\begin{aligned} & \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \|g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\|_2^2 ds \\ & \leq \beta_2^2 \Delta t \|X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\|_2^2. \end{aligned}$$

Conclude the above results to obtain

$$\mathbb{E} [B_2^T B_2] \leq \widehat{K}_9 (\Delta t)^2 + \frac{(1+\mu)\beta_2^2 \Delta t}{(1-\alpha\Delta t)^2} \|X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\|_2^2. \quad (3.1.18)$$

The fifth term of (3.1.14) can be estimated as follows

$$\begin{aligned} & \mathbb{E} \left[2 \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} \right) \left(\frac{e^{A\Delta t} - I - A\Delta t}{1-\alpha\Delta t} \right) \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) \right] \\ & \leq 2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \frac{\frac{1}{2}|A^2|(\Delta t)^2}{(1-\alpha\Delta t)^2} \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \\ & \leq \widehat{K}_{10} (\Delta t)^2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2. \end{aligned}$$

To estimate the sixth term of (3.1.14),

$$\begin{aligned} & \mathbb{E} \left[2 \left(\left(X_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} \right) - \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I + A\Delta t}{1-\alpha\Delta t} \right) \right) B_1 \right] \\ & = \mathbb{E} \left[2 \left(X_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} - \frac{I + A\Delta t}{1-\alpha\Delta t} \right) B_1 \right] \\ & \quad + \mathbb{E} \left[2 \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I + A\Delta t}{1-\alpha\Delta t} \right) B_1 \right]. \end{aligned} \quad (3.1.19)$$

Now we discuss these two terms separately. According to the result (3.1.16) and the positivity of the terms on the right hand side, we have

$$\begin{aligned}
& \mathbb{E} \left[2 \left(X_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} - \frac{I+A\Delta t}{1-\alpha\Delta t} \right) B_1 \right] \\
& \leq 2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \frac{\left| \frac{1}{2} A^2 (\Delta t)^2 \right|}{1-\alpha\Delta t} \|B_1\|_2 \\
& \leq \widehat{K}_{11}(\Delta t)^2 \left(\frac{\sqrt{\widehat{K}_7(\Delta t)^{3/2}}}{1-\alpha\Delta t} + \frac{\sqrt{1+\mu\beta_1}\Delta t}{(1-\alpha\Delta t)^2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \right) \\
& \leq \widehat{K}_{12}(\Delta t)^{7/2} + \frac{\sqrt{1+\mu\beta_1}\widehat{K}_{11}(\Delta t)^3}{(1-\alpha\Delta t)^2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2.
\end{aligned}$$

And,

$$\begin{aligned}
& \mathbb{E} \left[2 \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) B_1 \right] \\
& \leq 2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \frac{\|B_1\|_2}{1-\alpha\Delta t} (1+|A\Delta t|) \tag{3.1.20}
\end{aligned}$$

$$\begin{aligned}
& \leq \frac{2\sqrt{\widehat{K}_7(\Delta t)^{3/2}}}{1-\alpha\Delta t} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 (1+\Delta t|A|) \\
& \quad + \frac{2\sqrt{1+\mu\beta_1}\Delta t}{(1-\alpha\Delta t)^2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 (1+\Delta t|A|). \tag{3.1.21}
\end{aligned}$$

We use the conditional expectation to eliminate the seventh term

$$\begin{aligned}
& \mathbb{E} \left[\left(\left(X_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} \right) - \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) \right) B_2 \right] \\
& = \mathbb{E} \left[\left(\left(X_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} \right) - \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) \right) \mathbb{E} [B_2 | \mathcal{F}^{i\Delta t-k\tau}] \right] \\
& = 0.
\end{aligned}$$

For the last term,

$$\begin{aligned}
& \mathbb{E} [2B_1^T B_2] \\
& \leq 2 \|B_1^T\|_2 \cdot \|B_2\|_2 \\
& \leq 2 \left(\sqrt{\widehat{K}_7(\Delta t)^{3/2}} + \frac{\sqrt{1+\mu\beta_1}(\Delta t)}{1-\alpha\Delta t} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \right) \\
& \quad \times \left(\sqrt{\widehat{K}_9(\Delta t)} + \frac{\sqrt{1+\mu\beta_2}\sqrt{\Delta t}}{1-\alpha\Delta t} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \right)
\end{aligned}$$

$$\leq \widehat{K}_{13}(\Delta t)^{5/2} + \widehat{K}_{14}(\Delta t)^{3/2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2.$$

Combining all the estimation above, we have

$$\begin{aligned} & \left| \frac{X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau}}{(1-\alpha\Delta t)^2} \right|^2 - \left| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \\ & \leq \left(\frac{(1+\mu)\beta_2^2\Delta t}{(1-\alpha\Delta t)^2} + \frac{2\sqrt{(1+\mu)}\beta_1\Delta t}{(1-\alpha\Delta t)^2} + \widehat{K}_{16}(\Delta t)^{3/2} \right) \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \\ & \quad + \widehat{K}_{15}(\Delta t)^2 + \left(\frac{2\sqrt{\widehat{K}_7}(\Delta t)^{3/2}}{1-\alpha\Delta t} + \widehat{K}_{17}(\Delta t)^2 \right) \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2. \end{aligned}$$

Now we notice that the term $\left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2$ has coefficients, the largest of which contains a constant multiplied by Δt . The largest free term contains a constant multiplied by $(\Delta t)^2$.

Choosing μ and Δt small enough and applying Young's inequality for the term

$$(\Delta t)^{3/2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2,$$

and from (3.1.13) we get

$$\begin{aligned} & (1-\alpha\Delta t)^{-2M} \left\| X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right\|_2^2 \\ & \leq \sum_{i=0}^{M-1} (1-\alpha\Delta t)^{-2i} \left(\widehat{K}_{18}(\Delta t)^2 + \widehat{K}_{20}\Delta t \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \right) \\ & \leq \widehat{K}_{19}(\Delta t)(1-\alpha\Delta t)^{-2M} + \widehat{K}_{20}(\Delta t) \sum_{i=0}^{M-1} (1-\alpha\Delta t)^{-2i} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2, \end{aligned} \tag{3.1.22}$$

where

$$\begin{aligned} \widehat{K}_{19} &= \frac{\widehat{K}_{18}(1-\alpha\Delta t)^2}{2\alpha\Delta t - \alpha^2(\Delta t)^2}(\Delta t) = \frac{\widehat{K}_{18}(1-\alpha\Delta t)^2}{2\alpha - \alpha^2(\Delta t)}, \\ \widehat{K}_{20} &= \frac{(1+\mu)(2\beta_1 + \beta_2^2 + \varepsilon)}{(1-\alpha\Delta t)^2}. \end{aligned}$$

Here μ , ε and the time step Δt are chosen small enough such that

$$(1+\mu)(2\beta_1 + \beta_2^2 + \varepsilon) + \alpha^2\Delta t < 2\alpha.$$

Therefore,

$$\left(\widehat{K}_{20}\Delta t + 1\right)(1 - \alpha\Delta t)^2 < 1.$$

Now using the discrete time Gronwall inequality, from (3.1.22), we have

$$\begin{aligned} & (1 - \alpha\Delta t)^{-2M} \left\| X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right\|_2^2 \\ & \leq \widehat{K}_{19}\Delta t(1 - \alpha\Delta t)^{-2M} + \widehat{K}_{19}\widehat{K}_{20}(\Delta t)^2 \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(1 + \widehat{K}_{20}\Delta t\right)^{M-i-1} \\ & = \widehat{K}_{19}\Delta t(1 - \alpha\Delta t)^{-2M} + \widehat{K}_{19}\widehat{K}_{20}(\Delta t)^2 \frac{\left(1 + \widehat{K}_{20}\Delta t\right)^M - (1 - \alpha\Delta t)^{-2M}}{\left(1 + \widehat{K}_{20}\Delta t\right) - (1 - \alpha\Delta t)^{-2}}. \end{aligned}$$

So,

$$\begin{aligned} & \left\| X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right\|_2^2 \\ & \leq \widehat{K}_{19}\Delta t + \widehat{K}_{19}\widehat{K}_{20}(\Delta t)^2 \frac{1 - \left(\left(1 + \widehat{K}_{20}\Delta t\right)(1 - \alpha\Delta t)^2\right)^M}{1 - \left(1 + \widehat{K}_{20}\Delta t\right)(1 - \alpha\Delta t)^2} \\ & \leq \widehat{K}_{21}\Delta t. \end{aligned}$$

We can find a constant \widehat{K}_{21} which is independent of M and Δt . Finally we take $M = N + N'$, where $N\Delta t = k\tau$, $N' \in \mathbb{Z}$, then

$$\begin{aligned} & \limsup_{k \rightarrow \infty} \left\| X_{N'\Delta t}^{-k\tau} - \widehat{X}_{N'\Delta t}^{-k\tau} \right\|_2 \\ & = \limsup_{N \rightarrow \infty} \left\| X_{-k\tau+(N+N')\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(N+N')\Delta t}^{-k\tau} \right\|_2 \\ & \leq \sqrt{\widehat{K}_{21}}\sqrt{\Delta t}. \end{aligned}$$

So we get the result. \square

We have proved that the estimation of error from $-k\tau$ to $N'\Delta t$ as $k \rightarrow \infty$ can be controlled under the 1/2 order of the time-step. And the upper bound is uniform in time. The following theorem will give us a more general result, which is from $-k\tau$ to time r . Let $\widehat{X}_r^{-k\tau}$, $r > 0$ be given by (3.1.6).

Theorem 3.1.7. *Assume Conditions (A), (1) and (2). We choose $\Delta t = \tau/n$ for some $n \in \mathbb{N}$, $N = kn$. For any $r \geq 0$, there exists a constant $\widetilde{K} > 0$ such that for*

any sufficiently small fixed Δt ,

$$\limsup_{k \rightarrow \infty} \left\| X_r^{-k\tau} - \widehat{X}_r^{-k\tau} \right\|_2 \leq \widetilde{K} \sqrt{\Delta t},$$

where $X_r^{-k\tau}$ is the exact solution while $\widehat{X}_r^{-k\tau}$ is the numerical solution and \widetilde{K} is independent of Δt and r .

Proof. Assume for any $r \geq 0$, N' is the unique integer such that $N'\Delta t \leq r$, $(N' + 1)\Delta t > r$. According to the semi-flow property, we have,

$$X_r^{-k\tau}(\omega) - \widehat{X}_r^{-k\tau}(\omega) = X_r^{N'\Delta t}(\omega) \circ X_{N'\Delta t}^{-k\tau}(\omega) - \widehat{X}_r^{N'\Delta t}(\omega) \circ \widehat{X}_{N'\Delta t}^{-k\tau}(\omega),$$

where $\widehat{X}_r^{N'\Delta t}$ is finite time Euler-Maruyama approximation of solution of (2.0.1) from $N'\Delta t$ to r and $\widehat{X}_{N'\Delta t}^{-k\tau}$ is defined as before. So,

$$\begin{aligned} \left\| X_r^{-k\tau} - \widehat{X}_r^{-k\tau} \right\|_2 &\leq \left\| X_r^{N'\Delta t} \circ X_{N'\Delta t}^{-k\tau} - X_r^{N'\Delta t} \circ \widehat{X}_{N'\Delta t}^{-k\tau} \right\|_2 \\ &\quad + \left\| X_r^{N'\Delta t} \circ \widehat{X}_{N'\Delta t}^{-k\tau} - \widehat{X}_r^{N'\Delta t} \circ \widehat{X}_{N'\Delta t}^{-k\tau} \right\|_2. \end{aligned} \quad (3.1.23)$$

For the first term on the right-hand side, by Lemma 3.1.6, we have

$$\left\| X_{N'\Delta t}^{-k\tau} - \widehat{X}_{N'\Delta t}^{-k\tau} \right\| \leq K \sqrt{\Delta t}.$$

By the continuity of $X_r^{N'\Delta t}(\cdot)$ with respect to initial values in $L^2(\Omega)$ ([31]), then

$$\left\| X_r^{N'\Delta t} \circ X_{N'\Delta t}^{-k\tau} - X_r^{N'\Delta t} \circ \widehat{X}_{N'\Delta t}^{-k\tau} \right\|_2 \leq C \left\| X_{N'\Delta t}^{-k\tau} - \widehat{X}_{N'\Delta t}^{-k\tau} \right\|_2 \leq C_5 \sqrt{\Delta t},$$

where C_5 is independent of Δt . For the second term on the right-hand side of (3.1.23), it is finite time Euler-Maruyama approximation with same initial value. By Theorem 2.3.1 in Kloeden and Platen [29], there exists a constant $C_6 > 0$ such that for sufficiently $\Delta t > 0$,

$$\left\| X_r^{N'\Delta t} \circ \widehat{X}_{N'\Delta t}^{-k\tau} - \widehat{X}_r^{N'\Delta t} \circ \widehat{X}_{N'\Delta t}^{-k\tau} \right\|_2 \leq C_6 \sqrt{\Delta t},$$

where the choice of C_6 is independent of Δt . The result follows by taking $\widetilde{K} = C_5 + C_6$. \square

Corollary 3.1.8. *For any $r \geq 0$, the exact and numerical approximating random periodic solution of equation (2.0.1), X_r^* and \widehat{X}_r^* , given in Theorem 2.2.6 and Theorem 3.1.4 respectively satisfy*

$$\left\| X_r^* - \widehat{X}_r^* \right\|_2 \leq \widetilde{K} \sqrt{\Delta t}.$$

Proof. The result follows from

$$\left\| X_r^* - \widehat{X}_r^* \right\|_2 \leq \limsup_{k \rightarrow \infty} \left[\left\| X_r^* - X_r^{-k\tau} \right\|_2 + \left\| X_r^{-k\tau} - \widehat{X}_r^{-k\tau} \right\|_2 + \left\| \widehat{X}_r^{-k\tau} - \widehat{X}_r^* \right\|_2 \right].$$

□

3.2 Modified Milstein scheme

Next we consider the modified Milstein scheme which increases the convergence order for the infinite horizon problem. If we assume some additional conditions on the SDEs, the Milstein scheme will increase the strong order of error for finite horizon. In the following content, the scheme is modified by adding terms from higher order schemes to guarantee the result on infinite horizon. First we introduce the assumptions

Condition (1.a). *Assume there exists a constant $\tau > 0$ such that for any $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $f(t + \tau, x) = f(t, x)$, $g(t + \tau, x) = g(t, x)$, and there exist constants $C_0, \beta_1, \beta_2 > 0$ with $\beta_1 + \frac{\beta_2^2}{2} < |\lambda_1|$ such that for any $s, t \in \mathbb{R}$ and $x \in \mathbb{R}^m$,*

$$\begin{aligned} |f(s, x) - f(t, y)| &\leq C_0 |s - t| + \beta_1 |x - y|, \\ |g(s, x) - g(t, y)| &\leq C_0 |s - t| + \beta_2 |x - y|. \end{aligned}$$

Meanwhile, we need higher order of Lipschitz continuity of the function f and g and corresponding growth property.

Remark 3.2.1. *The main difference is about the functions f and g . In Condition (1.a), we assume higher order of continuity with respect to time. The higher order of Lipschitz continuity and linear growth are given by Kloeden and Platen [29] as we mentioned in Theoreme 2.3.3. To apply the modified Milstein scheme in the following section, later we give the corresponding condition of the higher order Lipschitz continuity as (3.2.2) and (3.2.3), which is the numerical interpretation for the above condition with the specified scheme.*

3.2.1 Numerical approximation for random periodic solution

Now we introduce the iteration formula for the modified Milstein scheme as follows,

$$\begin{aligned}
& \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \\
= & \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + A\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\Delta t + f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\Delta t \\
& + g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})(\Delta W_i) \\
& + \frac{\Delta Z_i}{2\sqrt{\Delta t}} \left[f\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - f\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right] \\
& + \frac{(\Delta W_i)^2 - \Delta t}{4\sqrt{\Delta t}} \left[g\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - g\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right], \quad (3.2.1)
\end{aligned}$$

with

$$\begin{aligned}
\widehat{\Upsilon}_{\pm}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) = & \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + A\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\Delta t + f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\Delta t \\
& \pm g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\sqrt{\Delta t}
\end{aligned}$$

and

$$\begin{aligned}
\Delta W_i &= \int_{-k\tau+i\Delta t}^{-k\tau+(i+1)\Delta t} dW_s, \\
\Delta Z_i &= \int_{-k\tau+i\Delta t}^{-k\tau+(i+1)\Delta t} \int_{-k\tau+i\Delta t}^s dW_u ds, \\
\frac{1}{2}((\Delta W_i)^2 - \Delta t) &= \int_{-k\tau+i\Delta t}^{-k\tau+(i+1)\Delta t} \int_{-k\tau+i\Delta t}^s dW_u dW_s
\end{aligned}$$

where $i = 0, 1, 2, \dots$, and $\widehat{X}_{-k\tau+0\Delta t}^{-k\tau} = \xi$. Here the terms

$$\frac{1}{2\sqrt{\Delta t}} \left[f\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - f\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right]$$

and

$$\frac{1}{2\sqrt{\Delta t}} \left[g\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - g\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right]$$

are the approximations for $g \frac{\partial f}{\partial x}$ and $g \frac{\partial g}{\partial x}$ respectively if we neglect higher terms.

We require the higher order of Lipschitz continuity for the function f and g in the modified Milstein scheme as follows. We assume that there exist some constants K_1^* and K_2^* independent of step-size Δt such that for any $x, y \in \mathbb{R}^d$ and $t \in \mathbb{R}$,

$$\frac{1}{2\sqrt{\Delta t}} \left| \widehat{F}_t^{(1)}(x) - \widehat{F}_t^{(1)}(y) \right| \leq K_1^* |x - y| \quad (3.2.2)$$

and

$$\frac{1}{2\sqrt{\Delta t}} \left| \widehat{G}_t^{(1)}(x) - \widehat{G}_t^{(1)}(y) \right| \leq K_2^* |x - y|, \quad (3.2.3)$$

where

$$\begin{aligned} \widehat{F}_t^{(1)}(x) &:= f(t, \widehat{Y}_+(x)) - f(t, \widehat{Y}_-(x)), \\ \widehat{G}_t^{(1)}(x) &:= g(t, \widehat{Y}_+(x)) - g(t, \widehat{Y}_-(x)). \end{aligned}$$

For the modified Milstein scheme, we can also set up a discrete semi-flow given by

$$\hat{u}_{i,j}(\xi) = \widehat{X}_{i\Delta t}^{j\Delta t}(\xi), \quad i \geq j, \quad i, j \in \{-kn, -kn + 1, \dots\}, \quad \hat{\theta} = \theta_{\Delta t}, \quad \hat{\theta}^n = \hat{\theta}\hat{\theta} \dots \hat{\theta}.$$

It is easy to check the semi-flow property of the process,

$$\hat{u}_{i,j}(\omega) \circ \hat{u}_{j,l}(\omega) = \hat{u}_{i,l}(\omega), \quad \text{for } i \geq j \geq l. \quad (3.2.4)$$

and the periodic property

$$\hat{u}_{i+n,j+n}(\omega) = \hat{u}_{i,j}(\hat{\theta}^n \omega), \quad \text{for } i \geq j.$$

From the iteration (3.2.1), we have that for any $M \geq 0$,

$$\begin{aligned} \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} &= (I + A\Delta t)^M \xi + \Delta t \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \\ &\quad + \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) (\Delta W_i) \\ &\quad + \sum_{i=0}^{M-1} \left\{ (I + A\Delta t)^{M-i-1} \frac{\Delta Z_i}{2\sqrt{\Delta t}} \right. \\ &\quad \quad \left. \times \left[f\left(i\Delta t, \widehat{Y}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - f\left(i\Delta t, \widehat{Y}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right] \right\} \\ &\quad + \sum_{i=0}^{M-1} \left\{ (I + A\Delta t)^{M-i-1} \frac{[(\Delta W_i)^2 - \Delta t]}{4\sqrt{\Delta t}} \right\} \end{aligned}$$

$$\times \left[g \left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) - g \left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) \right] \Big\}. \quad (3.2.5)$$

The convergence of the discretized processes relies on the boundedness under the modified Milstein scheme with additional terms. In the next lemma, we prove the conclusion still holds as Lemma 3.1.1 with higher order of approximation terms. Notice that the constants \widehat{K}_i are independent of those in the previous sections on the Euler-Maruyama scheme.

Lemma 3.2.2. *Assume Conditions (A), (1.a) and (2). Then there exists a constant $\widehat{C} > 0$ such that for any natural numbers $k \geq 0$, $0 \leq M \leq N$, and sufficiently small Δt , the numerical solution $\widehat{X}_{-k\tau+M\Delta t}^{-k\tau}$ defined by (3.1.2) satisfies*

$$\mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \leq \widehat{C}.$$

Proof. We still choose α such that $\beta_1 + \frac{\beta^2}{2} < \alpha < |\lambda_1|$. Then it is known that for any $M \leq N$,

$$\begin{aligned} & (1 - \alpha\Delta t)^{-2M} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \\ &= |\xi|^2 + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(\frac{\left| \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \right). \end{aligned} \quad (3.2.6)$$

When we consider each term in the sum, it becomes

$$\begin{aligned} & \frac{\left| \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \\ &= \left\{ \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} - I \right) + \frac{\Delta t}{1 - \alpha\Delta t} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \right. \\ & \quad + \frac{(\Delta W_i)^T}{1 - \alpha\Delta t} g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \\ & \quad + \frac{(\Delta Z_i)^T}{2(1 - \alpha\Delta t)\sqrt{\Delta t}} \left[f \left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) - f \left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) \right]^T \\ & \quad \left. + \frac{(\Delta W_i)^2 - \Delta t}{4(1 - \alpha\Delta t)\sqrt{\Delta t}} \left[g \left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) - g \left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) \right]^T \right\} \\ & \times \left\{ \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} + I \right) \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + \frac{\Delta t}{1 - \alpha\Delta t} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\} \end{aligned}$$

$$\begin{aligned}
& + \frac{g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) (\Delta W_i)}{1 - \alpha\Delta t} \\
& \left[f\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - f\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right] \frac{\Delta Z_i}{2(1 - \alpha\Delta t)\sqrt{\Delta t}} \\
& + \left[g\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - g\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right] \frac{(\Delta W_i)^2 - \Delta t}{4(1 - \alpha\Delta t)\sqrt{\Delta t}} \Big\} \\
& \tag{3.2.7}
\end{aligned}$$

Note $\left(\frac{I+A\Delta t}{1-\alpha\Delta t} - I\right) \left(\frac{I+A\Delta t}{1-\alpha\Delta t} + I\right)$ is non-positive definite, where Δt satisfies $0 < \Delta t \leq \frac{1}{\rho}$ as defined before, and for each i , $f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})$ and $g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})$ are both independent of (ΔW_i) and (ΔZ_i) . It is easy to verify the following properties of ΔZ_i with the Itô's isometry,

$$\begin{aligned}
\mathbb{E}[\Delta Z_i] &= 0, & \mathbb{E}[\Delta Z_i \Delta W_i] &= \frac{1}{2}(\Delta t)^2, \\
\mathbb{E}[\Delta Z_i]^2 &= \frac{1}{3}(\Delta t)^3, & \mathbb{E}[\Delta Z_i (\Delta W_i)^2] &= 0.
\end{aligned}$$

Taking expectation on both sides of (3.2.6), there are some vanished terms as we know that

$$\mathbb{E}((\Delta W_i)^2 - \Delta t) = 0, \quad \mathbb{E}[(\Delta W_i)((\Delta W_i)^2 - \Delta t)] = 0.$$

Considering (3.2.7), we have the following inequality with the linear growth property and Young's inequality,

$$\begin{aligned}
& (1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \\
& \leq \|\xi\|_2^2 + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{(\Delta t)^2}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{2\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T (I + A\Delta t) f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{(\Delta t)^{3/2}}{2(1 - \alpha\Delta t)^2} \mathbb{E} \left| g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
& \quad \times \mathbb{E} \left| f\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - f\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{(\Delta t)^2}{12(1 - \alpha\Delta t)^2}
\end{aligned}$$

$$\begin{aligned}
& \times \mathbb{E} \left| f \left(i\Delta t, \widehat{\Upsilon}_+ (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) - f \left(i\Delta t, \widehat{\Upsilon}_- (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{\Delta t}{8(1 - \alpha\Delta t)^2} \\
& \quad \times \mathbb{E} \left| g \left(i\Delta t, \widehat{\Upsilon}_+ (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) - g \left(i\Delta t, \widehat{\Upsilon}_- (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) \right|^2 \\
\leq & \|\xi\|_2^2 + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i-2} (\Delta t)^2 \left(C_1^2 + 2\beta_1 C_1 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 + \beta_1^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \right) \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i-2} (\Delta t) \left(C_2^2 + 2\beta_2 C_2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 + \beta_2^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \right) \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i-2} 2(\Delta t) \left(C_1 (1 + |A| \Delta t) \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \right. \\
& \quad \left. + \beta_1 (1 + |A| \Delta t) \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \right) \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i-2} \frac{(\Delta t)^2}{2} 2\beta_1 \left(C_2^2 + 2\beta_2 C_2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 + \beta_2^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \right) \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i-2} \frac{(\Delta t)^3}{12} (2\beta_1)^2 \left(C_2^2 + 2\beta_2 C_2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 + \beta_2^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \right) \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i-2} \frac{(\Delta t)^2}{8} (2\beta_2)^2 \left(C_2^2 + 2\beta_2 C_2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 + \beta_2^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \right) \\
\leq & \widehat{K}_2 + (1 - \alpha\Delta t)^{-2M} \widehat{K}_3 + \widehat{K}_4 \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \mathbb{E} \left| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2, \tag{3.2.8}
\end{aligned}$$

where

$$\begin{aligned}
\widehat{K}_1 &= \beta_1 + \frac{\beta_1^2}{3} \Delta t + \frac{\beta_2^2}{2}, \\
\widehat{K}_2 &= \|\xi\|_2^2, \\
\widehat{K}_3 &= \frac{\Delta t}{2\alpha\Delta t - \alpha^2 (\Delta t)^2} \times \frac{\left(C_1 + \beta_2 C_2 + \Delta t \left(\beta_1 C_1 + C_1 |A| + \beta_2 C_2 \widehat{K}_1 \right) \right)^2}{\widehat{\varepsilon} \left(2\beta_1 + \beta_2^2 + \Delta t \left(\beta_1^2 + 2\beta_1 |A| + \beta_2^2 \widehat{K}_1 \right) \right)} \\
& \quad + \frac{C_1^2 (\Delta t)^2 + C_2^2 \Delta t + C_2^2 (\Delta t)^2 \widehat{K}_1}{2\alpha\Delta t - \alpha^2 (\Delta t)^2}, \\
\widehat{K}_4 &= \frac{\Delta t (1 + \widehat{\varepsilon})}{(1 - \alpha\Delta t)^2} \left(2\beta_1 + \beta_2^2 + \Delta t \left(\beta_1^2 + 2\beta_1 |A| + \beta_2^2 \widehat{K}_1 \right) \right).
\end{aligned}$$

Here it is still possible to choose Δt and $\widehat{\varepsilon}$ small enough such that

$$(1 + \widehat{\varepsilon}) \left(2\beta_1 + \beta_2^2 + \Delta t \left(\beta_1^2 + 2\beta_1 |A| + \beta_2^2 \widehat{K}_1 \right) \right) + \alpha^2 \Delta t < 2\alpha.$$

This gives us that

$$(1 - \alpha\Delta t)^2 \left(1 + \widehat{K}_4 \right) < 1.$$

By the discrete Gronwall inequality,

$$\begin{aligned} & (1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \\ & \leq \widehat{K}_2 + \widehat{K}_3 (1 - \alpha\Delta t)^{-2M} + \sum_{i=0}^{M-1} \left(\widehat{K}_2 + \widehat{K}_3 (1 - \alpha\Delta t)^{-2i} \right) \widehat{K}_4 \left(1 + \widehat{K}_4 \right)^{M-i-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{E} \left| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 & \leq \widehat{K}_3 + \widehat{K}_2 \left(\left(1 + \widehat{K}_4 \right) (1 - \alpha\Delta t)^2 \right)^M \\ & \quad + \frac{\widehat{K}_3 \widehat{K}_4 (1 - \alpha\Delta t)^2 \left(1 - \left(\left(1 + \widehat{K}_4 \right) (1 - \alpha\Delta t)^2 \right)^M \right)}{1 - \left(1 + \widehat{K}_4 \right) (1 - \alpha\Delta t)^2} \leq \widehat{C}. \end{aligned}$$

The choice of the constant \widehat{C} is independent of k and we only need to choose the time-step Δt sufficiently small to conquer the influence of the additional terms. \square

Next we consider the convergence of the discretized processes with different initial values.

Lemma 3.2.3. *Denote by $\widehat{X}_{-k\tau+M\Delta t}^{-k\tau}$ and $\widehat{Y}_{-k\tau+M\Delta t}^{-k\tau}$ solutions of the modified Milstein scheme with initial values ξ and η respectively. Assume Conditions (A), (1.a) and Condition (2) for both initial values. Let $\Delta t = t/n$, $n \in \mathbb{Z}^+$, be sufficiently small. Then for any $\varepsilon > 0$, there exists an integer $M^* > 0$ such that for any $M \geq M^*$, we have*

$$\left\| \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} - \widehat{Y}_{-k\tau+M\Delta t}^{-k\tau} \right\|_2 < \varepsilon.$$

Proof. According to scheme (3.1.2) we have

$$\begin{aligned} & \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} - \widehat{Y}_{-k\tau+M\Delta t}^{-k\tau} \\ & = (I + A\Delta t)^M (\xi - \eta) + \Delta t \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} \widehat{F}_i \end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} \widehat{G}_i(\Delta W_i) \\
& + \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} \left[\widehat{F}_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - \widehat{F}_i^{(1)}(\widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}) \right] \frac{\Delta Z_i}{2\sqrt{\Delta t}} \\
& + \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} \left[\widehat{G}_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - \widehat{G}_i^{(1)}(\widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}) \right] \frac{(\Delta W_i)^2 - \Delta t}{4\sqrt{\Delta t}}.
\end{aligned}$$

Here

$$\begin{aligned}
\widehat{F}_i & := f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - f(i\Delta t, \widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}), \\
\widehat{G}_i & := g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - g(i\Delta t, \widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}), \\
\widehat{F}_i^{(1)}(x) & := f(i\Delta t, \widehat{Y}_+(x)) - f(i\Delta t, \widehat{Y}_-(x)), \\
\widehat{G}_i^{(1)}(x) & := g(i\Delta t, \widehat{Y}_+(x)) - g(i\Delta t, \widehat{Y}_-(x)).
\end{aligned}$$

Denote $\widehat{\zeta}_i = \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} - \widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}$. Then by Condition (1.a), we have

$$|\widehat{F}_i| \leq \beta_1 |\widehat{\zeta}_i|, \quad |\widehat{G}_i| \leq \beta_2 |\widehat{\zeta}_i|$$

and

$$\begin{aligned}
\left| \widehat{F}_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - \widehat{F}_i^{(1)}(\widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}) \right| & \leq 2K_1^* |\zeta_i| \sqrt{\Delta t}, \\
\left| \widehat{G}_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - \widehat{G}_i^{(1)}(\widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}) \right| & \leq 2K_2^* |\zeta_i| \sqrt{\Delta t}.
\end{aligned}$$

According to the method used in Lemma 3.1.1, we get the following result similar to inequality (3.2.8)

$$\begin{aligned}
& (1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{\zeta}_M \right|^2 \\
& \leq \|\xi - \eta\|_2^2 + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(\frac{\Delta t}{1 - \alpha\Delta t} \right)^2 \mathbb{E} \left| \widehat{F}_i \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| \widehat{G}_i \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{2\Delta t}{(1 - \alpha\Delta t)^2} \mathbb{E} \left| \left(\widehat{\zeta}_i \right)^T (I + A\Delta t) \widehat{F}_i \right| \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{(\Delta t)^{3/2}}{2(1 - \alpha\Delta t)^2} \mathbb{E} \left| \left(\widehat{G}_i \right)^T \left(\widehat{F}_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - \widehat{F}_i^{(1)}(\widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}) \right) \right|
\end{aligned}$$

$$\begin{aligned}
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{(\Delta t)^2}{12(1 - \alpha\Delta t)^2} \mathbb{E} \left| \widehat{F}_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - \widehat{F}_i^{(1)}(\widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
& + \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \frac{\Delta t}{8(1 - \alpha\Delta t)^2} \mathbb{E} \left| \widehat{G}_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) - \widehat{G}_i^{(1)}(\widehat{Y}_{-k\tau+i\Delta t}^{-k\tau}) \right|^2 \\
& \leq \|\xi - \eta\|_2^2 + \widehat{K}_5 \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \mathbb{E} \left| \widehat{\zeta}_i \right|^2,
\end{aligned}$$

where

$$\begin{aligned}
C &= \beta_1^2 + 2\beta_1 |A| + K_1^* \beta_2 + \frac{\Delta t}{3} (K_1^*)^2 + \frac{\Delta t}{2} (K_2^*)^2 \\
\widehat{K}_5 &= \frac{\Delta t}{(1 - \alpha\Delta t)^2} (2\beta_1 + \beta_2^2 + C\Delta t).
\end{aligned}$$

We choose Δt small enough such that

$$2\beta_1 + \beta_2^2 + C\Delta t + \alpha^2\Delta t < 2\alpha.$$

Therefore, we have

$$(1 - \alpha\Delta t)^2 (1 + \widehat{K}_5) < 1.$$

The discrete Gronwall inequality implies

$$(1 - \alpha\Delta t)^{-2M} \mathbb{E} \left| \widehat{\zeta}_M \right|^2 \leq \|\xi - \eta\|_2^2 \prod_{i=0}^{M-1} (1 + \widehat{K}_5) = \|\xi - \eta\|_2^2 (1 + \widehat{K}_5)^M.$$

Finally

$$\mathbb{E} \left| \widehat{\zeta}_M \right|^2 \leq \|\xi - \eta\|_2^2 \left((1 - \alpha\Delta t)^2 (1 + \widehat{K}_5) \right)^M < \varepsilon$$

with sufficiently large M . \square

In the numerical scheme, the process is considered as two parts, $[-k\tau, 0)$ and $[0, r]$. Define

$$\widehat{X}_r^{-k\tau} := \widehat{X}(r, 0, \omega) \circ \widehat{X}_0^{-k\tau}, \quad (3.2.9)$$

where $\widehat{X}(r, 0, \omega)$, $r \geq 0$, is finite time Milstein approximation of the solution of stochastic differential equation with time step size Δt , till $N'\Delta t \leq r$, where N' is the unique number such that $N'\Delta t \leq r$ and $(N' + 1)\Delta t > r$. If $N'\Delta t < r$, define

$$\begin{aligned}
\widehat{X}(r, 0, \omega)\xi &= \widehat{X}(N'\Delta t, 0, \omega) + f(N'\Delta t, \widehat{X}(N'\Delta t, 0, \omega))(r - N'\Delta t) \\
&\quad + g(N'\Delta t, \widehat{X}(N'\Delta t, 0, \omega))(W_r - W_{N'\Delta t})
\end{aligned} \quad (3.2.10)$$

Lemma 3.2.4. (*Continuity of the discrete semi-flow with respect to the initial value*) Denote by \tilde{X}_r^0 and \tilde{Y}_r^0 the solution of the finite Milstein scheme with the initial values $\tilde{\xi}$ and $\tilde{\eta}$ at time 0. Assume Conditions (A), (1.a) and Condition (2) for both initial values. Let Δt be sufficiently small, $p \geq 1$. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any $\left\| \tilde{\xi} - \tilde{\eta} \right\|_p < \delta$, we have

$$\left\| \tilde{X}_r^0(\omega, \tilde{\xi}) - \tilde{Y}_r^0(\omega, \tilde{\eta}) \right\|_p < \varepsilon.$$

Proof. Note that $\tilde{X}_{N'\Delta t}^0$ and $\tilde{Y}_{N'\Delta t}^0$ satisfy analogues of (3.2.5), with initial value $\tilde{\xi}$ and $\tilde{\eta}$ at time 0 instead of $-\kappa\tau$. Apply the Milstein scheme on the finite time $r' = N'\Delta t$ to obtain

$$\begin{aligned} & \left| \tilde{X}_{r'}^0(\omega, \tilde{\xi}) - \tilde{Y}_{r'}^0(\omega, \tilde{\eta}) \right|^p \\ & \leq 4^{p-1} \left| (I + A\Delta t)^{pN'} \left| \tilde{\xi} - \tilde{\eta} \right|^p + 4^{p-1} (\Delta t)^p \left| (I + A\Delta t)^{pN'} \left| \sum_{i=0}^{N'-1} (I + A\Delta t)^{-i-1} \tilde{F}_i \right|^p \right. \right. \\ & \quad + 4^{p-1} \left| (I + A\Delta t)^{pN'} \left| \sum_{i=0}^{N'-1} (I + A\Delta t)^{-i-1} \tilde{G}_i (W_{(i+1)\Delta t} - W_{i\Delta t}) \right|^p \right. \\ & \quad + 4^{p-1} \left| (I + A\Delta t)^{pN'} \left| \sum_{i=0}^{N'-1} (I + A\Delta t)^{-i-1} \right. \right. \\ & \quad \quad \left. \left. \times \left[\tilde{G}_i^{(1)}(\tilde{X}_{i\Delta t}^0) - \tilde{G}_i^{(1)}(\tilde{Y}_{i\Delta t}^0) \right] \frac{(\Delta W_i)^2 - dt}{4\sqrt{\Delta t}} \right|^p, \end{aligned} \quad (3.2.11)$$

where

$$\begin{aligned} \tilde{F}_i & := f(i\Delta t, \tilde{X}_{i\Delta t}^0) - f(i\Delta t, \tilde{Y}_{i\Delta t}^0), \\ \tilde{G}_i & := g(i\Delta t, \tilde{X}_{i\Delta t}^0) - g(i\Delta t, \tilde{Y}_{i\Delta t}^0), \\ \tilde{G}_i^{(1)}(x) & := g(i\Delta t, \hat{\Upsilon}_+(x)) - g(i\Delta t, \hat{\Upsilon}_-(x)). \end{aligned}$$

Denote $\tilde{\zeta}_i := \tilde{X}_{i\Delta t}^0 - \tilde{Y}_{i\Delta t}^0$. For convenience, we denote $C_p = 4^{p-1}$, $C_{p,N'} = 4^{p-1} N'^{p-1}$. Taking expectation on both sides of (3.2.11), and noting that the Lipschitz condition of function f and g , we have

$$\begin{aligned} & (1 - \alpha\Delta t)^{-pN'} \left\| \tilde{\zeta}_{N'} \right\|_p^p \\ & \leq C_p \left\| \tilde{\xi} - \tilde{\eta} \right\|_p^p + C_{p,N'} (\Delta t)^p \sum_{i=0}^{N'-1} (1 - \alpha\Delta t)^{-(i+1)p} \beta_1^p \left\| \tilde{\zeta}_i \right\|_p^p \end{aligned}$$

$$\begin{aligned}
& + C_{p,N'}(\Delta t)^{p/2} \sum_{i=0}^{N'-1} (1 - \alpha\Delta t)^{-(i+1)p} \beta_2^p \left\| \tilde{\zeta}_i \right\|_p^p \\
& + C_{p,N'}(\Delta t)^p \sum_{i=0}^{N'-1} (1 - \alpha\Delta t)^{-(i+1)p} \frac{(2K_2^*)^p (2p!! + 1)}{4} \left\| \tilde{\zeta}_i \right\|_p^p \\
& = C_p \left\| \tilde{\xi} - \tilde{\eta} \right\|_p^p + \tilde{K} \sum_{i=0}^{N'-1} (1 - \alpha\Delta t)^{-ip} \left\| \tilde{\zeta}_i \right\|_p^p,
\end{aligned}$$

where

$$\tilde{K} = \frac{C_{p,N'} \left((\Delta t)^p \left(\beta_1^p + \frac{(2K_2^*)^p (2p!! + 1)}{2} \right) + (\Delta t)^{p/2} \beta_2^p \right)}{(1 - \alpha\Delta t)^p},$$

which is bounded for any $1 \leq p < +\infty$. Then by the Gronwall inequality, we have

$$\left\| \tilde{\zeta}_{N'} \right\|_p^p \leq C_p \left\| \tilde{\xi} - \tilde{\eta} \right\|_p^p \left((1 + \tilde{K})(1 - \alpha\Delta t)^p \right)^{N'}.$$

Note

$$\begin{aligned}
& (1 + \tilde{K})(1 - \alpha\Delta t)^p \\
& \leq (1 - \alpha\Delta t)^p + C_{p,N'} \left((\Delta t)^p \left(\beta_1^p + \frac{(2K_2^*)^p (2p!! + 1)}{2} \right) + (\Delta t)^{p/2} \beta_2^p \right) \\
& \leq 1 + C_{p,N'}.
\end{aligned}$$

The result (3.2.4) at $r' = N'\Delta t$ follows

$$\delta = \frac{\varepsilon}{C_p} (1 + C'_{p,N'})^{-N'}.$$

Finally (3.2.4) at time r follows (3.2.10) and the estimate at $r' = N'\Delta t$. \square

Theorem 3.2.5. *Assume that Condition(A), (1.a) and Δt is fixed and small enough. The time domain is divided as $\tau = n\Delta t$. Then there exists $\hat{X}_r^* \in L^2(\Omega)$ such that for any fixed initial values ξ , the solution of the modified Milstein scheme satisfies*

$$\lim_{k \rightarrow \infty} \left\| \hat{X}_r^{-k\tau}(\xi) - \hat{X}_r^* \right\|_2 = 0, \quad (3.2.12)$$

and \hat{X}_r^* satisfies the random periodicity property.

The proof follows the construction of Cauchy sequence $\hat{X}_{-k\tau + M\Delta t}^{-k\tau}$ as in Theorem 3.1.4. We know $\hat{X}_{-k\tau + M\Delta t}^{-k\tau} \in L^2(\Omega)$ from Lemma 3.2.2 and the convergence of the sequence by Lemma 3.2.3. Then we obtain the existence of the limit and it is not hard to prove the random periodicity for the process.

3.2.2 The error estimation

We proved the existence of random periodic solutions of SDE (2.0.1) and its discretisations with modified Milstein scheme as the limits of semi-flows when the starting times were pushed to $-\infty$. Now we estimate the error between these two limits. It is natural to consider the difference between the discrete approximate solution and the exact solution. Let us recall the exact solution at time $-k\tau + N\Delta t$ as follows

$$\begin{aligned} X_{-k\tau+N\Delta t}^{-k\tau}(\omega, \xi) = & e^{AN\Delta t}\xi + e^{A(N\Delta t-k\tau)} \int_{-k\tau}^{N\Delta t-k\tau} e^{-As} f(s, X_s^{-k\tau}) ds \\ & + e^{A(N\Delta t-k\tau)} \int_{-k\tau}^{N\Delta t-k\tau} e^{-As} g(s, X_s^{-k\tau}) dW_s. \end{aligned} \quad (3.2.13)$$

Then we have the following theorem about the strong error of the modified Milstein scheme.

Theorem 3.2.6. *Assume Conditions (A), (1.a) and (2). Choose $\Delta t = t/n$ for some $n \in \mathbb{N}$ and $N = kn$. If $X_0^{-k\tau}$ and $\widehat{X}_0^{-k\tau}$ are the exact and the numerical solutions given by (3.2.13) and (3.2.5) respectively, then there exists a constant $K > 0$ such that for any sufficiently small fixed Δt , we have*

$$\limsup_{k \rightarrow \infty} \left\| X_0^{-k\tau} - \widehat{X}_0^{-k\tau} \right\|_2 \leq K\Delta t.$$

Proof. For any $M \leq N$, we have

$$\begin{aligned} & X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \\ = & \left(e^{AM\Delta t} - (I + A\Delta t)^M \right) \xi + e^{A(M\Delta t-k\tau)} \int_{-k\tau}^{M\Delta t-k\tau} e^{-As} f(s, X_s^{-k\tau}) ds \\ & - \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \Delta t \\ & - \sum_{i=0}^{M-1} \left\{ (I + A\Delta t)^{M-i-1} \frac{\Delta Z_i}{2\sqrt{\Delta t}} \right. \\ & \quad \left. \times \left[f\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - f\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right] \right\} \\ & + e^{A(M\Delta t-k\tau)} \int_{-k\tau}^{M\Delta t-k\tau} e^{-As} g(s, X_s^{-k\tau}) dW_s \\ & - \sum_{i=0}^{M-1} (I + A\Delta t)^{M-i-1} g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) (\Delta W_i) \end{aligned}$$

$$- \sum_{i=0}^{M-1} \left\{ (I + A\Delta t)^{M-i-1} \frac{[(\Delta W_i)^2 - \Delta t]}{4\sqrt{\Delta t}} \right. \\ \left. \times \left[g\left(i\Delta t, \widehat{\Upsilon}_+(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) - g\left(i\Delta t, \widehat{\Upsilon}_-(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})\right) \right] \right\}.$$

Applying the method of Lemma 3.2.2, we firstly consider

$$(1 - \alpha\Delta t)^{-2M} \left| X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right|^2 \\ = \sum_{i=0}^{M-1} (1 - \alpha\Delta t)^{-2i} \left(\frac{\left| X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \right). \quad (3.2.14)$$

For simplicity we denote

$$B_1 = \frac{1}{1 - \alpha\Delta t} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left[e^{-A(s+k\tau-(i+1)\Delta t)} f(s, X_s^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right. \\ \left. - \int_{i\Delta t - k\tau}^s F_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right] ds. \\ B_2 = \frac{1}{1 - \alpha\Delta t} \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left[e^{-A(s+k\tau-(i+1)\Delta t)} g(s, X_s^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right. \\ \left. - \int_{i\Delta t - k\tau}^s G_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right] dW_s,$$

with

$$F_i^{(1)}(x) = \frac{1}{2\sqrt{\Delta t}} \left(f\left(i\Delta t, \widehat{\Upsilon}_+(x)\right) - f\left(i\Delta t, \widehat{\Upsilon}_-(x)\right) \right), \\ G_i^{(1)}(x) = \frac{1}{2\sqrt{\Delta t}} \left(g\left(i\Delta t, \widehat{\Upsilon}_+(x)\right) - g\left(i\Delta t, \widehat{\Upsilon}_-(x)\right) \right).$$

Hence we have

$$X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \\ = e^{A\Delta t} X_{-k\tau+i\Delta t}^{-k\tau} - (I + A\Delta t) \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} + (1 - \alpha\Delta t) (B_1 + B_2).$$

Now we consider

$$\frac{\left| X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right|^2}{(1 - \alpha\Delta t)^2} - \left| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \\ = \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} - I \right) \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} + I \right)$$

$$\begin{aligned}
& \times \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) \\
& + \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right)^2 \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) + B_1^T B_1 + B_2^T B_2 \\
& + 2 \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} \right) \left(\frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right) \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) \\
& + 2 \left(\left(X_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} \right) - \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} \right) \right) B_1 \\
& + 2 \left(\left(X_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} \right) - \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{I + A\Delta t}{1 - \alpha\Delta t} \right) \right) B_2 + 2B_1^T B_2.
\end{aligned} \tag{3.2.15}$$

We know that the matrix $\left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} - I \right) \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} + I \right)$ can be non-positive-definite when we choose the Δt small enough. Now we consider each term in (3.2.15). First,

$$\begin{aligned}
& \mathbb{E} \left[\left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right)^2 \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right] \\
& \leq \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \left| \frac{\frac{1}{2}A^2 (\Delta t)^2}{1 - \alpha\Delta t} \right|^2 \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \\
& \leq \widehat{K}_6 (\Delta t)^4.
\end{aligned}$$

Next,

$$\begin{aligned}
& \mathbb{E} [B_1^T B_1] = \mathbb{E} |B_1|^2 \\
& \leq \frac{3(1 + \mu)}{\mu(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left| e^{-A(s+k\tau - (i+1)\Delta t)} - I \right| \left\| f(s, X_s^{-k\tau}) \right\|_2 ds \right)^2 \\
& + \frac{3(1 + \mu)}{\mu(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(s, X_s^{-k\tau}) - f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) \right. \right. \\
& \quad \left. \left. - \int_{i\Delta t - k\tau}^s F_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2 ds \right)^2 \\
& + \frac{1 + \mu}{(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 ds \right)^2 \\
& + \frac{3(1 + \mu)}{\mu(1 - \alpha\Delta t)^2} \left(\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| \int_{i\Delta t - k\tau}^s F_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) \right. \right. \\
& \quad \left. \left. - F_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2 ds \right)^2, \tag{3.2.16}
\end{aligned}$$

where μ is a small number from Young's inequality, which will be fixed later. By linear growth property of f and Lemma 2.2.3, we know that $\|f(s, X_s^{-k\tau})\|_2$ is bounded. So for the first term in (3.2.16) we only need to consider

$$\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} |e^{-A(s+k\tau-(i+1)\Delta t)} - I| ds \leq \frac{(\Delta t)^2}{2} Tr(-A).$$

Applying Itô's formula to $f(i\Delta t, X_s^{-k\tau})$, we have

$$\begin{aligned} & \left\| f(i\Delta t, X_s^{-k\tau}) - f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - \int_{i\Delta t - k\tau}^s g \frac{\partial f}{\partial x} dW_v \right\|_2 \\ & \leq \widehat{K}_7 |s - i\Delta t + k\tau| \end{aligned}$$

By Condition (1.a) and Taylor expansion as (3.2.2), the second term in (3.2.16) becomes

$$\begin{aligned} & \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(s, X_s^{-k\tau}) - f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - \int_{i\Delta t - k\tau}^s F_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2 ds \\ & \leq 3 \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \|f(s, X_s^{-k\tau}) - f(i\Delta t, X_s^{-k\tau})\|_2 ds \\ & \quad + 3 \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(i\Delta t, X_s^{-k\tau}) - f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - \int_{i\Delta t - k\tau}^s g \frac{\partial f}{\partial x} dW_v \right\|_2 ds \\ & \quad + 3 \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| \int_{i\Delta t - k\tau}^s g \frac{\partial f}{\partial x}(X_{-k\tau+i\Delta t}^{-k\tau}) - F_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2 ds \\ & \leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} 3(C_0 + \widehat{K}_7) |s - i\Delta t + k\tau| + 3C\sqrt{\Delta t}\sqrt{s - i\Delta t + k\tau} ds \\ & \leq \widehat{K}_8 (\Delta t)^2. \end{aligned}$$

Applying the global Lipschitz condition, the third term of (3.2.16) becomes

$$\begin{aligned} & \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| f(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - f(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2 ds \\ & \leq \beta_1 \Delta t \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2. \end{aligned}$$

Considering the last term of (3.2.16)

$$\begin{aligned} & \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| \int_{i\Delta t - k\tau}^s F_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) - F_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2 ds \\ & \leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} 2K_1^* \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \sqrt{s - i\Delta t + k\tau} ds \end{aligned}$$

$$\leq \widehat{K}_9(\Delta t)^{3/2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2.$$

We summarise the above inequalities to have

$$\begin{aligned} \mathbb{E} [B_1^T B_1] &\leq \widehat{K}_{10} (\Delta t)^4 + \frac{(1+\mu)\beta_1^2 (\Delta t)^2}{(1-\alpha\Delta t)^2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \\ &\quad + \widehat{K}_9^2 (\Delta t)^3 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2. \end{aligned}$$

This term is of the 4th order of Δt and 2nd order of Δt with $\left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2$. Similar to the $\mathbb{E} [B_1^T B_1]$, the following term can be estimated as

$$\begin{aligned} \mathbb{E} [B_2^T B_2] &= \mathbb{E} |B_2|^2 \\ &\leq \frac{3(1+\mu)}{\mu(1-\alpha\Delta t)^2} \int_{i\Delta t-k\tau}^{(i+1)\Delta t-k\tau} |e^{-A(s+k\tau-(i+1)\Delta t)} - I|^2 \|g(s, X_s^{-k\tau})\|_2^2 ds \\ &\quad + \frac{3(1+\mu)}{\mu(1-\alpha\Delta t)^2} \int_{i\Delta t-k\tau}^{(i+1)\Delta t-k\tau} \left\| g(s, X_s^{-k\tau}) - g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) \right. \\ &\quad \quad \left. - \int_{i\Delta t-k\tau}^s G_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2^2 ds \\ &\quad + \frac{1+\mu}{(1-\alpha\Delta t)^2} \int_{i\Delta t-k\tau}^{(i+1)\Delta t-k\tau} \left\| g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2^2 ds \\ &\quad + \frac{3(1+\mu)}{\mu(1-\alpha\Delta t)^2} \int_{i\Delta t-k\tau}^{(i+1)\Delta t-k\tau} \left\| \int_{i\Delta t-k\tau}^s G_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) - G_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2^2 ds, \end{aligned} \tag{3.2.17}$$

where μ is the same as it in the estimation of $\mathbb{E} [B_1^T B_1]$. By the linear growth property of g and Lemma 2.2.3, we know that $\|g(s, X_s^{-k\tau})\|_2^2$ and s are bounded. So we only need to consider

$$\int_{i\Delta t-k\tau}^{(i+1)\Delta t-k\tau} |e^{-A(s+k\tau-(i+1)\Delta t)} - I|^2 ds \leq \frac{2}{3} (\Delta t)^3 \text{Tr}(A^2).$$

Applying Itô's formula to $g(i\Delta t, X_s^{-k\tau})$, we have

$$\begin{aligned} &\left\| g(i\Delta t, X_s^{-k\tau}) - g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - \int_{i\Delta t-k\tau}^s g \frac{\partial g}{\partial x} dW_v \right\|_2^2 \\ &\leq \widehat{K}_{11} |s - i\Delta t + k\tau|^2 \end{aligned}$$

By Condition (1.a) and Taylor expansion as (3.2.3), the second term in (3.2.17) becomes

$$\int_{i\Delta t-k\tau}^{(i+1)\Delta t-k\tau} \left\| g(s, X_s^{-k\tau}) - g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - \int_{i\Delta t-k\tau}^s G_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2^2 ds$$

$$\begin{aligned}
&\leq 3 \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| g(s, X_s^{-k\tau}) - g(i\Delta t, X_s^{-k\tau}) \right\|_2^2 ds \\
&\quad + 3 \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| g(i\Delta t, X_s^{-k\tau}) - g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - \int_{i\Delta t - k\tau}^s g \frac{\partial g}{\partial x} dW_v \right\|_2^2 ds \\
&\quad + 3 \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| \int_{i\Delta t - k\tau}^s g \frac{\partial g}{\partial x} (X_{-k\tau+i\Delta t}^{-k\tau}) - G_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) dW_v \right\|_2^2 ds \\
&\leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} 2(C_0^2 + \widehat{K}_{11}) |s - i\Delta t + k\tau|^2 + C\Delta t |s - i\Delta t + k\tau| ds \\
&\leq \widehat{K}_{12} (\Delta t)^3.
\end{aligned}$$

The third term follows from the global Lipschitz condition

$$\begin{aligned}
&\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| g(i\Delta t, X_{-k\tau+i\Delta t}^{-k\tau}) - g(i\Delta t, \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right\|_2^2 ds \\
&\leq \beta_2^2 \Delta t \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2.
\end{aligned}$$

Using Itô isometry to estimate the last term

$$\begin{aligned}
&\int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} \left\| \int_{i\Delta t - k\tau}^s \left[G_i^{(1)}(X_{-k\tau+i\Delta t}^{-k\tau}) - G_i^{(1)}(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right] dW_v \right\|_2^2 ds \\
&\leq \int_{i\Delta t - k\tau}^{(i+1)\Delta t - k\tau} (2K_2^*)^2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 |s - i\Delta t + k\tau| ds \\
&\leq \widehat{K}_{13} (\Delta t)^2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2.
\end{aligned}$$

Conclude the above results to obtain

$$\begin{aligned}
\mathbb{E} [B_2^T B_2] &\leq \widehat{K}_{14} (\Delta t)^3 + \frac{(1 + \mu)\beta_2^2 \Delta t}{(1 - \alpha\Delta t)^2} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \\
&\quad + \widehat{K}_{13} (\Delta t)^2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2.
\end{aligned}$$

The fifth term of (3.2.15) can be estimate as follows

$$\begin{aligned}
&\mathbb{E} \left[2 \left(X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \left(\frac{e^{A\Delta t}}{1 - \alpha\Delta t} \right) \left(\frac{e^{A\Delta t} - I - A\Delta t}{1 - \alpha\Delta t} \right) \left(\widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right) \right] \\
&\leq 2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \frac{\frac{1}{2}|A^2|(\Delta t)^2}{(1 - \alpha\Delta t)^2} \left\| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2 \\
&\leq \widehat{K}_{15} (\Delta t)^2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2.
\end{aligned}$$

To estimate the sixth term of (3.2.15),

$$\begin{aligned} & \mathbb{E} \left[2 \left((X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} \right) - (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) \right) B_1 \right] \\ = & \mathbb{E} \left[2 (X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} - \frac{I+A\Delta t}{1-\alpha\Delta t} \right) B_1 \right] \\ & + \mathbb{E} \left[2 (X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) B_1 \right]. \end{aligned}$$

These two terms are considered separately,

$$\begin{aligned} & \mathbb{E} \left[2 (X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} - \frac{I+A\Delta t}{1-\alpha\Delta t} \right) B_1 \right] \\ \leq & 2 \|X_{-k\tau+i\Delta t}^{-k\tau}\|_2 \frac{|\frac{1}{2}A^2(\Delta t)^2|}{1-\alpha\Delta t} \|B_1\|_2 \\ \leq & \widehat{K}_{16}(\Delta t)^4 + \frac{\sqrt{1+\mu}\beta_1\widehat{K}_{17}(\Delta t)^3}{(1-\alpha\Delta t)^2} \|X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\|_2. \end{aligned}$$

And,

$$\begin{aligned} & \mathbb{E} \left[2 (X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) B_1 \right] \\ \leq & \frac{2\sqrt{\widehat{K}_{10}}(\Delta t)^2}{1-\alpha\Delta t} \|X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\|_2 (1+\Delta t|A|) \\ & + \frac{2\sqrt{1+\mu}\beta_1\Delta t}{(1-\alpha\Delta t)^2} \|X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\|_2^2 (1+\Delta t|A|) \\ & + 2\widehat{K}_9(\Delta t)^{3/2} \|X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\|_2^2 (1+\Delta t|A|). \end{aligned}$$

We use the conditional expectation to eliminate the seventh term

$$\begin{aligned} & \mathbb{E} \left[\left((X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} \right) - (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) \right) B_2 \right] \\ = & \mathbb{E} \left[\left((X_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{e^{A\Delta t}}{1-\alpha\Delta t} \right) - (\widehat{X}_{-k\tau+i\Delta t}^{-k\tau})^T \left(\frac{I+A\Delta t}{1-\alpha\Delta t} \right) \right) \mathbb{E} [B_2 | \mathcal{F}^{i\Delta t-k\tau}] \right] \\ = & 0. \end{aligned}$$

For the last term,

$$\begin{aligned} \mathbb{E} [2B_1^T B_2] & \leq 2 \|B_1^T\|_2 \cdot \|B_2\|_2 \\ & \leq \widehat{K}_{18}(\Delta t)^{7/2} + \widehat{K}_{19}(\Delta t)^{3/2} \|X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau}\|_2^2. \end{aligned}$$

Combining all the estimation above, we conclude the inequality with constants

$$\begin{aligned} & \left| \frac{X_{-k\tau+(i+1)\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau}}{(1-\alpha\Delta t)^2} \right|^2 - \left| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right|^2 \\ & \leq \left(\frac{(1+\mu)\beta_2^2\Delta t}{(1-\alpha\Delta t)^2} + \frac{2\sqrt{(1+\mu)}\beta_1\Delta t}{(1-\alpha\Delta t)^2} + \widehat{K}_{20}(\Delta t)^{3/2} \right) \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 \\ & \quad + \widehat{K}_{21}(\Delta t)^3 + \widehat{K}_{22}(\Delta t)^2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2. \end{aligned}$$

Now we notice that the term $\left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2$ has coefficients, the largest of which contains a constant multiplied by Δt . The largest free term contains a constant multiplied by $(\Delta t)^3$. Choosing μ and Δt small enough and applying Young's inequality for the term $(\Delta t)^2 \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2$, and from (3.2.14) we get

$$\begin{aligned} & (1-\alpha\Delta t)^{-2M} \left\| X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right\|_2^2 \\ & \leq \sum_{i=0}^{M-1} (1-\alpha\Delta t)^{-2i} \left(\widehat{K}_{23}\Delta t \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2 + \widehat{K}_{24}(\Delta t)^3 \right) \\ & \leq \widehat{K}_{25}(\Delta t)^2 (1-\alpha\Delta t)^{-2M} + \widehat{K}_{23}(\Delta t) \sum_{i=0}^{M-1} (1-\alpha\Delta t)^{-2i} \left\| X_{-k\tau+i\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+i\Delta t}^{-k\tau} \right\|_2^2, \end{aligned} \tag{3.2.18}$$

where

$$\begin{aligned} \widehat{K}_{23} &= \frac{(1+\mu)(2\beta_1 + \beta_2^2 + \varepsilon)}{(1-\alpha\Delta t)^2} \\ \widehat{K}_{25} &= \frac{\widehat{K}_{24}(1-\alpha\Delta t)^2}{2\alpha\Delta t - \alpha^2(\Delta t)^2}(\Delta t) = \frac{\widehat{K}_{24}(1-\alpha\Delta t)^2}{2\alpha - \alpha^2(\Delta t)}. \end{aligned}$$

Here μ , ε and the time step Δt are chosen small enough such that

$$\widehat{K}_{23}\Delta t + 1(1-\alpha\Delta t)^2 < 1. \tag{3.2.19}$$

Now using the discrete time Gronwall inequality, from (3.2.18), we have

$$\begin{aligned} & \left\| X_{-k\tau+M\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+M\Delta t}^{-k\tau} \right\|_2^2 \\ & \leq \widehat{K}_{25}(\Delta t)^2 + \widehat{K}_{25}\widehat{K}_{23}(\Delta t)^2 \frac{1 - \left((1 + \widehat{K}_{23}\Delta t) (1 - \alpha\Delta t)^2 \right)^M}{1 - \left((1 + \widehat{K}_{23}\Delta t) (1 - \alpha\Delta t)^2 \right)} \end{aligned}$$

$$\leq \widehat{K}_{26}(\Delta t)^2.$$

We can find a constant \widehat{K}_{26} which is independent of M and Δt . We take $M = N$, where $N\Delta t = k\tau$, then

$$\limsup_{k \rightarrow \infty} \left\| X_0^{-k\tau} - \widehat{X}_0^{-k\tau} \right\|_2 = \limsup_{N \rightarrow \infty} \left\| X_{-k\tau+N\Delta t}^{-k\tau} - \widehat{X}_{-k\tau+N\Delta t}^{-k\tau} \right\|_2 \leq \sqrt{\widehat{K}_{26}} \Delta t. \quad \square$$

Remark 3.2.7. *The reason we applied modified Milstein scheme instead of the classical one is because the local inaccuracy of the estimation for function f would decrease the order of error for $\mathbb{E}[B_1^T B_1]$. The corresponding consequence is the coefficient \widehat{K}_{23} in Gronwall inequality would involve more terms, which leads to the failure of inequality (3.2.19). It is not a problem for finite horizon as the constant $C(T)$ could be T -dependent, where T is the length of the approximation. To guarantee the convergence of the random periodic solution, we need to find a time independent scheme for the estimation. To avoid the complexity of modify the dissipative condition, our choice is to introduce the higher order terms to eliminate the influence of the inaccuracy from the estimation of the function f .*

We have proved the estimation of error from $-k\tau$ to 0 as $k \rightarrow \infty$ can be controlled under the order 1 of the time-step. And the upper bound is uniform in time. The following theorem will give us the more general result, which is from $-k\tau$ to time r . Let $\widehat{X}_r^{-k\tau}$, $r > 0$ be given by (3.2.9).

Theorem 3.2.8. *Assume Conditions (A), (1.a) and (2). We choose $\Delta t = t/n$ for some $n \in \mathbb{N}$, $N = kn$ and N' is the unique integer such that $N'\Delta t \leq r$, $(N'+1)\Delta t > r$ for $r \in [0, T]$. If $X_r^{-k\tau}$ is the exact solution while $\widehat{X}_r^{-k\tau}$ is the numerical solution given by (3.2.9). Then there exists a constant $\widetilde{K} > 0$ such that for any sufficiently small fixed Δt ,*

$$\limsup_{k \rightarrow \infty} \left\| X_r^{-k\tau} - \widehat{X}_r^{-k\tau} \right\|_2 \leq \widetilde{K} \Delta t,$$

for all $r \in [0, T]$, where \widetilde{K} is independent of Δt .

Proof. According to the semi-flow property, we have

$$X_r^{-k\tau}(\omega) - \widehat{X}_r^{-k\tau}(\omega) = X_r^0(\omega) \circ X_0^{-k\tau}(\omega) - \widehat{X}_r^0(\omega) \circ \widehat{X}_0^{-k\tau}(\omega),$$

where \widehat{X}_r^0 is finite time Milstein approximation of solution of (2.0.1) from 0 to r and $\widehat{X}_0^{-k\tau}$ is defined as before. So,

$$\begin{aligned} \left\| X_r^{-k\tau} - \widehat{X}_r^{-k\tau} \right\|_2 &\leq \left\| X_r^0 \circ X_0^{-k\tau} - X_r^0 \circ \widehat{X}_0^{-k\tau} \right\|_2 \\ &\quad + \left\| X_r^0 \circ \widehat{X}_0^{-k\tau} - \widehat{X}_r^0 \circ \widehat{X}_0^{-k\tau} \right\|_2. \end{aligned} \quad (3.2.20)$$

For the first term on the right-hand side, by Theorem 3.2.6, we have

$$\left\| X_0^{-k\tau} - \widehat{X}_0^{-k\tau} \right\| \leq K\Delta t.$$

By the continuity of $X_r^0(\cdot)$ with respect to initial values in $L^2(\Omega)$ ([31]), then

$$\left\| X_r^0 \circ X_0^{-k\tau} - X_r^0 \circ \widehat{X}_0^{-k\tau} \right\|_2 \leq C \left\| X_0^{-k\tau} - \widehat{X}_0^{-k\tau} \right\|_2 \leq C_5\Delta t,$$

where C_5 is independent of Δt . For the second term on the right-hand side of (3.2.20), it is finite time Milstein approximation with same initial value. By Theorem 10.3.5 in Kloeden and Platen [29], there exists a constant $C_6 > 0$ such that for sufficiently $\Delta t > 0$,

$$\left\| X_r^0 \circ \widehat{X}_0^{-k\tau} - \widehat{X}_r^0 \circ \widehat{X}_0^{-k\tau} \right\|_2 \leq C_6\Delta t,$$

where the choice of C_6 is independent of Δt . The result follows by taking $\widetilde{K} = C_5 + C_6$. \square

Corollary 3.2.9. *If we denote by X_r^* and \widehat{X}_r^* the exact and numerical approximating random periodic solution of equation (2.0.1) were given in Theorem 2.2.6 and Theorem 3.1.4 respectively, then*

$$\left\| X_r^* - \widehat{X}_r^* \right\|_2 \leq \widetilde{K}\sqrt{\Delta t}.$$

Proof. The result follows from

$$\left\| X_r^* - \widehat{X}_r^* \right\|_2 \leq \limsup_{k \rightarrow \infty} \left[\left\| X_r^* - X_r^{-k\tau} \right\|_2 + \left\| X_r^{-k\tau} - \widehat{X}_r^{-k\tau} \right\|_2 + \left\| \widehat{X}_r^{-k\tau} - \widehat{X}_r^* \right\|_2 \right]. \quad \square$$

Example 3.2.10. *To illustrate the errors in Theorems 3.1.7 and 3.2.8, we simulate the random periodic solution of Example 3.1.5 with 2000 different noise realisations by both Euler-Maruyama method and modified Milstein method.*

For the approximation of the increments ΔW_i and ΔZ_i , we use the method of Kloeden and Platen in [29] as follows,

$$\begin{aligned}\Delta W_i &= W_{-k\tau+(i+1)\Delta t}^1 - W_{-k\tau+i\Delta t}^1, \\ \Delta Z_i &= \frac{1}{2}\Delta t \left[(W_{-k\tau+(i+1)\Delta t}^1 - W_{-k\tau+i\Delta t}^1) + \frac{1}{\sqrt{3}} (W_{-k\tau+(i+1)\Delta t}^2 - W_{-k\tau+i\Delta t}^2) \right],\end{aligned}$$

where W^1 and W^2 are two independent Wiener processes.

We then apply Monte Carlo method to obtain the root mean square errors between the exact random periodic solution and the respective numerical schemes with 12 different step sizes:

$$\begin{aligned}1 \times 10^{-5}, 2 \times 10^{-5}, 3 \times 10^{-5}, 4 \times 10^{-5}, \\ 1 \times 10^{-4}, 2 \times 10^{-4}, 3 \times 10^{-4}, 4 \times 10^{-4}, \\ 1 \times 10^{-3}, 2 \times 10^{-3}, 3 \times 10^{-3}, 4 \times 10^{-3},\end{aligned}$$

where the exact one is given explicitly as

$$X_t^* = \int_{-\infty}^t e^{-(\pi+\frac{1}{2})(t-s)+W_t-W_s} \sin(\pi s) ds.$$

The relationship between the root mean square errors and the step size is shown in the log-log plot Fig. 3.3. The difference of the orders of convergence between the Euler-Maruyama method and Milstein method is clear from the numerical simulations.

3.3 Periodic measures

Let $\mathcal{P}(\mathbb{R}^m)$ denote all probability measures on \mathbb{R}^m . For $P_1, P_2 \in \mathcal{P}(\mathbb{R}^m)$, define metric $d_{\mathbb{L}}$ as follows:

$$d_{\mathbb{L}}(P_1, P_2) = \sup_{\varphi \in \mathbb{L}} \left| \int_{\mathbb{R}^m} \varphi(x) P_1(dx) - \int_{\mathbb{R}^m} \varphi(x) P_2(dx) \right|,$$

where

$$\mathbb{L} = \{\varphi : \mathbb{R}^m \rightarrow \mathbb{R} : |\varphi(x) - \varphi(y)| \leq |x - y| \text{ and } |\varphi(\cdot)| \leq 1\}.$$

From the result of Ikeda and Watanabe [25], it is not difficult to prove that the metric $d_{\mathbb{L}}$ is equivalent to the weak topology. This useful observation was made by Yuan and Mao [53].

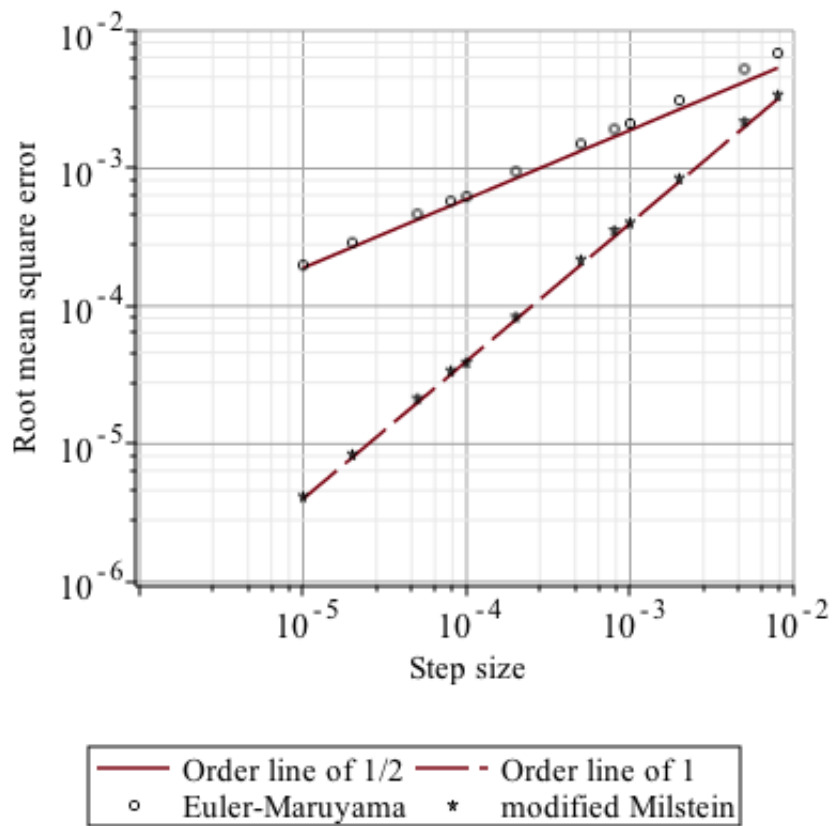


Figure 3.3: Root mean square error versus step size as log-log plot for the SDE (3.1.11)

We can define the transition probability of the semi-flow u which is generated by the solution of (2.0.2) as follows:

$$P(t+s, s, \xi, \Gamma) := P(\{\omega : u(t+s, s, \omega)\xi \in \Gamma\}) = P(X_{t+s}^s(\xi) \in \Gamma), \quad (3.3.1)$$

for any $\Gamma \in \mathcal{B}(\mathbb{R}^m)$. For any φ being bounded and measurable

$$P(t+s, s)\varphi(\xi) = \int_{\mathbb{R}^m} P(t+s, s, \xi, d\eta)\varphi(\eta) = E\varphi(X_{t+s}^s(\xi))$$

defines a semigroup satisfying

$$P(t+s+r, s+r) \circ P(s+r, s) = P(t+s+r, s), \quad r, t \geq 0, \quad s \in \mathbb{R}.$$

Recall the following definition of periodic measure given in [17].

Definition 3.3.1. ([17]) *The measure function $\rho : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}^m)$ is called periodic measure if it satisfies for any $s \in \mathbb{R}$, $t \geq 0$, and $\Gamma \in \mathcal{B}(\mathbb{R}^m)$,*

$$\rho_{s+\tau} = \rho_s, \quad \int_{\mathbb{R}^m} P(t+s, s, x, \Gamma)\rho_s(dx) = \rho_{t+s}(\Gamma).$$

From Theorem 2.2.6, we know that the random periodic solution of (2.0.2) exists. So by the result in [17], we know that the periodic measure ρ exists, which can be defined as the law of random periodic solutions, i.e.

$$\rho_r(\Gamma) = P(X_r^* \in \Gamma). \quad (3.3.2)$$

Similarly, we can define the transition probability of the discrete semi-flow \hat{u} from Euler-Maruyama scheme by

$$\hat{P}(t+s, s, \xi, \Gamma) := P(\{\omega : \hat{u}(t+s, s, \omega)\xi \in \Gamma\}) = P(\hat{X}_{t+s}^s(\xi) \in \Gamma). \quad (3.3.3)$$

For any φ being bounded and measurable

$$\hat{P}(t+s, s)\varphi(\xi) = \int_{\mathbb{R}^m} \hat{P}(t+s, s, \xi, d\eta)\varphi(\eta) = E\varphi(\hat{X}_{t+s}^s(\xi))$$

defines a semigroup satisfying

$$\hat{P}(t+s+r, s+r) \circ \hat{P}(s+r, s) = \hat{P}(t+s+r, s), \quad r, t \geq 0, \quad s \in \mathbb{R},$$

Similar to the result in [17], the measure function defined by

$$\hat{\rho}_r(\Gamma) = P(\hat{X}_r^* \in \Gamma), \quad (3.3.4)$$

is a periodic measure for Markov semigroup $\hat{P}(t+s, s)$. It satisfies for any $s \in \mathbb{R}$, $t \geq 0$, and $\Gamma \in \mathcal{B}(\mathbb{R}^m)$,

$$\hat{\rho}_{s+\tau} = \hat{\rho}_s, \int_{\mathbb{R}^m} \hat{P}(t+s, s, x, \Gamma) \hat{\rho}_s(dx) = \hat{\rho}_{t+s}(\Gamma).$$

We have following error estimate of ρ and $\hat{\rho}$. Consider the Euler-Maruyama scheme (3.1.1) first.

Theorem 3.3.2. *Assume Conditions (A), (1) and (2). Then periodic measures ρ and $\hat{\rho}$ of the Markov semigroup generated by the exact solution of (2.0.1) and the approximation (3.1.1) are weak limits of transition probabilities along integral multiples of period, i.e.*

$$P(r, -k\tau, \xi) \rightarrow \rho_r, \hat{P}(r, -k\tau, \xi) \rightarrow \hat{\rho}_r, \text{ as } k \rightarrow \infty, \quad (3.3.5)$$

weakly and the error estimate is

$$d_{\mathbb{L}}(\rho_r, \hat{\rho}_r) \leq \tilde{K} \sqrt{\Delta t}, \quad (3.3.6)$$

where \tilde{K} is independent of Δt and r .

Proof. To prove (3.3.5), by (3.3.1), (3.3.2), Theorem 2.2.6 and Jensen's inequality, we have

$$\begin{aligned} & d_{\mathbb{L}}(P(r, -k\tau, \xi), \rho_r) \\ &= \sup_{\varphi \in \mathbb{L}} \left| \int_{\mathbb{R}^m} \varphi(x) P(r, -k\tau, \xi, dx) - \int_{\mathbb{R}^m} \varphi(x) \rho_r(dx) \right| \\ &= \sup_{\varphi \in \mathbb{L}} \left| \mathbb{E}[\varphi(X_r^{-k\tau}) - \varphi(X_r^*)] \right| \\ &\leq \sup_{\varphi \in \mathbb{L}} \mathbb{E} |\varphi(X_r^{-k\tau}) - \varphi(X_r^*)| \\ &\leq \mathbb{E} |X_r^{-k\tau} - X_r^*| \\ &\leq \|X_r^{-k\tau} - X_r^*\|_2 \\ &\rightarrow 0, \end{aligned}$$

as $k \rightarrow \infty$. So $P(r, -k\tau, \xi) \rightarrow \rho_r$ weakly as $k \rightarrow \infty$ from the well known result in [25]. Similarly, we can have for the discrete system, $\hat{P}(r, -k\tau, \xi) \rightarrow \hat{\rho}_r$ weakly as $k \rightarrow \infty$. Now we consider the metric between these two periodic measures ρ and $\hat{\rho}$,

$$d_{\mathbb{L}}(\rho_r, \hat{\rho}_r)$$

$$\begin{aligned}
&= \sup_{\varphi \in \mathbb{L}} \left| \int_{\mathbb{R}^m} \varphi(x) \rho_r(dx) - \int_{\mathbb{R}^m} \varphi(x) \widehat{\rho}_r(dx) \right| \\
&\leq \sup_{\varphi \in \mathbb{L}} \left| \int_{\mathbb{R}^m} \varphi(x) \rho_r(dx) - \int_{\mathbb{R}^m} \varphi(x) P(r, -k\tau, \xi, dx) \right| \\
&\quad + \sup_{\varphi \in \mathbb{L}} \left| \int_{\mathbb{R}^m} \varphi(x) P(r, -k\tau, \xi, dx) - \int_{\mathbb{R}^m} \varphi(x) \widehat{P}(r, -k\tau, \xi, dx) \right| \quad (3.3.7) \\
&\quad + \sup_{\varphi \in \mathbb{L}} \left| \int_{\mathbb{R}^m} \varphi(x) \widehat{P}(r, -k\tau, \xi, dx) - \int_{\mathbb{R}^m} \varphi(x) \widehat{\rho}_r(dx) \right| \\
&= \sup_{\varphi \in \mathbb{L}} \left| \mathbb{E}[\varphi(X_r^*) - \varphi(X_r^{-k\tau})] \right| + \sup_{\varphi \in \mathbb{L}} \left| \mathbb{E}[\varphi(X_r^{-k\tau}) - \varphi(\widehat{X}_r^{-k\tau})] \right| \\
&\quad + \sup_{\varphi \in \mathbb{L}} \left| \mathbb{E}[\varphi(\widehat{X}_r^{-k\tau}) - \varphi(\widehat{X}_r^*)] \right| \\
&\leq \mathbb{E} |X_r^* - X_r^{-k\tau}| + \mathbb{E} |X_r^{-k\tau} - \widehat{X}_r^{-k\tau}| + \mathbb{E} |\widehat{X}_r^{-k\tau} - \widehat{X}_r^*| \\
&\leq \|X_r^* - X_r^{-k\tau}\|_2 + \|X_r^{-k\tau} - \widehat{X}_r^{-k\tau}\|_2 + \|\widehat{X}_r^{-k\tau} - X_r^*\|_2.
\end{aligned}$$

By Theorems 2.2.6, 3.1.4, 3.1.7, we have for any $\epsilon > 0$, there exists $N > 0$ such that when $k \geq N$,

$$\|X_r^* - X_r^{-k\tau}\|_2 \leq \frac{\epsilon}{3}, \quad \|\widehat{X}_r^{-k\tau} - X_r^*\|_2 \leq \frac{\epsilon}{3},$$

and

$$\|X_r^{-k\tau} - \widehat{X}_r^{-k\tau}\|_2 \leq \widetilde{K} \sqrt{\Delta t} + \frac{\epsilon}{3}.$$

Then taking $k \geq N$ in (3.3.7), we have

$$d_{\mathbb{L}}(\rho_r, \widehat{\rho}_r) \leq \widetilde{K} \sqrt{\Delta t} + \epsilon.$$

Note in the above inequality, the left hand side does not depend on k and ϵ is arbitrary. So (3.3.6) is obtained. \square

Remark 3.3.3. *There are a number of works about approximation of invariant measures for SDE using Euler-Maruyama method and Milstein method ([32], [45], [46], [53]). For finite horizon, the order of weak error with Euler-Maruyama method was proved to be 1.0, a significant improvement from the order 0.5 in the strong convergence (c.f. [29]). However, the order of 1.0 is not guaranteed in the infinite horizon case, see [32] for the case of the invariant measures. On the other hand, in some work such as [45], [46], the order of error of Euler-Maruyama method was managed to increase to 1.0 under the non-degenerate condition. Here we do not have such an assumption, and we have order 0.5 in the weak convergence formulation.*

However, in the case of the modified Milstein method, we will see that the error is of order 1.0 in the next theorem. Note that the error estimate with the Milstein scheme is also 1.0 in the weak convergence formulation even in the non-degenerate case ([45], [46]).

Theorem 3.3.4. *Assume Condition (A) and (1.a). Consider the modified Milstein scheme (3.2.1). Then the periodic measure $\hat{\rho}_r$ of the Markov semi-groups generated by the discretised semi-flow is the weak limit of its transition probability along integral multiples of period, i.e.*

$$\hat{P}(r, -k\tau, \xi) \rightarrow \hat{\rho}_r, \text{ as } k \rightarrow \infty,$$

weakly and the error estimate between the approximating periodic measure $\hat{\rho}_r$ and the exact periodic measure is

$$d_{\mathbb{L}}(\rho_r, \hat{\rho}_r) \leq K^* \Delta t,$$

where K^* is independent of Δt and r .

Proof. The proof is similar to the proof of Theorem 3.3.2, but using Theorem 3.2.8 instead of Theorem 3.1.7. □

3.4 Transformation of the periodic SDE via Lyapunov-Floquet transformation

In this section, we consider the following m -dimensional system

$$dX_t^{t_0} = A(t)X_t^{t_0}dt + \tilde{f}(t, X_t^{t_0})dt + \tilde{g}(t, X_t^{t_0})dW_t, \quad t \geq t_0, \quad (3.4.1)$$

with $X_{t_0}^{t_0} = \xi$. We assume that the matrix $A(t)$ is a continuous τ -periodic $m \times m$ real matrix and the functions \tilde{f} and \tilde{g} are both τ -periodic in time, i.e.

$$A(t + \tau) = A(t), \quad \tilde{f}(t + \tau, \cdot) = \tilde{f}(t, \cdot), \quad \tilde{g}(t + \tau, \cdot) = \tilde{g}(t, \cdot), \quad \text{for any } t \in \mathbb{R}.$$

To solve this problem we need to apply the Floquet theorem to transform this system to a system with the linear part having a time invariant generator.

3.4.1 The transformation

The well known Floquet theorem can be found in many books, such as [22]. It says that if $\Phi(t)$ is a fundamental matrix solution of the periodic system $\dot{X} = A(t)X$, then so is $\Phi(t + \tau)$. Moreover, there exists an invertible τ -periodic matrix $P(t)$ such that $\Phi(t) = P(t)e^{Rt}$, where R is a constant matrix. The matrix $P(t)$ is called the Lyapunov-Floquet transformation matrix and $X = P(t)Z$ is called the Lyapunov-Floquet transformation.

Proposition 3.4.1. *Under Lyapunov-Floquet transformation $X(t) = P(t)Z(t)$, the periodic system (3.4.1) is transformed to the following system with a constant coefficient matrix linear part*

$$dZ_t^{t_0} = RZ_t^{t_0}dt + P(t)^{-1}\tilde{f}(t, P(t)Z_t^{t_0})dt + P(t)^{-1}\tilde{g}(t, P(t)Z_t^{t_0})dW_t, \quad (3.4.2)$$

with $Z_{t_0}^{t_0} = P(t_0)^{-1}\xi$.

Proof. The proof follows some elementary calculations. □

From the periodicity of P , we know that

$$\Phi(t + \tau) = P(t + \tau)e^{R(t+\tau)} = P(t)e^{Rt}e^{R\tau} = \Phi(t)e^{R\tau}.$$

Since $e^{R+2\pi kiI} = e^R e^{2\pi kiI} = e^R$ for any $k \in \mathbb{Z}$, the constant matrix R is not unique. It is also not necessarily real, even if $e^{R\tau}$ is real. So we need the following corollary to guarantee such a real constant matrix exists.

Corollary 3.4.2. *Let $B = \frac{R+\bar{R}}{2}$, $S(t) = \Phi(t)e^{-Bt}$. Then $S(t)$ is real and 2τ -periodic. Under the transformation $X_t^{t_0} = S(t)Z_t^{t_0}$, the periodic system (3.4.1) is transformed to the following system with constant coefficient matrix linear part*

$$dZ_t^{t_0} = BZ_t^{t_0}dt + S(t)^{-1}\tilde{f}(t, S(t)Z_t^{t_0})dt + S(t)^{-1}\tilde{g}(t, S(t)Z_t^{t_0})dW_t, \quad (3.4.3)$$

with $Z_{t_0}^{t_0} = S(t_0)^{-1}\xi$,

Proof. Because $A(t)$ is real, so the matrix $C = e^{R\tau} = \Phi(\tau)\Phi^{-1}(0)$ is real. Thus for the real matrix $B = \frac{R+\bar{R}}{2}$, $C^2 = e^{R\tau}e^{\bar{R}\tau} = e^{2B\tau}$. Note $S(t)$ is real since B is real. And notice that

$$S(t + 2\tau) = \Phi(t + 2\tau)e^{-B(t+2\tau)} = \Phi(t)C^2e^{-2B\tau}e^{-Bt} = \Phi(t)e^{-Bt} = S(t).$$

Then we can obtain the time invariant system in a similar way as in the Corollary 3.4.1. The only difference is that the system with real constant coefficient matrix linear part becomes 2τ -periodic. \square

3.4.2 Convergence theorem of the periodic parameter matrix system

Condition (A'). *The matrix function $A(t)$ is τ -periodic, the corresponding matrix B is symmetric with eigenvalues satisfying $0 > \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$.*

Because $S(t)$ is continuous and periodic, we have the boundedness of it. The periodicity and continuity of $S(t)^{-1}$ is obtained from the properties of $S(t)$, it is concluded that $S(t)^{-1}$ is bounded as well. Thus there exists a constant γ such that

$$|S(t)^{-1}| |S(t)| \leq \gamma.$$

For the periodic parameter matrix system, we give the following condition

Condition (1'). *Assume there exists a constant $\tau > 0$ such that for any $t \in \mathbb{R}$, $x \in \mathbb{R}^m$, $\tilde{f}(t+\tau, x) = \tilde{f}(t, x)$, $\tilde{g}(t+\tau, x) = \tilde{g}(t, x)$. There exist constant $\tilde{C}_0, \tilde{\beta}_1, \tilde{\beta}_2 > 0$ with $\tilde{\beta}_1\gamma + \frac{\tilde{\beta}_2^2\gamma^2}{2} < |\lambda_1|$, such that for any $s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^m$,*

$$\begin{aligned} |\tilde{f}(s, x) - \tilde{f}(t, y)| &\leq \tilde{C}_0 |s - t|^{1/2} + \tilde{\beta}_1 |x - y|, \\ |\tilde{g}(s, x) - \tilde{g}(t, y)| &\leq \tilde{C}_0 |s - t|^{1/2} + \tilde{\beta}_2 |x - y|. \end{aligned}$$

From this condition it follows that for any $x \in \mathbb{R}^m$, the linear growth condition also holds

$$|\tilde{f}(t, x)| \leq \tilde{\beta}_1 |x| + \tilde{C}_1, \quad |\tilde{g}(t, x)| \leq \tilde{\beta}_2 |x| + \tilde{C}_2,$$

where the constants $\tilde{C}_1, \tilde{C}_2 > 0$, which are independent of time t .

Theorem 3.4.3. *Assume that Conditions (A'), (1'). Then there exists a unique random periodic solution $X_r^* \in L^2(\Omega)$ of period 2τ such that for any fixed initial value $\xi(\omega)$, the solution of (3.4.1) satisfies*

$$\lim_{k \rightarrow \infty} \|X_r^{-2k\tau}(\xi) - X_r^*\|_2 = 0.$$

Proof. We only need to verify that the corresponding time invariant system

$$dZ_t^{t_0} = BZ_t^{t_0}dt + f(t, Z_t^{t_0})dt + g(t, Z_t^{t_0})dW_t, \quad (3.4.4)$$

with $Z_{t_0}^{t_0} = S(t_0)^{-1}\xi$, where

$$f(t, x) = S(t)^{-1}\tilde{f}(t, S(t)x), \quad g(t, x) = S(t)^{-1}\tilde{g}(t, S(t)x),$$

satisfies the conditions of Theorem 2.2.6. It is easy to see that

$$f(t + 2\tau, x) = f(t, x), \quad g(t + 2\tau, x) = g(t, x).$$

For Condition (1), the largest eigenvalue of the matrix B is λ_1 . By the Lipschitz condition on function \tilde{f} and \tilde{g} , we have following result in the time invariant system

$$|f(t, x) - f(t, y)| \leq \tilde{\beta}_1\gamma|x - y|.$$

This means the function f will preserve the Lipschitz property with constant $\beta_1 = \tilde{\beta}_1\gamma$. Similarly we can prove that the function g possesses the Lipschitz condition with constant $\beta_2 = \tilde{\beta}_2\gamma$. Meanwhile, from Condition (1'), we have

$$\beta_1 + \frac{\beta_2^2}{2} < |\lambda_1|.$$

Moreover, for any $x \in \mathbb{R}^m$,

$$|f(t, x)| = \left| S(t)^{-1}\tilde{f}(t, S(t)x) \right| \leq \tilde{\beta}_1 |S(t)^{-1}| |S(t)x| + |S(t)^{-1}| \tilde{C}_1 \leq \beta_1 |x| + C_1.$$

Therefore we can verify the linear growth property of f and g with the constants $C_1, C_2 > 0$. The constants β_1 and β_2 are both independent of time t . The initial value of the time invariant system will preserve the boundedness because of the boundedness of $S(t)^{-1}$. According to Theorem 2.2.6, there exists a random periodic solution $Z_r^* \in L^2(\Omega)$ with period 2τ such that

$$\lim_{k \rightarrow \infty} \|Z_r^{-2k\tau}(\xi) - Z_r^*\|_2 = 0.$$

It turns out that

$$\lim_{k \rightarrow \infty} \|X_r^{-2k\tau}(\xi) - X_r^*\|_2 \leq \|S(r)\| \lim_{k \rightarrow \infty} \|Z_r^{-2k\tau}(\xi) - Z_r^*\|_2 = 0.$$

The 2τ -periodicity of $S(r)$ and $Z_r^{-2k\tau}$ give us the random periodicity of solution $X^*(r, \omega)$. So X_r^* is a random periodic solution of (3.4.1) of period 2τ . \square

3.4.3 Numerical approximation scheme and error estimate

With the existence of the random periodic solutions, we now consider the scheme to simulate the process Z of equation (3.4.3). Similar as before, we can consider strong and weak convergence in Euler-Maruyama and modified Milstein methods. We firstly consider strong convergence in the Euler-Maruyama scheme

$$\begin{aligned} & \widehat{Z}_{-2k\tau+(i+1)\Delta t}^{-2k\tau} \\ &= \widehat{Z}_{-2k\tau+i\Delta t}^{-2k\tau} + \left[B\widehat{Z}_{-2k\tau+i\Delta t}^{-2k\tau} + S(i\Delta t)^{-1}\widetilde{f}(i\Delta t, S(i\Delta t)\widehat{Z}_{-2k\tau+i\Delta t}^{-2k\tau}) \right] \Delta t \\ & \quad + S(i\Delta t)^{-1}\widetilde{g}(i\Delta t, S(i\Delta t)\widehat{Z}_{-2k\tau+i\Delta t}^{-2k\tau}) (W_{-2k\tau+(i+1)\Delta t} - W_{-2k\tau+i\Delta t}). \end{aligned} \quad (3.4.5)$$

Theorem 3.4.4. *Assume Conditions (A'), (1') and (2), $S(t) \in C^1(\mathbb{R})$. Then there exists \widehat{Z}_r^* , which is a random periodic solution of period 2τ for discrete random dynamical system generated from (3.4.4), such that for any $r \in [0, T]$*

$$\lim_{k \rightarrow \infty} \left\| X_r^{-2k\tau} - S(r)\widehat{Z}_r^{-2k\tau} \right\|_2 \leq \widetilde{K}\sqrt{\Delta t},$$

and

$$\left\| X_r^* - S(r)\widehat{Z}_r^* \right\|_2 \leq \widetilde{K}\sqrt{\Delta t},$$

for a constant $\widetilde{K} > 0$, which is independent of Δt , where X_r^* is the exact random periodic solution of (3.4.1).

Proof. By Theorem 3.1.4, there exists $\widehat{Z}_r^* \in L^2(\Omega)$ such that

$$\limsup_{k \rightarrow \infty} \left\| \widehat{Z}_r^{-2k\tau} - \widehat{Z}_r^* \right\|_2 = 0,$$

where \widehat{Z}_r^* is the random periodic solution of period 2τ for discrete random dynamical system generated from (3.4.4). According to Theorem 3.1.7, we have the conclusion that there exists a constant $K_1 > 0$ such that

$$\lim_{k \rightarrow \infty} \left\| X_r^{-2k\tau} - S(r)\widehat{Z}_r^{-2k\tau} \right\|_2 \leq K_1 \|S(r)\|_2 \sqrt{\Delta t} \leq \widetilde{K}\sqrt{\Delta t}.$$

Thus it follows that

$$\begin{aligned} & \left\| X_r^* - S(r)\widehat{Z}_r^* \right\|_2 \\ & \leq \limsup_{k \rightarrow \infty} \left\| X_r^* - X_r^{-2k\tau} \right\|_2 + \limsup_{k \rightarrow \infty} \left\| X_r^{-2k\tau} - S(r)\widehat{Z}_r^{-2k\tau} \right\|_2 \\ & \quad + \limsup_{k \rightarrow \infty} \left\| S(r)\widehat{Z}_r^{-2k\tau} - S(r)\widehat{Z}_r^* \right\|_2 \leq \widetilde{K}\sqrt{\Delta t}. \end{aligned} \quad \square$$

It is not hard to obtain the result for the modified Milstein scheme with Theorem 3.2.5 and Theorem 3.2.8.

Chapter 4

Weak approximations

4.1 Lifts of semi-flows, random periodic paths and periodic measures

Denote by $(\Omega, \mathcal{F}, P, (\theta(s))_{s \in \mathbb{R}})$ a metric dynamical system and $\theta(s) : \Omega \rightarrow \Omega$ is assumed to be measurably invertible for all $s \in \mathbb{R}$. Denote $\Delta := \{(t, s) \in \mathbb{R}^2, s \leq t\}$. The lift of a stochastic semi-flow $u : \Delta \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ to a cocycle on a cylinder and the corresponding lift of a random periodic path are introduced in following lemma.

Lemma 4.1.1. ([17]) *We lift the τ -periodic stochastic semi-flow $u : \Delta \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ to a random dynamical system on a cylinder $\tilde{\mathbb{X}} := [0, \tau) \times \mathbb{X}$ by the following:*

$$\tilde{\Phi}(t, \omega)(s, x) = (t + s \bmod \tau, u(t + s, s, \theta(-s)\omega)x), \quad (4.1.1)$$

for any $(s, x) \in \tilde{\mathbb{X}}$, $t \in \mathbb{R}^+$ and almost every $\omega \in \Omega$.

Then $\tilde{\Phi} : \mathbb{R}^+ \times \Omega \times \tilde{\mathbb{X}} \rightarrow \tilde{\mathbb{X}}$ is a cocycle on $\tilde{\mathbb{X}}$ over the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(s))_{s \in \mathbb{R}})$.

Moreover, assume $Y : \mathbb{R} \times \Omega \rightarrow \mathbb{X}$ is a random periodic solution of the semi-flow u with period $\tau > 0$. Then $\tilde{Y} : \mathbb{R} \times \Omega \rightarrow \tilde{\mathbb{X}}$ defined by

$$\tilde{Y}(s, \omega) := (s \bmod \tau, Y(s, \omega)), \quad (4.1.2)$$

is a random periodic solution of the cocycle $\tilde{\Phi}$ on $\tilde{\mathbb{X}}$.

Defined the skew product $\bar{\Theta} : \Delta \times \bar{\Omega} \rightarrow \bar{\Omega}$ of the metric dynamical system $(\Omega, \mathcal{F}, P, (\theta(s))_{s \in \mathbb{R}})$ and the semi-flow u by

$$\bar{\Theta}(t + s, s)(\omega, x) = (\theta(t)\omega, u(t + s, s, \theta(-s)\omega)x), \quad t \in \mathbb{R}^+, \quad s \in \mathbb{R}. \quad (4.1.3)$$

Here $\bar{\Omega} = \Omega \times \mathbb{X}$.

We can verify that for any $t_1, t_2 \in \mathbb{R}^+, s \in \mathbb{R}$,

$$\bar{\Theta}(t_2 + t_1 + s, t_1 + s) \circ \bar{\Theta}(t_1 + s, s) = \bar{\Theta}(t_2 + t_1 + s, s). \quad (4.1.4)$$

Theorem 4.1.2. [17] *Assume the τ -periodic stochastic semi-flow $u : \Delta \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ has a random periodic solution $Y : \mathbb{R} \times \Omega \rightarrow \mathbb{X}$. Define*

$$(\mu_s)_\omega(\Gamma) = \delta_{Y(s, \theta(-s)x)}(\Gamma).$$

Then

$$\mu_s(dx, d\omega) = (\mu_s)_\omega(dx) \times P(d\omega)$$

is a periodic measure of the skew product $\bar{\Theta}$ on the product measurable space $(\Omega \times \mathbb{X}, \mathcal{F} \otimes \mathcal{B}(\mathbb{X}))$, i.e.

$$\bar{\Theta}(t + s, s)\mu_s = \mu_{t+s}, \quad \mu_{\tau+s} = \mu_s,$$

for all $t \in \mathbb{R}^+, s \in \mathbb{R}$, which is equivalent to

$$u(t + s, s, \theta(-s)\omega)(\mu_s)_\omega = (\mu_{t+s})_{\theta(t)\omega} \text{ and } (\mu_{\tau+s})_\omega = (\mu_s)_\omega,$$

for all $t \in \mathbb{R}^+, s \in \mathbb{R}, \omega \in \Omega$.

Consider the case when $u(t + s, s, \cdot)$ is a Markovian semi-flow on a filtered dynamical system $(\Omega, \mathcal{F}, P, (\theta_t)_{t \in \mathbb{R}}, (\mathcal{F}_s^t)_{s \leq t})$, i.e. for any $s, t, u \in \mathbb{R}, s \leq t$, we have $\theta_u^{-1}\mathcal{F}_s^t = \mathcal{F}_{s+u}^{t+u}$ and $u(t + s, s, \cdot)$ is independent of $\mathcal{F}_{-\infty}^s$. The random periodic solution $Y(s, \omega)$ is assumed to be adapted, i.e. for any $s \in \mathbb{R}, Y(s, \cdot)$ is measurable with respect to $\mathcal{F}_{-\infty}^s := \bigvee_{r \leq s} \mathcal{F}_r^s$.

Denote the transition probability of u by

$$P(t + s, s, x, \Gamma) = P(\{\omega : u(t + s, s, \omega)x \in \Gamma\}),$$

for any $\Gamma \in \mathcal{B}(\mathbb{X}), t \in \mathbb{R}^+, s \in \mathbb{R}$. From (1.0.2) and the measure preserving property of θ_τ , the transition property $P(t + s, s, x, \Gamma)$ satisfies the periodic relation

$$P(t + s + \tau, s + \tau, x, \Gamma) = P(t + s, s, x, \Gamma), \quad (4.1.5)$$

for any $\Gamma \in \mathcal{B}(\mathbb{X}), t \in \mathbb{R}^+, s \in \mathbb{R}$.

For any probability measure ρ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, define

$$(P^*(t + s, s)\rho)(\Gamma) = \int_{\mathbb{X}} P(t + s, s, x, \Gamma)\rho(dx)$$

for any $\Gamma \in \mathcal{B}(\mathbb{X})$, $t \in \mathbb{R}^+$, $s \in \mathbb{R}$. Then the definition of periodic measure of the periodic Markov semigroup is given as follows,

Definition 4.1.3. [17] *The measure valued function $\rho : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{X})$ is called a τ -periodic measure of the τ -periodic Markov semigroup $P(t + s, s, x, \Gamma)$, $t \geq 0$, if for any $s \in \mathbb{R}$, $t \in \mathbb{R}^+$,*

$$P^*(t + s, s)\rho_s = \rho_{t+s}, \quad \rho_{s+\tau} = \rho_s. \quad (4.1.6)$$

One direction of the “equivalence” of random periodic paths and periodic measures are illustrated in the following theorem.

Theorem 4.1.4. [17] *Assume that the τ -periodic Markovian stochastic semi-flow $u : \Delta \times \Omega \times \mathbb{X} \rightarrow \mathbb{X}$ has an adapted random periodic solution $Y : \mathbb{R} \times \omega \rightarrow \mathbb{X}$. Then it has a periodic measure ρ_s on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ defined by*

$$\rho_s(\Gamma) = E\delta_{Y(s, \omega)}(\Gamma) = P(\{\omega : Y(s, \omega) \in \Gamma\}), \quad \Gamma \in \mathcal{B}(\mathbb{X}) \quad s \in \mathbb{R}. \quad (4.1.7)$$

Moreover, for any $t \in \mathbb{R}$,

$$E(m\{s \in [0, \tau) : Y(s, \cdot) \in \Gamma\}) = E(m\{s \in [t, t + \tau) : Y(s, \cdot) \in \Gamma\}). \quad (4.1.8)$$

This theorem gives us the fact that if we have a random periodic solution, then the law of the r.p.s. is a periodic measure. On the other direction, when there is a periodic measure, one can construct the corresponding random periodic solution in an enlarged probability space, details of which can be found in Feng and Zhao [17].

For the weak approximation, we consider following non-autonomous stochastic differential equation on \mathbb{R}^d .

$$\begin{aligned} dX(t) &= b(t, X(t))dt + \sigma(t, X(t))dW_t, \\ X(s) &= x, \end{aligned}$$

where $s \leq t$, W is a d_1 -dimensional Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. It is well-known that the semi-group is given by

$$u(t + s, s, x) = P(t + s, s)\phi(x) = \mathbb{E}\phi(X(t + s, s, x)), \quad t \geq 0.$$

Here we assume the existence of the periodic measure $\{\rho_s\}_{s \in \mathbb{R}}$ of the Markovian semi-group. By the consideration on the lifted cylinder $\tilde{X} = [0, \tau) \times \mathbb{R}^d$ in [17], the lifted cocycle with coordinates $\tilde{X} = (s, x)$ is given by

$$\tilde{X}(t)(x) = (t + s \pmod{\tau}, X(t + s, s, x(\theta(-s)\omega))),$$

where u is corresponding semi-flow. We follow the idea of the lifted case

$$d\tilde{X}(t) = \tilde{b}(\tilde{X}(t))dt + \tilde{\sigma}(\tilde{X}(t))d\tilde{W}(t),$$

where

$$\tilde{X}(0)\tilde{x} = \tilde{x} = (s, x), \quad \tilde{W} = (\tilde{W}_0, W),$$

\tilde{W}_0 is a one-dimensional Brownian motion which is independent of W ,

$$\tilde{b}(\tilde{X}) = \begin{pmatrix} 1 \\ b(\tilde{X}) \end{pmatrix} = \begin{pmatrix} 1 \\ b(\tilde{X}_0, X) \end{pmatrix},$$

and

$$\tilde{\sigma}(\tilde{X}) = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(\tilde{X}) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \sigma(\tilde{X}_0, X) \end{pmatrix}.$$

The infinitesimal generator of the process $X(t)$ is given by

$$\begin{aligned} \tilde{\mathcal{L}} &= \sum_{i=0}^d \tilde{b}_i(\tilde{x}) \frac{\partial}{\partial \tilde{x}_i} + \frac{1}{2} \sum_{i,j=0}^d \tilde{a}_{ij}(\tilde{x}) \frac{\partial^2}{\partial \tilde{x}_i \partial \tilde{x}_j}, \\ &= \sum_{i=1}^d b_i(s, x) \frac{\partial}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^d a_{ij}(s, x) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{\partial}{\partial s}, \end{aligned}$$

where $a(s, x) = \sigma(s, x)\sigma^*(s, x)$.

4.2 Assumptions

To obtain our result, we give the following conditions:

Condition (3) *The functions b, σ are of class C^∞ with bounded derivatives of any order. The function σ is bounded. The function b and σ are τ -period with respect to time, i.e.*

$$b(t + \tau, x) = b(t, x), \quad \sigma(t + \tau, x) = \sigma(t, x).$$

It is easy to obtain the periodicity of function $a(t + \tau, x) = a(t, x)$.

Condition (4) *The infinitesimal generator $\tilde{\mathcal{L}}$ is uniformly elliptic with respect to x , i.e. there exists a positive constant α such that for any $t, x, y \in \mathbb{R}^d$, we have*

$$\sum_{i,j} a_{ij}(t, y) x_i x_j \geq \alpha |x|^2.$$

Condition (5) *We assume the weak dissipative condition for the SDE, i.e. there exists a strictly positive constant β and a compact set K such that for any t and any $x \in K^c$, we have*

$$x \cdot b(t, x) \leq -\beta |x|^2.$$

4.3 Preliminary results and notations

Proposition 4.3.1. *Assume Conditions (3) and (5). Then for any $n \in \mathbb{N}$, there exist strictly positive constants C_n and γ_n , such that for any $t > 0$ and $x \in \mathbb{R}^d$, we have*

$$\mathbb{E} |X_{t+s}^{s,x}|^n \leq C_n (1 + |x|^n \exp(-\gamma_n t)).$$

Proof. As the constants in Conditions (3) and (5) are uniform with respect to initial condition, for simplicity, we omit them in this proof. We temporarily denote by $X_t := X_{t+s}^{s,x}$, and let $Y_t = |X_t|^2$, then we have

$$\begin{aligned} dY_t &= d|X_t|^2 = 2X_t dX_t + dX_t dX_t \\ &= 2X_t b(t, X_t) dt + 2X_t \sigma(t, X_t) dW_t + \sigma^2(t, X_t) dt. \end{aligned}$$

Apply the Itô's formula on $e^{\delta t} (Y_t)^{\frac{n}{2}}$ to obtain

$$\begin{aligned} de^{\delta t} (Y_t)^{\frac{n}{2}} &= \delta e^{\delta t} (Y_t)^{\frac{n}{2}} dt + e^{\delta t} d(Y_t)^{\frac{n}{2}} \\ &= \delta e^{\delta t} (Y_t)^{\frac{n}{2}} dt + \frac{n}{2} e^{\delta t} (Y_t)^{\frac{n}{2}-1} dY_t + \frac{n}{2} \left(\frac{n}{2} - 1 \right) e^{\delta t} \frac{(Y_t)^{\frac{n}{2}-2}}{2} dY_t dY_t \\ &= \delta e^{\delta t} |X_t|^n dt + n e^{\delta t} |X_t|^{n-1} b(t, X_t) dt + n e^{\delta t} |X_t|^{n-1} \sigma(t, X_t) dW_t \\ &\quad + \left(\frac{n}{2} + 4 \cdot \frac{n}{4} \left(\frac{n}{2} - 1 \right) \right) e^{\delta t} |X_t|^{n-2} \sigma^2(t, X_t) dt. \end{aligned}$$

By Condition (3), we denote the bound of function $\sigma(t, x)$ by C_σ , therefore, together with Condition (5), we have

$$\begin{aligned} de^{\delta t} |X_t|^n &= \delta e^{\delta t} |X_t|^n dt + ne^{\delta t} |X_t|^{n-1} b(t, X_t) dt + ne^{\delta t} |X_t|^{n-1} \sigma(t, X_t) dW_t \\ &\quad + \frac{n(n-1)}{2} e^{\delta t} |X_t|^{n-2} \sigma^2(t, X_t) dt \\ &\leq (\delta - n\beta) e^{\delta t} |X_t|^n dt + nC_\sigma e^{\delta t} |X_t|^{n-1} dW_t + \binom{n}{2} C_\sigma^2 e^{\delta t} |X_t|^{n-2} dt. \end{aligned}$$

Take expectation on both side after integrating from 0 to T , together with Young's inequality

$$|X_t|^{n-2} \leq \frac{(|X_t|^{n-2} \varepsilon)^{\frac{n}{n-2}}}{\frac{n}{n-2}} + \frac{\left(\frac{1}{\varepsilon}\right)^{\frac{n}{2}}}{\frac{n}{2}} = \frac{n-2}{n} \varepsilon^{\frac{n}{n-2}} |X_t|^n + \frac{2}{n\varepsilon^{\frac{n}{2}}},$$

we have

$$\begin{aligned} e^{\delta T} \mathbb{E} |X_T|^n &\leq |x|^n + (\delta - n\beta) \int_0^T e^{\delta t} \mathbb{E} |X_t|^n dt + \binom{n}{2} C_\sigma^2 \int_0^T e^{\delta t} \mathbb{E} |X_t|^{n-2} dt \\ &\leq |x|^n + \frac{(n-1)C_\sigma^2}{\delta\varepsilon^{\frac{n}{2}}} (e^{\delta T} - 1) \\ &\quad + \left(\delta - n\beta + \binom{n-1}{2} C_\sigma^2 \varepsilon^{\frac{n}{n-2}} \right) \int_0^T e^{\delta t} \mathbb{E} |X_t|^n dt. \end{aligned} \quad (4.3.1)$$

We denote by

$$K_6 = \delta - n\beta + \binom{n-1}{2} C_\sigma^2 \varepsilon^{\frac{n}{n-2}}$$

and choose the constant

$$\varepsilon < \left(\frac{2\beta}{(n-1)C_\sigma^2} \right)^{\frac{n-2}{n}}$$

to satisfy $K_6 - \delta < 0$. The choice of the constant δ would also guarantee that $K_6 > 0$.

Then we apply the Gronwall's inequality on (4.3),

$$\begin{aligned} &e^{\delta T} \mathbb{E} |X_T|^n \\ &\leq |x|^n + \frac{(n-1)C_\sigma^2}{\delta\varepsilon^{\frac{n}{2}}} (e^{\delta T} - 1) + K_6 \int_0^T e^{\delta t} \mathbb{E} |X_t|^n dt \\ &\leq |x|^n + \frac{(n-1)C_\sigma^2}{\delta\varepsilon^{\frac{n}{2}}} (e^{\delta T} - 1) + K_6 \int_0^T \left(|x|^n + \frac{(n-1)C_\sigma^2}{\delta\varepsilon^{\frac{n}{2}}} (e^{\delta t} - 1) \right) e^{\int_t^T K_6 dr} dt \\ &= |x|^n + \frac{(n-1)C_\sigma^2}{\delta\varepsilon^{\frac{n}{2}}} (e^{\delta T} - 1) + K_6 e^{K_6 T} \left(|x|^n - \frac{(n-1)C_\sigma^2}{\delta\varepsilon^{\frac{n}{2}}} \right) \int_0^T e^{-K_6 t} dt \end{aligned}$$

$$\begin{aligned}
& + \frac{(n-1)K_6 C_\sigma^2}{\delta \varepsilon^{\frac{n}{2}}} e^{K_6 T} \int_0^T e^{(\delta - K_6)t} dt \\
& = \frac{(n-1)C_\sigma^2}{\delta \varepsilon^{\frac{n}{2}}} e^{\delta T} + e^{K_6 T} \left(|x|^n - \frac{(n-1)C_\sigma^2}{\delta \varepsilon^{\frac{n}{2}}} \right) + \frac{(n-1)K_6 C_\sigma^2}{\delta(\delta - K_6)\varepsilon^{\frac{n}{2}}} (e^{\delta T} - e^{K_6 T}) \\
& = |x|^n e^{K_6 T} + \frac{(n-1)C_\sigma^2}{(\delta - K_6)\varepsilon^{\frac{n}{2}}} (e^{\delta T} - e^{K_6 T}).
\end{aligned}$$

To conclude,

$$\mathbb{E} |X_T|^n \leq |x|^n e^{(K_6 - \delta)T} + \frac{(n-1)C_\sigma^2}{(\delta - K_6)\varepsilon^{\frac{n}{2}}} (1 - e^{(K_6 - \delta)T}).$$

The existence of the positive constant γ_n is ensured by $K_6 - \delta < 0$. \square

Consider the sequence $\{X_{t_n}\}_{n \in \mathbb{N}}$ with $t_n = n\tau$, which is an ergodic Markov chain with property of contraction out of some set under our assumption.

Proposition 4.3.2. *Assume Condition (3), then there exists a constant $r > 1$ and a set $B(0, R) \supset K$, such that,*

$$\sup_{x \in B^c} \mathbb{E} \left[r |X_{t_{n+1}}|^2 - |X_{t_n}|^2 \mid X_{t_n} = x \right] < 0.$$

Proof. Apply the same idea for Proposition 4.3.1 with $n = 2$. In order to satisfy

$$\mathbb{E} \left[r |X_{t_{n+1}}|^2 - |X_{t_n}|^2 \mid X_{t_n} = x \right] \leq (|x|^2 e^{(K_6 - \delta)\tau} + C_{\beta, \sigma} (1 - e^{(K_6 - \delta)\tau})) r - |x|^2 < 0,$$

we need

$$(1 - r e^{(K_6 - \delta)\tau}) |x|^2 > r C_{\beta, \sigma} (1 - e^{(K_6 - \delta)\tau}).$$

As $K_6 - \delta < 0$, there always exists a constant r to ensure $1 - r e^{(K_6 - \delta)\tau} > 0$ for any positive constant τ . Then the ball B is determined by taking $R > \sqrt{\frac{r C_{\beta, \sigma} (1 - e^{(K_6 - \delta)\tau})}{1 - r e^{(K_6 - \delta)\tau}}}$. \square

Denote C_p^∞ is the space of the smooth function $\phi \in C^\infty$ with the property that itself and all its derivatives have at most polynomial growth at infinity. Let the function $\phi \in C_p^\infty$ and $u(t + s, s, x) = \mathbb{E} \phi(X_{t+s}^{s, x})$. It is well known that $u(t + s, s, x)$ is a classical solution of the PDE:

$$\begin{aligned}
\frac{d}{dt} u(t + s, s, x) & = \tilde{\mathcal{L}} u(t + s, s, x), \\
u(s, s, x) & = \phi(x).
\end{aligned}$$

By the Fokker-Planck equation, we have the density function of periodic measure $\tilde{\rho}$ satisfying:

$$\frac{d}{dt}p(s, x) = \tilde{\mathcal{L}}^*p(s, x) = 0, \quad (4.3.2)$$

where $\tilde{\mathcal{L}}^*$ is the conjugate operator of $\tilde{\mathcal{L}}$. The density function of the periodic measure is smooth with respect to the initial condition by our assumptions.

Consider the spatial differentiation of the solution with respect to the initial condition. Kunita showed in [30] that the function $u(t + s, s, x)$ satisfies that for any order $n \in \mathbb{N}$, there exist an integer $r_n \in \mathbb{N}$ such that for any $T > 0$, $\exists C_n(t) > 0$,

$$|D^n u(t + s, s, x)| \leq C_n(T)(1 + |x|^{r_n}), \quad \forall t < T. \quad (4.3.3)$$

Therefore the Proposition 4.3.1 gives us the property that the function ϕ and $D^n u(t + s, s, x)$ belong to $L^2(\mathbb{R}^{d+1}, \tilde{\rho})$.

The function $\frac{1}{\tau} \int_0^\tau u(t + s, s, x) ds$ has the same spatial derivatives as

$$\frac{1}{\tau} \int_0^\tau u(t + s, s, x) ds - \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} \phi(x) p(s, x) dx ds,$$

without loss of generality, in the following sections, we assume that

$$\int_{\tilde{\mathbb{X}}} \tilde{\phi}(\tilde{x}) d\tilde{\rho}(\tilde{x}) = \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} \phi(x) p(s, x) dx ds = 0, \quad (4.3.4)$$

where $\tilde{\phi}(\tilde{x}) = \phi(x)$, and the notation $\tilde{\rho}$ denote the average periodic measure, which is invariant in the case of lifted cocycle.

For simplicity, in the following sections, we may often write $\tilde{u}(t)$ or $u(t + s, s)$ to represent the function $u(t + s, s, x)$. We also often write b, a to represent $b(s, x)$ and $a(s, x)$ as we have the uniform conditions for these functions and any order of their derivatives in Condition (3). The operators ∂, ∇ and D on function $u(t + s, s, x)$ always refer to derivatives with respect to spatial coordinates. The derivatives with respect to initial time will stay as $\frac{\partial}{\partial s}$. Also, we also denote the infinitesimal generator by

$$\tilde{\mathcal{L}} = b_i \partial_i + \frac{1}{2} a_{ij} \partial_{ij} + \frac{\partial}{\partial s}.$$

Remark 4.3.3. *By the Proposition 4.3.1, for any compact set K and any $n \in \mathbb{N}$*

$$\frac{1}{\tau} \int_0^\tau \int_K |x|^n p(s, x) dx ds = \lim_{t \rightarrow \infty} \mathbb{E}(|X_{t+s}^{s,x}|^n 1_K(X_{t+s}^{s,x})) \leq C_n,$$

where C_n is determined from the proposition. Therefore the average periodic measure has the finite moments of any order.

4.4 Main results

4.4.1 Estimates on the average of $u(t + s, s)$ in any ball

Lemma 4.4.1. *Assume Conditions (3), (4) and (5), for any ball B , there exists strictly positive constants C and λ such that for any $t > 0$ and any $x \in B$,*

$$\frac{1}{\tau} \int_0^\tau |u(t + s, s, x)| ds \leq C \exp(-\lambda t).$$

Proof. Consider the Markov chain $\{X_{t_n}\}_{n \in \mathbb{N}}$ with $t_n = n\tau$ in our model. Its transition kernel $P(k\tau + s, s, x, \Gamma)$ is irreducible. The existence of small sets to satisfy the minorization condition can be found in some books (Meyn and Tweedie [34], Nummerlin [39]). By the Proposition 4.3.2 and Nummerlin's result, the Markov chain is geometrically recurrent. Then we follow the result of Tweedie in [49], we have that for any $\phi \in C_p^\infty$ with the property (4.3.4), there exist strictly positive constants C and λ such that for any n ,

$$\frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |\mathbb{E}\phi(X_{t_n+s}^{s,x})| p(s, x) dx ds \leq C \exp(-\lambda t_n). \quad (4.4.1)$$

By Proposition 4.3.1, we have that there exist strictly positive constants C_0 , γ and a integer $N \in \mathbb{N}$ such that

$$|u(t + s, s, x)| \leq C_0(1 + |x|^N \exp(-\gamma t)).$$

Apply this to (4.4.1) to obtain that for any n ,

$$\begin{aligned} & \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t_n + s, s, x)|^2 p(s, x) dx ds \\ & \leq \frac{C_0}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t_n + s, s, x)| (1 + |x|^N \exp(-\gamma t_n)) p(s, x) dx ds \\ & = \frac{C_0}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t_n + s, s, x)| p(s, x) dx ds \\ & \quad + \frac{C_0}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t_n + s, s, x)| |x|^N \exp(-\gamma t_n) p(s, x) dx ds \\ & \leq C_0 C \exp(-\lambda t_n) + \frac{C_0^2 \exp(-\gamma t_n)}{\tau} \int_0^\tau \int_{\mathbb{X}} (1 + |x|^N \exp(-\gamma t_n)) |x|^N p(s, x) dx ds \\ & \leq C_1 \exp(-\lambda_1 t_n). \end{aligned} \quad (4.4.2)$$

Then we prove the monotonicity of the function $\frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t+s, s, x)|^2 p(s, x) dx ds$ as follows. It is well known that

$$\begin{aligned} & \tilde{\mathcal{L}} |u(t+s, s)|^2 \\ &= b_i \partial_i |u(t+s, s)|^2 + \frac{1}{2} a_{ij} \partial_{ij} |u(t+s, s)|^2 + \frac{\partial}{\partial s} |u(t+s, s)|^2 \\ &= 2u(t+s, s) \tilde{\mathcal{L}} u(t+s, s) + a_{ij} \partial_i u(t+s, s) \partial_j u(t+s, s) \\ &= \frac{d}{dt} |u(t+s, s)|^2 + a_{ij} \partial_i u(t+s, s) \partial_j u(t+s, s). \end{aligned}$$

Therefore,

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t+s, s, x)|^2 p(s, x) dx ds \right) \\ &= \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} \frac{d}{dt} |u(t+s, s, x)|^2 p(s, x) dx ds \\ &= \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} \tilde{\mathcal{L}} |u(t+s, s, x)|^2 p(s, x) dx ds \\ & \quad - \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} a_{ij}(t+s, x) \partial_i u(t+s, s, x) \partial_j u(t+s, s, x) p(s, x) dx ds. \end{aligned}$$

From the property of density function of the average periodic measure (4.3.2) and Condition (4), we know that the function satisfies

$$\begin{aligned} & \frac{d}{dt} \left(\frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t+s, s, x)|^2 p(s, x) dx ds \right) \\ &= - \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} a_{ij}(t+s, x) \partial_i u(t+s, s, x) \partial_j u(t+s, s, x) p(s, x) dx ds \\ &\leq - \frac{\alpha}{\tau} \int_0^\tau \int_{\mathbb{X}} |\nabla u(t+s, s, x)|^2 p(s, x) dx ds \\ &\leq 0. \end{aligned}$$

From the decreasing property of the function and (4.4.2), we take $C_2 = C_1 \exp(\lambda_1 \tau)$, then for any $k\tau \leq t < (k+1)\tau$,

$$\begin{aligned} & \frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t+s, s, x)|^2 p(s, x) dx ds \\ & \leq C_1 \exp(-\lambda_1 k\tau) = C_2 \exp(-\lambda_1(k+1)\tau) \leq C_2 \exp(-\lambda_1 t). \end{aligned} \quad (4.4.3)$$

The above shows that the exponential contraction of $u(t+s, s, x)$ under the average of periodic measure holds for any t . Again we consider the result

$$\frac{d}{dt} |u(t+s, s)|^2 = \tilde{\mathcal{L}} |u(t+s, s)|^2 - a_{ij} \partial_i u(t+s, s) \partial_j u(t+s, s).$$

Multiplying the above inequality with $e^{\delta t}$, and integrating the both sides with respect to the average periodic measure $\bar{\rho}$, we obtain

$$e^{\delta t} \int_{\bar{\mathbb{X}}} \frac{d}{dt} |u(t+s, s)|^2 d\bar{\rho} + C_a e^{\delta t} \int_{\bar{\mathbb{X}}} |\nabla u(t+s, s)|^2 d\bar{\rho} \leq e^{\delta t} \int_{\bar{\mathbb{X}}} \tilde{\mathcal{L}} |u(t+s, s)|^2 d\bar{\rho} = 0,$$

where C_a is the bound of function $a = \sigma\sigma^*$, which comes from the boundedness of function σ . Therefore, taking integration from 0 to T on the both sides, we have

$$\int_0^T e^{\delta t} \int_{\bar{\mathbb{X}}} \frac{d}{dt} |u(t+s, s)|^2 d\bar{\rho} dt + C_a \int_0^T e^{\delta t} \int_{\bar{\mathbb{X}}} |\nabla u(t+s, s)|^2 d\bar{\rho} dt \leq 0. \quad (4.4.4)$$

The integration by parts on the first term of inequality (4.4.4) gives us

$$\begin{aligned} & \int_0^T e^{\delta t} \int_{\bar{\mathbb{X}}} d|u(t+s, s)|^2 d\bar{\rho} \\ &= e^{\delta T} \int_{\bar{\mathbb{X}}} |u(T+s, s)|^2 d\bar{\rho} - \int_{\bar{\mathbb{X}}} |u(s, s)|^2 d\bar{\rho} - \delta \int_0^T e^{\delta t} \int_{\bar{\mathbb{X}}} |u(t+s, s)|^2 d\bar{\rho} dt, \end{aligned}$$

where we have the initial condition that $u(s, s, x) = \tilde{\phi}(\tilde{x})$. By the Proposition 4.3.1 and the polynomial growth of the function ϕ , we have the constant $C_3 > 0$ that

$$\int_{\bar{\mathbb{X}}} |\tilde{\phi}(\tilde{x})|^2 d\bar{\rho} < C_3.$$

If we take $\delta < \lambda_1$, where λ_1 comes from (4.4.3), we also have the constant $C_4 > 0$ that for any T and any $s \in [0, \tau)$

$$\delta \int_0^T e^{\delta t} \int_{\bar{\mathbb{X}}} |u(t+s, s)|^2 d\bar{\rho} dt < C_4.$$

Applying these results on (4.4.4) to obtain that for any T and any $s \in [0, \tau)$,

$$\begin{aligned} & \int_0^T e^{\delta t} \int_{\bar{\mathbb{X}}} |\nabla u(t+s, s)|^2 d\bar{\rho} dt \\ & \leq \frac{1}{C_a} \left(\int_{\bar{\mathbb{X}}} |u(s, s)|^2 d\bar{\rho} + \delta \int_0^T e^{\delta t} \int_{\bar{\mathbb{X}}} |u(t+s, s)|^2 d\bar{\rho} dt \right) \\ & \leq C_5. \end{aligned} \quad (4.4.5)$$

Now we consider the following results

$$\begin{aligned} & \frac{d}{dt} |\nabla u(t+s, s)|^2 \\ &= 2(\partial_k u(t+s, s)) \partial_k \left(\frac{d}{dt} u(t+s, s) \right) \end{aligned}$$

$$\begin{aligned}
&= 2(\partial_k u(t+s, s)) \partial_k (\tilde{\mathcal{L}} u(t+s, s)) \\
&= 2(\partial_k u(t+s, s)) \partial_k (b_i \partial_i u(t+s, s)) + 2(\partial_k u(t+s, s)) \partial_k \left(\frac{\partial}{\partial s} u(t+s, s) \right) \\
&\quad + (\partial_k u(t+s, s)) \partial_k (a_{ij} \partial_{ij} u(t+s, s)) \\
&= 2b_i (\partial_k u(t+s, s)) (\partial_{ik} u(t+s, s)) + 2(\partial_k b_i) (\partial_k u(t+s, s)) (\partial_i u(t+s, s)) \\
&\quad + 2(\partial_k u(t+s, s)) \frac{\partial}{\partial s} (\partial_k u(t+s, s)) \\
&\quad + a_{ij} (\partial_k u(t+s, s)) (\partial_{ijk} u(t+s, s)) + (\partial_k a_{ij}) (\partial_k u(t+s, s)) (\partial_{ij} u(t+s, s)),
\end{aligned}$$

and

$$\begin{aligned}
&\tilde{\mathcal{L}} |\nabla u(t+s, s)|^2 \\
&= b_i \partial_i (\partial_k u(t+s, s))^2 + \frac{\partial}{\partial s} (\partial_k u(t+s, s))^2 + \frac{1}{2} a_{ij} \partial_{ij} (\partial_k u(t+s, s))^2 \\
&= 2b_i (\partial_k u(t+s, s)) (\partial_{ik} u(t+s, s)) + 2(\partial_k u(t+s, s)) \frac{\partial}{\partial s} (\partial_k u(t+s, s)) \\
&\quad + a_{ij} (\partial_k u(t+s, s)) (\partial_{ijk} u(t+s, s)) + a_{ij} (\partial_{ik} u(t+s, s)) (\partial_{jk} u(t+s, s)).
\end{aligned}$$

Therefore compare the difference between these two expansions, we obtain

$$\begin{aligned}
&\frac{d}{dt} |\nabla u(t+s, s)|^2 - \tilde{\mathcal{L}} |\nabla u(t+s, s)|^2 \\
&= -a_{ij} (\partial_{ik} u(t+s, s)) (\partial_{jk} u(t+s, s)) + 2(\partial_k b_i) (\partial_k u(t+s, s)) (\partial_i u(t+s, s)) \\
&\quad + (\partial_k a_{ij}) (\partial_k u(t+s, s)) (\partial_{ij} u(t+s, s)).
\end{aligned}$$

The elliptic condition for a_{ij} gives us a constant $C_5 > 0$ such that

$$-a_{ij} (\partial_{ik} u(t+s, s)) (\partial_{jk} u(t+s, s)) \leq -C_5 |D^2 u(t+s, s)|^2.$$

Meanwhile, we apply Young's inequality to the rest terms with Condition (3) to obtain

$$\begin{aligned}
&2(\partial_k b_i) (\partial_k u(t+s, s)) (\partial_i u(t+s, s)) + (\partial_k a_{ij}) (\partial_k u(t+s, s)) (\partial_{ij} u(t+s, s)) \\
&\leq C_6 \varepsilon |D^2 u(t+s, s)|^2 + \frac{C_7}{\varepsilon} |\nabla u(t+s, s)|^2,
\end{aligned}$$

where we will choose ε small enough to satisfy $-C_5 + C_6 \varepsilon < 0$.

Hence we have strictly positive constants C_7 and C_8 such that

$$\frac{d}{dt} |\nabla u(t+s, s)|^2 - \tilde{\mathcal{L}} |\nabla u(t+s, s)|^2 \leq -C_7 |D^2 u(t+s, s)|^2 + C_8 |\nabla u(t+s, s)|^2.$$

Here we choose $\gamma < \delta$ and multiply $e^{\gamma t}$ on both sides. After integration with respect to $\bar{\rho}$, for any large time T , the integration from 0 to T satisfies the inequality as follows:

$$\begin{aligned} & \int_0^T e^{\gamma t} \int_{\bar{\mathbb{X}}} \frac{d}{dt} |\nabla u(t+s, s)|^2 d\bar{\rho} dt - \int_0^T e^{\gamma t} \int_{\bar{\mathbb{X}}} \tilde{\mathcal{L}} |\nabla u(t+s, s)|^2 d\bar{\rho} dt \\ & \leq -C_7 \int_0^T e^{\gamma t} \int_{\bar{\mathbb{X}}} |D^2 u(t+s, s)|^2 d\bar{\rho} dt + C_8 \int_0^T e^{\gamma t} \int_{\bar{\mathbb{X}}} |\nabla u(t+s, s)|^2 d\bar{\rho} dt. \end{aligned}$$

Then we apply the result (4.4.5) with γ small enough and the property of $\tilde{\mathcal{L}}$ as (4.3.2) to obtain

$$e^{\gamma T} \int_{\bar{\mathbb{X}}} |\nabla u(T+s, s)|^2 d\bar{\rho} - \int_{\bar{\mathbb{X}}} |\nabla u(s, s)|^2 d\bar{\rho} \leq C_9.$$

By the boundedness of $\int_{\bar{\mathbb{X}}} |\nabla \tilde{\phi}|^2 d\bar{\rho}$, we have

$$\frac{1}{\tau} \int_0^\tau \int_{\bar{\mathbb{X}}} |\nabla u(t+s, s, x)|^2 p(s, x) dx ds \leq C \exp(-\gamma t). \quad (4.4.6)$$

We proved the basic case in the previous part of proof. It is natural that we continue to prove the induction step in the following content. Assume that for any $k \leq m$, there exist strictly positive constants C_k and γ_k such that for any $t > 0$,

$$\frac{1}{\tau} \int_0^\tau \int_{\bar{\mathbb{X}}} |D^k u(t+s, s, x)|^2 p(s, x) dx ds \leq C_k \exp(-\gamma_k t).$$

Here we need to compare the expansion of the operators $\frac{d}{dt}$ and $\tilde{\mathcal{L}}$ on the following term:

$$|D^m u(t+s, s, x)|^2 = \sum_{l(J)=m} (\partial_J u(t+s, s, x))^2,$$

where J is the multi-index with length $l(J)$. We also introduce the multi-indices K and L for the following relation between them,

$$\begin{aligned} & \frac{d}{dt} D_J u(t+s, s, x) \\ & = D_J \tilde{\mathcal{L}} u(t+s, s, x) \\ & = \partial_J \left(b_i \partial_i u(t+s, s, x) + \frac{\partial}{\partial s} u(t+s, s, x) + \frac{1}{2} a_{ij} \partial_{ij} u(t+s, s, x) \right) \\ & = b_i \partial_{J \cup \{i\}} u(t+s, s, x) + \frac{\partial}{\partial s} (\partial_J u(t+s, s, x)) + \frac{1}{2} a_{ij} \partial_{J \cup \{ij\}} u(t+s, s, x) \end{aligned}$$

$$+ \sum_{l(K)+l(L)\leq 2m+1} \Phi_{KL}^J \partial_K u(t+s, s, x) \partial_L u(t+s, s, x).$$

Here the notation Φ_{KL}^J contains all the combinations of spatial derivatives on the functions a and b for respect multi-indices K and L under some specified J . It is obvious the length of K and L will not exceed $m+1$. The boundedness of each elements in Φ_{KL}^J comes from Condition (3). Therefore we will always have the following result by Young's inequality with some constants small enough,

$$\begin{aligned} & \frac{d}{dt} |D^m u(t+s, s, x)|^2 - \tilde{\mathcal{L}} |D^m u(t+s, s, x)|^2 \\ &= -a_{ij} (\partial_{J \cup \{i\}} u(t+s, s, x)) (\partial_{J \cup \{j\}} u(t+s, s, x)) \\ & \quad + \sum_{l(K)+l(L)\leq 2m+1} \Phi_{KL}^J \partial_K u(t+s, s, x) \partial_L u(t+s, s, x) \\ & \leq -C_1^m |D^{m+1} u(t+s, s, x)|^2 + C_2^m \sum_{k\leq m} |D^k u(t+s, s, x)|^2. \end{aligned}$$

Then we choose a strictly positive constant δ_{m+1} small enough to proceed as the basic case. Multiplying $e^{\delta_{m+1}t}$ on both sides and integrating with respect to $\bar{\rho}$, we will have the following result after integration from 0 to T and let T tends to ∞ :

$$\int_0^\infty e^{\delta_{m+1}t} \left(\int_{\mathbb{X}} |D^{m+1} u(t+s, s)|^2 d\bar{\rho} \right) dt < \infty$$

Consider a higher order $D^{m+1}u(t+s, s, x)$, we have

$$\begin{aligned} & \frac{d}{dt} |D^{m+1} u(t+s, s, x)|^2 - \tilde{\mathcal{L}} |D^{m+1} u(t+s, s, x)|^2 \\ & \leq -C_1^{m+1} |D^{m+2} u(t+s, s, x)|^2 + C_2^{m+1} \sum_{k\leq m+1} |D^k u(t+s, s, x)|^2. \end{aligned}$$

By choosing $\gamma_{m+1} < \delta_{m+1}$ and following the same procedure as above, we have that there exist strictly positive constants C_{m+1} and γ_{m+1} such that for any $t > 0$,

$$\frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |D^{m+1} u(t+s, s, x)|^2 p(s, x) dx ds \leq C_{m+1} \exp(-\gamma_{m+1}t).$$

By math induction, we proved the above result holds for any order of spatial derivatives of $u(t+s, s, x)$.

Since the density function $p(s, x)$ is strictly positive continuous function on any ball $B = B(0, R)$, we have

$$\frac{1}{\tau} \int_0^\tau \|\partial_J u(t+s, s)\|_{L^2(B)}^2 ds \leq \frac{C}{\tau} \int_0^\tau \int_{\mathbb{X}} |\partial_J u(t+s, s, x)|^2 p(s, x) dx ds.$$

By the Sobolev embedding Theorem, we have that for any $x \in B$,

$$\frac{1}{\tau} \int_0^\tau |u(t+s, s, x)| ds \leq C \exp(-\lambda t),$$

which is the conclusion of this lemma. \square

4.4.2 Estimates on the average of $u(t+s, s)$ in $L^2(\pi_r)$

For our next lemma, we need to introduce the weight $\pi_r(s, x)$ with some integer r which is determined later,

$$\pi_r(s, x) = \frac{1}{(2 + |x|^2 + \cos(\frac{2\pi s}{\tau}))^r}.$$

If we consider the gradient of it,

$$\nabla \pi_r(s, x) = -\frac{2rx}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \pi_r(s, x),$$

and the partial derivatives with respect to initial time,

$$\frac{\partial}{\partial s} \pi_r(s, x) = \frac{\frac{2\pi r}{\tau} \sin(\frac{2\pi s}{\tau})}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \pi_r(s, x),$$

the following results will be concluded.

For any multi-index J and any integer r , we have a smooth function $\psi_{J,r}(s, x)$ and $\psi_{s,r}(s, x)$ such that,

$$\begin{aligned} \partial_J \pi_r(s, x) &= \psi_{J,r}(s, x) \pi_r(s, x), \\ \frac{\partial}{\partial s} \pi_r(s, x) &= \psi_{s,r}(s, x) \pi_r(s, x) \end{aligned}$$

where $\psi_{J,r}(s, x) \rightarrow 0$ and $\psi_{s,r}(s, x) \rightarrow 0$ when $|x| \rightarrow +\infty$. These properties will be used in the following lemma.

Lemma 4.4.2. *Assume Conditions (3), (4) and (5), there exists strictly positive constants C and λ such that for any $t > 0$, we have*

$$\frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t+s, s, x)|^2 \pi_r(s, x) dx ds \leq C \exp(-\lambda t).$$

Proof. We have the property (4.3.3):

$$\forall n \in \mathbb{N}, \exists r_n \in \mathbb{N}, \forall T > 0, \exists C_n(T) > 0 : \forall t < T, |D^n u(t + s, s, x)| \leq C_n(T)(1 + |x|^{r_n}).$$

Therefore, for any integer $n \geq 0$, it is possible to choose an integer r_n such that, for any $0 \leq m \leq n$ and any $t \geq 0$, we have

$$|D^m u(t + s, s, x)| \pi_{r_n}(s, x) \in L^2(\mathbb{R}^d).$$

When we consider the integer M_I defined by

$$l(I) = [M_I - d/2],$$

and the property of the weight π_r , we have that there exists an integer r_0 such that for any $t > 0$, any $r \geq r_0$ and any $m \leq M_I$,

$$D^m (u(t + s, s, x) \pi_r(s, x)) \in L^2(\mathbb{R}^d).$$

It is easy to get the periodicity of the function $u(t + s, s, x) \pi_r(s, x)$ with respect to the initial time s . Any order of its spatial derivatives are also τ -periodic in s . Then we have

$$\begin{aligned} & \int_0^\tau \int_{\mathbb{X}} \frac{d}{dt} |u(t + s, s)|^2 \pi_r dx ds \\ &= \int_0^\tau \int_{\mathbb{X}} 2u(t + s, s) \tilde{\mathcal{L}}u(t + s, s) \pi_r dx ds \\ &= \int_0^\tau \int_{\mathbb{X}} 2b_i u(t + s, s) (\partial_i u(t + s, s)) \pi_r dx ds \\ & \quad + \int_0^\tau \int_{\mathbb{X}} 2u(t + s, s) \left(\frac{\partial}{\partial s} u(t + s, s) \right) \pi_r dx ds \\ & \quad + \int_0^\tau \int_{\mathbb{X}} a_{ij} u(t + s, s) (\partial_{ij} u(t + s, s)) \pi_r dx ds \\ &= - \int_0^\tau \int_{\mathbb{X}} (\partial_i b_i) |u(t + s, s)|^2 \pi_r dx ds - \int_0^\tau \int_{\mathbb{X}} b_i |u(t + s, s)|^2 (\partial_i \pi_r) dx ds \\ & \quad - \int_0^\tau \int_{\mathbb{X}} |u(t + s, s)|^2 \left(\frac{\partial}{\partial s} \pi_r \right) dx ds \\ & \quad - \int_0^\tau \int_{\mathbb{X}} (\partial_i a_{ij}) u(t + s, s) (\partial_j u(t + s, s)) \pi_r dx ds \\ & \quad - \int_0^\tau \int_{\mathbb{X}} a_{ij} (\partial_i u(t + s, s)) (\partial_j u(t + s, s)) \pi_r dx ds \end{aligned}$$

$$- \int_0^\tau \int_{\mathbb{X}} a_{ij} u(t+s, s) (\partial_j u(t+s, s)) (\partial_i \pi_r) dx ds.$$

By Condition (4) and the property of the weight $\pi(s, x)$, we have that

$$\begin{aligned} & \int_0^\tau \int_{\mathbb{X}} \frac{d}{dt} |u(t+s, s)|^2 \pi_r dx ds \\ \leq & - \int_0^\tau \int_{\mathbb{X}} (\partial_i b_i) |u(t+s, s)|^2 \pi_r dx ds + \int_0^\tau \int_{\mathbb{X}} \frac{2r \cdot x \cdot b(s, x)}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} |u(t+s, s)|^2 \pi_r dx ds \\ & - \int_0^\tau \int_{\mathbb{X}} |u(t+s, s)|^2 \psi_s \pi_r dx ds \\ & + \frac{1}{2} \int_0^\tau \int_{\mathbb{X}} (\partial_{ij} a_{ij}) |u(t+s, s)|^2 \pi_r dx ds + \frac{1}{2} \int_0^\tau \int_{\mathbb{X}} (\partial_i a_{ij}) |u(t+s, s)|^2 \psi_j \pi_r dx ds \\ & - \alpha \int_0^\tau \int_{\mathbb{X}} |\nabla u(t+s, s)|^2 \pi_r dx ds \\ & + \frac{1}{2} \int_0^\tau \int_{\mathbb{X}} (\partial_j a_{ij}) |u(t+s, s)|^2 \psi_i \pi_r dx ds + \frac{1}{2} \int_0^\tau \int_{\mathbb{X}} a_{ij} |u(t+s, s)|^2 \psi_{ij} \pi_r dx ds \\ = & \int_0^\tau \int_{\mathbb{X}} \left(\Phi_{a,b}(s, x) + \Phi_\psi(s, x) + \frac{2r \cdot x \cdot b(s, x)}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \right) |u(t+s, s)|^2 \pi_r dx ds \\ & - \alpha \int_0^\tau \int_{\mathbb{X}} |\nabla u(t+s, s)|^2 \pi_r dx ds, \end{aligned}$$

where $\Phi_{a,b}$ is a bounded function depending on functions a, b and their derivatives, Φ_ψ is a function which could depend on functions ψ_I . It is easy to prove that $\Phi_{a,b}$ is independent of r . We also know that Φ_ψ tends to 0 when $|x|$ tending to ∞ . Therefore, we choose $r \geq r_0$ large enough to obtain

$$\limsup_{|x| \rightarrow \infty} \left(\Phi_{a,b}(s, x) + \Phi_\psi(s, x) + \frac{2r \cdot x \cdot b(s, x)}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \right) \leq C_\Phi - 2r\beta < 0.$$

Now choosing the ball $B = B(0, R)$ with R large enough, which depends on the integer r , we have following result by previous lemma,

$$\begin{aligned} & \int_0^\tau \int_{\mathbb{X}} \left(\Phi_{a,b}(s, x) + \Phi_\psi(s, x) + \frac{2r \cdot x \cdot b(s, x)}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \right) |u(t+s, s)|^2 \pi_r dx ds \\ = & \int_0^\tau \int_B \left(\Phi_{a,b}(s, x) + \Phi_\psi(s, x) + \frac{2r \cdot x \cdot b(s, x)}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \right) |u(t+s, s)|^2 \pi_r dx ds \\ & + \int_0^\tau \int_{B^c} \left(\Phi_{a,b}(s, x) + \Phi_\psi(s, x) + \frac{2r \cdot x \cdot b(s, x)}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \right) |u(t+s, s)|^2 \pi_r dx ds \end{aligned}$$

$$\begin{aligned} &\leq C_1 \int_0^\tau \int_B |u(t+s, s)|^2 \pi_r dx ds - C_2 \int_0^\tau \int_{B^c} |u(t+s, s)|^2 \pi_r dx ds \\ &\leq (C_1 + C_2) \int_0^\tau \int_B |u(t+s, s)|^2 \pi_r dx ds - C_2 \int_0^\tau \int_{\mathbb{X}} |u(t+s, s)|^2 \pi_r dx ds. \end{aligned}$$

Therefore,

$$\int_0^\tau \int_{\mathbb{X}} \frac{d}{dt} |u(t+s, s)|^2 \pi_r dx ds \leq -C_2 \int_0^\tau \int_{\mathbb{X}} |u(t+s, s)|^2 \pi_r dx ds + C_3 \exp(-\lambda t).$$

The basic result from ordinary differential equation gives us that there exist strictly constants C and λ such that for any $t > 0$,

$$\frac{1}{\tau} \int_0^\tau \int_{\mathbb{X}} |u(t+s, s)|^2 \pi_r dx ds \leq C \exp(-\lambda t).$$

□

4.4.3 Exponential decay in time of the spatial derivatives of the solution

Theorem 4.4.3. *Assume Conditions (3), (4) and (5), and let the function $\phi \in C_p^\infty$. Let $u(t+s, s, x) = \mathbb{E}\phi(X_{t+s}^{s,x})$. Then for any multi-index I , there exists an integer k_I , strictly positive constants Γ_I and γ_I such that the spatial derivative $\partial_I u(t+s, s, x)$ satisfies that*

$$\frac{1}{\tau} \int_0^\tau |\partial_I u(t+s, s, x)| ds \leq \Gamma_I (1 + |x|^{k_I}) \exp(-\gamma_I t).$$

Proof. The process of the proof is very similar to Lemma 4.4.1. We apply math induction on each order of spatial derivatives of $u(t+s, s, x)$, then it is possible to use Sobolev embedding Theorem to obtain the result. In Lemma 4.4.1, we applied the property of the density function $p(s, x)$ of the average periodic measure $\bar{\rho}$,

$$\tilde{\mathcal{L}}^* p(s, x) = 0.$$

It guaranteed the exponential contraction in any ball $B(0, R)$. When we consider the behaviour out of the ball, we have following result,

$$\begin{aligned} &\int \tilde{\mathcal{L}} |u(t+s, s)|^2 \pi_r d\tilde{x} \\ &= \int |u(t+s, s)|^2 \tilde{\mathcal{L}}^* \pi_r d\tilde{x} \end{aligned}$$

$$= \int \left(\Phi_{a,b} + \Phi_\psi + \frac{2r \cdot x \cdot b}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} \right) |u(t+s, s)|^2 \pi_r d\tilde{x}.$$

For some fixed r , it is possible to choose the ball large enough to make sure that for any $x \in B^c$, we have

$$\Phi_{a,b} + \Phi_\psi + \frac{2r \cdot x \cdot b}{2 + |x|^2 + \cos(\frac{2\pi s}{\tau})} < 0,$$

which we described in Lemma 4.4.2. Therefore we have some positive constants C_0 and λ_0 such that

$$\int_{\tilde{\mathbb{X}}} \tilde{\mathcal{L}} |u(t+s, s)|^2 \pi_r d\tilde{x} \leq C_0 \exp(-\lambda_0 t). \quad (4.4.7)$$

Then we compare the difference of operators $\frac{d}{dt}$ and $\tilde{\mathcal{L}}$ to obtain the estimation of higher order of spatial derivatives,

$$\int_{\tilde{\mathbb{X}}} \frac{d}{dt} |u(t+s, s)|^2 \pi_r d\tilde{x} - \int_{\tilde{\mathbb{X}}} \tilde{\mathcal{L}} |u(t+s, s)|^2 \pi_r d\tilde{x} \leq -\alpha \int_{\tilde{\mathbb{X}}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x}.$$

Multiplying $e^{\delta t}$ and integrating with respect to t from 0 to T , we have

$$\begin{aligned} & e^{\delta T} \int_{\tilde{\mathbb{X}}} |u(T+s, s)|^2 \pi_r d\tilde{x} + C \int_0^T e^{\delta t} \int_{\tilde{\mathbb{X}}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x} dt \\ & \leq \int_{\tilde{\mathbb{X}}} |\phi(\tilde{x})|^2 \pi_r d\tilde{x} + \delta \int_0^T e^{\delta t} \int_{\tilde{\mathbb{X}}} |u(t+s, s)|^2 \pi_r d\tilde{x} dt + \int_0^T e^{\delta t} \int_{\tilde{\mathbb{X}}} \tilde{\mathcal{L}} |u(t+s, s)|^2 \pi_r d\tilde{x} dt. \end{aligned}$$

By the property (4.4.3) and (4.4.7), we choose the constant δ small enough to obtain

$$\int_0^T e^{\delta t} \int_{\tilde{\mathbb{X}}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x} dt \leq C$$

Now we consider that

$$\begin{aligned} & \int_{\tilde{\mathbb{X}}} \frac{d}{dt} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x} - \int_{\tilde{\mathbb{X}}} \tilde{\mathcal{L}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x} \\ & \leq -C_1 \int_{\tilde{\mathbb{X}}} |D^2 u(t+s, s)|^2 \pi_r d\tilde{x} + C_2 \int_{\tilde{\mathbb{X}}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x}. \end{aligned}$$

Multiplying $e^{\gamma t}$ with γ small enough, we have the integration with respect to t ,

$$\begin{aligned} & e^{\gamma T} \int_{\tilde{\mathbb{X}}} |\nabla u(T+s, s)|^2 \pi_r d\tilde{x} + C \int_0^T e^{\gamma t} \int_{\tilde{\mathbb{X}}} |D^2 u(t+s, s)|^2 \pi_r d\tilde{x} dt \\ & \leq \int_{\tilde{\mathbb{X}}} |\nabla \phi(\tilde{x})|^2 \pi_r d\tilde{x} + (\gamma + C_2) \int_0^T e^{\gamma t} \int_{\tilde{\mathbb{X}}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x} dt \end{aligned}$$

$$+ \int_0^T e^{\gamma t} \int_{\tilde{\mathbb{X}}} \tilde{\mathcal{L}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x} dt,$$

which gives us the conclusion that

$$\int_{\tilde{\mathbb{X}}} |\nabla u(t+s, s)|^2 \pi_r d\tilde{x} \leq C e^{-\gamma t}.$$

It is easy to repeat the process for any $m \in \mathbb{N}$ with positive constants C_m and γ_m to obtain

$$\int_{\tilde{\mathbb{X}}} |D^m u(t+s, s)|^2 \pi_r d\tilde{x} \leq C_m e^{-\gamma_m t}.$$

As we showed, for any multi-index J ,

$$\partial_J \pi_r(s, x) = \psi_{J,r}(s, x) \pi_r(s, x),$$

where $\psi_{J,r}(s, x)$ is bounded for fixed r .

Then we prove the conclusion of the theorem by the weighted Sobolev embedding Theorem with $\pi_r(s, x) d\tilde{x}$ instead of the the density function of average periodic measure $p(s, x) d\tilde{x}$. \square

Remark 4.4.4. *The proof of the previous theorem also gives us the result that there exist some integer $l \in \mathbb{N}$ and strictly positive constants Γ and γ , such that for any t and x , we have*

$$\left| \frac{1}{\tau} \int_0^\tau u(t+s, s, x) ds - \int_{\tilde{\mathbb{X}}} \tilde{\phi}(\tilde{x}) d\tilde{\rho}(\tilde{x}) \right| \leq \Gamma(1 + |x|^l) \exp(-\gamma t).$$

4.5 Numerical Analysis

Here we consider the numerical approximation of our model with Euler-Maruyama method

$$\begin{aligned} \hat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} &= \hat{X}_{-k\tau+i\Delta t}^{-k\tau} + b(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau}) \Delta t \\ &\quad + \sigma(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau}) (W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t}), \end{aligned} \quad (4.5.1)$$

which followed the same notation as strong approximation in (3.1.1) except the functions b and σ . When considering the local error of the weak approximation, we firstly focus on the time interval from 0 to Δt and $k = 0$,

$$X_{\Delta t}^{0,x} = x + b(0, x) \int_0^{\Delta t} dt + \sigma(0, x) \int_0^{\Delta t} dWt + R,$$

where the expression of R can be determined by the stochastic Taylor expansions in Kloeden and Platen's book [29] as follows

$$\begin{aligned} R &= \int_0^t \int_0^r \tilde{\mathcal{L}}b(s, x) dz dr + \int_0^t \int_0^r \sigma(s, x) \frac{\partial}{\partial x} b(s, x) dW_z dr \\ &\quad + \int_0^t \int_0^r \tilde{\mathcal{L}}\sigma(s, x) dz dW_r + \int_0^t \int_0^r \sigma(s, x) \frac{\partial}{\partial x} \sigma(s, x) dW_z dW_r. \end{aligned}$$

Then we take the expectation to have

$$\mathbb{E} \left(X_{\Delta t}^{0,x} - \hat{X}_{\Delta t}^{0,x} \right) = \mathbb{E}(R) = C(\Delta t)^2,$$

where the existence of constant C comes from the smoothness of functions b and σ and the boundedness of their derivatives. In order to consider the weak error of function $\phi \in C_p^\infty$, we need to have the boundedness of any order of the discrete process's moments.

Proposition 4.5.1. *Assume Conditions (3) and (5), then for any integer n , there exist strictly positive constants C_n and γ_n such that for any step size Δt small enough, we have that for any initial x and any number of steps $N \in \mathbb{N}$*

$$\mathbb{E} \left| \hat{X}_{-k\tau+N\Delta t}^{-k\tau} \right|^n \leq C_n (1 + |x|^n \exp(-\gamma_n N \Delta t)).$$

Proof. By the Euler-Maruyama method, when the initial state $x \in K^c$, we have

$$\begin{aligned} &\left(\hat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right)^2 \\ &= \left(\hat{X}_{-k\tau+i\Delta t}^{-k\tau} + b(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau})\Delta t + \sigma(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau}) (W_{-k\tau+(i+1)\Delta t} - W_{-k\tau+i\Delta t}) \right)^2 \\ &= \left(\hat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^2 + \left(\hat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T b(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau})\Delta t \\ &\quad + \left(\hat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^T \sigma(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau})\Delta W_i \\ &\quad + \left(b(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right)^2 (\Delta t)^2 + \left(\sigma(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right)^2 (\Delta W_i)^2 \\ &\quad + \left(b(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right)^T \left(\sigma(i\Delta t, \hat{X}_{-k\tau+i\Delta t}^{-k\tau}) \right) (\Delta t)(\Delta W_i). \end{aligned}$$

Apply Conditions (3) and (5) after taking the expectation on both sides to obtain

$$\mathbb{E} \left(\hat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau} \right)^2 \leq \mathbb{E} \left(\hat{X}_{-k\tau+i\Delta t}^{-k\tau} \right)^2 (1 - \beta\Delta t) + C_1\Delta t$$

As the compactness of the set K , the above result still holds for some constant C_K instead of C_1 when $x \in K$. Therefore we choose $C = \max\{C_1, C_K\}$ and iterate the inequality,

$$\mathbb{E} \left(\widehat{X}_{-k\tau + N\Delta t}^{-k\tau} \right)^2 \leq |x|^2 (1 - \beta\Delta t)^N + C\Delta t \sum_{i=1}^{N-1} (1 - \beta\Delta t).$$

The conclusion is obtained when Δt is small enough for $n = 2$. It is easy to derive the higher orders of moment with same process. \square

To approximate the average periodic measure, we need to consider the average of different initial time s . Therefore, we also discretize the initial time with $\Delta t = \tau/N$. Then the numerical scheme are repeated with the initial conditions $\widehat{X}_0 = (n\Delta t, x)$, $n = 0, \dots, N - 1$. The choice of the constant N highly depends on the properties of the model. The difficulties to find the criterion come from the dimension of x and solving the general partial differential equations. The criterion for K is also extremely difficult by the same reason. But we have the long time behaviour for the numerical scheme as following theorem.

Theorem 4.5.2. *Assume Condition (3), (4) and (5). We Choose $\Delta t = \tau/N$ for some $N \in \mathbb{N}$. Then for any function $\phi \in \mathcal{C}_p^\infty$, if the scheme (4.5.1) is ergodic, it satisfies:*

$$\lim_{N, K \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{K} \sum_{k=1}^K \phi \left(\widehat{X}_{n\Delta t}^{-k\tau + n\Delta t}(x) \right) = \int \phi(x) d\bar{\rho}(x) + \mathcal{O}(\Delta t) \text{ a.s.}$$

Proof. Denote by $u(n\Delta t, s, x) = \mathbb{E}\phi(X_{n\Delta t}^{s,x})$ with $s < 0$. Therefore

$$u(n\Delta t, -k\tau + n\Delta t, x) = \mathbb{E}\phi(X_{n\Delta t}^{-k\tau + n\Delta t, x})$$

and

$$u(n\Delta t, n\Delta t, \widehat{X}_{n\Delta t}^{-k\tau + n\Delta t, x}) = \mathbb{E}\phi(\widehat{X}_{n\Delta t}^{-k\tau + n\Delta t, x}).$$

By the periodicity of $u(t + s, s, x)$ with respect to initial time s , it is always possible to move the initial time into $[0, \tau)$ like

$$u(n\Delta t, -k\tau + n\Delta t, x) = u(k\tau + n\Delta t, n\Delta t, x), \quad \forall n \in \{0, 1, \dots, N - 1\}.$$

Then we consider the following Itô's-Taylor expansions

$$u(k\tau + n\Delta t, (n + i)\Delta t, \widehat{X}_{i+1}^{n,x})$$

$$\begin{aligned}
&= u(k\tau + n\Delta t, (n+i)\Delta t, \widehat{X}_i^{n,x}) \\
&\quad + \mathcal{L}u(k\tau + n\Delta t, (n+i)\Delta t, \widehat{X}_i^{n,x})\Delta t + R_{1,i}^n(\Delta t)^2 \text{ a.s.}, \tag{4.5.2}
\end{aligned}$$

and

$$\begin{aligned}
&u(k\tau + n\Delta t, (n+i+1)\Delta t, \widehat{X}_i^{n,x}) \\
&= u(k\tau + n\Delta t, (n+i)\Delta t, \widehat{X}_i^{n,x}) \\
&\quad + \mathcal{L}u(k\tau + n\Delta t, (n+i)\Delta t, \widehat{X}_i^{n,x})\Delta t + R_{2,i}^n(\Delta t)^2 \text{ a.s.} \tag{4.5.3}
\end{aligned}$$

where $\widehat{X}_i^{n,x} := \widehat{X}_{-k\tau+n\Delta t+i\Delta t}^{-k\tau+n\Delta t,x}$. The coefficient $R_{1,i}^n$ before the local error $(\Delta t)^2$ have the form as follows

$$\mathbb{E} \left[\psi(\widehat{X}_i^{n,x}) \cdot \partial_{Ju} \left(k\tau + n\Delta t, (n+i)\Delta t, \widehat{X}_i^{n,x} + \theta \left(\widehat{X}_{i+1}^{n,x} - \widehat{X}_i^{n,x} \right) \right) \right], \tag{4.5.4}$$

where $0 < \theta < 1$ and the function $\psi(x)$ is a product of functions b, σ and their derivatives. The coefficient $R_{2,i}^n$ has the similar form as above. It is easy to obtain the boundedness of $\psi(x)$ with Condition (3). Considering the average periodic measure, when N goes to infinity, we have

$$\begin{aligned}
&\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \partial_{Ju} (k\tau + n\Delta t, (n+i)\Delta t, x) \\
&= \frac{1}{\tau} \int_0^\tau \partial_{Ju}(k\tau + s, s + i\Delta t, x) ds \\
&\leq C \exp(-\lambda(kN - i)\Delta t)(1 + |x|^l)
\end{aligned}$$

Combine this with Proposition 4.5.1 and Theorem 4.4.3, there exists a constant $\lambda > 0$ and integer $l \in \mathbb{N}$, such that

$$\begin{aligned}
&\sum_{i=0}^{kN-1} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |R_{1,i}^n| \\
&\leq \sum_{i=0}^{kN-1} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} C \mathbb{E} \left[\partial_{Ju} \left(k\tau + n\Delta t, (n+i)\Delta t, \widehat{X}_i^{n,x} + \theta \left(\widehat{X}_{i+1}^{n,x} - \widehat{X}_i^{n,x} \right) \right) \right] \\
&\leq \sum_{i=0}^{kN-1} C \mathbb{E} \left[\frac{1}{\tau} \int_0^\tau \partial_{Ju} \left(k\tau + s, s + i\Delta t, \widehat{X}_i^{n,x} + \theta \left(\widehat{X}_{i+1}^{n,x} - \widehat{X}_i^{n,x} \right) \right) ds \right] \\
&\leq \frac{C}{\tau} \sup_{i \geq 0} \mathbb{E} \left(1 + \left| \widehat{X}_{-k\tau+i\Delta t}^{-k\tau,x} \right|^l + \left| \widehat{X}_{-k\tau+(i+1)\Delta t}^{-k\tau,x} \right|^l \right) \sum_{i=0}^{kN-1} \exp(-\lambda(kN - i)\Delta t)
\end{aligned}$$

$$\leq \frac{1 - e^{-\lambda k N \Delta t}}{1 - e^{-\lambda \Delta t}} e^{\lambda \Delta t} C_l (1 + |x|^l).$$

Let k go to infinity and Δt be small enough, we have

$$\sum_{i=0}^{+\infty} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} |R_{1,i}^n| \leq \frac{C}{\Delta t} (1 + |x|^l).$$

The result also holds for $R_{2,i}^n$. Denote by $R_i^n = R_{1,i}^n - R_{2,i}^n$, we have

$$\lim_{N \rightarrow \infty} \sum_{i=0}^{+\infty} \frac{1}{N} \sum_{n=0}^{N-1} |R_i^n| (\Delta t)^2 \leq \tilde{C} (1 + |x|^s) (\Delta t).$$

By the successive comparison between (4.5.3) and (4.5.4), we have

$$\begin{aligned} & \lim_{N, K \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{K} \sum_{k=1}^K \phi \left(\widehat{X}_{n\Delta t}^{-k\tau + n\Delta t}(x) \right) \\ &= \lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \frac{1}{\tau} \int_0^\tau u(s, -k\tau + s, x) ds + \lim_{N \rightarrow \infty} \sum_{i=0}^{+\infty} \frac{1}{N} \sum_{n=0}^{N-1} |R_i^n| (\Delta t)^2 \text{ a.s.} \end{aligned} \quad (4.5.5)$$

The strong law of large numbers gives us

$$\lim_{K \rightarrow \infty} \frac{1}{K} \sum_{k=1}^K \frac{1}{\tau} \int_0^\tau u(s, -k\tau + s, x) ds = \frac{1}{\tau} \int_0^\tau \int \phi(x) d\rho_s(x) ds = \int \phi(x) d\bar{\rho}(x).$$

The left hand side of (4.5.5) gives us the measure under discretized average periodic measure

$$\int \phi(x) d\bar{\rho}_{\Delta t}(x) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=0}^{N-1} \frac{1}{K} \sum_{k=1}^K \phi \left(\widehat{X}_{n\Delta t}^{-k\tau + n\Delta t}(x) \right)$$

□

Chapter 5

Appendix

5.1 Useful inequalities for proofs

During the proofs of following lemmas and theorems, we need some inequalities to help us estimate, so here are some important ones for our estimation. We state them without detail proofs here.

5.1.1 Young's inequality

If a and b are non-negative real numbers and p and q are positive real numbers such that $1/p + 1/q = 1$ then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

This is the standard form of Young's inequality and now we introduce the Young's inequality with ϵ while $p = q = 2$. For any real non-negative numbers a and b , and for any $\epsilon > 0$, the following inequality holds:

$$ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$$

5.1.2 Continuous Gronwall inequality

Let I denote an interval of the real line. Let α, β and u be real-valued functions defined on I . Assume that β and u are continuous and that the negative part of α

is integrable on every closed and bounded subinterval of I . If β is non-negative and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_0^t \beta(s)u(s)ds, \quad \forall t \in I$$

then

$$u(t) \leq \alpha(t) + \int_0^t \alpha(s)\beta(s) \exp\left(\int_s^t \beta(r)dr\right) ds, \quad \forall t \in I$$

5.1.3 Discrete Gronwall inequality

Here we introduce a sharp Gronwall inequality from Holte[24](2009) for the proofs of discrete version of random periodic solution. Assume y_n, f_n and g_n are non-negative sequences and

$$y_n \leq f_n + \sum_{k=0}^n g_k y_k, \quad \text{for } n \geq 0$$

Then

$$y_n \leq f_n + \sum_{k=0}^n f_k g_k \prod_{k=j+1}^n (1 + g_j), \quad \text{for } n \geq 0$$

5.2 Proofs

Lemma 5.2.1. *Assume that the matrix A is symmetric and positive-definite as stated above. Then for any $\Delta t > 0$ the matrix*

$$e^{-A\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-A\Delta t)^i \tag{5.2.1}$$

is positive-definite for odd $p \in \mathbb{N}$ and negative-definite for even $p \in \mathbb{N}$ and $p = 0$.

Proof. We start with the one-dimensional statement:

For any $\alpha > 0$ and $\Delta t > 0$,

$$e^{-\alpha\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-\alpha\Delta t)^i \tag{5.2.2}$$

is positive-definite for odd $p \in \mathbb{N}$ and negative-definite for even $p \in \mathbb{N}$ and $p = 0$. The statement is valid for $p = 0$. Consider the function

$$f(t) := e^{-\alpha\Delta t} - \sum_{i=0}^{p+1} \frac{1}{i!} (-\alpha\Delta t)^i$$

and note that $f(0) = 0$. Then

$$f'(t) := -\alpha \left(e^{-\alpha\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-\alpha\Delta t)^i \right)$$

If

$$e^{-\alpha\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-\alpha\Delta t)^i$$

is positive, then $f'(t)$ is negative for all $\Delta t > 0$, and thus $f(t) < 0$ (and vice versa). Starting from $p = 0$, with each consecutive integer p the sign of (5.2.2) will reverse. So the one-dimensional statement is correct. Now, the matrix exponential of A is diagonalizable:

$$e^A = Qe^DQ^{-1}$$

where Q is invertible and D is diagonal with eigenvalues of A as its spectrum.

$$\begin{aligned} & Q^{-1} (e^{-A\Delta t}) Q \\ &= Q^{-1} \left(\sum_{i=0}^{\infty} \frac{1}{i!} (-A\Delta t)^i \right) Q \\ &= Q^{-1} \left(\sum_{i=0}^{\infty} \frac{1}{i!} (-QDQ^{-1}\Delta t)^i \right) Q \\ &= \sum_{i=0}^{\infty} \frac{1}{i!} (-D\Delta t)^i \\ &= e^{-D\Delta t} \end{aligned}$$

Therefore for any $p \geq 0$ we have

$$e^{-A\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-A\Delta t)^i$$

$$\begin{aligned}
&= e^{-A\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-QDQ^{-1}\Delta t)^i \\
&= Qe^{-D\Delta t}Q^{-1} - \sum_{i=0}^p \frac{1}{i!} Q(-D\Delta t)^i Q^{-1} \\
&= Q \left(e^{-D\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-D\Delta t)^i \right) Q^{-1}
\end{aligned}$$

Note the matrix

$$e^{-D\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-D\Delta t)^i$$

is diagonal with the following trace:

$$\begin{pmatrix}
e^{-\lambda_1\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-\lambda_1\Delta t)^i & 0 & \cdots & 0 \\
0 & e^{-\lambda_2\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-\lambda_2\Delta t)^i & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & e^{-\lambda_m\Delta t} - \sum_{i=0}^p \frac{1}{i!} (-\lambda_m\Delta t)^i
\end{pmatrix}$$

The diagonal elements are the eigenvalues of the matrix (5.2.1), and according to the one-dimensional statement proved earlier, they are all positive or all negative depending on the parity of p .

□

Lemma 5.2.2. *Assume that the matrix A is symmetric and positive-definite as stated above, and denote by λ the largest eigenvalue of A . Then, for $0 < \Delta t \leq \frac{1}{\lambda}$, the matrix*

$$e^{-A\Delta t j} - (I - A\Delta t)^j$$

is positive-definite for any $j \in \mathbb{N}$.

Proof. This lemma is proved by using the induction principle. The result for $j = 1$ follows from Lemma 5.2.1. Suppose now that for some j the matrix $e^{-A\Delta t j} - (I - A\Delta t)^j$ is positive-definite, and we need to discuss the following matrix

$$\begin{aligned}
&e^{-A\Delta t(j+1)} - (I - A\Delta t)^{j+1} \\
&= e^{-A\Delta t} (e^{-A\Delta t j} - (I - A\Delta t)^j) + (e^{-A\Delta t} - (I - A\Delta t)) (I - A\Delta t)^j
\end{aligned}$$

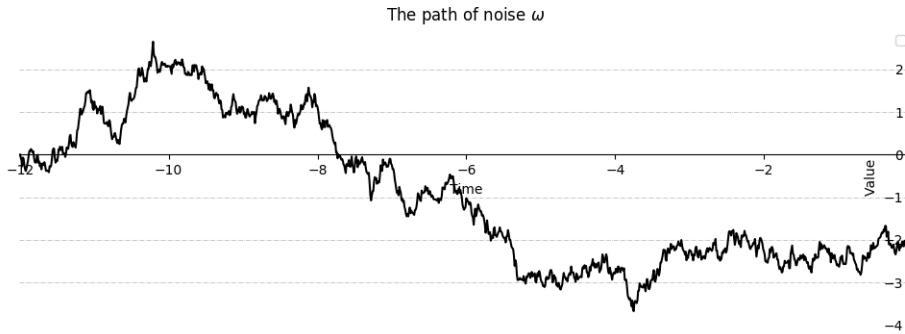


Figure 5.1: The path of noise ω used in the model

The four terms above on the right-hand side are all positive-definite matrices. Another useful result we can obtain is:

$$e^{-A\Delta t}(I - A\Delta t)^j = (I - A\Delta t)^j e^{-A\Delta t}$$

due to the property of the matrices of $e^{-A\Delta t}$ and $(I - A\Delta t)^j$. Therefore, we have a sum of two products of commuting positive-definite matrices, which is a positive-definite matrix. By the induction principle, all the matrices $e^{-A\Delta t j} - (I - A\Delta t)^j$ are positive-definite for any $j \in \mathbb{N}$

□

5.3 Numerical experiments

We will show the details of the numerical experiment here. Firstly, we attached the path of noise we used in the approximation as in Fig. 5.1. All the original numerical processes take the corresponding noise in the specified time intervals.

We used the same initial condition $x = 0.5$ for each approximation. The time intervals for each experiment differ from -6.5 to -12. The corresponding discretized process are illustrated as in Fig. 5.2 to 5.13. It is easy to notice the patterns are not all the same. But when we consider the pull-back with difference of $k\tau$, where the period $\tau = 2$, it is easy to notice the pattern looks similar. To show these pattern are the same, we move these graphs into the same figures as in Fig. 5.14 to Fig. 5.17.

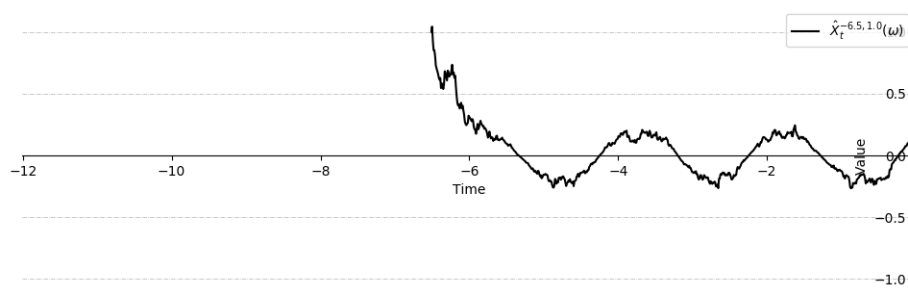


Figure 5.2: Simulations of the processes $\{\hat{X}_t^{-6.5}(\omega), -6.5 \leq t \leq 0\}$

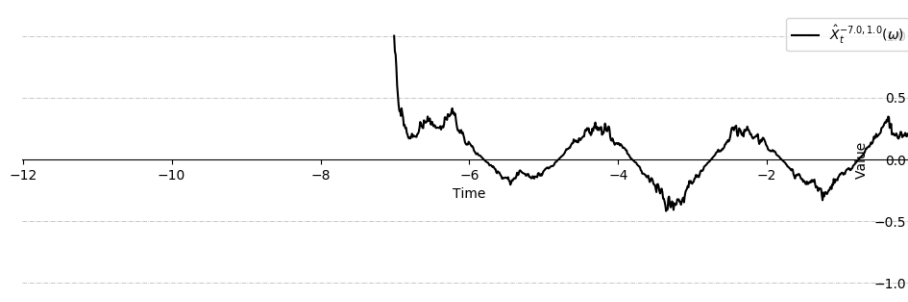


Figure 5.3: Simulations of the processes $\{\hat{X}_t^{-7}(\omega), -7 \leq t \leq 0\}$

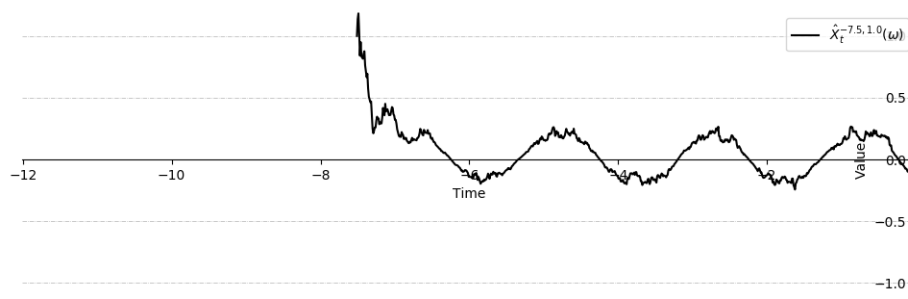


Figure 5.4: Simulations of the processes $\{\hat{X}_t^{-7.5}(\omega), -7.5 \leq t \leq 0\}$

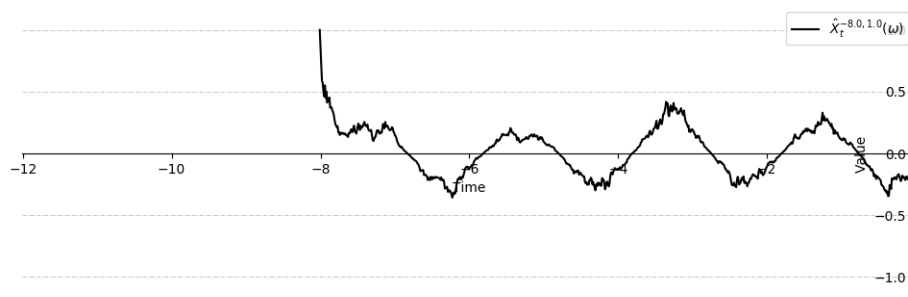


Figure 5.5: Simulations of the processes $\{\hat{X}_t^{-8}(\omega), -8 \leq t \leq 0\}$

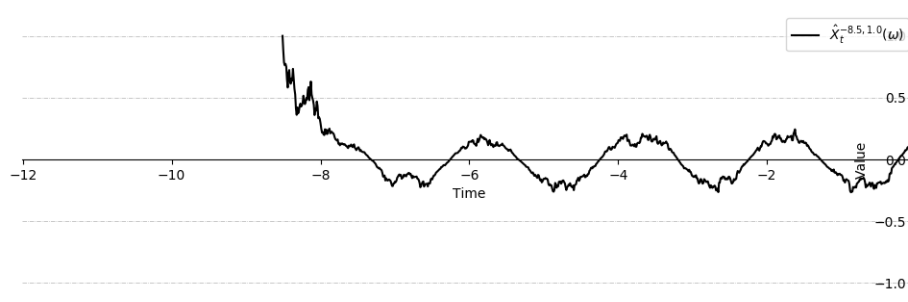


Figure 5.6: Simulations of the processes $\{\hat{X}_t^{-8.5}(\omega), -8.5 \leq t \leq 0\}$

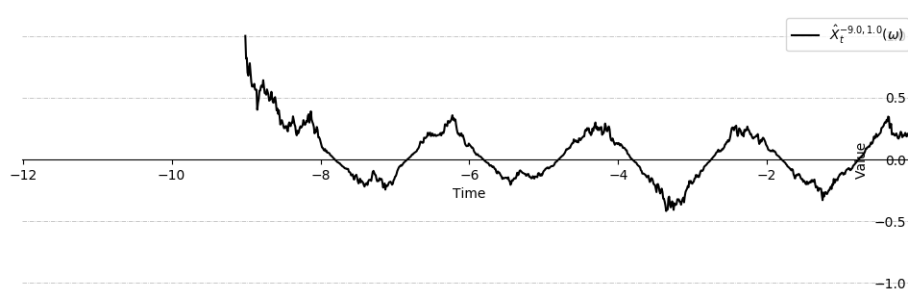


Figure 5.7: Simulations of the processes $\{\hat{X}_t^{-9}(\omega), -9 \leq t \leq 0\}$

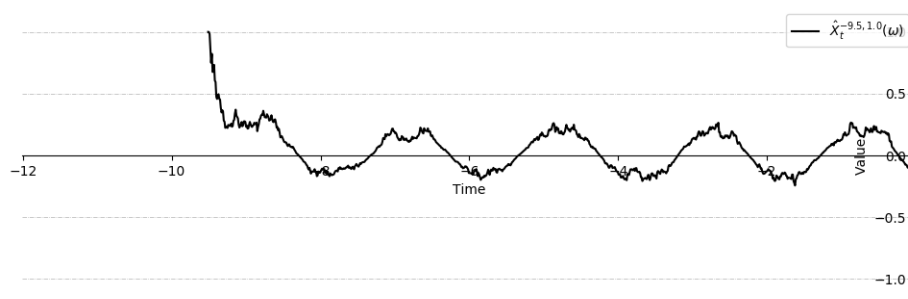


Figure 5.8: Simulations of the processes $\{\hat{X}_t^{-9.5}(\omega), -9.5 \leq t \leq 0\}$

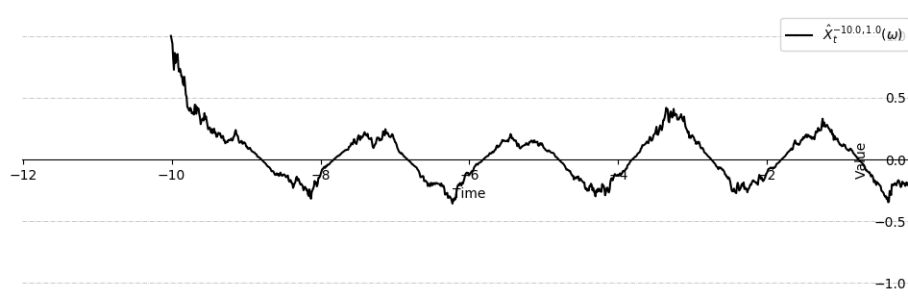


Figure 5.9: Simulations of the processes $\{\hat{X}_t^{-10}(\omega), -10 \leq t \leq 0\}$

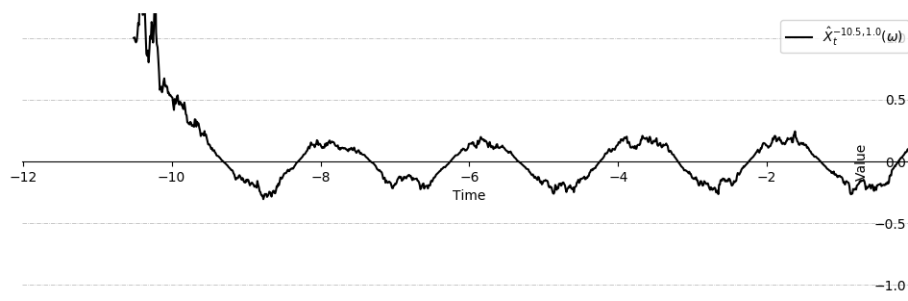


Figure 5.10: Simulations of the processes $\{\hat{X}_t^{-10.5}(\omega), -10.5 \leq t \leq 0\}$

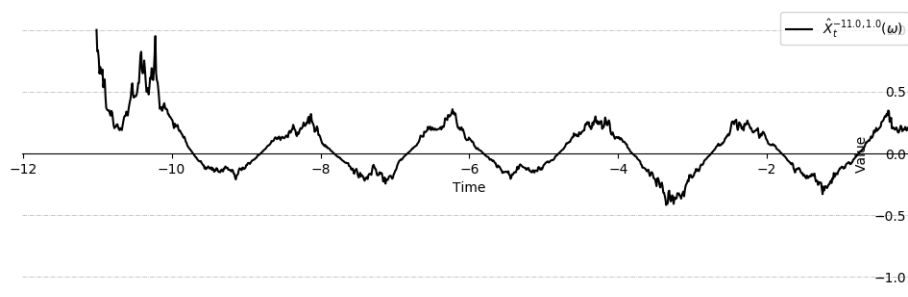


Figure 5.11: Simulations of the processes $\{\hat{X}_t^{-11}(\omega), -11 \leq t \leq 0\}$

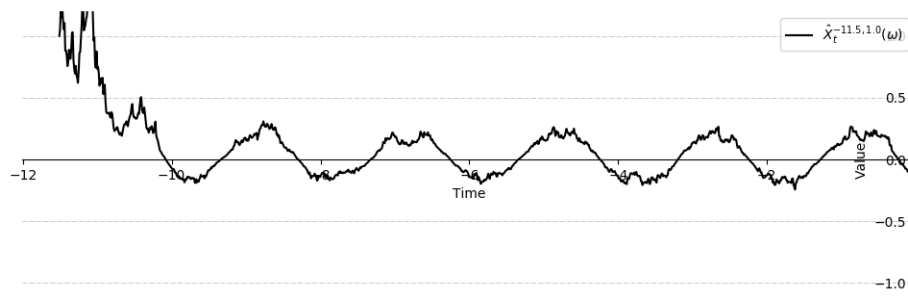


Figure 5.12: Simulations of the processes $\{\hat{X}_t^{-11.5}(\omega), -11.5 \leq t \leq 0\}$

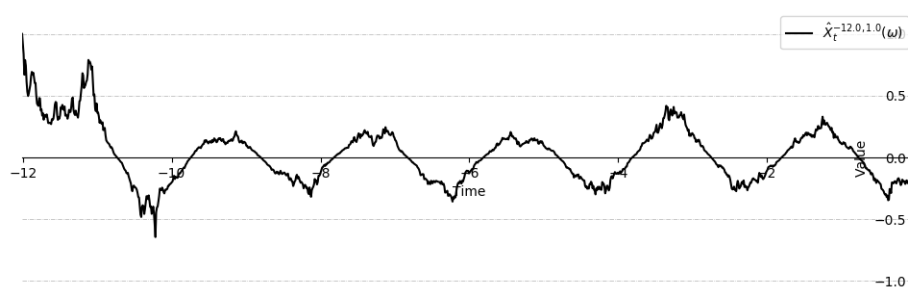


Figure 5.13: Simulations of the processes $\{\hat{X}_t^{-12}(\omega), -12 \leq t \leq 0\}$

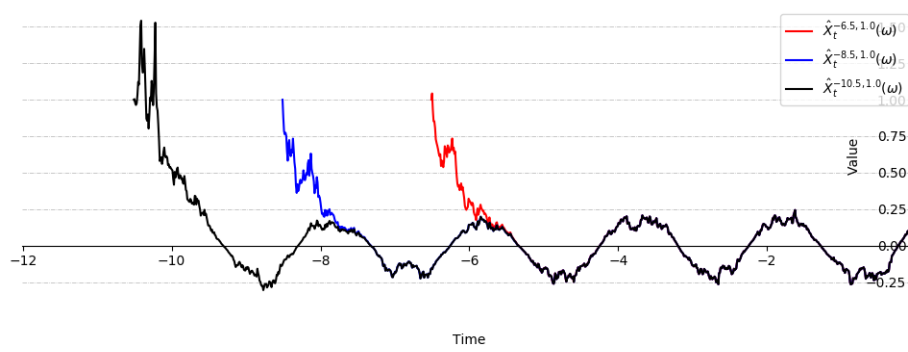


Figure 5.14: Simulations of the processes $\{\hat{X}_t^{-6.5}(\omega)\}$, $\{\hat{X}_t^{-8.5}(\omega)\}$, $\{\hat{X}_t^{-10.5}(\omega)\}$

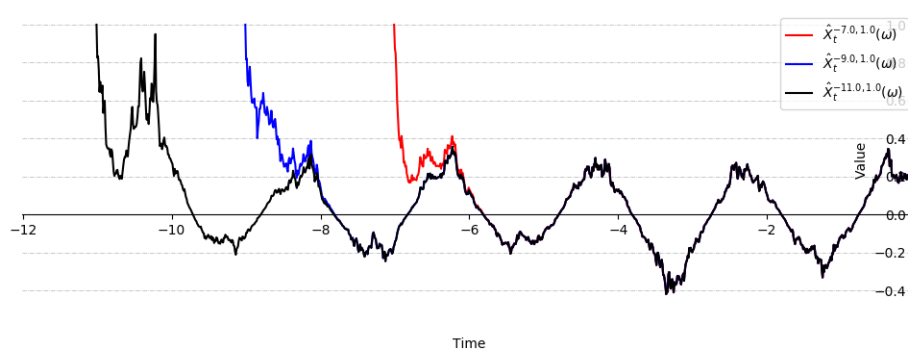


Figure 5.15: Simulations of the processes $\{\hat{X}_t^{-7}(\omega)\}$, $\{\hat{X}_t^{-9}(\omega)\}$, $\{\hat{X}_t^{-11}(\omega)\}$

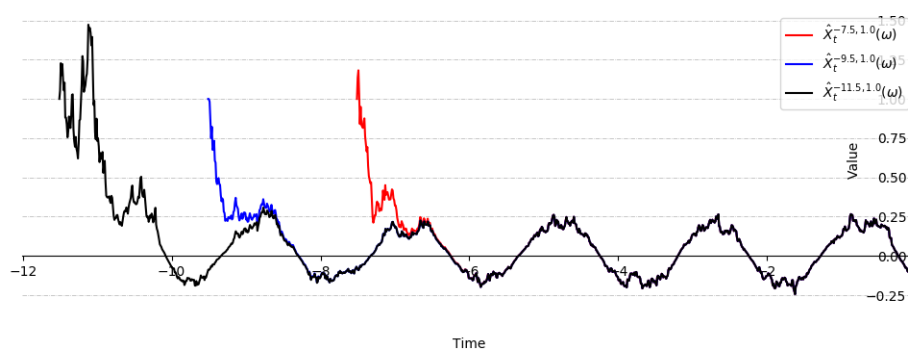


Figure 5.16: Simulations of the processes $\{\hat{X}_t^{-7.5}(\omega)\}$, $\{\hat{X}_t^{-9.5}(\omega)\}$, $\{\hat{X}_t^{-11.5}(\omega)\}$

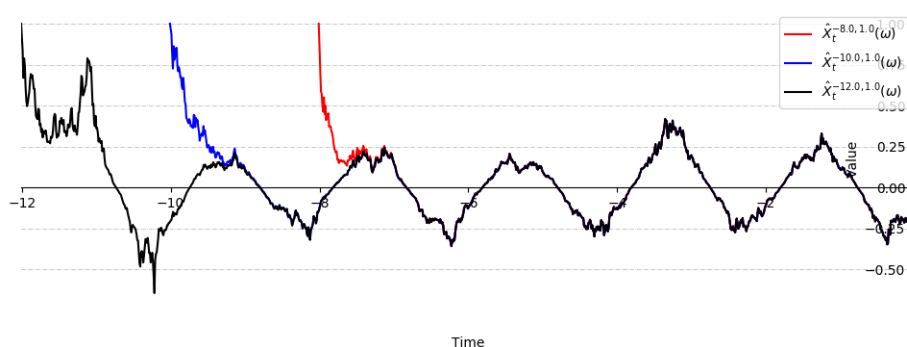


Figure 5.17: Simulations of the processes $\{\hat{X}_t^{-8}(\omega)\}$, $\{\hat{X}_t^{-10}(\omega)\}$, $\{\hat{X}_t^{-12}(\omega)\}$

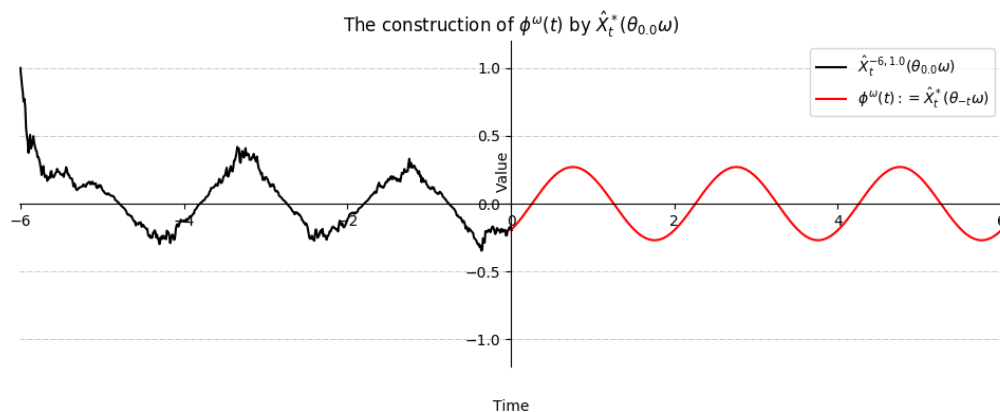


Figure 5.18: Simulations of the processes $\{\hat{X}_r^{-6}(\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

To show the random periodicity of the process, we defined the process $\{\hat{X}_t^*(\theta_{-t}\omega)\}$ in the above contents. Here we take the corresponding points of the process $\{\hat{X}_r^{-6}(\theta_t\omega)\}$ to form the $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$ as in Fig 5.18 to Fig 5.30

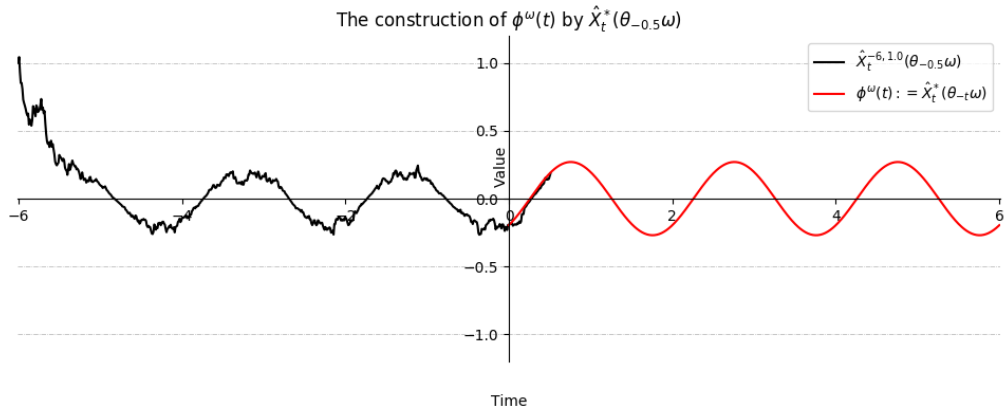


Figure 5.19: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-0.5}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

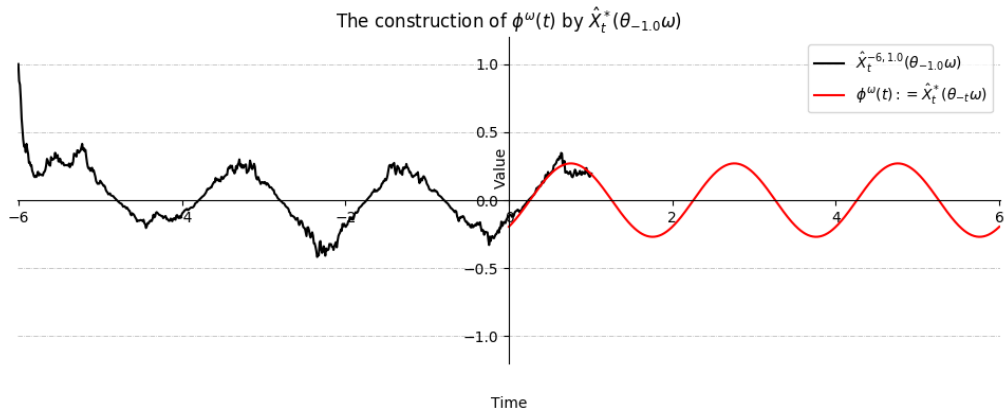


Figure 5.20: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-1.0}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

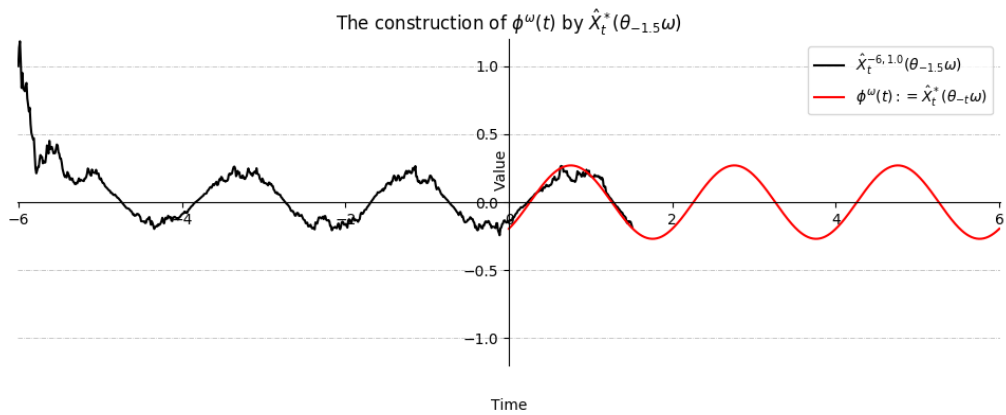


Figure 5.21: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-1.5}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

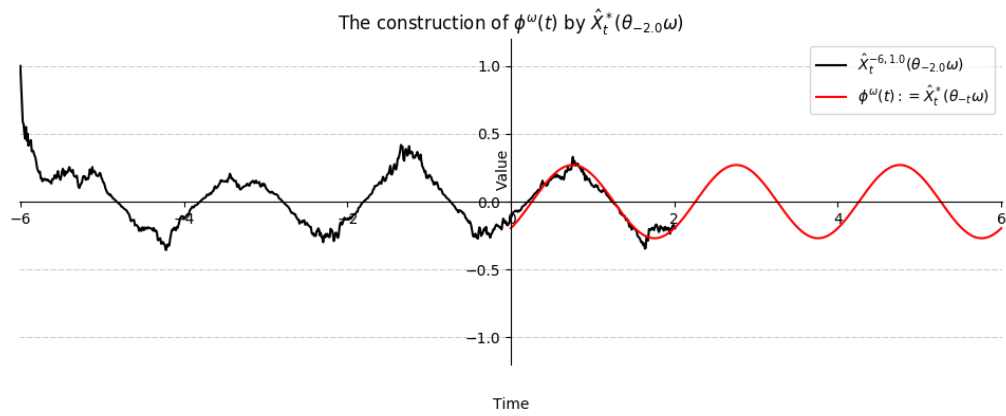


Figure 5.22: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-2}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

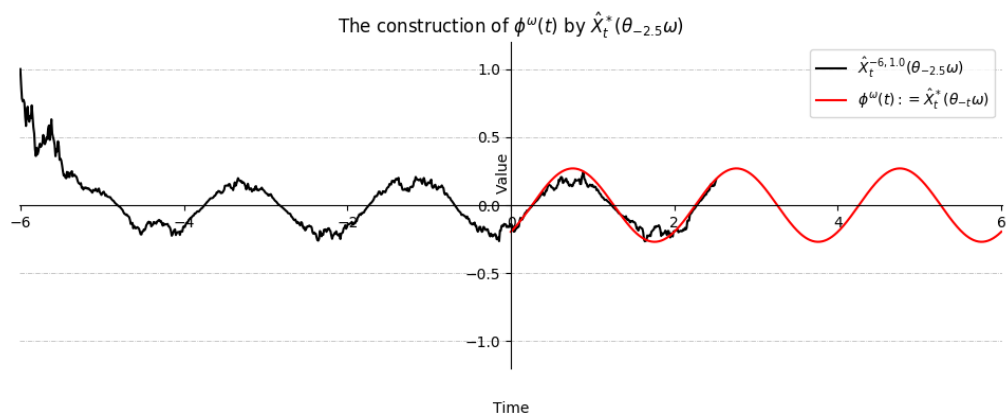


Figure 5.23: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-2.5}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

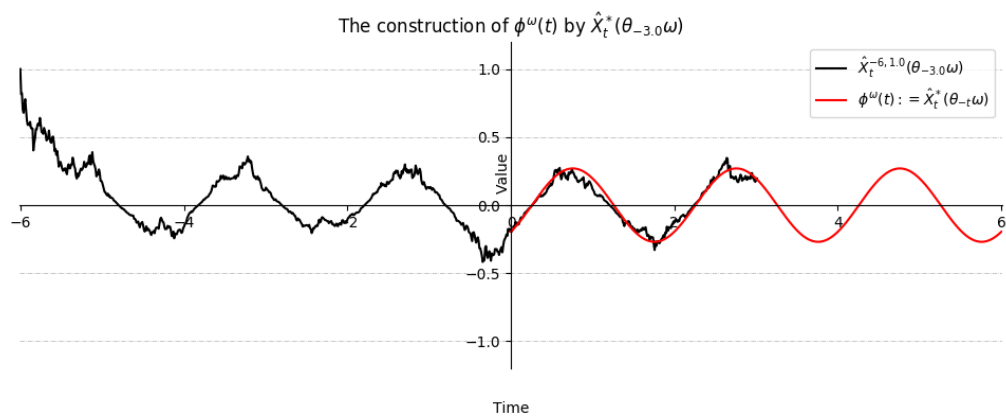


Figure 5.24: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-3}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

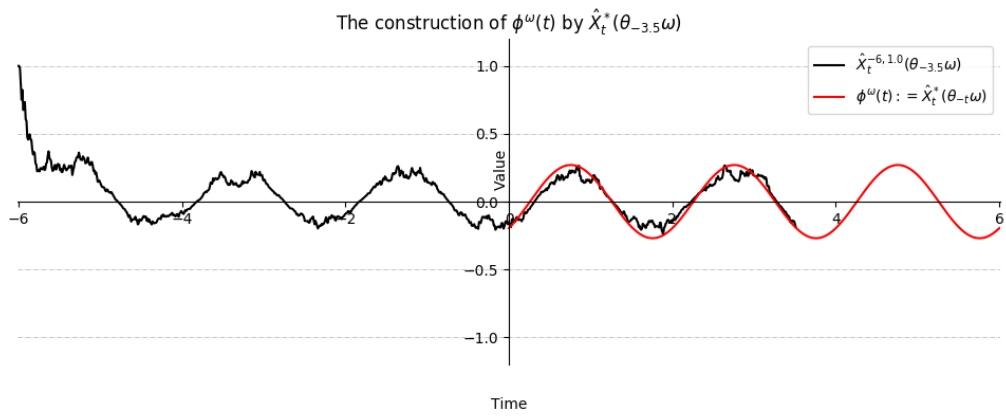


Figure 5.25: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-3.5}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

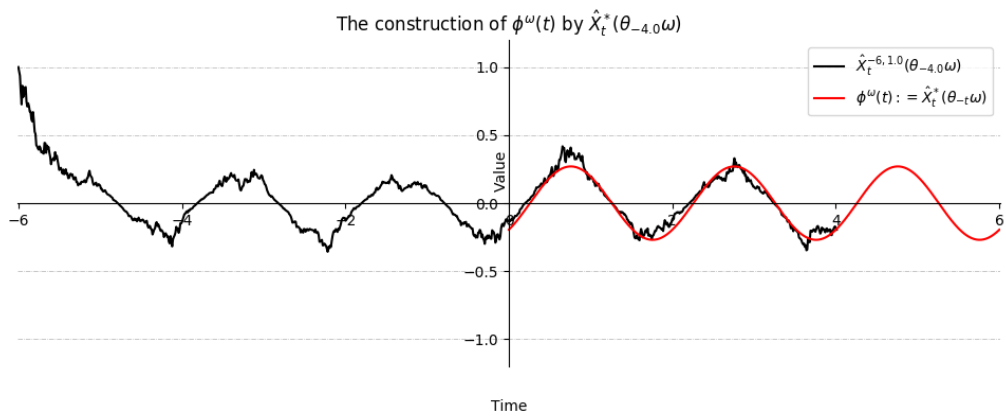


Figure 5.26: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-4.0}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

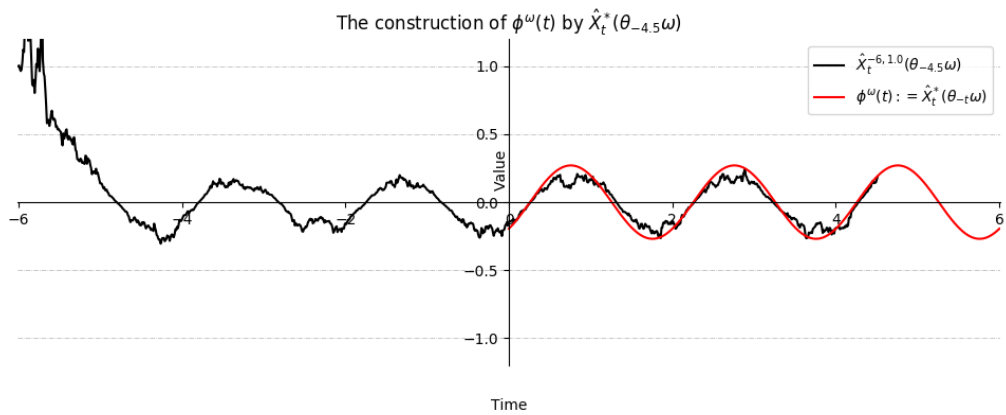


Figure 5.27: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-4.5}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

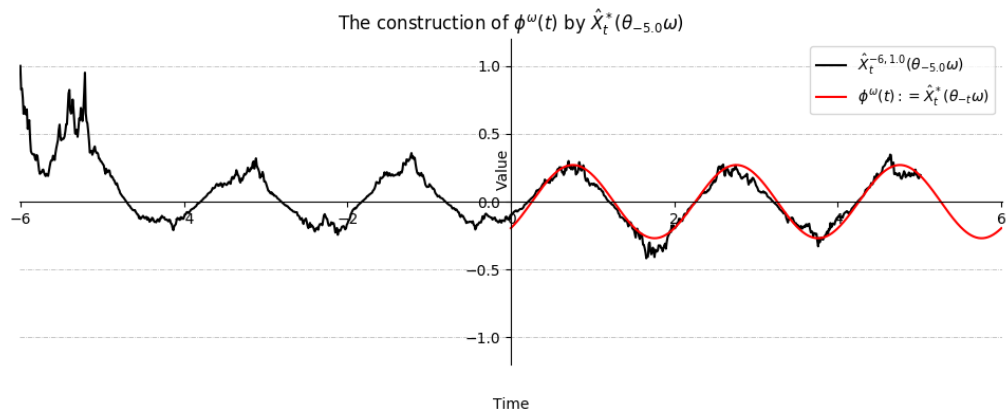


Figure 5.28: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-5.0}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

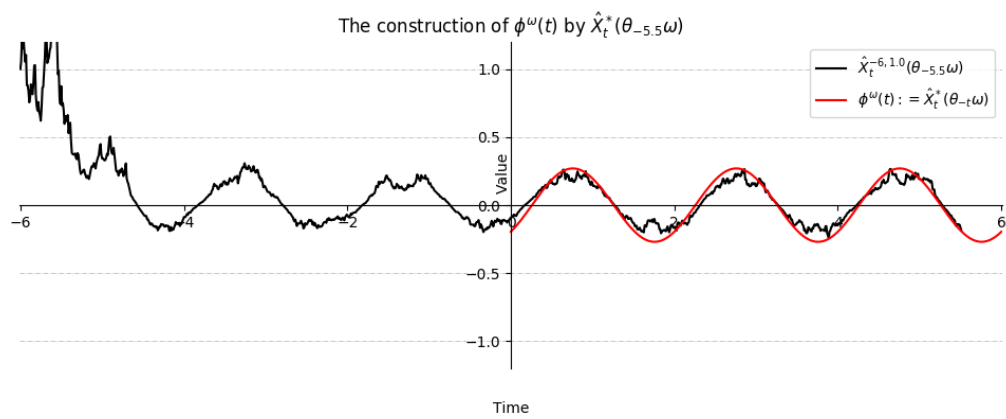


Figure 5.29: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-5.5}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

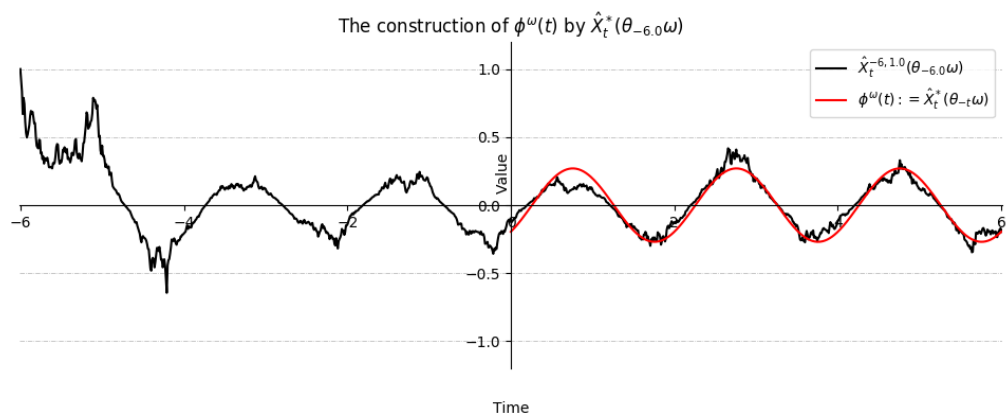


Figure 5.30: Simulations of the processes $\{\hat{X}_r^{-6}(\theta_{-6.0}\omega)\}$ and $\{\hat{X}_t^*(\theta_{-t}\omega), 0 \leq t \leq 6\}$

Bibliography

- [1] L. Arnold, *Random Dynamical System*, Springer-Verlag, Berlin, Heidelberg, New York, 1998.
- [2] J. Bao and C. Yuan, Numerical approximation of stationary distributions for stochastic partial differential equations, *J. Appl. Probab.* Vol 51, Number 3 (2014), 858-873.
- [3] P. W. Bates, K.N. Lu and B.X. Wang, Attractors of non-autonomous stochastic lattice systems in weighted spaces, *Physica D*, Vol. 289 (2014), 32-50.
- [4] P. Baxendale, Wiener processes on manifolds of maps, *Proc. Royal Soc. Edinburgh*, Vol. 87A (1980), 127-152.
- [5] J.-M. Bismut, A generalized formula of Ito and some other properties of stochastic flows, *Z. Wahrscheinlichkeitstheorie Verw. Gebiete*, Vol. 55 (1981), 331-350.
- [6] A. Borisyuk and F. Rassoul-Agha, Quasiperiodicity and phase locking in stochastic circle maps: A spectral approach. *Physica D: Nonlinear Phenomena*, Vol. 288 (2014), 30-44.
- [7] J. C. Butcher *Numerical methods for ordinary differential equations*. John Wiley and Sons, 2016.
- [8] T. Caraballo, P. E. Kloeden, B. Schmalfuss, Exponentially stable stationary solutions for stochastic evolution equations and their perturbation. *Appl. Math. Optim.*, Vol. 50 (2004), 183-207.
- [9] M. D. Chekroun, E. Simonnet and M. Ghil, Stochastic climate dynamics: random attractors and time-dependent invariant measures, *Physica D*, 240 (2011), 1685-1700.

-
- [10] A. M. Cherubini, J. S. W. Lamb, M. Rasmussen, and Y. Sato, A random dynamical systems perspective on stochastic resonance, 2015, *arXiv:1510.06973*.
- [11] K.D. Elworthy, Stochastic dynamical systems and their flows. In: *Stochastic Analysis*, ed. A. Friedman, M. Pinsky, London-New York: Academic Press 1978, pp. 79-95.
- [12] C. Feng, H. Zhao, and B. Zhou, Pathwise random periodic solutions of stochastic differential equations, *Journal of Differential Equations*, Vol. 251 (2011), 119-149.
- [13] C. Feng and H. Zhao, Random periodic solutions of SPDEs via integral equations and Wiener-Sobolev compact embedding, *Journal of Functional Analysis*, Vol. 262 (2012), 4377-4422.
- [14] C. Feng, Y. Wu and H. Zhao, Anticipating random periodic solutions–I. SDEs with multiplicative linear noise, 2015, *Journal of Functional Analysis*, Vol. 271 (2016), 365-417.
- [15] C. Feng, Y. Wu and H. Zhao, Anticipating random periodic solutions–II. SPDEs with multiplicative linear noise, preprint, 2015.
- [16] C. Feng, Y. Luo and H. Zhao, Random periodic solutions of semilinear nondissipative stochastic functional differential equations, in preparation, 2015.
- [17] C. Feng and H. Zhao, Random periodic processes, periodic measures and ergodicity, 2015, *arXiv:1408.1897*.
- [18] C. Feng, Y. Liu and H. Zhao, Numerical approximation of random periodic solutions of stochastic differential equations, *Z. Angew. Math. Phys.* (2017) 68: 119.
- [19] F. Flandoli, Stochastic flows for nonlinear second order parabolic SPDEs, *Ann. Probab.*, Vol. 24 (1996), 547-558.
- [20] M. J. Garrido-Atienza, K.N. Lu and B. Schmalfuss, Unstable invariant manifolds for stochastic PDEs driven by a fractional Brownian motion, *J. Differential Equations*, Vol. 248 (2010), 1637-1667.

-
- [21] B. Gess, Random attractors for stochastic porous media equations perturbed by space-time linear multiplicative noise, *C. R. Acad. Sci. Paris, Ser. I*, Vol. 350
- [22] P. Hartman, *Ordinary Differential Equations*, Society for Industrial and Applied Mathematics, Philadelphia, 2002.
- [23] D. Higham, X. Mao and A. Stuart, Exponential mean-square stability of numerical solutions to stochastic differential equations, *LMS J. Comput. Math.*, Vol. 6 (2003), 297-313.
- [24] J. M. Holte, Discrete Gronwall lemma and applications, *MAA-NCS meeting at the University of North Dakota*, 24 October 2009.
- [25] N. Ikeda, S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*. North Holland-Kodansha, Tokyo, 1981.
- [26] A. Jentzen and P. Kloeden, The Numerical Approximation of Stochastic Partial Differential Equations, *Milan J. Math.* Vol. 77 (2009) 205-244.
- [27] A. Jentzen and P. Kloeden, Taylor expansions of solutions of stochastic partial differential equations with additive noise, *Ann. Prob.*, Volume 38 (2010), 532-569.
- [28] W. E, K. Khanin, A. Mazel, Ya. Sinai, Invariant measures for Burgers equation with stochastic forcing, *Ann. of Math.*, Vol.151 (2000), 877-960.
- [29] P. E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, New York, 1991.
- [30] H. Kunita, Stochastic differential equations and stochastic flows of diffeomorphisms, *Ecole d'été de Probabilités de Saint-Flour XII-1982*, Lecture Notes in Mathematics, Springer, 1984.
- [31] H. Kunita, *Stochastic Flows and Stochastic Differential Equations*, Cambridge University Press, Cambridge, UK, 1990.

-
- [32] J. Mattingly, A.M. Stuart and D.J. Higham, Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise. *Stochastic Processes and their Applications* 101(2) (2002), 185-232.
- [33] P.-A. Meyer, Flot d'Une équation différentielle stochastique, *Séminaire de probabilités (Strasbourg)*, Vol. 15 (1981), 103-117.
- [34] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastic Stability*, Springer-Verlag, London, 1993.
- [35] G. N. Milstein, Approximation integration of stochastic differential equations. *Teor. Veroyatnost. i Primenen.* 19 (1974), 583-588
- [36] G. N. Milstein, *Numerical Integrations of Stochastic Differential Equations*, Kluwer, Dordrecht, 1995.
- [37] G. N. Milstein and M. V. Tretyakov, *Stochastic Numerics for Mathematical Physics*, Springer, 2004.
- [38] S.-E. A. Mohammed, T. Zhang, H.Z. Zhao, The stable manifold theorem for semilinear stochastic evolution equations and stochastic partial differential equations, *Mem. Amer. Math. Soc.*, Vol. 196 (2008), No.917, pp. 1-105.
- [39] E. Nummelin, *General irreducible Markov chains and non-negative operators*, Cambridge University Press, 1984.
- [40] H. Poincaré, Memoire sur les courbes definir par une equation differentiate, *J. Math. Pures Appli.*, Vol. 3 (1881), 375-442; Vol. 3 (1882), 251-296; Vol. 4 (1885), 167-244; Vol. 4 (1886), 151-217.
- [41] B. Schmalfuss, A fixed point theorem and the random graph transformation, *Journal of Mathematical Analysis and Applications*, Vol. 225 (1998), 91-113.
- [42] Ya. Sinai, Burgers system driven by a periodic stochastic flows. In: *Itô's stochastic calculus and probability theory*, Springer, Tokyo, 1996, pp. 347-353.
- [43] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis*, Vol. 12. Springer Science and Business Media, 2013.

-
- [44] A. Stuart and A. R. Humphries. *Dynamical systems and numerical analysis*. Vol. 2. Cambridge University Press, 1998.
- [45] D. Talay, Second order discretization schemes of stochastic differential systems for the computation of the invariant law, *Stochastics and Stochastic Reports*, Vol 29, 1990, 13-36.
- [46] D. Talay and L. Rubaro, Expansion of the global error for numerical schemes solving stochastic differential equations, *Stochastic Analysis and Applications*, Vol 8, 1990, 483-509.
- [47] A. Tocino, R. Ardanuy, Runge-Kutta methods for numerical solution of stochastic differential equations, *Journal of Computational and Applied Mathematics*, Vol. 138 (2002), 219-241.
- [48] R.L. Tweedie, Sufficient condition for ergodicity and recurrence of Markov chains on a general state space, *Stochastic Process and their Applications*, Vol. 3(4) (1975), 385-403.
- [49] R.L. Tweedie, The existence of moments for stationary Markov chains, *Journal of Applied Probability* Vol. 20, No. 1 (1983), pp. 191-196
- [50] N.G. Van Kampen, *Stochastic Processes in Physics and Chemistry*, Elsevier, 2007.
- [51] B.X. Wang, Existence, stability and bifurcation of random complete and periodic solutions of stochastic parabolic equations, *Nonlinear Analysis*, Vol. 103 (2014), 9-25.
- [52] A. Yevik and H. Zhao, Numerical approximations to the stationary solutions of stochastic differential equations, *SIAM Journal on Numerical Analysis*, Vol. 49 (2011), 1397-1416.
- [53] C. Yuan and X. Mao, Stability in distribution of numerical solutions for stochastic differential equations, *Stochastic Analysis and Applications*, Vol. 22, No. 5, pp. 1133-1150, 2004.
- [54] Q. Zhang, H. Zhao, Stationary solutions of SPDEs and infinite horizon BDS-DEs, *Journal of Functional Analysis*, Vol. 252 (2007), 171-219.

-
- [55] H. Zhao and Z. Zheng, Random periodic solutions of random dynamical systems, *Journal of Differential Equations*, Vol. 246 (2009), 2020-2038.