Students as Partners in Complex Number Task Design

Stephanie Treffert-Thomas, Barbara Jaworski, Dave Hewitt, Nikolaos Vlaseros and Marinos Anastasakis

Loughborough University, Loughborough, United Kingdom; s.thomas@lboro.ac.uk, b.jaworski@lboro.ac.uk, d.hewitt@lboro.ac.uk, n.vlaseros@lboro.ac.uk; m.anastasakis@lboro.ac.uk

We report on a collaborative project at university level involving students as partners in task design for a bridging mathematics module (known in the UK as a Foundation module) which is part of gaining access to first year degree studies. Three teacher-researchers met regularly with four student partners who developed a set of tasks on matrices and on complex numbers which were trialled with students on this Foundation module. We show the mediational processes by which the tasks developed from 'static' designs to more 'dynamic' designs using the software Autograph. Our analyses highlighted various tools in the mediation of the learning of mathematics, in the mediation of task development and in the mediation of the engagement of all team members in collaboration.

Keywords: Student-partner; computer-based task design; mediation and tool use; collaborative research; developmental research.

Background and literature

In 2011/12 findings from the ESUM project (Jaworski & Matthews, 2011; Jaworski, Robinson, Matthews & Croft, 2012) pointed to a problematic difference between teacher culture and expectations and those of students when researching students' use of mathematical tasks in a computer environment - when those tasks were designed for students by their teacher. In the current project we are trying to address this difference by engaging students in the design process and studying the emergent learning for the students and the researchers. The project was designed by three teacher-researchers (TRs, authors of this paper) to engage four students as partners in the design of mathematical tasks in a computer environment. Thus, the project is a collaboration between students (we refer to them as student-partners, SPs) and teacher-researchers (TRs) in the design of tasks. Several aspects were explored within this project: the development of mathematical tasks in a computer environment, SPs' perspectives on task design and participation in a research team, and the use of the tasks with current students studying on a Foundation module.

While there is research into the engagement of students as partners in course design at university level (see Mercer-Mapstone et al., 2017), few of these are within mathematics education (see however, Duah, Croft & Inglis, 2014; Fayowski & MacMillan, 2008). Our project builds on the work of Duah and Croft (2011) who involved student interns in the design of resources for two 2$^{\text{nd}}$ year mathematics modules experienced as difficult by students. Both the lecturer of the module and student interns learned considerably from this collaboration (Duah, 2017).

Our aim for initiating this project and recruiting SPs was to foster and study a deeper understanding of mathematics in a Foundation mathematics module, specifically the two topics of complex numbers and matrices. The Foundation Studies Programme is a one-year course intended for students who wish to study for a STEM subject at our university but do not satisfy the entry requirements for their chosen degree. All must take a Foundation mathematics module as part of their programme.

We focus on complex number tasks for this paper. The decision to use dynamic software in task design was taken by the TRs in order to connect algebraic and geometric representations of the arithmetic operations on complex numbers. We consider geometry and algebra as fundamental to all mathematics (Atiyah, 2001) and hence to its teaching and learning. While we are aware that providing geometric insights can potentially be problematic (it can both help and hinder, see Gueudet-Chartier, 2004), we align ourselves with those who value these as a means of connecting with students' more intuitive notions of a concept (Stewart & Thomas, 2009; Uhlig, 2003).

Theoretical and Methodological Perspectives

We take a socio-cultural perspective on teaching and learning. Central are the Vygotskian notions of mediation and tool use – the idea that attaining an object of activity is achieved better through the mediation of some tool, artefact, person or process (Vygotsky, 1978, Wertsch, 1991). All members of the project were learners: the SPs learned how to design mathematical tasks and work with the other members in a research environment; the TRs learned about student perspectives and the process of involving students in task development. Mediation was similarly diverse: all participants mediated the learning of others through the collaborative process. There were various tools: computer-based resources in the form of *Autograph*[1] files that were accessed by Foundation students in the tutorials; paper-based resources such as lecture notes and problem sheets, additional instructions on the use of Autograph, and questions in relation to the Autograph files. We return to mediation and tools later.

The research team consisted of nine people: three teacher-researchers (TR1, TR2 and TR3, the first three authors), two doctoral students (the last two authors who helped with data collection and analyses) and four SPs. All were engaged in developmental research within a community of inquiry, seeking to create knowledge, improve practice and the mathematical learning experience of students (see Jaworski, 2006). All participants took part in design meetings and were involved in an iterative design-research approach where participants *inquire* into the processes in which they engage. The SPs learned mathematics through inquiry into task design. They particularly learned task design through several iterations of the design process (inherently an inquiry process) in which drafts of their tasks were critiqued by other members of the team.

Our approach used ethnographic methods with data collected that could help answer our research question. Data collected included (1) audio-recordings of project design meetings, (2) written reports by SPs reflecting on their experience of task design, (3) interviews with SPs, (4) computer-based tasks (Autograph files), (5) various field notes. We took a grounded approach to data analyses. As part of this process data reductions were made of all design meetings. For a data reduction one of the researchers listened to the audio-recording while simultaneously making time-related factual notes

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[1] Autograph https://www.autograph-maths.com is a piece of software which dynamically links 'objects'. This allows the user to 'see' the result of moving/changing one object – other objects are linked and move/change with the first object.

on the content. This resulted in a factual summary in tabular form with interpretative comments written in a separate column alongside suggestions for areas of transcription. With these summaries we aimed to capture key points of the task design in order to chart their development, written reports were read and used to support the analysis in respect of the recordings; interview data were partially transcribed and key points summarised; computer-based tasks were recorded as screenshots; field notes were read and integrated into summaries.

In this paper, we focus on the SPs' design of tasks in complex numbers and specifically on their transition from thinking in terms of the nature of tasks with which they were familiar from their past mathematical experience, to developing more dynamic forms of the tasks. We ask:

What was the process by which SPs' generation of tasks developed?

The development of the complex number tasks

The tasks were developed for use with students taking a mathematics module in their Foundation Studies Programme. The SPs were recruited from former Foundation students via an interview process. The four students, chosen from the sixteen who applied, were those who had achieved a good grade in their Foundation mathematics module and who showed interest and initiative in thinking about their potential involvement. All had begun their first year of an Engineering or Science degree (Chemistry, Physics, Mechanical and Chemical Engineering).

The SPs' first task in the project was to review their notes from the Foundation module on two topics, matrices and complex numbers. To initiate the project, the whole team met for an introduction to the computer environment (Autograph) by an expert in its design and use. The SPs knew TR2, the lecturer of the Foundation mathematics module and were introduced to the other project members. Audio data from the meeting shows that SPs participated in a discussion on the use of Autograph and expressed views on experiences from their Foundation studies. These were of particular value and interest to the TRs, who gained insight into student perspectives on teaching and mathematical learning. Following this initial meeting, SPs engaged in designing tasks and bringing their current designs to successive 'design meetings' (DMs) where designs were discussed and modifications suggested.

The computer environment, Autograph, can be seen as a *tool* in two ways: first, it was to be a tool for the design of tasks as we discuss below. Second, and more immediate in the initial meeting, it became a catalyst for drawing the SPs into a discussion with the team in a non-threatening way. The team was relatively new to Autograph and so all could ask questions and talk about the use of the software. TRs engaged in a mediational process of drawing the SPs into the team via questions to the expert about how to use Autograph and the dynamic nature of its commands. The SPs were asked to use Autograph to produce first (and successive) drafts of tasks to be used with the new cohort of Foundation students. Gradually, roles and relationships were established, with the researchers taking the lead initially and the SPs showing a growing confidence in participation in the project.

Design of tasks and the Design Meetings – an iterative process

The first design meeting (DM1) was a significant staging post in the developmental process. It involved the team and the expert in a discussion of the tasks which the SPs had brought to the meeting. Each task had a mathematical element and an Autograph element. The SPs presented their initial ideas, one of which involved a multiple-choice style task on addition of complex numbers based on the type of questions that they were used to on problem sheets, such as 'If $z_1=2+5i$ and $z_2=3-6i$, what is z_1+z_2?' The Autograph file displayed the complex numbers as vectors including also the correct answer (one variation was to include some incorrect answers with Foundation students asked to choose the correct one). The initial task made use of the commands within Autograph such as 'hiding' and 'unhiding' lines and objects and 'dragging' a complex number z to see what happened to a linked object such as z_1+z_2, z_1z_2 or z^3.

In SPs' initial designs, the mathematical element came first and the Autograph element followed. The SPs' expectation was for Foundation students to work out solutions on paper first and then use the software to view complex numbers in an Argand diagram and verify their results. Thus, the Autograph element was a tool to illustrate the mathematical relationship but it was left up to Foundation students to make the link. In the meetings the whole team discussed the tasks. In DM2, one member (TR1) commented that a task seemed rather "static". Discussion followed as to what aspects contributed to the task being static, and what a task that was more dynamic and exploratory might look like. TR1 wrote in a reflection afterwards: "So far, we are seeing quite 'static' tasks – tasks in which there is something to find, with a right answer." In this sense, the initial tasks mirrored typical questions on problem sheets that the SPs had been used to. TR1 continued writing, "A challenge now is to use the power of the software to offer more interactive, open-ended, exploratory tasks".

The word "static" acted as a tool in the mediational process. It was used briefly in DM2 with TR1 commenting that tasks might follow a "sequence from straightforward 'How do you work out this…?', straightforward questions like that, … but work towards something that is more open-ended and more exploratory, so that they [the Foundation students] have to actually do something themselves, to explore a situation" (DM2, 40:08). Discussions in the subsequent design meetings ensued. As team members expressed their view on a 'dynamic' task, a sense emerged of the mathematics of the task becoming more integrated with its representation(s) in Autograph and Foundation students engaging with these representations and the computer environment. In an end-of-project interview when reflecting on the early design process, one of the SPs (SP1) recalled how the word "static" had been a catalyst for new ways of thinking about the tasks. He said that, as a Foundation student, he had been familiar with procedural tasks on the problem sheets. He and his partner (SP2) started from such a point of view to represent the tasks in Autograph with lecture notes and associated problem sheets acting as mediational tools, guiding their initial perceptions.

Working on the complex number tasks and communicating electronically, the two SPs had shared potential examples of dynamic tasks. As they exchanged ideas, they said, a clearer sense of possibilities emerged. These exchanges were themselves mediational; we interpret the oral and written words of the SPs as saying that different versions of the tasks acted as tools to promote new thinking and subsequent modifications. Discussion of static versus dynamic tasks in design meetings opened up new ideas and dimensions, again a mediational process. For example, the first iteration of the design process produced procedural tasks with expected answers while subsequent iterations shifted the nature of tasks to involve SPs in inquiry processes as they explored the questions asked. Team members expressed their ideas in different ways, encouraging a group perception of the nature of dynamical tasks (exemplified below). We see here a significantly new mode of mediation. The discussion of tasks allowed team members to question and share their own views with the SPs contributing alongside the other members of the team.

Finalising the tasks

The SPs developed six complex numbers tasks: adding and subtracting two complex numbers, multiplying together two complex numbers, multiplying a complex number and its complex conjugate and raising a complex number to a power. All final designs had their genesis in the ideas presented in DM1. SPs undertook major revisions of the tasks between the first and the second design meeting. Critiquing the tasks in the second design meeting, team members spent a lot of time discussing mathematical ideas, ways of viewing complex numbers (as a number, a point or a vector) and ways of solving. Thus in discussions involving all member of the team, the tasks were tools that mediated SPs' mathematical learning as part of the iterative design methodology and the mode of inquiry in our project. SPs offered Autograph files and mathematical explanations that gave insights into their development of mathematical sophistication – for example, working fluently with vector representations in constructing complex number addition geometrically – with the teacher-researchers learning from this activity about student expectations and culture. TRs saw the tasks as designed to mediate Foundation students' learning of mathematics in tutorials and, to this end, constructed written instructions to accompany the Autograph files for the Foundation students.

In subsequent design meetings the team focussed almost entirely on technical aspects for improving the Autograph files. All tasks presented after DM2 had a 'dynamic' element such as 'dragging' a complex number around the screen until 'it fitted' a desired location, or exploring the relationship between the complex numbers by varying one or more of the linked objects. We provide examples of two tasks in more detail below.

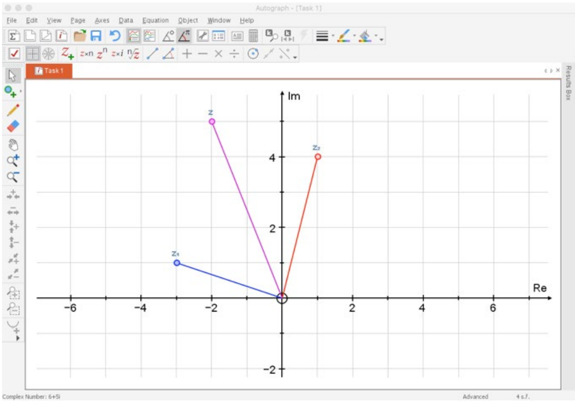

Figure 1: Task 1

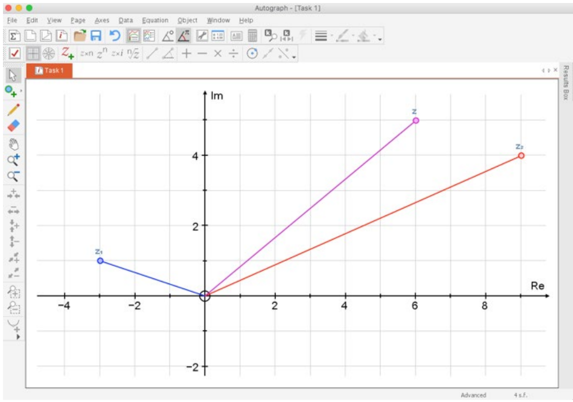

Figure 2: Task 1 with z_2 moved

The first task centred on the addition of complex numbers. SPs designed the initial addition task by following closely the type of question they had encountered on problem sheets: Given z_1 and z_2, find z_1+z_2 and (making explicit the Autograph element) use Autograph to check that your answer is correct. After receiving feedback in DM1 the SPs set about 'reverse-engineering' the questions which SP1 described as looking at the answer and working backwards in order to design the task with the visualisation (Autograph) providing the information. Hence, for addition of complex numbers, the final task was set out to display three complex numbers, z_1, z_2 and z where z was equal to z_1+z_2 (see Figure 1). The instruction for Foundation students was to keep z_1 fixed and to move z_2 until z reached the position $6+5i$ (see Figure 2). At this point several lines on the screen moved as the linked object z_1+z_2 moved dynamically with the movement of z_2. The task was to find z_2 and determine the (arithmetic) relationship between z_1 and z_2 (addition) and its geometric representation (parallelogram law). Foundation students followed instructions on a separate hand-out on how to proceed with the Autograph files and could 'check' final answers by using the 'Unhide All' command in Autograph.

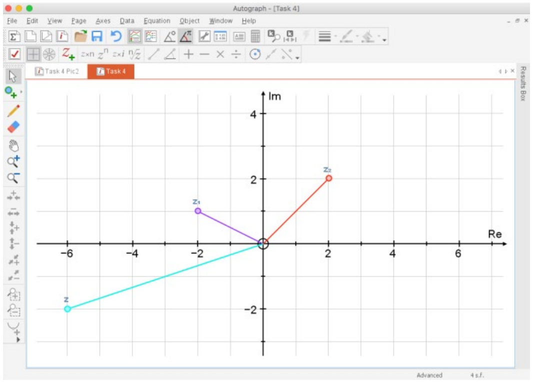

Figure 3: Task 4

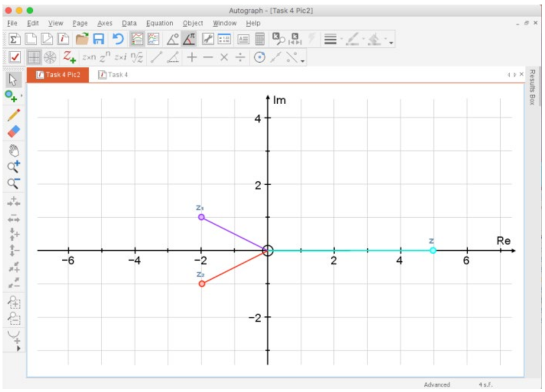

Figure 4: Task 4 with z_2 moved

The task in our second example centred on the multiplication of a complex number by its complex conjugate. Presenting their ideas in DM2 one of the SPs (SP2) said that "there are some [of our tasks] that are quite similar. [But] we purposely put some in to try and be tricky" (DM2, 54:07). He was referring to the complex conjugate task. The idea originating from the design meeting was to display a complex number and its complex conjugate together on one screen as well as their product. One of the researchers (TR2) commented how nice this task was exclaiming "Oooh, I like that!" (DM2, 55:10) because you could see the angles cancelling out – one angle taking a positive value while the other (while equal in size) was negative. Foundation students were asked to explore and comment on this task with TR1 suggesting to use the polar grid representation because "it would be quite nice to show the angles there" (DM2, 55:32). Hence, in the final design there were three complex numbers, z_1, z_2 and z where z was equal to z_1z_2 (see Figure 3). The task was to notice what happened to z when z_2 was moved into the position of the complex conjugate of z_1 (see Figure 4). In this task design the SPs made no reference to any numerical value for z_1 or for z_2, or their product. This task has a very different character and is more general, inviting exploration.

In the end-of-project interview, one of the SPs (SP1) suggested that "these tasks invited a deeper understanding than standard [problem] sheets" and that while many students may know that the multiplication of a complex number and its complex conjugate results in a real number, they, as partners in the project – knew 'why' that was the case.

All six tasks were trialled with Foundation students over a two-year period with two different cohorts of students. Tasks were integral to the syllabus and formed part of the Foundation students' undergraduate learning experience. Analyses of these data are ongoing.

Summary of the project collaboration

The project involved students as partners in the design of tasks in complex numbers (as well as matrices - not reported here). Few examples in the literature exist but those that do point to the benefit of such a collaboration for both staff and students (see Bovill, Cook-Sather & Felten, 2011). Duah &

Croft, (2011) found as we did that the SPs gained a deeper understanding of the mathematics studied, in our case the mathematics of complex numbers. For example, SP1 recalled that "When I came across complex numbers in my third year [of degree study]…, it was immediately clear to me why the solutions appeared as complex conjugate pairs while many students had to spend time revising the principal." In our analyses we see multi-layered mediational actions. One layer relates to the mathematical learning of the SPs mediated by tools. These are the tasks themselves, the software used to create the tasks, the SPs' discussions when working together as a pair, and the inquiry mode in design meetings when all members of the team discussed drafts of the tasks.

A second layer relates to the work of the research team as a community of inquiry where we inquired into the design of computer-based tasks and into the learning processes of SPs and other team members. The SPs were drawn into the inquiry-based activity of designing dynamic tasks through the mediational nature of becoming familiar with Autograph and discussing its use together with other members of the team. Autograph acted as a tool for stimulating thinking and sharing of mathematical ideas and possibilities; it also acted as a tool for integrating SPs as partners in the task design process within the team.

We observed how the response of a team member using the word "static" was mediational in exerting a shift in SPs' perception of mathematics tasks: from static forms towards consideration of more dynamic forms of tasks using Autograph. Thus 'static' itself acted as a tool to promote a new perception of the nature of tasks. The word "static" was not planned. It emerged through the interactivity, and resulted in a stimulus for the SPs towards more dynamic forms of tasks.

At a third level, our engagement in research led to additional data being collected outside the design meetings. Interviews with SPs, where they were encouraged to reflect on their activity and learning at different stages in the project were revealing for the researchers. The interview process itself was mediational in allowing the researchers to perceive the stages in development of the SPs' thinking and perception - from their experiences with questions on problem sheets towards their design of dynamic tasks to engage Foundation students in mathematical inquiry.

In summary, we have characterised learning as a mediational process involving a variety of tools. Our analyses contributed to our understanding of the task design process where we see a new way of mediation that drew SPs into the collaboration as partners. Our project and analyses are contributing new knowledge in the field of students as partners in curriculum and course design and to the scarce body of literature in this area.

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