

A note about integrable systems on low dimensional Lie groups and Lie algebras

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1 Introduction

This paper was motivated by the following natural question: are there any natural analogs of integrable cases in rigid body dynamics (e.g. Euler, Lagrange and Kovalevskaya tops) if one replaces the rotation group $SO(3)$ with another 3-dimensional group?

We do not discuss this question here (see [10] for recent progress in this direction) and the present paper can be considered as a preparatory work for further study of integrable cases on low-dimensional Lie groups. Our first goal is to explain why any left-invariant Hamiltonian system on (the cotangent bundle of) a 3-dimensional Lie group G (for instance, geodesic flows of Riemannian and sub-Riemannian metrics) is Liouville integrable. We show that integrability of such systems easily follows from the fact that the coadjoint orbits of G are two-dimensional (Theorem 1) so that the dimension of G and other properties, e.g., unimodularity (cf. [20]) are less important. Notice that if coadjoint orbits of G have

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dimension ≥ 4 , it may well happen that G admits no integrable left-invariant geodesic flows at all [5, 16, 17, 18].

Next we give normal forms of left-invariant Riemannian and sub-Riemannian metrics on 3-dimensional Lie groups. Recall that in general a left-invariant metric on a Lie group of dimension 3 is defined by six constants $g_{ij} = g_{ji}$, $i, j = 1, 2, 3$, but some of them “can be killed” by means of the automorphism group. For instance, in the case of $SO(3)$ every left-invariant metric (equivalently, the inertia tensor) can be reduced to a diagonal form so that there are only three essential parameters known as principal moments of inertia. We will provide a similar reduction for all simply connected Lie groups of dimension 3 (Theorem 3). We will focus, however, on the case of solvable groups as the cases of $SO(3)$ and $SL(2)$ have been already extensively studied: the rotation group $SO(3)$ leading to classical Euler and Euler-Poisson equations is a fundamental object in Geometry, Mechanics and Mathematical Physics [1, 9, 28], for $SL(2)$ we refer to recent papers [3, 19, 21, 22]. Our description is explicit and will be given in global coordinates on G (Theorem 5).

We do not want to say that these results are essentially new. Geodesic flows of Lie groups is a very popular subject (in dimension 3, see [2, 4, 6, 11, 12, 19, 20]) and the integrability mechanisms are now well understood (see e.g., [7, 8, 23]). We especially would like to refer to the paper [2] by Barrett *et al.*, devoted to classification of left-invariant sub-Riemannian metrics on 3-dimensional Lie groups. The authors obtain classification in different terms which makes the comparison of their results with ours a non-obvious task. We definitely prefer our approach as we present the answer in a short explicit form (Theorems 3 and 5) that seems to be quite suitable for further studies of integrable systems on these Lie groups as illustrated in Section 5.

To avoid possible misunderstanding and confusion with the notation we are using below, we would like to emphasise that throughout the paper we identify the following objects related to Lie groups and Lie algebras:

- left-invariant vector fields on a Lie group G ;
- left-invariant functions on the cotangent bundle T^*G ;
- elements of the Lie algebra \mathfrak{g} of the Lie group G ;
- linear functions on the dual space \mathfrak{g}^* .

Notice that each of the corresponding vector spaces carries a natural structure of a Lie algebra (w.r.t. the Lie bracket of vector field, the canonical Poisson bracket on T^*G , the commutator on \mathfrak{g} and the Lie-Poisson bracket on \mathfrak{g}^* respectively). These Lie algebras are *canonically* isomorphic. For example, by f_0, X_1, \dots, X_{n-1} we denote a basis of left-invariant vector fields on G , but we equally may think of them as a basis of the Lie algebra \mathfrak{g} , or Cartesian coordinates on \mathfrak{g}^* , or linear functions on T^*G .

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2 Integrability of left-invariant systems on Lie groups with 2-dimensional coadjoint orbits

The purpose of this section is the following

Theorem 1. *Let G be a connected Lie group such that its (generic) coadjoint orbits are two-dimensional. Then any left-invariant Hamiltonian system on T^*G is Liouville integrable, i.e., possesses $n = \dim G$ independent commuting integrals $F_0 = H, F_1, \dots, F_{n-1}$. Moreover, F_1, \dots, F_{n-1} are polynomial in momenta.*

*In particular, if G is a three-dimensional Lie group, then every left-invariant Hamiltonian system on T^*M is Liouville integrable.*

Proof. The proof is based on the description of Lie groups G satisfying the above condition obtained by A.Konyaev [15] and the classical Noether theorem that states the following:

Theorem 2 (Noether theorem). *Consider a Hamiltonian system on T^*M with a Hamiltonian H and let ξ be a vector field on M that preserves the Hamiltonian¹. Then ξ , as a linear function on T^*M , is a first integral of this Hamiltonian system, called Noether integral, i.e., $\{\xi, H\} = 0$.*

Let H be an arbitrary left-invariant function on T^*G and

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}. \quad (1)$$

be the corresponding Hamiltonian system on T^*G endowed with the canonical Poisson structure.

Recall the following simple fact: if Y is a right-invariant vector field on G , then its flow $\Phi_Y^t : G \rightarrow G$ is given by

$$\Phi_Y^t(x) = \exp(tY_e) \cdot x = L_{\exp(tY_e)}(x), \quad x \in G,$$

where $Y_e = Y(e) \in T_eG$ is the value of Y at the identity element $e \in G$. In other words, the flow of a *right*-invariant vector field is given by *left* translations. In the context of the Noether theorem this means that *left*-invariant objects are preserved by *right*-invariant vector fields.

Let Y_1, \dots, Y_n denote a basis of right-invariant vector fields on G . In view of the above remark, Y_i preserves H and, according to the Noether theorem, Y_1, \dots, Y_n span an algebra $\mathcal{F}_{\text{right}}$ of (right-invariant) first integrals for the Hamiltonian system (1) (recall that we consider Y_i as a function on T^*G linear in momenta, i.e., we set $Y_i(q, p) = \langle p, Y_i(q) \rangle$). This property is, basically, equivalent to the fact that every left-invariant vector field commute with every right-invariant vector field on a Lie group G and, similarly, every left-invariant function on T^*G Poisson commute with every right-invariant function.

¹This condition can be understood in the following way. Let $\Phi_\xi^t : M \rightarrow M$ be the (local) flow of ξ on M . This map can be naturally lifted to the cotangent bundle $\widehat{\Phi}_\xi^t : T^*M \rightarrow T^*M$. Then “ ξ preserves H ” means $H(\widehat{\Phi}_\xi^t(q, p)) = H(q, p)$ for all $t \in \mathbb{R}$ and $(q, p) \in T^*M$. In other words, H is preserved by the flow of ξ naturally extended to T^*M .

This Lie algebra $\mathcal{F}_{\text{right}}$ is obviously isomorphic to the Lie algebra \mathfrak{g} of the Lie group G . In addition to that, the Hamiltonian H itself is a first integral. To construct n commuting integrals, we have to distinguish two different cases:

- (i) H is not right-invariant;
- (ii) H is right-invariant (that is, H is in fact bi-invariant since by our assumption H is left-invariant).

In the first case, H is functionally independent of Y_1, \dots, Y_n and therefore can be added to the algebra of integrals so that we obtain a larger algebra of first integrals $\mathcal{F} = \mathcal{F}_{\text{right}} \oplus \langle H \rangle$.

In the second case, H is a certain function of Y_1, \dots, Y_n and therefore cannot be considered as an essentially new first integral. However, if we take any left-invariant vector field X which is not right invariant (such a vector field obviously exists as G is not commutative), this vector field gives another Noether integral for our Hamiltonian system and the algebra of first integrals can be taken in the form $\mathcal{F} = \mathcal{F}_{\text{right}} \oplus \langle X \rangle$.

Thus, in the both cases we obtain a non-commutative finite-dimensional algebra of first integrals \mathcal{F} that contains $n + 1$ independent integrals. Moreover, from the algebraic viewpoint \mathcal{F} is isomorphic to the direct sum of \mathfrak{g} and a one-dimensional commutative Lie algebra (generated by either H or X), i.e. $\mathcal{F} \simeq \mathfrak{g} \oplus \mathbb{R}$.

Our first remark is that \mathcal{F} provides the integrability of (1) in the non-commutative sense (see, Mischenko, Fomenko [24]) or, in a slightly different terminology, (1) is superintegrable. Indeed, if the generators of the (finite-dimensional) algebra \mathcal{F} of first integrals are functionally independent, then the non-commutative integrability condition takes the form

$$\dim G = n = \frac{1}{2}(\dim \mathcal{F} + \text{ind } \mathcal{F})$$

and in our case we have $\dim \mathcal{F} = \dim \mathfrak{g} + \dim \mathbb{R} = n + 1$ and $\text{ind } \mathcal{F} = \text{ind } \mathfrak{g} + \text{ind } \mathbb{R} = (n - 2) + 1 = n - 1$. (Recall that the index of \mathfrak{g} is the codimension of a generic coadjoint orbit which equals two in our case so that $\text{ind } \mathfrak{g} = n - 2$). In particular, this implies that the invariant isotropic integral submanifolds² have dimension $n - 1$ ³.

Now to complete the proof it remains to construct $n - 1$ independent commuting polynomials F_1, \dots, F_{n-1} in Y_1, Y_2, \dots, Y_n . In the theory of integrable system on finite-dimensional Lie algebras, such a collection of polynomials is known as a complete commutative set of polynomials on the dual space \mathfrak{g}^* . The number of independent polynomials in this set must, by definition, be equal to $\frac{1}{2}(\dim \mathfrak{g} + \text{ind } \mathfrak{g})$. Mischenko and Fomenko conjectured in [25] that such a collection exists for any finite-dimensional Lie algebra \mathfrak{g} , and this conjecture was proved by Sadetov [27] in 2004. This remark basically completes the proof, as we obtain n commuting integrals of the form F_1, \dots, F_{n-1} and either H (in case (i)) or X (in case (ii)). All of these integrals (except perhaps for H) are polynomial in momenta by construction.

²We cannot say ‘‘invariant tori’’ as they are not necessarily compact.

³For example, invariant integral surfaces for left-invariant metrics on $SO(3)$ are not three- but two-dimensional.

We want, however, to describe commuting polynomials F_1, \dots, F_{n-1} explicitly without referring to a rather non-trivial construction from [27]. To that end, we use the classification of Lie algebras with two-dimensional generic coadjoint orbits obtained by Konyaev [15].

According to his classification there is an infinite series of such algebras that are semidirect sums of a one-dimensional Lie algebra and a commutative ideal of an arbitrary dimension (see a more detailed description below) and six “exceptional” Lie algebras listed below:

Case 1: 3-dimensional Lie algebra $so(3)$ with relations

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, \quad [\mathbf{e}_1, \mathbf{e}_3] = -\mathbf{e}_2, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1.$$

Case 2: 3-dimensional Lie algebra $sl(2)$ with relations

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_1, \quad [\mathbf{e}_1, \mathbf{e}_3] = -2\mathbf{e}_2, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_3.$$

Case 3: 4-dimensional Lie algebra $A_{4,8}$ with relations⁴

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_4] = \mathbf{e}_2, \quad [\mathbf{e}_3, \mathbf{e}_4] = -\mathbf{e}_3.$$

Case 4: 4-dimensional Lie algebra $A_{4,10}$ with relations

$$[\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_1, \quad [\mathbf{e}_2, \mathbf{e}_4] = -\mathbf{e}_3, \quad [\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_2.$$

Case 5: 5-dimensional Lie algebra $A_{5,3}$ with relations

$$[\mathbf{e}_3, \mathbf{e}_4] = \mathbf{e}_5, \quad [\mathbf{e}_3, \mathbf{e}_5] = \mathbf{e}_1, \quad [\mathbf{e}_4, \mathbf{e}_5] = \mathbf{e}_3.$$

Case 6: 6-dimensional Lie algebra $A_{6,3}$ with relations

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_6, \quad [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_4, \quad [\mathbf{e}_2, \mathbf{e}_3] = \mathbf{e}_5.$$

First we consider the infinite series mentioned above. Each n -dimensional Lie algebra from this series admits the following n -dimensional matrix representation:

$$\begin{pmatrix} x_0 A & \bar{x} \\ 0 \dots 0 & 0 \end{pmatrix}, \quad \text{where } \bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}, \quad x_0, x_1, \dots, x_{n-1} \in \mathbb{R}$$

and A is a certain $(n-1) \times (n-1)$ matrix which determines the type of \mathfrak{g} . Two Lie algebras of this kind with different matrices A_1 and A_2 are isomorphic if A_1 and A_2 satisfy the relation $A_2 = \lambda P A_1 P^{-1}$ for some nonzero $\lambda \in \mathbb{R}$ and invertible matrix P . As already noticed, this Lie algebra is a semidirect product of the one-dimensional Lie

⁴The notation $A_{n,k}$ is taken from [26]. Here n denotes the dimension of \mathfrak{g} and k the number of this Lie algebra in the list of n -dimensional Lie algebras given in [26], $n \leq 6$.

(sub)algebra generated by $\begin{pmatrix} x_0 A & \bar{0} \\ 0 \dots 0 & 0 \end{pmatrix}$ and the $(n - 1)$ -dimensional commutative ideal

$$\mathfrak{h} = \left\{ \begin{pmatrix} 0_{n-1} & \bar{x} \\ 0 \dots 0 & 0 \end{pmatrix} \right\}.$$

A complete set consisting of $n - 1$ independent commuting polynomials for this Lie algebra is easy to construct: one can just take the linear functions corresponding to the natural basis $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ of this commutative ideal. In other words, the first integrals on T^*G are linear functions of the form $F_i(q, p) = \langle p, Y_i(q) \rangle$, where $Y_i(q)$ is the right-invariant vector field on G generated by one the basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_{n-1}$ of the commutative ideal $\mathfrak{h} \subset \mathfrak{g} \simeq T_e G = \text{Lie}(G)$. This shows, in particular, that all additional integral in this case are in fact linear. In particular, this is the case for all solvable Lie algebras of dimension 3.

To complete the proof, we need to construct a complete set of commuting polynomials for the remaining six “exceptional” Lie algebras from Konyaev’s list. Recall that this set should contain $\dim \mathfrak{g} - 1$ polynomials in $\mathbf{e}_1, \dots, \mathbf{e}_n$ (the basis of \mathfrak{g} which was used above to define commutation relations). This can be done in several different ways (and of course was done by many authors). Below we present one of possible answers for each of these Lie algebras individually:

Case 1, $so(3)$: $F_1 = \mathbf{e}_1^2 + \mathbf{e}_2^2 + \mathbf{e}_3^2$, $F_2 = \mathbf{e}_1$.
(F_1 is Casimir).

Case 2, $sl(2)$: $F_1 = \mathbf{e}_2^2 + \mathbf{e}_3 \mathbf{e}_1$, $F_2 = \mathbf{e}_1$.
(F_1 is Casimir).

Case 3, $A_{4,8}$: $F_1 = \mathbf{e}_2 \mathbf{e}_3 - \mathbf{e}_1 \mathbf{e}_4$, $F_2 = \mathbf{e}_1$, $F_3 = \mathbf{e}_2$
(F_1, F_2 are Casimirs).

Case 4, $A_{4,10}$: $F_1 = 2\mathbf{e}_1 \mathbf{e}_4 + \mathbf{e}_2^2 + \mathbf{e}_3^2$, $F_2 = \mathbf{e}_1$, $F_3 = \mathbf{e}_2$
(F_1, F_2 are Casimirs).

Case 5, $A_{5,3}$: $F_1 = \mathbf{e}_3^2 + 2\mathbf{e}_2 \mathbf{e}_5 - 2\mathbf{e}_1 \mathbf{e}_4$, $F_2 = \mathbf{e}_1$, $F_3 = \mathbf{e}_2$, $F_4 = \mathbf{e}_3$
(F_1, F_2, F_3 are Casimirs).

Case 6, $A_{6,3}$: $F_1 = \mathbf{e}_1 \mathbf{e}_5 + \mathbf{e}_3 \mathbf{e}_6 - \mathbf{e}_2 \mathbf{e}_4$, $F_2 = \mathbf{e}_4$, $F_3 = \mathbf{e}_5$, $F_4 = \mathbf{e}_6$, $F_5 = \mathbf{e}_3$
(F_1, F_2, F_3, F_4 are Casimirs).

To get commuting functions F_1, \dots, F_{n-1} on T^*G we just need to replace \mathbf{e}_i in the above formulas by the corresponding linear function $Y_i(q, p) = \langle p, Y_i(q) \rangle$ where Y_i denotes the right-invariant vector field on G corresponding to the basis vector $\mathbf{e}_i \in \mathfrak{g}$. □

3 Classification of left-invariant Riemannian and sub-Riemannian metrics on three dimensional Lie groups

Since we are working with both Riemannian and sub-Riemannian left-invariant metrics, it is more convenient to describe them in terms of the corresponding Hamiltonians on the cotangent bundle T^*G (which can also be understood as quadratic forms on T^*G). Recall that for every left-invariant metric (Riemannian or sub-Riemannian) on a Lie group G ,

the corresponding Hamiltonian (quadratic form on T^*G) can be written as

$$\frac{1}{2} \sum_{\alpha, \beta=1}^n g^{\alpha\beta} X_\alpha X_\beta,$$

where X_1, \dots, X_n ($n = \dim G$) is a basis of left-invariant vector fields on G and $g^{\alpha\beta}$ are constants satisfying symmetry and (semi)-positive definiteness conditions.

Equivalently, X_1, \dots, X_n can be understood as the Cartesian coordinates on \mathfrak{g}^* dual to a certain basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ of the Lie algebra \mathfrak{g} , and in fact by using the canonical identification of \mathfrak{g} with $(\mathfrak{g}^*)^*$ we may assume that $\mathbf{e}_i = X_i$. In this view, a left-invariant metric on G is defined by a quadratic form (with constant coefficients) on the dual space \mathfrak{g}^* . Our goal is to classify left-invariant metrics on G (equivalently, positive (semi)-definite quadratic forms on \mathfrak{g}^*) up to the following natural equivalence relation.

Definition 1. Two quadratic forms (Hamiltonians)

$$H = \frac{1}{2} \sum g^{\alpha\beta} X_\alpha X_\beta \quad \text{and} \quad \tilde{H} = \frac{1}{2} \sum \tilde{g}^{\alpha\beta} X_\alpha X_\beta$$

are said to be *equivalent*, if there is an automorphism $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ of the Lie algebra \mathfrak{g} such that

$$\frac{1}{2} \sum \tilde{g}^{\alpha\beta} X_\alpha X_\beta = \frac{1}{2} \sum g^{\alpha\beta} \phi(X_\alpha) \phi(X_\beta).$$

This equivalence relation guarantees that the corresponding automorphism $\Phi : G \rightarrow G$ is an isometry between the (sub-)Riemannian metrics corresponding to H and \tilde{H} . In particular, the corresponding (canonical map) map $\Phi^* : T^*G \rightarrow T^*G$ transforms H to \tilde{H} so that the Hamiltonians H and \tilde{H} as well as the corresponding Hamiltonian systems are equivalent in the strongest possible sense.

Thus, for each 3-dimensional Lie group we just need to reduce a given $H = \frac{1}{2} \sum g^{\alpha\beta} X_\alpha X_\beta$ to a certain canonical form by means of transformations from the automorphism group $\text{Aut}(\mathfrak{g})$ which is well known for each 3-dimensional Lie algebra \mathfrak{g} . This can be done by elementary algebraic manipulations similar to those used in undergraduate Linear Algebra courses like “completing the square”. We will demonstrate this procedure in detail for the algebra \mathfrak{g}_{IV} . Transferring it to all the other cases is just an easy exercise, but first we need to agree about the notation in order for the final result (Theorem 3) to make sense.

Below is the list of Lie algebras in dimension 3 (known as Bianchi classification) with a fixed basis X_0, X_1, X_2 and the automorphism group $\text{Aut}(\mathfrak{g})$ explicitly written in terms of this basis ⁵ (cf. [13, 14]).

For each Lie algebra we indicate non-trivial commutation relations between basis elements X_0, X_1, X_2 and then give an explicit matrix form \mathbf{A}_ϕ for the transformations $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ from the automorphism group $\text{Aut}(\mathfrak{g})$. The parameters $a, b, \alpha, \beta, \gamma, \delta$ in matrices \mathbf{A}_ϕ below take arbitrary real values satisfying the only restriction that $\det \mathbf{A}_\phi \neq 0$. In the cases of solvable Lie algebras I – VII, X_0, X_1 and X_2 denote a basis of \mathfrak{g} such that X_1 and X_2 generate a commutative ideal on which X_0 acts in a certain way.

⁵The full automorphism group could be slightly larger, but $\text{Aut}(\mathfrak{g})$ definitely contains its connected identity component, which is sufficient for our purposes.

- \mathfrak{g}_I : commutative Lie algebra

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = A_\phi \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad A_\phi \in GL(3, \mathbb{R}), \quad \dim \text{Aut}(\mathfrak{g}_I) = 9.$$

- \mathfrak{g}_{II} : (Heisenberg Lie algebra) $[X_0, X_2] = X_1$;

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} \alpha & a & \beta \\ 0 & \alpha\delta - \gamma\beta & 0 \\ \gamma & b & \delta \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_{II}) = 6.$$

- \mathfrak{g}_{III} : $[X_0, X_1] = X_1$;

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_{III}) = 4.$$

- \mathfrak{g}_{IV} : $[X_0, X_1] = X_1$, $[X_0, X_2] = X_1 + X_2$;

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & \alpha & 0 \\ 0 & \beta & \alpha \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_{IV}) = 4.$$

- \mathfrak{g}_V : (book Lie algebra) $[X_0, X_1] = X_1$, $[X_0, X_2] = X_2$;

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & \alpha & \beta \\ 0 & \gamma & \delta \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_V) = 6.$$

- \mathfrak{g}_{VI_0} : (semidirect sum $\mathfrak{e}(1, 1) = \mathfrak{so}(1, 1) + \mathbb{R}^2$) $[X_0, X_1] = X_1$, $[X_0, X_2] = -X_2$;

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_{VI_0}) = 4.$$

- \mathfrak{g}_{VI} : $[X_0, X_1] = X_1$, $[X_0, X_2] = aX_2$ with $a \neq \pm 1$, $a \neq 0$;

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_{VI}) = 4.$$

- \mathfrak{g}_{VII_0} : (semidirect sum $\mathfrak{e}(2) = \mathfrak{so}(2) + \mathbb{R}^2$) $[X_0, X_1] = -X_2$, $[X_0, X_2] = X_1$;

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_{VII_0}) = 4.$$

- $\mathfrak{g}_{\text{VII}}$: $[X_0, X_1] = aX_1 - X_2$, $[X_0, X_2] = X_1 + aX_2$.

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = \begin{pmatrix} 1 & a & b \\ 0 & \alpha & \beta \\ 0 & -\beta & \alpha \end{pmatrix} \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad \dim \text{Aut}(\mathfrak{g}_{\text{VII}}) = 4.$$

- $\mathfrak{g}_{\text{VIII}}$: (simple Lie algebra $sl(2, \mathbb{R})$) $[X_0, X_1] = 2X_1$, $[X_0, X_2] = -2X_2$, $[X_1, X_2] = X_0$.

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = A_\phi \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad A_\phi \in SO(2, 1), \quad \dim \text{Aut}(\mathfrak{g}_{\text{VIII}}) = 3.$$

More specifically, $A_\phi^\top \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

- \mathfrak{g}_{IX} : (simple Lie algebra $so(3)$) $[X_0, X_1] = X_2$, $[X_1, X_2] = X_0$, $[X_2, X_0] = X_1$.

$$\begin{pmatrix} X'_0 \\ X'_1 \\ X'_2 \end{pmatrix} = A_\phi \begin{pmatrix} X_0 \\ X_1 \\ X_2 \end{pmatrix}, \quad A_\phi \in SO(3), \text{ i.e. } A_\phi^\top = A_\phi^{-1}, \quad \dim \text{Aut}(\mathfrak{g}_{\text{IX}}) = 3.$$

As a typical example, we now demonstrate the reduction-to-canonical-form process for the Lie algebra \mathfrak{g}_{IV} . We start with an arbitrary quadratic form

$$H = \sum_{i,j=0}^2 g^{ij} X_i X_j.$$

Notice that due to the positive (semi-)definiteness of H , we have $g^{00} > 0$. In the Riemannian case this fact is obvious. In the sub-Riemannian case, $g^{00} = 0$ would imply that $H = g^{11} X_1^2 + 2g^{12} X_1 X_2 + g^{22} X_2^2$ which is forbidden as X_1 and X_2 generate a proper ideal in \mathfrak{g}_{IV} but not the whole Lie algebra. Thus, we can write

$$H = g^{00} \left(X_0 + \frac{g^{01}}{g^{00}} X_1 + \frac{g^{02}}{g^{00}} X_2 \right)^2 + \text{quadratic form in } X_1 \text{ and } X_2.$$

Since the transformation $X'_0 = X_0 + \frac{g^{01}}{g^{00}} X_1 + \frac{g^{02}}{g^{00}} X_2$, $X'_1 = X_1$, $X'_2 = X_2$ belongs to the automorphism group, we see that H can be reduced to the form

$$\tilde{H} = g^{00} X_0^2 + aX_1^2 + bX_2^2 + 2cX_1X_2$$

for some $a, b, c \in \mathbb{R}$.

Next for the same reason as above we notice that $b > 0$ (in the sub-Riemannian case, $b = 0$ would imply that $H = g^{00} X_0^2 + aX_1^2$ which is forbidden as X_0 and X_1 generate a proper subalgebra of \mathfrak{g}_{IV}). Hence, we can rewrite \tilde{H} in the form

$$\tilde{H} = g^{00} X_0^2 + aX_1^2 + bX_2^2 + 2cX_1X_2 = g^{00} X_0^2 + \left(\sqrt{b}X_2 + \frac{c}{\sqrt{b}}X_1 \right)^2 + \left(a - \frac{c^2}{b} \right) X_1^2$$

Now applying the automorphism $X'_0 = X_0$, $X'_1 = \sqrt{b}X_1$, $X'_2 = \frac{c}{\sqrt{b}}X_1 + \sqrt{b}X_2$, we conclude that \tilde{H} is equivalent to

$$g^{00}X_0^2 + X_2^2 + \left(\frac{ab - c^2}{b}\right)X_1^2$$

or, by using simpler notation,

$$AX_0^2 + X_2^2 + CX_1^2, \quad A, C > 0.$$

In the sub-Riemannian case the last term in this expression has to automatically disappear and we come to the following conclusion:

Proposition 1. *Every left-invariant Riemannian metric on the 3-dimensional Lie group G_{IV} is defined up to equivalence by the quadratic form*

$$H = AX_0^2 + X_2^2 + CX_1^2, \quad \text{for some } A, C > 0.$$

Every left-invariant sub-Riemannian metric on the 3-dimensional Lie group G_{IV} is defined up to equivalence by the quadratic form

$$H = AX_0^2 + X_2^2, \quad \text{for some } A > 0.$$

Similar elementary computations for all the other 3-dimensional Lie algebras lead to the following final result.

Let G be a simply connected three-dimensional group and X_0, X_1, X_2 be a basis of left-invariant vector fields. This basis can be treated as a basis of the corresponding Lie algebra, one of those from Bianchi classification. In each case, we will assume that this basis is canonical, i.e. coincides with the basis described above for each algebra from the Bianchi list $\mathfrak{g}_I, \dots, \mathfrak{g}_{VII}$. As already discussed, it will be more convenient for us to define left-invariant metrics by means of the corresponding Hamiltonians being quadratic forms in X_0, X_1, X_2 with constant coefficients. In this setting we have

Theorem 3. *The canonical forms of left-invariant Riemannian and sub-Riemannian metrics on simply connected three dimensional Lie groups are*

| | Riemannian | sub-Riemannian |
|---------------------------|-------------------------------------------|-------------------------------------------------------------------|
| G_I | $X_0^2 + X_1^2 + X_2^2$ | none |
| G_{II} | $X_0^2 + B X_1^2 + X_2^2$ | $X_0^2 + X_2^2$ |
| G_{III} | $A X_0^2 + X_1^2 + X_2^2 + C X_1 X_2$ | $A X_0^2 + (X_1 + X_2)^2$ |
| G_{IV} | $A X_0^2 + B X_1^2 + X_2^2$ | $A X_0^2 + X_2^2$ |
| G_V | $A X_0^2 + X_1^2 + X_2^2$ | none |
| G_{VI} and G_{VI_0} | $A X_0^2 + X_1^2 + X_2^2 + C X_1 X_2$ | $A X_0^2 + (X_1 + X_2)^2$ |
| G_{VII} and G_{VII_0} | $A X_0^2 + B X_1^2 + X_2^2$ | $A X_0^2 + X_2^2$ |
| G_{VIII} | $A X_0^2 + B (X_1^2 + X_2^2) + C X_1 X_2$ | $A X_0^2 + B (X_1 + X_2)^2$ or $B (X_1^2 + X_2^2) + C X_1 X_2$ |
| G_{IX} | $A X_0^2 + B X_1^2 + C X_2^2$ | $A X_0^2 + B X_1^2$ |

Here $A, B, C \in \mathbb{R}$ are arbitrary parameters satisfying the positive (semi-)definiteness assumption (more precisely $A > 0$ and $B > 0$ in all the cases, $|C| < 2$ for G_{III} , G_{VI} and G_{VI_0} , $|C| < 2B$ for G_{VIII} , and $C > 0$ for G_{IX}).

The assumption that G is simply connected can sometimes be important for the following reason. We consider the automorphism group $\text{Aut}(\mathfrak{g})$ and then extend its action to the group G . If G is simply connected, then such an extension always exists. Otherwise, there might be some topological obstructions. In other words, the automorphism group of G can be smaller than that of \mathfrak{g} . For example, if instead of the abelian group \mathbb{R}^3 we consider the torus \mathbb{R}^3/Γ which is still an abelian Lie group with the same Lie algebra \mathfrak{g} , then in order for an automorphism $\phi : \mathfrak{g} \rightarrow \mathfrak{g}$ to induce an automorphism $\Phi : \mathbb{R}^3/\Gamma \rightarrow \mathbb{R}^3/\Gamma$, we need an additional condition that ϕ preserves the lattice Γ .

In the next section we give explicit formulas for left- and right-invariant vector fields on solvable three-dimensional Lie groups in local coordinates. This will give us a possibility to study the corresponding geodesic flows from the analytic viewpoint and, in particular, to explicitly integrate them.

4 Explicit description of left-invariant geodesic flows on non-semisimple 3-dimensional Lie groups

Without loss of generality we may assume that the corresponding Lie group G takes the following matrix form:

$$G = G_A = \left\{ \begin{pmatrix} \exp(q_0 A) & \bar{q} \\ 0 & \dots & 0 & 1 \end{pmatrix}, \text{ where } \bar{q} = (q_1, \dots, q_{n-1})^\top \in \mathbb{R}^{n-1} \right\} \subset GL(n, \mathbb{R}). \quad (2)$$

The parameters q_0, q_1, \dots, q_{n-1} are treated as global coordinates on the group. Topologically, this group is diffeomorphic to \mathbb{R}^n . Then we can just consider the coordinates (q_0, \bar{x}) to study this group.

It is easy to prove that the multiplication in G in these coordinates can be written as follows:

$$(q_0, \bar{q}) * (y_0, \bar{y}) = (q_0 + y_0, \exp(q_0 A)\bar{y} + \bar{q}). \quad (3)$$

Each right- and left-invariant vector field is given by $n = \dim G$ arbitrary parameters. For right-invariant vector fields, we will denote them by $\eta^0, \eta^1, \dots, \eta^{n-1}$, for left-invariant by $\xi^0, \xi^1, \dots, \xi^{n-1}$. We now prove

Proposition 2. *A left-invariant vector field X_ξ on G takes the form:*

$$X_{(\xi^0, \bar{\xi})} = (\xi^0, \exp(q_0 A)\bar{\xi}) = \xi^0 \frac{\partial}{\partial q_0} + \sum_{i,j=1}^{n-1} B_j^i(q_0) \xi^j \frac{\partial}{\partial q_i}, \text{ where } \bar{\xi} = \begin{pmatrix} \xi^1 \\ \vdots \\ \xi^{n-1} \end{pmatrix} \quad (4)$$

and $B_j^i(q_0)$ are the components of the matrix $\exp(q_0 A)$. A right-invariant vector field Y_η on G takes the form

$$Y_{(\eta^0, \bar{\eta})} = (\eta^0, \eta^0 \cdot A\bar{q} + \bar{\eta}) = \eta^0 \frac{\partial}{\partial x^0} + \sum_{i,j=1}^{n-1} (\eta^0 A_j^i q_j + \eta^i) \frac{\partial}{\partial x^i}, \quad \text{where } \bar{\eta} = \begin{pmatrix} \eta^1 \\ \vdots \\ \eta^{n-1} \end{pmatrix}. \quad (5)$$

Proof. For verification we will use the following standard method for constructing left- and right-invariant vector fields on Lie groups. Let $\gamma(t)$ be an arbitrary curve in G such that $\gamma(0) = e$ (the identity of the group) and $\frac{d}{dt}\gamma(0) = \xi \in T_e G \simeq \mathfrak{g}$. Then the left-invariant vector field generated by ξ is defined by the formula

$$X_\xi(x) = \left. \frac{d}{dt} \right|_{t=0} (x * \gamma(t)), \quad x \in G.$$

Similarly, for right-invariant vector fields: $Y_\xi = \left. \frac{d}{dt} \right|_{t=0} (\gamma(t) * x)$.

In our case, we set $\gamma(t) = (\xi^0 t, \bar{\xi} t)$. Hence, by using formula (3):

$$X_{(\xi^0, \bar{\xi})} = \left. \frac{d}{dt} \right|_{t=0} ((q_0, \bar{q}) * (\xi^0 t, \bar{\xi} t)) = \left. \frac{d}{dt} \right|_{t=0} (q_0 + \xi^0 t, \exp(q_0 A) \bar{\xi} t + \bar{q}) = (\xi^0, \exp(q_0 A) \bar{\xi}).$$

Similarly, for $\gamma(t) = (\eta^0 t, \bar{\eta} t)$ we have

$$Y_{(\eta^0, \bar{\eta})} = \left. \frac{d}{dt} \right|_{t=0} ((\eta^0 t, \bar{\eta} t) * (q_0, \bar{q})) = \left. \frac{d}{dt} \right|_{t=0} (\eta^0 t + q_0, \exp(\eta^0 A t) \bar{q} + \bar{\eta} t) = (\eta^0, \eta^0 A \bar{q} + \bar{\eta}),$$

as stated. \square

Remark 1. Notice that the vector fields $\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{n-1}}$ are right-invariant. One more right-invariant vector field takes the form $\frac{\partial}{\partial x^0} + \sum_{i,j=1}^{n-1} A_j^i q_j \frac{\partial}{\partial x^i}$.

Remark 2. It is straightforward to check that left- and right-invariant vector fields given by (4) and (5) commute, i.e., $[X_{(\xi^0, \bar{\xi})}, X_{(\eta^0, \bar{\eta})}] = 0$ for any $(\xi^0, \bar{\xi})$ and $(\eta^0, \bar{\eta})$.

Now if we choose an arbitrary basis in the space of left-invariant vector fields X_0, \dots, X_{n-1} then by treating them as linear functions on the cotangent bundle T^*G , we can have the corresponding Hamiltonian in terms of canonical coordinates (x, p) . Observe that the Hamiltonian H does not contain variables q_1, \dots, q_{n-1} so that p_1, \dots, p_{n-1} are commuting integrals of the corresponding geodesic flow (which, of course, correspond to right-invariant vector fields $\frac{\partial}{\partial q_1}, \dots, \frac{\partial}{\partial q_{n-1}}$). In particular, we are led to the following conclusion:

Theorem 4. *The geodesic flow of any left-invariant Riemannian or sub-Riemannian metric on a Lie group G defined by (2) is Liouville integrable. As commuting integrals, one can consider the momenta p_1, \dots, p_{n-1} and the Hamiltonian H itself.*

As we see, explicit formulas for left-invariant vector fields (as well as for left-invariant metrics) depend on the matrix A that defines the group $G = G_A$. For the three-dimensional group of this type, these formulas are summarised in the following table (for each group

we indicate the corresponding 2×2 matrix A and its exponent $\exp(q_0 A)$ used in the general formula (4) for left-invariant vector fields). Notice that the commutation relations between the left-invariant X_0, X_1, X_2 described below agree with those we used in Section 3 to introduce the list of non-semisimple Lie algebras $\mathfrak{g}_I, \dots, \mathfrak{g}_{VII}$.

| Group | A | $\exp(q_0 A)$ | Basis of left-invariant vector fields |
|---------------|-------------------------------------------------|-----------------------------------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| G_I : | $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = \partial_{q_1} \\ X_2 = \partial_{q_2} \end{cases}$ |
| G_{II} : | $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} 1 & q_0 \\ 0 & 1 \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = \partial_{q_1} \\ X_2 = q_0 \partial_{q_1} + \partial_{q_2} \end{cases}$ |
| G_{III} : | $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ | $\begin{pmatrix} e^{q_0} & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = e^{q_0} \partial_{q_1} \\ X_2 = \partial_{q_2} \end{cases}$ |
| G_{IV} : | $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} e^{q_0} & q_0 e^{q_0} \\ 0 & e^{q_0} \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = e^{q_0} \partial_{q_1} \\ X_2 = q_0 e^{q_0} \partial_{q_1} + e^{q_0} \partial_{q_2} \end{cases}$ |
| G_V : | $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ | $\begin{pmatrix} e^{q_0} & 0 \\ 0 & e^{q_0} \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = e^{q_0} \partial_{q_1} \\ X_2 = e^{q_0} \partial_{q_2} \end{cases}$ |
| G_{VI_0} : | $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ | $\begin{pmatrix} e^{q_0} & 0 \\ 0 & e^{-q_0} \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = e^{q_0} \partial_{q_1} \\ X_2 = e^{-q_0} \partial_{q_2} \end{cases}$ |
| G_{VI} : | $\begin{pmatrix} 1 & 0 \\ 0 & a \end{pmatrix}$ | $\begin{pmatrix} e^{q_0} & 0 \\ 0 & e^{aq_0} \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = e^{q_0} \partial_{q_1} \\ X_2 = e^{aq_0} \partial_{q_2} \end{cases}$ |
| G_{VII_0} : | $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ | $\begin{pmatrix} \cos q_0 & \sin q_0 \\ -\sin q_0 & \cos q_0 \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = \cos q_0 \partial_{q_1} - \sin q_0 \partial_{q_2} \\ X_2 = \sin q_0 \partial_{q_1} + \cos q_0 \partial_{q_2} \end{cases}$ |
| G_{VII} : | $\begin{pmatrix} a & 1 \\ -1 & a \end{pmatrix}$ | $\begin{pmatrix} e^{aq_0} \cos q_0 & e^{aq_0} \sin q_0 \\ -e^{aq_0} \sin q_0 & e^{aq_0} \cos q_0 \end{pmatrix}$ | $\begin{cases} X_0 = \partial_{q_0} \\ X_1 = e^{aq_0} \cos q_0 \partial_{q_1} - e^{aq_0} \sin q_0 \partial_{q_2} \\ X_2 = e^{aq_0} \sin q_0 \partial_{q_1} + e^{aq_0} \cos q_0 \partial_{q_2} \end{cases}$ |

Combining these formulas with Theorem 3 we obtain

Theorem 5. *The canonical forms of left-invariant Riemannian and sub-Riemannian metrics on solvable simply connected Lie groups of dimension 3 (in local coordinates q_0, q_1, q_2 introduced in (2)) are presented in the following table:*

| | Riemannian case | sub-Riemannian case |
|---------------|-----------------------------------------------------------------------------------------------------------------------------|-----------------------------------------------------------------|
| G_I : | $p_0^2 + p_1^2 + p_2^2$ | none |
| G_{II} : | $p_0^2 + Bp_1^2 + (q_0p_1 + p_2)^2$ | $p_0^2 + (q_0p_1 + p_2)^2$ |
| G_{III} : | $Ap_0^2 + e^{2q_0}p_1^2 + p_2^2 + Ce^{q_0}p_1p_2$ | $Ap_0^2 + (e^{q_0}p_1 + p_2)^2$ |
| G_{IV} : | $Ap_0^2 + e^{2q_0}(Bp_1^2 + (q_0p_1 + p_2)^2)$ | $Ap_0^2 + e^{2q_0}(q_0p_1 + p_2)^2$ |
| G_V : | $Ap_0^2 + e^{2q_0}(p_1^2 + p_2^2)$ | none |
| G_{VI_0} : | $Ap_0^2 + e^{2q_0}p_1^2 + e^{-2q_0}p_2^2 + Cp_1p_2$ | $Ap_0^2 + (e^{q_0}p_1 + e^{-q_0}p_2)^2$ |
| G_{VI} : | $Ap_0^2 + e^{2q_0}p_1^2 + e^{2aq_0}p_2^2 + Ce^{(a+1)q_0}p_1p_2$ | $Ap_0^2 + (e^{q_0}p_1 + e^{aq_0}p_2)^2$ |
| G_{VII_0} : | $Ap_0^2 + B(\cos q_0 \cdot p_1 - \sin q_0 \cdot p_2)^2 + (\sin q_0 \cdot p_1 + \cos q_0 \cdot p_2)^2$ | $Ap_0^2 + (\sin q_0 \cdot p_1 + \cos q_0 \cdot p_2)^2$ |
| G_{VII} : | $Ap_0^2 + e^{2aq_0}\left(C(\cos q_0 \cdot p_1 - \sin q_0 \cdot p_2)^2 + (\sin q_0 \cdot p_1 + \cos q_0 \cdot p_2)^2\right)$ | $Ap_0^2 + e^{2aq_0}(\sin q_0 \cdot p_1 + \cos q_0 \cdot p_2)^2$ |

Here $A, B, C \in \mathbb{R}$ are arbitrary parameters satisfying the positive (semi-)definiteness assumption (more precisely $A > 0$, $B > 0$ and $C^2 < 4$).

5 Example: the group $G_{VII_0} = \widetilde{E(2)}$

Let us consider the geodesic flow for a left-invariant sub-Riemannian metric on G_{VII_0} in more detail.

The Hamiltonian takes the form $H = \frac{1}{2}\left(Ap_0^2 + (\sin q_0 \cdot p_1 + \cos q_0 \cdot p_2)^2\right)$ and it admits three non-commuting linear integrals (that represent basis right-invariant vector fields, see Remark 1).

$$\begin{aligned} F_0 &= p_0 + q_2p_1 - q_1p_2, \\ F_1 &= p_1, \\ F_2 &= p_2. \end{aligned}$$

Together with H , all these integrals form a four-dimensional (non-commutative) algebra of first integrals with two Casimir functions H and $F = F_1^2 + F_2^2$.

The corresponding sub-Riemannian geodesics can be easily found in quadratures. We are going to describe the geodesics through the identity of the group which is the origin of our coordinate system, i.e., $(q_0, q_1, q_2) = (0, 0, 0)$.

Indeed, let us fix the values of F_0 , F_1 and F_2 :

$$F_0 = p_0 + q_2p_1 - q_1p_2 = c_0, \quad F_1 = p_1 = c_1, \quad F_2 = p_2 = c_2.$$

Then the Hamiltonian system with the Hamiltonian H can be rewritten in variables q_1, q_2 and q_3 only:

$$\begin{aligned}\frac{dq_0}{dt} &= \frac{\partial H}{\partial p_0} = Ap_0 = A(c_0 - q_2c_1 - q_1c_2), \\ \frac{dq_1}{dt} &= \frac{\partial H}{\partial p_1} = \sin q_0 (p_1 \sin q_0 + p_2 \cos q_0) = \sin q_0 (c_1 \sin q_0 + c_2 \cos q_0), \\ \frac{dq_2}{dt} &= \frac{\partial H}{\partial p_2} = \cos q_0 (p_1 \sin q_0 + p_2 \cos q_0) = \cos q_0 (c_1 \sin q_0 + c_2 \cos q_0).\end{aligned}\tag{6}$$

This dynamical system in $\mathbb{R}^3(q_1, q_2, q_3)$ now depends on c_0, c_1, c_2 as parameters and admits a non-trivial energy integral that can now be written as

$$H = \frac{1}{2} \left(A(c_0 - c_1q_2 + c_2q_1)^2 + (c_1 \sin q_0 + c_2 \cos q_0)^2 \right).\tag{7}$$

For simplicity, set $c_0 = 1, c_1 = 1, c_2 = 0$ (the general case is not essentially different). Then (8) becomes

$$\begin{aligned}\frac{dq_0}{dt} &= A(1 - q_2), \\ \frac{dq_1}{dt} &= \sin^2 q_0, \\ \frac{dq_2}{dt} &= \cos q_0 \sin q_0,\end{aligned}\tag{8}$$

and

$$H = \frac{1}{2} (A(1 - q_2)^2 + \sin^2 q_0).\tag{9}$$

It is interesting to notice that the first and third equations of this system form a one-degree of freedom Hamiltonian system with the Hamiltonian (9). The level lines of this Hamiltonian (i.e., in fact solutions of this subsystem) are shown in Figure 1.

This observation shows that sub-Riemannian geodesics can be of several different types:

- Type 1. Trivial geodesics, corresponding to the minima of H located at the points $q_2 = 1, q_0 = \pi k, k \in \mathbb{Z}$. For these geodesics $H = 0$ and hence therefore there is no motion.
- Type 2. The geodesics corresponding to the saddle equilibria of H located at points $q_2 = 1, q_0 = \pi/2 + \pi k, k \in \mathbb{Z}$. For such geodesics, q_0 and q_2 remain constant, but $q_1(t) = t$. Geometrically, these are straight lines.
- Type 3. Geodesics that are periodic in variables q_0 and q_2 , whereas $q_1(t)$ is strictly increasing. They correspond to closed level lines $\{H = h \leq 1/2\}$ shown in Figure 1. From the view point of the dynamics in $\mathbb{R}^3(q_0, q_1, q_2)$, they are located on invariant cylinders.

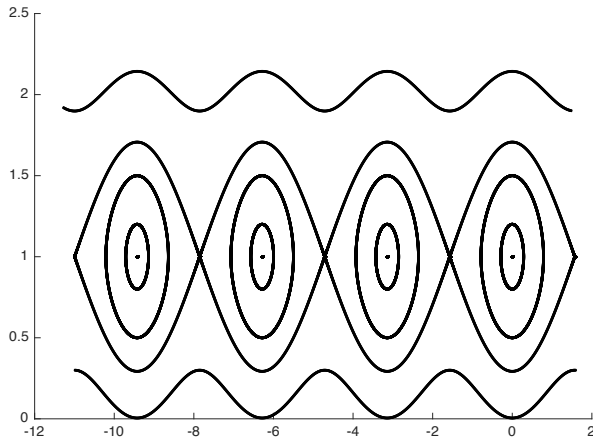


Figure 1: Level lines of $H(q_0, q_2)$

- Type 4. Geodesics that correspond to non-closed level lines of $\{H = h \geq 1/2\}$ shown in Figure 1.
- Type 5. Geodesics that corresponds to separatrices connecting two saddle equilibrium points in Figure 1. They are located at the critical level $\{H = 1/2\}$. From the view point of the dynamics in $\mathbb{R}^3(q_0, q_1, q_2)$, they asymptotically approach two “critical” geodesics of type 2 as $t \rightarrow \pm\infty$.

Typical geodesics of types 3, 4 and 5 are shown in Figure 2 in projection to the coordinate plane $\mathbb{R}(q_1, q_2)$.

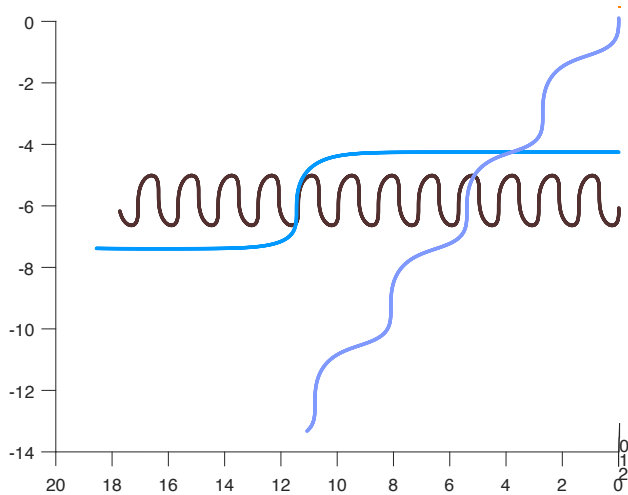


Figure 2: Three types of sub-Riemannian geodesics.

Locally, system of ODEs (8) can be easily solved in quadratures. For instance, if we consider $\tau = q_0$ as a new parameter on geodesics (8), then we have $q_0(\tau) = \tau$, $q_2(\tau) = 1 \pm \sqrt{\frac{1}{A}(2h - \sin^2 \tau)}$, where h is a constant of integration. Here we simply use the fact that in terms of q_0 and q_2 , the solutions coincides with the levels of the Hamiltonian (9), i.e., satisfy the relation $\frac{1}{2}(A(1 - q_2(\tau))^2 + \sin^2 q_0(\tau)) = h$. To recover $q_1(\tau)$, it remains to solve the equation

$$\frac{dq_1}{d\tau} = \frac{dq_1}{dt} \frac{dt}{d\tau} = \frac{dq_1}{dt} \left(\frac{dq_0}{dt} \right)^{-1} = \frac{\sin^2 q_0(\tau)}{A(1 - q_2(\tau))} = \frac{\sin^2 \tau}{\sqrt{A(2h - \sin^2 \tau)}}$$

so that finally we get the following parametric equation for sub-Riemannian geodesics (with the fixed values of integrals F_0 , F_1 and F_2)

$$q_0(\tau) = \tau, \quad q_1(\tau) = \int \frac{\sin^2 \tau d\tau}{\sqrt{A(2h - \sin^2 \tau)}}, \quad q_2(\tau) = 1 \pm \sqrt{\frac{1}{A}(2h - \sin^2 \tau)}.$$

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