Nonlinearity Estimator-Based Control of A Class of Uncertain Nonlinear Systems

Jun Yang, Ting Li, Cunjia Liu, Shihua Li, Wen-Hua Chen

Abstract—The robust control problem of a class of nonlinear systems subject to external disturbances, control gain uncertainty and nonlinear uncertainties is investigated in this paper using a nonlinearity estimator-based control approach. Different from the existing results, the crucial but highly restrictive hypothesis on the boundedness of nonlinear uncertainties is removed from this paper by means of the tools of semi-global stabilization. By delicately constructing a specific composite Lyapunov function for the closed-loop system as well as several useful level sets, the rigorous qualitative robustness performance is presented for the closed-loop system. Finally, an example of a single-link manipulator is utilized to demonstrate the performance specification claimed by the theoretical analysis.

Index Terms—Nonlinearity estimator, control gain uncertainty, qualitative robustness, uncertain nonlinear systems, semi-global stability.

I. INTRODUCTION

A fundamental and crucial task of control systems is to deal with external disturbances and uncertainties [1], [2], [3]. The traditional robust control using high control gains is recognized as a major design tool to suppress disturbances/uncertainties in nonlinear control theory [4]. Nevertheless, robust control is mostly achieved at a price of sacrificing the nominal control performance since the control performance in the nominal case is usually not directly taken into account [15]. Such a design philosophy may cause unsatisfactory overall performance since most practical systems usually operate around their nominal operation point, and rarely operate far away from their nominal operation point [5].

As an alternative approach to traditional robust control, various disturbance/uncertainty estimation and attenuation (DUEA) approaches have been proposed for disturbance rejection and uncertainty attenuation [6], [7], [8], [9], [10], [11]. In the context of DUEA, the total uncertainties including external disturbances and plant-model mismatch are usually estimated and compensated online within the closed-loop system [12], [13]. A remarkable merit of DUEA against the traditional robust control is the essential nature of non-worst-case-oriented design. Within the framework of DUEA, an estimator acting as a patch is utilized to online observe the disturbances/uncertainties. In the absence of disturbances/uncertainties, the patch estimator would be inactivated

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and a baseline controller is able to preserve the nominal control performances. Otherwise, the estimator is activated for uncertainty estimation and compensation, which improves robustness without involving excessive control energy. To conclude, the DUEA framework provides a good balance between the robustness and nominal performance [14], [15].

Note that the DUEA was initially put forward to purely estimate and compensate external disturbances. Later, it has been discovered that DUEA approaches exhibiting promising and powerful ability in estimating and compensating uncertainties caused by plant-model mismatch. Closed-loop stability of DUEA has been thoroughly developed for nonlinear systems subject to various external disturbances [16]. In [12], [13], the total uncertainties including control gain uncertainty and nonlinear uncertainties have been estimated by DUEA, however, few rigorous stability is available when DUEA is used as a robust control method to deal with model uncertainties. Unlike the external disturbance case, the influence of the model uncertainties is actually a function of system states. It is unrealistic to suppose that the uncertainties are bounded for stability analysis. The coupling between the system dynamics and the estimator error dynamics poses great challenges for rigorous robustness stability analysis. In [14], an extended high-gain observer-based control approach was proposed to establish rigorous stability of DUEA with performance recovery. In the field of active disturbance rejection control (ADRC) known as a kind of DUEA approaches, the robustness stability of the closed-loop system under uncertainties is developed by using an observer bandwidth factor evaluation approach [17], [18]. Recently, the work in [19] further established a semi-global practical stability of ADRC approach for systems with unknown control gain b by using adaptive estimators.

In this paper, a nonlinearity estimator-based control approach is developed for robust control of a class of uncertain nonlinear systems. What is more, we attempt to establish rigorous robustness stability of the resultant closed-loop system. The nominal dynamics of the uncertain nonlinear systems under consideration are nonlinear, which differs from [17], [18], [19]. A preliminary result of this paper has been published in [5], which provides a simple local result. However, to surmount the limitation of local result, it is quite challenging to conduct stability analysis as it is hard to evaluate the bounds of coupling terms between the system dynamics and the estimator error dynamics outside a local region. This issue will be addressed in this paper by means of modifying the tools of semi-global stabilization. To this end, we first construct several interesting level sets which depend on the set of the initial values of both system and observer states. The delicate

construction of the useful level sets admits their separation from the observer gain factor to be assigned, which opens the door to semi-global stabilization of the closed-loop system in question. It has been shown that the proposed control method guarantees that the system with arbitrary large initial states converges to a prescribed arbitrary small set by assigning large enough observer gain factor. It has been further proved that in the absence of external disturbances, the closed-loop system is semi-global asymptotical stable in the sense that with arbitrary large initial values, the system states will converge to zero by the proposed controller. Finally, a practical example of a single-link manipulator is investigated to demonstrate the theoretical claims presented in the paper. The results show that with certain quantity of uncertainties it is able to tune the observer gain factor such that the stability of the closed-loop systems is guaranteed. The relationship between the observer gain factor and the amount of uncertainties is explicitly demonstrated by the simulation results.

II. PRELIMINARIES AND PROBLEM FORMULATION

A. Problem Formulation

We consider a class of nonlinear systems subject to external disturbances and nonlinear uncertainties, given by

$$\dot{x} = f(x) + G(x)[d_1(x)u + d_2(x,\omega)], \qquad (1)$$

which can be rewritten as follows

$$\dot{x} = f(x) + G(x)[d_0 u + d(x,\omega,u)], \qquad (2)$$

where $x \in \mathbb{R}^n$ is the system state vector, $u \in \mathbb{R}$ is the control input, $d(x,\omega,u) = (d_1(x) - d_0)u + d_2(x,\omega) \in \mathbb{R}$ denotes the lumped uncertainty including both external disturbances and model uncertainties, $d_1(x)$ is an unknown control coefficient whose known nominal part is $d_0 \neq 0$ that is a constant, $d_2(x,\omega)$ is an unknown function denoting the uncertain effects caused by nonlinear uncertainties and external disturbances, ω is the external disturbance satisfying that ω and $\dot{\omega}$ are defined on the compact set $\mathcal{D} \subset \mathbb{R}^q$. $f(x)$ and $G(x)$ are known smooth nonlinear functions with $\|G(x)\| \neq 0$ for any $x \in \mathbb{R}^n$ where $\| \cdot \|$ denotes the Euclidean norm. Suppose that the lumped uncertainty $d(x,\omega,u)$ satisfies the following assumption.

Assumption 1: *The lumped uncertainty $d(x,\omega,u)$ is partially continuously differentiable in terms of the arguments x and ω, and $|d_1(x)| \neq 0$ for any $x \in \mathbb{R}^n$. Moreover, there exists a known positive constant $\delta_0 < 1$ and known nonnegative continuous scalar functions $\bar{\delta}_1(x)$ and $\bar{\delta}_2(x)$ such that*

$$
\begin{aligned}
&\left| \frac{d_1(x) - d_0}{d_0} \right| \leq \delta_0, \\
&\left\| \frac{\partial d_2}{\partial x} \right\| \leq \bar{\delta}_1(x), \ \left\| \frac{\partial d_2}{\partial \omega} \right\| \leq \bar{\delta}_2(x),
\end{aligned}
\qquad (3)
$$

for any $x \in \mathbb{R}^n$ and $\omega \in \mathcal{D}$.

Remark 1: The lumped uncertainty $d(x,\omega,u)$ is unknown which could include control coefficient uncertainty, external disturbances and model uncertainties including unmodelled nonlinear dynamics, parametric perturbations, as well as other static structured/unstructured modeling errors. The assumptions on the lumped uncertainties are quite different between

that in this paper and the existing ones. In most of existing works on DUEA, it is generally required that both the lumped uncertainty and its derivative with respect to time are bounded by some constants, that is, $|d(x,\omega,u)| < c_1$ and $|\dot{d}(x,\omega,u)| < c_2$. This is quite restrictive because $d(x,\omega,u)$ and $\dot{d}(x,\omega,u)$, which are functions in terms of their arguments x, ω and u, may violate the restrictions $|d(x,\omega,u)| < c_1$ and $|\dot{d}(x,\omega,u)| < c_2$ as x and u go toward infinity. For a simple example that $d(x,\omega,u) = x$, the condition $|d(x,\omega,u)| < c_1$ means that $x < c_1$, which implies that the state x is always bounded and stable regardless of the control. Based on this statement, the existing works can only obtain local stability result. However, in this paper, the assumption on the lumped uncertainties is supposed to be bounded by some nonlinear functions, that is, $\|\frac{\partial d_2}{\partial x}\| \leq \bar{\delta}_1(x)$ and $\|\frac{\partial d_2}{\partial \omega}\| \leq \bar{\delta}_2(x)$. Under the framework of semi-global stabilization, we only require that the initial values of the states are within a compact set, while the real states can go beyond this compact set. Since the lumped uncertainty are functions in terms of system states, they generally can not be bounded by a fixed compact set. This imposes great challenges for the design and analysis of the closed-loop system under the proposed control approach in the context of semi-global stabilization rather than local one.

Remark 2: The hypothesis $|(d_1(x) - d_0)/d_0| \leq \delta_0$ in Assumption 1 (which indicates that $d_1(x)$ and d_0 must have the same signs) should be taken into account for theoretical proof of the paper. However, the nominal value of d_0 generally can be chosen in a wide range such that the hypothesis holds. Actually, for any unknown control gain satisfying $C_1 < d_1(x) < C_2$ (or $-C_2 < d_1(x) < -C_1$), where C_1 and C_2 are arbitrary positive constants. By selecting $d_0 > C_2/2$ (or $d_0 < -C_2/2$), the condition $|(d_1(x) - d_0)/d_0| \leq \delta_0$ is satisfied.

Assumption 2: *Suppose that the equilibrium $x = 0$ of the nominal system of (2), i.e., $\dot{x} = f(x) + d_0 G(x)u$ is globally asymptotically stabilized by a static state feedback controller $\bar{u}(x) = \alpha(x)$, where $\alpha(x)$ is continuously differentiable and can be designed by some nonlinear control method. That is, the closed-loop nominal system under the baseline controller $\bar{u}(x)$, governed by $\dot{x} = f_c(x)$ is globally asymptotically stable with $f_c(x) = f(x) + d_0 G(x)\alpha(x)$.*

The objective of this paper is to develop a nonlinearity estimator-based control approach for nonlinear system (2) such that the equilibrium of the closed-loop system is semi-globally stable even in the presence of nonlinear uncertainties and external disturbances.

B. Preliminaries

We list some crucial definitions and lemmas for synthesis and analysis of the main result of the paper.

Definition 1: [20] A point $x = 0$ of system $\dot{x} = f(x,u)$ is said to be semi-globally practically stabilizable by dynamic state feedback if, for each pair of compact sets (X_s, X_l) that are the neighborhoods of $(0,0)$ and satisfy $X_s \subset X_l$, there exists a locally Lipschitz dynamic state feedback $u = h_1(x,\xi), \dot{\xi} = h_2(x,\xi)$ and a pair of compact sets $(X_{\xi,s}, X_{\xi,l})$ such that all the solutions of the closed-loop system starting

from the set $X_l \times X_{\xi,l}$ are captured by the set $X_s \times X_{\xi,s}$. If the set $X_s \times X_{\xi,s} = \{(0,0)\}$, the point $x = 0$ is said to be semi-globally stabilizable by dynamic state feedback.

Lemma 1[21] For given real numbers $m > 0, n > 0$ and a, there exists $c > 0$ such that

$$|ax^m y^n| \le c|x|^{m+n} + \frac{n}{m+n}\left(\frac{m}{(m+n)c}\right)^{\frac{m}{n}}|a|^{\frac{m+n}{n}}|y|^{m+n},$$

for all $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

Lemma 2[13] Suppose that a continuously differentiable function $F(x,y,z) \in \mathbb{R}^p$ satisfies

$$|\mathfrak{h}|\left\|\frac{\partial F}{\partial z}\right\| \le \epsilon, \quad \forall (x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p, \qquad (4)$$

where $0 < \epsilon < 1$ is a positive constant, $\mathfrak{h}$ is a constant. Then for any function $\text{sat}(\cdot)$ whose derivative is dominated by one, i.e.

$$\left\|\frac{\partial \text{sat}}{\partial s}(s)\right\| \le 1, \ \forall s \in \mathbb{R}^p,$$

there exists a unique continuously differentiable function $\Phi_{\text{sat}}(\theta, \varrho, x, y)$ solving the equation

$$z = \theta - \mathfrak{h}\,\text{sat}(F(x,y,z) - \varrho),$$

where $\theta \in \mathbb{R}^p$ and $\varrho \in \mathbb{R}^p$.

Lemma 3[1] Suppose that the equilibrium $x = 0$ of the system

$$\dot{x} = H(x) \qquad (5)$$

is globally asymptotically stable. Then there exists a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ such that

$$\alpha_1(\|x\|) \le V(x) \le \alpha_2(\|x\|),$$
$$\frac{\partial V}{\partial x} H(x) \le -\Phi(x), \qquad (6)$$

where α_1, α_2 and Φ are continuous positive definite and class $\mathcal{K}_\infty$ functions.

Lemma 4[1] Let $x = 0$ be an equilibrium point for the nonlinear system $\dot{x} = f(x)$, where $f : \mathcal{D}_0 \to \mathbb{R}^n$ is continuously differentiable, $\mathcal{D}_0 = \{x \in \mathbb{R}^n | \ \|x\| < r_0\}$, and $\frac{\partial f(x)}{\partial x}$ is bounded on $\mathcal{D}_0$. If the equilibrium $x = 0$ is locally exponentially stable; that is, the trajectories of the system satisfy

$$\|x(t)\| \le k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall x(t_0) \in \mathcal{D}_1, \quad \forall t \ge t_0 \ge 0,$$

where λ is a positive constant, $\mathcal{D}_1 = \{x \in \mathbb{R}^n | \ \|x\| < r_1\}$, and r_1 is a positive constant satisfying $r_1 \le r_0/k$ with $k > 0$. Then, there is a $\mathcal{C}_1$ function $V : \mathcal{D}_1 \to \mathbb{R}$ such that

$$c_1\|x\|^2 \le V(x) \le c_2\|x\|^2,$$
$$\frac{\partial V}{\partial x}f(x) \le -c_3\|x\|^2, \qquad (7)$$
$$\left\|\frac{\partial V}{\partial x}\right\| \le c_4\|x\|,$$

for some positive constants c_1, c_2, c_3, and c_4.

III. Main Results

A. Nonlinearity Estimator-Based Control Design

To begin with, a nonlinearity estimator is designed below to estimate the lumped uncertainties $d(x,u,\omega)$ in system (2), which is depicted by

$$\dot{\xi} = -Lh(x)\left[f(x) + G(x)(d_0 u + \xi + p(x))\right],$$
$$\hat{d} = \xi + p(x), \frac{\partial p(x)}{\partial x} = Lh(x), \qquad (8)$$

where $\xi \in \mathbb{R}$ is the internal state of the observer, $\hat{d} \in \mathbb{R}$ is the estimate of the lumped uncertainty $d(x,\omega,u)$, L is a positive observer gain factor to be designed, and $h(x)$ is the continuous observer gain which is regardless of the scalar L. The error vector of the nonlinearity estimator (8) is defined as follows

$$\zeta = d - \hat{d}. \qquad (9)$$

With the condition given in Assumption 2, one can obtain from Lemma 3 that there exist continuous positive definite and $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\bar{\alpha}$, and ϕ, and a $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}_{\ge 0}$ satisfying

$$\underline{\alpha}(\|x\|) \le V(x) \le \bar{\alpha}(\|x\|),$$
$$\frac{\partial V}{\partial x}f_c(x) \le -\phi(x). \qquad (10)$$

Define a n-dimensional super cube set as $\Gamma_r^n \triangleq [-r,r]^n$. Let $C = \max\{V(x)\big| x \in \Gamma_r^n\}$. We then define the level sets Δ_C and Δ as follows

$$\Delta_C = \{x \in \mathbb{R}^n : V(x) \le C\},$$
$$\Delta = \{(x,\zeta,\omega) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^q : V(x) \le C+1, |\zeta| \le 1, \omega \in D\}. \quad (11)$$

Similarly we can construct a level set Δ_{C+1}. By construction we have $\Gamma_r^n \subset \Delta_C \subset \Delta_{C+1}$. As per the above construction, the nonlinearity estimator-based control law is designed as follows

$$u = \alpha(x) - \text{SAT}_M(\hat{d})/d_0, \qquad (12)$$

where $M = \sup_{(x,\zeta,\omega)\in\Delta} |((d_1(x) - d_0)\alpha(x) + d_2(x,\omega) - \zeta)d_0/d_1|$, $\text{SAT}_M(\cdot)$ is an odd function defined by

$$\text{SAT}_M(s) = \begin{cases} s, & \text{if } 0 \le s \le M, \\ (s-M)(1+M-s)+M, & \text{if } M < s < M+0.5, \\ M + 0.25, & \text{if } s \ge 0.5 + M. \end{cases} \qquad (13)$$

The function $\text{SAT}_M(\cdot)$ is nondecreasing, continuously differentiable with a locally Lipschitz derivative, bounded uniformly in M, and its derivative satisfies $0 \le \frac{d\text{SAT}_M}{ds} \le 1$.

B. Stability Analysis

Thanks to Lemma 2, the control law (12) can be rewritten as

$$u = \Xi(\alpha(x), \zeta, x, \omega), \qquad (14)$$

and $\Xi(\cdot)$ is a continuously differentiable function. Denote $d_\Xi(x,\omega,\zeta) \triangleq d(x,\omega,\Xi(\alpha(x),\zeta,x,\omega))$. Substituting the proposed control law (14) with the nonlinearity estimator (8) into

the nonlinear system (2), one obtains the closed-loop system below

$$\dot{x} = f_c(x) + G(x)(d_\Xi(x,\omega,\zeta) - \text{SAT}_M(d_\Xi(x,\omega,\zeta) - \zeta)),$$
$$\dot{\zeta} = -Lh(x)G(x)\zeta + \dot{d}_\Xi(x,\omega,\zeta),$$

$$(15)$$

where

$$\dot{d}_\Xi = \frac{\partial d_\Xi}{\partial x}\dot{x} + \frac{\partial d_\Xi}{\partial \zeta}\dot{\zeta} + \frac{\partial d_\Xi}{\partial \omega}\dot{\omega}. \qquad (16)$$

Note that

$$\frac{\partial d_\Xi}{\partial x} = \left(1 + \frac{1}{d_0}\frac{\partial d}{\partial u}\frac{\partial \text{SAT}_M}{\partial s}\right)^{-1}\left(\frac{\partial d}{\partial x} + \frac{\partial d}{\partial u}\frac{\partial \alpha}{\partial x}\right),$$

$$\frac{\partial d_\Xi}{\partial \zeta} = \left(1 + \frac{1}{d_0}\frac{\partial d}{\partial u}\frac{\partial \text{SAT}_M}{\partial s}\right)^{-1}\frac{1}{d_0}\frac{\partial d}{\partial u}\frac{\partial \text{SAT}_M}{\partial s}, \qquad (17)$$

$$\frac{\partial d_\Xi}{\partial \omega} = \frac{\partial d_2}{\partial \omega}.$$

Inserting (17) into (15), one obtains the closed-loop system having the following form

$$\dot{x} = f_c(x) + G(x)(d_\Xi(x,\omega,\zeta) - \text{SAT}_M(d_\Xi(x,\omega,\zeta) - \zeta)),$$
$$\dot{\zeta} = -L\left(1 + \frac{1}{d_0}\frac{\partial d}{\partial u}\frac{\partial \text{SAT}_M}{\partial s}\right)h(x)G(x)\zeta + \psi_1(x,\zeta,\omega)$$
$$+ \psi_2(x,\zeta,\omega,\dot{\omega}) + \psi_3(x,\zeta,\omega)(d_\Xi(x,\omega,\zeta)$$
$$- \text{SAT}_M(d_\Xi(x,\omega,\zeta) - \zeta)).$$

$$(18)$$

with

$$\psi_1(x,\zeta,\omega) = \left(\frac{\partial d}{\partial x} + \frac{\partial d}{\partial u}\frac{\partial \alpha}{\partial x}\right)f_c(x),$$

$$\psi_2(x,\zeta,\omega,\dot{\omega}) = \left(1 + \frac{1}{d_0}\frac{\partial d}{\partial u}\frac{\partial \text{SAT}_M}{\partial s}\right)\frac{\partial d_2}{\partial \omega}\dot{\omega}, \qquad (19)$$

$$\psi_3(x,\zeta,\omega) = \left(\frac{\partial d}{\partial x} + \frac{\partial d}{\partial u}\frac{\partial \alpha}{\partial x}\right)G(x).$$

For convenient of expression, we further define a super cube set Π_x and a super compact set Π_ξ as $\Pi_x \triangleq \Gamma_r^n = [-r,r]^n$ and $\Pi_\xi \triangleq \Gamma_r = [-r,r]$, respectively. The semi-global practical stability of the closed-loop system (18) is given by the following theorem.

Theorem 1: Suppose that Assumptions 1 and 2 are satisfied for the nonlinear system (2). Given any arbitrarily large number $r > 0$ and any arbitrarily small number $\varepsilon > 0$, there exist a positive scalar L and a finite time $T_f > 0$, such that, for $\forall t \geq T_f$, all the solutions of the closed-loop system (2)-(8)-(12), with initial states in $\Pi_x \times \Pi_\xi$, are captured by the set $\bar{Q}_\varepsilon^n \times \bar{Q}_\varepsilon \triangleq \{x \in \mathbb{R}^n : \|x\| \leq \varepsilon\} \times \{\zeta \in \mathbb{R} : |\zeta| \leq \varepsilon\}$ if the observer gain $h(x)$ is selected such that $h(x)G(x) > 0$ for any $x \in \mathbb{R}^n$.

Proof. The proof of the theorem is divided into two steps. For the first one, we prove that the solution of the closed-loop system is well defined on $[0,\infty)$ and furthermore the system states will enter the set Δ in finite time and never escape once enter. Then, it is proved that the trajectories of the closed-loop system will be captured by an arbitrarily small set $\bar{Q}_\varepsilon^n \times \bar{Q}_\varepsilon$.

Step 1. There exist a positive scalar L and a finite time $T > 0$ such that the trajectories of the closed-loop system (8)-(12)-(18) enter the set Δ and remain therein for all $t \geq T$.

Keeping the estimator (8) and error definition (9) in mind, it follows from $(x(0), \xi(0)) \in \Pi_x \times \Pi_\xi$ that the initial condition satisfies

$$V(x(0)) \leq C \quad \text{and} \quad |\zeta(0)| \leq b(1 + L), \qquad (20)$$

where b is a positive constant depending on r.

Now consider any solution of (18) defined on the open set $\Lambda \triangleq \{(x,\zeta) : V(x) \leq C + 1, \zeta \in \mathbb{R}\}$ with the initial condition satisfying (20). Then, we conclude that there exists a positive scalar L such that a solution of (18) is well defined on a right maximal interval $[0, T_c)$ with $T_c = \infty$. We prove the above conclusion by contradiction. Toward that end, we suppose that for $\forall L \in \mathbb{R}$, T_c were finite. As the fact that $\zeta \in \mathbb{R}$ for $\forall L \in \mathbb{R}$ and $\forall t \in [0, T_c)$ whether T_c is finite or ∞, we know $\lim_{t \to T_c} V(x(t)) = C + 1$. Then, from (10), (18) and the boundedness of $\text{SAT}_M(\cdot)$, we know that the functions $\Xi(\cdot)$ and $d_\Xi(\cdot)$ are bounded functions. Hence, there exists a positive constant η_1 (independent of L) such that, for $\forall(x,\zeta) \in \Lambda$,

$$\dot{V} \leq -\phi(x) + \left|\frac{\partial V}{\partial x}(G(x)(d_\Xi(x,\omega,\zeta) - \text{SAT}_M(d_\Xi(x,\omega,\zeta) - \zeta)))\right|$$

$$\leq \eta_1.$$

$$(21)$$

Denote $T_1 = \frac{1}{2\eta_1}$. From (20) and (21), we know that

$$V(x(t)) \leq V(x(0)) + \eta_1 t \leq C + \frac{1}{2}, \quad \forall t \in [0, T_1], \qquad (22)$$

which implies that $T_c > T_1$. It follows from Assumption 1 that, for $\forall(x,\zeta) \in \Lambda$,

$$1 + \frac{1}{d_0}\frac{\partial d}{\partial u}\frac{\partial \text{SAT}_M}{\partial s} \geq 1 - \left|\frac{d_1(x) - d_0}{d_0}\right| \geq 1 - \delta_0. \qquad (23)$$

Therefore, by Lemma 1 and (18), one obtains that there are three positive real numbers η_0, η_2 and η_3 (independent of L) such that

$$\frac{d\zeta^2}{dt} \leq -2(L - \eta_2)\eta_0\zeta^2 + \eta_3, \quad \forall(x,\zeta) \in \Lambda. \qquad (24)$$

Then it follows from (20) and (24) that, for $\forall t \in [0, T_c)$,

$$|\zeta(t)| \leq b(1 + L)e^{-(L-\eta_2)\eta_0 t} + \sqrt{\frac{\eta_3}{2(L - \eta_2)\eta_0}}. \qquad (25)$$

To this end, there exists a positive constant L_1 such that, for $L \geq L_1$,

$$|\zeta(t)| \leq \min\left\{1, \frac{1}{2\eta_4 T_c}\right\}, \quad \forall t \in [T_1, T_c), \qquad (26)$$

where η_4 is a positive constant ensuring

$$\dot{V} \leq -\phi(x) + \eta_4|\zeta|, \quad \forall(x,\zeta,\omega) \in \Delta. \qquad (27)$$

It follows from (20), (26) and (27) that

$$V(x(t)) \leq V(x(0)) + \eta_4|\zeta|T_c \leq C + \frac{1}{2}, \quad \forall t \in [T_1, T_c), \qquad (28)$$

which is a contradiction. Thus, we can find a positive scalar L to ensure that $T_c = \infty$ and the closed-loop solution is bounded and remains in the set Λ. Then, by (24) and (25), we obtain

that there exist a positive constant L_2 and a finite time T such that, for $L \geq L_2$,

$$|\zeta(t)| \leq 1, \quad \forall t \in [T, \infty), \tag{29}$$

which implies that the trajectories of the closed-loop system enter the set Δ and remain therein for all $t \geq T$.

Step 2. The trajectories of the closed-loop system will finally enter an arbitrarily small set $\bar{Q}_\varepsilon^n \times \bar{Q}_\varepsilon$. By the previous argument in Step 1, we know that when $L \geq L_2$, the trajectories of the closed-loop system enter the set Δ and remain therein for all $t \geq T$. Therefore, we only need to consider the case that $(x, \zeta, \omega) \in \Delta$ for $t \geq T$ and $L \geq L_2$.

Define a Lyapunov function for the closed-loop system as $W(x, \zeta) = V(x) + U(\zeta)$ with $U(\zeta) = \frac{1}{2}\zeta^2$. For any given $\varepsilon > 0$, there exists a constant $\rho \in (0, 1)$ such that

$$\Omega_\rho \subset \bar{Q}_\varepsilon^n \times \bar{Q}_\varepsilon, \tag{30}$$

where Ω_ρ is a compact level set defined by $\Omega_\rho \triangleq \{(x, \zeta) \in \Lambda : W(x, \zeta) \leq \rho\}$. To facilitate the following analysis, we further define two sets as follows

$$S_1 = \{(x, \zeta) \in \Lambda : \rho \leq W(x, \zeta)\},$$
$$S_2 = \{(x, \zeta) : V(x) \leq C + 1, \quad U(\zeta) \leq 1\} \cap S_1.$$

Define a compact set $S_0 = S_2 \cap \{(x, \zeta) : \zeta = 0\}$. It then can be derived from the construction of S_2 and $\zeta = 0$ for $(x, \zeta) \in S_0$ that $x \neq 0$ for $(x, \zeta) \in S_0$. As such, it follows from Assumption 2 and Lemma 3 that

$$\dot{W}(x, \zeta) \leq -\phi(x) < 0, \ \forall (x, \zeta) \in S_0. \tag{31}$$

By continuity, $\dot{W}(x, \zeta)$ is negative at each point of some open set $S_0^- \supset S_0$. We then consider the compact set $\bar{S}_0 = S_2 \setminus S_0^-$. Since $\zeta \neq 0$ at each point of $\bar{S}_0$, there exist two constants $\bar{e} > \underline{e} > 0$ such that $\underline{e} \leq |\zeta| \leq \bar{e}$ for $(x, \zeta) \in \bar{S}_0$.

Then for all $(x, \zeta) \in \bar{S}_0$, $\dot{W}(x, \zeta)$ can be estimated from (18) and given by

$$\dot{W}(x, \zeta) \leq -\phi(x) + \eta_4 \bar{e} - (L - \eta_2)\underline{e}^2 + \frac{\eta_3}{2}. \tag{32}$$

By (32), one can get a constant $L_3 > 0$ such that, for $L > L^* = \max\{L_2, \ L_3\}$, $\dot{W}(x, \zeta) < 0$ at each point of S_2.

Invoking the statements in both Step 1 and Step 2, it can be concluded that there exists the time $T_f > 0$ such that, for $t \geq T_f$ and $L > L^*$, the trajectories of the closed-loop will enter the set $\Omega_\rho \subset \bar{Q}_\varepsilon^n \times \bar{Q}_\varepsilon$. □

In the following, we further establish the semi-global asymptotical stability of the closed-loop systems without the external disturbances.

Theorem 2: Suppose that Assumptions 1 and 2 are satisfied for the nonlinear system (2) without the external disturbances (e.g., $\omega = 0$), $d(x, u)$ is twice partially continuously differentiable on x, and the equilibrium $x = 0$ for $\dot{x} = f_c(x)$ is locally exponentially stable. Given any arbitrarily large number $r > 0$, there exists a positive scalar L such that all the solutions of the closed-loop system (2)-(8)-(12), with initial states in $\Pi_x \times \Pi_\xi$, will converge to the origin if the observer gain $h(x)$ is selected such that $h(x)G(x) > 0$ for any $x \in \mathbb{R}^n$.

Proof. The proof of this theorem can be proved by firstly addressing the semi-global practical stability, and thereafter

the locally asymptotical stability of the closed-loop system. Note that the semi-global practical stability of the problem in question is a special case of that in Theorem 1. As such, we mainly focus on the development of the locally asymptotical stability.

Let $\sigma_1 = \max\left\{\left\|\left[x^T, \zeta^T\right]^T\right\| \Big| (x, \zeta) \in \Omega_\rho\right\}$ and $\sigma = \min\{\sigma_1, \varepsilon\}$. It follows from Assumption 2 and Lemma 4 that there exist some positive numbers b_0, k_0, and a $\mathcal{C}^1$ function $V_\sigma(x)$ satisfies

$$\frac{\partial V_\sigma}{\partial x} f_c(x) \leq -b_0 \|x\|^2, \ \left\|\frac{\partial V_\sigma}{\partial x}\right\| \leq k_0 \|x\|, \tag{33}$$

for all $x \in \bar{Q}_\sigma^n \triangleq \{x \in \mathbb{R}^n : \|x\| \leq \sigma\}$.

Since $\omega(t) = 0$, by Assumptions 1 and 2, we can conclude that $\psi_1(x, \zeta)$ is continuously differentiable and satisfies $\psi_1(0, 0) = 0$. Thus there exist positive constants $k_1 > 0$ and $k_2 > 0$ such that

$$\|\psi_1(x, \zeta)\| = \left\|\int_0^x \frac{\partial \psi_1(s, \zeta)}{\partial s} ds\right\| \leq k_1 \|x\|,$$
$$\psi_3(x, \zeta)(d_\Xi(x, \zeta) - \text{SAT}_M(d_\Xi(x, \zeta) - \zeta)) \leq k_2 |\zeta|. \tag{34}$$

for $(x, \zeta) \in \bar{Q}_\sigma^n \times \bar{Q}_\sigma$ and $\omega \in \mathcal{D}$. By inequalities (20), (33), (34) and Lemma 1, for $(x, \zeta) \in \bar{Q}_\sigma^n \times \bar{Q}_\sigma$, one can obtain the derivatives of $V_\sigma(x)$ and $U(\zeta)$ along the closed-loop system (18) as

$$\dot{V}_\sigma(x) \leq -b_0 \|x\|^2 + 2k_0 \nu_1 \|x\||\zeta|$$
$$\leq -\frac{b_0}{2}\|x\|^2 + \frac{2k_0^2 \nu_1^2}{b_0}|\zeta|^2, \tag{35}$$

and

$$\dot{U}(\zeta) \leq -L|\zeta|^2 + k_1 \|x\||\zeta| + k_2 |\zeta|^2$$
$$\leq -L|\zeta|^2 + \frac{b_0}{4}\|x\|^2 + \left(\frac{k_1^2}{b_0} + k_2\right)|\zeta|^2, \tag{36}$$

where ν_1 is a positive constant. Define a composite Lyapunov function $W_\sigma(x, \zeta) = V_\sigma(x) + U(x, \zeta)$ for the closed-loop system (18), whose derivative is obtained from (35) and (36), and given by

$$\dot{W}_\sigma(x, \zeta) \leq -\frac{b_0}{4}\|x\|^2 - \left(L - k_2 - \frac{k_1^2}{b_0} - \frac{2k_0^2 \nu_1^2}{b_0}\right)|\zeta|^2,$$

for $(x, \zeta) \in \bar{Q}_\sigma^n \times \bar{Q}_\sigma$. Clearly, by selecting $L > \max\left\{L^*, (k_2 b_0 + k_1^2 + 2k_0^2 \nu_1^2)/b_0\right\}$, the locally asymptotical stability of the closed-loop system is guaranteed in the local region $(x, \zeta) \in \bar{Q}_\sigma^n \times \bar{Q}_\sigma$.

To conclude, in the presence of nonlinear uncertainties, the proposed control approach renders the asymptotical stability of the closed-loop system with a basin of attraction that could be arbitrarily large. □

Remark 3: It is worthy noting that the asymptotic stability for the nonlinear systems with constant disturbances in Theorem 2 can also be achieved in a similar way, where the detailed proof is omitted for space.
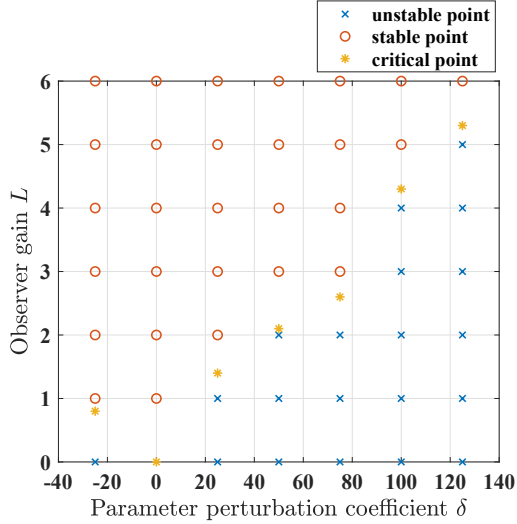
Fig. 1. Diagram of stability region in terms of the observer gain factor L and parameter perturbation coefficient δ.

IV. A SINGLE-LINK MANIPULATOR EXAMPLE

To validate the statements claimed previously, the control problem for a single-link manipulator systems is considered in this section. The dynamic model is described, which is given by [22]

$$
\begin{aligned}
\mathcal{M}\ddot{q} + \mathcal{B}\dot{q} + \mathcal{N}\sin(q) &= I \\
\mathcal{D}\dot{I} + \mathcal{H}I &= V - k_m\dot{q},
\end{aligned}
\tag{37}
$$

where q, $\dot{q}$ and $\ddot{q}$ denote the link angular position, velocity, and acceleration, respectively. I denotes the motor armature current, and V is the voltage. $\mathcal{D} = \mathcal{D}_0 + \delta\mathcal{D}_0$ is the uncertain parameter, where $\mathcal{D}_0$ denotes the nominal value of $\mathcal{D}$ and δ is a parameter perturbation coefficient sketching the uncertainty of parameter $\mathcal{D}$. The value of parameters are given as $\mathcal{M} = 1, \mathcal{D}_0 = 0.5, \mathcal{B} = 1, k_m = 10, \mathcal{H} = 0.5$, and $\mathcal{N} = 10$. Let $x_1 = q$, $x_2 = \dot{q}$, $x_3 = I$, and $u = V$. A single-link manipulator system is expressed as

$$
\begin{aligned}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{1}{\mathcal{M}}x_3 + f_2(x), \\
\dot{x}_3 &= \frac{1}{\mathcal{D}_0}u + f_3(x,u),
\end{aligned}
\tag{38}
$$

where $f_2(x) = -\frac{\mathcal{N}}{\mathcal{M}}\sin(x_1) - \frac{\mathcal{B}}{\mathcal{M}}x_2$, $f_3(x,u) = -\frac{k_m}{\mathcal{D}_0}x_2 - \frac{\mathcal{H}}{\mathcal{D}_0}x_3 - \frac{\mathcal{D}-\mathcal{D}_0}{\mathcal{D}_0}\left(\frac{1}{\mathcal{D}}u - \frac{k_m}{\mathcal{D}}x_2 - \frac{\mathcal{H}}{\mathcal{D}}x_3\right)$.

To this end, the lumped uncertainty of the system (38) is collected and given by

$$
d(x,u) = -\frac{\mathcal{D}-\mathcal{D}_0}{\mathcal{D}_0}\left(\frac{1}{\mathcal{D}}u - \frac{k_m}{\mathcal{D}}x_2 - \frac{\mathcal{H}}{\mathcal{D}}x_3\right).
$$

Assume that the measurable states of the system with measurement noises are expressed as:

$$
\begin{aligned}
y_1 &= x_1 + \varsigma_1, \\
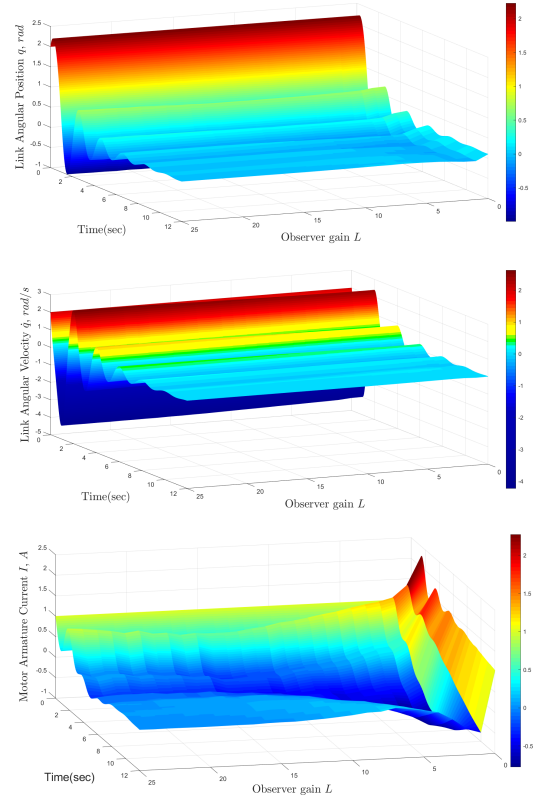y_2 &= x_2 + \varsigma_2, \\
y_3 &= x_3 + \varsigma_3,
\end{aligned}
$$

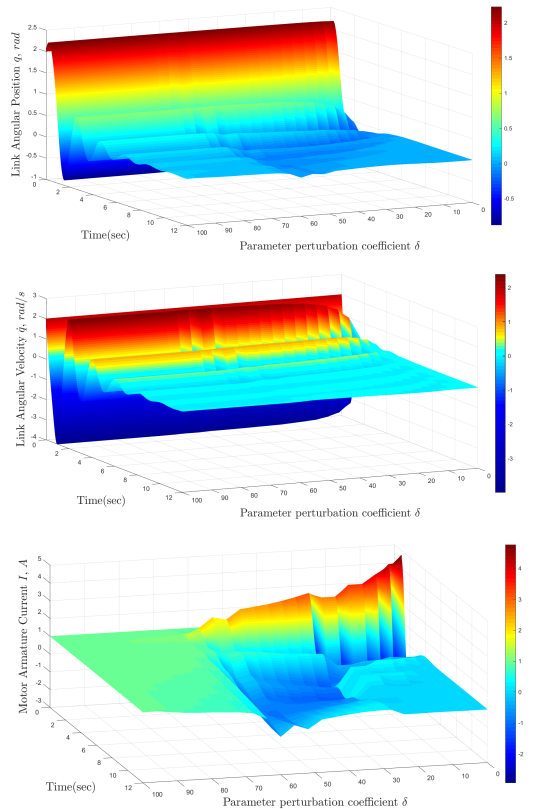Fig. 2. System state x for different observer gain factor L.

Fig. 3. System state x for different parameter perturbation coefficient δ.

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where $\varsigma_i(i = 1, 2, 3)$ are measurement noises and obey norm distribution in 3σ rules $\left(\text{i.e. } \varsigma_i \sim N\left(0, \frac{1}{9}\right)\right)$. By applying the proposed control method in this paper, the disturbance estimation and the control law are designed as

$$\dot{\xi} = -L\left(-\frac{k_m}{\mathcal{D}_0}y_2 - \frac{\mathcal{H}}{\mathcal{D}_0}y_3 + \frac{1}{\mathcal{D}_0}u + \hat{d}\right),$$
$$u = \mathcal{D}_0\left(-k_1 y_1 - k_2 y_2 - k_3 y_3 - \text{SAT}_M(\hat{d})\right), \tag{39}$$

where k_1, k_2 and k_3 are the feedback control gains, and $\hat{d}$ is the estimate of the lumped uncertainty generated by the nonlinearity estimator (8).

For simulation studies, the controller gains in (39) are chosen as $k_1 = 2$, $k_2 = 1.5$, $k_3 = 1$. The initial state values are taken as $x = [2, 2, 1]^T$. Since this paper mainly focuses on uncertainty estimation and compensation, we assume that there is no external disturbance during the simulation. The threshold of the saturation function is designed as $M = 50$.

To evaluate the qualitative robustness of the propose method, we focus on investigating the relationship between the observer gain factor L and parameter perturbation coefficient δ on the stability of the system (37). In the simulation, δ is supposed to range from -25 to 125, whereas the observer gain L is assumed to range from 0 to 6. The diagram of stability region of the system in terms of the observer gain factor L and parameter perturbation coefficient δ is shown in Fig. 1. It has been clearly shown by Fig. 1 that the observer gain factor L should be substantially increased to render stability as the increase of uncertainty parameter perturbation coefficient δ. We further conduct two cases of simulations to qualitatively demonstrate the robust stability of the presented control approach. For the first one, the parameter perturbation coefficient δ is fixed as $\delta = 120$, and the tracking control performance under different observer gain is investigated. The results in this case are shown in Fig. 2, which shows that the closed-loop system tends to be unstable as the observer gain L decreases to about 5. For the second one, the robust stability of the controller with fixed observer gain $L = 2$ is investigated, where the result is shown by Fig. 3. It can be observed from the figure that the stability as well as control performance is getting worse with the increase of the parameter perturbation coefficient δ.

V. CONCLUSION

A nonlinearity estimator-based control approach has been proposed in this paper to address the robust control problem of a class of nonlinear systems subject to both disturbances and uncertainties. The qualitative robustness stability has been established for the closed-loop system in the sense of semi-global practical and asymptotical stabilization. The results obtained have provided theoretical justifications for DUEA approaches to estimate and compensate nonlinearities for nonlinear systems in practice. The simulation results on a single-link manipulator have demonstrated the qualitative relationship between observer gain factor and quantity of nonlinearities.

APPENDIX: PROOF OF LEMMA 2

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