ISVCS11, 12th International Symposium on Vibrations of Continuous Systems
Sporthotel Panorama, Str. Sciuz, 1, 39033 Corvara In Badia BZ – Italy, July 28 - August 2, 2019

A Pinned-Pinned Beam with and without a Distributed Foundation: A simple exact relationship between their eigenvalues

William P. Howson* and **Andrew Watson#**

* Independent Consultant
Gwanwyn, Craig Penlline, CF71 7RT, UK
w.p.howson@gmail.com

Dept. of Aero and Auto Engineering
Loughborough University
Loughborough, LE11 3TU, UK
a.watson@lboro.ac.uk

Introduction

The body of this paper considers a pinned-pinned Bernoulli-Euler beam, from which the core natural frequencies and critical buckling loads corresponding to in-plane flexure, can be determined easily. The theory is then developed to yield an exact relationship between the static axial load in the beam and the frequency of vibration. This enables the core eigenvalues to be related exactly to their counterparts when the beam is additionally supported on a two parameter elastic foundation. The relationship is simple, exact and obviates the complex problems involved in solving the foundation problem using more traditional techniques. A number of illustrative problems are solved to confirm the accuracy and efficacy of the approach.

Theory

Consider first the exact, fourth order differential equation governing the harmonic motion of an axially loaded Bernoulli-Euler beam of length, L, that is supported on a two parameter, distributed foundation, whose transverse and rotational restraining stiffnesses per unit length are k_y and k_θ, respectively. The resulting equation is well known, can be deduced easily from Howson and Watson [1] and can be written in the following non-dimensional form

$$[D^4 + p^{*2}D^2 - b^{*2}]V = 0 \tag{1}$$

where $D = d/d\xi$, $\xi = x/L$ is the non-dimensional length parameter and V is the amplitude of the transverse displacement

$$p^{*2} = p^2 - k_\theta^{*2} b^{*2} = b^2 - k_y^{*2} \tag{2}$$

$$p^2 = PL^2/EIk_\theta^* = k_\theta L^2/EIb^2 = \rho AL^4\omega^2/EIk_y^* = k_y L^4/EI \tag{3}$$

ρ and E are the density and Young's modulus of the member material respectively, A and I are the area and second moment of area of the cross-section, ω is the radian frequency of vibration and P is the static axial load in the member, which is positive for compression, zero, or negative for tension. Equations (2) and (3) establish the non-dimensional member parameters p^2 and b^2, which uniquely define the member effects of static axial load and frequency, respectively [2,3], together with p^{*2} and b^{*2} which define their interaction with the non-dimensional foundation parameters.

Imposing pinned-pinned boundary conditions enables Equation (1) to be solved by assuming a general solution of the form

$$V = C\sin(i\pi\xi) \qquad m = 1, 2, \ldots, \infty \tag{4}$$

where C is an arbitrary constant, V defines the modal (displaced) shape, which also satisfies the boundary conditions. Substituting for V in Equation (1) then yields

$$(i\pi)^4 - p^{*2}(i\pi)^2 - b^{*2} = 0 \tag{5}$$

or

$$b^{*2}/(i\pi)^4 + p^{*2}/(i\pi)^2 = 1 \tag{6}$$

It is now helpful to introduce the notion of ‘member environment’ which, for the remainder of this paper, will be defined as follows. An environment will relate to either vibration or buckling and can be established by allocating constant values to the appropriate independent parameters in Equation (6). The core vibration environment will be defined by $p^2 = k_y^* = k_\theta^* = 0$ and will yield the classical natural frequency parameters

$$b_{c,i} = (i\pi)^2 \qquad i = 1,2,\ldots,\infty \tag{7}$$

In similar fashion, the core buckling environment will be defined by $b^2 = k_y^* = k_\theta^* = 0$ and will yield the classical buckling parameters

$$p_{c,i} = (i\pi) \qquad i = 1,2,\ldots,\infty \tag{8}$$

and hence that

$$b_{c,i} = p_{c,i}^2 \qquad i = 1,2,\ldots,\infty \tag{9}$$

A further result of this is to enable Equation (6) to be written as

$$b^{*2}/b_{c,i}^2 + p^{*2}/p_{c,i}^2 = 1 \tag{10}$$

It is interesting to note in passing that solutions to Equation (10) will lie on the arc of an ellipse when b^{*2} and p^{*2} are both positive and on the arc of the adjoining hyperbola when they are of opposite sign. Simpler solutions prevail, of course, when one or other of them is zero. Equation (10) can now be used to model a range of vibration or buckling problems in which any appropriate combination of the non-dimensional effects can be neglected by setting the relevant parameter to zero.

Discussion and numerical examples

The remainder of this paper now seeks to highlight aspects of Equation (10) while demonstrating its simplicity and effectiveness when applied to practical structures. This is best achieved by expanding it out in symbolic form to its most general vibration and buckling environments, as given in Equation (11), respectively, i.e.

$$b_i^2 = b_{c,i}^2[1 - (p^2/p_{c,i}^2)] + [(b_{c,i}^2/p_{c,i}^2)k_\theta^* + k_y^*] \qquad i = 1,2,\ldots,\infty \tag{11}$$

and hence that

$$p_i^2 = p_{c,i}^2[1 - (b^2/b_{c,i}^2)] + [(p_{c,i}^2/b_{c,i}^2)k_y^* + k_\theta^*] \qquad i = 1,2,\ldots,\infty \tag{12}$$

where the subscript i has now been introduced on the dependent variable to denote modal rank, since there will be an infinite number of solutions for each new environment created.

Consider first the asymmetric relationship between Equations (11), which can be put into context as follows. Assume a vibration environment in which $k_y = k_\theta = 0$ and $p^2 = 0.4p_{c,1}^2$. Then from

Table 1: Relationship given by Equations (11) between the core eigenvalues and their counterparts in the required environment.

	Environment			Modal Rank	Core Eigenvalues		Solution
	k_y^*	k_θ^*	p^2	i	$p_{c,i}^2$	$b_{c,i}^2$	p_i^2
Vibration	80	0	0	1	9.86960	97.4091	177.409
	80	0	0	3	88.8264	7890.14	7970.14
	0	50	-1	2	39.4784	1558.55	3571.94
	80	50	-1	1	9.86960	97.4091	680.759
Buckling	80	0	0	1	9.86960	97.4091	17.9753
	0	50	0	2	39.4784	1558.55	89.4784
	80	50	0	4	157.914	24936.7	208.420

Equation (11) the frequency of vibration that would reduce the member stiffness to zero would correspond to $b_1^2 = 0.6b_{c,1}^2$. A similar buckling environment could be written as $k_y = k_\theta = 0$ and $b^2 = 0.6b_{c,1}^2$ then from Equation (12) the compressive axial load that would reduce the member stiffness to zero would correspond to $p_1^2 = 0.4p_{c,1}^2$. The same problem is thus solved both through a vibration and a buckling context. Closer inspection of Equations (11) enable a number of helpful points to be made. Firstly, it is clear that b_i^2 and p_i^2 must always be zero or positive and that the values of b^2 and p^2 shape their respective (constant) environments. Hence, when $k_y^* = k_\theta^* = 0$; $0 \leq b^2 \leq b_(c,1)^2$ and $p^2 \leq p_(c,1)^2$. When $k_y^* > 0$ and/or $k_\theta^* > 0$, the values of $b^2(\geq 0)$ and p^2 are only (additionally) constrained by the requirement that b_i^2 and p_i^2 remain positive in their respective environments. More generally it is clear that in both vibration and buckling problems, the rotational stiffness becomes more influential as the modal rank increases. The data for the remaining examples are given below so that the hand solutions developed from Equations (11) and given in Table 1 can be checked by alternative means.

$E = 2.0\times10^{11}N/m^2$ $\quad I = 1.6\times10^{-5}m^4$ $\quad \rho = 8\times10^3 kg/m^3$ $\quad A = 10^{-2}m^2$ $\quad L = 4m$
$k_y = 10^6 N/m^2$ $\quad k_\theta = 10^7 N$ $\quad P = 2\times10^5 N$ for compression and negative for tension.

The problem parameters and solutions are given in Table 1 below.

Conclusions

A simple formula that can be manipulated easily by hand and which can predict exactly the change in core eigenvalues of a simple pinned-pinned beam to their counterparts in any other allowable environment has been presented and its efficacy demonstrated.

References

[1] Howson, W.P.; Watson, A.: On the provenance of hinged-hinged frequencies in Timoshenko beam theory. Computers and Structures, Vol. 197, pp. 71–81, 2018.

[2] Howson, W.P.; Williams, F.W.: Natural frequencies of frames with axially loaded Timoshenko members. Journal of Sound and Vibration, Vol. 96, pp. 503–515, 1973.

[3] Howson, W.P.; Banerjee, J.R.; Williams, F.W.: Concise equations and program for exact eigensolutions of plane frames including member shear. Advances in Engineering Software, Vol. 96, pp. 137-141, 1983.