TYPICAL LONG TIME BEHAVIOUR OF GROUND STATE-TRANSFORMED JUMP PROCESSES

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ABSTRACT. We consider a class of Lévy-type processes derived via a Doob-transform from Lévy processes conditioned by a control function called potential. These ground state transformed-processes (also called $P(\phi)_1$-processes) have position-dependent and generally unbounded components, with stationary distributions given by the ground states of the Lévy generators perturbed by the potential. We derive precise lower and upper envelopes for the almost sure long time behaviour of these ground state-transformed Lévy processes, characterized through escape rates and integral tests. We also highlight the role of the parameters by specific examples.

Key-words: fluctuations of jump processes, $P(\phi)_1$-processes, stationary distributions, Feynman-Kac semigroups, non-local Schrödinger operators, potentials, ground states

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1. Introduction

Given a random process $(X_t)_{t\geq 0}$, a fundamental question is what is its typical sample path behaviour on the long run. This generally involves a statement of the form

$$\liminf_{t\to\infty} \frac{X_t}{\tau_1(t)} = C_1 \quad \text{and} \quad \limsup_{t\to\infty} \frac{X_t}{\tau_2(t)} = C_2, \qquad \mathbb{P}-\text{a.s.}, \tag{1.1}$$

where τ_1, τ_2 are positive functions on the positive semi-axis, C_1, C_2 are finite non-zero constants, and $\mathbb{P}$ is the probability measure of the process. The functions τ_1, τ_2 provide lower and upper almost sure envelopes, and thus give a characterization of the time-scale on which the process typically evolves in the long time limit.

In the present paper our aim is to consider this problem for a large class of Lévy-type jump processes obtained from Lévy processes conditioned by Kato-class potentials, assuming that the so obtained processes have a stationary distribution. Such processes arise from the Feynman-Kac representation of non-local Schrödinger operators of the form $H = -L + V$, where L is the L^2-generator of a Lévy process $(X_t)_{t\geq 0}$ on a suitable probability space, and V is a multiplication operator called potential. This representation reads

$$\left(e^{-tH} f\right)(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s)ds} f(X_t)], \quad f \in L^2(\mathbb{R}^d),\ x \in \mathbb{R}^d,\ t \geq 0, \tag{1.2}$$

where the expectation is taken with respect to the probability measure of the process $(X_t)_{t\geq 0}$. Since the semigroup defined by the right hand side is not measure preserving, using the ground state (i.e., eigenfunction at the bottom of the spectrum) φ_0 of H one can change the space $L^2(\mathbb{R}^d)$ to the weighted Hilbert space $L^2(\mathbb{R}^d, \varphi_0^2 dx)$ on which the correspondingly

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transformed semigroup becomes a Markov semigroup, and thus by a change of measure the right hand side in (1.2) turns into an expectation with respect to a random process $(\widetilde{X}_t)_{t\geq 0}$ derived from $(X_t)_{t\geq 0}$ (for further details see Section 2 below). We call such processes *ground state-transformed (GST) processes* (also known as $P(\phi)_1$*-processes*), which are thus a case of Doob h-transformed processes, where the function h is φ_0. The properties of such a process will then be relevant in a probabilistic study of the semigroup $\{e^{-tH} : t \geq 0\}$.

The ground state-transformed processes $(\widetilde{X}_t)_{t\geq 0}$ make a class of independent interest, even when the above relevance is ignored. The generator of $(\widetilde{X}_t)_{t\geq 0}$ is

$$
\begin{aligned}
(\widetilde{H}f)(x) \;=\; & -\frac{1}{2}\sigma\nabla\cdot\sigma\nabla f(x) - \sigma\nabla\ln\varphi_0(x)\cdot\sigma\nabla f(x) \\
& - \int_{0<|z|\leq 1} \frac{\varphi_0(x+z)-\varphi_0(x)}{\varphi_0(x)} z\cdot\nabla f(x)\nu(z)dz \\
& - \int_{\mathbb{R}^d\setminus\{0\}} \big(f(x+z)-f(x)-z\cdot\nabla f(x)1_{\{|z|\leq 1\}}\big)\frac{\varphi_0(x+z)}{\varphi_0(x)}\nu(z)dz,
\end{aligned}
\tag{1.3}
$$

where ν is the Lévy intensity and $A = \sigma\sigma^T$ is the diffusion matrix of $(X_t)_{t\geq 0}$, and where we use the notation $\sigma\nabla\cdot\sigma\nabla f(x) = \sum_{i,j=1}^d (\sigma\sigma^T)_{ij}\partial_{x_i}\partial_{x_j} f(x)$. Under suitable conditions (see a discussion in [29]), the GST process satisfies a stochastic differential equation with jumps of the form

$$
\begin{aligned}
\widetilde{X}_t \;=\; & \widetilde{X}_0 + \sigma B_t + \int_0^t \sigma\nabla\ln\varphi_0(\widetilde{X}_s)\,ds + \int_0^t\int_{|z|\leq 1}\frac{\varphi_0(\widetilde{X}_s+z)-\varphi_0(\widetilde{X}_s)}{\varphi_0(\widetilde{X}_s)} z\nu(z)dzds \\
& + \int_0^t\int_{|z|\leq 1}\int_0^\infty z 1_{\left\{v\leq\frac{\varphi_0(\widetilde{X}_{s-}+z)}{\varphi_0(\widetilde{X}_{s-})}\right\}}\widetilde{N}(ds,dz,dv) \\
& + \int_0^t\int_{|z|>1}\int_0^\infty z 1_{\left\{v\leq\frac{\varphi_0(\widetilde{X}_{s-}+z)}{\varphi_0(\widetilde{X}_{s-})}\right\}} N(ds,dz,dv),
\end{aligned}
\tag{1.4}
$$

where $(B_t)_{t\geq 0}$ is standard Brownian motion, N is a Poisson random measure on $[0,\infty)\times\mathbb{R}^d\times[0,\infty)$ with intensity $dt\nu(z)dzdv$, and $\widetilde{N}$ is the related compensated Poisson measure.

From the above two observations it is seen that the potential V perturbing the Lévy process enters the GST process via the ground state φ_0 of the operator H, and in general gives rise to a position-dependent drift and a position-dependent bias in the jump kernel, i.e., a Lévy-type process. Such processes are currently much researched on various levels of generality [39, 37, 26]. Our focus on GST processes has the advantage that they have a definite structure while being a rich class, and the analysis depends on the properties of a control function V through φ_0. Also, from the expression in terms of the SDE above we note that GST processes have unbounded coefficients, while most results on Lévy-type processes have been established so far for bounded coefficients only (i.e., for cases when the symbol of the generator is uniformly bounded with respect to the position x in space). Our goal in this paper is to describe the profile function τ and the constant C in function of the properties of L and V.

The long term behaviour for the free processes, i.e., when the potential $V\equiv 0$, is described by classic results. When $(X_t)_{t\geq 0}$ is an $\mathbb{R}^d$-valued Brownian motion, Khinchin's law of iterated logarithm (LIL) says [24] that the common envelope is described by

$$\tau(t) = \sqrt{2t \log\log t} \quad \text{and} \quad C = 1.$$

There is an abundant literature on related results (e.g., the running maximum, characterization of limit points, local times, other functionals of Brownian motion and random walk, large deviations, and similar problems on the typical short time behaviour, etc), for some standard summaries see, e.g., [8, 35, 13].

This behaviour becomes very different in the case of heavy-tailed purely jump processes. Khinchin has also shown [25] (see important improvements in [14, 6]) that for non-Gaussian stable processes no similar LIL holds in a very severe sense. If $(X_t)_{t\geq 0}$ is an isotropic α-stable process with $0 < \alpha < 2$, then C is either zero or infinite for any positive increasing function τ on the positive semi-axis, according to whether $\int_1^\infty \tau(t)^{-\alpha} dt$ is finite or infinite. In contrast, for a real-valued Lévy process $(X_t)_{t\geq 0}$ having a finite variance and zero mean Gnedenko proved [17] that τ is the same as for Brownian motion and $C = \sqrt{\operatorname{var} X_1}$. For processes which are spectrally one-sided or contain stable components etc, see [46, 6, 33], and a standard modern summary is [38]. For more recent results using Dirichlet forms see [42]. We note that there is a large literature on short time LIL-type behaviour of Lévy jump processes, however, the $t \downarrow 0$ limit is beyond the scope of our paper.

Loosely speaking, the above results indicate that for a symmetric process the structure of the almost sure long time profile τ is determined by the standard deviation and a small margin given by a slowly varying correction factor. This margin can be further refined by integral tests. Recall that τ is said to be in the upper resp. lower class at infinity with respect to $(X_t)_{t\geq 0}$ whenever $\mathbb{P}(X_t < \tau(t) : \text{as } t \to \infty)$ is 1 or 0. For Brownian motion, the so called Kolmogorov-Petrovsky integral test says [32] that if g is a positive increasing function, then

$$\mathbb{P}\left(|B_t| \leq \sqrt{t} g(t) : \text{as } t \to \infty\right) = 0 \text{ or } 1$$

according to $\int_1^\infty \frac{g^d(t)}{t} e^{-\frac{g^2(t)}{2}} dt$ being finite or infinite. Also, the Dvoretzky-Erdős integral test says [12] that if h is a positive function, decreasing to zero, and $d \geq 3$, then

$$\mathbb{P}\left(|B_t| \geq \sqrt{t} h(t) : \text{as } t \to \infty\right) = 0 \text{ or } 1$$

as $\int_1^\infty \frac{h^{d-2}(t)}{t} dt$ is finite or infinite. In particular, it follows that for some $n \in \mathbb{N}$ and $d = 3$,

$$\tau(t) = \sqrt{2t\left(\log_2 t + \frac{3}{2}\log_3 t + \log_4 t + ... + \log_{n-1} t + (1+\varepsilon)\log_n t\right)}$$

where $\log_n$ means n-fold iterated logarithm, is in the upper or lower class at infinity, if ε positive or negative, respectively. For further integral tests related to Brownian motion and some jump processes we refer to [45, 44, 23].

The problem of long time behaviour has also been addressed for diffusions. In the works [1, 2] conditions have been obtained for diffusions defined by stochastic differential equations such that the solutions continue to obey a LIL behaviour; see also the classic paper by Motoo [31], and [30] and the references therein. For GST processes obtained by conditioning Brownian motion, Rosen and Simon [36] considered polynomial potentials increasing to infinity at infinity and diffusions generated by the Schrödinger operator $-\frac{1}{2}\Delta + V$. They showed that if the degree of this polynomial is $2m \geq 2$, and the coefficient of the leading term is $a_{2m} > 0$, then the a.s. long-time profile of the GST process (called by the authors $P(\phi)_1$-process) is

$$\tau(t) = (\log t)^{\frac{1}{m+1}} \quad \text{and} \quad C = 1/a_{2m}^{2(m+1)}. \tag{1.5}$$

In [5], more generally, Kato-class potentials V were considered to study the support of Gibbs measures on Brownian paths, see also [28]. Here it is shown that whenever the Schrödinger operator has a ground state $\varphi_0 \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and a spectral gap Λ, then the profile function of the so obtained two-sided diffusion is determined by the condition

$$\frac{e^{-\Lambda|t|}}{\varphi_0(X_t)} \to 0 \quad \text{as} \quad |t| \to \infty,$$

from which explicit expressions can be derived for specific (classes of) examples. While this result has the advantage to deal with a large class of potentials, it overestimates τ to large or small degrees dependent on V.

Long time behaviour for ground state-transformed jump Lévy processes has been explored only for isotropic stable processes so far, in the context of the fractional Laplacian $(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, see [18]. In this paper we go far beyond this class. Our main results are as follows. First we present an integral test for GST processes derived from a general underlying Lévy process conditioned by a general Kato-class potential (Theorem 3.1 and Corollary 3.1 below). This will be achieved in terms of a functional directly featuring the ground state (escape rates), to which we will be able to use the detailed information on their decay/concentration properties recently obtained in [19, 20]. Next we restrict to a subclass of jump processes for which multiple large jumps are dominated by single large jumps (which we call jump-paring Lévy processes), and split the discussion to confining potentials (V increasing to infinity at infinity) and decaying potentials (V decreasing to zero at infinity), allowing us to get sharp characterisations of the time evolution envelopes. For confining potentials we present an integral test in Theorem 4.1 and its implication on the long time behaviour in Corollary 4.1, and a similar pair of results for decaying potentials in Theorem 4.4 and Corollary 4.2. We refine even further by assuming regular variation in Theorems 4.2-4.3 in the case of confining potentials, and slow variation in Corollaries 4.3-4.4 in the case of decaying potentials. We also prove the intuition that a faster decaying potential should imply tighter long time evolution profiles (Theorem 3.2), and illustrate all these results by specific examples (Section 4.4) highlighting the interplay of the Lévy intensity and the potential in determining the growth of paths.

2. **The underlying and the ground state-transformed processes**

2.1. **Symmetric jump-paring Lévy processes**

Let $(X_t)_{t\geq 0}$ be a symmetric, $\mathbb{R}^d$-valued, $d \geq 1$, Lévy process on a suitable probability space. We use the notations $\mathbb{P}^x$ and $\mathbb{E}^x$ for the probability measure and expected value of the process starting in $x \in \mathbb{R}^d$, respectively. The process $(X_t)_{t\geq 0}$ is determined by the characteristic function

$$\mathbb{E}^0\left[e^{i\xi\cdot X_t}\right] = e^{-t\psi(\xi)}, \quad \xi \in \mathbb{R}^d,\ t > 0,$$

with exponent given by the Lévy-Khintchin formula

$$\psi(\xi) = A\xi\cdot\xi + \int_{\mathbb{R}^d} (1-\cos(\xi\cdot z))\nu(dz). \tag{2.1}$$

Here A is a symmetric non-negative definite $d\times d$ matrix, and ν is a symmetric Lévy measure on $\mathbb{R}^d\setminus\{0\}$, i.e., $\int_{\mathbb{R}^d}(1\wedge|z|^2)\nu(dz) < \infty$ and $\nu(E) = \nu(-E)$, for all measurable $E \subset \mathbb{R}^d\setminus\{0\}$, thus the Lévy triplet of the process is $(0, A, \nu)$. We assume throughout that the Lévy measure is an infinite measure and it is absolutely continuous with respect to Lebesgue measure with density (Lévy intensity) $\nu(x) > 0$, i.e.,

$$\nu(\mathbb{R}^d\setminus\{0\}) = \infty \quad \text{and} \quad \nu(dx) = \nu(x)dx. \tag{2.2}$$

When $A \equiv 0$ and $\nu \not\equiv 0$, the Lévy process $(X_t)_{t\geq 0}$ is a purely jump process, when $\nu \equiv 0$ and and $A \not\equiv 0$, it is purely continuous. Recall that $(X_t)_{t\geq 0}$ is a Markov process with respect to its natural filtration, satisfying the strong Markov property and having càdlàg paths. Moreover, under (2.2) the process has the strong Feller property, i.e., its transition semigroup satisfies $P_t(L^\infty(\mathbb{R}^d)) \subset C_{\rm b}(\mathbb{R}^d)$, for all $t > 0$. Equivalently, the one-dimensional distributions of $(X_t)_{t\geq 0}$ are absolutely continuous with respect to Lebesgue measure, i.e., there exist the transition probability densities $p(t,x,y) = p(t, y-x, 0) =: p(t, y-x)$. Its infinitesimal generator L is uniquely determined by its Fourier symbol

$$\widehat{Lf}(\xi) = -\psi(\xi)\widehat{f}(\xi), \quad \xi \in \mathbb{R}^d,\ f \in \operatorname{Dom} L, \tag{2.3}$$

with domain $\operatorname{Dom} L = \left\{ f \in L^2(\mathbb{R}^d) : \psi\widehat{f} \in L^2(\mathbb{R}^d)\right\}$. It is a negative non-local self-adjoint operator such that

$$Lf(x) = \sum_{i,j=1}^d a_{ij}\frac{\partial^2 f}{\partial x_j \partial x_i}(x) + \int \big(f(x+z) - f(x) - \mathbf{1}_{B(0,1)}(z) z\cdot\nabla f(x)\big)\, \nu(z)dz, \quad x \in \mathbb{R}^d,$$

for $f \in C_0^\infty(\mathbb{R}^d)$. For more details on Lévy processes we refer to [38, 4].

In what follows we will also consider a more restricted class of symmetric Lévy processes defined by a condition on the large jumps. Recall the following standard notations. For given functions f, g the notation $f \asymp Cg$ means that $C^{-1}g \leq f \leq Cg$ with a constant C, and $f \asymp g$ means that there is a constant C such that this relation holds. Also, we write $f \approx g$ when $\lim_{r\to\infty} f(r)/g(r) = 1$. The constants will be assumed to be dependent on the dimension d by default, while dependence of C on the process $(X_t)_{t\geq 0}$ will be indicated by $C(X)$.

Assumption 2.1. *The following conditions hold:*

1. *There exist a non-increasing function $f : (0,\infty) \to (0,\infty)$ and constants $C_1, C_2, C_3 > 0$ such that*

$$C_1 f(|x|) \leq \nu(x) \leq C_2 f(|x|), \quad x \neq 0, \tag{2.4}$$

and

$$\int_{|y|>1/2,\,|x-y|>1/2} f(|x-y|)f(|y|)dy \leq C_3\, f(|x|), \quad |x| \geq 1. \tag{2.5}$$

(2) *There exist* $t_{\rm b} > 0$ *and* $C_4 = C_4(X, t_{\rm b})$ *such that* $0 < p(t_{\rm b}, x) \leq C_4$*, for all* $x \in \mathbb{R}^d$*.*
(3) *For all* $0 < p < q < R \leq 1$ *we have* $\sup_{x\in B(0,p)} \sup_{y \in B_q(0)^c} G_{B_R(0)}(x, y) < \infty$*, where* $G_{B_R(0)}(x, y) = \int_0^\infty p_{B_R(0)}(t, x, y)dt$ *denotes the Green function of the process* $(X_t)_{t\geq 0}$ *in the ball* $B_R(0)$*.*

We refer to the class of Lévy processes satisfying Assumption 2.1 as *symmetric jump-paring Lévy processes*, and to condition (2.5) as the *jump-paring property*. It means that double (and by iteration, all multiple) large jumps are stochastically dominated by single large jumps. This condition has been introduced in [19], for its further uses see also [22, 20].

Example 2.1. The jump-paring class has a non-trivial overlap with subordinate Brownian motions in the sense that neither contains the other class. Some landmark examples include:

(1) isotropic α-stable processes, generated by $L = (-\Delta)^{\alpha/2}$, $0 < \alpha < 2$
(2) isotropic relativistic α-stable processes, generated by $L = (-\Delta + m^{2/\alpha})^{\alpha/2} - m$, $0 < \alpha < 2$, $m > 0$
(3) isotropic geometric α-stable processes, generated by $L = \log(1+(-\Delta)^{\alpha/2})$, $0 < \alpha < 2$
(4) jump-diffusion processes obtained as the sum of a mutually independent Brownian motion and an isotropic α-stable process, generated by $L = -a\Delta + b(-\Delta)^{\alpha/2}$, $0 < \alpha < 2$, $a, b > 0$.

In contrast, the variance gamma process corresponding to an $\alpha = 2$ geometric stable process does not belong to the jump-paring class. For a more detailed discussion of special cases and examples we refer to [19].

The restricted class of processes given by Assumption 2.1 will be used only in Section 4 below. For the remainder of this section $(X_t)_{t\geq 0}$ denotes a general symmetric Lévy process corresponding to the Lévy-Khintchin exponent (2.1).

2.2. Ground state-transformed processes

2.2.1. **Potentials and Feynman-Kac semigroup.** Below we will consider Lévy processes conditioned by appropriate potentials. Recall that a Borel measurable function $V : \mathbb{R}^d \to \mathbb{R}$ is an *X-Kato class potential* whenever for its positive and negative parts

$$V_- \in \mathcal{K}^X \quad \text{and} \quad V_+ 1_C \in \mathcal{K}^X \text{ for every compact subset } C \subset \mathbb{R}^d, \tag{2.6}$$

holds, where $h \in \mathcal{K}^X$ means that

$$\lim_{t\downarrow 0} \sup_{x\in\mathbb{R}^d} \mathbb{E}^x \left[\int_0^t |h(X_s)|ds \right] = 0. \tag{2.7}$$

By an extension of Khasminskii's lemma [28, Lem. 3.37] to X-Kato potentials, it follows that the random variables $\int_0^t V(X_s)ds$ are exponentially integrable for all $t \geq 0$, and thus we can define the Feynman-Kac semigroup

$$T_t f(x) = \mathbb{E}^x \left[e^{-\int_0^t V(X_s)ds} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d),\ t \geq 0,\ x \in \mathbb{R}^d. \tag{2.8}$$

Using the Markov property and stochastic continuity of the process $(X_t)_{t\geq 0}$ it can be shown that $\{T_t : t \geq 0\}$ is a strongly continuous one-parameter semigroup of symmetric operators on $L^2(\mathbb{R}^d)$. Moreover, by the Hille-Yoshida theorem there exists a self-adjoint operator H bounded from below such that $e^{-tH} = T_t$. The generator can be identified as the non-local Schrödinger operator $H = -L + V$ defined as a form sum, where L is the infinitesimal generator of the Lévy process $(X_t)_{t\geq 0}$.

The following will be a basic standing assumption for the whole paper.

Assumption 2.2. *The potential V is in X-Kato class, chosen such that $\lambda_0 := \inf \operatorname{Spec} H \in \mathbb{R}$ is an isolated eigenvalue of H.*

We denote the corresponding eigenfunction (called *ground state*) by φ_0, i.e.,

$$H\varphi_0 = \lambda_0 \varphi_0, \quad \varphi_0 \not\equiv 0, \quad \varphi_0 \in \operatorname{Dom} H \subset L^2(\mathbb{R}^d)$$

holds. By standard arguments [34, Th.XIII.43], [28, Sect. 3.4.3] it follows that φ_0 is unique and has a strictly positive version, which we will use throughout below.

Both from the perspective of existence of a ground state and for the purposes of the discussion below, it is useful to single out two large classes of potentials.

Example 2.2 (**Confining potentials**)**.** A potential V is *confining* if $V(x) \to \infty$ as $|x| \to \infty$. In this case $\operatorname{Spec} H$ is purely discrete, and a (unique) ground state φ_0 exists. Some examples include:

(1) *Harmonic and anharmonic oscillators:* Let $V(x) = |x|^{2n}$, $n \in \mathbb{N}$. The case $n = 1$ describes the potential of the harmonic oscillator, and $n \geq 2$ give anharmonic oscillators.

(2) *Double and multiple well potentials:* The potential $V(x) = |x|^4 - b|x|^2$, $b > 0$, is a symmetric double well potential. Multiple well potentials can be obtained by higher order polynomials.

Example 2.3 (**Decaying potentials**)**.** A potential V is *decaying* if $V(x) \to 0$ as $|x| \to \infty$. In this case $\operatorname{Spec} H$ contains the essential spectrum $\operatorname{Spec}_{\text{ess}} H = \operatorname{Spec}_{\text{ess}} L = [0, \infty)$, and whether it also contains a non-empty discrete component depends on further details of V. Some decaying X-Kato class potentials of special interest in mathematical physics are:

(1) *Potential wells:* Let $V(x) = -v(x)$ with a compactly supported, non-negative bounded Borel function $v \not\equiv 0$. Specifically, we can choose $V(x) = -a\mathbf{1}_{B(0,1)}(bx)$, for $a, b > 0$.

(2) *Coulomb-type potentials:* Let f in Assumption 2.1 be such that $f(r) = r^{-d-\alpha}$, $r \in (0, 1]$, for some $\alpha \in (0, 2)$, and let $V(x) = -(a_1|x|^{-\beta_1} \wedge a_2|x|^{-\beta_2})$, with $\beta_1 \in (0, \alpha \wedge d]$, $\beta_2 \in [\beta_1, \infty)$ and $a_1, a_2 > 0$.

(3) *Yukawa-type potentials:* Let f in Assumption 2.1 be as in (2) above and $V(x) = -(a_1|x|^{-\beta_1} \wedge a_2|x|^{-\beta_2}e^{-b|x|})$, with $\beta_1 \in (0, \alpha \wedge d]$, $\beta_2 \in [\beta_1, \infty)$ and $a_1, a_2, b > 0$.
(4) *Pöschl-Teller potential:* $V(x) = -a/\cosh^2(b|x|)$, with $a, b > 0$.
(5) *Morse potential:* $V(x) = a((1 - e^{-b(|x|-r_0)})^2 - 1)$, with $a, b, r_0 > 0$.

2.2.2. **Ground state-transformed process.** By using φ_0, we define the ground state transform as the unitary map

$$U : L^2(\mathbb{R}^d, \varphi_0^2 dx) \to L^2(\mathbb{R}^d, dx), \quad f \mapsto \varphi_0 f.$$

Also, we define the intrinsic Feynman-Kac semigroup

$$\widetilde{T}_t f(x) = \frac{e^{\lambda_0 t}}{\varphi_0(x)} T_t(\varphi_0 f)(x) \tag{2.9}$$

associated with $\{T_t : t \geq 0\}$. Using the integral kernel $u(t, x, y)$ of T_t we have then that $\widetilde{T}_t f(x) = \int_{\mathbb{R}^d} \tilde{u}(t, x, y) f(y) \varphi_0^2(y) dy$ with the integral kernel given by

$$\widetilde{u}(t, x, y) = \frac{e^{\lambda_0 t} u(t, x, y)}{\varphi_0(x)\varphi_0(y)}, \tag{2.10}$$

and infinitesimal generator $\tilde{H} = U^{-1}(H - \lambda_0)U$, with domain

$$\operatorname{Dom} \widetilde{H} = \{f \in L^2(\mathbb{R}^d, \varphi_0^2 dx) : Uf \in \operatorname{Dom} H\}.$$

A calculation then shows that $\widetilde{H}$ is given by the expression (1.4). Furthermore, the operators $\widetilde{T}_t$ are contractions and we have $\widetilde{T}_t \mathbf{1}_{\mathbb{R}^d} = \mathbf{1}_{\mathbb{R}^d}$ for all $t \geq 0$, thus $\{\widetilde{T}_t : t \geq 0\}$ is a Markov semigroup on $L^2(\mathbb{R}^d, \varphi_0^2 dx)$.

The self-adjoint operator $\widetilde{H}$ generates a stationary Markov process, which we call a *ground state-transformed (GST) process.* (In the terminology of [36] it is called a $P(\phi)_1$-process associated with potential V.) To define GST processes, we need two-sided underlying processes. Denote by Ω_{r} the space of right continuous functions from $[0, \infty)$ to $\mathbb{R}^d$ with left limits (i.e., càdlàg functions), and by Ω_{l} the space of left continuous functions from $[0, \infty)$ to $\mathbb{R}^d$ with right limits (i.e., càglàd functions). Denote the corresponding Borel σ-fields by $\mathcal{B}(\Omega_{\mathrm{r}})$ and $\mathcal{B}(\Omega_{\mathrm{l}})$, respectively. Let $(X_t^{\mathrm{r}})_{t\geq 0}$ be a Lévy process on the space $(\Omega_{\mathrm{r}}, \mathcal{B}(\Omega_{\mathrm{r}}), \mathbb{P}_{\mathrm{r}}^x)$, where $X_t^{\mathrm{r}}(\omega) = \omega(t)$ is the coordinate process on Ω_{r}, and let $(X_t^{\mathrm{l}})_{t\geq 0}$ be a Lévy process on the space $(\Omega_{\mathrm{l}}, \mathcal{B}(\Omega_{\mathrm{l}}), \mathbb{P}_{\mathrm{l}}^x)$, where $X_t^{\mathrm{l}}(\varpi) = \varpi(t)$ is the coordinate process on Ω_{l}. Consider the product probability space $(\Omega_{\mathrm{r}} \times \Omega_{\mathrm{l}}, \mathcal{B}(\Omega_{\mathrm{r}}) \times \mathcal{B}(\Omega_{\mathrm{l}}), \mathbb{P}_{\mathrm{r}}^x \otimes \mathbb{P}_{\mathrm{l}}^x)$, and for every $\hat{\omega} = (\omega, \varpi) \in \Omega_{\mathrm{r}} \times \Omega_{\mathrm{l}}$ define

$$\hat{X}_t(\hat{\omega}) = \begin{cases} \omega(t) & \text{if } t \geq 0, \\ \varpi(-t) & \text{if } t < 0. \end{cases} \tag{2.11}$$

Then $t \mapsto \hat{X}_t(\cdot)$ is a càdlàg function for all $t \in \mathbb{R}$. Denote by Ω the space of càdlàg functions $\mathbb{R} \to \mathbb{R}^d$, with Borel σ-field by $\mathcal{B}(\Omega)$. Consider the image measure $\mathbb{Q}^x = (\mathbb{P}_{\mathrm{r}}^x \otimes \mathbb{P}_{\mathrm{l}}^x) \circ \hat{X}_{\cdot}^{-1}$. Then the coordinate process $(Y_t)_{t\in\mathbb{R}}$ on $(\Omega, \mathcal{B}(\Omega), \mathbb{Q}^x)$ is a Lévy process such that $\mathbb{Q}^x(Y_0 = x) = 1$, the increments $(Y_{t_i} - Y_{t_{i-1}})_{1\leq i\leq n}$ are independent and stationary for every $0 = t_0 < ... < t_n$, $n \in \mathbb{N}$, the increments $(Y_{-t_{i-1}} - Y_{-t_i})_{1\leq i\leq n}$ are independent and stationary for every $0 = -t_0 > ... > -t_n$, $n \in \mathbb{N}$, and the function $\mathbb{R} \ni t \mapsto Y_t(\cdot) \in \mathbb{R}^d$ is $\mathbb{Q}^x$-a.s. càdlàg.

Using two-sided càdlàg path space, we can now define GST processes. The following result gives the existence and fundamental properties of GST processes for general underlying Lévy processes and general Kato-class potentials. A first variant for jump processes has been obtained in [18, Th. 5.1] for GST processes derived from isotropic stable processes, but the argument is generic and it applies directly to the present settings, see for further details [29, Th. 2.1]. For an initial variant of the concept defined for an underlying Brownian motion and allowing simplifications due to path continuity we refer to [43, 5]. For infinite dimensional GST processes we refer to [15, 16]; see also a detailed discussion in [28].

Theorem 2.1 (**Ground state-transformed process**). *Let V be an X-Kato class potential and $\{\widetilde{T}_t : t \geq 0\}$ be the corresponding intrinsic Feynman-Kac semigroup. For all $x \in \mathbb{R}^d$ there exists a probability measure $\widetilde{\mathbb{P}}^x$ on $(\Omega, \mathcal{B}(\Omega))$ and a random process $(\widetilde{X}_t)_{t\in\mathbb{R}}$ satisfying the following properties:*

(1) *Let $-\infty < t_0 \leq t_1 \leq ... \leq t_n < \infty$ be an arbitrary division of the real line, for any $n \in \mathbb{N}$. The initial distribution of the process is*

$$\widetilde{\mathbb{P}}^x(\widetilde{X}_0 = x) = 1,$$

and the finite dimensional distributions of $\widetilde{\mathbb{P}}^x$ with respect to the stationary distribution $\varphi_0^2 dx$ are given by

$$\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^n f_j(\widetilde{X}_{t_j})\Big]\varphi_0^2(x)dx = \Big(f_0,\ \widetilde{T}_{t_1-t_0}\, f_1 ...\ \widetilde{T}_{t_n-t_{n-1}}\, f_n\Big)_{L^2(\mathbb{R}^d,\varphi_0^2 dx)} \tag{2.12}$$

for all $f_0, f_n \in L^2(\mathbb{R}^d, \varphi_0^2 dx)$, $f_j \in L^\infty(\mathbb{R}^d)$, $j = 1, ..., n-1$.

(2) *The finite dimensional distributions are time-shift invariant, i.e.,*

$$\int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^n f_j(\widetilde{X}_{t_j})\Big]\varphi_0^2(x)dx = \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\Big[\prod_{j=0}^n f_j(\widetilde{X}_{t_j+s})\Big]\varphi_0^2(x)dx, \quad s \in \mathbb{R},\, n \in \mathbb{N}.$$

(3) *$(\widetilde{X}_t)_{t\geq 0}$ and $(\widetilde{X}_t)_{t\leq 0}$ are independent, and $\widetilde{X}_{-t} \stackrel{\mathrm{d}}{=} \widetilde{X}_t$, for all $t \in \mathbb{R}$.*

(4) *With the filtrations $(\mathcal{F}_t^+)_{t\geq 0} = \sigma\{\widetilde{X}_s : 0 \leq s \leq t\}$ and $(\mathcal{F}_t^-)_{t\leq 0} = \sigma\{\widetilde{X}_s : t \leq s \leq 0\}$, the random process $(\widetilde{X}_t)_{t\geq 0}$ is a Markov process with respect to $\big(\mathcal{F}_t^+\big)_{t\geq 0}$, and $(\widetilde{X}_t)_{t\leq 0}$ is a Markov process with respect to $\big(\mathcal{F}_t^-\big)_{t\leq 0}$.*

Furthermore, we have for all $f, g \in L^2(\mathbb{R}^d, \varphi_0^2 dx)$ the change-of-measure formula

$$(f, \widetilde{T}_t g)_{L^2(\mathbb{R}^d,\varphi_0^2 dx)} = (f\varphi_0, e^{-t(H-\lambda_0)} g\varphi_0)_{L^2(\mathbb{R}^d,dx)} = \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}[f(\widetilde{X}_0)g(\widetilde{X}_t)]\varphi_0^2(x)dx, \quad t \geq 0. \tag{2.13}$$

In particular, we have the path measure

$$\widetilde{\mathbb{P}}(A) = \int_{\mathbb{R}^d} \mathbb{E}_{\widetilde{\mathbb{P}}^x}\left[1_A\right]\varphi_0^2(x)dx, \quad A \in \mathcal{B}(\Omega). \tag{2.14}$$

Remark 2.1.

(1) As shown in [29, Th. 3.1], under the condition that $x \mapsto \nabla \log \varphi_0(x)$ is locally bounded, a GST process $(\widetilde{X}_t)_{t\geq 0}$ satisfies the SDE given in (1.4). We will discuss some specific cases below.

(2) The probability measure $\widetilde{\mathbb{P}}$ can be seen as a Gibbs measure on the space of two-sided càdlàg paths. Consider a regular version of the conditional probability measure $\widetilde{\mathbb{P}}^{x,s}_{y,t}(\,\cdot\,) = \widetilde{\mathbb{P}}(\,\cdot\,|\widetilde{X}_s = x, \widetilde{X}_t = y)$, $x, y \in \mathbb{R}^d$, $s < t \in \mathbb{R}$, and the (not normalized) measure on $(\Omega|_{[s,t]}, \mathcal{B}(\Omega|_{[s,t]}))$ corresponding to the Lévy bridge process $(X_r)_{s\leq r\leq t}$ given by

$$b^{x,y}_{[s,t]}(\,\cdot\,) = p(t-s, y-x)\mathbb{P}^{x,s}_{y,t}(\,\cdot\,).$$

Then (2.14) can be equivalently written as

$$\widetilde{\mathbb{P}}(A) = \int_{\mathbb{R}^d} dx \varphi_0(x) \int_{\mathbb{R}^d} dy \varphi_0(y) \int_\Omega e^{-\int_s^t (V(X_r(\omega))-\lambda_0)dr} 1_A db^{x,y}_{[s,t]}(\omega)$$

for all $A \in \mathcal{B}(\Omega|_{[s,t]})$ and all $s < t \in \mathbb{R}$. It can be shown that the family of conditional probabilities indexed by the family of intervals $[s,t]$ and given by the last integral above satisfies the Dobrushin-Lanford-Ruelle consistency relations, and thus $\widetilde{\mathbb{P}}$ is a Gibbs measure on $(\Omega, \mathcal{B}(\Omega))$ with respect to the potential V. The details are left to the interested reader; for a discussion of Gibbs measures relative to stable processes see [18, Sect. 5.3], which can be extended through similar steps. Our results below on the almost sure long time behaviour of GST processes will then also characterize the supports of these Gibbs measures.

(3) When V is a confining potential, the process $(\widetilde{X}_t, \widetilde{\mathbb{P}}^x)_{t\geq 0, x\in\mathbb{R}^d}$ is typically $\varphi_0^2 dx$-recurrent. In other words, for every $x \in \mathbb{R}^d$ and Borel set $A \subset \mathbb{R}^d$ such that $\int_A \varphi_0^2(y)dy > 0$ (or, equivalently, with positive Lebesgue measure) we have $\int_0^\infty \widetilde{\mathbb{P}}^x(\widetilde{X}_t \in A)dt = \infty$. If there exists $g : [0,\infty) \to [0,\infty)$, $g(r) \nearrow \infty$ as $r \to \infty$, $g(r+1) \asymp g(r)$, $r \geq 1$, such that $V(x) \asymp g(|x|)$, then it follows from the estimates of the kernel $u(t,x,y)$ [21, Cor. 4.7] that there exists $t_0 > 0$ such that for every $t \geq t_0$, $x \in \mathbb{R}^d$ and $A \subset \mathbb{R}^d$ as above it holds that $\widetilde{\mathbb{P}}^x(\widetilde{X}_t \in A) \geq c$, with a constant $c = c(x,A) > 0$.

For cases when φ_0 is explicitly known, we can construct specific GST processes which give further insight.

Example 2.4 (GST Brownian motion). First consider the underlying Lévy process $(X_t)_{t\geq 0}$ to be a standard Brownian motion. Though we discuss the one-dimensional cases only, the first two examples below can be extended to arbitrary finite dimension.

(1) *Ornstein-Uhlenbeck process:* Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$, with confining potential $V(x) = \frac{\gamma^2}{2}x^2 - \frac{\gamma}{2}$, $\gamma > 0$. A calculation gives

$$\varphi_0(x) = \sqrt[4]{\frac{\gamma}{\pi}}\, e^{-\frac{\gamma x^2}{2}} \quad \text{and} \quad \widetilde{H} = -\frac{1}{2}\frac{d^2}{dx^2} + \gamma x \frac{d}{dx}.$$

Hence we have the GST process satisfying the SDE

$$dX_t = -\gamma X_t dt + dB_t, \quad X_0 = a,$$

i.e., the Ornstein-Uhlenbeck process $X_t = ae^{-\gamma t} + \int_0^t e^{-\gamma(t-s)} dB_s$. The role of the potential appears in a strong killing, which makes the process favour the region around the origin, and spend proportionally less time further away.

(2) *Brownian motion in a finite potential well:* Let $H = -\frac{1}{2}\frac{d^2}{dx^2} + V$, with compactly supported potential $V(x) = -v 1_{\{|x|\le a\}}$, $a, v > 0$. We have

$$\varphi_0(x) = A_0 e^{-\sqrt{2|\lambda_0|}|x|} 1_{\{|x|>a\}} + B_0 \cos(\sqrt{2(v-|\lambda_0|)}x) 1_{\{|x|\le a\}},$$

where A_0, B_0 can be determined by the normalization condition $\|\varphi_0\|_2 = 1$, and the ground state eigenvalue is the smallest solution $\lambda = \lambda_0$ of the transcendental equation $\tan(a\sqrt{2(v-|\lambda|)}) = \sqrt{\frac{\lambda}{v-\lambda}}$. Using that $a\sqrt{2(v-|\lambda_0|)} < \frac{\pi}{2}$, we obtain that the GST process satisfies the equation $dX_t = b(X_t)dt + dB_t$, with drift term

$$b(X_t) = -\sqrt{2|\lambda_0|}\ \mathrm{sgn}(X_t) 1_{\{|x|>a\}} - \sqrt{2(v-|\lambda_0|)} \tan\left(\sqrt{2(v-|\lambda_0|)} X_t\right) 1_{\{|x|\le a\}}.$$

From the above it can be seen that as soon as a path exits the potential well, it is pulled back by the drift at a constant speed $\sqrt{2|\lambda_0|}$, which will act as a basic mechanism preventing explosion.

(3) *Diffusions with Pearson distributions:* It is a yet little explored though notable fact that the six classes of Pearson distribution correspond to classical Schrödinger operators with Pöschl-Teller, Morse etc potentials given in Example 2.3 above. For further details see [3, Table 10].

Example 2.5 (**GST Cauchy process**)**.** Let $H = (-\frac{1}{2}\frac{d^2}{dx^2})^{1/2} + V$. Using the results in [27], in which explicit solutions have been obtained for the harmonic potential $V(x) = x^2$, and in [11] for the anharmonic potential $V(x) = x^4$, one can construct related GST processes for the one-dimensional 1-stable (i.e., Cauchy) process generated by the square root of the one-dimensional negative Laplacian. In this case we have

$$\begin{aligned}\widetilde{H} f(x) = &-c_d \int_{\mathbb{R}^d\setminus\{0\}} \left(f(x+z) - f(x) - z\cdot\nabla f(x) 1_{\{|z|\le 1\}}\right) \frac{\varphi_0(x+z)}{\varphi_0(x)} |z|^{-d-1} dz \\ &- c_d \int_{0<|z|\le 1} \frac{\varphi_0(x+z)-\varphi_0(x)}{\varphi_0(x)} z\cdot\nabla f(x) |z|^{-d-1} dz,\end{aligned}$$

from which a specific case of (1.4) can be obtained.

3. Integral tests and long time behaviour for general jump GST-processes

3.1. Technical lemmas

In this section we consider general underlying Lévy processes defined by the exponent (2.1), i.e., do not make the restriction to the jump-paring class given by Assumption 2.1.

We start by an extension of the Borel-Cantelli lemma, which also extends a result in [36]. The first statement is a direct consequence of the classical Borel-Cantelli lemma, while the second uses the concept of h-mixing (see below).

Lemma 3.1. *Suppose a function $\tau : \mathbb{N} \to (0,\infty)$ is given.*

(1) *If* $\sum_{n=1}^{\infty} \widetilde{\mathbb{P}}\big(|\widetilde{X}_n| \geq \tau(n)\big) < \infty$*, then* $|\widetilde{X}_n| < \tau(n)$ *for almost every* $n \in \mathbb{N}$*,* $\widetilde{\mathbb{P}}$*-a.s.*

(2) *If* $\sum_{n=1}^{\infty} \widetilde{\mathbb{P}}\big(|\widetilde{X}_n| > \tau(n)\big) = \infty$*, then* $|\widetilde{X}_n| \geq \tau(n)$ *for infinitely many* $n \in \mathbb{N}$*,* $\widetilde{\mathbb{P}}$*-a.s.*

Proof. Recall the following concept used in [36, Section 2]: Given a probability space $(\Omega, \mathcal{F}, P)$ and a function $h : \mathbb{N} \cup \{0\} \to \mathbb{R}^+$, a sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of sub-σ-fields of $\mathcal{F}$ is called h-mixing whenever for every $\mathcal{F}_m$-measurable function f and $\mathcal{F}_n$-measurable function g, $m, n \in \mathbb{N}$, the estimate on the covariance $\big|\mathbb{E}_P[fg] - \mathbb{E}_P[f]\mathbb{E}_P[g]\big| \leq h(|n-m|)\|f\|_2\|g\|_2$ holds.

Coming to our context, notice that when $\sum_{n=1}^{\infty} \widetilde{\mathbb{P}}\big(|\widetilde{X}_n| > \tau(n)\big) < \infty$, the Borel-Cantelli lemma gives $|\widetilde{X}_n| \leq \tau(n)$ for almost all $n \in \mathbb{N}$, $\widetilde{\mathbb{P}}$-a.s., and thus (1) holds. To obtain (2), let $\mathcal{F}_n = \sigma\{\widetilde{X}_t : n \leq t \leq n+1\}$, for $n \in \mathbb{N}$. By using that $\Lambda := \inf\left(\mathrm{Spec}(H) \setminus \{\lambda_0\}\right) - \lambda_0 > 0$ and the same argument as in [36, Th.3], we find that the family of σ-fields $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is h-mixing with the function $h(n) := e^{-\Lambda n}$, $n \in \mathbb{N}$. Therefore, if $\sum_{n=1}^{\infty} \widetilde{\mathbb{P}}\big(|\widetilde{X}_n| > \tau(n)\big) = \infty$, then by [36, Th.2(8b)] it follows that $|\widetilde{X}_n| \geq \tau(n)$ for infinitely many $n \in \mathbb{N}$, $\widetilde{\mathbb{P}}$-a.s., and (2) holds. □

Next we establish an estimate needed to control the series appearing in the previous lemma, which will play an essential role below.

Lemma 3.2. *Let $(X_t)_{t\geq 0}$ be a Lévy process determined by (2.1), V a potential satisfying Assumption 2.2, and $(\widetilde{X}_t)_{t\geq 0}$ the corresponding GST-process with probability measure $\widetilde{\mathbb{P}}$. Then for every non-decreasing function $\tau : [0,\infty) \to (0,\infty)$ we have*

$$\sum_{n=1}^{\infty} \widetilde{\mathbb{P}}\left(|\widetilde{X}_n| \geq \tau(n)\right) < \infty \iff \int_1^{\infty} dr \int_{|x|\geq \tau(r)} \varphi_0^2(x)dx < \infty.$$

Proof. First notice that by monotonicity of τ we have

$$\int_1^{\infty} \widetilde{\mathbb{P}}\left(|\widetilde{X}_t| \geq \tau(r)\right) dr < \infty \iff \sum_{n=1}^{\infty} \widetilde{\mathbb{P}}\left(|\widetilde{X}_t| \geq \tau(n)\right) < \infty, \quad t \geq 0.$$

Let $A_r = \big\{y \in \mathbb{R}^d : |y| \geq \tau(r)\big\}$. By (2.14), (2.12) and (2.9), we have

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_t| \geq \tau(r)\right) = \int_{\mathbb{R}^d} \widetilde{T}_t(\mathbf{1}_{A_r}\varphi_0)(x)\varphi_0^2(x)dx = \int_{\mathbb{R}^d} \varphi_0(x)e^{\lambda_0 t}T_t(\mathbf{1}_{A_r}\varphi_0)(x)dx$$

and by using the symmetry of the operator T_t and the eigenvalue equation $T_t\varphi_0 = e^{-\lambda_0 t}\varphi_0$, $t \geq 0$, we get

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_t| \geq \tau(r)\right) = \int_{A_r} e^{\lambda_0 t}T_t\varphi_0(x)\varphi_0(x)dx = \int_{A_r} \varphi_0^2(x)dx.$$

Thus

$$\int_1^{\infty} \widetilde{\mathbb{P}}\left(|\widetilde{X}_t| \geq \tau(r)\right) dr = \int_1^{\infty}\int_{|x|>\tau(r)} \varphi_0^2(x)dxdr,$$

which completes the proof. □

3.2. General integral test and almost sure long-time behaviour

First we present an integral test to GST-processes obtained for general Lévy processes in the above framework. For $c > 0$ and a non-decreasing function $\tau : [0, \infty) \to (0, \infty)$ define

$$I_{\varphi_0}(c, \tau) := \int_1^\infty dr \int_{|x| \geq \tau(r)} \varphi_0^2(cx)dx = \int_1^\infty \tau^d(r) \int_{|x|\geq 1} \varphi_0^2(c\tau(r)x)dxdr$$

and

$$c_{\varphi_0}(\tau) := \inf \{c > 0 : I_{\varphi_0}(c, \tau) < \infty\}.$$

Clearly, in general, $c_{\varphi_0}(\tau) \in [0, \infty]$, and the integral $I_{\varphi_0}(c, \tau)$ can be seen as an escape rate for given τ.

The following 0-1 criterion holds.

Theorem 3.1 (**Integral test: general underlying process**). *Let $(X_t)_{t\geq 0}$ be a Lévy process determined by (2.1), V a potential satisfying Assumption 2.2, and $(\widetilde{X}_t)_{t\geq 0}$ the corresponding GST-process with probability measure $\widetilde{\mathbb{P}}$. Then for every non-decreasing function $\tau : [0, \infty) \to (0, \infty)$ we have*

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_n| \geq \tau(n) \text{ for infinitely many } n \in \mathbb{N}\right) = \begin{cases} 0 & \text{if} \quad I_{\varphi_0}(1, \tau) < \infty, \\ 1 & \text{if} \quad I_{\varphi_0}(1, \tau) = \infty. \end{cases} \tag{3.1}$$

Proof. The equalities in (3.1) follow directly from Lemmas 3.2 and 3.1. □

Corollary 3.1 (**Long-time behaviour: general underlying process**). *Under the conditions of Theorem 3.1 we have that*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\tau(n)} = c_{\varphi_0}(\tau), \quad \widetilde{\mathbb{P}} - \text{a.s.} \tag{3.2}$$

Proof. For every $c > 0$ and for a non-decreasing function τ as in the statement of the theorem the test (3.1) gives

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_n| \geq c\tau(n) \text{ for infinitely many } n \in \mathbb{N}\right) = \begin{cases} 0 & \text{if} \quad I_{\varphi_0}(c, \tau) < \infty, \\ 1 & \text{if} \quad I_{\varphi_0}(c, \tau) = \infty. \end{cases} \tag{3.3}$$

The result then follows directly from (3.3). □

Below we will rewrite the integral I_{φ_0} in a more suitable way to investigate the explicit dependence of the result on the Lévy triplet of the underlying process and the potential.

Remark 3.1. In this paper we identify the upper envelope profiles for the traces of the GST-processes on the positive integers, i.e., $(\widetilde{X}_n)_{n\geq 1}$, rather than for the full paths $(\widetilde{X}_t)_{t\geq 0}$. An extension of our results to the full time-set would require some precise estimates for the suprema of the process $(|\widetilde{X}|_t)_{t\geq 0}$ on unit time intervals, which are currently not available. However, similarly as in the classical case (see e.g. [36]), it is reasonable to expect that even our results for integers give a full picture of how the asymptotic behaviour of paths of the jump GST-processes depends on the input data like the Lévy intensity of the underlying Lévy processes and the external potential.

As a second type of result of general character we show a comparison principle. Intuitively, a more pinning potential gives rise to a ground state which decays faster, and so the corresponding GST should fluctuate less. The following result proves this intuition.

Theorem 3.2. *Let* $(\widetilde{X}_t^{(1)})_{t\geq 0}$ *and* $(\widetilde{X}_t^{(2)})_{t\geq 0}$ *be the two GST-processes corresponding to the ground states* $\varphi_0^{(1)}$ *and* $\varphi_0^{(2)}$*, respectively. Suppose that there exists* $c_0 > 0$ *such that for every* $c \geq c_0$ *we have*

$$\liminf_{|x|\to\infty} \frac{\varphi_0^{(1)}(cx)}{\varphi_0^{(2)}(x)} > 0. \tag{3.4}$$

Then the following hold.

(1) *For every non-decreasing function* $\tau^{(2)}$ *such that*

$$(0,\infty) \ni c_{\varphi_0^{(2)}} = \limsup_{n\to\infty} \frac{|\widetilde{X}_n^{(2)}|}{\tau^{(2)}(n)}, \quad \widetilde{\mathbb{P}} - \text{a.s.} \tag{3.5}$$

it follows that

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n^{(1)}|}{\tau^{(2)}(n)} = \infty, \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

(2) *If* $\tau^{(2)}$ *is a non-decreasing function satisfying* (3.5) *and* $\tau^{(1)}$ *is a non-decreasing function such that*

$$\widetilde{\mathbb{P}}\left(\limsup_{n\to\infty} \frac{|\widetilde{X}_n^{(1)}|}{\tau^{(1)}(n)} < \infty\right) > 0, \tag{3.6}$$

then also

$$\limsup_{n\to\infty} \frac{\tau^{(1)}(n)}{\tau^{(2)}(n)} = \infty.$$

Proof. Suppose that condition (3.4) holds. Also, let (3.5) be satisfied for a given non-decreasing function $\tau^{(2)}$ and denote $c_2 := c_{\varphi_0^{(2)}}$. By a change of variable in the inner integral, for every $c \geq c_0 c_2$ and $\varepsilon \in (0, c_2)$ it follows that

$$\begin{aligned} I_{\varphi_0^{(1)}}(c, \tau^{(2)}) &= \int_1^\infty dr \int_{|x|\geq \tau^{(2)}(r)} \left(\varphi_0^{(1)}(cx)\right)^2 dx \\ &= \left(\frac{1}{c_2 - \varepsilon}\right)^d \int_1^\infty dr \int_{|x|\geq (c_2-\varepsilon)\tau^{(2)}(r)} \left(\varphi_0^{(1)}\left(\frac{cx}{c_2-\varepsilon}\right)\right)^2 dx. \end{aligned}$$

By (3.4) there exist $C, R > 0$ such that

$$\varphi_0^{(1)}\left(\frac{cx}{c_2-\varepsilon}\right) \geq C\varphi_0^{(2)}(x), \quad |x| \geq R.$$

Thus the above estimate implies

$$I_{\varphi_0^{(1)}}(c, \tau^{(2)}) \geq C^2 \left(\frac{1}{c_2-\varepsilon}\right)^d \int_{r_0}^\infty dr \int_{|x|\geq (c_2-\varepsilon)\tau^{(2)}(r)} \left(\varphi_0^{(2)}(x)\right)^2 dx,$$

for every $r_0 \geq 1$ such that $(c_2 - \varepsilon)\tau^{(2)}(r_0) \geq R$. By (3.5) and the test (3.3), we have $I_{\varphi_0^{(2)}}(c_2 - \varepsilon, \tau^{(2)}) = \infty$ and the latter integral cannot be convergent. This means that for every $c \geq c_0 c_2$ we also have $I_{\varphi_0^{(1)}}(c, \tau^{(2)}) = \infty$. Thus the integral test (3.3) again yields that for every $K \in \mathbb{N}$ such that $K \geq c_0 c_2$

$$\widetilde{\mathbb{P}}\left(\limsup_{n\to\infty} \frac{|\widetilde{X}_n^{(1)}|}{\tau^{(2)}(n)} \geq K\right) = 1.$$

This then gives

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n^{(1)}|}{\tau^{(2)}(n)} = \infty, \quad \widetilde{\mathbb{P}} - \text{a.s.},$$

which completes the proof of (1). The assertion (2) is a direct consequence of (1). □

Remark 3.2. We note the following in relation to the condition (3.4) above.

(1) The decay rates of the ground state eigenfunctions for confining and decaying potentials are determined by (4.5) and (4.21)-(4.22), respectively. Thus condition (3.4) can be efficiently checked for a large class of underlying Lévy processes and potentials.
(2) Condition (3.4) is immediately satisfied if the order of the decay rate of $\varphi_0^{(2)}(x)$ at infinity is substantially greater than that of $\varphi_0^{(1)}(x)$, e.g., when $\varphi_0^{(2)}(x) \leq c_1 e^{-c_2|x|^{\beta_1}}$ and $\varphi_0^{(1)}(x) \geq c_3 e^{-c_4|x|^{\beta_2}}$ with $0 < \beta_2 < \beta_1$, or $\varphi_0^{(1)}(x) \geq c_3 |x|^{-\gamma}$ with $\gamma > d$, for large $|x|$. For examples we refer to Section 5.

4. Almost sure long time behaviour of GST-processes arising from jump-paring Lévy processes

4.1. Sharp tail estimates for stationary distributions

First we prove a technical lemma which will be applied to derive sharp tail estimates for the stationary distributions of the GST processes.

Lemma 4.1. *Let $r_0 \geq 1$ and let $h : [r_0, \infty) \to (0, \infty)$ be a given function such that*

(i) $h(r) r^d \to 0$ *as* $r \to \infty$,
(ii) $h(r) r^{d-1} \in L^1(r_0, \infty)$,
(iii) $h \in \mathcal{C}^1(r_0, \infty)$.

Consider the following conditions:

(L) *There exist a non-decreasing $\mathcal{C}^1$-class function $\kappa : (r_0, \infty) \to (0, \infty)$ and constants $A_1 \geq 0$ and $B_1 > 0$ such that*

$$-r \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\kappa(r)} \leq A_1 \quad \text{and} \quad -r \frac{\mathrm{d}}{\mathrm{d}r} \log h(r) - d \leq B_1 \kappa(r), \quad r > r_0. \tag{4.1}$$

(U) *There exist a non-decreasing $\mathcal{C}^1$-class function $\kappa : (r_0, \infty) \to (0, \infty)$ and constants $A_2 \geq 0$ and $B_2 > 0$ such that*

$$-r \frac{\mathrm{d}}{\mathrm{d}r} \frac{1}{\kappa(r)} \geq A_2 \quad \text{and} \quad -r \frac{\mathrm{d}}{\mathrm{d}r} \log h(r) - d \geq B_2 \kappa(r), \quad r > r_0. \tag{4.2}$$

The following hold.

(1) *Under assumptions (i)-(iii) and condition (L) we have*

$$\int_{s>r} h(s)s^{d-1}ds \leq \frac{1}{A_1+B_1}\frac{h(r)r^d}{\kappa(r)}, \quad r > r_0.$$

(2) *Under assumptions (i)-(iii) and condition (U) we have*

$$\int_{s>r} h(s)s^{d-1}ds \geq \frac{1}{A_2+B_2}\frac{h(r)r^d}{\kappa(r)}, \quad r > r_0.$$

Proof. We only prove (1) as the proof of (2) goes in the same way. Since

$$\begin{aligned}\frac{\frac{\mathrm{d}}{\mathrm{d}r}\left(h(r)\frac{r^d}{\kappa(r)}\right)}{\frac{\mathrm{d}}{\mathrm{d}r}\int_{s>r} h(s)s^{d-1}ds} &= \frac{h'(r)\frac{r^d}{\kappa(r)} + h(r)\frac{dr^{d-1}\kappa(r)-r^d\kappa'(r)}{\kappa^2(r)}}{-h(r)r^{d-1}} \\ &= \frac{-r\frac{\mathrm{d}}{\mathrm{d}r}\log h(r) - d}{\kappa(r)} - r\frac{\mathrm{d}}{\mathrm{d}r}\frac{1}{\kappa(r)},\end{aligned}$$

for almost every $r > r_0$. By using (4.1) we see that

$$(A_1+B_1)\frac{\mathrm{d}}{\mathrm{d}r}\int_{s>r} h(s)s^{d-1}ds \leq \frac{\mathrm{d}}{\mathrm{d}r}\left(h(r)\frac{r^d}{\kappa(r)}\right), \quad r > r_0.$$

Then by integrating on the two sides of the above inequality over the interval (r,∞), $r > r_0$, and using assumptions (i)-(ii), the result follows. □

4.2. **The case of confining potentials**

In this section we consider the class of symmetric jump-paring Lévy processes defined by Assumption 2.1, and subject them to appropriate potentials.

Denote

$$V_{\mathrm{U}}(x) := \sup_{y\in B(x,1)} V(y) \quad \text{and} \quad V_{\mathrm{L}}(x) := \inf_{y\in B(x,1)} V(y), \quad x \in \mathbb{R}^d. \tag{4.3}$$

When $V_{\mathrm{U}}(x) \asymp V_{\mathrm{L}}(x)$ for $|x| > R$ with some $R > 0$, then we say that the values of V are *almost constant on unit balls outside a bounded set* or, in short, that V *is almost constant on unit balls.*

We impose the following regularity condition on the potentials.

Assumption 4.1. *Let* $V \in \mathcal{K}_{\pm}^X$ *be a confining potential, i.e.* $V(x) \to \infty$ *as* $|x| \to \infty$*. Moreover, we assume that there exist functions* $g^{\mathrm{U}}, g^{\mathrm{L}} : (1,\infty) \to (0,\infty)$ *such that*

(4.4)

$$g^{\mathrm{U}}(r) \asymp \left(\int_{\mathbf{S}^{d-1}} \left(\frac{1}{(1 \vee V_{\mathrm{U}}(r\theta))}\right)^2 d\theta\right)^{1/2} \quad \text{and} \quad g^{\mathrm{L}}(r) \asymp \left(\int_{\mathbf{S}^{d-1}} \left(\frac{1}{(1 \vee V_{\mathrm{L}}(r\theta))}\right)^2 d\theta\right)^{1/2}$$

for all $r > 1$.

Under Assumption 4.1, the ground state $0 < \varphi_0 \in C_{\rm b}(\mathbb{R}^d)$ and there exist constants $C_1, C_2 > 0$ such that (see [18, Th.2.4, Cor.2.2])

$$C_1 \frac{1 \wedge \nu(x)}{1 \vee V_{\rm U}(x)} \leq \varphi_0(x) \leq C_2 \frac{1 \wedge \nu(x)}{1 \vee V_{\rm L}(x)}, \quad x \in \mathbb{R}^d. \tag{4.5}$$

To make some direct computations and find the direct profile functions for paths of the processes for specific jump intensities and given potentials V it is useful to rewrite the integral test in a more explicit way. Let $\kappa : [1, \infty) \to (0, \infty)$ be a given function. For $c > 0$ and a non-decreasing function $\tau : [0, \infty) \to (0, \infty)$, we denote

$$I^{\rm U}_{\nu,V,\kappa}(c, \tau) = \int^{\infty} \left(g^{\rm U}(c\tau(r)) f(c\tau(r))\right)^2 \frac{\tau^d(r)}{\kappa(\tau(r))} dr$$

and

$$I^{\rm L}_{\nu,V,\kappa}(c, \tau) = \int^{\infty} \left(g^{\rm L}(c\tau(r)) f(c\tau(r))\right)^2 \frac{\tau^d(r)}{\kappa(\tau(r))} dr.$$

Also, define

$$c^{\rm L}_{\nu,V,\kappa}(\tau) := \inf \left\{c > 0 : I^{\rm L}_{\nu,V,\kappa}(c, \tau) < \infty\right\} \quad \text{and} \quad c^{\rm U}_{\nu,V,\kappa}(\tau) := \sup \left\{c > 0 : I^{\rm U}_{\nu,V,\kappa}(c, \tau) = \infty\right\}.$$

Since $I^{\rm U}_{\nu,V,\kappa}(c, \tau) \leq I^{\rm L}_{\nu,V,\kappa}(c, \tau)$ for every $c > 0$ and τ, we always have $c^{\rm U}_{\nu,V,\kappa}(\tau) \leq c^{\rm L}_{\nu,V,\kappa}(\tau)$.

We are now ready to state the first main result in this section.

Theorem 4.1 (**Integral test: jump-paring underlying process**). *Let Assumptions 2.1-4.1 hold. Assume, in addition, that the profiles $g^{\rm U}, g^{\rm L}$ appearing in Assumption 4.1 are $\mathcal{C}^1$-class functions. Then we have the following.*

(1) *If condition (L) in Lemma 4.1 holds for the function $h = (g^{\rm L} f)^2$ with $r_0 = 1$ and some κ, then for every non-decreasing function $\tau : [0, \infty) \to (0, \infty)$ we have*

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_n| \geq \tau(n) \text{ for infinitely many } n \in \mathbb{N}\right) = 0 \quad \textit{whenever} \quad I^{\rm L}_{\nu,V,\kappa}(1, \tau) < \infty. \tag{4.6}$$

(2) *If condition (U) in Lemma 4.1 holds for the function $h = (g^{\rm U} f)^2$ with $r_0 = 1$ and some κ, then for every non-decreasing function $\tau : [0, \infty) \to (0, \infty)$ we have*

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_n| \geq \tau(n) \text{ for infinitely many } n \in \mathbb{N}\right) = 1 \quad \textit{whenever} \quad I^{\rm U}_{\nu,V,\kappa}(1, \tau) = \infty. \tag{4.7}$$

Proof. First we prove (1). Using the general Theorem 3.1, it suffices to show that $I_{\varphi_0}(1, \tau) < \infty$ whenever $I^{\rm L}_{\nu,V,\kappa}(1, \tau) < \infty$. Note that when the latter integral is finite, necessarily $\tau(r) \to \infty$ as $r \to \infty$. We have

$$I_{\varphi_0}(1, \tau) = \int_1^{\infty} dr \int_{|x| \geq \tau(r)} \varphi_0^2(x) dx.$$

According to (4.5), by the fact that under our assumptions $\nu(x) \to 0$ and $V(x) \to \infty$ as $|x| \to \infty$, there exists $R \geq 1$ such that there is a constant $C > 0$ satisfying

$$\varphi_0(x) \leq C \frac{\nu(x)}{V^{\rm L}(x)}, \quad |x| \geq R.$$

Let now $r_0 > 1$ be large enough such that $\tau(r) \geq R$ for $r \geq r_0$. With this, by Assumptions 2.1 (i) and 4.1, we clearly have

$$\begin{aligned} \int_{r_0}^{\infty} dr \int_{|x|\geq \tau(r)} \varphi_0^2(x)dx &\leq C \int_{r_0}^{\infty} dr \int_{|x|\geq \tau(r)} \left(\frac{\nu(x)}{V^{\mathrm{L}}(x)}\right)^2 dx \\ &\leq C_1 \int_{r_0}^{\infty} dr \int_{s\geq \tau(r)} \left(f(s)g^{\mathrm{L}}(s)\right)^2 s^{d-1} ds. \end{aligned} \tag{4.8}$$

To conclude, it suffices to apply Lemma 4.1 (1) to the inner integral in (4.8). We first check the assumptions (i)-(iii) of this lemma for $h(s) := (f(s)g^{\mathrm{L}}(s))^2$, $s \geq R$. Since $g^{\mathrm{L}}(s)$ is bounded for large s and f is the profile for the Lévy measure far from the origin, the first two conditions (i)-(ii) follow immediately. Moreover, g^{L} is assumed to be a $\mathcal{C}^1$-class function in (R, ∞). If the same is true for f, then the condition (iii) holds as well and, by applying Lemma 4.1 (1) to such $h(s)$, we get

$$\int_{r_0}^{\infty} dr \int_{|x|\geq \tau(r)} \varphi_0^2(x)dx \leq C_2 \int_{r_0}^{\infty} \left(f(\tau(r))g^{\mathrm{L}}(\tau(r))\right)^2 \frac{\tau^d(r)}{\kappa(\tau(r))} dr \leq C_3 I^{\mathrm{L}}_{\nu,V,\kappa}(1,\tau) < \infty.$$

Since the integral $\int_1^{r_0} dr \int_{|x|\geq \tau(r)} \varphi_0^2(x)dx$ is convergent, we conclude that $I_{\varphi_0}(1,\tau) < \infty$.

On the other hand, if f is not a $\mathcal{C}^1$-class function, then due to the convolution condition (2.5) we can show that there is a constant $C > 0$ such that $f(s) \leq Cf(s+1)$ for all $s \geq 1$ [22, Lem.1, Lem.3]. With this, we can construct a $\mathcal{C}^1$-class function f_0 such that $f_0(r) \asymp f(r)$, $r \geq 1$ (this can be done by putting $f_0(r) := f(r)$, for $r \in \mathbb{N}$, and by $\mathcal{C}^1$-interpolation). Then, the function $h(s) := (f(s)g^{\mathrm{L}}(s))^2$ under the integral in (4.8) above can be replaced with $h_0(s) := (f_0(s)g^{\mathrm{L}}(s))^2$ to which Lemma 4.1 (1) applies directly as above.

To see (2), it suffices to check that $I_{\varphi_0}(1,\tau) = \infty$ whenever $I^{\mathrm{L}}_{\nu,V,\kappa}(1,\tau) = \infty$. The proof of this again uses the general integral test in Theorem 3.1 and similar arguments as above based on the converse inequalities. □

Corollary 4.1 (**Long time behaviour: jump-paring underlying process**). *Under the conditions of part (1) in Theorem 4.1 it follows that*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\tau(n)} \leq c^{\mathrm{L}}_{\nu,V,\kappa}(\tau), \quad \widetilde{\mathbb{P}} - a.s., \tag{4.9}$$

and under the conditions in part (2) it follows that

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\tau(n)} \geq c^{\mathrm{U}}_{\nu,V,\kappa}(\tau), \quad \widetilde{\mathbb{P}} - a.s. \tag{4.10}$$

Proof. This is an immediate consequence of Theorem 4.1. □

Remark 4.1. If V is almost constant on unit balls, then for every $c > 0$ and τ we have $I^{\mathrm{U}}_{\nu,V,\kappa}(c,\tau) \asymp I^{\mathrm{L}}_{\nu,V,\kappa}(c,\tau)$ (i.e., both integrals are convergent or divergent at the same time) and $c^{\mathrm{L}}_{\nu,V,\kappa}(\tau) = c^{\mathrm{U}}_{\nu,V,\kappa}(\tau)$. In this case the integral tests (4.6)-(4.7) and the limsup resulting constants in (4.9)-(4.10) are sharp. For more specific examples of potentials we will see that this holds in an essentially greater generality. Moreover, if φ_0 decays polynomially at infinity (cf. Theorem 4.3), then the resulting constants $c^{\mathrm{L}}_{\nu,V,\kappa}(\tau)$ and $c^{\mathrm{U}}_{\nu,V,\kappa}(\tau)$ are necessarily 0 or ∞.

In Section 5 we illustrate our Theorem 4.1 and Corollary 4.1 by various choices of the Lévy density ν and the confining potential V. Note that all these results apply to general non-decreasing test functions τ. This is a consequence of the sharp estimates in Lemma 4.1, which requires some initial smoothness of the profiles for V.

4.3. **Almost sure behaviour profiles for confining potentials with regular variation**

If one is interested in constructing explicit almost sure long time behaviour profiles corresponding to specific types of Lévy measures and potentials, but not in the study of integral tests for general non-decreasing functions τ as in Theorem 4.1, then one can use more direct argument. Such profiles are typically strictly increasing functions and therefore one can use the Fubini theorem instead of the tail estimates in Lemma 4.1.

In this section we construct directly the explicit almost sure long time behaviour profiles for the paths of the GST process in the case when $-\log(f(s)g^{\mathrm{L}}(s))$ and $-\log(f(s)g^{\mathrm{U}}(s))$ are asymptotically equivalent with strictly increasing regularly varying functions at infinity.

Recall that a function $\mathcal{R} : (r_0, \infty) \to (0, \infty)$ is said to be regularly varying at infinity with index $\lambda \in \mathbb{R}$ if

$$\lim_{r\to\infty} \frac{\mathcal{R}(sr)}{\mathcal{R}(r)} = s^{\lambda}, \quad s > 0,$$

and $\mathcal{L} : (r_0, \infty) \to (0, \infty)$ is called slowly varying at infinity if it is regularly varying with index $\lambda = 0$. Every function $\mathcal{R}$ regularly varying at infinity with index $\lambda \in \mathbb{R}$ can be represented in the form

$$\mathcal{R}(r) = r^{\lambda}\mathcal{L}(r),$$

where $\mathcal{L}$ is slowly varying at infinity. It is known that $\mathcal{L}$ can be assumed to be a continuous function. For $r > \mathcal{R}(r_0)$ define

$$\mathcal{R}^*(r) := \inf\{s \in [r_0, \infty) : \mathcal{R}(s) \geq r\}.$$

We have $\mathcal{R}^*(r) = r^{1/\lambda}\mathcal{L}^*(r)$ and $\mathcal{R}^*$ is the asymptotic inverse function of $\mathcal{R}$ in the sense of the relation

$$\mathcal{R}(\mathcal{R}^*(r)) \approx \mathcal{R}^*(\mathcal{R}(r)) \approx r. \tag{4.11}$$

The notation $f(r) \approx g(r)$ means that $\lim_{r\to\infty} f(r)/g(r) = 1$. In this case the functions f and g are called asymptotically equivalent at infinity. The function $\mathcal{L}^*$ is slowly varying at infinity and is called the conjugate slowly varying function of $\mathcal{L}$. It is known that if $\mathcal{R}^*$ is an asymptotic inverse function of $\mathcal{R}$, then it is unique in the sense that if there is another slowly varying function $\mathcal{L}'$ satisfying $\mathcal{R}(r^{1/\lambda}\mathcal{L}'(r)) \approx r$, then $\mathcal{L}' \approx \mathcal{L}^*$. By (4.11) we also have

$$\lim_{r\to\infty} (\mathcal{L}^*(r))^{\lambda}\mathcal{L}(r^{1/\lambda}\mathcal{L}^*(r)) = 1. \tag{4.12}$$

Recall that the function $\mathcal{R}$ is called to be ultimately increasing if there exists $r_0 > 0$ such that $\mathcal{R}$ is increasing on (r_0, ∞). For further properties and details we refer to e.g. [40, Ch.1].

Theorem 4.2 (**Regularly varying Lévy intensities and potentials**)**.** *Let Assumptions 2.1-4.1 hold and suppose that there exists* $A \in (0,\infty)$ *such that for* $g_1 = g^{\mathrm{U}}$ *and* $g_2 = g^{\mathrm{L}}$

$$\log g_i(r) + \log f(r) = -A\,\mathcal{R}(r) + o(\mathcal{R}(r)) \quad \text{as} \quad r \to \infty, \quad i = 1,2 \tag{4.13}$$

holds with $\mathcal{R}(r) = r^{\lambda}\mathcal{L}(r)$*, where* $\lambda > 0$ *and* $\mathcal{L} : [r_0,\infty) \to (0,\infty)$ *is a slowly varying function at infinity. If* $\mathcal{R}$ *is ultimately increasing, then*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{1/\lambda}\mathcal{L}^*(\log n)} = \frac{1}{(2A)^{1/\lambda}}, \quad \widetilde{\mathbb{P}} - a.s., \tag{4.14}$$

where $\mathcal{L}^*$ *is the conjugate slowly varying function of* $\mathcal{L}$*.*

Remark 4.2. With the settings of the above theorem, whenever

$$\mathcal{L}(r) \approx \mathcal{L}\left(\frac{r}{(\mathcal{L}(r))^{1/\lambda}}\right), \tag{4.15}$$

we can take

$$\mathcal{L}^*(r) = \left(\mathcal{L}(r^{1/\lambda})\right)^{-1/\lambda}. \tag{4.16}$$

Indeed, under (4.15) we have $\mathcal{R}\left(r^{1/\lambda}\left(\mathcal{L}\left(r^{1/\lambda}\right)\right)^{-1/\lambda}\right) \approx r$, i.e., the function $r^{1/\lambda}\left(\mathcal{L}\left(r^{1/\lambda}\right)\right)^{-1/\lambda}$ is the asymptotic inverse of $\mathcal{R}$ and (4.16) holds by the asymptotic uniqueness of $\mathcal{L}^*$.

Proof of Theorem 4.2. Let $\theta(r) := e^{r^{\lambda}\mathcal{L}(r)}$, $r > 0$. We may assume that r_0 is large enough such that $\mathcal{R}$ is increasing and continuous on $[r_0,\infty)$. In particular, there exists an inverse function $\theta^{-1} : [\theta(r_0),\infty) \to [r_0,\infty)$. For a shorthand notation write $F_i(r) := (f(r)g_i(r))^2 r^{d-1}$. By similar argument as in (4.8) (using two sided estimates (4.5)), we have for $c > 0$

$$C_1 I^{(1)}_{\nu,V}(c,\theta^{-1}) \le \int_{\theta(r_0/c)}^{\infty} dr \int_{|x|\ge c\theta^{-1}(r)} \varphi_0^2(x)dx \le C_2 I^{(2)}_{\nu,V}(c,\theta^{-1}), \tag{4.17}$$

where

$$I^{(i)}_{\nu,V}(c,\theta^{-1}) = \int_{\theta(r_0/c)}^{\infty} dr \int_{s\ge c\theta^{-1}(r)} F_i(s)ds, \quad i = 1,2,$$

and the constants C_1, C_2 do not depend on c and θ. Moreover, by Fubini's theorem,

$$I^{(i)}_{\nu,V}(c,\theta^{-1}) = \int_{r_0}^{\infty} \theta(r/c)F_i(r)dr \quad i = 1,2.$$

It follows from (4.13) that for every $\varepsilon \in (0,1)$ there is $r_\varepsilon > 0$ such that for all $r > r_\varepsilon$

$$F_1(r) \ge e^{-2(1+\varepsilon)Ar^{\lambda}\mathcal{L}(r)} \quad \text{and} \quad F_2(r) \le e^{-2(1-\varepsilon)Ar^{\lambda}\mathcal{L}(r)}. \tag{4.18}$$

With this we have for every $c > 0$

$$\theta(r/c)F_2(r) \le \exp\left(\left(\frac{1}{c^{\lambda}}\frac{\mathcal{L}(r/c)}{\mathcal{L}(r)} - 2(1-\varepsilon)A\right)r^{\lambda}\mathcal{L}(r)\right), \quad r > r_\varepsilon,$$

and

$$\theta(r/c)F_1(r) \ge \exp\left(\left(\frac{1}{c^{\lambda}}\frac{\mathcal{L}(r/c)}{\mathcal{L}(r)} - 2(1+\varepsilon)A\right)r^{\lambda}\mathcal{L}(r)\right), \quad r > r_\varepsilon.$$

Therefore, by (4.18) and by slow variation of $\mathcal{L}$, for every $c > (2A)^{-1/\lambda}$ there exist $\varepsilon \in (0,1)$ and $R > 0$ such that

$$\frac{1}{c^\lambda}\frac{\mathcal{L}(r/c)}{\mathcal{L}(r)} - 2(1+\varepsilon)A < 0, \quad r > R.$$

Hence $I^{(2)}_{\nu,V}(c,\theta^{-1}) < \infty$ whenever $c > (2A)^{-1/\lambda}$. Due to (4.17) also $I_{\varphi_0}(c,\theta^{-1}) < \infty$ for this range of c. By similar argument we can also show that $I^{(1)}_{\nu,V}(c,\theta^{-1}) = \infty$ (and, therefore, $I_{\varphi_0}(c,\theta^{-1}) = \infty$), for every $c < (2A)^{-1/\lambda}$. We then have $c_{\varphi_0}(\tau) = (2A)^{-1/\lambda}$ and, by Corollary 3.1 with $\tau = \theta^{-1}$, we finally get

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\theta^{-1}(n)} = \frac{1}{(2A)^{1/\lambda}}, \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

To complete the proof it suffices to observe that by asymptotic uniqueness of $\mathcal{R}^*$ it follows that $\mathcal{R}^*(\log r) \approx \theta^{-1}(r)$. □

The next theorem involves the Lévy intensities and potentials of slow variation at infinity. For convenience, denote k-fold iterated logarithm by $\log_k$.

Theorem 4.3 (**Slowly varying Lévy intensities and potentials**). *Let Assumptions 2.1-4.1 hold and suppose that there exist $\gamma \in [d,\infty)$, $l \in \mathbb{N}$ and $\beta_1, ..., \beta_l \in \mathbb{R}$ such that for $g_1 = g^{\mathrm{U}}$ and $g_2 = g^{\mathrm{L}}$ we have*

$$f(r)g_i(r) \asymp r^{-\gamma}(\log r)^{\beta_1}(\log_2 r)^{\beta_2}\cdots(\log_l r)^{\beta_l} \quad \text{as} \quad r\to\infty, \quad i=1,2. \tag{4.19}$$

For natural numbers $k \geq l$ and any $\delta > 0$ denote

$$\tau_k(r) := r^{\frac{1}{2\gamma-d}}\left((\log r)^{2\theta_1+1}(\log_2 r)^{2\theta_2+1}\cdots(\log_k r)^{2\theta_k+\delta}\right)^{\frac{1}{2\gamma-d}},$$

where $\theta_i = \beta_i$ for $1 \leq i \leq l$ and $\theta_i = 0$ for $l < i \leq k$, whenever $k > l$. Then for every $k \geq l$ we have $\widetilde{\mathbb{P}}$-almost surely

$$\limsup_{n\to\infty}\frac{|\widetilde{X}_n|}{\tau_k(n)} = \begin{cases} 0 & \text{if} \quad \delta > 1, \\ \infty & \text{if} \quad \delta \leq 1. \end{cases} \tag{4.20}$$

Proof. Let $\gamma \in [d,\infty)$, $l \in \mathbb{N}$ and $\beta_1, ..., \beta_l \in \mathbb{R}$ be given and let $F_i(r) := (f(r)g_i(r))^2 r^{d-1}$. Fix $k \geq l$ and for $\delta > 0$ consider the function

$$\vartheta(r) := r^{2\gamma-d}(\log r)^{-2\theta_1-1}(\log_2 r)^{-2\theta_2-1}\cdots(\log_k r)^{-2\theta_k-\delta}, \quad r > \exp_k e,$$

where $\exp_k$ denotes the k-fold iterated exponential function, $\theta_i = \beta_i$ for $1 \leq i \leq l$, and $\theta_i = 0$ for $l < i \leq k$, whenever $k > l$. Clearly, ϑ is continuous on $(\exp_k e, \infty)$. We can also check that there exists $R = R(k,\gamma,\theta_1,...,\theta_k,\delta) \geq \exp_k e$ such that ϑ is an increasing function on (R,∞). Similarly as in the previous proof, it is enough to consider the integrals

$$I^{(i)}_{\nu,V}(c,\vartheta^{-1}) = \int_R^\infty \vartheta(r/c)F_i(r)dr, \quad i=1,2, \quad c>0.$$

By (4.19), for every $c > 0$ there is $R_c \geq R$ such that for $i = 1,2$ and every $r > R_c$ we have

$$\vartheta(r/c)F_i(r) \asymp r^{-1}(\log r)^{-1}(\log_2 r)^{-1}\cdots(\log_k r)^{-\delta}.$$

From this we see that $I^{(2)}_{\nu,V}(c,\theta^{-1}) < \infty$ for every $c > 0$, whenever $\delta > 1$, and similarly, $I^{(1)}_{\nu,V}(c,\theta^{-1}) = \infty$ for every $c > 0$, whenever $\delta \in (0,1]$. By Corollary 3.1 with $\tau = \vartheta^{-1}$, we finally get that $\widetilde{\mathbb{P}}$-almost surely

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\vartheta^{-1}(n)} = \begin{cases} 0 & \text{if} \quad \delta > 1, \\ \infty & \text{if} \quad \delta \leq 1. \end{cases}$$

Similarly as in the previous theorem, it suffices to check that $\vartheta^{-1}(r) \approx \tau_k(r)$. Since $\vartheta(r)$ is regularly varying with index $2\gamma - d$, its asymptotic inverse function is of the form $r^{1/(2\gamma-d)}\mathcal{L}^*(r)$. Hence by asymptotic uniqueness of $\mathcal{L}^*(r)$ and by Remark 4.2 we obtain $\vartheta^{-1}(r) \approx \tau_k(r)$. □

4.4. **The case of decaying potentials**

Next we consider potentials satisfying the following condition.

Assumption 4.2. *Let $V \in \mathcal{K}^X_{\pm}$ be a decaying potential, i.e. $V(x) \to 0$ as $|x| \to \infty$, and let $\lambda_0 < 0$ be an isolated eigenvalue of H.*

As shown in [20] (see also [7]), the fall-off of φ_0 depends now on the rate of the decay of ν at infinity and the distance of λ_0 from the essential spectrum of H. Typically, the following three different situations may occur:

(1) If the Lévy density ν decays strictly sub-exponentially at infinity (cf. [20, Th. 4.1 and 4.3]), then

$$C_1\,(1 \wedge \nu(x)) \leq \varphi_0(x) \leq C_2\,(1 \wedge \nu(x)), \tag{4.21}$$

with constants $C_1 = C_1(X,\lambda_0)$ and $C_2 = C_2(X,\lambda_0)$ (note that the estimates (4.21) depend on $\lambda_0 < 0$ only via the multiplicative constants C_1 and C_2).

(2) If the Lévy density ν decays exponentially at infinity and there exists $\eta_0 = \eta_0(X) > 0$, independent of V, such that if $\lambda_0 \in (-\infty, -\eta_0)$ (i.e. λ_0 is a sufficiently low-lying eigenvalue), then the estimate (4.21) continues to hold (see [20, (3.3)] and [20, Th. 4.2]).

(3) If the Lévy density ν decays exponentially at infinity and $\lambda_0 \in [-\eta_0, 0)$, then there is a constant $\theta > 0$ such that for every $\varepsilon \in (0,1)$ there exists a constant C such that

$$\varphi_0(x) \geq C\,\left(e^{-\theta\sqrt{|\lambda_0|+\varepsilon}\,|x|} \vee (1 \wedge \nu(x))\right). \tag{4.22}$$

We refer the reader to [20, Sec. 4.3-4.4] for further discussion.

We now analyze the cases (1)-(2) and (3) separately, and illustrate them by specific examples. For simplicity, in our results below we refer directly to the estimates (4.21)-(4.22).

As in the previous subsection, let $\kappa : [1,\infty) \to (0,\infty)$ be a given function. For $c > 0$, a non-decreasing function $\tau : [0,\infty) \to (0,\infty)$ and $\varepsilon \in (0,1)$, we denote

$$I_{\nu,\kappa}(c,\tau) = \int^{\infty} (f(c\tau(r)))^2\,\frac{\tau^d(r)}{\kappa(\tau(r))}dr \quad \text{and} \quad I^{\varepsilon}_{\lambda_0,\kappa}(c,\tau) = \int^{\infty} e^{-2c\theta\sqrt{|\lambda_0|+\varepsilon}\,\tau(r)}\tau^{d-1}(r)dr$$

Also, let

$$c_{\nu,\kappa}(\tau) := \inf\{c > 0 : I_{\nu,\kappa}(c,\tau) < \infty\} \quad \text{and} \quad c^{\varepsilon}_{\lambda_0,\kappa}(\tau) := \sup\left\{c > 0 : I^{\varepsilon}_{\lambda_0,\kappa}(c,\tau) = \infty\right\}.$$

We are now ready to state the version of Theorem 4.1 for decaying potentials in cases (1)-(2) and (3) above.

Theorem 4.4 (**Integral test: jump-paring underlying process**). *Let Assumptions 2.1 and 4.2 hold. Then we have the following.*

(1) *If* (4.21) *holds, and conditions (L) and (U) in Lemma 4.1 hold for the function* $h = f^2$ *with* $r_0 = 1$ *and some* κ*, then for every non-decreasing function* $\tau : [0, \infty) \to (0, \infty)$ *we have*

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_n| \geq \tau(n) \text{ for infinitely many } n \in \mathbb{N}\right) = \begin{cases} 0 & \text{if} \quad I_{\nu,\kappa}(1, \tau) < \infty, \\ 1 & \text{if} \quad I_{\nu,\kappa}(1, \tau) = \infty. \end{cases} \tag{4.23}$$

(2) *If* (4.22) *holds, then for every non-decreasing function* $\tau : [0, \infty) \to (0, \infty)$ *we have*

$$\widetilde{\mathbb{P}}\left(|\widetilde{X}_n| \geq \tau(n) \text{ for infinitely many } n \in \mathbb{N}\right) = 1, \tag{4.24}$$

whenever $I^{\varepsilon}_{\lambda_0,\kappa}(1, \tau) = \infty$*, for some* $\varepsilon \in (0, 1)$*.*

Proof. We can use the same arguments as in the proof of Theorem 4.1. The difference is that now the decay of the ground state φ_0 at infinity is determined by (4.21) and (4.22), respectively. Thus we take $h(r) = f^2(r)$ and $h(r) = e^{-2\theta\sqrt{|\lambda_0|+\varepsilon}\, r}$ in parts (1) and (2) above. □

Corollary 4.2 (**Long time behaviour: jump-paring underlying process**). *Under the assumptions of Theorem 4.4 we have*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\tau(n)} = c_{\nu,\kappa}(\tau), \quad \widetilde{\mathbb{P}} - a.s., \tag{4.25}$$

when (4.21) *holds, and for every* $\varepsilon \in (0, 1)$*,*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\tau(n)} \geq c^{\varepsilon}_{\lambda_0,\kappa}(\tau), \quad \widetilde{\mathbb{P}} - a.s., \tag{4.26}$$

when (4.22) *holds.*

Proof. This is an immediate consequence of Theorem 4.4. □

In the case of decaying potentials, we can also formulate versions of Theorems 4.2-4.3. Since the proofs of these results are similar, we leave them to the reader.

Corollary 4.3 (**Regularly varying Lévy intensities**). *Let Assumptions 2.1 and 4.2 hold.*

(1) *If* (4.21) *holds and there exists* $A \in (0, \infty)$ *such that*

$$\log f(r) = -A\,\mathcal{R}(r) + o(\mathcal{R}(r)) \quad as \quad r \to \infty, \tag{4.27}$$

with increasing $\mathcal{R}(r) = r^{\lambda}\mathcal{L}(r)$*, where* $\lambda \in (0, 1]$ *and* $\mathcal{L} : [r_0, \infty) \to (0, \infty)$ *is a slowly varying function at infinity, then*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{1/\lambda}\mathcal{L}^*(\log n)} = \frac{1}{(2A)^{1/\lambda}}, \quad \widetilde{\mathbb{P}} - a.s., \tag{4.28}$$

where $\mathcal{L}^*$ *is the conjugate slowly varying function of* $\mathcal{L}$ *(cf. Remark 4.2).*

(2) *If* (4.22) *holds, then*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\log n} \geq \frac{1}{2\theta\sqrt{|\lambda_0|}}, \quad \widetilde{\mathbb{P}} - a.s. \tag{4.29}$$

Corollary 4.4 (**Slowly varying Lévy intensities**). *Let Assumptions 2.1 and 4.2 hold and suppose that there exist* $\gamma \in [d,\infty)$, $l \in \mathbb{N}$ *and* $\beta_1, ..., \beta_l \in \mathbb{R}$ *such that*

$$f(r) \asymp r^{-\gamma} (\log r)^{\beta_1} (\log_2 r)^{\beta_2} \cdots (\log_l r)^{\beta_l} \quad as \quad r \to \infty. \tag{4.30}$$

For natural numbers $k \geq l$ *and any* $\delta > 0$ *denote*

$$\tau_k(r) := r^{\frac{1}{2\gamma-d}} \left((\log r)^{2\theta_1+1} (\log_2 r)^{2\theta_2+1} \cdots (\log_k r)^{2\theta_k+\delta} \right)^{\frac{1}{2\gamma-d}},$$

where $\theta_i = \beta_i$ *for* $1 \leq i \leq l$ *and* $\theta_i = 0$ *for* $l < i \leq k$, *whenever* $k > l$. *Then for every* $k \geq l$ *we have* $\widetilde{\mathbb{P}}$*-almost surely*

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\tau_k(n)} = \begin{cases} 0 & \text{if} \quad \delta > 1, \\ \infty & \text{if} \quad \delta \leq 1. \end{cases} \tag{4.31}$$

5. **Examples**

Now we discuss the asymptotic behaviour of paths of ground state-transformed processes with underlying Lévy processes having absolutely continuous Lévy measures with densities $\nu(x) \asymp f(|x|)$ such that

$$f(r) = \mathbf{1}_{(0,1]}(r)\, r^{-d-\alpha} + \mathbf{1}_{(1,\infty)}(r)\, e^{-\mu r^{\beta}} r^{-\gamma}, \quad r > 0, \tag{5.1}$$

where $d \geq 1$, $\alpha \in (0,2)$, $\mu \geq 0$, $\beta \geq 0$ and $\gamma \geq 0$. As proven in [22, Prop. 2], condition (2.5) holds exactly in the following three cases:

(L1) $\mu = 0$ and $\gamma > d$

(L2) $\mu > 0$, $\beta \in (0,1)$ and $\gamma \geq 0$

(L3) $\mu > 0$, $\beta = 1$ and $\gamma > (d+1)/2$.

All the other assumptions are satisfied as well. Notice that this choice of the profile f leads naturally to the following important classes of the underlying Lévy processes. In particular, (L1) includes the *isotropic* α*-stable processes* ($\gamma = d + \alpha$) and *layered* α*-stable processes* ($\gamma > d + \alpha$), and (L3) includes *relativistic* α*-stable processes* ($\mu = m^{1/\alpha}$, $\gamma = (d+1+\alpha)/2$, for $m > 0$) and *tempered stable processes* ($\mu > 0$, $\gamma = d + \alpha$). The processes satisfying (L2) make an intermediate class between the families of processes with polynomially and exponentially large jumps, and are now increasingly studied in the literature; they include the so-called *Weibull-type Lévy processes* or *Lévy processes with Weibull-distributed large jumps.*

First we consider confining potentials $V(x) \asymp g(|x|)$ with

$$g(r) = e^{\eta r^{\vartheta}} r^{\rho} \log(1+r)^{\sigma}, \quad r \geq 0, \tag{5.2}$$

where $\eta, \vartheta, \rho, \sigma \geq 0$ are chosen in a way that $g(r) \to \infty$ as $r \to \infty$. Observe that with this choice of f and g the integral tests in Theorem 4.1 hold with $I^{\mathrm{U}}_{\nu,V,\kappa}(1,\tau) = I^{\mathrm{L}}_{\nu,V,\kappa}(1,\tau) = I(1,\tau)$, where

$$I(1,\tau) := \int^{\infty} \frac{e^{-2\mu\tau(r)^{\beta} - 2\eta\tau(r)^{\vartheta}} \tau(r)^{d-2(\gamma+\rho)}}{\log(1+\tau(r))^{\sigma}\kappa(\tau(r))}\, dr$$

and $\kappa(r) = r^{\beta \vee \vartheta}$ if $\mu, \beta > 0$, or $\eta, \vartheta > 0$ and $\kappa(r) \equiv$ const otherwise.

Moreover, the following illustrates the facts established in Corollary 4.1 and Theorems 4.2-4.3, highlighting the parameters of ν and V that determine the long time behaviour.

Example 5.1 (**Envelopes for confining potentials**). Suppose the profiles f and g for the density of the Lévy measure ν and the potential V are given by (5.1) and (5.2), respectively. Then we have the following.

(1) `Stretched exponential and exponential jump intensity:` Let (L2) or (L3) hold.

(1.1) If $\eta > 0$ and $\vartheta > \beta$, then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{1/\vartheta}} = \frac{1}{(2\eta)^{1/\vartheta}}, \quad \widetilde{\mathbb{P}} - a.s.$$

(1.2) If $\eta > 0$ and $\vartheta = \beta$, then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{1/\vartheta}} = \frac{1}{(2(\mu+\eta))^{1/\vartheta}}, \quad \widetilde{\mathbb{P}} - a.s.$$

(1.3) If $\eta > 0$ and $\beta > \vartheta$ or $\eta = 0$, then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{1/\beta}} = \frac{1}{(2\mu)^{1/\beta}}, \quad \widetilde{\mathbb{P}} - a.s.$$

(2) `Polynomial jump intensity:` Let (L1) hold.

(2.1) If $\eta, \vartheta > 0$, then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{1/\vartheta}} = \frac{1}{(2\eta)^{1/\vartheta}}, \quad \widetilde{\mathbb{P}} - a.s.$$

(2.2) If $\eta = 0$ and $\rho \geq 0$, then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\left(n(\log n)^{2\sigma+\delta}\right)^{\frac{1}{2(\gamma+\rho)+d}}} = \begin{cases} 0 & \text{if} \quad \delta > 1, \\ \infty & \text{if} \quad \delta \leq 1, \end{cases} \quad \widetilde{\mathbb{P}} - a.s.$$

Note that Example 5.1 (2.2) applies directly to the fractional GST-processes related to $H = (-\Delta)^{\alpha/2} + V$, where $\alpha \in (0,2)$ and V is a confining potential from Example 2.2. If $V(x) = |x|^{2n}$, $n \in \mathbb{N}$ (harmonic and anharmonic oscillators), then this result holds with $\gamma = d + \alpha$, $\rho = 2n$ and $\sigma = 0$. If V is a double or multiple potential well, then a similar result holds with a suitable ρ.

Next we illustrate our results obtained in Section 4.4 for decaying potentials. Corollaries 4.3-4.4 imply the following.

Example 5.2 (**Envelopes for decaying potentials**). Suppose the profile f for the density of the Lévy measure ν is given by (5.1), and V is a decaying potential such that Assumption 4.2 holds. Then we have the following.

(1) If (L1) holds, then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(n(\log n)^\delta)^{\frac{1}{2\gamma+d}}} = \begin{cases} 0 & \text{if} \quad \delta > 1, \\ \infty & \text{if} \quad \delta \le 1, \end{cases} \qquad \widetilde{\mathbb{P}} - a.s.$$

(2) If (L2) holds, then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{1/\beta}} = \frac{1}{(2\mu)^{1/\beta}}, \quad \widetilde{\mathbb{P}} - a.s.$$

(3) If (L3) holds and the ground state eigenvalue $\lambda_0 < 0$ is sufficiently low-lying (so that (4.21) holds), then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\log n} = \frac{1}{2\mu}, \quad \widetilde{\mathbb{P}} - a.s.$$

(4) If (L3) holds and the ground state eigenvalue $\lambda_0 < 0$ is close to zero (so that (4.22) holds), then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\log n} \ge \frac{1}{2\theta\sqrt{|\lambda_0|}}, \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

Recall that some classes of decaying potentials of special importance are listed in Example 2.3, to which these results can be applied.

Interestingly, our general results obtained in Section 3 apply directly to diffusive GST-proceses as well. Indeed, in many cases the behaviour of the ground state φ_0 at infinity is known explicitly and we can analyze the test integrals $I_{\varphi_0}(1,\tau)$ by similar methods as in Sections 4.2-4.3. For instance, this can be done for some of the GST-Brownian motions. Below we give the limsup-almost sure behaviour profiles for the two important models discussed in Example 2.4. The details are left to the reader.

Example 5.3 (**Envelopes for GST Brownian motion**).

(1) *Ornstein-Uhlenbeck process:* If $(\widetilde{X}_t)_{t\ge 0}$ is a GST-process described in Example 2.4 (1), then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\sqrt{\log n}} = \frac{1}{\sqrt{\gamma}}, \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

This result is well-known (see [36, Th. 4] and references therein), and reproduced by our results above. Similarly, if $H = -\Delta + V$, where $V(x) = |x|^{2\beta}$, $\beta > 1$, then it is well-known [9, Sect. 4] that

$$\varphi_0(x) \asymp |x|^{-(\beta/2)+(d-1)/2} e^{-|x|^{1+\beta}/(1+\beta)}$$

for large enough $|x|$, and our approach again directly applies giving

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{(\log n)^{\frac{1}{1+\beta}}} = \left(\frac{1+\beta}{2}\right)^{\frac{1}{1+\beta}}, \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

This can be compared with (1.5) and [36, Th. 12].

(2) *Brownian motion in a finite potential well:* If $(\widetilde{X}_t)_{t\geq 0}$ is a GST-process described in Example 2.4 (2), then

$$\limsup_{n\to\infty} \frac{|\widetilde{X}_n|}{\log n} = \frac{1}{2\sqrt{2|\lambda_0|}}, \quad \widetilde{\mathbb{P}} - \text{a.s.}$$

Note that the almost sure asymptotics for this case is close to that obtained for the jump type GST processes constructed for decaying potentials in the case when the ground state eigenvalue λ_0 is close to zero (cf. Example 5.2 (4)).

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