

MAXIMUM PRINCIPLES FOR TIME-FRACTIONAL CAUCHY PROBLEMS WITH SPATIALLY NON-LOCAL COMPONENTS

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ABSTRACT. We show a strong maximum principle and an Alexandrov-Bakelman-Pucci estimate for the weak solutions of a Cauchy problem featuring Caputo time-derivatives and non-local operators in space variables given in terms of Bernstein functions of the Laplacian. To achieve this, first we propose a suitable meaning of a weak solution, show their existence and uniqueness, and establish a probabilistic representation in terms of time-changed Brownian motion. As an application, we also discuss an inverse source problem.

1. Introduction

Evolution equations featuring fractional time-derivatives currently receive much attention in both pure and applied mathematics due to, on the one hand, new qualitative properties not encountered in the realm of PDE and, on the other hand, for their novel modelling capabilities in science [1, 2, 3, 7, 11, 19, 24, 25, 27, 28]. There are a number of inequivalent concepts of fractional derivatives in use, which are presently the object of a wide-ranging study. One such concept is the Caputo-derivative, which for a given number $\alpha \in (0, 1)$ and a suitable function f is defined by

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{1}{(t-s)^\alpha} \frac{df(s)}{ds} ds,$$

where Γ is the usual Gamma-function. Equations with Caputo-derivatives occur, for instance, in the context of anomalous transport theory, arising as scaling limits of continuous-time random walk (CTRW) models. Fractional diffusion models described by the equation

$$\partial_t^\alpha p(t, x) = D(-\Delta)^{\nu/2} p(t, x),$$

with Caputo time-derivatives of the order $\alpha \in (0, 1)$ and fractional Laplacians in space variables of index $\nu \in (0, 2)$, and with diffusion constant $D > 0$, have been much studied in the literature; for a discussion see [26] and the references therein. This equation is the continuum limit of a CTRW model in which the random walker makes jumps $z \in \mathbb{R}^d$ whose probability distribution has a tail proportional to $|z|^{-\nu/2}$, separated by random waiting times T with tail distributions proportional to $T^{-\alpha}$, leading to a non-Gaussian behaviour of the limit process. This, in particular, captures more realistically the empirically observed anomalous spread of contaminants in groundwater flow through a porous soil, see e.g. [12], which is just one of the multiple applications of such equations.

In the present paper our goal is to consider a whole class of integro-differential equations of the above type, in which we maintain Caputo time-derivative but allow many other choices of non-local operators in the space variable. Let $\mathcal{D} \subset \mathbb{R}^d$ be a bounded domain, and $F : (0, \infty) \times \mathcal{D} \rightarrow \mathbb{R}$ and $V, \varphi_0 : \mathbb{R}^d \rightarrow \mathbb{R}$ be given functions, subject to conditions to be specified below. Also, let Ψ be a

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so called Bernstein function (see Section 2 for the details), which we will use to define non-local (pseudo-differential) operators of the form $\Psi(-\Delta)$, where Δ is the usual Laplacian. A specific choice is not only the usual or the fractional Laplacian above, but many others of interest such as their sum (describing jump-diffusion), or relativistic, geometric etc type of stable operators covering applications in relativistic quantum theory, laser physics etc (for a discussion see [16]). With these ingredients, in this paper our interest is to establish maximum principles for the solutions of integro-differential equations of the form

$$\begin{cases} \partial_t^\alpha \varphi + \Psi(-\Delta) \varphi + V\varphi = F(t, x) & \text{in } (0, \infty) \times \mathcal{D} \\ \varphi(t, x) = 0 & \text{in } (0, \infty) \times \mathcal{D}^c \\ \varphi(0, x) = \varphi_0(x) & \text{in } \{0\} \times \mathcal{D}. \end{cases} \quad (1.1)$$

Maximum principles and related Alexandrov-Bakelman-Pucci (ABP) estimates are results of fundamental relevance in the analysis of PDE or, more recently, integro-differential equations. In [5] we have recently obtained so-called refined and weak maximum principles, anti-maximum principles, ABP estimates, Liouville-type theorems and related results for elliptic non-local equations of the type as the space-dependent part in (1.1), as well as a parabolic ABP estimate for the sub-solutions for the case when time dependence entered via usual derivatives corresponding to $\alpha = 1$ above. (We also refer to the introduction of this paper for a review of the state of the art in the PDE literature.) In particular, the latter has the form

$$\sup_{[0, T] \times \mathcal{D}} \varphi^+ \leq \left(\sup_{[0, T] \times \mathcal{D}^c} \varphi^+ \vee \sup_{\{T\} \times \mathcal{D}} \varphi^+ \right) + C \left(1 + \frac{1}{(\Psi([\text{diam } \mathcal{D}]^{-2}))^{\frac{2}{p'}}} \right) \|F\|_{p, Q_T},$$

for (the positive part of) a bounded weak sub-solution φ of (1.1) with $\alpha = 1$, where \mathcal{D} is assumed to have a regularity property, Ψ satisfies a one-sided weak scaling condition with parameter $\underline{\mu}$ (see stated precisely in Assumption 3.2 below), $F \in L^p([0, T] \times \mathcal{D})$, $C = C(p, d, \Psi)$ is a constant, p, p' are Hölder-conjugate exponents, and where $p > \frac{d}{2\underline{\mu}} + 1$.

For time-fractional equations maximum principles have been considered in [20, 21, 22, 23], however, only for the cases when instead of a non-local operator $\Psi(-\Delta)$ the Laplacian or a second order elliptic differential operator in divergence form is used. Further developing our approach proposed in [5] to time-fractional equations, in this paper we consider also non-local spatial dependence, going well beyond the results established by these authors. A counterpart of the ABP estimate for the time-fractional case leads us to

$$p > \frac{d}{2\underline{\mu}} + \frac{1}{\alpha},$$

which gives then an explanation of the bound on p above, with a neat separation of space-time contributions.

The remainder of this paper is organized as follows. In Section 2 we briefly discuss Bernstein functions and subordinate Brownian motion, which are our main tools in describing the spatial part in the equations. In our main Section 3 first we show existence and uniqueness of weak solutions of the Cauchy problem (Theorem 3.1), and then derive a probabilistic representation of the solution in terms of expectations over time-changed Brownian motion (Theorem 3.2). Next we prove a strong maximum principle (Theorem 3.3) for any bounded domain with regular boundary. In Theorem 3.4 we present a result on the stability of solutions under varying the data. Then we derive an Alexandrov-Bakelman-Pucci type estimate for the solutions under a one-sided weak scaling property on the Bernstein functions used (Theorem 3.5). Finally, we present an application to the inverse source problem (Theorem 3.6), which has also a practical relevance.

2. Bernstein functions of the Laplacian and subordinate Brownian motions

Consider the set of non-negative, completely monotone functions

$$\mathcal{B} = \left\{ f \in \mathcal{C}^\infty((0, \infty)) : f \geq 0 \text{ and } (-1)^n \frac{d^n f}{dx^n} \leq 0, \text{ for all } n \in \mathbb{N} \right\}$$

and its subset

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{u \downarrow 0} f(u) = 0 \right\}.$$

An element of \mathcal{B} is called a Bernstein function, in particular, they are non-negative, increasing, and concave. Let \mathcal{M} be the set of Borel measures \mathbf{m} on $\mathbb{R} \setminus \{0\}$ with the property that

$$\mathbf{m}((-\infty, 0)) = 0 \quad \text{and} \quad \int_{\mathbb{R} \setminus \{0\}} (y \wedge 1) \mathbf{m}(dy) < \infty.$$

It is known that Bernstein functions $\Psi \in \mathcal{B}_0$ can be represented in the form

$$\Psi(u) = bu + \int_{(0, \infty)} (1 - e^{-yu}) \mathbf{m}(dy)$$

with $b \geq 0$, and the map $[0, \infty) \times \mathcal{M} \ni (b, \mathbf{m}) \mapsto \Psi \in \mathcal{B}_0$ is bijective.

Example 2.1. Some key examples of Ψ include:

- (i) $\Psi(u) = u^{\nu/2}$, $\nu \in (0, 2]$
- (ii) $\Psi(u) = (u + m^{2/\nu})^{\nu/2} - m$, $\nu \in (0, 2)$, $m > 0$
- (iii) $\Psi(u) = u^{\nu/2} + u^{\tilde{\nu}/2}$, $\nu, \tilde{\nu} \in (0, 2]$
- (iv) $\Psi(u) = u^{\nu/2} (\log(1+u))^{-\tilde{\nu}/2}$, $\nu \in (0, 2]$, $\tilde{\nu} \in [0, \nu]$
- (v) $\Psi(u) = u^{\nu/2} (\log(1+u))^{\tilde{\nu}/2}$, $\nu \in (0, 2)$, $\tilde{\nu} \in (0, 2 - \nu)$.

Bernstein functions give the Laplace exponents of subordinators. Recall that an \mathbb{R}^+ -valued Lévy process $(S_t)_{t \geq 0}$ on a probability space $(\Omega_S, \mathcal{F}_S, \mathbb{P}_S)$ is called a subordinator whenever $\mathbb{P}_S(S_s \leq S_t) = 1$ for $s \leq t$. There is a bijection between the set of subordinators on a given probability space and Bernstein functions in \mathcal{B}_0 , and the relationship

$$\mathbb{E}_{\mathbb{P}_S}[e^{-uS_t^\Psi}] = e^{-t\Psi(u)}, \quad u, t \geq 0,$$

holds, where $\Psi \in \mathcal{B}_0$, and where we have written $(S_t^\Psi)_{t \geq 0}$ for the unique subordinator associated with Bernstein function Ψ . Corresponding to the examples of Bernstein functions above, the related processes are (i) $\nu/2$ -stable subordinator, (ii) relativistic $\nu/2$ -stable subordinator, (iii) mixtures of independent subordinators of different indices etc.

Let $(B_t)_{t \geq 0}$ be \mathbb{R}^d -valued Brownian motion on Wiener space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$, running twice at its normal speed so that it has variance $\mathbb{E}_{\mathbb{P}_W}[B_t^2] = 2t$, $t \geq 0$. Also, let $(S_t^\Psi)_{t \geq 0}$ be an independent subordinator. The random process

$$\Omega_W \times \Omega_S \ni (\omega, \varpi) \mapsto B_{S_t^\Psi(\varpi)}(\omega) \in \mathbb{R}^d$$

is called subordinate Brownian motion under $(S_t^\Psi)_{t \geq 0}$, and

$$\mathbb{E}_{\mathbb{P}_S \times \mathbb{P}_W}[e^{iu \cdot B_{S_t^\Psi}}] = e^{t\Psi(|u|^2)}, \quad t > 0, u \in \mathbb{R}^d,$$

holds. Except for the trivial case generated by $\Psi(u) = u$ giving Brownian motion, every subordinate Brownian motion is a jump Lévy process, satisfying the strong Markov property. For simplicity, we will denote a subordinate Brownian motion by $(X_t)_{t \geq 0}$, its probability measure for the process

starting at $x \in \mathbb{R}^d$ by \mathbb{P}^x , and expectation with respect to this measure by \mathbb{E}^x . The subordination procedure gives, in particular, the expression

$$\mathbb{P}(X_t \in A) = \int_0^\infty \mathbb{P}_W(B_s \in A) \mathbb{P}_S(S_t \in ds),$$

for every measurable set A . For a detailed discussion of Bernstein functions, subordinators and subordinate Brownian motion we refer to [30].

By subordination it also follows that the infinitesimal generator of subordinate Brownian motion with a subordinator corresponding to a Bernstein function $\Psi \in \mathcal{B}_0$ is the pseudo-differential operator $-\Psi(-\Delta)$ defined by the Fourier multiplier

$$(\widehat{\Psi(-\Delta)f})(y) = \Psi(|y|^2)\widehat{f}(y), \quad y \in \mathbb{R}^d, \quad f \in \text{Dom}(\Psi(-\Delta)),$$

with domain $\text{Dom}(\Psi(-\Delta)) = \{f \in L^2(\mathbb{R}^d) : \Psi(|\cdot|^2)\widehat{f} \in L^2(\mathbb{R}^d)\}$. By general arguments it can be seen that $-\Psi(-\Delta)$ is a negative, self-adjoint operator with core $C_c^\infty(\mathbb{R}^d)$.

In what follows, we will use the Hartman-Wintner condition

$$\lim_{|u| \rightarrow \infty} \frac{\Psi(|u|^2)}{\log|u|} = \infty. \quad (2.1)$$

It is known that under this condition the subordinate Brownian motion $(X_t)_{t \geq 0}$ has a bounded continuous transition probability density $q_t(x, y) = q_t(x - y)$, and $q_t(\cdot)$ is radially decreasing, see [18].

3. Maximum principles

3.1. Weak solution and stochastic representation

Let \mathcal{D} be a bounded connected domain in \mathbb{R}^d , $V \in \mathcal{C}(\bar{\mathcal{D}})$, and consider the first exit time

$$\tau_{\mathcal{D}} = \inf\{t > 0 : X_t \notin \mathcal{D}\}$$

of subordinate Brownian motion $(X_t)_{t \geq 0}$ from \mathcal{D} . The killed Feynman-Kac semigroup in \mathcal{D} is given by

$$T_t^{\mathcal{D}, V} f(x) = \mathbb{E}^x[e^{-\int_0^t V(X_s) ds} f(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}], \quad x \in \mathcal{D}, \quad t > 0, \quad f \in L^2(\mathcal{D}). \quad (3.1)$$

It is shown in [4, Lem. 3.1] that under condition (2.1) $T_t^{\mathcal{D}, V}$ is a Hilbert-Schmidt operator on $L^2(\mathcal{D})$ for every $t > 0$. Also, it is a strongly continuous semigroup in $L^2(\mathcal{D})$ with generator $-H^{\mathcal{D}, V}$, where

$$H^{\mathcal{D}, V} = \Psi(-\Delta) + V,$$

defined in form sense. This implies that the spectrum of $H^{\mathcal{D}, V}$ is purely discrete, and there exists a countable set $(\lambda_n)_{n \in \mathbb{N}}$ of eigenvalues of finite multiplicity each, and corresponding square integrable eigenfunctions $(\varphi_n)_{n \in \mathbb{N}}$ of $H^{\mathcal{D}, V}$, which form an orthonormal basis in $L^2(\mathcal{D})$. Furthermore, the eigenfunctions are bounded continuous functions in \mathcal{D} . Assuming that $V \geq 0$, it follows that

$$0 < \lambda_1 < \lambda_2 \leq \dots$$

Every $T_t^{\mathcal{D}, V}$ is an integral operator, with integral kernel given by the eigenfunction expansion

$$T_t^{\mathcal{D}, V}(t, x, y) = \sum_{n=1}^{\infty} e^{-\lambda_n t} \varphi_n(x) \varphi_n(y), \quad t > 0, \quad x, y \in \mathcal{D}.$$

These properties hold for the $V \equiv 0$ case as well, and we will denote $H^{\mathcal{D}, 0} = H^{\mathcal{D}}$. For further details on general facts we refer to [13, Ch. XIX, Th. 6.2, Cor. 6.3].

Let $\{\lambda_1^0 < \lambda_2^0 \leq \dots\}$ be the eigenvalues of the operator $H^{\mathcal{D}} = \Psi(-\Delta)$. Since $V \geq 0$, we can see by the min-max principle that

$$\lambda_n \geq \lambda_n^0 - \|V\|_\infty, \quad n \in \mathbb{N}. \quad (3.2)$$

Indeed, note that for any $f \in L^2(\mathcal{D})$

$$\begin{aligned} & \int_{\mathcal{D}} \left(\frac{1}{t} \mathbb{E}^x \left[(e^{-\int_0^t V(X_s) ds} - 1) f(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right] - \mathbb{E}^x \left[V(X_t) f(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right] \right)^2 dx \\ &= \int_{\mathcal{D}} \mathbb{E}^x \left[\frac{1}{t} (e^{-\int_0^t V(X_s) ds} - 1 - tV(X_t)) f(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right]^2 dx \\ &\leq \int_{\mathcal{D}} \mathbb{E}^x [f^2(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] \mathbb{E}^x [\Xi_t^2] dx, \end{aligned} \quad (3.3)$$

where

$$\Xi_t = \frac{1}{t} \left(e^{-\int_0^t V(X_s) ds} - 1 - tV(X_t) \right).$$

Let $\varepsilon > 0$ be arbitrary, and take a compactly supported continuous function $\psi \in L^2(\mathcal{D})$ such that $\int_{\mathcal{D}} |f^2(y) - \psi^2(y)| dy < \varepsilon$. Note that Ξ_t is bounded uniformly in $t > 0$ almost surely, and $\mathbb{E}^x[\Xi_t] \rightarrow 0$ as $t \rightarrow 0$ for all $x \in \mathcal{D}$. Therefore, by the dominated convergence theorem we have

$$\int_{\mathcal{D}} \mathbb{E}^x [\psi^2(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] \mathbb{E}^x [\Xi_t^2] dx \rightarrow 0, \quad \text{as } t \rightarrow 0.$$

On the other hand

$$\begin{aligned} \int_{\mathcal{D}} \mathbb{E}^x [(f^2(X_t) - \psi^2(X_t)) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] \mathbb{E}^x [\Xi_t^2] dx &\leq \kappa \int_{\mathcal{D}} \left[\int_{\mathcal{D}} |f^2(y) - \psi^2(y)| q_t(x-y) dy \right] dx \\ &\leq \kappa \int_{\mathcal{D}} |f^2(y) - \psi^2(y)| dy = \kappa \varepsilon, \end{aligned}$$

with a constant κ dependent on $\|V\|_{\infty}$, where q_t is the transition density of $(X_t)_{t \geq 0}$ at time t . Hence (3.3) tends to zero as $t \rightarrow 0$. Clearly, this implies

$$\frac{1}{t} \mathbb{E}^x \left[(e^{-\int_0^t V(X_s) ds} - 1) f(X_t) \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right] \rightarrow V(x) f(x) \quad \text{as } t \rightarrow 0,$$

in $L^2(\mathcal{D})$. Hence $\text{Dom}(H^{\mathcal{D},V}) = \text{Dom}(H^{\mathcal{D}})$, and for every $f \in \text{Dom}(H^{\mathcal{D}})$ we have

$$H^{\mathcal{D},V} f = H^{\mathcal{D}} f + V f.$$

Let now L_n be any n -dimensional subspace of $\text{Dom}(H^{\mathcal{D},V}) = \text{Dom}(H^{\mathcal{D}})$. By the min-max representation of the eigenvalues it follows that

$$\begin{aligned} \lambda_n &= \inf_{L_n} \sup \{ \langle H^{\mathcal{D},V} f, f \rangle : f \in L_n \text{ and } \|f\|_{L^2(\mathcal{D})} = 1 \} \\ &= \inf_{L_n} \sup \{ \langle H^{\mathcal{D}} f, f \rangle + \langle V f, f \rangle : f \in L_n \text{ and } \|f\|_{L^2(\mathcal{D})} = 1 \} \\ &\geq \inf_{L_n} \sup \{ \langle H^{\mathcal{D}} f, f \rangle : f \in L_n \text{ and } \|f\|_{L^2(\mathcal{D})} = 1 \} - \|V\|_{\infty} \\ &= \lambda_n^0 - \|V\|_{\infty}, \end{aligned}$$

which shows (3.2).

To define a mild or weak solution of the Cauchy problem (1.1) we will use the operators

$$S_t \psi = \sum_{n \geq 1} E_{\alpha,1}(-\lambda_n t^{\alpha}) \langle \varphi_n, \psi \rangle \varphi_n,$$

and

$$K_t \psi = \sum_{n \geq 1} t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) \langle \varphi_n, \psi \rangle \varphi_n,$$

where the pointed brackets mean scalar product in $L^2(\mathcal{D})$ and

$$E_{\alpha,\beta}(z) = \sum_{n \geq 0} \frac{z^n}{\Gamma(\alpha k + \beta)}, \quad z \in \mathbb{C},$$

denotes the Mittag-Leffler function (see [14]). Recall from [28, p. 35] that for a suitable constant $C = C(\alpha, \beta)$, the estimate

$$|E_{\alpha,\beta}(-s)| \leq \frac{C}{1+s}, \quad s \geq 0, \quad (3.4)$$

holds for $\alpha \in (0, 2)$ and $\beta \in \mathbb{R}$. Thus we have for every $t \geq 0$ that

$$\left(\sup_{n \in \mathbb{N}} |E_{\alpha,1}(-\lambda_n t^\alpha)| \right) \vee \left(\sup_{n \in \mathbb{N}} t^{\alpha-1} |E_{\alpha,\alpha}(-\lambda_n t^\alpha)| \right) < \infty,$$

ensuring that S_t and K_t are well-defined linear operators on $L^2(\mathcal{D})$ for all $t > 0$.

To study the existence of a weak solution, we define powers of the operator $H^{\mathcal{D},V}$. For $\gamma \in \mathbb{R}$ let

$$H_\gamma^{\mathcal{D},V} \psi = \sum_{n=1}^{\infty} \lambda_n^\gamma \langle \varphi_n, \psi \rangle \varphi_n, \quad \psi \in \text{Dom}(H_\gamma^{\mathcal{D},V})$$

with

$$\text{Dom}(H_\gamma^{\mathcal{D},V}) = \left\{ \psi \in L^2(\mathcal{D}) : \sum_{n=1}^{\infty} \lambda_n^{2\gamma} \langle \varphi_n, \psi \rangle^2 < \infty \right\} := \mathcal{H}_\gamma.$$

Note that \mathcal{H}_γ is a Hilbert space with norm

$$\|\psi\|_{\mathcal{H}_\gamma} = \left(\sum_{n=1}^{\infty} \lambda_n^{2\gamma} \langle \varphi_n, \psi \rangle^2 \right)^{1/2}.$$

We also have for $\gamma > 0$ that

$$\text{Dom}(H_\gamma^{\mathcal{D},V}) \subset L^2(\mathcal{D}) \simeq (L^2(\mathcal{D}))' \subset (\text{Dom}(H_\gamma^{\mathcal{D},V}))'.$$

For notational economy we set $(\text{Dom}(H_\gamma^{\mathcal{D},V}))' = \text{Dom}(H_{-\gamma}^{\mathcal{D},V}) =: \mathcal{H}_{-\gamma}$. For every $f \in \mathcal{H}_{-\gamma}$ we denote its action on $\psi \in \mathcal{H}_\gamma$ by ${}_{-\gamma}\langle f, \psi \rangle_\gamma$. The space $\mathcal{H}_{-\gamma}$ is again a Hilbert space with norm

$$\|f\|_{\mathcal{H}_{-\gamma}} = \left(\sum_{n=1}^{\infty} \lambda_n^{-2\gamma} {}_{-\gamma}\langle \varphi_n, f \rangle_\gamma^2 \right)^{1/2}.$$

Now we define weak solutions in the spirit of [29]. Throughout this paper we use Caputo-time derivative as defined in the Introduction.

Definition 3.1. We say that φ is a weak solution of (1.1) if

- (1) $\varphi(t, \cdot) \in \text{Dom}(H^{\mathcal{D},V})$ for almost every $t \in (0, T)$, and $\partial^\alpha \varphi \in L_{\text{loc}}^2((0, T), L^2(\mathcal{D}))$,
- (2) the equality

$$\partial_t^\alpha \varphi + \Psi(-\Delta) \varphi + V \varphi = F(t, \cdot) \quad (3.5)$$

holds in $L^2(\mathcal{D})$ for almost every $t \in (0, T)$,

- (3) for a $\gamma > 0$ we have $\varphi \in \mathcal{C}([0, T]; \mathcal{H}_{-\gamma})$ with

$$\lim_{t \rightarrow 0} \|\varphi(t, \cdot) - \varphi_0\|_{\mathcal{H}_{-\gamma}} = 0.$$

Our first goal in this section is to establish existence of a weak solution for (1.1). We will use the following assumption on the Bernstein function.

Assumption 3.1. There exists $\beta \in (0, 1]$ such that

$$\liminf_{|u| \rightarrow \infty} \frac{\Psi(u)}{u^\beta} > 0.$$

Theorem 3.1 (Existence and uniqueness of weak solution). *Let Assumption 3.1 hold and \mathcal{D} be a bounded domain in \mathbb{R}^d satisfying the exterior cone condition. If $F \in L^\infty((0, T); L^2(\mathcal{D}))$ and $\varphi_0 \in L^2(\mathcal{D})$, then*

$$\varphi(t, x) = S_t \varphi_0(x) + \int_0^t K_{t-s} F(s, x) ds, \quad x \in \mathcal{D}, t > 0, \quad (3.6)$$

is the unique weak solution of (1.1) such that $\varphi \in L^2((0, T); \mathcal{H}_1)$ and $\partial_t^\alpha \varphi \in L^2((0, T) \times \mathcal{D})$. Moreover, we have

$$\lim_{t \rightarrow 0} \|\varphi(t, \cdot) - \varphi_0\|_{\mathcal{H}_{-\gamma}} = 0,$$

for every $\gamma > \frac{d}{4\beta} - 1$.

Proof. The arguments below are inspired by [29] and some details are left to the reader to avoid repetitions.

Step 1: First consider $F = 0$. In this case we have

$$\varphi(t, x) = \sum_{n=1}^{\infty} \langle \varphi_n, \varphi_0 \rangle E_{\alpha, 1}(-\lambda_n t^\alpha) \varphi_n(x).$$

By similar arguments as in [29, Th. 2.1(i)] we obtain

$$\lim_{t \rightarrow 0} \|\varphi(t, \cdot) - \varphi\|_{L^2(\mathcal{D})} = 0.$$

Step 2: Next consider $\varphi_0 = 0$. In this case we have

$$\varphi(t, x) = \sum_{n=1}^{\infty} \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n (t-s)^\alpha) \langle \varphi_n, F(s, \cdot) \rangle ds \right] \varphi_n(x).$$

As observed in [29], it follows that

$$\begin{aligned} & \partial_t^\alpha \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n (t-s)^\alpha) \langle \varphi_n, F(s, \cdot) \rangle ds \right] \\ &= -\lambda_n \left[\int_0^t (t-s)^{\alpha-1} E_{\alpha, \alpha}(-\lambda_n (t-s)^\alpha) \langle \varphi_n, F(s, \cdot) \rangle ds \right] + \langle F(t, \cdot), \varphi_n \rangle. \end{aligned} \quad (3.7)$$

The arguments in [29, Th. 2.2] also give that

$$\|\partial_t^\alpha \varphi\|_{L^2((0, T) \times \mathcal{D})}^2 + \|H^{\mathcal{D}, V} \varphi\|_{L^2((0, T) \times \mathcal{D})} \leq \kappa \|F\|_{L^2((0, T) \times \mathcal{D})},$$

with a constant κ , independently of F . Hence, applying (3.7) we see that (3.5) holds. Next we show that

$$\lim_{t \rightarrow 0} \|\varphi(t, \cdot)\|_{\mathcal{H}_{-\gamma}} = 0. \quad (3.8)$$

By a similar type of calculation as in [29, p. 434] we obtain

$$\|\varphi(t, \cdot)\|_{\mathcal{H}_{-\gamma}} \leq \kappa_1 \|F\|_{L^\infty((0, T); L^2(\mathcal{D}))} \sum_{n=1}^{\infty} \frac{1}{\lambda_n^{2\gamma+2}} (1 - E_{\alpha, 1}(-\lambda_n t^\alpha)), \quad (3.9)$$

for a constant κ_1 . By the assumption on \mathcal{D} and [8, Th. 4.5], there exists $\delta \in (0, 1)$ satisfying

$$\lambda_n^0 \geq \delta \Psi(\lambda_{n, \text{Lap}}),$$

where $\lambda_{n,\text{Lap}}$ denotes the eigenvalues of the Laplacian with Dirichlet boundary condition in \mathcal{D} , arranged in increasing order and including multiplicity. On the other hand, from [9] we have $\lambda_{n,\text{Lap}} \gtrsim n^{2/d}$. Combining this with (3.2) and (3.9), and using Assumption 3.1, we find

$$\|\varphi(t, \cdot)\|_{\mathcal{H}_{-\gamma}} \leq \kappa_2 \|F\|_{L^\infty((0,T);L^2(\mathcal{D}))} \sum_{n=1}^{\infty} \frac{1}{n^{\frac{2\beta}{d}(2\gamma+2)}} (1 - E_{\alpha,1}(-\lambda_n t^\alpha)),$$

with a constant κ_2 . By our assumption on γ we have $\frac{2\beta}{d}(2\gamma+2) > 1$. Since $1 - E_{\alpha,1}(-\lambda_n t^\alpha)$ is bounded, see [29, Lem. 3.1], and tends to zero as $t \rightarrow 0$, we readily obtain (3.8).

Finally, to show uniqueness a reasoning similar to [29, pp.432-433] can be used. \square

Remark 3.1. It is easily seen from (3.6) and (3.4) that for $F = 0$ and $\varphi \in L^2(\mathcal{D})$ we obtain the decay rate

$$\|\varphi(t, \cdot)\|_{L^2(\mathcal{D})} \leq \frac{C}{1 + \lambda_1 t^\alpha}, \quad t > 0,$$

with a constant $C > 0$, which can be compared with [29, Cor. 2.6].

In the remaining part of this paper our main tool will be formula (3.6) and its probabilistic representation which we derive next. Let $(\xi_t)_{t \geq 0}$ be a stable subordinator with exponent $\alpha \in (0, 1)$, and denote by g_t its smooth transition probability density at time t . The function g_t is bounded for every $t > 0$, see [17]. Moreover, by [32, Th. 1.1] there exists a constant $c_1 > 0$ such that

$$g_1(x) \leq \frac{c_1}{(1+x)^{1+\alpha}}, \quad x > 0. \quad (3.10)$$

By usual scaling it is also known that

$$g_t(u) = t^{-1/\alpha} g_1(t^{-1/\alpha} u), \quad t > 0.$$

Let $(\eta_t)_{t \geq 0}$ be the inverse of $(\xi_t)_{t \geq 0}$, i.e., the process

$$\eta_t = \inf\{s > 0 : \xi_s > t\}, \quad t > 0.$$

It is known [25] that $(\eta_t)_{t \geq 0}$ has density

$$\eta_t(du) = t\alpha^{-1} u^{-1-1/\alpha} g_1(u^{-1/\alpha} t) du, \quad t > 0. \quad (3.11)$$

We choose the process $(\eta_t)_{t \geq 0}$ to be independent of the subordinate Brownian motion $(X_t)_{t \geq 0}$, and continue to make the assumption $V \geq 0$ without specifying it explicitly.

Theorem 3.2 (Probabilistic representation). *Let φ satisfy (3.6) with given $\varphi_0 \in L^2(\mathcal{D})$ and $F \in L^2((0, T) \times \mathcal{D})$. Then we have*

$$\varphi(t, x) = \int_0^\infty T_u^{\mathcal{D}, V} \varphi_0(x) \eta_t(du) + \int_0^t \left[\int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) T_l^{\mathcal{D}, V} F(s, x) dl \right] ds, \quad (3.12)$$

where $\{T_t^{\mathcal{D}, V} : t \geq 0\}$ is the semigroup defined by (3.1).

Proof. First we show that

$$S_t \varphi_0(x) = \mathbb{E}^x \left[e^{-\int_0^{\eta_t} V(X_s) ds} \varphi_0(X_{\eta_t}) \mathbf{1}_{\{\tau_{\mathcal{D}} > \eta_t\}} \right], \quad (3.13)$$

which gives the first term at the right hand side of (3.12). Denote Laplace transform by \mathcal{L} , and the transition probability density of $(X_t)_{t \geq 0}$ at time $t > 0$ by q_t . We use the fact that the Laplace transform of $E_{\alpha,1}(-\lambda t^\alpha)$ is $\frac{s^\alpha - 1}{s^\alpha + \lambda}$. Also, for every $\psi \in L^2(\mathcal{D})$, extending it to \mathbb{R}^d by zero, we have

$$\begin{aligned} \|T_l^{\mathcal{D}, V} \psi\|_{L^2(\mathcal{D})}^2 &\leq \int_{\mathcal{D}} [\mathbb{E}^x[\psi(X_l) \mathbf{1}_{\{\tau_{\mathcal{D}} > l\}}]]^2 dx \\ &\leq \int_{\mathcal{D}} \mathbb{E}^x[\psi^2(X_l)] \mathbb{P}^x(\tau_{\mathcal{D}} > l) dx \end{aligned}$$

$$\leq \left[\sup_{\mathcal{D}} \mathbb{P}^x(\tau_{\mathcal{D}} > l) \right] \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \psi^2(y) q_l(x, y) dy dx \leq \|\psi\|_{L^2(\mathcal{D})}^2. \quad (3.14)$$

Thus we have

$$\begin{aligned} \mathcal{L}(S.\varphi_0)(s) &= \sum_{n=1}^{\infty} \frac{s^{\alpha-1}}{s^{\alpha} + \lambda_n} \langle \varphi_n, \varphi_0 \rangle \varphi_n, \\ &= \sum_{n=1}^{\infty} s^{\alpha-1} \left[\int_0^{\infty} e^{-(s^{\alpha} + \lambda_n)l} dl \right] \langle \varphi_n, \varphi_0 \rangle \varphi_n, \\ &= \int_0^{\infty} s^{\alpha-1} e^{-s^{\alpha} l} \left[\sum_{n=1}^{\infty} e^{-\lambda_n l} \langle \varphi_n, \varphi_0 \rangle \varphi_n \right] dl \\ &= \int_0^{\infty} s^{\alpha-1} e^{-s^{\alpha} l} T_l^{\mathcal{D},V} \varphi_0(x) dl. \end{aligned}$$

The above equality should be understood in $L^2(\mathcal{D})$, which is justified by (3.14).

Recall the equality

$$s^{\alpha-1} e^{-s^{\alpha} l} = \frac{1}{\alpha} \int_0^{\infty} u e^{-su} g_1(ul^{-1/\alpha}) l^{-1-1/\alpha} du,$$

see [24]. Combining it with the above gives

$$\mathcal{L}(S.\varphi_0)(s) = \int_0^{\infty} e^{-su} \left[\frac{1}{\alpha} \int_0^{\infty} u g_1(ul^{-1/\alpha}) l^{-1-1/\alpha} T_l^{\mathcal{D},V} \psi_0(x) dl \right] du$$

Comparing the Laplace transforms on both sides and using (3.11), we obtain (3.13). We note that a similar argument appeared in [25] in another context.

Next we calculate the second term at the right hand side of (3.12). We will use that for $f_{\lambda}(t) = t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha})$, $\lambda > 0$, the expression

$$\mathcal{L}(f_{\lambda})(s) = \frac{1}{s^{\alpha} + \lambda}, \quad s > \lambda^{1/\alpha},$$

holds, see [14, p. 312]. Let $\psi \in \mathcal{C}_c(\mathcal{D})$, i.e., a continuous function with compact support. Define

$$K_t^m \psi = \sum_{n=1}^m t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n t^{\alpha}) \langle \varphi_n, \psi \rangle \varphi_n,$$

and

$$T_l^{\mathcal{D},V,m} \psi = \sum_{n=1}^m e^{-\lambda_n l} \langle \varphi_n, \psi \rangle \varphi_n.$$

Note that $K_t^m \psi$ and $T_l^{\mathcal{D},V,m} \psi$ are continuous functions in $(t, x) \in (0, \infty) \times \mathcal{D}$. Thus for every $s^{\alpha} > \max\{\lambda_1, \dots, \lambda_m\}$ we have

$$\begin{aligned} \mathcal{L}(K_t^m \psi)(s) &= \sum_{n=1}^m \frac{1}{s^{\alpha} + \lambda_n} \langle \varphi_n, \psi \rangle \varphi_n = \sum_{n=1}^m \int_0^{\infty} e^{-(s^{\alpha} + \lambda_n)l} dl \langle \varphi_n, \psi \rangle \varphi_n \\ &= \int_0^{\infty} e^{-s^{\alpha} l} \left[\sum_{n=1}^m e^{-\lambda_n l} \langle \varphi_n, \psi \rangle \varphi_n \right] dl = \int_0^{\infty} e^{-s^{\alpha} l} T_l^{\mathcal{D},V,m} \psi dl \\ &= \int_0^{\infty} \left[\int_0^{\infty} e^{-su} g_l(u) du \right] T_l^{\mathcal{D},V,m} \psi dl \\ &= \int_0^{\infty} e^{-su} \left[\int_0^{\infty} l^{-1/\alpha} g_1(l^{-1/\alpha} u) T_l^{\mathcal{D},V,m} \psi dl \right] du. \end{aligned}$$

Again comparing the two Laplace transforms, we get

$$K_t^m \psi = \int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha} t) T_l^{\mathcal{D}, V, m} \psi \, dl.$$

Letting $m \rightarrow \infty$ and using a denseness argument, we obtain

$$K_t \psi = \int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha} t) T_l^{\mathcal{D}, V} \psi \, dl, \quad \psi \in L^2(\mathcal{D}).$$

Applying this equality to F in (3.6), the result follows. \square

Remark 3.2. Note that for every $\kappa > 0$ and $s > 0$ we have by (3.10)

$$\begin{aligned} \int_0^\kappa l^{-1/\alpha} g_1(sl^{-1/\alpha}) \, dl &\leq c_1 \int_0^\kappa l^{-1/\alpha} \frac{1}{(1 + sl^{-1/\alpha})^{1+\alpha}} \, dl \\ &= c_1 s^{1-\alpha} \int_0^{\kappa s^{-\alpha}} \frac{u}{(1 + u^{1/\alpha})^{1+\alpha}} \, du, \\ &\leq c_1 s^{1-\alpha} \int_0^\infty \frac{u}{(1 + u^{1/\alpha})^{1+\alpha}} \, du = c_2 s^{1-\alpha}, \end{aligned}$$

with a constant c_2 , and in the second line we used the substitution $l = s^\alpha u$. The above integral is finite since $\alpha \in (0, 1)$. The estimate also implies that the rightmost term in (3.12) is finite.

As a consequence of Theorem 3.2 we have the following results which generalize Theorems 1 and 3 in [23] substantially.

Corollary 3.1. *Let $\varphi_0^i \in L^2(\mathcal{D})$ and $F_i \in L^2((0, T) \times \mathcal{D})$ for $i = 1, 2$. Furthermore, assume that $\varphi_0^1 \geq \varphi_0^2$ and $F_1 \geq F_2$. Then we have $\varphi^1(t, x) \geq \varphi^2(t, x)$ almost surely in $(0, T) \times \mathcal{D}$.*

Corollary 3.2. *Let $V_i \in \mathcal{C}(\bar{\mathcal{D}})$ and satisfy $V_1 \geq V_2$. Let φ_i denote the solution of (3.6) with non-negative data (φ_0, F) and V_i , $i = 1, 2$. Then we have $\varphi_1 \leq \varphi_2$, almost surely in $(0, T) \times \mathcal{D}$.*

Proof. This follows from the fact that $T_t^{\mathcal{D}, V_1} \psi \leq T_t^{\mathcal{D}, V_2} \psi$ for all t whenever $\psi \geq 0$. \square

3.2. Maximum principles and an Aleksandrov-Bakelman-Pucci estimate

Next we turn to proving maximum principles. For the remainder of the paper we assume that the domain \mathcal{D} is such that all its boundary points are regular with respect to $(X_t)_{t \geq 0}$, i.e.,

$$\mathbb{P}^z(\tau_{\mathcal{D}} = 0) = 1, \quad z \in \partial\mathcal{D}.$$

In [4] we have shown that every bounded convex set is regular with respect to subordinate Brownian motion. When $(X_t)_{t \geq 0}$ is an isotropic α -stable process, any domain \mathcal{D} with the exterior cone condition is regular with respect to this process.

Recall the notation

$$h \geq 0 \quad \text{meaning} \quad h(x) \geq 0 \text{ for all } x \in \mathcal{D} \text{ and } h \not\equiv 0.$$

Theorem 3.3 (Strong maximum principle). *Let $\varphi_0 \geq 0$ and $F = 0$, and suppose that \mathcal{D} has a regular boundary with respect to $(X_t)_{t \geq 0}$. Furthermore, assume that $\varphi_0 \in L^\infty(\mathcal{D})$. If φ is a weak solution satisfying (3.6), then $\varphi(t, \cdot) \in \mathcal{C}(\bar{\mathcal{D}})$ and $\varphi(t, x) > 0$ for every $t > 0$ and $x \in \mathcal{D}$.*

Proof. From [4, Lem. 3.1] we know that

$$T_t^{\mathcal{D},V} \varphi_0(x) = \int_{\mathcal{D}} T^{\mathcal{D},V}(t, x, y) \varphi_0(y) dy, \quad t > 0, \quad (3.15)$$

with kernel

$$T^{\mathcal{D},V}(t, x, y) = \mathbb{E}_{\mathbb{P}_S} \left[p_{S_t^\Psi}(x - y) \mathbb{E}_{0, S_t^\Psi}^{x,y} \left[e^{-\int_0^t V(B_{S_s^\Psi}) ds} \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right] \right],$$

where $p_t(x - y) = (4\pi t)^{-d/2} \exp(-\frac{|x-y|^2}{4t})$, and $\mathbb{E}_{0, S_t^\Psi}^{x,y}[\cdot]$ denotes expectation with respect to the Brownian bridge measure from x at time 0 to y at time s , evaluated at random time $s = S_t^\Psi$. It is also shown in [4, Lem. 3.1] that $T^{\mathcal{D},V}(t, x, y) = T^{\mathcal{D},V}(t, y, x)$ for $t > 0$ and $T^{\mathcal{D},V}(t, x, y)$ is continuous in $(0, \infty) \times \mathcal{D} \times \mathcal{D}$. This implies that $x \mapsto T_t^{\mathcal{D},V} \varphi_0(x)$ is continuous in \mathcal{D} for $t > 0$. Furthermore, by [4, Lem. 3.1(v)], we see that $x \mapsto T_t^{\mathcal{D},V} 1(x)$ is continuous in $\bar{\mathcal{D}}$ and vanishes on the boundary. Since

$$|T_t^{\mathcal{D},V} \varphi_0(x)| \leq \|\varphi_0\|_{L^2(\mathcal{D})} \|T^{\mathcal{D},V}(t, x, \cdot)\|_{L^\infty(\mathcal{D})}^{1/2} |T_t^{\mathcal{D},V} 1(x)|^{1/2},$$

it follows that $x \mapsto T_t^{\mathcal{D},V} \varphi_0(x)$ is continuous in $\bar{\mathcal{D}}$ for every $t > 0$ and the function vanishes on $\partial\mathcal{D}$. Again, for every $t > 0$,

$$|T_t^{\mathcal{D},V} \varphi_0(x)| \leq \|\varphi_0\|_{L^\infty(\mathcal{D})} T_t^{\mathcal{D},V} 1(x) \leq \|\varphi_0\|_{L^\infty(\mathcal{D})} \mathbb{P}^x(\tau_{\mathcal{D}} > t).$$

Since

$$\varphi(t, x) = \int_0^\infty T_u^{\mathcal{D},V} \varphi_0(x) \eta_t(du), \quad t > 0$$

by (3.12), it follows from the dominated convergence theorem that $\varphi(t, \cdot) \in \mathcal{C}(\bar{\mathcal{D}})$. The second claim follows by (3.15) and the observation that $T^{\mathcal{D},V}(t, x, y) > 0$ for $t > 0$ and $x, y \in \mathcal{D}$. Indeed, note that for every $\kappa > 0$, by the support theorem of Brownian paths there are paths inside \mathcal{D} starting from x at time 0 and ending at y at time κ with positive probability. This implies that for a given S_t^Ψ ,

$$\mathbb{E}_{0, S_t^\Psi}^{x,y} \left[e^{-\int_0^t V(B_{S_s^\Psi}) ds} \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right] > 0$$

holds which, in turn, implies $T^{\mathcal{D},V}(t, x, y) > 0$. \square

Remark 3.3. A much weaker version of Theorem 3.2 has been obtained in [21, Th. 1.1] and [23, Cor. 4] where $H^{\mathcal{D},V}$ was given by a divergence form elliptic operator. A strong maximum principle was conjectured in [21]. Our result applies to a much larger class of non-local operators and establishes the full strong maximum principle.

We also have a stability result.

Theorem 3.4 (Stability of solutions). *Suppose that $V_i \in \mathcal{C}(\bar{\mathcal{D}})$, $V_i \geq 0$, $\varphi_0^i \in L^2(\mathcal{D})$ and $F_i \in L^\infty((0, T); L^2(\mathcal{D}))$ for $i = 1, 2$. Let φ_i be the corresponding weak solution given by (3.12). Then for a constant $C = C(T, d, \Psi, \mathcal{D}, \alpha)$ we have*

$$\begin{aligned} \operatorname{ess\,sup}_{(0,T)} \|\varphi_1(t, \cdot) - \varphi_2(t, \cdot)\|_{L^2(\mathcal{D})} &\leq C \left[\mathfrak{A} \|V_1 - V_2\|_{L^\infty(\mathcal{D})} + \|\varphi_0^1 - \varphi_0^2\|_{L^2(\mathcal{D})} \right. \\ &\quad \left. + \operatorname{ess\,sup}_{(0,T)} \|F_1(s, \cdot) - F_2(s, \cdot)\|_{L^2(\mathcal{D})} ds \right], \end{aligned} \quad (3.16)$$

where

$$\mathfrak{A} = \max \left\{ \|\varphi_0^1\|_{L^2(\mathcal{D})}, \|\varphi_0^2\|_{L^2(\mathcal{D})}, \operatorname{ess\,sup}_{(0,T)} \|F_1(s, \cdot)\|_{L^2(\mathcal{D})}, \operatorname{ess\,sup}_{(0,T)} \|F_2(s, \cdot)\|_{L^2(\mathcal{D})} \right\}.$$

Proof. We begin with the following estimate. Consider $\psi_i \in L^2(\mathcal{D})$, $i = 1, 2$, and extend them outside of \mathcal{D} by zero. Then for every $x \in \mathcal{D}, t > 0$, we note that

$$\begin{aligned}
|T_t^{\mathcal{D}, V_1} \psi_1(x) - T_t^{\mathcal{D}, V_2} \psi_2(x)| &\leq \mathbb{E}^x \left[\left| e^{-\int_0^t V_1(X_s) ds} - e^{-\int_0^t V_2(X_s) ds} \right| |\psi_1(X_t)| \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right] \\
&\quad + \mathbb{E}^x \left[e^{-\int_0^t V_2(X_s) ds} |\psi_1(X_t) - \psi_2(X_t)| \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}} \right] \\
&\leq t \|V_1 - V_2\|_{L^\infty(\mathcal{D})} \mathbb{E}^x [|\psi_1(X_t)| \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] + \mathbb{E}^x [|\psi_1(X_t) - \psi_2(X_t)| \mathbf{1}_{\{\tau_{\mathcal{D}} > t\}}] \\
&\leq t \|V_1 - V_2\|_{L^\infty(\mathcal{D})} \left(\sup_{x \in \mathcal{D}} \mathbb{P}^x(\tau_{\mathcal{D}} > t) \right)^{1/2} \mathbb{E}^x [|\psi_1(X_t)|^2]^{1/2} \\
&\quad + \left(\sup_{x \in \mathcal{D}} \mathbb{P}^x(\tau_{\mathcal{D}} > t) \right)^{1/2} \mathbb{E} [|\psi_1(X_t) - \psi_2(X_t)|^2]^{1/2},
\end{aligned}$$

where in the second inequality we used the fact that $x \mapsto e^{-x}$ is Lipschitz continuous in $[0, \infty)$ and $V_2 \geq 0$. Thus, following a calculation similar to (3.14), we obtain

$$\begin{aligned}
\int_{\mathcal{D}} |T_t^{\mathcal{D}, V_1} \psi_1(x) - T_t^{\mathcal{D}, V_2} \psi_2(x)|^2 dx &\leq \kappa_1 t^2 \left(\sup_{x \in \mathcal{D}} \mathbb{P}^x(\tau_{\mathcal{D}} > t) \right) \|V_1 - V_2\|_{L^\infty(\mathcal{D})}^2 \|\psi_1\|_{L^2(\mathcal{D})}^2 \\
&\quad + \left(\sup_{x \in \mathcal{D}} \mathbb{P}^x(\tau_{\mathcal{D}} > t) \right) \|\psi_1 - \psi_2\|_{L^2(\mathcal{D})}^2,
\end{aligned} \tag{3.17}$$

with a constant κ_1 . By [5, Lem. 3.1], for every $k \in \mathbb{N}$ there exists a constant c_k , dependent on d, k , satisfying

$$\sup_{\mathcal{D}} \mathbb{E}^x [\tau_{\mathcal{D}}^k] \leq \frac{c_k}{(\Psi([\text{diam } \mathcal{D}]^{-2}))^k}, \tag{3.18}$$

where $\text{diam } \mathcal{D}$ denotes the diameter of \mathcal{D} . Therefore choosing $k = 5$ in (3.18) and putting it in (3.17), with a constant κ_2 we have

$$\begin{aligned}
\int_{\mathcal{D}} |T_t^{\mathcal{D}, V_1} \psi_1(x) - T_t^{\mathcal{D}, V_2} \psi_2(x)|^2 dx \\
\leq \kappa_2 t^2 (1 \wedge t^{-5}) \|V_1 - V_2\|_{L^\infty(\mathcal{D})}^2 \|\psi_1\|_{L^2(\mathcal{D})}^2 + \kappa_2 (1 \wedge t^{-5}) \|\psi_1 - \psi_2\|_{L^2(\mathcal{D})}^2.
\end{aligned} \tag{3.19}$$

We use (3.19) to estimate the left hand side of (3.16). Our main ingredient is formula (3.12). By Minkowski's integral inequality we note that

$$\begin{aligned}
&\left[\int_{\mathcal{D}} \left| \int_0^\infty T_u^{\mathcal{D}, V_1} \varphi_0^1(x) - T_u^{\mathcal{D}, V_1} \varphi_0^2(x) \eta_t(du) \right|^2 dx \right]^{1/2} \\
&\leq \int_0^\infty \left[\int_{\mathcal{D}} |T_u^{\mathcal{D}, V_1} \varphi_0^1(x) - T_u^{\mathcal{D}, V_1} \varphi_0^2(x)|^2 dx \right]^{1/2} \eta_t(du) \\
&\leq \sqrt{\kappa_2} \int_0^\infty \left[u(1 \wedge u^{-5/2}) \mathfrak{A} \|V_1 - V_2\|_{L^\infty(\mathcal{D})} + (1 \wedge u^{-5/2}) \|\varphi_0^1 - \varphi_0^2\|_{L^2(\mathcal{D})} \right] \eta_t(du) \\
&\leq \kappa_3 \left(\mathfrak{A} \|V_1 - V_2\|_{L^\infty(\mathcal{D})} + \|\varphi_0^1 - \varphi_0^2\|_{L^2(\mathcal{D})} \right),
\end{aligned} \tag{3.20}$$

with a constant κ_3 , where in the third line we used (3.19). Now we compute the difference for the rightmost term in (3.12). We again apply Minkowski's integral inequality to get

$$\begin{aligned}
&\left[\int_{\mathcal{D}} \left[\int_0^t \int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) |T_l^{\mathcal{D}, V_1} F_1(s, x) - T_l^{\mathcal{D}, V_2} F_2(s, x)| dl ds \right]^2 dx \right]^{1/2} \\
&\leq \int_0^t \int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) \left[\int_{\mathcal{D}} |T_l^{\mathcal{D}, V_1} F_1(s, x) - T_l^{\mathcal{D}, V_2} F_2(s, x)|^2 dx \right]^{1/2} dl ds
\end{aligned}$$

$$\begin{aligned}
 &\leq \int_0^t \int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) \left[\int_{\mathcal{D}} |T_l^{\mathcal{D}, V_1} F_1(s, x) - T_l^{\mathcal{D}, V_2} F_2(s, x)|^2 dx \right]^{1/2} dl ds \\
 &\leq \kappa_4 \int_0^t \int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) \left[l(1 \wedge l^{-5/2}) \|F_1(s, \cdot)\|_{L^2(\mathcal{D})} \|V_1 - V_2\|_{L^\infty(\mathcal{D})} + \right. \\
 &\quad \left. (1 \wedge l^{-5/2}) \|F_1(s, \cdot) - F_2(s, \cdot)\|_{L^2(\mathcal{D})} \right] dl ds \\
 &\leq \kappa_5 \left(\mathfrak{A} \|V_1 - V_2\|_{L^\infty(\mathcal{D})} + \operatorname{ess\,sup}_{(0, T)} \|F_1(s, \cdot) - F_2(s, \cdot)\|_{L^2(\mathcal{D})} ds \right), \tag{3.21}
 \end{aligned}$$

with constants κ_4, κ_5 , where in the fourth line we used again (3.19). In (3.21) we made use of Remark 3.2 showing that the integral in l converges. Then (3.16) follows by combining (3.12), (3.20) and (3.21). \square

The next result gives an Aleksandrov-Bakelman-Pucci (ABP) estimate for the time-fractional Cauchy problem (1.1). We will use a class of Bernstein functions with the following property, for an introduction see [6].

Assumption 3.2. The function Ψ is said to satisfy a weak lower scaling (WLSC) property with parameters $\underline{\mu} > 0$, $\underline{c} \in (0, 1]$ and $\underline{\theta} \geq 0$, if

$$\Psi(\gamma u) \geq \underline{c} \gamma^{\underline{\mu}} \Psi(u), \quad u > \underline{\theta}, \quad \gamma \geq 1.$$

We note that the WLSC property implies that the Hartman-Wintner condition (2.1) holds.

Example 3.1. The specific cases of Ψ in Example 2.1 satisfy Assumption 3.2 with $\underline{\theta} = 0$ and the following values:

- (i) $\underline{\mu} = \frac{\nu}{2}$
- (ii) $\underline{\mu} = \frac{\nu}{2}$
- (iii) $\underline{\mu} = \frac{\nu}{2} \wedge \frac{\tilde{\nu}}{2}$
- (iv) $\underline{\mu} = \frac{\nu - \tilde{\nu}}{2}$
- (v) $\underline{\mu} = \frac{\nu}{2}$,

where the order of cases listed above corresponds to the enumeration in Example 2.1.

We recall the standing assumption $V \geq 0$.

Theorem 3.5 (ABP estimate). *Let Ψ satisfy Assumption 3.2, and φ be a weak solution of (1.1) given by representation (3.6). Furthermore, assume $\varphi_0^+ \in L^\infty(\mathcal{D})$ and $F \in L^p((0, T) \times \mathcal{D})$ with $p > \frac{d}{2\underline{\mu}} + \frac{1}{\alpha}$. Then for almost every $(t, x) \in (0, T) \times \mathcal{D}$ we have*

$$\varphi^+(t, x) \leq \|\varphi_0^+\|_{L^\infty(\mathcal{D})} + C \|F^+\|_{L^p((0, T) \times \mathcal{D})}, \tag{3.22}$$

with a constant C , dependent on \mathcal{D}, p, d, Ψ . (Here the plus superscript means positive part.)

Proof. As before, we denote by q_t the transition density of $(X_t)_{t \geq 0}$. Extending φ_0 by 0 outside of \mathcal{D} , we note that for $t > 0$

$$T_t^{\mathcal{D}, V} \varphi(x) \leq \mathbb{E}^x [\varphi(X_t)] \leq \int_{\mathbb{R}^d} \varphi_0^+(y) q_t(x - y) dy \leq \|\varphi_0^+\|_{L^\infty(\mathcal{D})},$$

where the last estimate follows by the Hölder inequality. Thus also

$$\int_0^\infty T_u^{\mathcal{D}, V} \varphi_0(x) \eta_t(du) \leq \|\varphi_0^+\|_{L^\infty(\mathcal{D})}. \tag{3.23}$$

Taking into account (3.12) and (3.23), we only need to estimate the rightmost term in (3.12). Due to Assumption 3.2 and [5, Lem. 2.1] there exist positive constants κ_1, κ_2 satisfying

$$q_t(x) \leq \kappa_1 t^{-\frac{d}{2\mu}}, \quad x \in \mathbb{R}^d, \quad t \in (0, \kappa_2]. \quad (3.24)$$

Consider $\psi \in L^p(\mathcal{D})$ with $p > \frac{d}{2\mu} + \frac{1}{\alpha}$, and extend it outside \mathcal{D} by 0. Then for $t \in (0, \kappa_2]$ we obtain

$$T_t^{\mathcal{D},V} \psi(x) \leq \int_{\mathbb{R}^d} \psi^+(y) q_t(x-y) dy \leq \left[\int_{\mathbb{R}^d} \psi^+(y)^p q_t(x-y) dy \right]^{1/p} \leq \kappa_1^{1/p} t^{-\frac{d}{2p\mu}} \|\psi^+\|_{L^p(\mathcal{D})},$$

where in the last line we used (3.24). Therefore,

$$\begin{aligned} \int_0^{\kappa_2} l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) T_l^{\mathcal{D},V} \psi(x) dl &\leq \kappa_1^{1/p} \|\psi\|_{L^p(\mathcal{D})} \int_0^{\kappa_2} l^{-1/\alpha} l^{-\frac{d}{2p\mu}} g_1(l^{-1/\alpha}(t-s)) dl \\ &\leq \kappa_3 (t-s)^{\alpha-1-\frac{d\alpha}{2p\mu}} \|\psi^+\|_{L^p(\mathcal{D})}, \end{aligned} \quad (3.25)$$

with a constant κ_3 . To obtain the last line, note that by (3.10) (see also Remark 3.2) with $\gamma = \frac{1}{\alpha} + \frac{d}{2p\mu}$ and $s > 0$ we have

$$\begin{aligned} \int_0^{\kappa_2} l^{-\gamma} g_1(l^{-1/\alpha}s) dl &\leq c_1 \int_0^{\kappa_2} l^{-\gamma} \frac{1}{(1+l^{-1/\alpha}s)^{1+\alpha}} dl \\ &= s^{-\alpha\gamma+\alpha} \int_0^{\kappa_2 s^{-\alpha}} \frac{u^{1-\frac{d}{2p\mu}}}{(1+u^{1/\alpha})^{1+\alpha}} du \\ &\leq s^{-\alpha\gamma+\alpha} \int_0^\infty \frac{u^{1-\frac{d}{2p\mu}}}{(1+u^{1/\alpha})^{1+\alpha}} du, \end{aligned} \quad (3.26)$$

where the integral above is finite and $-\alpha\gamma + \alpha = \alpha - 1 - \frac{d\alpha}{2p\mu}$. Observe that the assumption $p > \frac{d}{2\mu} + \frac{1}{\alpha}$ gives $(\alpha - 1 - \frac{d\alpha}{2p\mu})\frac{p}{p-1} + 1 > 0$. Thus by using Young's inequality and (3.25) we obtain that

$$\begin{aligned} \int_0^t \int_0^{\kappa_2} l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) T_l^{\mathcal{D},V} F^+(s, x) dl ds &\leq \kappa_3 \int_0^t (t-s)^{\alpha-1-\frac{d\alpha}{2p\mu}} \|F^+(s, \cdot)\|_{L^p(\mathcal{D})} ds \\ &\leq \kappa'_3 \|F^+\|_{L^p((0,T) \times \mathcal{D})}, \end{aligned} \quad (3.27)$$

with a constant κ'_3 .

Next consider $t \in (\kappa_2, \infty)$. Since for $t \geq \kappa_2$ we have

$$\sup_{x \in \mathbb{R}^d} q_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-ix \cdot y} e^{-t\Psi(|y|^2)} dy \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\Psi(|y|^2)} dy \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-\kappa_2\Psi(|y|^2)} dy,$$

we obtain

$$\sup_{t \geq \kappa_2} \sup_{x, y \in \mathbb{R}^d} q_t(x, y) \leq q_{\kappa_2}(0). \quad (3.28)$$

Then for $t \geq \kappa_2$ we get

$$\begin{aligned} T_t^{\mathcal{D},V} \psi(x) &\leq \mathbb{E}^x [\psi^+(X_t) \mathbb{1}_{\{\tau_{\mathcal{D}} > t\}}] \leq \mathbb{E}^x [(\psi^+(X_t))^p]^{\frac{1}{p}} \mathbb{P}^x(\tau_{\mathcal{D}} > t)^{\frac{p-1}{p}} \\ &\leq q_{\kappa_2}(0)^{\frac{1}{p}} \|\psi^+\|_{L^p(\mathcal{D})}, \end{aligned}$$

using (3.28) in the last line. Since $\frac{1}{\alpha} > 1$, we have then

$$\int_{\kappa_2}^\infty l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) T_l^{\mathcal{D},V} \psi(x) dl \leq \kappa_4 \|\psi^+\|_{L^p(\mathcal{D})}, \quad (3.29)$$

with constant κ_4 . From (3.29) we get

$$\int_0^t \int_{\kappa_2}^{\infty} l^{-1/\alpha} g_1(l^{-1/\alpha}(t-s)) T_l^{\mathcal{D},V} F(s,x) dl \leq \kappa_5 \|F^+\|_{L^p((0,T)\times\mathcal{D})},$$

with a constant κ_5 . Thus (3.22) follows by using (3.12), (3.23) and (3.27). \square

Remark 3.4. It is interesting to point out the similarity of the bound in Theorem 3.5 and the result obtained in [5, Th. 3.2]. For $\alpha = 1$ a similar parabolic ABP estimate is obtained in [5, Th. 3.2] for $p > \frac{d}{2\mu} + 1$, and now it is seen that the second term equal to 1 is a contribution due to the usual time-derivative.

The above results are useful in the study of the inverse source problem discussed in [21]. This problem can be roughly stated as follows: Given $x_0 \in \mathcal{D}$, $T > 0$ fixed, and $\varphi_0 = 0$, if the inhomogeneity is given in the form $F(t,x) = \rho_1(t)\rho_2(x)$, is it possible to determine ρ_1 by single point observation data $\varphi(t, x_0)$ for $t \in [0, T]$? The spatial component ρ_2 simulates, for instance, a source of contaminants which may be hazardous, see [15, 21, 29] for further discussion.

Theorem 3.6. *Let $\varphi_0 = 0$ and $F(t,x) = \rho_1(s)\rho_2(x)$. We assume that either $\rho_2 \in L^\infty(\mathcal{D})$ or, in case Assumption 3.2 holds, $\rho_2 \in L^p(\mathcal{D})$ for some $p > \frac{d}{2\mu}$. Then for φ satisfying (3.12) we have that the map*

$$\mathcal{D} \ni x \mapsto \varphi(\cdot, x) \in L^1(0, T)$$

is continuous. Moreover, if $\rho_2 \geq 0$ and for some $x_0 \in \mathcal{D}$ we have $\varphi(\cdot, x_0) = 0$ in $L^1(0, T)$, then $\rho_1 = 0$.

Proof. Recall from (3.15)

$$T_t^{\mathcal{D},V} f(x) = \int_{\mathcal{D}} T^{\mathcal{D},V}(t,x,y) f(y) dy, \quad t > 0,$$

where the kernel $T(t,x,y)$ is bounded for every $t > 0$ and continuous in the variables $x, y \in \mathcal{D}$. This implies for every $f \in L^1(\mathcal{D})$ that the map $x \mapsto T_t^{\mathcal{D},V} f(x)$ is continuous in \mathcal{D} , for every $t > 0$. For $x \in \mathcal{D}$ and $t > 0$ define

$$\chi_t(x) = \int_0^\infty l^{-1/\alpha} g_1(l^{-1/\alpha}t) T_l^{\mathcal{D},V} \rho_2(x) dl.$$

If $\rho_2 \in L^\infty(\mathcal{D})$, then we have $|T_l^{\mathcal{D},V} \rho_2(x)| \leq \|\rho_2\|_{L^\infty(\mathcal{D})}$, as $V \geq 0$, and therefore, by Remark 3.2 and the dominated convergence theorem it follows that χ_t is continuous in \mathcal{D} for $t > 0$. Also, observe that in this situation

$$|\chi_t(x)| \leq \kappa_3 \|\rho_2\|_{L^\infty(\mathcal{D})} (t^{\alpha-1} + 1), \quad t > 0, x \in \mathcal{D},$$

with a constant κ_3 ; see also Remark 3.2. Now suppose that Assumption 3.2 holds and $\rho_2 \in L^p(\mathcal{D})$ for some $p > \frac{d}{2\mu}$. Then from the proof of Theorem 3.5 we see that for every $x \in \mathcal{D}$

$$|T_l^{\mathcal{D},V} \rho_2(x)| \leq C \left[l^{-\frac{d}{2p\mu}} \mathbf{1}_{(0,\kappa_2]}(l) + \mathbf{1}_{(\kappa_2,\infty)} \right] \|\rho_2\|_{L^p(\mathcal{D})},$$

with a constant C . Thus by (3.26) it is seen that χ_t is continuous in \mathcal{D} for every $t > 0$. Again by the proof of Theorem 3.5 we find

$$|\chi_t(x)| \leq \kappa_3 \|\rho_2\|_{L^p(\mathcal{D})} (t^{\alpha-1-\frac{d\alpha}{2p\mu}} + 1), \quad t > 0, x \in \mathcal{D},$$

with a constant κ_3 . Observe from (3.12) that

$$\varphi(t,x) = \int_0^t \rho_1(t-s) \chi_s(x) ds.$$

Let $x_n \rightarrow x \in \mathcal{D}$. Then

$$\int_0^T |\varphi(t, x_n) - \varphi(t, x)| dt \leq \int_0^T \int_0^t |\rho_1(t-s)| |\chi_s(x_n) - \chi_s(x)| ds dt.$$

Using the above bounds on χ_t and its continuity in \mathcal{D} , it follows by Young's inequality and dominated convergence that the right hand side above tends to zero as $n \rightarrow \infty$. This shows the first part of the theorem.

To obtain the second part, note that $\chi \cdot (x_0) \in L^1(0, T)$. By the given condition we also have

$$\int_0^t \rho_1(s) \chi_{t-s}(x_0) ds = 0, \quad t \in [0, T].$$

The above equality makes sense due to the continuity result we proved above. By Titchmarsh's theorem [10, 31] there exist non-negative κ_1, κ_2 with $\kappa_1 + \kappa_2 \geq T$ and $\rho_1 = 0$ almost everywhere in $(0, \kappa_1)$ and $\chi \cdot (x_0) = 0$ almost everywhere in $(0, \kappa_2)$. However, by our assumption on ρ_2 it follows that $T_l^{\mathcal{D}, V} \rho_2 > 0$ on \mathcal{D} (see the proof of Theorem 3.3) and $g_1 > 0$ on $(0, \infty)$, implying $\chi \cdot (x_0) > 0$ in every $(0, \kappa_2)$ for $\kappa_2 > 0$. Thus $\kappa_1 \geq T$, which completes the proof of the theorem. \square

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