

Contributions to the Derivation and Well-posedness Theory of Kinetic Equations



Megan Kate Griffin-Pickering

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge

This thesis is submitted for the degree of
Doctor of Philosophy

Jesus College

August 2019

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. It does not exceed the prescribed word limit for the relevant Degree Committee.

Megan Kate Griffin-Pickering
August 2019

Abstract

This thesis is concerned with certain partial differential equations, of kinetic type, that are involved in the modelling of many-particle systems.

The Vlasov-Poisson system is a model for a dilute plasma in an electrostatic regime. The classical version describes the electrons in the plasma. The first part of this thesis focuses on a variant known as the Vlasov-Poisson system with massless electrons (VPME), which instead describes the ions. Compared to the classical system, VPME includes an additional exponential nonlinearity, with the consequence that several results known for the classical system were not previously available for VPME.

In particular, global well-posedness had not been proved. In this thesis, we prove that VPME has unique global-in-time solutions in two and three dimensions, for a general class of initial data matching results currently available for the classical system.

The quasi-neutral limit is an important approximation of Vlasov equations in plasma physics, in which the Debye screening length of the plasma tends to zero; the formal limiting system is a kinetic Euler equation. For a rigorous passage to the limit, a restriction on the initial data is required. In this thesis, we prove the quasi-neutral limit from the VPME system to the kinetic isothermal Euler system, for a certain class of rough data.

We then investigate the rigorous connection between these Vlasov equations and the associated particle systems. We derive VPME and the two kinetic Euler models associated respectively to the classical Vlasov-Poisson and VPME systems rigorously from systems of extended charges.

For my parents

Preface

Statement on Collaboration. Chapter 1 is a literature review completed under the supervision and guidance of Prof Mikaela Iacobelli and Prof Clément Mouhot.

The problems considered in Chapters 2, 3, 4, 5 and Appendix A were suggested by Prof Mikaela Iacobelli. These chapters consist of my own work completed under the supervision and guidance of Prof Mikaela Iacobelli.

List of Works. This thesis contains the following works:

- The contents of Chapter 2 are submitted for publication in the form of article [35].
- The contents of Chapters 3, 4 and the part of Chapter 5 dealing with the KIsE system are accepted for publication in *Journal de Mathématiques Pures et Appliquées*, in the form of article [36].
- The contents of Chapter 5 on the KInE system are published in *SIAM Journal on Mathematical Analysis*, in the form of article [37].

Acknowledgements

This thesis would not have been possible without the efforts and support of many people. I am very grateful to them all.

I would like to thank first of all my supervisors, Mikaela Iacobelli and Clément Mouhot, for providing inspiration, guidance, support and encouragement over the course of my PhD and the writing of this thesis. It has been a pleasure and a privilege.

I thank the many people in the Maths department who have helped me along the way. Thanks to the kinetic group for a lively and supportive working environment: Ludovic Cesbron, Helge Dietert, Amit Einav, Jo Evans, Jessica Guerand, Franca Hoffmann, Tom Holding, Harsha Hutridurga, Andrei Ichim, Angeliki Menegaki, Iván Moyano, Ariane Trescases and Renato Velozo. Thanks also to Tessa Blackman and Arti Sheth Thorne for administrative support. I thank my CCA cohort: Kweku Abraham, Nicolai Baldin, Fritz Hiesmayr, Lisa Kreusser, Eardi Lila, Matthias Löffler, Erlend Riis, Andrew Swan, Maxime Van de Moortel and Mo Dick Wong.

I thank everyone at ETH Zürich who welcomed me on a research visit there during the course of this work.

I thank Maxime Hauray and Claude Warnick for examining this thesis, and for a very interesting discussion.

I would like to close by expressing my deep gratitude to my friends and family for their invaluable support over the years. A special mention must go to my wonderful housemates, Em Black, Nate Dunmore and Sophie Ip: thank you for everything; this thesis would not exist without you. Finally, I thank my parents for their unconditional and tireless support over many years and hurdles.

Table of contents

1	Introduction	1
1.1	Kinetic Equations in Physics	2
1.2	Plasma Models	6
1.3	Well-Posedness Theory for Vlasov Equations	21
1.4	Well-Posedness Theory for the Vlasov-Poisson System	40
1.5	The Quasi-Neutral Limit	51
1.6	Mean Field Limits	61
1.7	Summary of Results	67
2	VPME: Global Well-posedness in 2D and 3D	75
2.1	Statement of Results	77
2.2	Strategy	78
2.3	Basic Estimates	79
2.4	Regularity of the Electric Field	81
2.5	Stability of the Electric Field	90
2.6	Wasserstein Stability and Uniqueness	93
2.7	A Priori Estimates on the Mass Density	101
2.8	A Regularised System	110
2.9	Construction of Solutions	114
3	The Quasi-Neutral Limit for the VPME System With Rough Data	135
3.1	Statement of Results	136
3.2	Estimates on the Electric Field	139
3.3	Wasserstein Stability	146
3.4	Growth Estimates	152
3.5	Quasi-Neutral Limit	160

4	Derivation of the VPME System from a System of Extended Ions	163
4.1	Introduction	163
4.2	Statement of Results	165
4.3	Preliminary Estimates	168
4.4	W_2 stability for the regularised VPME system	174
4.5	Convergence from the Regularised System to VPME	180
4.6	Typicality	184
5	Particle Derivations for Kinetic Euler Systems	187
5.1	Introduction	188
5.2	Statement of Results	191
5.3	Strategy	195
5.4	Preliminaries	197
5.5	Proof for General Configurations	201
5.6	Typicality	219
	References	223
	Appendix A Differential Inequalities	229

Chapter 1

Introduction

Contents

1.1	Kinetic Equations in Physics	2
1.2	Plasma Models	6
1.2.1	The Vlasov-Poisson System	6
1.2.2	The Massless Electron Model	9
1.2.3	Quasi-Neutrality	13
1.2.4	Derivation from a Particle System	18
1.3	Well-Posedness Theory for Vlasov Equations	21
1.3.1	Vlasov Equations	21
1.3.2	Regular Potentials	23
1.3.3	Coulomb Interaction	36
1.4	Well-Posedness Theory for the Vlasov-Poisson System	40
1.4.1	Basic Estimates	41
1.4.2	Uniqueness for Solutions with Bounded Density	47
1.4.3	Global Existence of Solutions	48
1.4.4	The Vlasov-Poisson System with Massless Electrons	49
1.5	The Quasi-Neutral Limit	51
1.5.1	Challenges	51
1.5.2	Known Results	52
1.5.3	Existence Theory for Kinetic Euler Systems	53
1.5.4	Rough Data Results	54

1.6	Mean Field Limits	61
1.6.1	Lipschitz Forces	62
1.6.2	Singular Forces	63
1.6.3	Mean Field Limits for VPME	65
1.6.4	Derivation of Kinetic Euler Systems	66
1.7	Summary of Results	67
1.7.1	Well-Posedness for the VPME System	67
1.7.2	Quasi-Neutral Limit for VPME With Rough Data	68
1.7.3	Derivation of VPME from a System of Extended Ions	70
1.7.4	Derivation of Kinetic Euler Systems from Systems of Extended Charges	70

1.1 Kinetic Equations in Physics

This thesis provides a contribution to the mathematical theory of kinetic equations. Kinetic equations are a class of partial differential equations used in the modelling of large many-particle systems.

A fundamental concrete example of such a system is a gas. The atomic theory of matter posits that a gas consists of a very large number of particles. However, it is not possible for humans to observe these particles directly. Instead, we experience a gas through its macroscopic properties, such as its temperature and pressure. It is a fundamental physical problem to understand how the observable behaviours of a gas emerge from the underlying dynamics of its constituent particles. The ‘kinetic theory of gases’ is a physical theory that aims to explain these behaviours as the consequences of the motion of the gas particles. The mathematical field of kinetic theory deals with a class of PDEs that arise from this theory.

The different scales of description of a gas can be translated into different types of mathematical models. In a ‘microscopic’ model, one tracks the states of all particles in the system individually. Typically this results in a classical mechanical N -body problem. Of course, for the physical applications we have in mind, N will be far too large for such a model to be practical. For example, for gases at room temperature, N is of order 10^{23} . It is therefore desirable to replace this model with a coarser description of the system.

A ‘macroscopic’ model describes the evolution of *observable* quantities associated to the system, such as the density, velocity and temperature. For gas modelling, equations from fluid

mechanics are typically used. In this case, the gas is modelled as a continuum with a single velocity at each point in space.

Kinetic equations offer a ‘mesoscopic’ description of physical systems, at a scale in between the particle and continuum models. A kinetic model uses a statistical description of the system under consideration. The state of the system is described using a density function $f = f(t, x, v)$. The key structural feature is that f is allowed to depend not only on the time t and the spatial position x of the particles, but also on their velocity v . The model retains the information that gas particles at a particular spatial position may have different velocities, in contrast to a continuum model. The function $f(t, x, v)$ is then interpreted as the density of particles with position x and velocity v at time t .

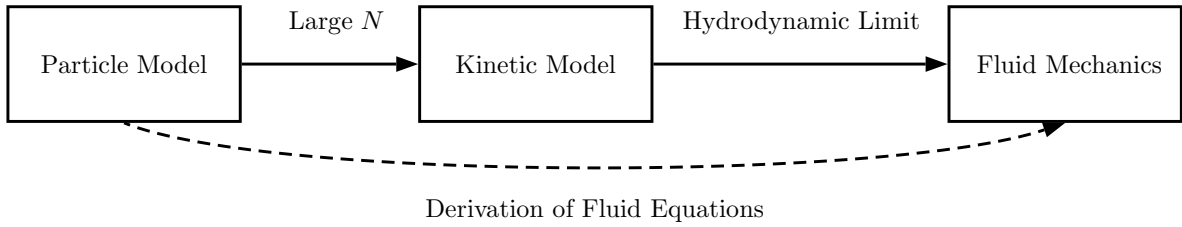


Fig. 1.1 Microscopic to macroscopic hierarchy for kinetic equations

It is a key problem in mathematical kinetic theory to justify the place of a kinetic equation in this hierarchy of models. On the one hand, we wish to derive kinetic equations from the underlying models they describe; this validates the use of the equation from a mathematical perspective. On the other hand, we wish to connect kinetic models to higher level macroscopic models, through hydrodynamic limits. This provides a possible strategy for deriving macroscopic models, such as the equations of fluid dynamics, from particle models - by passing via an intervening kinetic model.

The origins of modern kinetic theory are usually traced to the work of Maxwell [69] on the modelling of gases. In this work, Maxwell derived a version of the Boltzmann equation for dilute gases and identified its equilibrium distribution. Boltzmann [14] generalised this work and derived the ‘H-theorem’, which says that the (physical) entropy of solutions of the Boltzmann equation that are not in equilibrium must increase over time. The Boltzmann equation gives a mesoscopic description of a system of particles that interact with each other through collisions and otherwise move freely. It takes the form

$$\partial_t f + v \cdot \nabla_x f = Q(f),$$

where Q is a nonlinear integral operator, acting in v only, that describes the change in f due to collisions. Notice that the interaction between particles in this model is localised in x .

Subsequently, Jeans [58] proposed a kinetic model to describe large systems of stars interacting through gravity. This model is now referred to as the *gravitational* Vlasov-Poisson system. In contrast to the Boltzmann equation, this is a collisionless model. In this thesis, we will consider the *electrostatic* Vlasov-Poisson system, which is very closely related to the equation proposed by Jeans.

The principal focus of this thesis will be on kinetic equations that arise in the modelling of plasma. A plasma is an ionised gas, which means that it consists of charged particles. The interaction between these particles is thus electromagnetic in nature. An important difference between electromagnetic interactions and interaction through collisions is that electromagnetic forces are long range. Charged particles influence each other even if the spatial separation between them is large. For this reason, the models typically used for ionised gases have a different structure from those typically used for electrically neutral gases.

Landau [60] proposed a kinetic model to describe the evolution of a plasma, now known as the Landau-Coulomb equation. This model is based on adapting the Boltzmann equation to the case of Coulomb interaction, and thus focuses on describing collisions within a plasma. It takes the form

$$\partial_t f + v \cdot \nabla_x f = Q_L(f), \quad (1.1)$$

where $Q_L(f)$ denotes the nonlinear Landau collision operator.

Vlasov [84, 85] proposed an alternative kinetic model for plasma. This equation takes the form

$$\partial_t f + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0, \quad (1.2)$$

where E and B are, respectively, the electric and magnetic fields generated by the plasma itself. The fields E and B are found using the Maxwell equations. This interaction is therefore non-local in space, representing the fact that particles feel the influence of other particles in the system even if their spatial separation is large. The system (1.2) is known as the (non-relativistic) Vlasov-Maxwell system.

In [84], Vlasov first considers a version of (1.2) that includes a term accounting for collisions. However, he then argues that on the physical scales relevant to plasma, the collision terms can be neglected. This illustrates an important principle: the ‘correct’ model to use for a physical system is not fixed solely based on the type of interactions in the system, but rather depends also on the physical regime in which the system is considered. For this reason, quantitative results are important in the study of hierarchies such as Figure 1.1. They allow us to identify, from a mathematical perspective, the timescales and regimes of physical parameters on which the models are valid.

The central equation we consider in this thesis is a kinetic model for the ions in an unmagnetised plasma. It is known as the Vlasov-Poisson system with massless electrons, or VPME. This is a non-collisional model, similar in structure to (1.2). It is a variant of the better-known Vlasov-Poisson system, which models the electrons in an unmagnetised plasma. Our goal is to consider a hierarchy of models similar to the one shown in Figure 1.1, centred around the VPME system.

The macroscopic limit considered in this thesis is known as the quasi-neutral limit. This is not a hydrodynamic limit, because the limiting equation is still a kinetic model. However, our study is intended to be in a similar spirit to this framework: we will derive the macroscopic model from an underlying kinetic model, and use this to connect the macroscopic model to a particle system. The macroscopic limit we consider is motivated by a widely used approximation in plasma physics, known as quasi-neutrality. In this approximation, a key characteristic parameter of the plasma, known as the Debye length, is set to zero. Under this approximation, equations of the form (1.2) are replaced by kinetic equations with a more singular type of interaction. In this thesis we refer to these equations as ‘kinetic Euler’ systems. The mathematical study of the quasi-neutral limit is motivated by the need to identify the physical regime in which this approximation is indeed valid.

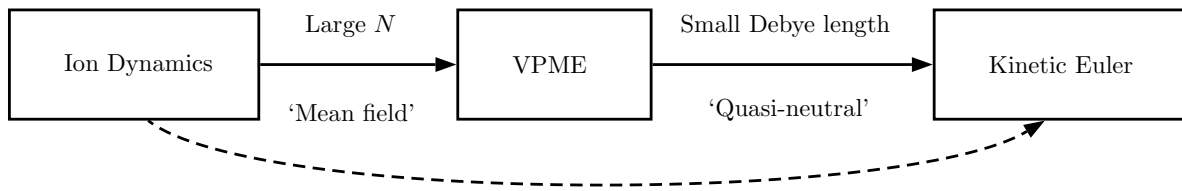


Fig. 1.2 Microscopic to macroscopic hierarchy for the VPME system

An important step in this investigation is to study the well-posedness of the VPME system. Before studying solutions of the VPME system, we wish to prove that such solutions exist and are unique for a given initial datum. Previously, global-in-time well-posedness results were not available for the VPME system in any dimension higher than one. In Chapter 2 of this thesis, we fill this gap by proving global well-posedness for the VPME system in dimension two and three.

Moreover, we develop a key toolbox of estimates on the electric field for the VPME system, which allow many results for the Vlasov-Poisson system to be adapted to the VPME case. We demonstrate this in our study of the hierarchy shown in Figure 1.2, where these estimates are key element. In Chapter 3, we prove a rigorous limit from the VPME system to a kinetic Euler model. In Chapter 4, we derive the VPME system from a microscopic system of extended charges. In Chapter 5, we combine these limits to identify a physical regime in which the kinetic Euler system can be derived from a system of extended charges.

1.2 Plasma Models

1.2.1 The Vlasov-Poisson System

In this thesis, we will focus primarily on kinetic equations arising in the modelling of plasma. Plasma is a state of matter that forms when an electrically neutral gas is subjected to high temperatures or a strong electromagnetic field. This causes some of the gas particles to dissociate, splitting apart into charged particles. The resulting system is an ionised gas.

Plasma is abundant in the universe. The study of plasma is important in astrophysics - in space, plasma is found for example in stars, the solar wind and the interstellar medium. Plasma is also studied as part of research into nuclear fusion reactors.

The degree of ionisation - that is, the fraction of gas particles that dissociate - varies in real plasmas. In this thesis, we will concentrate on models that describe only the charged species in the plasma, neglecting the neutral species that do not dissociate. The interactions with the neutral species are very weak in comparison to the interactions of the charged species. There are two types of charged particle in a plasma: negatively charged electrons and positively charged ions.

The relevant physical situation to keep in mind is therefore a coupled system of ions and electrons. Since these particles are charged, they will interact with each other principally through electromagnetic forces. In fact, it is usual to make an assumption which decouples the dynamics of the two species. This assumption is based on the fact that the mass of an electron is much smaller than the mass of an ion. Consequently, an electron typically moves much more quickly than an ion. This results in a separation between the timescales on which each species evolves.

Consider first the point of view of the electrons. The ions are, relatively speaking, much more massive and therefore slow moving. For this reason, it is common to assume that the ions are stationary over the interval of time on which the plasma is observed. It remains to model the dynamics of the electrons.

The **Vlasov-Poisson** system is a well-known kinetic equation describing this situation. The electrons are described by a density function $f = f(t, x, v)$, which is the unknown in the

following system of equations:

$$(VP) := \begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q_e}{m_e} E \cdot \nabla_v f = 0, \\ \nabla_x \times E = 0, \\ \varepsilon_0 \nabla_x \cdot E = Q_i + q_e \rho_f, \\ \rho_f(t, x) := \int_{\mathbb{R}^d} f(t, x, v) dv \\ f|_{t=0} = f_0 \geq 0. \end{cases} \quad (1.3)$$

Here q_e is the charge on each electron, m_e is the mass of an electron and ε_0 is the electric permittivity. $Q_i : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is the charge density contributed by the ions, which is independent of time since we assume that the ions are stationary. The electrons experience a force $q_e E$, where E is the **electric field** induced by the whole plasma. This is found from the Gauss law

$$\nabla_x \times E = 0, \quad \varepsilon_0 \nabla_x \cdot E = Q_i + q_e \rho_f,$$

which arises as an electrostatic approximation of the full Maxwell equations.

The name ‘Poisson’ comes from the following rewriting of the system. Since $\nabla_x \times E = 0$, it is possible to write E as a gradient. Thus there exists a function U such that $E = -\nabla_x U$. Then U must satisfy the Poisson equation

$$-\varepsilon_0 \Delta U = Q_i + q_e \rho_f.$$

The function U is known as the **electrostatic potential**.

The system (1.3) expresses the fact that each electron in the plasma feels the influence of the other particles in the plasma in an averaged sense, through the electric field E induced collectively by the whole plasma. This is a long-range interaction between particles. In particular, this equation does not account for collisions between particles, which would require the Landau-Coulomb operator Q_L mentioned above in equation (1.1).

The system (1.3) needs to be equipped with boundary conditions. For applications to nuclear fusion, it is typical to consider plasmas confined by a strong magnetic field. For astrophysical applications, we might consider a condition of decay at infinity. In this thesis, we will focus on the case of periodic boundary conditions. That is, we assume that the spatial variable x lies in the d -dimensional flat torus \mathbb{T}^d , which can be identified with the space $[-\frac{1}{2}, \frac{1}{2}]^d$. The velocity variable v lies in the whole Euclidean space \mathbb{R}^d .

Moreover, we restrict in particular to the case where the background ion density Q_i is spatially uniform. This is considered to be a reasonable approximation under the assumption

that the scale of fluctuations in the ion density is much larger than the scale of fluctuations in the electron density. This results in the system

$$(VP) := \begin{cases} \partial_t f + v \cdot \nabla_x f + \frac{q_e}{m_e} E \cdot \nabla_v f = 0, \\ \nabla_x \times E = 0, \\ \varepsilon_0 \nabla_x \cdot E = q_e (\rho_f - \int_{\mathbb{T}^d \times \mathbb{R}^d} f \, dx \, dv), \\ f|_{t=0} = f_0 \geq 0. \end{cases} \quad (1.4)$$

The ion charge density is chosen to be

$$Q_i \equiv -q_e \int_{\mathbb{T}^d \times \mathbb{R}^d} f \, dx \, dv$$

so that the system is globally neutral. This is required by the conservation of charge, since the plasma forms from an electrically neutral gas. In mathematical treatments, it is common to see (1.4) written in the rescaled form

$$(VP) := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ \nabla_x \times E = 0, \\ -\nabla_x \cdot E = \rho_f - 1, \\ f|_{t=0} = f_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 \, dx \, dv = 1. \end{cases} \quad (1.5)$$

Another commonly considered form of the system is the case where the spatial variable x lies in the whole space \mathbb{R}^d . In this case the boundary condition is that the electric field should decay at infinity. The electric field can then be represented as the convolution of ρ_f with the Coulomb kernel

$$K(x) = C_d \frac{x}{|x|^d}.$$

The system then reads as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E(x) = C_d \int_{y \in \mathbb{R}^d} \frac{x-y}{|x-y|^d} \rho_f(y) \, dy, \\ f|_{t=0} = f_0 \geq 0, \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0 \, dx \, dv = 1. \end{cases} \quad (1.6)$$

In this case, if we included a uniformly distributed background of ions this would result in a system with infinite mass. It is therefore usual to consider a system describing only the dynamics of electrons, resulting in a uniformly vanishing background.

By changing the sign in the equation for E in (1.6), we obtain a model for a system with gravitational interaction:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E(x) = -C_d \int_{y \in \mathbb{R}^d} \frac{x-y}{|x-y|^d} \rho_f(y) dy, \\ f|_{t=0} = f_0 \geq 0, \int_{\mathbb{R}^d \times \mathbb{R}^d} f_0 dx dv = 1. \end{cases}$$

This model is used to describe, for example, the dynamics of large collections of stars. It was discovered by Jeans [58]. We do not consider gravitational Vlasov-Poisson systems in this thesis.

1.2.2 The Massless Electron Model

The classical version of the Vlasov-Poisson system, which was presented above, describes the electrons in a dilute, unmagnetised plasma. We may instead wish to model the ions in the plasma. This leads to a variant of the Vlasov-Poisson system that will be the second main equation considered in this thesis.

From the point of view of the ions, the electrons have a very small mass and so are very fast moving. It is not possible to assume that they are stationary. Instead, we observe that, since the electrons move quickly relative to the ions, the frequency of electron-electron collisions will be high in comparison to other kinds of collisions, and electron-electron collisions are relevant and frequent on the typical timescale of evolution of the ions. It is therefore common in physics literature to assume that the electrons are close to thermal equilibrium.

The limit of **massless electrons** is the limit in which the ratio between the masses of the electrons and ions, m_e/m_i , tends to zero. Here m_e is the mass of an electron and m_i is the mass of an ion. This limit is physically relevant due to the large disparity in mass between the ions and electrons. In this limit, it is assumed that the electrons instantaneously assume the equilibrium distribution.

The equilibrium distribution can be identified by studying the equation for the evolution of electrons. Let the ion density $\rho[f_i]$ be fixed, and assume that all ions carry the same charge q_i . We have discussed that the evolution of the electron density can be modelled by the Vlasov-Poisson system (1.3). As discussed above, in the long time regime we consider we expect the effect of electron-electron collisions to be significant. We therefore augment the system (1.3)

with a collision operator. Bellan [11, Chapter 13, Equation (13.46)] suggests the following rescaling of the Landau-Coulomb operator for modelling collisions in a plasma:

$$Q_{\text{elec}}(f) = \frac{C_e}{m_e^2} Q_L(f),$$

where C_e is a constant depending on physical quantities such as the electron charge q_e and number density n_e , but *not* on the electron mass m_e . The Landau-Coulomb operator Q_L is defined as follows: for a given function $g = g(v) : \mathbb{R}^d \rightarrow \mathbb{R}$,

$$Q_L(g) := \nabla_v \cdot \int_{\mathbb{R}^d} a(v - v_*) : [g(v_*) \nabla_v g(v) - g(v) \nabla_v g(v_*)] dv_*.$$

The tensor a is defined by

$$a(z) = \frac{|z|^2 - z \otimes z}{|z|^3}.$$

We thus consider the following model for the electron density f_e , here posed on the torus $x \in \mathbb{T}^d$:

$$\begin{cases} \partial_t f_e + v \cdot \nabla_x f_e + \frac{q_e}{m_e} E \cdot \nabla_v f_e = \frac{C_e}{m_e^2} Q_L(f_e), \\ \nabla_x \times E = 0, \quad \varepsilon_0 \nabla_x \cdot E = q_i \rho[f_i] + q_e \rho[f_e]. \end{cases} \quad (1.7)$$

Consider the rescaling in velocity

$$F_e(t, x, v) = m_e^{-\frac{d}{2}} f_e \left(t, x, \frac{v}{\sqrt{m_e}} \right).$$

Notice that $\rho[F_e] = \rho[f_e]$. Then F_e satisfies

$$\begin{cases} \sqrt{m_e} \partial_t F_e + v \cdot \nabla_x F_e + q_e E \cdot \nabla_v F_e = C_e Q_L(F_e), \\ \nabla \times E = 0, \quad \varepsilon_0 \nabla \cdot E = q_i \rho[f_i] + q_e \rho[F_e]. \end{cases} \quad (1.8)$$

We assume that F_e converges to a stationary distribution $\bar{f}_e = \bar{f}_e(x, v)$ as m_e tends to zero, and focus on formally identifying \bar{f}_e .

We expect that \bar{f}_e should satisfy the equation

$$v \cdot \nabla_x \bar{f}_e + q_e E \cdot \nabla_v \bar{f}_e = C_e Q_L(\bar{f}_e).$$

By considering the following entropy functional

$$H[f] = \int f \log f \, dx \, dv,$$

which is non-increasing for solutions of (1.8), it is possible to show that \tilde{f}_e should be a **local Maxwellian** of the form

$$\tilde{f}_e(x, v) = \rho_e(x) (\pi \beta_e(x))^{d/2} \exp \left[-\beta_e(x) |v - u_e(x)|^2 \right], \quad (1.9)$$

since these are precisely the distributions for which the time derivative of the entropy vanishes. The electron density ρ_e , mean velocity u_e and inverse temperature β_e can then be studied using an argument similar to the one given in the proof of [7, Theorem 1.1].

Substituting the form (1.9) into equation (1.7), we obtain the following identity for all x such that $\rho_e(x) \neq 0$ and all $v \in \mathbb{R}^d$:

$$\begin{aligned} & -\nabla_x \beta_e \cdot (v - u_e) |v - u_e|^2 - u_e \cdot \nabla_x \beta_e |v - u_e|^2 + \beta_e (v - u_e)^\top \nabla_x u_e (v - u_e) \\ & + (v - u_e) \cdot \left[\nabla_x \log(\rho_e \beta_e^{d/2}) - q_e \beta_e E + u_e \cdot \nabla_x u_e \right] + u_e \cdot \nabla_x \log(\rho_e \beta_e^{d/2}) = 0. \end{aligned}$$

For each fixed x , the left hand side is a polynomial in $v - u_e(x)$, whose coefficients must all be equal to zero. For example, by looking at the cubic term we see that $\nabla_x \beta_e = 0$ and thus β_e must be a constant independent of x .

The quadratic term then gives

$$v^\top \nabla_x u_e v = 0 \quad \text{for all } v \in \mathbb{R}^d,$$

which implies that $\nabla_x u_e$ is skew-symmetric. On a spatial domain for which a Korn inequality holds, it is possible to restrict which u_e can occur. For example, in the case of the torus $x \in \mathbb{T}^d$ considered, the fact that the symmetric part of $\nabla_x u_e$ vanishes implies that u_e is constant [25, Proposition 13].

Finally, from the linear term we obtain that

$$\nabla_x \log(\rho_e \beta_e^{d/2}) - q_e \beta_e E = 0.$$

Since $\nabla_x \times E = 0$, E is a gradient - that is, it can be written as $E = -\nabla U$ for some function U . Then

$$\nabla_x \log(\rho_e \beta_e^{d/2}) = -q_e \beta_e \nabla_x U.$$

From this we deduce that ρ_e should be of the form

$$\rho_e(x) = A \exp(-q_e \beta_e U),$$

for some constant $A > 0$. This is known as a **Maxwell-Boltzmann** law.

Since $\nabla_x \times E = 0$, we can write $E = -\nabla_x U$, where U is the electrostatic potential within the plasma. Then

$$\nabla_x \rho_e = -\frac{q_e}{k_B T_e} \rho_e \nabla_x U.$$

From this we deduce that ρ_e should be of the form

$$\rho_e(x) = A \exp\left(-\frac{q_e U(x)}{k_B T_e}\right),$$

which is known as a **Maxwell–Boltzmann** law.

See Bardos et al. [7] for rigorous results on identifying the Maxwell-Boltzmann law as the limiting distribution of the electrons in the massless limit, for Vlasov-Poisson models with collision operators of Boltzmann or BGK type.

From equation (1.7), we see that the electrostatic potential U should satisfy the following semilinear elliptic PDE:

$$-\varepsilon_0 \Delta U = q_i \rho[f_i] + A q_e \exp\left(-\frac{q_e U}{k_B T_e}\right). \quad (1.10)$$

The normalising constant A is chosen so that the system is globally neutral, that is, the total charge is zero:

$$\int_{\mathbb{T}^d} q_i \rho[f_i] + A q_e \exp\left(-\frac{q_e U}{k_B T_e}\right) dx = 0.$$

The equation (1.10) replaces the standard Poisson equation for the electrostatic potential in the Vlasov-Poisson system (1.3). After a suitable normalisation of physical constants, this leads to the following system for the ions:

$$(VPME) := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ \Delta U = e^U - \rho_f, \\ f|_{t=0} = f_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 dx dv = 1. \end{cases} \quad (1.11)$$

This is known as the Vlasov-Poisson system **with massless electrons**, or **VPME**.

Note that solutions of the system (1.11) always satisfy global neutrality, since on the torus the Poisson equation

$$\Delta U = h$$

can only be solved if h has total integral zero.

The VPME system has been used in the physics literature in, for instance, numerical studies of the formation of ion-acoustic shocks [68, 77] and the development of phase-space vortices behind such shocks [15], as well as in studies of the expansion of plasma into vacuum [70]. A physically oriented introduction to the model (1.11) may be found in [40].

This system is one of the central equations considered in this thesis. We will investigate several mathematical questions related to this system. Chapter 2 focuses on the problem of showing global well-posedness for this equation in dimension two and three. We then look at two limits relating the VPME system to other models for the ions in a plasma. One is the **quasi-neutral** limit, which connects the VPME system (1.11) to a macroscopic model for the plasma. The other is the **mean field** limit, in which the aim is to derive the VPME system from an underlying system of interacting particles.

1.2.3 Quasi-Neutrality

Quasi-neutrality is a concept from plasma physics, referring to a situation in which a certain characteristic scale of the plasma is small. This property occurs frequently in real plasmas. We will use this idea to motivate a mathematical problem known as the quasi-neutral limit. This is a limit in which one can derive other plasma models from the Vlasov-Poisson systems, under a certain rescaling.

1.2.3.1 The Debye Length

Plasmas have several important characteristic scales, one of which is the **Debye (screening) length**, λ_D . The Debye length is important in describing electrostatic phenomena in the plasma. For example, it characterises charge separation within the plasma, describing the scale at which it can be observed that the plasma contains areas with a net positive or negative charge, and so is not microscopically neutral.

In physics textbooks, such as [11, 23, 63], the Debye length is usually motivated by a description of charge screening within plasmas. Electric fields applied to plasmas are damped, because the mobile charges within the plasma move to oppose the field. For example, if a positive test charge is placed into a plasma, the electrons will be attracted to and so move towards it, consequently neutralising the charge density. The Debye length describes the scale beyond which such fields are damped.

To see this more explicitly, consider the aforementioned situation of placing a point test charge into a plasma. Assume that the plasma is in equilibrium before the test charge is placed. We consider a regime in which the ions can be assumed to be fixed and uniformly distributed,

and we only take the motion of the electrons into account. We assume that their motion is fast enough that the electrons are always thermalised.

We consider placing a point test charge of charge β at the origin. When the charge is added, it induces a potential Φ . In response, the electrons take on a Maxwell-Boltzmann distribution. Their spatial density is therefore

$$\rho_e = n_e \exp\left(\frac{q_e \Phi}{k_B T_e}\right),$$

where q_e is the charge on an electron, k_B is the Boltzmann constant and T_e is the electron temperature and n_e is the spatial density of electrons prior to the introduction of the test charge. The normalisation n_e is chosen because the potential Φ should decay in the far field. The potential Φ should solve the following equation:

$$\varepsilon_0 \Delta \Phi = n_e q_e \left(\exp\left(\frac{q_e \Phi}{k_B T_e}\right) - 1 \right) - \beta \delta_0,$$

where δ_0 denotes the Dirac distribution.

By rescaling this equation, we can identify key parameters. Let $\tilde{\Phi} := \frac{q_e \Phi}{k_B T_e}$. Then

$$\frac{\varepsilon_0 k_B T_e}{n_e q_e^2} \Delta \tilde{\Phi} = e^{\tilde{\Phi}} - 1 - \frac{\beta}{n_e q_e} \delta_0. \quad (1.12)$$

This motivates defining a scale λ_D by

$$\lambda_D := \left(\frac{\varepsilon_0 k_B T_e}{n_e q_e^2} \right)^{1/2}. \quad (1.13)$$

This scale is the Debye length associated to the electrons. Equation (1.12) then becomes

$$\lambda_D^2 \Delta \tilde{\Phi} = e^{\tilde{\Phi}} - 1 - \frac{\beta}{n_e q_e} \delta_0.$$

From the structure of this equation we can see that λ_D is important in determining the shape of the potential. More explicitly, $\tilde{\Phi}$ is then of the form $\tilde{\Phi}(x) = \Psi\left(\frac{x}{\lambda_D}\right)$, where Ψ is a solution of the following equation:

$$\Delta \Psi = e^{\Psi} - 1 - \frac{\beta}{n_e q_e \lambda_D^d} \delta_0. \quad (1.14)$$

To see the screening effect, in standard physics presentations it is common to linearise equation (1.14) to obtain

$$\Delta\Psi = \Psi - \frac{\beta}{n_e q_e \lambda_D^d} \delta_0.$$

Using the symmetry of the problem, it is possible to derive an explicit formula for Ψ (see for example [11]). In dimension $d = 3$ this is

$$\Psi(x) = \frac{\beta}{4\pi} \frac{1}{n_e q_e \lambda_D^3} \frac{e^{-|x|}}{|x|}.$$

Then Φ takes the form

$$\Phi(x) = \frac{\beta}{4\pi} \frac{k_B T_e}{n q_e^2} \lambda_D^{-2} \frac{e^{-\frac{|x|}{\lambda_D}}}{|x|} = \frac{\beta}{4\pi \epsilon_0} \frac{e^{-\frac{|x|}{\lambda_D}}}{|x|},$$

which is known as the *Yukawa potential*. The decay of Φ demonstrates the shielding effect described earlier: the typical length of spatial decay of Φ is of order λ_D .

If we consider a timescale on which the ions are significantly mobile, it is possible to perform a similar analysis in which the motion of ions also plays a role in this screening effect. The ions will then have an associated Debye length, which may differ from the electron Debye length. It is defined by the formula (1.13), replacing the the electron density, temperature and charge with the corresponding values for the ions.

Since the Debye length is related to observable quantities such as the density and temperature, it can be found for a real plasma. Typically, λ_D is much smaller than the typical length scale of observation L . The parameter

$$\varepsilon := \frac{\lambda_D}{L}$$

is therefore expected to be small. In this case the plasma is called **quasi-neutral** - since the scale of charge separation is small, the plasma appears to be neutral at the scale of observation.

Quasi-neutrality is a very common property of real plasmas, to the point that some references include quasi-neutrality as part of the definition of a plasma. For example, Chen [23, Section 1.2] includes quasi-neutrality as one of the key properties distinguishing plasmas from ionised gases more generally. In plasma physics literature, the approximation that $\varepsilon \approx 0$ is widely used. For this reason, it is important to understand what happens to the Vlasov-Poisson system in the limit as ε tends to zero. This is known as the **quasi-neutral limit**.

1.2.3.2 The Debye Length in the Vlasov-Poisson System

When written in appropriate dimensionless variables, the classical Vlasov-Poisson system (1.5) takes the form

$$(VP)_\varepsilon := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ -\varepsilon^2 \Delta U = \rho_f - 1, \\ f|_{t=0} = f_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) dx dv = 1. \end{cases} \quad (1.15)$$

In equation (1.15) we can see that the relative Debye length ε indeed appears as a parameter describing the scale of the electric field E . In physics literature, it is common to work under the assumption that $\varepsilon \approx 0$. This leads to another model.

1.2.3.3 Kinetic Euler Systems

Formally setting $\varepsilon = 0$ in the system (1.15) results in the following system:

$$(KInE) := \begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \\ \rho_f = 1, \\ f|_{t=0} = f_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) dx dv = 1. \end{cases} \quad (1.16)$$

This is an example of a **kinetic Euler** system. In system (1.16) the force $-\nabla_x U$ is now defined implicitly through the constraint $\rho_f = 1$, rather than explicitly through the Poisson equation, as in (1.15). It is akin to a pressure term in a fluid equation.

The system (1.16) was discussed by Brenier [19] as a kinetic formulation of the incompressible Euler equations. The correspondence can be seen clearly by considering monokinetic solutions of (1.16), which are solutions of the form

$$f(t, x, v) = \rho(t, x) \delta_0(v - u(t, x)) \quad (1.17)$$

for some density ρ and velocity field u . If f of the form (1.17) is a solution of (1.16), then u is in turn a solution of the incompressible Euler equations:

$$(InE) := \begin{cases} \partial_t u + u \cdot \nabla_x u - \nabla_x U = 0, \\ \nabla_x \cdot u = 0. \end{cases} \quad (1.18)$$

This observation illustrates the reason for calling (1.16) an ‘Euler’ equation. We will refer to (1.16) as the **kinetic incompressible Euler** system, to distinguish it from other kinetic Euler systems we will introduce below.

It is also possible to consider the quasi-neutral limit for the VPME system. In this case, the scaled system is

$$(VPME)_\varepsilon := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ \varepsilon^2 \Delta U = e^U - \rho_f, \\ f|_{t=0} = f_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) dx dv = 1. \end{cases} \quad (1.19)$$

The formal limit is another kinetic Euler system:

$$(KIsE) := \begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \\ U = \log \rho_f, \\ f|_{t=0} = f_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) dx dv = 1. \end{cases} \quad (1.20)$$

This system was introduced and studied in a physics context in [38–40]. For monokinetic solutions of (1.20), the pair (ρ, u) satisfies the following isothermal Euler system:

$$(IsE) := \begin{cases} \partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla_x : (\rho u \otimes u) - \nabla_x \rho = 0. \end{cases}$$

We therefore refer to (1.20) as the **kinetic isothermal Euler** system.

It is worth observing that, although we have introduced both KInE and KIsE under the umbrella of ‘kinetic Euler systems’, these systems are structurally different. In the KIsE system (1.20), the force $-\nabla_x U$ depends on ρ_f in an explicit, albeit singular, way. In the KInE system, the force is defined implicitly through the incompressibility constraint. This distinction is similar to the difference between compressible and incompressible Euler equations, which is not surprising considering the connection between these systems through the monokinetic case.

To understand the KIsE system, it is often instructive to consider a related system, named **Vlasov–Dirac–Benney** by Bardos [4]. This system can also be thought of as a kinetic Euler

equation. The VDB system reads as follows:

$$(VDB) := \begin{cases} \partial_t f + v \cdot \nabla_x f - \nabla_x U \cdot \nabla_v f = 0, \\ U = \rho_f - 1 \\ f|_{t=0} = f_0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) \, dx \, dv = 1. \end{cases} \quad (1.21)$$

It can be obtained by linearising the coupling $U = \log \rho_f$ between U and ρ_f . As well as being interesting in its own right, the VDB system provides useful intuition for the KIsE system, since they have similar structures but the coupling between $\nabla_x U$ and ρ_f is linear for VDB.

1.2.3.4 The Quasi-Neutral Limit

Formal identification of the limiting system does not guarantee that equations (1.16) and (1.20) are good approximations for (1.15) and (1.19) when ε is small but non-zero. To show this, it is necessary to study the quasi-neutral limit from each Vlasov-Poisson system to the corresponding kinetic Euler system. It is not guaranteed that this approximation will always be valid. Note, for example, that Medvedev [70] describes a situation in which the quasi-neutral approximation $U = \log \rho$ is not valid everywhere. This provides a physical motivation for a study of the transition between (1.19) and (1.20). In particular we would like to identify conditions on the data and quantitative ranges of the physical parameters for which the limit holds. Results of this type identify regimes in which the limiting systems (1.16) and (1.20) can be validated mathematically.

The rigorous mathematical justification of the quasi-neutral limit turns out to be a challenging problem, requiring quite stringent restrictions on the initial data. The reasons for this are at least in part of physical origin. This will be discussed further in Section 1.5. In Chapter 3, we consider the quasi-neutral limit from VPME to the kinetic isothermal Euler system. We are able to prove a rigorous quasi-neutral limit for a class of rough data, in dimension two and three.

1.2.4 Derivation from a Particle System

It is a fundamental problem in the theory of kinetic equations to derive the PDE models, in a rigorous way, from the underlying physical system they are meant to describe. For instance, consider the classical Vlasov-Poisson system (1.5), which we motivated as a model for the electrons in a plasma. A lower level description of this system would be to consider a system of N electrons, each modelled as a point particle with mass m_e and charge q_e . The dynamics of the electrons can be described using the laws of classical mechanics. In this case, the state of

the i^{th} electron can be characterised by its position X_i and velocity V_i . For ease of presentation, here we consider this system evolving in the whole space without a background of ions. The evolution of $(X_i, V_i)_{i=1}^N$ should then be described by the following system of ODEs:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{m_e} E(X_i; \{X_j\}_{j \neq i}), \end{cases} \quad (1.22)$$

where $E(X_i; \{X_j\}_{j \neq i})$ is the electrostatic force exerted on electron i by the other electrons in the plasma. The force exerted on electron i by electron j is given by the Coulomb force between point charges; in three dimensions this is

$$\frac{q_e^2}{4\pi\epsilon_0} \frac{X_i - X_j}{|X_i - X_j|^3},$$

where ϵ_0 denotes the vacuum permittivity. We use the notation K for the function

$$K(x) = \frac{x}{4\pi|x|^3}.$$

Then the particle system (1.22) becomes

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = -\frac{q_e^2}{m_e} \sum_{j \neq i} K(X_i - X_j). \end{cases}$$

After an appropriate non-dimensionalisation, we reach the system

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = -\alpha \sum_{j \neq i} K(X_i - X_j), \end{cases} \quad (1.23)$$

where the parameter α is a function of the physical constants of the plasma.

We would like to show that the PDE model (1.6) gives a good description of the limiting behaviour of the particle system (1.23) as N tends to infinity. To do this it is necessary to rescale the system, which means choosing α to vary with N . The choice of scaling $\alpha = \alpha(N)$ affects the kind of limit we can obtain.

In order to derive the Vlasov-Poisson system (1.6), the appropriate choice is the **mean field** scaling $\alpha(N) = 1/N$. With this choice, formally speaking, the Vlasov-Poisson system appears to describe the limiting behaviour of (1.23). This connection between the particle system (1.23) and the PDE (1.6) is called the **mean field limit**. However, whether this limit truly holds is

an open problem, and it may be false in some regimes. In Section 1.6, we will discuss the technical obstacles to this limit in greater detail, and give an overview of recent approaches to this problem.

Similar issues affect the derivation of the VPME model for ions. In this case, for the microscopic model we consider a system of ions, modelled as point particles, in a background of thermalised electrons. The assumption of thermalisation is justified by the difference between the typical timescales of the ions and electrons. This is modelled by an ODE system of the form

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = -\frac{1}{N} \sum_{j \neq i} K(X_i - X_j) + K * e^U, \end{cases} \quad (1.24)$$

where U is the electrostatic potential induced by the ions and the background of electrons. That is, U satisfies

$$\Delta U = e^U - \frac{1}{N} \sum_{i=1}^N \delta_{X_i}. \quad (1.25)$$

As in the classical case, the rigorous derivation of the VPME system from the particle system (1.24) remains an open problem. However, in Chapter 4 we will prove a rigorous derivation of the VPME system from a regularised version of (1.24).

If we consider a different choice of scaling $\alpha(N)$, then it may be possible to obtain different PDE models in the limit. Recall for example the hierarchy of plasma models that we laid out in Figure 1.2. By using a different scaling of α than the mean field scaling, it is possible to pass from the particle model to the kinetic Euler system KIsE (1.20), rather than the VPME system. We investigate this direction in Chapter 5, where we derive the kinetic Euler systems from regularised particle systems, under an alternative scaling. For example, for the KInE system (1.16) we use a scaling of the form

$$\alpha(N) = C \frac{(\log N)^\kappa}{N},$$

for some exponent κ stated in Theorem 1.26 in Section 1.7. For the KIsE system (1.20) we use a scaling of the form

$$\alpha(N) = C \frac{\log \log \log N}{N}.$$

1.3 Well-Posedness Theory for Vlasov Equations

The Vlasov-Poisson system is an example of a nonlinear scalar transport equation. In particular it belongs to a class of PDEs known as **Vlasov equations**. The aim of this section is to introduce this class of equations and to review some of the fundamental concepts involved in their well-posedness theory. For further details, we refer to the notes of Golse [31].

1.3.1 Vlasov Equations

Vlasov equations are used to model large systems of interacting particles. Below is a general example:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + F[f] \cdot \nabla_v f = 0, \\ F[f](t, x) = -\nabla_x W * \rho_f, \\ \rho_f(t, x) = \int_{\mathbb{R}^d} f(t, x, v) dv, \\ f(0, x, v) = f_0(x, v) \geq 0. \end{cases} \quad (1.26)$$

The unknown $f = f(t, x, v)$ represents the density of particles at time t which have position x and velocity v . In this thesis, we will only discuss the problem posed on domains without boundary, letting $(x, v) \in \mathcal{X} \times \mathbb{R}^d$, where either $\mathcal{X} = \mathbb{R}^d$ (the **whole space** case) or $\mathcal{X} = \mathbb{T}^d$ (the **periodic** case).

Equation (1.26) describes a system in which particles influence each other through pairwise interactions. The interaction between two particles is described by a pair potential W depending only on the separation between the particles in position space. The force exerted on a particle at position x by another particle at position y is therefore $-\nabla_x W(x - y)$. For example, the Vlasov-Poisson case can be obtained by choosing W to be the Green's function of the Laplacian on \mathcal{X} . In the limit as the number of particles tends to infinity, this results in an effective force $F[f]$ as defined in (1.26). In Section 1.6 we will discuss the connection between the PDE (1.26) and the underlying particle system in more detail. In this section, we will focus on the well-posedness theory of the PDE (1.26).

1.3.1.1 Transport Equations

The basic underlying structure of (1.26) is that of a scalar transport equation. Letting $z = (x, v)$, equation (1.26) is of the form

$$\begin{cases} \partial_t f + b \cdot \nabla_z f = 0, \\ f(0, z) = f_0(z). \end{cases} \quad (1.27)$$

where

$$b(t, z) := (v, F[f](t, x)).$$

The fact that the force F , and therefore b , depends on f is what creates the nonlinearity in (1.26).

In this section, we will discuss some aspects of the theory of the linear equation (1.27), which will be useful for understanding the nonlinear Vlasov equation. The properties of the transport equation (1.27) clearly depend on the choice of the vector field b . For proving well-posedness, what is important to understand is the regularity of b . One way to understand this is through the classical method of characteristics.

The transport equation (1.27) is associated with a family of characteristics. A path $(Z(t))_{t \in \mathbb{R}}$ in phase space is a characteristic trajectory for equation (1.27) if it satisfies the ODE

$$\frac{dZ}{dt} = b(t, Z). \quad (1.28)$$

The characteristics are a useful tool for understanding the behaviour of equations like (1.27) - see for example [24] for a more detailed exposition of this theory. We will make use of characteristics in many places in this thesis. We will use the notation $Z(t; s, z)$ to denote a solution of (1.28) which has phase space position z at time s ; that is, $Z(s; s, z) = z = (x, v)$.

If b is sufficiently regular, then the system (1.28) has a unique global-in-time solution for any choice of data (s, x, v) . For example, this will be the case if b is continuous in all variables and Lipschitz with respect to z , with less than linear growth:

$$|b(t, z_1) - b(t, z_2)| \leq L|z_1 - z_2|, \quad |b(t, z)| \leq C(1 + |z|). \quad (1.29)$$

In this case the characteristic trajectories can be used to construct solutions of the transport equation (1.27). Suppose for example that f_0 is a C^1 function with compact support. Then (1.27) has a unique solution f_t which can be represented by the formula

$$f(t, z) = f_0(Z(0; t, z)).$$

This shows that the equation (1.27) indeed models the transport of mass along this family of curves.

The representation of the solution f in terms of characteristics is useful for understanding its properties. For example, it immediately implies that the equation preserves sign: if $f_0 \geq 0$, then also $f_t \geq 0$ for all t . Similarly, the $L^\infty(\mathcal{X} \times \mathbb{R}^d)$ norm of the solution cannot grow over

time: if $f_0 \in L^\infty(\mathcal{X} \times \mathbb{R}^d)$, then also $f_t \in L^\infty(\mathcal{X} \times \mathbb{R}^d)$ with

$$\|f(t, \cdot)\|_{L^\infty(\mathcal{X} \times \mathbb{R}^d)} \leq \|f_0\|_{L^\infty(\mathcal{X} \times \mathbb{R}^d)}.$$

If b is less regular than specified in condition (1.29), then for both the ODE (1.28) and the PDE (1.27) solutions may not exist globally and may not be unique without further assumptions. The well-posedness theory of the ODE (1.28) under weaker regularity conditions than (1.29) was considered by DiPerna and Lions [26], by making use of the connection between the ODE and the corresponding transport equation (1.27). This strategy was then extended in many other works; for more details on this subject see for example Ambrosio [1].

In the context of Vlasov equations, the lesson is that the well-posedness theory of the transport equation (1.27) depends crucially on the regularity of the vector field b . Consequently, for the nonlinear Vlasov equation, it will be important to understand the regularity of the force $F[f]$ and how that regularity depends on properties of f . This depends on the regularity of the interaction potential W .

1.3.2 Regular Potentials

When W is sufficiently regular, the Vlasov equation (1.26) has a well developed well-posedness theory. In particular, when $\nabla_x W$ is a Lipschitz function, it is possible to construct unique global solutions to the Vlasov equation (1.26), assuming very little regularity on the initial datum f_0 . In fact it will be enough to suppose only that f_0 is a probability measure with moments of sufficiently high order.

The case where $\nabla_x W$ is Lipschitz is thought of as the ‘regular’ case for Vlasov equations of the form (1.26). We will discuss this case here in some detail, as this will allow us to introduce several useful ideas in Vlasov equation theory that we will return to in the Vlasov-Poisson case. In particular, the aim is to understand how the assumption that $\nabla_x W$ is Lipschitz is used in the well-posedness results.

The immediate consequence of the Lipschitz assumption is that $\nabla_x W * \rho$ is then a Lipschitz function, for any finite measure ρ , without further regularity assumptions. This means that even measure solutions of (1.26) will have an associated classical characteristic flow. We will see subsequently that this assumption on W also allows the construction of unique solutions for the nonlinear problem (1.26).

Notation: Spaces of Measures. We are going to look for measure solutions of equation (1.26). We begin by specifying some notation for the class of solutions we are interested in.

At each fixed time t , the solution should be measure on the phase space $\mathcal{X} \times \mathbb{R}^d$, where the position space \mathcal{X} is either \mathbb{R}^d or the torus \mathbb{T}^d . We denote the space of finite signed Borel measures on $\mathcal{X} \times \mathbb{R}^d$ by \mathcal{M} , and the space of finite (non-negative) measures by \mathcal{M}_+ . We equip these spaces with the topology of weak convergence of measures, in which a sequence $(\mu_n)_n$ converges to μ as n tends to infinity if and only if

$$\lim_{n \rightarrow \infty} \int_{\mathcal{X} \times \mathbb{R}^d} \phi \, d\mu_n = \int_{\mathcal{X} \times \mathbb{R}^d} \phi \, d\mu, \quad \forall \phi \in C_b(\mathcal{X} \times \mathbb{R}^d).$$

Furthermore, \mathcal{M}_1 denotes the subspace of \mathcal{M} consisting of all finite signed measures with a finite first moment:

$$\int_{\mathcal{X} \times \mathbb{R}^d} (|x| + |v|) \, d|\mu|(x, v) < +\infty.$$

Similarly $\mathcal{M}_{+,1} = \mathcal{M}_1 \cap \mathcal{M}_+$.

The space $C([0, \infty); \mathcal{M})$ consists of paths $t \mapsto \mu(t) \in \mathcal{M}$ taking values in the space of signed measures, where the continuity of the path is defined in terms of the topology of weak convergence of measures. The spaces $C([0, \infty); \mathcal{M}_+)$, $C([0, \infty); \mathcal{M}_1)$ and $C([0, \infty); \mathcal{M}_{+,1})$ are defined similarly. In the next section we will look for solutions of the Vlasov equation (1.26) in the space $C([0, \infty); \mathcal{M}_{+,1})$. We refer to such solutions informally as ‘measure-valued solutions’.

1.3.2.1 Weak Formulation

To make sense of measure solutions for equation (1.26), it is necessary to understand the equation in a weak sense.

Definition 1 (Weak Solutions). $f \in C([0, \infty); \mathcal{M}_+)$ is a weak solution of the Vlasov equation (1.26) if, for all test functions $\phi \in C_c^1([0, \infty) \times \mathcal{X} \times \mathbb{R}^d)$,

$$\int_0^\infty \int_{\mathcal{X} \times \mathbb{R}^d} \left[\partial_t \phi + v \cdot \nabla_x \phi - (\nabla_x W *_x \rho_f) \cdot \nabla_v \phi \right] f(t, dx, dv) \, dt + \int_{\mathcal{X} \times \mathbb{R}^d} \phi(0, x, v) f_0(dx, dv) = 0. \quad (1.30)$$

In order for (1.30) to make sense, $\nabla_x W *_x \rho_f$ must be regular enough that $(\nabla_x W *_x \rho_f) \cdot \nabla_v \phi$ is integrable with respect to f . This is ensured by the following lemma, which shows that $\nabla_x W *_x \rho_f$ will be Lipschitz provided that $\nabla_x W$ is Lipschitz.

Lemma 1.1. *Let $\mu \in \mathcal{M}(\mathcal{X})$, and let ∇W be a Lipschitz function. Assume that either (a) $\nabla W \in L^\infty(\mathcal{X})$ or that (b) $\mu \in \mathcal{M}_1(\mathcal{X})$. Then $\nabla W * \mu$ is a Lipschitz function. Moreover, we have the quantitative estimate*

$$|\nabla W * \mu(x) - \nabla W * \mu(y)| \leq \|\nabla W\|_{Lip} |\mu|(\mathcal{X}) |x - y|. \quad (1.31)$$

Proof. First, we check that the function

$$\nabla W * \mu(x) := \int_{\mathcal{X}} \nabla W(x-y) d\mu(y)$$

is well-defined. Under condition (a),

$$|\nabla W(x-y)| \leq \|\nabla W\|_{L^\infty(\mathcal{X})} \in L^1(\mu).$$

Under condition (b),

$$|\nabla W(x-y)| \leq |\nabla W(x)| + \|\nabla W\|_{\text{Lip}}|y| \in L^1(|\mu|).$$

Next we prove the estimate (1.31). By definition,

$$\nabla W * \mu(x) - \nabla W * \mu(y) = \int_{\mathcal{X}} [\nabla W(x-z) - \nabla W(y-z)] d\mu(z).$$

We estimate this using the fact that ∇W is Lipschitz:

$$\begin{aligned} |\nabla W * \mu(x) - \nabla W * \mu(y)| &\leq \int_{\mathcal{X}} |\nabla W(x-z) - \nabla W(y-z)| d|\mu|(z) \\ &\leq \|\nabla W\|_{\text{Lip}}|x-y| \int_{\mathcal{X}} 1 d|\mu|(z), \end{aligned}$$

which completes the proof. \square

1.3.2.2 Representation via Characteristics

We will again find it useful to represent solutions of the Vlasov equation (1.26) in terms of their characteristics. When f_0 is a measure we cannot define the composition $f_0(Z(0;t,z))$. Instead we look at another representation, which uses duality.

Observe that $b(t,z) := (v, F[f](t,x))$ is divergence free, so that the transport equation can also be written as a continuity equation:

$$\begin{cases} \partial_t f + \text{div}_z(bf) = 0, \\ f|_{t=0} = f_0. \end{cases} \quad (1.32)$$

We have already used this fact when defining weak solutions (Definition 1). Solutions of the continuity equation (1.32) can be characterised as the **pushforward** of their initial data along the characteristic flow induced by b .

Definition 2 (Pushforward). For $i = 1, 2$, let $(\Omega_i, \mathcal{F}_i)$ be a measurable space. Let μ be a measure on $(\Omega_1, \mathcal{F}_1)$. Let $\mathcal{T} : \Omega_1 \rightarrow \Omega_2$ be a measurable map. The pushforward of μ along \mathcal{T} , denoted by $\mathcal{T}_\# \mu$, is the measure on $(\Omega_2, \mathcal{F}_2)$ defined by the relation

$$\mathcal{T}_\# \mu(A) = \mu(\mathcal{T}^{-1}(A)), \quad \forall A \in \mathcal{F}_2.$$

In particular, for all measurable functions g on Ω_2 for which the composition $g \circ \mathcal{T}$ is integrable with respect to μ , the following relation holds:

$$\int_{\Omega_2} g \, d\mathcal{T}_\# \mu = \int_{\Omega_1} g \circ \mathcal{T} \, d\mu.$$

To represent solutions of the linear continuity equation (1.32), we will take \mathcal{T} to be the **flow map** induced by the characteristic system (1.28). The flow is a family of maps $(\Phi^{s,t} : \mathcal{X} \times \mathbb{R}^d \rightarrow \mathcal{X} \times \mathbb{R}^d)_{s,t}$, defined by the property

$$\Phi^{s,t}(z) = Z(t; s, z).$$

Then, if b satisfies (1.29), the continuity equation (1.32) has a unique solution f for any initial datum $f_0 \in \mathcal{M}_+$. Moreover, f has the representation $f_t = \Phi_\#^{0,t} f_0$. That is, for all $\phi \in C_c$,

$$\int_{\mathcal{X} \times \mathbb{R}^d} \phi(z) f(t, dz) = \int_{\mathcal{X} \times \mathbb{R}^d} \phi(\Phi^{0,t}(z)) f_0(dz) = \int_{\mathcal{X} \times \mathbb{R}^d} \phi(Z(t; 0, z)) f_0(dz). \quad (1.33)$$

For proofs and further details, see [1, Proposition 4].

Conservation of Mass From the representation (1.33) it is clear that the equation conserves mass: for a continuous path $f \in C([0, \infty); \mathcal{M}_+)$, f_t has finite mass for all times t , and therefore (1.33) can be extended to the function $\phi \equiv 1$. This implies that f_t has the same total mass as f_0 . The assumptions $f_0 \geq 0$ and $\int_{\mathcal{X} \times \mathbb{R}^d} f_0 \, dz = 1$ thus ensure that, for each fixed t , f_t is a probability measure on $\mathcal{X} \times \mathbb{R}^d$.

We let \mathcal{P} denote the space of probability measures on $\mathcal{X} \times \mathbb{R}^d$, and \mathcal{P}_1 the space of probability measures with a finite first moment. The discussion above shows that it is enough to consider measure-valued solutions in the space $C([0, \infty); \mathcal{P})$.

1.3.2.3 Existence of Solutions

We now turn to the existence of solutions for the nonlinear Vlasov equation (1.26). We refer to the works [18, 27, 72, 31] for the following result.

Theorem 1.2. *Assume that $\nabla W : \mathcal{X} \rightarrow \mathbb{R}^d$ is a Lipschitz function. Let $f_0 \in \mathcal{P}_1$. Then there exists a solution $f \in C([0, \infty); \mathcal{P})$ of the Vlasov equation (1.26), in the sense of Definition 1.*

The proof of this result presented here follows the presentation of [31, Theorem 1.3.1]. The proof can be formulated as a fixed point problem: first, we decouple the nonlinearity and consider the equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + F[f] \cdot \nabla_v g = 0, \\ g|_{t=0} = f_0. \end{cases} \quad (1.34)$$

In equation (1.34), g is the unknown and f should be thought of as a fixed ‘input’. We have discussed above that, for any $f_0 \in \mathcal{P}$, the solution g of equation (1.34) has the representation $g = \Phi^{0, \cdot}[f]_{\#} f_0$. A solution of the nonlinear Vlasov equation (1.26) can therefore be constructed by proving the existence of a fixed point of the map $f \mapsto \Phi^{0, \cdot}[f]_{\#} f_0$.

In fact, we will look at the corresponding map on the flow $\Phi^{0, \cdot}[f]$. We can think of $\Phi^{0, \cdot}[f]$ as an element of the space $C(\mathbb{R} \times \mathcal{X} \times \mathbb{R}^d)$. This can be made into a Banach space with an appropriate choice of norm. For flows we want to consider the map \mathcal{T} defined by

$$\mathcal{T}[\phi] = \Phi^{0, \cdot}[\phi_{\#} f_0].$$

If $\phi = \mathcal{T}[\phi]$, then $\phi_{\#} f_0$ is a solution of the Vlasov equation (1.26) with initial datum f_0 .

To find such a fixed point, we can use a standard iteration argument. Set ϕ_0 to be the identity map on $\mathcal{X} \times \mathbb{R}^d$ for all t , and consider the sequence $(\phi_n)_n$ where $\phi_{n+1} = \mathcal{T}[\phi_n]$. To prove that this iteration converges to a limit, it is enough to show that \mathcal{T} is a contraction on $C([0, T]; \mathcal{X} \times \mathbb{R}^d)$ for some T . We can then conclude that a fixed point exists by the standard arguments for a Picard iteration.

We equip $C([0, T]; \mathcal{X} \times \mathbb{R}^d)$ with the norm

$$\|\phi\|_{Y_T} := \sup_{t \in [0, T]} \sup_{z \in \mathcal{X} \times \mathbb{R}^d} \frac{|\phi(t, z)|}{1 + |z|}.$$

This is chosen instead of the usual uniform norm, because the identity map is not bounded. However, it does have finite $\|\cdot\|_{Y_T}$ norm. Moreover, any flow induced by a vector field b satisfying the condition (1.29) can be shown to satisfy a bound of the form $|\Phi^{0, t}(z)| \leq C(t)(1 + |z|)$ for a continuous function $C(t)$. Hence it is reasonable to use the norm $\|\cdot\|_{Y_T}$ for such flows.

Lemma 1.3 (\mathcal{T} is a contraction). *Let ∇W be a Lipschitz function and assume that $f_0 \in \mathcal{P}_1$. Then there exists T such that \mathcal{T} is a contraction on Y_T :*

$$\|\mathcal{T}[\phi_1] - \mathcal{T}[\phi_2]\|_{Y_T} \leq C(T, W, f_0) \|\phi_1 - \phi_2\|_{Y_T},$$

where $C(T, W, f_0)$ depends only on $\|\nabla W\|_{Lip}$ and $M_1(f_0)$, the first moment of f_0 .

Proof. We introduce the notation $Z_i = (X_i, V_i)$ to denote trajectories of $\mathcal{T}[\phi_i]$:

$$Z_i(t; z) = \Phi^{0,t}[\phi_i \# f_0](z).$$

By definition, for each $z \in \mathcal{X} \times \mathbb{R}^d$, Z_i satisfies the ODE

$$\dot{X}_i = V_i, \quad \dot{V}_i = F[\phi_i \# f_0](X_i), \quad Z(0; z) = z. \quad (1.35)$$

Then

$$\|\mathcal{T}[\phi_1] - \mathcal{T}[\phi_2]\|_{Y_T} = \sup_{t \in [0, T]} \frac{|Z_1(t, z) - Z_2(t, z)|}{1 + |z|}.$$

Our strategy will be to control the quantity $|Z_1(t; z) - Z_2(t; z)|$, using the fact that Z_i solves the ODE (1.35). For the x coordinate, clearly

$$|X_1(t) - X_2(t)| \leq \int_0^t |V_1(s) - V_2(s)| \, ds.$$

In the v coordinate, we have

$$|V_1(t) - V_2(t)| \leq \int_0^t |F[\phi_1 \# f_0](X_1(s)) - F[\phi_2 \# f_0](X_2(s))| \, ds.$$

We split the integrand into two:

$$\int_0^t |F[\phi_1 \# f_0](s, X_1(s)) - F[\phi_2 \# f_0](s, X_2(s))| \, ds \leq I_1 + I_2,$$

where

$$I_1 := \int_0^t |F[\phi_1 \# f_0](s, X_1(s)) - F[\phi_1 \# f_0](s, X_2(s))| \, ds$$

and

$$I_2 := \int_0^t |F[\phi_1 \# f_0](s, X_2(s)) - F[\phi_2 \# f_0](s, X_2(s))| \, ds.$$

The control of the quantities I_1 and I_2 depends on two key estimates. For I_1 we need to understand the regularity of $F[f]$ for fixed f . For I_2 we need to understand the stability of $F[f]$ with respect to f .

The regularity estimate is provided by Lemma 1.1: since $\phi_1 \# f_0 \in C([0, T]; \mathcal{P})$,

$$\|F[\phi_1 \# f_0](t, \cdot)\|_{\text{Lip}} \leq \|\nabla W\|_{\text{Lip}}.$$

Then

$$I_1 \leq \|\nabla W\|_{\text{Lip}} \int_0^t |X_1(s) - X_2(s)| \, ds.$$

The stability estimate is proved below in Lemma 1.4. It implies that, for all $x \in \mathcal{X}$,

$$|F[\phi_1 \# f_0](t, x) - F[\phi_2 \# f_0](t, x)| \leq M_1(f_0) \|\nabla W\|_{\text{Lip}} \|\phi_1 - \phi_2\|_{Y_t}.$$

Then

$$I_2 \leq M_1(f_0) \|\nabla W\|_{\text{Lip}} \int_0^t \|\phi_1 - \phi_2\|_{Y_s} \, ds.$$

Then

$$\begin{aligned} |X_1(t) - X_2(t)| + |V_1(t) - V_2(t)| &\leq C_W \int_0^t |X_1(s) - X_2(s)| + |V_1(s) - V_2(s)| \, ds \\ &\quad + M_1(f_0) \|\nabla W\|_{\text{Lip}} \int_0^t \|\phi_1 - \phi_2\|_{Y_s} \, ds, \end{aligned}$$

where

$$C_W := \max\{\|\nabla W\|_{\text{Lip}}, 1\}.$$

This implies that

$$|Z_1(t; z) - Z_2(t; z)| \leq M_1(f_0) \left(e^{C_W t} - 1 \right) \|\phi_1 - \phi_2\|_{Y_t},$$

which proves the result. \square

The proof above relies on Lemma 1.1 and the following stability estimate.

Lemma 1.4 (Stability Estimate for F). *Let ∇W be a Lipschitz function, and $f_0 \in \mathcal{M}_{+,1}$. Then, for any $\phi_1, \phi_2 \in C(\mathcal{X} \times \mathbb{R}^d; \mathcal{X} \times \mathbb{R}^d)$,*

$$\sup_{x \in \mathcal{X}} |F[\phi_1 \# f_0] - F[\phi_2 \# f_0]| \leq M_1(f_0) \|\nabla W\|_{\text{Lip}} \sup_{z \in \mathcal{X} \times \mathbb{R}^d} \frac{|\phi_1(z) - \phi_2(z)|}{1 + |z|}.$$

Proof. By definition,

$$|F[\phi_1 \# f_0](x) - F[\phi_2 \# f_0](x)| = \left| \int_{\mathcal{X} \times \mathbb{R}^d} \nabla W(x - P_X \phi_1(z)) - \nabla W(x - P_X \phi_2(z)) \, df_0(z) \right|,$$

where P_X denotes the projection onto the x coordinate. Since ∇W is Lipschitz,

$$\begin{aligned} |F[\phi_1 \# f_0](x) - F[\phi_2 \# f_0](x)| &\leq \|\nabla W\|_{\text{Lip}} \int_{\mathcal{X} \times \mathbb{R}^d} |P_X \phi_1(z) - P_X \phi_2(z)| \, d f_0(z) \\ &\leq \|\nabla W\|_{\text{Lip}} \left(\sup_{z \in \mathcal{X} \times \mathbb{R}^d} \frac{|\phi_1(z) - \phi_2(z)|}{1 + |z|} \right) \int_{\mathcal{X} \times \mathbb{R}^d} (1 + |z|) f_0(dz). \end{aligned}$$

□

1.3.2.4 Uniqueness

We present a uniqueness argument due to Dobrushin [27].

Theorem 1.5 (Uniqueness of Solutions for Equation (1.26)). *Assume that $\nabla W : \mathcal{X} \rightarrow \mathbb{R}^d$ is a bounded, Lipschitz function. Let $f_0 \in \mathcal{P}_1$. The solution f constructed in Theorem 1.2 is the unique solution of (1.26) in the class $C([0, \infty); \mathcal{M})$.*

The idea is to prove a quantitative stability estimate between solutions with respect to their initial data. To do this, we will make use of a certain distance on probability measures which metrises the topology of weak convergence of measures. These are known as Wasserstein distances.

Wasserstein Distances. The Wasserstein distances, also known as Monge–Kantorovich distances, are a family of distances on probability measures. They are defined in terms of couplings of measures.

Definition 3 (Coupling). Let (Ω, \mathcal{F}) be a measurable space. Let μ and ν be two probability measures on Ω . A **coupling** of μ and ν is a measure π on the product space such that, for all $A \in \mathcal{F}$, the following two equalities hold:

$$\pi(A \times \Omega) = \mu(A), \quad \pi(\Omega \times A) = \nu(A).$$

The set of couplings of μ and ν is denoted by $\Pi(\mu, \nu)$.

Given this definition, it is possible to define the Wasserstein distances $(W_p)_{p \in [1, \infty]}$.

Definition 4 (Wasserstein Distances). Let (Ω, d) be a Polish space, equipped with its Borel σ -algebra \mathcal{F} .

Let $p \in [1, \infty)$. Let μ and ν be probability measures on (Ω, \mathcal{F}) such that, for some $x_0 \in \Omega$,

$$\int_{\Omega} d(x, x_0)^p \, d\mu(x) < \infty, \quad \int_{\Omega} d(x, x_0)^p \, d\nu(x) < \infty. \quad (1.36)$$

Then the p^{th} order Wasserstein distance between μ and ν , $W_p(\mu, \nu)$, is defined by

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{(x, y) \in \Omega \times \Omega} d(x, y)^p d\pi(x, y) \right)^{1/p}, \quad p \in [1, +\infty). \quad (1.37)$$

The property (1.36) ensures that the quantity (1.37) is well defined.

In the case $p = +\infty$,

$$W_\infty(\mu, \nu) = \inf_{\pi \in \Pi(\mu, \nu)} \pi - \text{ess sup}_{x, y \in \Omega \times \Omega} d(x, y).$$

Each distance W_p satisfies the triangle inequality, and provides a metric on the space of probability measures satisfying (1.36) (see for example [83, Theorem 7.3]).

$$W_p(\mu_1, \mu_3) \leq W_p(\mu_1, \mu_2) + W_p(\mu_2, \mu_3).$$

For $p \in [1, \infty)$, convergence with respect to W_p is equivalent to weak convergence (of measures) along with convergence of the p^{th} moment (see e.g. [83, Theorem 7.12]). W_p therefore metrises the topology of weak convergence of measures.

Wasserstein distances have a monotonicity property: if $p \leq q$, then

$$W_p(\mu, \nu) \leq W_q(\mu, \nu). \quad (1.38)$$

This follows from the monotonicity of L^p norms on finite measure spaces.

We also recall an important duality property - see for example [83, Theorem 1.3].

Lemma 1.6 (Kantorovich duality). *Let $\mu, \nu \in \mathcal{P}(\Omega)$ be probability measures satisfying (1.36) for $p \in [1, \infty)$. Then*

$$W_p^p(\mu, \nu) = \sup_{(\phi, \psi) \in \mathcal{F}} \left\{ \int_{\Omega} \phi d\mu - \int_{\Omega} \psi d\nu \right\},$$

where

$$\mathcal{F} := \{(\phi, \psi) \in L^1(\mu) \times L^1(\nu) : \forall x, y \in \Omega, \phi(x) - \psi(y) \leq d(x, y)^p\}.$$

An important specific case of this property arises in the case $p = 1$, where the duality can be phrased in terms of Lipschitz functions.

Lemma 1.7 (Kantorovich duality for W_1). *Let $\mu, \nu \in \mathcal{P}(\Omega)$ be probability measures satisfying (1.36) for $p = 1$. Then*

$$W_1(\mu, \nu) = \sup_{\phi, \|\phi\|_{Lip} \leq 1} \left\{ \int_{\Omega} \phi \, d\mu - \int_{\Omega} \phi \, d\nu \right\}, \quad \text{where} \quad \|\phi\|_{Lip} := \sup_{x \neq y} \frac{|\phi(x) - \phi(y)|}{|x - y|}.$$

Stability Estimate. Theorem 1.5 is proved by controlling the Wasserstein distance between two solutions of the Vlasov equation (1.26), in terms of the Wasserstein distance between their initial data. This will imply the uniqueness of measure solutions for (1.26) as an immediate corollary. In the following, we will make use of the first order Wasserstein distance W_1 on the space $\mathcal{X} \times \mathbb{R}^d$ equipped with the metric

$$d(z_1, z_2) = |x_1 - x_2| + |v_1 - v_2|,$$

where $z_i = (x_i, v_i)$.

Lemma 1.8 (Stability Estimate for (1.26)). *Assume that ∇W is a Lipschitz function. For $i = 1, 2$, let $f_i \in C([0, \infty); \mathcal{P}_1)$ be a solution of the Vlasov equation (1.26) with initial datum $f_{0,i} \in \mathcal{P}_1$. Then*

$$W_1(f_1(t), f_2(t)) \leq \exp(2\|\nabla W\|_{Lip} t) W_1(f_{0,1}, f_{0,2}).$$

Corollary 1.9. *For $i = 1, 2$, let $f_i \in C([0, \infty); \mathcal{P}_1)$ be a solution of the Vlasov equation (1.26), with the same initial datum $f_0 \in \mathcal{P}_1$. Then $f_1 = f_2$.*

Remark 1. The requirement for f_i to have a first moment can be removed by considering the truncated distance

$$d(z_1, z_2) = \min\{|x_1 - x_2| + |v_1 - v_2|, 1\}.$$

In order to streamline the arguments, we present the proof without this truncation.

We now discuss the proof of Lemma 1.8. The overall strategy will be used many times throughout this thesis.

The proof is based on, firstly, the choice of a particular coupling of the solutions. This coupling is constructed using the characteristic flows $\Phi^{0,\cdot}[f_i]$ induced by the solutions. Given any $\pi_0 \in \Pi(f_{0,1}, f_{0,2})$, let π_t be defined by the relation, for any $\phi \in C_b((\mathcal{X} \times \mathbb{R}^d)^2)$,

$$\int_{(\mathcal{X} \times \mathbb{R}^d)^2} \phi(z_1, z_2) \, d\pi_t := \int_{(\mathcal{X} \times \mathbb{R}^d)^2} \phi(\Phi^{0,t}[f_1](z_1), \Phi^{0,t}[f_2](z_2)) \, d\pi_0(z_1, z_2).$$

That is,

$$\pi_t := (\Phi^{0,t}[f_1] \otimes \Phi^{0,t}[f_2])_{\#} \pi_0.$$

Using this coupling π_t , a functional can be constructed which controls the first order Wasserstein distance between the solutions:

$$D := \int_{(\mathcal{X} \times \mathbb{R}^d)^2} (|x_1 - x_2| + |v_1 - v_2|) d\pi_t.$$

The control of this functional follows a strategy similar to the one used in the construction of solutions, in the proof of Lemma 1.3. The aim is to prove a Grönwall estimate on D . This again relies on two key estimates, one expressing the regularity of the force $F[f]$ and the other its stability with respect to f . The regularity estimate is the same Lipschitz estimate from Lemma 1.1: for $f \in \mathcal{P}$,

$$\|F[f]\|_{\text{Lip}} \leq \|\nabla W\|_{\text{Lip}}.$$

The stability estimate in this case needs to be quantified in the Wasserstein distance W_1 . It is a consequence of Kantorovich duality (Lemma 1.7).

Lemma 1.10. *Let ∇W be a Lipschitz function on \mathcal{X} . For $i = 1, 2$, let $\rho_i \in \mathcal{P}(\mathcal{X})$. Then, for all \mathcal{X} ,*

$$|\nabla W * [\rho_1 - \rho_2](x)| \leq \|\nabla W\|_{\text{Lip}} W_1(\rho_1, \rho_2).$$

Using these estimates, we are able to prove Lemma 1.8.

Proof of Lemma 1.8. We introduce the notation $Z_i(t, z) := \Phi^{0,t}[f_i](z)$ for the characteristics corresponding to the solution f_i . Note that

$$D(t) = \int_{(\mathcal{X} \times \mathbb{R}^d)^2} |X_1(t, z_1) - X_2(t, z_2)| + |V_1(t, z_1) - V_2(t, z_2)| d\pi_0(z_1, z_2).$$

The aim is to prove a Grönwall-type estimate on D . Using the ODE satisfied by the flow we obtain

$$\begin{aligned} |X_1(t, z_1) - X_2(t, z_2)| &= \left| x_1 - x_2 + \int_0^t V_1(s, z_1) - V_2(s, z_2) ds \right| \\ &\leq |x_1 - x_2| + \int_0^t |V_1(s, z_1) - V_2(s, z_2)| ds. \end{aligned}$$

Similarly,

$$|V_1(t, z_1) - V_2(t, z_2)| \leq |v_1 - v_2| + \int_0^t |\nabla_x W * \rho_{f_1}(X_1(s, z_1)) - \nabla_x W * \rho_{f_2}(X_2(s, z_2))| ds.$$

Altogether this implies the inequality

$$D(t) \leq D(0) + \int_0^t \int_{(\mathcal{X} \times \mathbb{R}^d)^2} |V_1(s, z_1) - V_2(s, z_2)| \\ + |\nabla_x W * \rho_{f_1}(X_1(s, z_1)) - \nabla_x W * \rho_{f_2}(X_2(s, z_2))| d\pi_0(z_1, z_2) ds.$$

We can rewrite this as

$$D(t) \leq D(0) + \int_0^t \int_{(\mathcal{X} \times \mathbb{R}^d)^2} |v_1 - v_2| + |\nabla_x W * \rho_{f_1}(x_1) - \nabla_x W * \rho_{f_2}(x_2)| d\pi_s(z_1, z_2) ds.$$

The important quantity to control is

$$\int_{(\mathcal{X} \times \mathbb{R}^d)^2} |\nabla_x W * \rho_{f_1}(X_1(s)) - \nabla_x W * \rho_{f_2}(X_2(s))| d\pi_s.$$

We split this into two parts:

$$\int_{(\mathcal{X} \times \mathbb{R}^d)^2} |\nabla_x W * \rho_{f_1}(x_1) - \nabla_x W * \rho_{f_2}(x_2)| d\pi_s \leq I_1 + I_2,$$

where

$$I_1 = \int_{(\mathcal{X} \times \mathbb{R}^d)^2} |\nabla_x W * \rho_{f_1}(x_1) - \nabla_x W * \rho_{f_1}(x_2)| d\pi_s$$

and

$$I_2 = \int_{(\mathcal{X} \times \mathbb{R}^d)^2} |\nabla_x W * \rho_{f_1}(x_2) - \nabla_x W * \rho_{f_2}(x_2)| d\pi_s.$$

Controlling I_1 depends on understanding the regularity of $\nabla_x W * \rho_{f_1}$. In this case, since $\nabla_x W$ is Lipschitz, Lemma 1.1 implies that $\nabla_x W * \rho_{f_1}$ is also Lipschitz. We therefore have the estimate

$$I_1 \leq \|\nabla_x W\|_{\text{Lip}} \int_{(\mathcal{X} \times \mathbb{R}^d)^2} |x_1 - x_2| d\pi_s.$$

For I_2 , we use the stability estimate from Lemma 1.10:

$$\|\nabla_x W * \rho_{f_1} - \nabla_x W * \rho_{f_2}\|_{L^\infty(\mathcal{X})} \leq \|\nabla_x W\|_{\text{Lip}} W_1(\rho_{f_1}, \rho_{f_2}).$$

Then

$$I_2 \leq \|\nabla_x W\|_{\text{Lip}} W_1(\rho_{f_1}, \rho_{f_2}) \int_{(\mathcal{X} \times \mathbb{R}^d)^2} d\pi_s = \|\nabla_x W\|_{\text{Lip}} W_1(\rho_{f_1}, \rho_{f_2}).$$

Then, since $W_1(\rho_{f_1}, \rho_{f_2}) \leq W_1(f_1, f_2)$, it follows that

$$I_2 \leq \|\nabla_x W\|_{\text{Lip}} D.$$

We conclude that

$$D(t) \leq D(0) + 2\|\nabla_x W\|_{\text{Lip}} \int_0^t D(s) \, ds,$$

which implies the result. \square

1.3.2.5 Summary: Role of the Lipschitz Assumption

We collect together the main points where the assumption that ∇W is Lipschitz was used in the well-posedness theory presented above.

Regularity of the Force. The fact that $\nabla_x W$ is Lipschitz implies that $F[f] = \nabla_x W * \rho$ is Lipschitz for any measure ρ , without requiring any additional regularity assumptions: for any probability measure f ,

$$\|F[f]\|_{\text{Lip}} \leq \|\nabla_x W\|_{\text{Lip}}. \quad (1.39)$$

This property is used to prove existence of the characteristic trajectories for measure solutions. It also plays an important role in the construction of solutions (Lemma 1.3) and the stability estimate in Lemma 1.8.

Existence of the Characteristic Flow. The fact that $F[f]$ is Lipschitz was used to show that the characteristic system for the Vlasov equation (1.26) is well-posed. This ensures that the characteristic flow exists. This flow was used extensively in the estimates above.

Stability of the Force. Another key ingredient is that difference between the forces induced by two different solutions f_1 and f_2 can be controlled in terms of a weak distance between ρ_{f_1} and ρ_{f_2} . We used this in the proof of the stability estimate in Lemma 1.8 in the form

$$\|F[f_1] - F[f_2]\|_{L^\infty(\mathcal{X})} \leq \|\nabla W\|_{\text{Lip}} W_1(\rho_{f_1}, \rho_{f_2}). \quad (1.40)$$

This estimate also implies Lemma 1.4, which was used in the construction of solutions. This is due to the estimate

$$\begin{aligned} W_1(\phi_1 \# f_0(t), \phi_2 \# f_0(t)) &\leq \int_{\mathcal{X} \times \mathbb{R}^d} |\phi_1^{0,t}(z) - \phi_2^{0,t}(z)| f_0(dz) \\ &\leq \|\phi_1 - \phi_2\|_{Y_t} \int_{\mathcal{X} \times \mathbb{R}^d} (1 + |z|) f_0(dz). \end{aligned}$$

In conclusion, the well-posedness theory for Vlasov equations with regular interaction depends on the two key estimates (1.39) and (1.40). When we discuss the well-posedness

theory of more general equations of Vlasov type, one of the recurring ideas will be to look for suitable versions of or replacements for these estimates. In the following sections of this introduction, we will discuss this idea in the case of the Vlasov-Poisson system (1.5). In Chapter 2, where we prove the well-posedness of the VPME system (1.11), the key step of the proof will be to prove suitable regularity and stability estimates on the electric field.

1.3.3 Coulomb Interaction

The Vlasov-Poisson system is an example of a Vlasov equation (1.26), for which the interaction potential W is singular. Consider for example the whole space case $\mathcal{X} = \mathbb{R}^d$, where the Vlasov-Poisson system reads as follows:

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ -\Delta U = \rho_f, \\ f|_{t=0} = f_0 \geq 0, \int_{\mathbb{R}^{2d}} f_0 = 1. \end{cases} \quad (1.41)$$

The electrostatic potential U is a solution of the Poisson equation on \mathbb{R}^d . It can therefore be represented using the Green's function of the Laplacian. Let G denote the fundamental solution of the Poisson equation on \mathbb{R}^d :

$$-\Delta G = \delta_0.$$

Then $U = G * \rho$ and $E = -\nabla(G * \rho)$. Thus (1.41) formally has the form of a Vlasov equation (1.26), with the choice $W = G$. A similar representation is possible in the periodic case $\mathcal{X} = \mathbb{T}^d$.

In Section 1.3.2 we discussed the well-posedness theory of Vlasov equations when the interaction potential W is sufficiently regular. The main difficulty in analysing the Vlasov-Poisson system comes from the fact that the Coulomb potential G has a singularity, which means that the theory for regular potentials does not apply. In order to make progress, it is necessary to understand the properties of G . This will depend on the dimension d of the system. In this section we will focus on the cases $d = 2, 3$.

In the whole space case $\mathcal{X} = \mathbb{R}^d$, explicit formulae for G are available - see for example Hörmander [50, Theorem 3.3.2]:

$$G(x) = \begin{cases} -\frac{1}{2\pi} \log |x|, & d = 2, \\ \frac{1}{4\pi|x|}, & d = 3. \end{cases} \quad (1.42)$$

Similarly, the force $K = -\nabla G$ is given by the formulae

$$K(x) = \begin{cases} \frac{x}{2\pi|x|^2}, & d = 2, \\ \frac{x}{4\pi|x|^3}, & d = 3. \end{cases} \quad (1.43)$$

On the torus $\mathcal{X} = \mathbb{T}^d$, the fundamental solution has the form $G_{\text{per}} = G + G_0$, where $G_0 \in C^\infty(\mathbb{T}^d)$ is a smooth function - see for example [37, Lemma 2.1]. Similarly, we write $K_{\text{per}} = -\nabla_x G_{\text{per}}$ in the form

$$K_{\text{per}} = K + K_0. \quad (1.44)$$

From the formulae (1.43) it is clear that K is not a Lipschitz kernel, due to the point singularity at $x = 0$. This means that $K * \rho$ will not in general be a Lipschitz force if we only know that ρ is a measure - consider for example the case where ρ is a Dirac mass. This causes problems for the theory outlined in Subsection 1.3.2. The way to deal with this is to consider solutions for which ρ has higher regularity.

In fact ∇K is not integrable near zero, so we cannot even show that $K * \rho$ is Lipschitz for general $\rho \in L^\infty$. This is a critical regularity estimate for solutions of Poisson's equation. However, it is possible to prove a log-Lipschitz estimate, which is enough to guarantee the existence of a unique characteristic flow.

This regularity estimate suggests that the class of f for which ρ_f lies in L^∞ is interesting for the study of the Vlasov-Poisson system. We will see in Section 1.4 that solutions with $\rho \in L^\infty$ have a uniqueness property, and that such solutions exist globally in time provided that the initial datum is compactly supported.

Lemma 1.11 (Log-Lipschitz regularity of the electric field). *Let U be a solution of*

$$-\Delta U = \rho$$

for $\rho \in L^\infty(\mathbb{T}^d)$. Then

$$|\nabla U(x) - \nabla U(y)| \leq C_d \|\rho\|_{L^\infty(\mathbb{T}^d)} |x - y| \left(1 + \log \left(\frac{\sqrt{d}}{|x - y|} \right) \mathbb{1}_{|x - y| \leq \sqrt{d}} \right).$$

The proof of this well-known result can be found for instance in [66, Lemma 8.1] for the case where the spatial domain is \mathbb{R}^2 . It was used by Yudovich [86] in the proof of the existence and uniqueness of global weak solutions the 2D incompressible Euler equations with bounded vorticity. For completeness we briefly recall the proof below on the torus \mathbb{T}^d for general d , since this case will be used several times in this thesis.

Proof. We use the representation $\nabla U = K_{\text{per}} * \rho$. Since $K_{\text{per}} = K + K_0$, we have

$$\begin{aligned} |\nabla U(x) - \nabla U(y)| &= \left| \int_{\mathbb{T}^d} [K_{\text{per}}(x-z) - K_{\text{per}}(y-z)] \rho(z) dz \right| \\ &\leq C_d \left| \int_{\mathbb{T}^d} \left[\frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right] \rho(z) dz \right| + \left| \int_{\mathbb{T}^d} [K_0(x-z) - K_0(y-z)] \rho(z) dz \right| \end{aligned}$$

For the second term, we use the fact that $K_0 \in C^1(\mathbb{T}^d)$ to deduce that

$$\left| \int_{\mathbb{T}^d} [K_0(x-z) - K_0(y-z)] \rho(z) dz \right| \leq \|\nabla K_0\|_{L^\infty(\mathbb{T}^d)} |x-y| \int_{\mathbb{T}^d} |\rho(z)| dz \leq C_d \|\rho\|_{L^\infty} |x-y|.$$

For the first term we split the integral by defining the two regions:

$$A_1 = \{z \in \mathbb{T}^d : |x-z| \leq 2|x-y|\}, \quad A_2 = \{z \in \mathbb{T}^d : 2|x-y| \leq |x-z| \leq 2\sqrt{d}\}.$$

Then let

$$I_i := \int_{z \in A_i} \left[\frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right] \rho(z) dz.$$

For $z \in A_1$,

$$|x-z| \leq 2|x-y|, \quad |y-z| \leq 3|x-y|.$$

Thus, letting $u = x-z$,

$$I_1 \leq \|\rho\|_{L^\infty(\mathbb{T}^d)} \left(\int_{|u| \leq 2|x-y|} |u|^{-(d-1)} du + \int_{|u| \leq 3|x-y|} |u|^{-(d-1)} du \right) \leq 5\|\rho\|_{L^\infty(\mathbb{T}^d)} |x-y|.$$

For I_2 , we first assume that $|x-y| \leq \sqrt{d}$, since otherwise A_2 is empty. For $z \in A_2$, set $K^{(z)}(w) := \frac{w-z}{|w-z|^d}$. Since

$$|\nabla K^{(z)}(w)| \leq \frac{C_d}{|w-z|^d}$$

for some dimension dependent constant $C_d > 0$, we deduce that

$$\begin{aligned} \left| \frac{x-z}{|x-z|^d} - \frac{y-z}{|y-z|^d} \right| &= |K^{(z)}(x) - K^{(z)}(y)| \leq \left(\sup_{\theta \in [0,1]} |\nabla K^{(z)}((1-\theta)x + \theta y)| \right) |x-y| \\ &\leq C_d \left(\sup_{\theta \in [0,1]} |(1-\theta)x + \theta y - z|^{-d} \right) |x-y| \\ &= C_d \left(\sup_{\theta \in [0,1]} |x-z + \theta(y-x)|^{-d} \right) |x-y|. \end{aligned}$$

Since for $z \in A_2$,

$$|x - y| \leq \frac{1}{2}|x - z|,$$

it follows that for all $\theta \in [0, 1]$,

$$|x - z + \theta(y - x)| \geq |x - z| - \frac{\theta}{2}|x - z| \geq \frac{1}{2}|x - z|.$$

Hence

$$\left| \frac{x - z}{|x - z|^d} - \frac{y - z}{|y - z|^d} \right| \leq C_d |x - z|^{-d} |x - y|.$$

Then

$$\begin{aligned} I_2 &\leq C_d |x - y| \int_{z \in A_2} |x - z|^{-d} \rho(z) \, dz \leq C_d |x - y| \|\rho\|_{L^\infty(\mathbb{T}^d)} \int_{z \in A_2} |x - z|^{-d} \, dz \\ &= C_d |x - y| \|\rho\|_{L^\infty(\mathbb{T}^d)} \int_{2|x-y|}^{2\sqrt{d}} \frac{1}{r} \, dr \leq C_d |x - y| \|\rho\|_{L^\infty(\mathbb{T}^d)} \log \frac{\sqrt{d}}{|x - y|}. \end{aligned}$$

Altogether we obtain

$$|\nabla U(x) - \nabla U(y)| \leq C_d \|\rho\|_{L^\infty(\mathbb{T}^d)} |x - y| \left(1 + \log \left(\frac{\sqrt{d}}{|x - y|} \right) \mathbb{1}_{|x-y| \leq \sqrt{d}} \right).$$

□

1.4 Well-Posedness Theory for the Vlasov-Poisson System

The theory of existence and uniqueness of solutions for the Vlasov-Poisson system for electrons is nowadays well understood. For instance, it is known that solutions exist globally in time, without a smallness or perturbative condition on the initial data f_0 . In this section, we review some of the more recent well-posedness results for the Vlasov-Poisson system for electrons. We place a particular emphasis on those results upon which our study of the VPME system in Chapter 2 will be based.

Iordanskii [56] proved that, in dimension $d = 1$, (1.5) has a unique smooth solution for initial data f_0 decaying sufficiently quickly at infinity. In the three-dimensional case $d = 3$, Arsen'ev [2] introduced a notion of weak solution for the Vlasov-Poisson system (1.5) and proved their existence, globally in time, for initial data f_0 belonging to the space $L^1 \cap L^\infty(\mathbb{R}^6)$. Horst and Hunze [55] then showed that the boundedness condition $f_0 \in L^\infty(\mathbb{R}^6)$ could be relaxed to $f_0 \in L^p(\mathbb{R}^6)$ for $p \geq (12 + 3\sqrt{5})/11$.

In Vlasov-Poisson theory, the terminology ‘strong’ or ‘classical’ solution usually refers to a solution for which there is an associated classical characteristic flow, according to the definitions in Section 1.3.1.1, rather than necessarily to a C^1 solution. The techniques involved in proving the existence of such solutions differ depending on the dimension d considered. The mathematical reason for this is that the estimates available for the electric field depend on the dimension d . We will discuss this in greater depth below.

Ukai and Okabe [81] proved the existence of global-in-time strong solutions in the two-dimensional case $d = 2$, for initial data $f_0 \in C^1(\mathbb{R}^4)$ decaying sufficiently fast at infinity.

In dimension $d = 3$, local-in-time existence was established by Kurth [59]. Global-in-time existence was proved under various symmetry or near-symmetry conditions in [9, 52, 78], and for small data, with no symmetry condition, by Bardos–Degond [6].

For large data, global-in-time strong solutions in dimension $d = 3$ were constructed by Pfaffelmoser [74], for any compactly supported initial datum $f_0 \in L^\infty(\mathbb{R}^6)$. The proof is based on controlling the rate of growth of the support of the solution, and showing that the solution remains compactly supported in \mathbb{R}^6 at all times. Schaeffer gave a streamlined proof of the same result in [79]. Horst [54] extended these results to include non-compactly supported initial data with sufficiently fast decay at infinity. These results are valid for the whole space case $(x, v) \in \mathbb{R}^6$ and are based on an analysis of the characteristic trajectories of the solution. To prove a version of this result on the torus (that is, for $(x, v) \in \mathbb{T}^3 \times \mathbb{R}^3$), it is necessary to deal with the fact that characteristic trajectories wrap around the torus. The method was adapted to the torus by Batt and Rein [10], who proved the existence of global-in-time strong solutions for (1.5) in the three-dimensional case $d = 3$, for initial data $f_0 \in L^1 \cap L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ with compact support.

An alternative approach to the construction of global-in-time solutions in 3D was provided by Lions and Perthame [64]. Their method is based on proving the propagation of moments. They showed global existence of solutions, provided that the initial datum $f_0 \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ has moments in velocity of sufficiently high order. However, their strategy is for the whole space case $x \in \mathbb{R}^d$, and it is not currently known how to adapt it to the torus.

Pallard [73] combined the two approaches by proving the propagation of moments using a method reminiscent of [74, 79, 10]. This result extends the range of moments beyond that covered by the result from Lions–Perthame [64], as well as including a result of propagation of moments for the equation posed on the torus.

Lions and Perthame [64] also proved a uniqueness criterion for their solutions under an additional technical condition requiring Lipschitz continuity of the initial datum f_0 . Robert [76] then proved uniqueness for solutions that are compactly supported in phase space for all time. Loeper [65] proved a general uniqueness result which requires only boundedness of the mass density ρ_f , and therefore includes the compactly supported case. Loeper’s result is similar in style to the Dobrushin result quoted above as Theorem 1.5, in that it relies on a stability estimate on solutions in a Wasserstein distance, in Loeper’s case W_2 . In the one dimensional case, Hauray [47] proved a weak-strong uniqueness principle, showing that if a bounded density solution exists, then this solution is unique among measure-valued solutions. This result is also based on a Wasserstein stability result.

In the remainder of this section, we introduce in greater detail two of the more recent results in the well-posedness theory for the Vlasov-Poisson system for electrons. We begin by discussing Loeper’s [65] uniqueness result for solutions for which ρ_f is bounded in L^∞ . Then, in Subsection 1.4.3, we discuss the approach of Pfaffelmoser [74], Schaeffer [79] and Batt and Rein [10] based on controlling the support. In Chapter 2, we will prove analogues of these results for the VPME system, which are stated in Section 1.7 below as Theorems 1.21 and 1.22.

1.4.1 Basic Estimates

We begin by recalling some basic properties of solutions of the Vlasov-Poisson system (1.5). The aim is to understand the basic a priori estimates on solutions that are implied by the conservation laws of the system. In this case, the conserved quantities are the energy and the mass.

1.4.1.1 L^p Estimates

In Subsections 1.3.1.1 and 1.3.2.2 above, we discussed the conservation of L^p norms for solutions of transport and continuity equations.

Here, we use the fact that the Vlasov-Poisson system is a transport equation to derive $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ estimates on smooth solutions. Note first that the vector field

$$b(x, v) = (v, E(x))$$

is divergence free in $\mathbb{T}^d \times \mathbb{R}^d$. Thus, if f is a solution of the Vlasov-Poisson system (1.5), then f satisfies both a transport and a continuity equation.

We assume that f is a smooth solution of (1.5) and therefore induces a classical characteristic flow. We therefore have the following $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ estimates.

Firstly, for the transport equation (1.27), if the initial data f_0 is in the space $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, the solution f satisfies, for $t \geq 0$,

$$\|f(t, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} = \|f_0\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}.$$

This was a consequence of the representation via characteristics.

Secondly, for the continuity equation (1.32) we showed that, if f_0 is a finite measure, then the total mass of the solution f_t is non-increasing. In particular, if $f_0 \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$, then

$$\|f(t, \cdot, \cdot)\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)} = \|f_0\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)}.$$

This was a consequence of the pushforward representation (1.33).

For strong solutions of the Vlasov-Poisson system (1.5) with initial data $f_0 \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, we therefore expect to have uniform in time L^p estimates, for any $p \in [1, +\infty]$: for any $t \geq 0$ and any $p \in [1, +\infty]$,

$$\|f(t, \cdot, \cdot)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq \|f_0\|_{L^1}^{\frac{1}{p}} \|f_0\|_{L^\infty}^{1-\frac{1}{p}}. \quad (1.45)$$

1.4.1.2 Moments

In the theory of kinetic equations, it is important to understand how to extract information about the spatial density ρ_f from properties of the full phase-space density f . The following lemma is a well-known useful result showing that control of the moments of f implies $L^p(\mathbb{T}^d)$ integrability for ρ_f .

Lemma 1.12. *Let $f \in L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ satisfy, for some $q > 0$,*

$$M_q[f] = \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^q f(x, v) \, dx \, dv < \infty.$$

Then, the mass density $\rho[f]$, defined by

$$\rho[f](x) := \int_{\mathbb{R}^d} f(x, v) \, dv$$

belongs to the space $L^{\frac{q+d}{d}}(\mathbb{T}^d)$. Moreover, we have the following estimate on the norm:

$$\|\rho[f]\|_{L^{\frac{q+d}{d}}(\mathbb{T}^d)} \leq C_{d,q} \|f\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}^{\frac{q}{q+d}} M_q[f]^{\frac{d}{q+d}}.$$

Proof. The proof uses an interpolation argument. We split the integral defining $\rho[f]$ into two parts - one involving small velocities and another involving large velocities. For some R , to be fixed later, we have

$$\int_{\mathbb{R}^d} f(x, v) \, dv = \int_{|v| \leq R} f(x, v) \, dv + \int_{|v| > R} f(x, v) \, dv.$$

The small velocity term is bounded in terms of the volume of the ball of radius R in \mathbb{R}^d , since $f \in L^\infty$:

$$\int_{|v| \leq R} f(x, v) \, dv \leq C_d \|f\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R^d.$$

The large velocity term is bounded using the higher moment:

$$\int_{|v| > R} f(x, v) \, dv \leq R^{-q} \int_{\mathbb{R}^d} |v|^q f(x, v) \, dv.$$

Thus

$$\int_{\mathbb{R}^d} f(x, v) \, dv \leq C_d \|f\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R^d + R^{-q} \int_{\mathbb{R}^d} |v|^q f(x, v) \, dv.$$

We choose R so as to minimise this bound. The optimal value is

$$R = R(t, x) = C_d \left(\frac{\int_{\mathbb{R}^d} |v|^q f(x, v) \, dv}{\|f\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}} \right)^{\frac{1}{q+d}}.$$

This implies that

$$\int_{\mathbb{R}^d} f(x, v) \, dv \leq C_{d,q} \|f\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}^{\frac{q}{q+d}} \left(\int_{\mathbb{R}^d} |v|^q f(x, v) \, dv \right)^{\frac{d}{q+d}}.$$

Next, take the $L^{\frac{q+d}{d}}(\mathbb{T}^d)$ norm of these quantities:

$$\left\| \int_{\mathbb{R}^d} f(x, v) \, dv \right\|_{L^{\frac{q+d}{d}}(\mathbb{T}^d)} \leq C_{d,q} \|f\|_{L^\infty(\mathbb{T}^d)}^{\frac{q}{q+d}} \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^q f(x, v) \, dv \, dx \right)^{\frac{d}{q+d}}.$$

This completes the proof. \square

This estimate is used to deduce properties of the electric field E from estimates on the moments of the solution f . It is particularly useful in conjunction with the conservation of energy, as we will explain in the next section.

1.4.1.3 The Role of the Energy

The Vlasov-Poisson system has an associated energy functional, which is defined as follows:

$$\mathcal{E}^{\text{VP}}[f] := \underbrace{\frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv}_{=:\mathcal{K}} + \underbrace{\frac{1}{2} \int_{\mathbb{T}^d} |\nabla U|^2 \, dx}_{=:\mathcal{P}}. \quad (1.46)$$

The first term \mathcal{K} is the total kinetic energy of f , while the second term \mathcal{P} is the electrostatic potential energy. This functional is formally conserved by the equation.

Lemma 1.13 (Conservation of Energy). *Let f be a smooth solution of the Vlasov-Poisson system (1.5). Then, for all t ,*

$$\mathcal{E}^{\text{VP}}[f(t, \cdot, \cdot)] = \mathcal{E}^{\text{VP}}[f_0].$$

Proof. Consider the time derivative of the kinetic energy \mathcal{K} . Using the equation (1.5) satisfied by f , we find that

$$\begin{aligned} \frac{d}{dt} \mathcal{K}(t) &= \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(t, x, v) \, dx \, dv \\ &= \int_{\mathbb{T}^d \times \mathbb{R}^d} [-v \cdot \nabla_x f - E \cdot \nabla_v f] |v|^2 \, dx \, dv. \end{aligned}$$

We apply integration by parts. The term involving $\nabla_x f$ vanishes because the function $v|v|^2$ does not depend on x . We are left with

$$\frac{d}{dt} \mathcal{K}(t) = 2 \int_{\mathbb{T}^d \times \mathbb{R}^d} [v \cdot E(t, x)] f(t, x, v) \, dx \, dv.$$

We define the **current density** $J_f : [0, \infty) \times \mathbb{T}^d \rightarrow \mathbb{R}^d$ by

$$J_f(t, x) := \int_{\mathbb{R}^d} v f(t, x, v) \, dv.$$

Then

$$\frac{d}{dt}\mathcal{K}(t) = 2 \int_{\mathbb{T}^d} E(t, x) \cdot J_f(t, x) dx.$$

Since E is defined by the relation $E = -\nabla_x U$, we may write

$$\frac{d}{dt}\mathcal{K} = -2 \int_{\mathbb{T}^d} \nabla_x U \cdot J_f dx.$$

We integrate by parts one more time to obtain

$$\frac{d}{dt}\mathcal{K} = 2 \int_{\mathbb{T}^d} U \operatorname{div}_x J_f dx.$$

We next consider the time derivative of \mathcal{P} . We calculate that

$$\frac{d}{dt}\mathcal{P} = \frac{d}{dt} \left(\int_{\mathbb{T}^d} |\nabla_x U|^2 dx \right) = 2 \int_{\mathbb{T}^d} \partial_t \nabla_x U \cdot \nabla_x U dx$$

Integration by parts gives

$$\frac{d}{dt}\mathcal{P} = -2 \int_{\mathbb{T}^d} U \partial_t \Delta_x U dx.$$

We use the Poisson equation to substitute for ΔU :

$$\frac{d}{dt}\mathcal{P} = 2 \int_{\mathbb{T}^d} U \partial_t \rho_f dx. \quad (1.47)$$

To obtain an expression for $\partial_t \rho_f$, we use the transport equation:

$$\partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0.$$

Integrating with respect to v gives

$$\partial_t \rho_f + \int_{\mathbb{R}^d} v \cdot \nabla_x f dv + \int_{\mathbb{R}^d} E \cdot \nabla_v f dv = 0.$$

Since E is independent of v , $E \cdot \nabla_v f$ is a total derivative in v and the third term vanishes. The second term may be expressed as

$$\begin{aligned} \int_{\mathbb{R}^d} v \cdot \nabla_x f dv &= \int_{\mathbb{R}^d} \operatorname{div}_x (vf) dv \\ &= \operatorname{div}_x \left(\int_{\mathbb{R}^d} vf dv \right) \\ &= \operatorname{div}_x J_f. \end{aligned}$$

We conclude that $\partial_t \rho_f = -\operatorname{div}_x J$. Substituting this into (1.47) gives

$$\frac{d}{dt} \mathcal{P} = -2 \int_{\mathbb{T}^d} U \operatorname{div}_x J_f \, dx = -\frac{d}{dt} \mathcal{K}.$$

□

Since both \mathcal{K} and \mathcal{P} are non-negative, control of \mathcal{E}^{VP} implies a bound on \mathcal{K} , which is exactly $M_2[f]$, the second order velocity moment of f . By Lemma 1.12, this implies an L^p estimate on ρ_f .

Lemma 1.14. *Let $f \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, such that $\mathcal{E}^{\text{VP}}[f]$ is finite. Then*

$$\|\rho_f\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C_d \|f\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}^{\frac{2}{d+2}} \mathcal{E}^{\text{VP}}[f]^{\frac{d}{d+2}}.$$

It is therefore useful to be able to prove estimates for the Vlasov-Poisson system that involve the quantity $\|\rho_f\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}$. We will use a similar property in the study of the VPME system.

1.4.1.4 Electric Field

An important consequence of these estimates is that solutions of the Vlasov-Poisson system with bounded energy automatically satisfy certain uniform-in-time integrability bounds on their electric fields E . This follows from the fact that $E = -\nabla U$, where

$$-\Delta U = \rho_f - 1, \quad x \in \mathbb{T}^d.$$

We then apply standard estimates for the Poisson equation. By Calderón-Zygmund estimates, if $\rho_f \in L^p(\mathbb{T}^d)$ for $p \in [1, \infty)$, then

$$\|E\|_{W^{1,p}(\mathbb{T}^d)} \leq \|U\|_{W^{2,p}(\mathbb{T}^d)} \leq C_p \|\rho_f\|_{L^p(\mathbb{T}^d)}.$$

For the choice $p = \frac{d+2}{d}$, we therefore have a uniform-in-time estimate on E in $W^{1,p}(\mathbb{T}^d)$.

Moreover, Sobolev embedding estimates can be used to deduce uniform-in-time L^q estimates on E for some $q = q(p, d)$. This explains why the theory of the Vlasov-Poisson system differs depending on the dimension. As d increases, there are two effects. One is that the gain of integrability on E with respect to ρ_f due to Sobolev embedding is smaller. At the same time, the integrability on ρ_f obtained from the conservation of energy is also of lower order. Overall, the a priori uniform estimate on E is weaker for higher d .

Explicitly, in dimension $d = 2$ we have the critical case of Sobolev embedding, where the Lebesgue exponent is equal to the dimension of the space. We therefore expect a uniform-in-time L^p estimate on E for any $p \in [1, \infty)$: for all $t \geq 0$,

$$\|E(t, \cdot)\|_{L^p(\mathbb{T}^d)} \leq C(p, f_0).$$

In dimension $d = 3$, we only have this estimate up to $p = \frac{15}{4}$: for all $t \geq 0$,

$$\|E(t, \cdot)\|_{L^{\frac{15}{4}}(\mathbb{T}^d)} \leq C(f_0).$$

This is a key difference between the cases $d = 2$ and $d = 3$.

1.4.2 Uniqueness for Solutions with Bounded Density

We discuss a uniqueness result due to Loeper [65], dealing with solutions of the Vlasov-Poisson system for electrons (1.5) with bounded density - that is, solutions for which the spatial density ρ_f lies in $L^\infty(\mathbb{T}^d)$. The following theorem comes from [65, Theorem 1.2].

Theorem 1.15. *Let $f_0 \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ and let $T > 0$. There exists at most one solution f of (1.5), in the sense of Definition 1, for which $\rho_f \in L^\infty([0, T]; L^\infty(\mathbb{T}^d \times \mathbb{R}^d))$.*

Remark 2. The definition of weak solutions given in Definition 1 makes sense for solutions of the Vlasov-Poisson system with bounded density, since in this case E is a log-Lipschitz function by Lemma 1.11.

This result is based on a version of the Wasserstein stability estimate in the smooth case that we discussed in Lemma 1.8. In Section 1.3.3 we discussed the fact that Lemma 1.8 does not hold for Vlasov equations with singular interactions such as the Coulomb force. However, we can recover a similar estimate by assuming additional regularity on the solutions we consider. In fact, a version of the stability estimate is possible for solutions with bounded density. The following theorem is a consequence of the proof of [65, Theorem 1.2].

Theorem 1.16. *Let $f_1(0), f_2(0)$ be probability measures on $\mathbb{T}^d \times \mathbb{R}^d$. For each i , let f_i be a solution of the Vlasov-Poisson system (1.5). Assume that, for some $T > 0$, for $i = 1, 2$,*

$$\sup_{t \in [0, T]} \|\rho[f_i(t)]\|_{L^\infty(\mathbb{R}^d)} \leq C_0.$$

Then the second order Wasserstein distance between the two solutions satisfies the following estimate: for all $t \in [0, T]$

$$W_2(f_1(t), f_2(t)) \leq \begin{cases} C \exp \left[C \left(1 + \log \frac{W_2(f_1(0), f_2(0))}{4\sqrt{d}} \right) e^{-Ct} \right] & \text{if } W_2(f_1(0), f_2(0)) \leq d \\ W_2(f_1(0), f_2(0)) e^{Ct} & \text{if } W_2(f_1(0), f_2(0)) > d. \end{cases}$$

The constant C depends on C_0 .

This result is proved following a similar method to the proof of Lemma 1.8. It requires suitable replacements for the regularity estimate (1.39) and stability estimate (1.40). For the regularity estimate we use the log-Lipschitz property from Lemma 1.11. The necessary stability estimate was proved by Loeper [65]. The following result comes from [65, Theorem 2.9]. It can also be adapted to the torus.

Theorem 1.17. *For each $i = 1, 2$, let $\rho_i \in \mathcal{P} \cap L^\infty(\mathbb{R}^d)$ and let U_i satisfy*

$$-\Delta U_i = \rho_i, \quad \lim_{|x| \rightarrow \infty} U_i(x) = 0.$$

Then

$$\|\nabla U_1 - \nabla U_2\|_{L^2(\mathbb{R}^d)} \leq \max_i \|\rho_i\|_{L^\infty(\mathbb{T}^d)}^{1/2} W_2(\rho_1, \rho_2).$$

In the one-dimensional case $d = 1$, it is possible to replace Theorem 1.16 with a weak-strong stability estimate due to Hauray [47, Theorem 1.9], in which only one of the solutions needs to have bounded density. This results in a weak-strong uniqueness principle. This was extended to the VPME case in [44].

In Chapter 2, we will prove an analogue of Theorem 1.16 for VPME. This result is stated below as Theorem 1.21.

1.4.3 Global Existence of Solutions

We have seen above that bounded density solutions of the Vlasov-Poisson system have a uniqueness property. It is also possible to show that solutions in this class exist globally in time, for compactly supported initial data. This was proved by Pfaffelmoser [74] and Schaeffer [79] for the whole space case (1.6), and subsequently for the torus by Batt and Rein [10]. For example, the following result is a consequence of the estimates of Batt and Rein [10].

Theorem 1.18 (Existence of solutions on the torus). *Let $d = 3$, and suppose that $f_0 \in C_c^1(\mathbb{T}^d \times \mathbb{R}^d)$, with $f_0 \geq 0$ and $\|f_0\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)} = 1$. Then there exists a global-in-time solution $f \in C^1([0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d)$ of the Vlasov-Poisson system (1.5).*

The approach of [74, 79, 10] is based on showing that f_t is compactly supported for all t , with the size of the support controlled by a continuous function:

$$\text{supp } f_t \subset \mathbb{T}^d \times B_{R(t)}(0), \quad R \in C(\mathbb{R}_+; \mathbb{R}_+).$$

This implies global-in-time existence because the solution can always be continued locally from a compactly supported initial datum [51, 74].

Moreover, these solutions have bounded density. Indeed, $\rho[f_t]$ satisfies the estimate

$$\|\rho[f_t]\|_{L^\infty(\mathbb{T}^d)} \leq C_d \|f_t\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R_t^d,$$

where C_d is the volume of the unit ball in \mathbb{R}^d . Thus the uniform bound on the L^∞ norm of f_t (1.45) implies that

$$\|\rho[f_t]\|_{L^\infty(\mathbb{T}^d)} \leq C_d \|f_0\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R_t^d \leq C(d, f_0) R_t^d.$$

The uniqueness result of Loeper (Theorem 1.15) thus applies to these solutions.

In Chapter 2, we will prove an analogous result on the global-in-time existence of solutions for the VPME system. Our method makes use of the estimates of Batt and Rein [10], which control the electric field for the classical Vlasov-Poisson system (1.5). We will revisit their method in detail in Chapter 2, in Section 2.7, where we adapt these estimates to the VPME case.

1.4.4 The Vlasov-Poisson System with Massless Electrons

In Chapter 2 of this thesis, we investigate the well-posedness theory of the VPME system (1.11). The main difficulty in analysing the VPME model compared to the classical system is that the coupling between the density f and the electric field E is nonlinear.

In the one-dimensional case, global existence of solutions was proved in Han-Kwan–Iacobelli [44, Theorem 1.1]. In dimension $d = 3$, weak solutions were constructed globally in time by Bouchut [17].

In this thesis, we extend the main well-posedness results outlined above for the system (1.5) to the VPME case. We prove a uniqueness result in the style of Loeper [65], which is stated in Section 1.7 as Theorem 1.21, and proved in Chapter 2, Section 2.6. We also prove the global existence of solutions following a method in the style of Batt and Rein [10]. This result is stated in Section 1.7 as Theorem 1.22, and proved in Chapter 2, Sections 2.7–2.9.

The key step in our analysis is to prove appropriate regularity and stability estimates on the electric field in the VPME system. These estimates are proved in Chapter 2, Sections 2.4 and 2.5.

1.5 The Quasi-Neutral Limit

1.5.1 Challenges

In this section we describe some of the challenges involved in proving a rigorous quasi-neutral limit. These challenges can be related to known phenomena in plasma physics.

To understand the issues, it is useful to observe that the quasi-neutral limit can be related to a long time limit for the Vlasov-Poisson system. If f is a solution of the unscaled Vlasov-Poisson system (1.5), then $f_\varepsilon = f\left(\frac{t}{\varepsilon}, \frac{x}{\varepsilon}, v\right)$ is a solution of the system with quasi-neutral scaling (1.15).

One source of difficulty therefore arises from dealing with instabilities in the Vlasov-Poisson system, which are known to exist. An important example is the **two-stream instability**, which we describe below. In order to avoid this instability, it is necessary to restrict the class of data considered.

Another source of difficulty is that solutions of the Vlasov-Poisson system exhibit an oscillatory behaviour in their electric field. This oscillation is related to the **plasma oscillations**, which are a well-known phenomenon in plasma physics. The oscillation does not become small in the quasi-neutral limit, and must therefore be removed.

1.5.1.1 Two-Stream Instability

The two-stream instability is an instability mechanism for the Vlasov-Poisson system that is well-known in plasma physics. The model problem is to consider firing two jets of electrons at each other - these jets are the ‘two streams’. This physical situation is known to be unstable – see for example [11, Section 5.1].

In a kinetic model, configurations of this type can be represented by choosing an initial datum f_0 whose velocity distribution consists of two bumps. Solutions beginning from such configurations develop an instability taking the form of vortex-like behaviour in phase space – see for example [12] for simulations and experimental results on this phenomenon. Such behaviour prevents the quasi-neutral limit from holding. It is therefore necessary to place some condition on the initial data for the Vlasov-Poisson system that rules out this instability.

1.5.1.2 Plasma Oscillations

Another key issue in proving the quasi-neutral limit is the presence of a known oscillatory behaviour that does not vanish in the limit. This is related to the ‘plasma oscillations’ which are well-known in plasma physics. This behaviour was first described by Tonks and Langmuir [80]. The origin of these oscillations can be understood by considering a plasma in an equilibrium state, so that both ions and electrons are uniformly distributed. Consider perturbing the electrons

by a small amount. The ions then exert a restoring force on the perturbed electrons which pulls them back towards their starting position. The electrons will then overshoot their starting position, and the continued influence of the restoring force will set up an oscillation.

This behaviour is seen mathematically in solutions of the Vlasov-Poisson system as an oscillation in the electric field. These oscillations were analysed by Grenier [33] using the fact that the electric field satisfies the following equation:

$$\varepsilon^2 \partial_t^2 \operatorname{div} E + \operatorname{div} E = \operatorname{div} [(1 - \rho)E] + \nabla_x^2 : \int_{\mathbb{R}^d} v \otimes v f_\varepsilon \, dv. \quad (1.48)$$

In physics presentations, one typically considers the situation described above of a system slightly perturbed out of equilibrium. In this case the spatial density ρ is close to uniform. If we assume further that the electrons are cold, then the density f_ε is roughly of the form $f_\varepsilon \approx \delta_{u(t,x)}$. Since the average velocity u should be small, it is possible to make the approximation

$$\varepsilon^2 \partial_t^2 E + E \approx 0.$$

From this it can be derived that E oscillates with frequency $\frac{1}{\varepsilon}$.

Grenier [33] showed that similar oscillations occur for the full equation (1.48) coupled with the Vlasov-Poisson system. These oscillations do not vanish in the quasi-neutral limit. However, they can be accounted for by introducing a ‘corrector’ function. This corrector can be built from the solutions of known equations - see Section 1.5.4.1 below for details.

1.5.2 Known Results

The mathematical study of the quasi-neutral limit from Vlasov-Poisson to kinetic incompressible Euler received significant attention in the 90s. We refer for instance to the papers of Brenier–Grenier [21] and Grenier [32], using an approach based on defect measures, and the result of Grenier [34] for the one-dimensional case. Brenier [20] and Masmoudi [67] considered the ‘cold electrons’ regime, in which f_ε converges to a Dirac mass in velocity as $\varepsilon \rightarrow 0$, and the limit is the incompressible Euler system (1.18).

A general result of interest for our purposes is the work of Grenier [33], who proved the quasi-neutral limit rigorously under the condition that the initial data for (1.15) are uniformly analytic, in a sense based on a reformulation of the Vlasov-Poisson and kinetic incompressible Euler systems as multi-fluid pressureless Euler systems. We will describe this precisely in Section 1.5.4.1.

In Sobolev regularity, the quasi-neutral limit does not hold in general. This is due to the possible occurrence of the two-stream instability, which would prevent strong convergence

to the limit. Counterexamples to the quasi-neutral limit were constructed by Han-Kwan and Hauray [42], in arbitrarily high Sobolev regularity.

An alternative way of relaxing the regularity constraint was investigated by Han-Kwan and Iacobelli [43, 44]. These works extended the quasi-neutral limit to a class of rough data. The data must be a very small perturbation of the uniformly analytic case, but may be as rough as L^∞ (or a measure in the case $d = 1$). The smallness of the perturbation is measured in a Wasserstein distance. The papers [43, 44] cover the cases where the dimension $d = 1, 2, 3$.

For the massless electrons model, similar results are available. The cold ions case was considered in [41]. In analytic regularity, it is possible to prove a version of Grenier's result [33] - see the discussion in [44] after Proposition 4.1. Han-Kwan and Iacobelli [44] showed a rigorous limit in dimension one, again for rough perturbations of analytic data. However, in higher dimensions, a similar result was not previously available. This gap is filled in this thesis - in Chapter 3, we present a proof of a rigorous quasi-neutral limit for the VPME system for a class of rough data in dimension $d = 2, 3$.

A positive result is available in Sobolev regularity for the VDB system (1.21) by Han-Kwan and Rousset [46]. The VDB system can also be derived in a quasi-neutral limit, from the following linearised version of the VPME system:

$$(VPME)_\varepsilon := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ \varepsilon^2 \Delta_x U = 1 + U - \rho_f, \\ f|_{t=0} = f_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0(x, v) \, dx \, dv = 1 \end{cases}$$

Han-Kwan and Rousset [46] proved this limit rigorously for data with sufficiently high Sobolev regularity, under an additional stability criterion.

1.5.3 Existence Theory for Kinetic Euler Systems

A proof of the quasi-neutral limit will either require or imply a result of existence of solutions for the relevant kinetic Euler equation. Local existence of solutions was shown in analytic regularity in one dimension by Bossy et al. [16] for the kinetic incompressible Euler system, as part of a study of more general kinetic equations with an incompressibility constraint. The corresponding result for the Vlasov–Dirac–Benney equation was proved by Jabin and Nouri [57]; see also Mouhot and Villani [71, Section 9]. Local-in-time existence in analytic regularity in higher dimensions is a corollary of the quasi-neutral limit results of Grenier [33].

In Sobolev regularity the systems are in general ill-posed. Instability results were shown for the VDB system by Bardos and Nouri [8] and by Bardos and Besse [5] – see also the report of Bardos [4] on this work. These results were later extended by Han-Kwan–Nguyen [45] and Baradat [3], with results covering both the VDB and kinetic incompressible Euler systems.

In Sobolev regularity, local-in-time existence has been proved under a further stability criterion in [5, 46] for the VDB system. No global-in-time existence results are available for any of the kinetic Euler systems we consider.

1.5.4 Rough Data Results

In Chapter 3, we will prove a result on the quasi-neutral limit from the VPME system (1.11) to the KIsE system (1.20). Our result holds for rough data that are small perturbations of the analytic case, in dimension $d = 2, 3$. It is analogous to the results of Han-Kwan–Iacobelli [43] for the classical Vlasov-Poisson system, and an extension of the one dimensional result of Han-Kwan–Iacobelli [44].

In the following sections, we discuss the results of Grenier [33] for the analytic case and Han-Kwan–Iacobelli [44, 43] for small rough perturbations of the analytic case.

1.5.4.1 Analytic Case

In this section, we discuss a result by Grenier [33] dealing with the quasi-neutral limit for the classical Vlasov-Poisson system (1.15), in the case of analytic initial data.

Multi-Fluid Formulation. Grenier’s approach to the quasi-neutral limit is phrased in terms of a representation of the Vlasov-Poisson system (1.15) as a multi-fluid pressureless Euler-Poisson system. The idea is to think of the electron density f_ε as a superposition of layers of fluid, each described by an associated velocity field. To be more precise, assume that f_ε may be represented in the form

$$f_\varepsilon(t, x, v) = \int_{\Theta} \rho_\varepsilon^\theta(t, x) \delta_{v_\varepsilon^\theta(t, x)}(v) \mu(d\theta), \quad (1.49)$$

for some probability space $(\Theta, \mathcal{C}, \mu)$ and a family of functions $(\rho_\varepsilon^\theta, v_\varepsilon^\theta)_{\theta \in \Theta}$.

The representation (1.49) allows a general class of initial data $f_\varepsilon(0)$. For example, any continuous initial datum $f_\varepsilon(0)$ can be represented in this form. Choose $\Theta = \mathbb{R}^d$, and let

$$\mu(d\theta) = \frac{1}{c_k(1 + |\theta|^k)} d\theta,$$

for suitable values of k and c_k so that μ is a probability measure. Then let

$$\rho_\varepsilon^\theta(0, x) = c_k(1 + |\theta|^k) f_\varepsilon(0, x, \theta), \quad v_\varepsilon^\theta(0, x) = \theta.$$

The formula (1.49) is also able to represent distributions with much lower regularity in v , such as sums of Dirac masses in velocity.

The idea of the multi-fluid representation of the Vlasov-Poisson system is to formulate a representation (1.49) of f_ε where the $(\rho_\varepsilon^\theta, v_\varepsilon^\theta)_{\theta \in \Theta}$ satisfy fluid equations. The multi-fluid system for $(\rho_\varepsilon^\theta, v_\varepsilon^\theta)_{\theta \in \Theta}$ associated to (1.15) is the following:

$$(VP - MF)_\varepsilon := \begin{cases} \partial_t \rho_\varepsilon^\theta + \operatorname{div}_x(\rho_\varepsilon^\theta v_\varepsilon^\theta) = 0, \\ \partial_t v_\varepsilon^\theta + (v_\varepsilon^\theta \cdot \nabla_x) v_\varepsilon^\theta = E_\varepsilon, \\ \nabla_x \times E_\varepsilon = 0, \\ \varepsilon^2 \nabla_x \cdot E_\varepsilon = \int_\Theta \rho_\varepsilon^\theta \mu(d\theta) - 1. \end{cases} \quad (1.50)$$

If the collection $(\rho_\varepsilon^\theta, v_\varepsilon^\theta)_{\theta \in \Theta}$ is a solution of system (1.50), then the formula (1.49) defines a weak solution of the Vlasov-Poisson system (1.15).

We can consider the quasi-neutral limit of the multi-fluid system by formally setting $\varepsilon = 0$ in the system (1.50). This results in the system

$$(KInE - MF) := \begin{cases} \partial_t \rho^\theta + \operatorname{div}_x(\rho^\theta v^\theta) = 0, \\ \partial_t v^\theta + (v^\theta \cdot \nabla_x) v^\theta = E, \\ \nabla_x \times E = 0, \\ \int_\Theta \rho^\theta \mu(d\theta) = 1. \end{cases} \quad (1.51)$$

If $(\rho^\theta, v^\theta)_{\theta \in \Theta}$ satisfies the system (1.51), then the density g defined by

$$g(t, x, v) := \int_\Theta \rho^\theta(t, x) \delta_{v^\theta(t, x)}(v) \mu(d\theta) \quad (1.52)$$

is a weak solution of the kinetic incompressible Euler system (1.16). The connection between systems (1.50) and (1.51) is thus a multi-fluid version of the quasi-neutral limit.

Quantifying Analyticity. To measure analyticity the norms $\|\cdot\|_{B_\delta}$ are used. These are defined for $\delta > 1$ by

$$\|g\|_{B_\delta} := \sum_{k \in \mathbb{Z}^d} |\hat{g}(k)| \delta^{|k|},$$

where $\hat{g}(k)$ denotes the Fourier coefficient of g of index k . Here δ is a (large) parameter quantifying the exponential rate of decay of the Fourier coefficients of g - the larger δ is, the faster $\hat{g}(k)$ must decay with respect to k in order for the series to converge, and therefore the smoother g must be in order to have finite norm.

Correctors. To obtain convergence in the quasi-neutral limit, it is necessary to account for the plasma oscillations described from a physical perspective in Section 1.5.1.2. This is done by introducing correctors, which describe these oscillations.

The correctors are defined as follows. For a solution $(\rho^\theta, v^\theta)_{\theta \in \Theta}$ of (1.51), we also define the corresponding current density

$$J(t, x) := \int_{\Theta} v^\theta(t, x) \rho^\theta(t, x) \mu(d\theta).$$

If g is defined by (1.52), then J satisfies

$$J(t, x) = \int_{\mathbb{R}^d} v g(t, x, v) dv.$$

Similarly, for a solution $f_\varepsilon(0, x, v)$ for the system (1.15), we define the current density J_ε :

$$J_\varepsilon(t, x) := \int_{\mathbb{R}^d} v f_\varepsilon(t, x, v) dv = \int_{\Theta} v_\varepsilon^\theta(t, x) \rho_\varepsilon^\theta(t, x) \mu(d\theta).$$

Note also that the electric field E_ε is defined in (1.50).

The corrector \mathcal{R}_ε is defined by

$$\mathcal{R}_\varepsilon(t, x) := \frac{1}{i} \left(d_+(t, x) e^{\frac{it}{\varepsilon}} - d_-(t, x) e^{-\frac{it}{\varepsilon}} \right), \quad (1.53)$$

where d_\pm are solutions of the following equations:

$$\begin{aligned} \nabla_x \times d_\pm &= 0, \quad \nabla_x \cdot (\partial_t d_\pm + J \cdot \nabla_x d_\pm) = 0 \\ \nabla_x \cdot d_\pm(0) &= \lim_{\varepsilon \rightarrow 0} \nabla_x \cdot \left(\frac{\varepsilon E_\varepsilon(0) \pm i J_\varepsilon(0)}{2} \right). \end{aligned}$$

Note that this system only depends on f_ε through its initial data $f_\varepsilon(0, x, v)$.

The corrector is used to ‘filter’ the velocity fields, after which convergence can be proved. Instead of looking at v_ε^θ , one shows that $v_\varepsilon^\theta - \mathcal{R}_\varepsilon$ converges to a limit.

Quasi-Neutral Limit. The main result of Grenier [33] is a quasi-neutral limit from system (1.50) to (1.51), under the assumption of uniform analyticity of the data with respect to x . The following result is one case of the main result of [33].

Theorem 1.19 (Quasi-neutral Limit for Analytic Data). *For each $\varepsilon \in (0, 1)$, let $g_\varepsilon(0)$ be a choice of initial datum for the Vlasov-Poisson system (1.15). Assume that the $\{g_\varepsilon(0)\}_{\varepsilon \in (0, 1)}$ are uniformly analytic in x , in the sense that, for some $\delta_0 > 1$,*

$$\sup_{\varepsilon \in (0, 1), v \in \mathbb{R}^d} (1 + |v|^{d+1}) \|g_\varepsilon(0, \cdot, v)\|_{B_{\delta_0}} \leq C.$$

Suppose further that

$$\sup_{\varepsilon \in (0, 1)} \left\| \int_{\mathbb{R}^d} g_\varepsilon(0, x, v) dv - 1 \right\|_{B_{\delta_0}} \leq \eta,$$

where η is sufficiently small.

Let $\{\rho_\varepsilon^\theta(0, x), v_\varepsilon^\theta(0, x)\}_{\theta \in \mathbb{R}^d, \varepsilon \in (0, 1)}$ be defined by

$$\rho_\varepsilon^\theta(0, x) = c_d(1 + |\theta|^{d+1})g_\varepsilon(0, x, \theta), \quad v_\varepsilon^\theta(0, x) = \theta. \quad (1.54)$$

Assume that, for each $\theta \in \mathbb{R}^m$, $\rho_\varepsilon^\theta(0, x)$ and $v_\varepsilon^\theta(0, x)$ have limits $\rho^\theta(0, x)$ and $v^\theta(0, x)$ as ε tends to zero, in the sense of distributions. Assume also that $\operatorname{div} J_\varepsilon(0)$ and $\varepsilon E_\varepsilon(0)$ have limits in the sense of distributions as ε tends to zero. Let δ_1 satisfy $\delta_0 > \delta_1 > 1$. Then there exists $T > 0$ such that:

- (i) For each $\varepsilon \in (0, 1)$, there exists a solution $\{\rho_\varepsilon^\theta, v_\varepsilon^\theta\}_{\theta \in \mathbb{R}^d}$ of the multi-fluid system (1.50) with initial data given by (1.54), with each ρ_ε^θ and v_ε^θ bounded in $C([0, T]; B_{\delta_1})$.
- (ii) There exists a solution $(\rho^\theta, v^\theta, E)$ to the limiting system (1.51) for $t \in [0, T]$, with initial data

$$\begin{aligned} \rho^\theta(0) &= \lim_{\varepsilon \rightarrow 0} \rho_\varepsilon^\theta(0) \\ v^\theta(0) &= \lim_{\varepsilon \rightarrow 0} \left[v_\varepsilon^\theta(0) - \nabla \Delta^{-1} \operatorname{div} J_\varepsilon(0) \right] \end{aligned}$$

- (iii) Define the corrector function \mathcal{R}_ε by (1.53), and let

$$\tilde{v}_\varepsilon^\theta := \mathcal{R}_\varepsilon,$$

then as ε tends to zero, for all $s \in \mathbb{N}$,

$$\sup_{t \in [0, T]} \left[\|\rho_\varepsilon^\theta - \rho_\theta\|_{H^s} + \|(v_\varepsilon^\theta - \tilde{v}_\varepsilon^\theta) - v^\theta\|_{H^s} \right] \rightarrow 0.$$

In fact, Grenier proves [33, Theorem 1.1.3] that the quasi-neutral limit for the multi-fluid systems holds for any sufficiently smooth solutions of (1.50), satisfying for some $s > \frac{d}{2} + 2$ and some $T > 0$,

$$\sup_{\theta \in \Theta, t \in [0, T], \varepsilon \in (0, 1)} \left[\|\rho_\varepsilon^\theta(t)\|_{H^s(\mathbb{T}^d)} + \|v_\varepsilon^\theta(t)\|_{H^s(\mathbb{T}^d)} + \|\varepsilon E_\varepsilon(t)\|_{H^s(\mathbb{T}^d)} \right] < +\infty. \quad (1.55)$$

The analyticity is needed to prove that solutions satisfying (1.55) exist. It is used to get a time interval of existence that is uniform in ε . Grenier proves the existence of such solutions in [33, Theorem 1.1.2], using a strategy similar to the proof of the Cauchy-Kovalevskaya theorem by Caflisch [22].

VPME. A similar strategy can be used to prove the quasi-neutral limit for VPME in analytic regularity. The necessary modifications of the proof are discussed in [44].

The multi-fluid system for VPME is

$$(VPME - MF)_\varepsilon := \begin{cases} \partial_t \rho_\varepsilon^\theta + \operatorname{div}_x(\rho_\varepsilon^\theta v_\varepsilon^\theta) = 0, \\ \partial_t v_\varepsilon^\theta + (v_\varepsilon^\theta \cdot \nabla_x) v_\varepsilon^\theta = -\nabla_x U_\varepsilon, \\ \varepsilon^2 \Delta_x U_\varepsilon = e^{U_\varepsilon} - \int_\Theta \rho_\varepsilon^\theta \mu(d\theta). \end{cases}$$

In the quasineutral limit, we formally obtain

$$(KIsE - MF) := \begin{cases} \partial_t \rho^\theta + \operatorname{div}_x(\rho^\theta v^\theta) = 0, \\ \partial_t v^\theta + (v^\theta \cdot \nabla_x) v^\theta = -\nabla_x U, \\ U = \log \left(\int_\Theta \rho^\theta \mu(d\theta) \right). \end{cases}$$

In analytic regularity, we have the following equivalent of Theorem 1.19. The main difference between the results is that correctors are not necessary in the VPME case, as explained in [44, Section 4.1].

Theorem 1.20. *For each $\varepsilon \in (0, 1)$, let $g_\varepsilon(0)$ be a choice of initial datum for the VPME (1.19). Assume that the $\{g_\varepsilon(0)\}_{\varepsilon \in (0, 1)}$ are uniformly analytic in x , in the sense that, for some $\delta_0 > 1$,*

$$\sup_{\varepsilon \in (0, 1), v \in \mathbb{R}^d} (1 + |v|^{d+1}) \|g_\varepsilon(0, \cdot, v)\|_{B_{\delta_0}} \leq C.$$

Suppose further that

$$\sup_{\varepsilon \in (0,1)} \left\| \int_{\mathbb{R}^d} g_\varepsilon(0, x, v) dv - 1 \right\|_{B_{\delta_0}} \leq \eta,$$

where η is sufficiently small.

Let $\{\rho_\varepsilon^\theta(0, x), v_\varepsilon^\theta(0, x)\}_{\theta \in \mathbb{R}^d, \varepsilon \in (0,1)}$ be defined by

$$\rho_\varepsilon^\theta(0, x) = c_d(1 + |\theta|^{d+1})g_\varepsilon(0, x, \theta), \quad v_\varepsilon^\theta(0, x) = \theta.$$

Assume that, for each $\theta \in \mathbb{R}^m$, $\rho_\varepsilon^\theta(0, x)$ has a limit $\rho^\theta(0, x)$ as ε tends to zero, in the sense of distributions.

Let δ_1 satisfy $\delta_0 > \delta_1 > 1$. Then there exists $T > 0$ such that:

- (i) For each $\varepsilon \in (0, 1)$, there exists a solution $((\rho_\varepsilon^\theta, v_\varepsilon^\theta)_{\theta \in \mathbb{R}^d}, E_\varepsilon)$ of the multi-fluid system (1.50) with initial data given by (1.54), with each ρ_ε^θ and v_ε^θ bounded in $C([0, T]; B_{\delta_1})$.
- (ii) There exists a solution $(\rho^\theta, v^\theta, E)$ to the limiting system (1.51) for $t \in [0, T]$, with initial data $(\rho^\theta(0), v^\theta(0))$ such that, as ε tends to zero, for all $s \in \mathbb{N}$,

$$\sup_{t \in [0, T]} \left[\|\rho_\varepsilon^\theta - \rho^\theta\|_{H^s(\mathbb{T}^d)} + \|v_\varepsilon^\theta - v^\theta\|_{H^s(\mathbb{T}^d)} \right] \rightarrow 0.$$

1.5.4.2 Rough Perturbations

The works of Han-Kwan and Iacobelli [44, 43] deal with small, rough perturbations of these analytic results. The idea is to consider initial data $f_\varepsilon(0)$ for the Vlasov-Poisson system satisfying

$$W_p(f_\varepsilon(0), g_\varepsilon(0)) \leq \phi(\varepsilon).$$

where the functions $g_\varepsilon(0)$ satisfy the conditions of Theorem 1.19 or Theorem 1.20 as appropriate. The function ϕ must converge to zero sufficiently quickly as ε tends to zero. If the rate is too slow, then the Sobolev counterexamples from [42] will not be excluded and the limit is false in general. Thus $f_\varepsilon(0)$ is a small perturbation of the analytic regime, in the sense of a Wasserstein distance.

In [44], the authors prove a quasi-neutral limit of this kind in the one-dimensional case $d = 1$, for both the classical Vlasov-Poisson and VPME cases. This one-dimensional result is valid even for measure data. In [43], they prove a result in higher dimensions $d = 2, 3$, for the limit from the classical Vlasov-Poisson system (1.15) to the KInE system, for a class of L^∞ data. However, previously a result in dimensions $d = 2, 3$ was not available for the VPME system.

In both works [44, 43], the method of proof is to consider the solutions f_ε and g_ε of the relevant Vlasov-Poisson system starting respectively from the initial data $f_\varepsilon(0)$ and $g_\varepsilon(0)$. Stability estimates are used to show that f_ε and g_ε are sufficiently close to each other in a Wasserstein distance, so that the limit that holds for g_ε can also be extended to f_ε . The key ingredient is the stability estimate of Loeper [65], quoted above as Theorem 1.16, or, in one dimension, a weak-strong estimate in the style of Hauray [47]. The obstacle to proving a rough data quasi-neutral limit for VPME in dimensions $d = 2, 3$ was the previous absence of suitable Loeper-style stability estimates in this case.

In this thesis, we are able to fill this gap, proving a quasi-neutral limit for the VPME system in dimensions $d = 2, 3$ for data that are small perturbations of functions satisfying the assumptions of Theorem 1.20. This result is stated below in Section 1.7 as Theorem 1.23, and proved in Chapter 3. The key ingredient is the stability estimates that we develop for the VPME system in Chapter 2, Sections 2.4-2.5.

1.6 Mean Field Limits

The problem of deriving a Vlasov equation from its underlying particle system is known as the **mean field limit**. In a typical formulation of this problem, one considers a system of N point particles evolving under the influence of binary interactions between the particles, described by an interaction force $-\nabla W$ derived from a potential $-W$, and possibly an external force $-\nabla V$ arising from an external potential $-V$. The dynamics are modelled by a system of ODEs describing the phase space position $(X_i, V_i)_{i=1}^N$ of each of the N particles:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum_{j \neq i} \nabla_x W(X_i - X_j) + \nabla V(X_i). \end{cases} \quad (1.56)$$

The choice of scaling $1/N$ is designed to be the appropriate one to obtain a Vlasov equation in the limit. The formal limiting system is

$$\partial_t f + v \cdot \nabla_x f + (\nabla W * \rho_f + \nabla V) \cdot \nabla_v f = 0. \quad (1.57)$$

The aim of this section is to discuss what is known rigorously about the limit from the ODE system (1.56) to the Vlasov equation (1.57).

In order to make sense of the limit, we need a way to compare solutions of the ODE system (1.56), which are trajectories in the space $(\mathcal{X} \times \mathbb{R}^d)^N$, with solutions of the PDE (1.57), which are functions or distributions. The connection is formulated by using **empirical measures**. The empirical measure μ^N associated to a configuration $(X_i, V_i)_{i=1}^N$ is defined by the formula

$$\mu^N := \frac{1}{N} \sum_{i=1}^N \delta_{(X_i, V_i)}. \quad (1.58)$$

The normalisation $1/N$ ensures that μ^N is always a probability measure.

Then, any solution of the ODE system (1.56) can be represented as a path $t \mapsto \mu^N(t)$ in the space of probability measures. μ^N can then be compared with solutions of the PDE (1.57). As a mathematical statement, the mean field limit corresponds to showing that,

$$\text{if } \lim_{N \rightarrow \infty} \mu_0^N = f_0, \text{ then } \lim_{N \rightarrow \infty} \mu_t^N = f_t \text{ for } t > 0.$$

This convergence should hold in the sense of weak convergence of measures. Whether it is possible to prove such a statement depends on the choice of the potentials W and V . In particular, what matters is their regularity.

1.6.1 Lipschitz Forces

Early works on the mathematical justification of the mean field limit include, among others, Braun-Hepp [18], Neunzert-Wick [72] and Dobrushin [27]. These results deal with the case of Lipschitz force fields, that is, under the assumption that

$$\nabla W, \nabla V \in W^{1,\infty}, \quad \nabla W(0) = 0.$$

We highlight in particular the method of proof used by Dobrushin [27], which provides a framework still used in more recent works on the subject. The approach has two key steps:

- (i) First, observe that, if $(X_i, V_i)_{i=1}^N$ is a solution of (1.56), then the resulting empirical measure μ^N is a weak solution of the Vlasov equation (1.57), in the sense of Definition 1. Indeed, we have already discussed that Definition 1 makes sense for measure solutions when the force is Lipschitz. For empirical measures, the weak form requires that, for all $\phi \in C_c^\infty([0, T] \times \mathcal{X} \times \mathbb{R}^d)$,

$$\begin{aligned} & \int_0^T \frac{1}{N} \sum_{i=1}^N \left\{ \partial_t \phi(t, Z_i) + V_i \cdot \nabla_x \phi(t, Z_i) + \sum_{j=1}^N \nabla W(X_i - X_j) \cdot \nabla_v \phi(t, Z_i) \right\} dt \\ & + \int_0^T \frac{1}{N} \sum_{i=1}^N \nabla V(X_i) \cdot \nabla_v \phi(t, Z_i) dt + \frac{1}{N} \sum_{i=1}^N \phi(0, Z_i(0)) = 0. \end{aligned} \quad (1.59)$$

The ODE (1.56) implies that the left hand side of (1.59) is equal to

$$\frac{1}{N} \sum_{i=1}^N \phi(T, Z_i(T)),$$

which is equal to zero since ϕ is compactly supported in time. Thus μ^N is a weak solution of (1.57).

- (ii) Then, the mean field limit is proved by showing a stability result for solutions of the Vlasov equation (1.57). In the case of Lipschitz forces, the necessary result is provided by Lemma 1.8. Notice in particular that this is a quantitative stability estimate. This fact will be useful later when we discuss regularised limits.

In plasma models, we consider interactions between charged particles. Electrostatic interactions are described by the Coulomb force K , which we defined in equation (1.43). This situation is not covered by these results for Lipschitz forces, because K has a strong singularity at the origin. Next, we discuss what is known in the theory of mean field limits for interactions with a singularity.

1.6.2 Singular Forces

In many physical systems, the interaction force follows an inverse power law $|\nabla W| \sim |x|^{-\alpha}$. For example, in dimension $d = 3$ the Coulomb force corresponds to the case $\alpha = d - 1$. Because of the singularity at the origin, these forces are not regular enough to apply Dobrushin's results. From now on, we will discuss forces satisfying a growth condition of the following form: for all $x \in \mathbb{R}^d \setminus \{0\}$,

$$\frac{|\nabla W(x)|}{|x|^\alpha} \leq C, \frac{|\nabla^2 W(x)|}{|x|^{\alpha+1}} \leq C \quad \alpha \in (0, d-1]. \quad (1.60)$$

1.6.2.1 Typicality

For singular forces, it is not necessarily expected that a mean field limit result should hold for any choice of initial data for the ODE (1.56). Instead, we seek to prove results that hold for a 'large' class of initial data.

In order to discuss the notion of the size of a set of configurations, it is necessary to specify a measure on the space of configurations. This can be done by considering choosing initial configurations randomly. Given a fixed probability measure $f_0 \in \mathcal{P}(\mathcal{X} \times \mathbb{R}^d)$, the initial data for (1.56) are chosen by drawing independent samples from f_0 . The (random) empirical measures μ^N constructed by this procedure will converge weakly to f_0 as N tends to infinity, for example in probability. This is a consequence of a generalised Glivenko–Cantelli theorem – see for example Van der Vaart and Wellner [82] for an in depth discussion of results of this type. We look for mean field limit results that hold with a high probability with this method of choosing initial data.

1.6.2.2 Regularised Limits

One approach to the mean field limit problem for singular forces is to consider a regularisation of the limit. The idea is to replace the singular potential W with a regularised function W_r . ∇W_r should be smooth enough that a mean field limit holds with this interaction for fixed $r > 0$. The regularisation parameter r quantifies the degree of approximation; as r tends to zero, W_r converges to W . The regularised mean limit involves taking simultaneously the limits $r \rightarrow 0$ and $N \rightarrow \infty$, ultimately deriving the Vlasov equation (1.57) with singular interaction W from a sequence of regularised particle systems (1.56) with regularised interaction ∇W_r .

The aim in this scheme is to optimise the rate at which r may converge to zero as N tends to infinity. The faster r converges to zero, the closer the regularised particle systems are to the original interaction. An important benchmark is to be able to take $r \lesssim N^{-\frac{1}{d}}$, which is the typical spatial separation between particles if they are distributed uniformly in the spatial domain.

There have been several works aimed at deriving Vlasov equations with singular forces from regularised particle systems. For instance, Hauray and Jabin [49] considered a truncation method in which the force is cut off below a certain distance from the origin r_N , dependent on the number of particles N . They showed that the mean field limit holds for a large set of initial configurations, for forces satisfying (1.60) with $\alpha < d - 1$ (in particular not the Vlasov-Poisson case), from a particle system with force truncated at r_N , provided that r_N converges to zero sufficiently slowly as N tends to infinity. In [48, 49], they are also able to prove a mean field limit without truncation for the case of ‘weakly singular’ forces in which $\alpha < 1$.

1.6.2.3 Vlasov-Poisson

The Vlasov-Poisson system (1.6) gives a kinetic description of a system of electrons interacting through electrostatic interaction:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum_{j \neq i} K(X_i - X_j). \end{cases}$$

This system is of the form (1.56) with the choice $W = -G$, with G defined in (1.42), and $V = 0$.

A mean field limit for the Vlasov-Poisson system was proved in the one dimensional case $d = 1$ by Hauray [47]. In higher dimensions, the true mean field limit for Vlasov-Poisson remains a major open problem, due to the strong singularity of K .

Instead, one may consider regularised limits. Observe that K is of the form (1.60), with $\alpha = d - 1$. This endpoint case was not covered in [49]. We highlight two recent results on regularised mean field limits for this system, due to Lazarovici [61] and Lazarovici and Pickl [62]. The two results are of slightly kinds and use different regularisation methods.

Lazarovici and Pickl [62] proved a regularised mean field limit for the Vlasov-Poisson system, using a regularisation by truncation. The truncation radius can be chosen to satisfy $r_N \sim N^{-\frac{1}{d} + \eta}$ for any $\eta > 0$. Their results show that there exists a large set of initial configurations for which the mean field limit holds. However, these configurations are identified in a non-explicit way which does not rely on the initial condition alone. This is because their argument relies on a law of large numbers argument throughout the evolution to compare the mean field force from the particle system to the limiting force.

Lazarovici [61] considered a method of regularisation by convolution. In this approach, the point particles are replaced by delocalised packets of charge, with some smooth, compactly supported shape χ , fixed throughout the evolution. For the classical Vlasov-Poisson case, this

results in the particle system

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum_{j \neq i} [\chi *_{\mathbf{x}} K *_{\mathbf{x}} \chi] (X_i - X_j). \end{cases} \quad (1.61)$$

This is known as a system of ‘extended charges’. The shape is then allowed to depend on a regularisation parameter r by taking

$$\chi_r(x) := r^{-d} \chi\left(\frac{x}{r}\right). \quad (1.62)$$

Lazarovici showed that a mean field limit holds with high probability, provided that $r_N \geq CN^{-\frac{1}{d(d+2)} + \eta}$ for some $\eta > 0$. The admissible configurations are identified by a condition on the initial configuration alone.

The double regularisation $\chi_r *_{\mathbf{x}} \nabla_{\mathbf{x}} G *_{\mathbf{x}} \chi_r$ is used because it offers several useful technical features. This type of regularisation was previously considered by Horst [53] in the Vlasov-Maxwell case and later used by Rein [75]; a version also appears in Bouchut [17]. One advantage is that the microscopic dynamics correspond to a Hamiltonian system, for which the corresponding energy converges as r tends to zero to the energy of the true Vlasov-Poisson system.

1.6.3 Mean Field Limits for VPME

In Chapter 4, we consider the problem of deriving the VPME system (1.11) in the mean field limit from an underlying particle system. A natural choice for a particle system related to VPME is to consider the dynamics of N ions, modelled as point charges, in a background of thermalised electrons. On the torus, this is modelled by an ODE system of the form

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum_{j \neq i}^N K_{\text{per}}(X_i - X_j) - K * e^U, \end{cases} \quad (1.63)$$

where U satisfies

$$\Delta U = e^U - \frac{1}{N} \sum_{i=1}^N \delta_{X_i}.$$

We can think of this system as being of the form (1.56) by taking $-W = G_{\text{per}}$ and an ‘external’ potential $V = G_{\text{per}} * e^U$. Of course V is not truly an external potential because U depends nonlinearly on f .

The mean field limit for VPME was proved in the one dimensional case $d = 1$ in [44]. In higher dimensions, once again there are no results available, due to the singularity of the Coulomb interaction. Instead, we consider a regularised system. We use a regularisation based on the approach of Lazarovici [61]. We consider a system of ‘extended ions’ represented by the following system of ODEs:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = -\chi_r * \nabla_x U_r(X_i), \end{cases} \quad (1.64)$$

where U_r satisfies

$$\Delta U_r = e^{U_r} - \frac{1}{N} \sum_{i=1}^N \chi_r(X_i), \quad x \in \mathbb{T}^d. \quad (1.65)$$

We are able to derive the VPME system (1.11) from this regularised system, under a condition on the initial data that is satisfied with high probability for $r_N \geq CN^{-\frac{1}{d(d+2)} + \eta}$. This matches the rate found in Lazarovici’s result for the Vlasov-Poisson system. This result is stated below as Theorem 1.24. The proof is given in Chapter 4.

1.6.4 Derivation of Kinetic Euler Systems

In Chapter 5, we consider the derivation of the kinetic incompressible Euler and kinetic isothermal Euler systems introduced in Section 1.2.3.3. Our approach is based on a combined quasi-neutral and mean field limit. We are able to derive the KInE system from a version of (1.61) with a modified scaling. Similarly, we derive the KIsE system from a version of (1.64) with a modified scaling. These results are stated below as Theorems 1.25 and 1.26.

For the KInE system, we obtain a scaling of the form

$$\alpha(N) = C \frac{(\log N)^\kappa}{N},$$

for some κ specified in the statement of Theorem 1.25. For the KIsE system, we obtain a scaling of the form

$$\alpha(N) = C \frac{\log \log \log N}{N}.$$

1.7 Summary of Results

In this section, we summarise the main results of this thesis.

1.7.1 Well-Posedness for the VPME System

In Chapter 2, we will study the well-posedness theory of the Vlasov-Poisson system with massless electrons. We prove the existence and uniqueness of bounded density solutions for compactly supported initial data.

The key technique is a decomposition of the electrostatic potential U . We write U in the form $U = \bar{U} + \hat{U}$, where \bar{U} and \hat{U} satisfy the following equations:

$$-\Delta \bar{U} = \rho_f - 1, \quad \Delta \hat{U} = e^{\bar{U} + \hat{U}} - 1.$$

Thus \bar{U} is the electrostatic potential used in the classical Vlasov-Poisson system, and can therefore be handled using known methods. The key step is to study \hat{U} , in particular its regularity and stability with respect to ρ_f . This decomposition was previously considered in the one dimensional case $d = 1$ in [44].

We are able to show that \hat{U} is in general smoother than \bar{U} , including in the higher dimensional cases $d = 2, 3$. This fact allows known methods for the classical Vlasov-Poisson system to be adapted to the VPME case. In this thesis, we demonstrate this principle in two cases. In one case, we adapt the proof of uniqueness for bounded density solutions, due to Loeper [65], to the VPME case. We then prove the existence of bounded density solutions by adapting the methods of Batt and Rein [10].

1.7.1.1 Uniqueness

We prove a uniqueness result for solutions of the VPME system (1.11) with bounded density. This is an analogue of Loeper's result [65, Theorem 1.2] for the Vlasov-Poisson system (1.5).

Theorem 1.21 (Uniqueness of bounded density solutions for VPME). *For each $i = 1, 2$, let f_i be a solution of (1.11) in the space $C([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$, with bounded density*

$$\sup_{t \in [0, T]} \|\rho[f_i]\|_{L^\infty(\mathbb{T}^d)} \leq C_i,$$

for the same initial datum $f(0) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. Then $f_1 = f_2$.

The proof of this result is given in Chapter 2, in Section 2.6. As for Loeper's result, the proof of Theorem 1.21 is based on a stability estimate for solutions of the VPME system in

W_2 . The key step is to prove a stability estimate for \widehat{U} with respect to ρ_f , analogous to the one proved for \bar{U} by Loeper (Theorem 1.17).

1.7.1.2 Existence of Solutions

We prove the existence of solutions for the VPME system (1.11) with bounded density, for compactly supported initial data.

Theorem 1.22 (Existence of solutions with bounded density). *Let $d = 2, 3$. Consider an initial datum $f_0 \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ with compact support in $\mathbb{T}^d \times \mathbb{R}^d$. Then there exists a global in time solution $f_t \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ of (1.11) with initial data f_0 , which has conserved energy and a mass density bounded locally in time:*

$$\mathcal{E}^{ME}[f_t] = \mathcal{E}^{ME}[f_0], \quad \rho_{f_t} \in L_{loc}^\infty([0, \infty); L^\infty(\mathbb{T}^d)).$$

The proof of this result is given in Chapter 2, Sections 2.7–2.9. The main idea is to control the size of the support of the solution. This is the method used by Pfaffelmoser [74], Schaeffer [79] and Batt–Rein [10] for the classical Vlasov-Poisson system. Since we work with the VPME system posed on the torus, we follow the methods of Batt and Rein [10]. The key step is to prove suitable bounds on $\widehat{E} = -\nabla \widehat{U}$, the smooth part of the electric field.

1.7.2 Quasi-Neutral Limit for VPME With Rough Data

In Chapter 3, we prove the quasi-neutral limit from the VPME system (1.11) to the KIsE system (1.20), for a class of rough data. The starting point is analogous to the approach of Han-Kwan and Iacobelli [43] for the classical Vlasov-Poisson system, which we described in Section 1.5.4.2. The idea is to work with initial data for the VPME system that are small perturbations of analytic functions satisfying the assumptions of Theorem 1.20.

Iterated Exponentials. In the following statement, we will use the notation $\overline{\exp}_n$ to denote the n -fold iteration of the exponential function. For example

$$\overline{\exp}_3(x) := \exp \exp \exp(x).$$

Theorem 1.23 (Quasi-neutral limit). *Let $d = 2, 3$. Consider initial data $f_\varepsilon(0)$ satisfying the following conditions:*

- (Uniform bounds) $f_\varepsilon(0)$ is bounded and has bounded energy, uniformly with respect to ε : for some constant $C_0 > 0$,

$$\|f_\varepsilon(0)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq C_0 \quad (1.66)$$

and

$$\frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv + \frac{\varepsilon^2}{2} \int_{\mathbb{T}^d} |\nabla U|^2 \, dx + \int_{\mathbb{T}^d} U e^U \, dx \leq C_0.$$

- (Control of support) There exists $C_1 > 0$ such that

$$f_\varepsilon(0, x, v) = 0 \quad \text{for } |v| > \exp(C_1 \varepsilon^{-2}). \quad (1.67)$$

- (Perturbation of an analytic function) There exist $g_\varepsilon(0)$ satisfying, for some $\delta > 1$, $\eta > 0$, and $C > 0$,

$$\begin{aligned} \sup_{\varepsilon > 0} \sup_{v \in \mathbb{R}^d} (1 + |v|^{d+1}) \|g_\varepsilon(0, \cdot, v)\|_{B_\delta} &\leq C, \\ \sup_{\varepsilon > 0} \left\| \int_{\mathbb{R}^d} g_\varepsilon(0, \cdot, v) \, dv - 1 \right\|_{B_\delta} &\leq \eta, \end{aligned}$$

as well as the support condition (1.67), such that, for all $\varepsilon > 0$,

$$W_2(f_\varepsilon(0), g_\varepsilon(0)) \leq [\overline{\exp}_4(C\varepsilon^{-2})]^{-1}$$

for C sufficiently large with respect to C_0, C_1 .

- (Convergence of data) $g_\varepsilon(0)$ has a limit $g(0)$ in the sense of distributions as $\varepsilon \rightarrow 0$.

Let f_ε denote the unique solution of (3.1) with bounded density and initial datum $f_\varepsilon(0)$. Then there exists a time horizon $T_* > 0$, independent of ε but depending on the collection $\{g_{0,\varepsilon}\}_\varepsilon$, and a solution g of (3.2) on the time interval $[0, T_*]$ with initial datum $g(0)$, such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T_*]} W_1(f_\varepsilon(t), g(t)) = 0.$$

The proof of this result is given in Chapter 3. The main idea is to develop a version of the W_2 stability estimate for the VPME system that is quantified with respect to the Debye length ε . This estimate is proved in Section 3.3. This estimate is used to control the distance between the solution f_ε with rough data and the analytic solution with data $g_\varepsilon(0)$, which is already known to converge in the limit $\varepsilon \rightarrow 0$ due to Theorem 1.20.

1.7.3 Derivation of VPME from a System of Extended Ions

In Chapter 4, we consider the derivation of the VPME system from an underlying microscopic system. We use the ‘extended ions’ system (1.64)–(1.25) introduced in Section 1.6.3. Our main result is the following theorem.

Theorem 1.24 (Regularised mean field limit). *Let $d = 2, 3$, and let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be a compactly supported choice of initial datum for (4.2). Let f denote the unique bounded density solution of the VPME system (1.11) with initial datum $f(0)$. Fix $T_* > 0$.*

Assume that $r = r(N)$ and the initial configurations for (1.64)–(1.65) are chosen such that the corresponding empirical measures satisfy, for some sufficiently large constant $C > 0$, depending on T_ and the support of $f(0)$,*

$$\limsup_{N \rightarrow \infty} \frac{W_2^2(f(0), \mu_r^N(0))}{r^{d+2+C} |\log r|^{-1/2}} < 1.$$

Then the empirical measure μ_r^N associated to the particle system dynamics starting from this configuration converges to f :

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T_*]} W_2(f(t), \mu_r^N(t)) = 0. \quad (1.68)$$

In particular, suppose that f_0 has a finite k^{th} moment for some $k > 4$:

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (|x|^k + |v|^k) f_0(\mathrm{d}x \mathrm{d}v) < +\infty.$$

Choose $r(N) = N^{-\gamma}$ for some γ satisfying

$$\gamma < \frac{1}{d+2} \min \left\{ \frac{1}{d}, 1 - \frac{4}{k} \right\}.$$

For each N , let the initial configurations for the regularised N -particle system (4.4) be chosen by taking N independent samples from f_0 . Then (1.68) holds with probability one.

1.7.4 Derivation of Kinetic Euler Systems from Systems of Extended Charges

In Chapter 5, we derive the KInE (1.16) and KIsE (1.20) systems from underlying microscopic systems of extended charges.

1.7.4.1 Kinetic Isothermal Euler

For the KIsE system, we consider the following system of extended ions, which is a version of (1.64)-(1.65) with quasi-neutral scaling:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = -\chi_r * \nabla_x U(X_i), \end{cases} \quad (1.69)$$

where U satisfies

$$\varepsilon^2 \Delta U = e^U - \frac{1}{N} \sum_{i=1}^N \chi_r(x - X_i).$$

Recall that χ_r is the scaled mollifier defined in (1.62). For this system, we prove the following convergence result. We state the result here for the case where the initial data for (1.69) are chosen randomly, by taking independent samples from a given law $f_\varepsilon(0)$. See Theorem 5.3 in Chapter 5, Subsection 5.2.2 for a statement dealing with other initial configurations.

Theorem 1.25 (From extended ions to kinetic isothermal Euler). *Let $d = 2$ or 3 , and let $f_\varepsilon(0), g_\varepsilon(0)$ and $g(0)$ satisfy the assumptions of Theorem 1.23. Let $T_* > 0$ be the maximal time of convergence from Theorem 1.23 and let g denote the solution of the KIsE system (5.5) with initial data $g(0)$ on the time interval $[0, T_*]$ appearing in the conclusion of Theorem 1.23.*

Let $r = r(N)$ be of the form

$$r(N) = cN^{-\frac{1}{d(d+2)} + \eta}, \quad \text{for some } \eta > 0, c > 0.$$

There exists a constant C , depending on d, η, c and $\{f_\varepsilon(0)\}_\varepsilon$, such that the following holds.

Let $\varepsilon = \varepsilon(N)$ satisfy

$$\varepsilon(N) \geq \frac{C}{\sqrt{\log \log \log N}}, \quad \lim_{N \rightarrow \infty} \varepsilon(N) = 0.$$

For each N , let the initial conditions for the regularised and scaled N -particle ODE system (1.69) be chosen randomly with law $f_{\varepsilon(N)}(0)^{\otimes N}$. Let $\mu_{\varepsilon, r}^N(t)$ denote the empirical measure associated to the solution of (1.69).

Then, with probability one,

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T_*]} W_1(\mu_{\varepsilon, r}^N(t), g(t)) = 0.$$

1.7.4.2 Kinetic Incompressible Euler

We also prove a similar result for the KInE system. For the microscopic model, we consider the following system of extended electrons:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{\varepsilon^{-2}}{N} \sum_{j \neq i} \chi_r * K_{\text{per}} * \chi_r(X_i - X_j). \end{cases}$$

In this case, we choose configurations for the particle system that approximate initial data $f_\varepsilon(0)$ for the Vlasov-Poisson system (1.5) for which the quasi-neutral limit holds. Specifically, we assume that $f_\varepsilon(0)$ satisfy the assumptions of the rough data result of Han-Kwan–Iacobelli [43]. These assumptions are stated below.

Assumption 1 (KInE, $d = 2, 3$). The data $f_\varepsilon(0)$ satisfy the following:

- (i) There exists C_0 independent of ε such that

$$\|f_\varepsilon(0)\|_{L^\infty} \leq C_0, \quad \mathcal{E}_\varepsilon^{\text{VP}}(f_\varepsilon(0)) \leq C_0.$$

- (ii) (Control of support) For some $\gamma > 0$,

$$f_\varepsilon(0, x, v) = 0 \quad \text{for } |v| > \varepsilon^{-\gamma}. \quad (1.70)$$

- (iii) (Analytic + perturbation) There exist functions $g_\varepsilon(0)$ satisfying (1.70) and

$$\sup_{\varepsilon \in (0,1)} \sup_{v \in \mathbb{R}^d} (1 + |v|^{d+1}) \|g_\varepsilon(0, \cdot, v)\|_{B_{\delta_0}} \leq C,$$

such that

$$W_2(f_\varepsilon(0), g_\varepsilon(0)) \leq \left[\overline{\exp}_2 \left(C \varepsilon^{-(2+d\zeta)} \right) \right]^{-1},$$

for $C > 0$ sufficiently large, where $\zeta = \zeta(\gamma)$ is defined as follows:

- For $d = 2$, we fix any $\delta > 2$ and let

$$\zeta = \max\{\gamma, \delta\}.$$

- For $d = 3$ we let

$$\zeta = \max\left\{\gamma, \frac{38}{3}\right\}.$$

We then have the following result.

Theorem 1.26. *Let $d = 2$ or 3 . For each $\varepsilon > 0$, let $f_\varepsilon(0)$ satisfy Assumption 1. Suppose that $g_\varepsilon(0)$ has a limit $g(0)$ in the sense of distributions as ε tends to zero. There exists $T_* > 0$ and a solution $g(t)$ of (1.16) on the time interval $[0, T_*]$ such that the following holds.*

Fix $T \leq T_$ and $\alpha > 0$. There exist constants A_T, C_T depending on T, α and $\{f_\varepsilon(0)\}_\varepsilon$ such that the following holds. Let $r = r(N)$, $\varepsilon = \varepsilon(N)$ satisfy*

$$r = A_T N^{-\frac{1}{d(d+2)} + \alpha}, \quad \varepsilon \geq C_T (\log N)^{-\frac{1}{2+d\zeta}}$$

Then, if the initial N -particle configurations $[Z_i^\varepsilon(0)]_{i=1}^N$ are chosen by taking N independent samples from $f_\varepsilon(0)$, with probability one the following limit holds:

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} W_1(\tilde{\mu}_{\varepsilon, r}^N(t), g(t)) = 0,$$

where

- $\mu_{\varepsilon, r}^N(t)$ denotes the empirical measure corresponding to the solution of (5.4) with initial datum $[Z_i^\varepsilon(0)]_{i=1}^N$;
- $\tilde{\mu}_{\varepsilon, r}^N(t)$ is the measure constructed by filtering $\mu_{\varepsilon, r}^N(t)$ using the corrector \mathcal{R}_ε defined in (1.53), according to Definition 5 in Chapter 5, Subsection 5.2.1.1, using the given choice of $f_\varepsilon(0)$ and $g(0)$.

Chapter 2

The Vlasov-Poisson System with Massless Electrons

Global Well-posedness in Two and Three Dimensions

Contents

2.1	Statement of Results	77
2.1.1	Uniqueness of Solutions With Bounded Density	77
2.1.2	Global Existence of Solutions	77
2.2	Strategy	78
2.3	Basic Estimates	79
2.3.1	$L^p(\mathbb{T}^d \times \mathbb{R}^d)$ Estimates	79
2.3.2	Energy	79
2.3.3	Estimates on the Mass Density	80
2.4	Regularity of the Electric Field	81
2.4.1	Regularity of \bar{U}	82
2.4.2	Existence of \hat{U}	83
2.4.3	Regularity of \hat{U}	87
2.5	Stability of the Electric Field	90
2.6	Wasserstein Stability and Uniqueness	93
2.7	A Priori Estimates on the Mass Density	101
2.7.1	Two-dimensional Case	101

2.7.2	Three-dimensional Case	105
2.8	A Regularised System	110
2.9	Construction of Solutions	114
2.9.1	Energy conservation	131

In this chapter, we prove global-in-time well-posedness results for the Vlasov-Poisson system with massless electrons (VPME) in dimension $d = 2, 3$. Recall that VPME is a kinetic model for the ions in a plasma. We introduced this model from a physical perspective in Section 1.2.2. On the torus, the system reads as follows:

$$(VPME) := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ \Delta_x U = e^U - \int_{\mathbb{R}^d} f dv = e^U - \rho_f, \\ f|_{t=0} = f(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f(0, x, v) dx dv = 1. \end{cases} \quad (2.1)$$

Global well-posedness of the VPME system was shown in the one dimensional case in Han-Kwan–Iacobelli [43]. In three dimensions, Bouchut [17] proved the global-in-time existence of weak solutions. However, global existence of strong solutions was not proved in dimensions higher than one.

In this chapter, we prove global-in-time existence of strong solutions of the VPME system in dimensions $d = 2$ and $d = 3$, for compactly supported data. We also prove the uniqueness of solutions with bounded density.

These results provide analogues of two important results for the classical Vlasov-Poisson system (1.5). Our uniqueness result is comparable to the result of Loeper [65] for the classical system. We also prove a global existence result in the style of Batt and Rein [10] for solutions on the torus for $d = 2, 3$.

The key step in our analysis is a decomposition of the electric field. We are able to show that the electric field in the VPME system behaves like a perturbation of the electric field in the classical Vlasov-Poisson system (1.5). This perturbation is general more regular than the remainder of the electric field. This key observation allows known methods for the classical system to be adapted to the VPME case. We demonstrate this in full for the results mentioned above.

2.1 Statement of Results

The aim of this chapter is to prove the following results. Together they provide, for $d = 2, 3$, global existence of solutions of the VPME system (2.1), for a general class of data, as well as uniqueness for solutions with bounded density.

2.1.1 Uniqueness of Solutions With Bounded Density

Theorem 2.1 (Uniqueness for solutions with bounded density). *Let $d = 2, 3$. Let $f_0 \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$ with $\rho[f_0] \in L^\infty(\mathbb{T}^d)$. Fix a final time $T > 0$. Then there exists at most one solution $f \in C([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ of the system (2.1), with initial datum f_0 , such that $\rho[f] \in L^\infty([0, T]; L^\infty(\mathbb{T}^d))$.*

Moreover, we have a quantitative stability estimate. Let f, g be solutions of the VPME system (2.1) such that

$$\sup_{t \in [0, T]} \|\rho[f(t)]\|_{L^\infty(\mathbb{T}^d)}, \quad \sup_{t \in [0, T]} \|\rho[g(t)]\|_{L^\infty(\mathbb{T}^d)} \leq M$$

for some constant $M > 0$. Then there exists $C > 0$ depending on M such that for all $t \in [0, T]$,

$$W_2(f(t), g(t)) \leq \begin{cases} \exp \left[C \left(1 + \log \frac{W_2(f(0), g(0))}{4\sqrt{d}} \right) e^{-Ct} \right] & \text{if } W_2(f(0), g(0)) \leq d \\ W_2(f(0), g(0)) e^{Ct} & \text{if } W_2(f(0), g(0)) > d. \end{cases}$$

The proof of this theorem is given in Section 2.6.

2.1.2 Global Existence of Solutions

We prove a result of global-in-time existence, analogous to that of Batt and Rein [10] for the Vlasov-Poisson system (1.5). The following theorem is proved in Sections 2.7 and 2.9.

Theorem 2.2 (Existence of solutions with bounded density). *Let $d = 2, 3$. Consider an initial datum $f_0 \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ with compact support in $\mathbb{T}^d \times \mathbb{R}^d$. Then there exists a global in time solution $f \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ of (1.11) with initial data f_0 . This solution has bounded density, locally in time:*

$$\rho_f \in L_{loc}^\infty([0, \infty); L^\infty(\mathbb{T}^d)).$$

2.2 Strategy

The core of this chapter is an analysis of the electric field E in the VPME system, which is carried out in Sections 2.4 and 2.5. We prove integrability and regularity estimates on E in terms of integrability estimates on ρ_f , as well as estimates on the stability of E with respect to ρ_f . We have discussed in Sections 1.3 and 1.4 that this is the key information needed when studying well-posedness for a nonlinear equation of Vlasov type.

The fundamental step in our analysis is a decomposition of E . For example, in the torus case we write E as $\bar{E} + \hat{E}$ where

$$\bar{E} = -\nabla \bar{U}, \quad \hat{E} = -\nabla \hat{U},$$

and \bar{U} and \hat{U} solve respectively

$$\Delta \bar{U} = 1 - \rho_f, \quad \Delta \hat{U} = e^{\bar{U} + \hat{U}} - 1.$$

This decomposition was previously used in [44] for the one dimensional case.

The advantage of this decomposition is that it allows us to think of E as a perturbation of the electric field \bar{E} that we would have in the classical Vlasov-Poisson system (1.5). Moreover, the perturbation \hat{E} depends on ρ_f only through \bar{U} , which will be smoother than ρ_f due to the regularising properties of the Poisson equation. We might therefore expect \hat{U} to have higher order regularity than \bar{U} , and this is indeed what we find (see Proposition 2.4). For this reason we sometimes refer to \bar{U} as the ‘singular’ part of the potential, and \hat{U} as the ‘regular’ or ‘smooth’ part.

The challenging part of the analysis of \hat{U} is that it satisfies an equation with an exponential nonlinearity. It turns out that, even though the nonlinearity is exponential, it has a helpful sign which allows it to be controlled. We show this using techniques from Calculus of Variations. This strategy was also used in the one-dimensional case in [44]. Here we are able to extend these results to higher dimensions.

These regularity estimates allow us to adapt methods developed for the classical Vlasov-Poisson system to the VPME case. In Section 2.6, we prove Theorem 2.1 on the uniqueness of bounded density solutions. The strategy is to prove a W_2 stability estimate on solutions, with respect to their initial data. This proof follows the strategy of Loeper [65].

In Section 2.9, we construct global-in-time solutions of the VPME system on the torus (2.1). Our approach follows the method of Batt and Rein [10] for the classical Vlasov-Poisson system, which is in itself an adaptation of the methods of Pfaffelmoser [74] and Schaeffer [79] to the torus. The key step is to control the growth of the size of the support of f_t in $\mathbb{T}^d \times \mathbb{R}^d$.

In Section 2.8, we prove the existence of unique solutions to a regularised version of the VPME system. This will allow us to complete the existence proof by an approximation argument. This result is not quite covered in the existing literature to our knowledge, because the electric field E in the VPME system cannot be written in the form

$$E = F * \rho$$

for some F . Instead, the coupling between E and ρ is nonlinear. However, the estimates in Sections 2.4 and 2.5 allow us to adapt well-known methods for Vlasov equations with smooth interactions, which we presented in Section 1.3, to our approximating system.

2.3 Basic Estimates

In this section, we summarise some basic estimates for solutions of the VPME system. In a similar spirit to Section 1.4.1, we look at the a priori estimates we can expect to have on solutions, based on the conservation laws of the system.

2.3.1 $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ Estimates

Solutions f of the VPME system solve a transport equation with a divergence free vector field. If $f_0 \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, we therefore have uniform in time estimates on the $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ norms of f .

Lemma 2.3. *Assume that $f_0 \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. Let f be a solution of (2.1). Then, for all $p \in [1, \infty]$,*

$$\|f_t\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq \|f_0\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)}^{\frac{1}{p}} \|f_0\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}^{1 - \frac{1}{p}}.$$

2.3.2 Energy

The system (2.1) has an associated energy functional:

$$\mathcal{E}^{\text{ME}}[f] := \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla U|^2 \, dx + \int_{\mathbb{T}^d} U e^U \, dx. \quad (2.2)$$

This functional is conserved by the evolution for smooth solutions.

The conservation of the energy functional provides a uniform estimate on the second moment of f in velocity, or, in other words, the kinetic energy of f . In Section 1.4.1, we discussed a similar property for the classical Vlasov-Poisson system. For the classical system

(1.5) this implication is immediate because both terms in the classical energy (1.46) are non-negative.

For the VPME energy functional (2.2), we need to deal with the fact that some of the terms may be negative. In particular, we need to consider possible values of the function $x \mapsto xe^x$ for $x \in \mathbb{R}$. Since

$$\frac{d}{dx}(xe^x) = (x+1)e^x,$$

it follows that, for all $x \in \mathbb{R}$,

$$xe^x \geq (xe^x)|_{x=-1} = -e^{-1}.$$

Consequently, bounds on the total energy \mathcal{E}^{ME} imply bounds on the kinetic energy:

$$\frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv \leq \mathcal{E}^{\text{ME}}[f] + e^{-1}. \quad (2.3)$$

2.3.3 Estimates on the Mass Density

As discussed in Section 1.4.1, it is possible to deduce $L^p(\mathbb{T}^d)$ estimates on the mass density ρ_f from estimates on the moments of the full phase-space density f . Using the bound (2.3) on the kinetic energy, we can therefore obtain the same $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ estimate on ρ_f as in the classical case, using the interpolation estimate from Lemma 1.12. Since the energy functionals are formally conserved by their respective equations, this estimate is uniform in time. That is, we can expect a uniform bound of the form

$$\|\rho_f(t, \cdot)\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C(d, f_0). \quad (2.4)$$

The constant $C(f_0)$ depends on the initial datum f_0 through the relation

$$C(d, f_0) = C_d \left(\mathcal{E}^{\text{ME}}[f_0] + e^{-1} \right)^{\frac{d}{d+2}} \|f_0\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}^{\frac{2}{d+2}},$$

where C_d is a dimension dependent constant.

In the next section, we will derive regularity estimates on the electric field in terms of integrability estimates on ρ_f . The crucial point is that we will be able to derive estimates on the smooth part of the potential \widehat{U} that depend on the $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ norm of ρ_f , and are therefore uniform in time.

2.4 Regularity of the Electric Field

In this section we prove several key regularity estimates on the electric field in the VPME system. These estimates will ultimately show that the electric field in the VPME case is a smooth perturbation of the field in the classical case. This fact will allow us to adapt the existing techniques for proving well-posedness of the Vlasov-Poisson system to the massless electron model.

The strategy is based on the following decomposition of the electric field. We let $E = \bar{E} + \hat{E}$, where

$$\bar{E} = -\nabla \bar{U}, \quad \hat{E} = -\nabla \hat{U},$$

and \bar{U} and \hat{U} solve respectively

$$\Delta \bar{U} = 1 - \rho_f, \quad \Delta \hat{U} = e^{\bar{U} + \hat{U}} - 1.$$

We impose without loss of generality that \bar{U} has zero mean over the torus:

$$\int_{\mathbb{T}^d} \bar{U} \, dx = 0.$$

Notice that in this way $U := \bar{U} + \hat{U}$ solves

$$\Delta U = e^U - \rho_f.$$

We consider charge densities $\rho_f \in L^\infty(\mathbb{T}^d)$, since the solutions of VPME we will work with will have densities in this class. We want to derive integrability and regularity estimates on E , assuming that ρ_f has this degree of integrability. In particular, the aim is to give estimates in terms of the $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ and $L^\infty(\mathbb{T}^d)$ norms of ρ_f . The $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ norm is of interest because the conservation of energy for the VPME system provides a uniform-in-time bound on this quantity.

Our aim is to prove the following proposition.

Proposition 2.4 (Regularity estimates on \bar{U} and \hat{U}). *Let $d = 2, 3$. Let $h \in L^\infty(\mathbb{T}^d)$. Then there exist unique $\bar{U} \in W^{1,2}(\mathbb{T}^d)$ with zero mean and $\hat{U} \in W^{1,2}(\mathbb{T}^d)$ satisfying*

$$\Delta \bar{U} = 1 - h, \quad \Delta \hat{U} = e^{\bar{U} + \hat{U}} - 1.$$

Moreover we have the following estimates: for some constant $C_{\alpha,d} > 0$,

$$\begin{aligned} \|\bar{U}\|_{C^{0,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}\right), & \alpha \in \begin{cases} (0,1) & \text{if } d=2 \\ (0, \frac{1}{5}] & \text{if } d=3. \end{cases} \\ \|\bar{U}\|_{C^{1,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \left(1 + \|h\|_{L^\infty(\mathbb{T}^d)}\right), & \alpha \in (0,1) \\ \|\hat{U}\|_{C^{1,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \exp \left(C_{\alpha,d} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}\right) \right), & \alpha \in (0,1) \\ \|\hat{U}\|_{C^{2,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \exp \exp \left(C_{\alpha,d} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}\right) \right), & \alpha \in \begin{cases} (0,1) & \text{if } d=2 \\ (0, \frac{1}{5}] & \text{if } d=3. \end{cases} \end{aligned}$$

The remainder of this section is structured as follows:

- (i) In Section 2.4.1, we discuss the existence and regularity of \bar{U} , which is based on standard theory for the Poisson equation.
- (ii) In Section 2.4.2, we prove the existence of \hat{U} , given \bar{U} with the regularity derived above. Our proof uses techniques from the calculus of variations.
- (iii) In Section 2.4.3, we prove energy estimates on \hat{U} . These estimates imply regularity for \hat{U} using the regularity theory of the Poisson equation.

2.4.1 Regularity of \bar{U}

The regularity estimates on \bar{U} are based on the analysis of the following Poisson equation

$$\Delta \bar{U} = 1 - h.$$

The existence of a solution $\bar{U} \in W^{1,2}(\mathbb{T}^d)$, unique up to additive constants, for $h \in L^2(\mathbb{T}^d) \subset L^\infty(\mathbb{T}^d)$ is well-known - see for example [28, Chapter 6]. Choosing a solution with zero mean then specifies \bar{U} uniquely. In the following lemma, we recall some standard elliptic regularity estimates for this solution, that follow from Calderón-Zygmund estimates for the Laplacian [30, Section 9.4], and Sobolev inequalities.

Lemma 2.5. *Let $\bar{U} \in W^{1,2}(\mathbb{T}^d)$ have zero mean and satisfy*

$$\Delta \bar{U} = h.$$

- (i) If $h \in L^{\frac{d+2}{d}}(\mathbb{T}^d)$, then for all $\alpha \in (0, 1)$, if $d = 2$, or $\alpha \in (0, \frac{1}{5}]$ if $d = 3$, there exists a constant $C_{\alpha,d} > 0$ such that

$$\|\bar{U}\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq C_{\alpha,d} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right).$$

- (ii) If $h \in L^\infty(\mathbb{T}^d)$, then for any $\alpha \in (0, 1)$, there exists a constant $C_{\alpha,d}$ such that

$$\|\bar{U}\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C_{\alpha,d} \|h\|_{L^\infty(\mathbb{T}^d)}.$$

In order to prove estimates on the VPME system, we would ideally like to have good control of the regularity of the electric field, especially the singular part $\nabla \bar{U}$. The estimates in Lemma 2.5 are not quite strong enough to provide Lipschitz regularity for $\nabla \bar{U}$. However, a log-Lipschitz estimate is available. We stated this as Lemma 1.11 in the introduction and gave a proof there. Below, we recall the statement.

Lemma 2.6 (Log-Lipschitz regularity of \bar{E}). *Let \bar{U} be a solution of*

$$\Delta \bar{U} = h$$

for $h \in L^\infty(\mathbb{T}^d)$. Then

$$|\nabla \bar{U}(x) - \nabla \bar{U}(y)| \leq C_d \|h\|_{L^\infty(\mathbb{T}^d)} |x - y| \left(1 + \log \left(\frac{\sqrt{d}}{|x - y|} \right) \mathbb{1}_{|x - y| \leq \sqrt{d}} \right).$$

2.4.2 Existence of \hat{U}

In this section we prove the existence of a solution \hat{U} of the equation

$$\Delta \hat{U} = e^{\bar{U} + \hat{U}} - 1, \tag{2.5}$$

under the condition that $\bar{U} \in \dot{W}^{1,2} \cap L^\infty(\mathbb{T}^d)$. By Lemma 2.5, when we consider solutions of the VPME system (2.1), we expect \bar{U} to have this regularity. Our strategy is to seek a minimiser of the functional

$$h \mapsto \mathcal{J}[h] := \int_{\mathbb{T}^d} \frac{1}{2} |\nabla h|^2 + \left(e^{\bar{U} + h} - h \right) dx$$

in the space $h \in W^{1,2}(\mathbb{T}^d)$. We then use the fact that equation (2.5) is the Euler-Lagrange equation of this minimisation problem to conclude that the minimiser is a solution of (2.5).

The following lemma holds in any dimension d .

Lemma 2.7 (Existence of a Unique Minimiser). *Let $\bar{U} \in \dot{W}^{1,2} \cap L^\infty(\mathbb{T}^d)$. Then there exists a unique $\hat{U} \in W^{1,2}(\mathbb{T}^d)$ such that*

$$\mathcal{J}[\hat{U}] = \inf_{h \in W^{1,2}(\mathbb{T}^d)} \mathcal{J}[h].$$

Proof. The uniqueness of minimisers follows from the fact that \mathcal{J} is a strictly convex functional.

We prove existence of a minimiser using the direct method of the calculus of variations. Consider a minimising sequence $(h_k)_{k \in \mathbb{N}}$ such that, as k tends to infinity,

$$\mathcal{J}[h_k] \rightarrow \inf_{h \in W^{1,2}(\mathbb{T}^d)} \mathcal{J}[h].$$

We will show that h_k is uniformly bounded in $W^{1,2}(\mathbb{T}^d)$. This will allow us to extract a convergent subsequence. Since $\bar{U} \in \dot{W}^{1,2} \cap L^\infty(\mathbb{T}^d)$ and the torus has finite measure, in fact $\bar{U} \in W^{1,2}(\mathbb{T}^d)$ by monotonicity of L^p norms. Therefore, for k sufficiently large,

$$\begin{aligned} \mathcal{J}[h_k] &\leq \mathcal{J}[-\bar{U}] \\ &= \|\nabla \bar{U}\|_{L^2(\mathbb{T}^d)}^2 + \int_{\mathbb{T}^d} (1 + \bar{U}) \, dx \\ &\leq \|\nabla \bar{U}\|_{L^2(\mathbb{T}^d)}^2 + 1 + \|\bar{U}\|_{L^\infty(\mathbb{T}^d)} =: C_1. \end{aligned} \tag{2.6}$$

The constant C_1 is finite since $\bar{U} \in \dot{W}^{1,2} \cap L^\infty(\mathbb{T}^d)$.

We observe that

$$e^{\bar{U}+s} - s = e^{\bar{U}} e^s - s \geq e^{-\|\bar{U}\|_{L^\infty(\mathbb{T}^d)}} e^s - s \geq s^2 - C_2,$$

for some constant C_2 depending on $\|\bar{U}\|_{L^\infty(\mathbb{T}^d)}$ only. Thus

$$\int_{\mathbb{T}^d} e^{\bar{U}+h_k} - h_k \, dx \geq \int_{\mathbb{T}^d} h_k^2 - C_2 \, dx. \tag{2.7}$$

By equations (2.6) and (2.7),

$$\int_{\mathbb{T}^d} \frac{1}{2} |\nabla h_k|^2 + (h_k^2 - C_2) \, dx \leq \mathcal{J}[h_k] \leq C_1 + 1.$$

Thus the sequence $(h_k)_{k \in \mathbb{N}}$ is bounded in $W^{1,2}(\mathbb{T}^d)$, uniformly in k :

$$\|\nabla h_k\|_{L^2(\mathbb{T}^d)} + \|h_k\|_{L^2(\mathbb{T}^d)} \leq C_3,$$

where the constant C_3 is independent of k .

By weak compactness of $W^{1,2}(\mathbb{T}^d)$, up to a subsequence h_k converges weakly in $W^{1,2}(\mathbb{T}^d)$ to a function \widehat{U} :

$$h_k \rightharpoonup \widehat{U} \quad \text{in } W^{1,2}(\mathbb{T}^d).$$

Since $W^{1,2}(\mathbb{T}^d)$ is compactly embedded in $L^2(\mathbb{T}^d)$, we also have strong convergence:

$$h_k \rightarrow \widehat{U} \quad \text{in } L^2(\mathbb{T}^d).$$

Then, up to a further subsequence, we have

$$h_k \rightarrow \widehat{U} \quad a.e.$$

To show that \widehat{U} is indeed a minimiser, we prove that \mathcal{J} is lower semicontinuous. By the weak convergence of h_k to \widehat{U} in $W^{1,2}(\mathbb{T}^d)$ and by strong convergence in $L^2(\mathbb{T}^d)$,

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{T}^d} \frac{1}{2} |\nabla h_k|^2 dx \geq \int_{\mathbb{T}^d} \frac{1}{2} |\nabla \widehat{U}|^2 dx,$$

since the norm is lower semicontinuous under weak convergence. Strong convergence in $L^2(\mathbb{T}^d)$ implies strong convergence in $L^1(\mathbb{T}^d)$, and therefore

$$\lim_{k \rightarrow \infty} \int_{\mathbb{T}^d} h_k dx = \int_{\mathbb{T}^d} \widehat{U} dx.$$

By Fatou's Lemma,

$$\liminf_{k \rightarrow \infty} \int_{\mathbb{T}^d} e^{\widehat{U} + h_k} dx \geq \int_{\mathbb{T}^d} \liminf_{k \rightarrow \infty} e^{\widehat{U} + h_k} dx = \int_{\mathbb{T}^d} e^{\widehat{U} + \widehat{U}} dx.$$

We conclude that

$$\lim_{k \rightarrow \infty} \mathcal{J}[h_k] \geq \mathcal{J}[\widehat{U}],$$

which proves that \widehat{U} is a minimiser. □

Next, we check that the minimiser \widehat{U} solves the Euler-Lagrange equation (2.5). Again this holds in any dimension d .

Lemma 2.8 (Euler-Lagrange Equation). *Let $\bar{U} \in \dot{W}^{1,2} \cap L^\infty(\mathbb{T}^d)$ and let $\widehat{U} \in W^{1,2}(\mathbb{T}^d)$ be the unique minimiser of \mathcal{J} whose existence was proved in Lemma 2.7. Then \widehat{U} is a weak solution of equation (2.5).*

Proof. First observe that, by definition of a minimiser,

$$\mathcal{J}[\widehat{U}] \leq \mathcal{J}[-\bar{U}] < +\infty.$$

Explicitly, by definition of \mathcal{J} ,

$$\int_{\mathbb{T}^d} \frac{1}{2} |\nabla \widehat{U}|^2 + \left(e^{\bar{U}} e^{\widehat{U}} - \widehat{U} \right) dx \leq \mathcal{J}[-\bar{U}] < +\infty.$$

Since $\widehat{U} \in W^{1,2}(\mathbb{T}^d)$, by monotonicity of norms on the torus we have $\widehat{U} \in L^1(\mathbb{T}^d)$. From this we deduce that

$$\int_{\mathbb{T}^d} e^{\bar{U}} e^{\widehat{U}} dx < +\infty.$$

Since $\|\bar{U}\|_{L^\infty(\mathbb{T}^d)}$ is finite, it follows that

$$\int_{\mathbb{T}^d} e^{\widehat{U}} dx < +\infty.$$

Altogether, we have $\nabla \widehat{U} \in L^2(\mathbb{T}^d)$, $\widehat{U} \in L^1(\mathbb{T}^d)$, $e^{\bar{U}} \in L^\infty(\mathbb{T}^d)$, and $e^{\widehat{U}} \in L^1(\mathbb{T}^d)$.

We now show that \widehat{U} is a weak solution of (2.5). Let $\phi \in C_c^\infty$, $\eta > 0$. By minimality of \widehat{U} ,

$$\mathcal{J}[\widehat{U}] \leq \mathcal{J}[\widehat{U} + \eta \phi].$$

Then,

$$\begin{aligned} 0 &\leq \frac{\mathcal{J}[\widehat{U} + \eta \phi] - \mathcal{J}[\widehat{U}]}{\eta} = \frac{1}{\eta} \left(\int_{\mathbb{T}^d} \frac{1}{2} |\nabla \widehat{U} + \eta \nabla \phi|^2 - \frac{1}{2} |\nabla \widehat{U}|^2 dx \right) \\ &\quad + \frac{1}{\eta} \left(\int_{\mathbb{T}^d} e^{\bar{U}} e^{\widehat{U} + \eta \phi} - e^{\bar{U}} e^{\widehat{U}} dx \right) + \frac{1}{\eta} \left(\int_{\mathbb{T}^d} -(\widehat{U} + \eta \phi) + \widehat{U} dx \right) \\ &= \int_{\mathbb{T}^d} \nabla \widehat{U} \cdot \nabla \phi + \frac{\eta |\nabla \phi|^2}{2} dx + \int_{\mathbb{T}^d} e^{\bar{U} + \widehat{U}} \frac{e^{\eta \phi} - 1}{\eta} dx - \int_{\mathbb{T}^d} \phi dx. \end{aligned}$$

In the limit as η goes to 0 we obtain, for all $\phi \in C_c^\infty(\mathbb{T}^d)$,

$$0 \leq \lim_{\eta \rightarrow 0} \frac{\mathcal{J}[\widehat{U} + \eta \phi] - \mathcal{J}[\widehat{U}]}{\eta} = \int_{\mathbb{T}^d} \nabla \widehat{U} \cdot \nabla \phi dx + \int_{\mathbb{T}^d} e^{\bar{U} + \widehat{U}} \phi dx - \int_{\mathbb{T}^d} \phi dx.$$

Since the latter inequality is valid both for ϕ and for $-\phi$, we have that, for all $\phi \in C_c^\infty(\mathbb{T}^d)$,

$$0 = \int_{\mathbb{T}^d} \nabla \widehat{U} \cdot \nabla \phi + \left(e^{\bar{U} + \widehat{U}} - 1 \right) \phi \, dx. \quad (2.8)$$

This implies that \widehat{U} is a weak solution of the equation

$$\Delta \widehat{U} = e^{\bar{U} + \widehat{U}} - 1 \quad \text{on } \mathbb{T}^d,$$

which completes the proof. \square

2.4.3 Regularity of \widehat{U}

In this section, we prove regularity estimates on \widehat{U} .

Lemma 2.9 (Regularity of \widehat{U}). *(i) Let $\bar{U} \in \dot{W}^{1,2} \cap L^\infty(\mathbb{T}^d)$, with the estimate*

$$\|\bar{U}\|_{L^\infty(\mathbb{T}^d)} \leq M_1.$$

Let $\widehat{U} \in W^{1,2}(\mathbb{T}^d)$ be the solution of (2.5) constructed in Section 2.4.2. Then $\widehat{U} \in C^{1,\alpha}(\mathbb{T}^d)$ for any $\alpha \in (0, 1)$, with the estimate

$$\|\widehat{U}\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C(1 + e^{2M_1}).$$

(ii) Let $\bar{U} \in \dot{W}^{1,2} \cap C^{0,\alpha}(\mathbb{T}^d)$ for some $\alpha \in (0, 1)$, with the estimate

$$\|\bar{U}\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq M_2.$$

Then $\widehat{U} \in C^{2,\alpha}(\mathbb{T}^d)$ with

$$\|\widehat{U}\|_{C^{2,\alpha}(\mathbb{T}^d)} \leq C \exp \left[C(M_1 + (1 + e^{2M_1})) \right] (M_2 + (1 + e^{2M_1})).$$

Proof. Our general strategy will be to consider the equation satisfied by \widehat{U} :

$$\Delta \widehat{U} = e^{\bar{U} + \widehat{U}} - 1, \quad (2.9)$$

which we think of as a Poisson equation with source $e^{\bar{U} + \widehat{U}} - 1$. The aim is to make use of standard regularity estimates for the Poisson equation in order to deduce the required regularity for \widehat{U} . The first part of the statement follows from Calderón-Zygmund estimates and Sobolev embedding, while the second part makes use of Schauder estimates. However, in order to

implement this strategy, we must first prove suitable L^p and Hölder estimates on the source term $e^{\bar{U}+\hat{U}} - 1$. We will obtain these using energy estimates on the equation (2.9).

To prove an L^p estimate on $e^{\bar{U}+\hat{U}}$ for $p > 1$, the formal argument is to choose $e^{(p-1)\hat{U}}$ as a test function in the weak form of equation (2.9). Since

$$\nabla e^{(p-1)\hat{U}} = (p-1)e^{(p-1)\hat{U}} \nabla \hat{U},$$

from (2.9) we obtain

$$-(p-1) \int_{\mathbb{T}^d} |\nabla \hat{U}|^2 e^{(p-1)\hat{U}} dx = \int_{\mathbb{T}^d} \left(e^{\bar{U}} e^{p\hat{U}} - e^{(p-1)\hat{U}} \right) dx.$$

This implies that

$$\int_{\mathbb{T}^d} e^{\bar{U}} e^{p\hat{U}} dx \leq \int_{\mathbb{T}^d} e^{(p-1)\hat{U}} dx.$$

Bounding $e^{\bar{U}}$ from below using the L^∞ norm of \bar{U} , we obtain

$$e^{-M_1} \int_{\mathbb{T}^d} e^{p\hat{U}} dx \leq \int_{\mathbb{T}^d} e^{(p-1)\hat{U}} dx.$$

In other words,

$$\|e^{\hat{U}}\|_{L^p(\mathbb{T}^d)}^p \leq e^{M_1} \|e^{\hat{U}}\|_{L^{p-1}(\mathbb{T}^d)}^{p-1}. \quad (2.10)$$

To this we add an $L^1(\mathbb{T}^d)$ estimate, which follows from the fact that $e^U - 1$ must have total integral zero: since \hat{U} is a solution of (2.5), we have

$$0 = \int_{\mathbb{T}^d} \Delta \hat{U} dx = \int_{\mathbb{T}^d} e^U - 1 dx.$$

Since $U = \bar{U} + \hat{U}$, it follows that

$$1 = \int_{\mathbb{T}^d} e^{\bar{U}+\hat{U}} dx \geq e^{-M_1} \int_{\mathbb{T}^d} e^{\hat{U}} dx.$$

Thus

$$\|e^{\hat{U}}\|_{L^1(\mathbb{T}^d)} = \int_{\mathbb{T}^d} e^{\hat{U}} dx \leq e^{M_1}.$$

Applying (2.10) in the case $p = 2$, we find that

$$\|e^{\hat{U}}\|_{L^2(\mathbb{T}^d)} \leq e^{M_1}.$$

By induction we then obtain

$$\|e^{\hat{U}}\|_{L^n(\mathbb{T}^d)} \leq e^{M_1} \quad (2.11)$$

for any integer n .

We now check that (2.10) can be obtained rigorously. The weak form of equation (2.9) is given in equation (2.8); that is, \widehat{U} satisfies (2.8) for test functions $\phi \in C^\infty(\mathbb{T}^d)$. Since $e^U \in L^1(\mathbb{T}^d)$, this further extends to all $\phi \in L^\infty \cap W^{1,2}(\mathbb{T}^d)$ by a density argument. We therefore define a test function in $L^\infty \cap W^{1,2}(\mathbb{T}^d)$ that approximates $e^{(p-1)\widehat{U}}$.

Consider the truncated function

$$\widehat{U}_k := (\widehat{U} \wedge k), \quad \text{for all } k \in \mathbb{N}.$$

Since $e^{(p-1)\widehat{U}_k} \in L^\infty(\mathbb{T}^d)$ and $\nabla \widehat{U} \in L^2(\mathbb{T}^d)$,

$$\nabla e^{(p-1)\widehat{U}_k} = (p-1)e^{(p-1)\widehat{U}_k} \nabla \widehat{U}_k = (p-1)e^{\widehat{U}_k} \nabla \widehat{U} \chi_{\{\widehat{U} < k\}} \in L^2(\mathbb{T}^d);$$

thus $e^{(p-1)\widehat{U}_k} \in L^\infty \cap W^{1,2}(\mathbb{T}^d)$, and we can use it as a test function in equation (2.8):

$$\begin{aligned} 0 &= \int_{\mathbb{T}^d} \nabla \widehat{U} \cdot \nabla e^{(p-1)\widehat{U}_k} dx + \int_{\mathbb{T}^d} \left(e^{\widehat{U}} e^{(\widehat{U} + (p-1)\widehat{U}_k)} - e^{(p-1)\widehat{U}_k} \right) dx \\ &= (p-1) \int_{\mathbb{T}^d} \nabla \widehat{U} \cdot e^{(p-1)\widehat{U}_k} \nabla \widehat{U} \chi_{\{\widehat{U} < k\}} dx + \int_{\mathbb{T}^d} \left(e^{\widehat{U}} e^{(\widehat{U} + (p-1)\widehat{U}_k)} - e^{(p-1)\widehat{U}_k} \right) dx \\ &= (p-1) \int_{\mathbb{T}^d} |\nabla \widehat{U}|^2 e^{(p-1)\widehat{U}_k} \chi_{\{\widehat{U} < k\}} dx + \int_{\mathbb{T}^d} e^{\widehat{U}} e^{(\widehat{U} + (p-1)\widehat{U}_k)} dx - \int_{\mathbb{T}^d} e^{(p-1)\widehat{U}_k} dx. \end{aligned} \quad (2.12)$$

Since $\int_{\mathbb{T}^d} |\nabla \widehat{U}|^2 e^{(p-1)\widehat{U}_k} \chi_{\{\widehat{U} < k\}} dx \geq 0$, and $e^{-M_1} \leq e^{\widehat{U}} \leq e^{M_1}$, (2.12) implies that

$$e^{-M_1} \int_{\mathbb{T}^d} e^{\widehat{U} + (p-1)\widehat{U}_k} \leq \int_{\mathbb{T}^d} e^{(p-1)\widehat{U}_k}.$$

By definition of \widehat{U}_k we have that $e^{(p-1)\widehat{U}_k}$ is increasing and converges monotonically to $e^{(p-1)\widehat{U}}$, hence by the Monotone Convergence Theorem

$$e^{-M_1} \int_{\mathbb{T}^d} e^{\widehat{U} + (p-1)\widehat{U}} = e^{-M_1} \int_{\mathbb{T}^d} e^{p\widehat{U}} \leq \int_{\mathbb{T}^d} e^{\widehat{U}}.$$

Thus we obtain (2.10), which implies the L^n estimate (2.11) for any integer n .

We now prove the desired estimates on \widehat{U} . For part (i) of the statement, we apply (2.11) for some $n > d$. Since

$$\|e^U\|_{L^n(\mathbb{T}^d)} = \|e^{\widehat{U} + \widehat{U}}\|_{L^n(\mathbb{T}^d)} \leq e^{\|\widehat{U}\|_{L^\infty(\mathbb{T}^d)}} \|e^{\widehat{U}}\|_{L^n(\mathbb{T}^d)} \leq e^{2M_1},$$

we have

$$\Delta \widehat{U} = e^{\widehat{U} + \widehat{U}} - 1 \in L^n(\mathbb{T}^d),$$

with

$$\|e^{\bar{U}+\hat{U}} - 1\|_{L^n(\mathbb{T}^d)} \leq 1 + e^{2M_1}.$$

By Calderón-Zygmund estimates for the Poisson equation [30, Section 9.4],

$$\|\hat{U}\|_{W^{2,n}(\mathbb{T}^d)} \leq C_{n,d} (1 + e^{2M_1}).$$

Using Sobolev embedding for n sufficiently large, we deduce that for any $\alpha \in (0, 1)$, $\hat{U} \in C^{1,\alpha}(\mathbb{T}^d)$, with

$$\|\hat{U}\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C_{\alpha,d} (1 + e^{2M_1}).$$

For part (ii) of the statement, it is enough to prove a bound on the $C^{0,\alpha}(\mathbb{T}^d)$ norm of \hat{U} . This bound is provided by part (ii). Then, if

$$\|\bar{U}\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq M_2,$$

we have

$$\|U\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq M_2 + C_{\alpha,d} (1 + e^{2M_1}),$$

and so

$$\|e^U\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq C \exp [C (M_1 + (1 + e^{2M_1}))] (M_2 + (1 + e^{2M_1})).$$

Thus $\Delta \hat{U} \in C^{0,\alpha}(\mathbb{T}^d)$.

We may then apply Schauder estimates [30, Chapter 4] to obtain $\hat{U} \in C^{2,\alpha}$ with the following estimate:

$$\begin{aligned} \|\hat{U}\|_{C^{2,\alpha}(\mathbb{T}^d)} &\leq C \left(\|\hat{U}\|_{L^\infty(\mathbb{T}^d)} + \|e^U - 1\|_{C^{0,\alpha}(\mathbb{T}^d)} \right) \\ &\leq C \exp [C (M_1 + (1 + e^{2M_1}))] (M_2 + (1 + e^{M_1})). \end{aligned}$$

This completes the proof of part (ii). □

2.5 Stability of the Electric Field

In this section we study the stability of the electric field $E = -\nabla U$ with respect to the charge density ρ_f . This is one of the key estimates needed to prove the uniqueness of solutions with bounded density. Our aim is to prove a version of the following estimate for the singular part \bar{E} , due to Loeper [65].

Lemma 2.10 (Loeper-type estimate for Poisson's equation). *For each $i = 1, 2$, let $h_i \in L^\infty(\mathbb{T}^d)$ be a probability density function:*

$$h_i \geq 0, \quad \int_{\mathbb{T}^d} h_i(x) \, dx = 1.$$

Let \bar{U}_i be a solution of

$$\Delta \bar{U}_i = h_i - 1.$$

Then

$$\|\nabla \bar{U}_1 - \nabla \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2 \leq \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2).$$

The main result of this section is the following proposition:

Proposition 2.11. *For each $i = 1, 2$, let \bar{U}_i be a solution of*

$$\Delta \bar{U}_i = h_i - 1,$$

where $h_i \in L^\infty \cap L^{(d+2)/d}(\mathbb{T}^d)$. Then

$$\|\nabla \bar{U}_1 - \nabla \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2 \leq \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2).$$

Now, in addition, let \hat{U}_i be a solution of

$$\Delta \hat{U}_i = e^{\bar{U}_i + \hat{U}_i} - 1. \tag{2.13}$$

Then

$$\begin{aligned} \|\nabla \hat{U}_1 - \nabla \hat{U}_2\|_{L^2(\mathbb{T}^d)}^2 &\leq \exp \exp \left[C_d \left(1 + \max_i \|h_i\|_{L^{(d+2)/d}(\mathbb{T}^d)} \right) \right] \\ &\quad \times \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2). \end{aligned}$$

Proof. We will show that

$$\|\nabla \hat{U}_1 - \nabla \hat{U}_2\|_{L^2(\mathbb{T}^d)}^2 \leq C \|\bar{U}_1 - \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2. \tag{2.14}$$

The proposition then follows from the Poincaré inequality for zero mean functions:

$$\|\bar{U}_1 - \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2 \leq C_d \|\nabla \bar{U}_1 - \nabla \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2.$$

We then apply the Loeper estimate from Lemma 2.10.

We now prove (2.14). For convenience, we define the constant

$$A := \exp \left[\max_i \|\bar{U}_i\|_{L^\infty(\mathbb{T}^d)} + \max_i \|\widehat{U}_i\|_{L^\infty(\mathbb{T}^d)} \right]$$

which will be fixed throughout the proof. Note that A can be controlled using Proposition 2.4,

$$\|\bar{U}_i\|_{L^\infty(\mathbb{T}^d)} \leq C_d \left(1 + \|h_i\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right), \quad \|\widehat{U}_i\|_{L^\infty(\mathbb{T}^d)} \leq \exp \left(C_d \left(1 + \|h_i\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right).$$

Subtracting the two equations (2.13), we deduce that $\widehat{U}_1 - \widehat{U}_2$ satisfies

$$\Delta(\widehat{U}_1 - \widehat{U}_2) = e^{\bar{U}_1 + \widehat{U}_1} - e^{\bar{U}_2 + \widehat{U}_2} = e^{\bar{U}_1} (e^{\widehat{U}_1} - e^{\widehat{U}_2}) + e^{\widehat{U}_2} (e^{\bar{U}_1} - e^{\bar{U}_2}). \quad (2.15)$$

The weak form of (2.15) extends by density to test functions in $L^\infty \cap W^{1,2}(\mathbb{T}^d)$. Since $\widehat{U}_1 - \widehat{U}_2$ has this regularity by assumption, it is an admissible test function. Hence

$$\begin{aligned} - \int_{\mathbb{T}^d} |\nabla \widehat{U}_1 - \nabla \widehat{U}_2|^2 dx &= \int_{\mathbb{T}^d} e^{\bar{U}_1} (e^{\widehat{U}_1} - e^{\widehat{U}_2}) (\widehat{U}_1 - \widehat{U}_2) dx \\ &\quad + \int_{\mathbb{T}^d} e^{\widehat{U}_2} (e^{\bar{U}_1} - e^{\bar{U}_2}) (\widehat{U}_1 - \widehat{U}_2) dx =: I_1 + I_2. \end{aligned} \quad (2.16)$$

Observe that $(e^x - e^y)(x - y)$ is always non-negative. Furthermore, by the Mean Value Theorem applied to the function $x \mapsto e^x$ we have a lower bound

$$(e^x - e^y)(x - y) \geq e^{\min\{x,y\}}(x - y)^2.$$

We use this to bound I_1 from below:

$$I_1 \geq e^{-\|\bar{U}_1\|_{L^\infty(\mathbb{T}^d)} - \max_i \|\widehat{U}_i\|_{L^\infty(\mathbb{T}^d)}} \|\widehat{U}_1 - \widehat{U}_2\|_{L^2(\mathbb{T}^d)}^2 \geq A^{-1} \|\widehat{U}_1 - \widehat{U}_2\|_{L^2(\mathbb{T}^d)}^2. \quad (2.17)$$

For I_2 we use the fact that, again by the Mean Value Theorem,

$$|e^x - e^y| \leq e^{\max\{x,y\}} |x - y|.$$

Therefore

$$I_2 \leq e^{\|\widehat{U}_2\|_{L^\infty(\mathbb{T}^d)} + \max_i \|\bar{U}_i\|_{L^\infty(\mathbb{T}^d)}} \int_{\mathbb{T}^d} |\bar{U}_1 - \bar{U}_2| |\widehat{U}_1 - \widehat{U}_2| dx \leq A \int_{\mathbb{T}^d} |\bar{U}_1 - \bar{U}_2| |\widehat{U}_1 - \widehat{U}_2| dx.$$

By the Cauchy-Schwarz inequality, for any choice of $\alpha > 0$

$$I_2 \leq A \left(\alpha \|\bar{U}_1 - \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2 + \frac{1}{4\alpha} \|\widehat{U}_1 - \widehat{U}_2\|_{L^2(\mathbb{T}^d)}^2 \right). \quad (2.18)$$

Substituting (2.17) and (2.18) into (2.16), we obtain

$$\begin{aligned} \int_{\mathbb{T}^d} |\nabla \widehat{U}_1 - \nabla \widehat{U}_2|^2 dx &\leq A \left(\alpha \|\bar{U}_1 - \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2 + \frac{1}{4\alpha} \|\widehat{U}_1 - \widehat{U}_2\|_{L^2(\mathbb{T}^d)}^2 \right) \\ &\quad - A^{-1} \|\widehat{U}_1 - \widehat{U}_2\|_{L^2(\mathbb{T}^d)}^2. \end{aligned} \quad (2.19)$$

We wish to choose α as small as possible such that

$$\frac{A}{4\alpha} - A^{-1} \leq 0.$$

Thus the optimal choice is $\alpha = \frac{A^2}{4}$. Substituting this into (2.19) gives

$$\int_{\mathbb{T}^d} |\nabla \widehat{U}_1 - \nabla \widehat{U}_2|^2 dx \leq \frac{1}{4} A^3 \|\bar{U}_1 - \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2.$$

This completes the proof of (2.14). □

2.6 Wasserstein Stability and Uniqueness

In this section, we prove a quantitative stability estimate between solutions with bounded density. This is a version of Loeper's [65] estimate for the classical Vlasov-Poisson system, which we recalled as Theorem 1.16.

Proposition 2.12 (Stability for solutions with bounded density). *For $i = 1, 2$, let f_i be solutions of (2.1) satisfying, for some constant $M > 0$ and all $t \in [0, T]$,*

$$\rho[f_i(t)] \leq M. \quad (2.20)$$

Then there exists a constant $C > 0$, depending on M , such that, for all $t \in [0, T]$,

$$W_2(f_1(t), f_2(t)) \leq \beta(t, W_2(f_1(0), f_2(0))),$$

where β denotes the function

$$\beta(t, x) = \begin{cases} 16de \exp \left[\left(1 + \log \frac{x}{16d} \right) e^{-Ct} \right] & t \leq T_0 \\ (d \vee x) e^{C(1+\log 16)(t-T_0)} & t > T_0, \end{cases}$$

where

$$T_0 = \inf \left\{ t > 0 : 16de \exp \left[\left(1 + \log \frac{x}{16d} \right) e^{-Ct} \right] > d \right\}.$$

This estimate immediately implies a uniqueness result for solutions of VPME with bounded density.

Corollary 2.13 (Uniqueness of solutions with bounded density). *For each $i = 1, 2$, let f_i be a solution of (2.1) in the space $C([0, T]; \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$, with bounded density*

$$\sup_{t \in [0, T]} \|\rho[f_i]\|_{L^\infty(\mathbb{T}^d)} \leq C_i,$$

for the same initial datum $f(0) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$. Then $f_1 = f_2$.

Proof. We will prove Proposition 2.12 by means of a Gronwall type estimate. To do this, we will first consider the evolution of particular specially constructed couplings $\pi_t \in \Pi(f_1(t), f_2(t))$. First, observe that f_i can be represented as the pushforward of the initial datum $f_i(0)$ along the characteristic flow associated to (2.1). That is, for each $i = 1, 2$, given f_i , consider the flow $Z_i(t; 0, z) = (X_i(t; 0, z), V_i(t; 0, z))$ defined by the system of ODEs

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = E_i(X_i) \\ (X_i(0; 0, z), V_i(0; 0, z)) = z = (x, v), \end{cases} \quad (2.21)$$

where E_i is the electric field induced by f_i :

$$E_i = -\nabla U_i, \quad \Delta U_i = e^{U_i} - \rho[f_i].$$

We again use the decomposition $E_i = \widehat{E}_i + \bar{E}_i$. Since $\rho[f_i] \in L^\infty(\mathbb{T}^d)$ by assumption (2.20), Lemma 2.6 implies that \bar{E}_i has log-Lipschitz regularity. Since $L^\infty(\mathbb{T}^d) \subset L^{\frac{d+2}{d}}(\mathbb{T}^d)$, we have $\rho[f_i] \in L^\infty \cap L^{\frac{d+2}{d}}(\mathbb{T}^d)$. Thus we may apply Proposition 2.4 to deduce Lipschitz regularity of \widehat{E}_i . Overall this implies that E_i has log-Lipschitz regularity, which is sufficient to guarantee the existence of a unique solution to the system (2.21). The uniqueness of the flow implies that the

linear Vlasov equation

$$\partial_t g + v \cdot \nabla_x g + E_i \cdot \nabla_v g = 0, \quad g|_{t=0} = f_i(0) \quad (2.22)$$

has a unique measure-valued solution g (see for instance [1, Theorem 3.1]). This solution can be represented as the pushforward of the initial data along the characteristic flow, as defined in Definition 2:

$$g_t = Z(t; 0, \cdot)_\# f_i(0). \quad (2.23)$$

Since f_i is also a solution of (2.22), and the solution is unique, it follows that $g = f_i$. We deduce that f_i has the representation (2.23). Note that here we are not yet asserting any *nonlinear* uniqueness, because we already fixed E_i to be the electric field corresponding to f_i .

We use the representation above to construct π_t . First, fix an arbitrary initial coupling $\pi_0 \in \Pi(f_1(0), f_2(0))$. We then build a coupling π_t for which each marginal evolves along the appropriate characteristic flow. That is, we let

$$\pi_t = (Z_1(t; 0, \cdot) \otimes Z_2(t; 0, \cdot))_\# \pi_0. \quad (2.24)$$

This means that, for all $\phi \in C_b((\mathbb{T}^d \times \mathbb{R}^d)^2)$,

$$\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \phi(z_1, z_2) d\pi_t(z_1, z_2) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(Z_1(t; 0, z_1), Z_2(t; 0, z_2)) d\pi_0(z_1, z_2). \quad (2.25)$$

We now consider the quantity

$$D(t) := \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} (|X_1(t; 0, z_1) - X_2(t; 0, z_2)|^2 + |V_1(t; 0, z_2) - V_2(t; 0, z_2)|^2) d\pi_0(z_1, z_2). \quad (2.26)$$

By definition (2.24) we have

$$D(t) = \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x_1 - x_2|^2 + |v_1 - v_2|^2 d\pi_t(z_1, z_2).$$

It then follows from the definition of the Wasserstein distances (Definition 4) that

$$W_2^2(f_1(t), f_2(t)) \leq D(t). \quad (2.27)$$

Moreover, since π_0 was arbitrary, we have

$$W_2^2(f_1(0), f_2(0)) = \inf_{\pi_0} D(0). \quad (2.28)$$

We will therefore focus next on controlling the growth of $D(t)$. This amounts to performing a Gronwall estimate along the trajectories of the characteristic flow. We give the details in Lemma 2.14 below. We obtain a bound

$$D(t) \leq \beta(t, D(0)),$$

where the function β is defined by

$$\beta(t, x) \leq \begin{cases} 16de \exp \left[\left(1 + \log \frac{x}{16d} \right) e^{-Ct} \right] & t \leq T_0(x) \\ (d \vee x) e^{C(1+\log 16)(t-T_0)} & t > T_0(x), \end{cases} \quad (2.29)$$

where

$$T_0(x) = \inf \left\{ t > 0 : C \exp \left[\left(1 + \log \frac{x}{16d} \right) e^{-Ct} \right] > d \right\}.$$

From (2.27) it follows that

$$W_2^2(f_1(t), f_2(t)) \leq \beta(t, D(0)).$$

Finally, taking infimum over π_0 and applying (2.28) concludes the proof. \square

Lemma 2.14 (Control of D). *Let D be defined by (2.26). Then*

$$D(t) \leq \beta(t, D(0)),$$

where β is defined by (2.29) for some $C > 0$ depending on M .

Proof. Differentiating with respect to t gives

$$\dot{D} = 2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} (X_1(t) - X_2(t)) \cdot (V_1(t) - V_2(t)) + (V_1(t) - V_2(t)) \cdot [E_1(X_1(t)) - E_2(X_2(t))] d\pi_0. \quad (2.30)$$

We split the electric field into four parts:

$$\begin{aligned} E_1(X_1) - E_2(X_2) &= [\bar{E}_1(X_1) - \bar{E}_1(X_2)] + [\bar{E}_1(X_2) - \bar{E}_2(X_2)] \\ &\quad + [\hat{E}_1(X_1) - \hat{E}_1(X_2)] + [\hat{E}_1(X_2) - \hat{E}_2(X_2)]. \end{aligned}$$

Applying Hölder's inequality to (2.30), we obtain

$$\dot{D} \leq D + 2\sqrt{D} \sum_{i=1}^4 I_i^{1/2},$$

where

$$\begin{aligned} I_1 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\bar{E}_1(X_1) - \bar{E}_1(X_2)|^2 d\pi_0, & I_2 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\bar{E}_1(X_2) - \bar{E}_2(X_2)|^2 d\pi_0; \\ I_3 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\hat{E}_1(X_1) - \hat{E}_1(X_2)|^2 d\pi_0, & I_4 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\hat{E}_1(X_2) - \hat{E}_2(X_2)|^2 d\pi_0. \end{aligned} \quad (2.31)$$

We estimate the above terms in Lemmas 2.15-2.18 below. Altogether we obtain

$$\dot{D}(t) \leq \begin{cases} CD(t) \left(1 + \left|\log \frac{D(t)}{16d}\right|\right) & \text{if } D(t) < d \\ C(1 + \log 16)D(t) & \text{if } D(t) \geq d. \end{cases} \quad (2.32)$$

We use this to derive a Gronwall type bound on $D(t)$. If $D(0) < d$ then, since D is a continuous function of t , there is an interval $t \in [0, t_1)$ on which $D(t) < d$. On this interval D satisfies

$$\dot{D}(t) \leq CD(t) \left(1 + \left|\log \frac{D(t)}{16d}\right|\right) = CD(t) \left(1 - \log \frac{D(t)}{16d}\right).$$

We then note that the function $\zeta : \mathbb{R} \rightarrow \mathbb{R}$ defined by

$$\zeta(t) = K \exp \left[A e^{-Ct} \right]$$

has the property

$$\dot{\zeta} = C\zeta \log \frac{K}{\zeta}.$$

Thus for $t \in [0, t_1]$, D satisfies

$$D(t) \leq 16de \exp \left[\log \frac{D(0)}{16de} e^{-Ct} \right]. \quad (2.33)$$

Let

$$T_0 := \inf \left\{ t > 0 : 16de \exp \left(\log \frac{D(0)}{16de} e^{-Ct} \right) > d \right\}.$$

Then $t_1 \geq T_0$ and the bound (2.33) holds for $t \in [0, T_0]$.

For $t > T_0$, we claim that D satisfies

$$D(t) \leq de^{C(1+\log 16)(t-T_0)}. \quad (2.34)$$

Indeed, if $D(t) > d$, we define

$$\tau(t) = \sup \{ s < t : D(s) \leq d \}.$$

Then

$$D(t) = d + \int_{\tau(t)}^t \dot{D}(s) \, ds.$$

Then, by (2.32), we have the bound

$$\begin{aligned} D(t) &\leq d + C(1 + \log 16) \int_{\tau(t)}^t D(s) \, ds \\ &\leq d + C(1 + \log 16) \int_{T_0}^t D(s) \, ds. \end{aligned}$$

This bound clearly also holds for those $t > T_0$ for which $D(t) \leq d$. Then (2.34) follows from the integral form of Gronwall's inequality. If $D(0) \geq d$ then a similar argument shows that

$$D(t) \leq D(0)e^{C(1+\log 16)t}.$$

Note that $T_0 = 0$ in this case. This completes the proof. □

Lemma 2.15 (Control of I_1). *Let I_1 be defined as in (2.31). Then*

$$I_1 \leq C(M+1)^2 H(D),$$

where D is defined as in (2.26) and

$$H(x) := \begin{cases} x \left(\log \frac{x}{16d} \right)^2 & \text{if } x \leq d \\ d (\log 16)^2 & \text{if } x > d. \end{cases}$$

Proof. First we use the regularity estimate for \bar{E}_1 from Lemma 2.6:

$$\begin{aligned} I_1 &\leq \|\rho_{f_1} - 1\|_{L^\infty(\mathbb{T}^d)}^2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |X_1(t) - X_2(t)|^2 \left(\log \frac{4\sqrt{d}}{|X_1(t) - X_2(t)|} \right)^2 d\pi_0 \\ &= \frac{1}{4} \|\rho_{f_1} - 1\|_{L^\infty(\mathbb{T}^d)}^2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |X_1(t) - X_2(t)|^2 \left(\log \frac{|X_1(t) - X_2(t)|^2}{16d} \right)^2 d\pi_0. \end{aligned}$$

The function

$$a(x) = x \left(\log \frac{x}{16d} \right)^2$$

is concave on the set $x \in [0, 16de^{-1}]$. Since $X_i(t) \in \mathbb{T}^d$, we have $|X_1 - X_2|^2 \leq d$. Note that

$$a'(d) = -\log 16(2 - \log 16) > 0;$$

hence the function $H(x)$ defined in the statement is concave on \mathbb{R}^+ , and

$$I_1 \leq \frac{1}{4} \|\rho_{f_1} - 1\|_{L^\infty(\mathbb{T}^d)}^2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} H(|X_1(t) - X_2(t)|^2) d\pi_0.$$

Then, since π_0 is a probability measure, we may apply Jensen's inequality to deduce that

$$I_1 \leq \frac{1}{4} \|\rho_{f_1} - 1\|_{L^\infty(\mathbb{T}^d)}^2 H \left(\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |X_1(t) - X_2(t)|^2 d\pi_0 \right) \leq \frac{1}{4} \|\rho_{f_1} - 1\|_{L^\infty(\mathbb{T}^d)}^2 H(D).$$

□

Lemma 2.16 (Control of I_2). *Let I_2 be defined as in (2.31). Then*

$$I_2 \leq M^2 D,$$

where D is defined as in (2.26).

Proof. From (3.16), for all $\phi \in C(\mathbb{T}^d)$ we have

$$\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \phi[X_i(t)] d\pi_0 = \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(x) f_i(t, x, v) dx dv = \int_{\mathbb{T}^d} \phi(x) \rho_{f_i}(t, x) dx. \quad (2.35)$$

Thus

$$\begin{aligned} I_2 &= \int_{\mathbb{T}^d} |\bar{E}_1(x) - \bar{E}_2(x)|^2 \rho_{f_2}(t, x) dx \\ &\leq \|\rho_{f_2}\|_{L^\infty(\mathbb{T}^d)} \|\bar{E}_1 - \bar{E}_2\|_{L^2(\mathbb{T}^d)}^2 \\ &= \|\rho_{f_2}\|_{L^\infty(\mathbb{T}^d)} \|\nabla \bar{U}_1 - \nabla \bar{U}_2\|_{L^2(\mathbb{T}^d)}^2. \end{aligned}$$

We use the Loeper-type stability estimate from Lemma 2.10 to control the difference between different electric fields. Then

$$I_2 \leq \max_i \|\rho_{f_i}\|_{L^\infty(\mathbb{T}^d)}^2 W_2^2(\rho_{f_1}, \rho_{f_2}) \leq \max_i \|\rho_{f_i}\|_{L^\infty(\mathbb{T}^d)}^2 D.$$

□

Lemma 2.17 (Control of I_3). *Let I_3 be defined as in (2.31). Then*

$$I_3 \leq C_{M,d} D,$$

where D is defined as in (2.26) and $C_{M,d}$ depends on M and d .

Proof. Observe that

$$\begin{aligned} I_3 &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \left| \widehat{E}_1[X_1(t)] - \widehat{E}_1[X_2(t)] \right|^2 d\pi_0 \\ &\leq \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \|\widehat{E}_1\|_{C^1(\mathbb{T}^d)}^2 |X_1(t) - X_2(t)|^2 d\pi_0 \\ &\leq \|\widehat{U}_1\|_{C^{2,\alpha}(\mathbb{T}^d)}^2 D \end{aligned}$$

for any $\alpha > 0$. To this we apply the regularity estimate on \widehat{U}_1 from Proposition 2.4 with $\alpha \in A_d$:

$$\|\widehat{U}_1\|_{C^{2,\alpha}(\mathbb{T}^d)} \leq C_{\alpha,d} \exp \exp \left(C_{\alpha,d} \left(1 + \|\rho_{f_1}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right) \leq C_{\alpha,M,d},$$

since

$$\|\rho_{f_i}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq \|\rho_{f_i}\|_{L^\infty(\mathbb{T}^d)} \leq M.$$

Thus we have

$$I_3 \leq C_{M,d} D.$$

□

Lemma 2.18 (Control of I_4). *Let I_4 be defined as in (2.31). Then*

$$I_4 \leq C_{M,d} M^2 D,$$

where D is defined as in (2.26) and $C_{M,d}$ depends on M and d .

Proof. Using (2.35) again, we deduce that

$$\begin{aligned} I_4 &= \int_{\mathbb{T}^d} |\widehat{E}_1(x) - \widehat{E}_2(x)|^2 \rho_{f_2}(t, x) dx \\ &\leq \|\rho_{f_2}\|_{L^\infty(\mathbb{T}^d)} \|\widehat{E}_1 - \widehat{E}_2\|_{L^2(\mathbb{T}^d)}^2 \\ &= \|\rho_{f_2}\|_{L^\infty(\mathbb{T}^d)} \|\nabla \widehat{U}_1 - \nabla \widehat{U}_2\|_{L^2(\mathbb{T}^d)}^2. \end{aligned}$$

To control the $L^2(\mathbb{T}^d)$ distance between the electric fields we use the stability estimate in Proposition 2.11:

$$\|\nabla \widehat{U}_1 - \nabla \widehat{U}_2\|_{L^2(\mathbb{T}^d)}^2 \leq \exp \exp \left[C_d \left(1 + \max_i \|\rho_{f_i}\|_{L^{(d+2)/d}(\mathbb{T}^d)} \right) \right] \max_i \|\rho_{f_i}\|_{L^\infty} W_2^2(\rho_{f_1}, \rho_{f_2}).$$

Therefore

$$\begin{aligned} I_4 &\leq \exp \exp \left[C_d \left(1 + \max_i \|\rho_{f_i}\|_{L^{(d+2)/d}(\mathbb{T}^d)} \right) \right] \max_i \|\rho_{f_i}\|_{L^\infty}^2 W_2^2(\rho_1, \rho_2) \\ &\leq \exp \exp \left[C_d \left(1 + \max_i \|\rho_{f_i}\|_{L^{(d+2)/d}(\mathbb{T}^d)} \right) \right] \max_i \|\rho_{f_i}\|_{L^\infty}^2 D \leq C_{M,d} M^2 D. \end{aligned}$$

□

2.7 A Priori Estimates on the Mass Density

In this section, we prove a priori $L^\infty(\mathbb{T}^d)$ bounds on the mass density ρ_f of a solution f of the VPME system (2.1). The idea is to control the maximum possible growth of a characteristic trajectory. This gives a bound on how far the support of f_t can spread over time. We will need to use different methods depending on whether $d = 2$ or $d = 3$.

2.7.1 Two-dimensional Case

In this section we fix $d = 2$. Our aim is to prove the following a priori growth estimate on the charge density of a solution of (2.1) with bounded density.

Proposition 2.19. *Let $d = 2$. Let f be a solution of (2.1) with bounded density $\rho_f \in L_{loc}^\infty([0, +\infty); L^\infty(\mathbb{T}^2))$, satisfying for some constant $C_0 > 0$,*

$$\|f\|_{L^\infty([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2)} \leq C_0, \quad \sup_{t \in [0,T]} \mathcal{E}^{ME}[f(t)] \leq C_0.$$

Assume that $f(0)$ has compact support contained in $\mathbb{T}^2 \times B_{\mathbb{R}^2}(0, R_0)$, for some R_0 . Then

$$\sup_{t \in [0,T]} \|\rho_f(t)\|_{L^\infty(\mathbb{T}^2)} \leq C_T (1 + R_0)^2.$$

The constant C_T depends on C_0 and T .

2.7.1.1 Control of \bar{E}

Lemma 2.20. *Let $d = 2$. Let f be a solution of (2.1) satisfying for some constant C_0 ,*

$$\|f\|_{L^\infty([0,T] \times \mathbb{T}^2 \times \mathbb{R}^2)} \leq C_0, \quad \sup_{t \in [0,T]} \mathcal{E}^{ME}[f(t)] \leq C_0,$$

and such that the support of f at any time t is contained in $\mathbb{T}^2 \times B_{\mathbb{R}^2}(0; R(t))$ for some function $R : [0, +\infty) \rightarrow [0, +\infty)$. Then for all $x \in \mathbb{T}^2$ and all $t \in [0, T]$,

$$|\bar{E}(t, x)| \leq C \left(1 + [\log(1 + R(t))]^{1/2}\right).$$

Proof. We follow the methods of [43, Proposition 3.3]. From the equation we have the representation

$$\bar{E}(x) = K_{\text{per}} * [\rho_f - 1](x) = K * [\rho_f - 1](x) + K_0 * [\rho_f - 1](x), \quad (2.36)$$

where $K_0 = -\nabla G_0$ is the smooth function defined in (1.44).

For the second term, since K_0 is continuous on \mathbb{T}^2 and thus bounded, by Young's inequality we have

$$\|K_0 * (\rho_f - 1)\|_{L^\infty(\mathbb{T}^2)} \leq \|K_0\|_{L^\infty(\mathbb{T}^2)} \|\rho_f - 1\|_{L^1(\mathbb{T}^2)} \leq 2\|K_0\|_{L^\infty(\mathbb{T}^2)}. \quad (2.37)$$

The second inequality follows because ρ_f has unit mass.

We estimate the first term by splitting the integral into a part close to the origin and a part far from the origin: for any $\delta \in (0, 1)$ and any $x \in \mathbb{T}^2$, we have

$$\begin{aligned} |K * [\rho_f - 1](x)| &\leq C \left| \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \frac{y}{|y|^2} (\rho_f(x - y) - 1) dy \right| \\ &\leq C \left| \int_{|y| \leq \delta} \frac{y}{|y|^2} (\rho_f(x - y) - 1) dy \right| + C \left| \int_{2 \geq |y| > \delta} \frac{y}{|y|^2} (\rho_f(x - y) - 1) dy \right| \\ &\leq C\delta \|\rho_f - 1\|_{L^\infty(\mathbb{T}^2)} + C \left(\int_{2 \geq |y| > \delta} \frac{1}{|y|^2} dy \right)^{1/2} \|\rho_f - 1\|_{L^2(\mathbb{T}^2)}. \\ &\leq C\delta \|\rho_f - 1\|_{L^\infty(\mathbb{T}^2)} + C(-\log(\delta))^{1/2} \|\rho_f - 1\|_{L^2(\mathbb{T}^2)}. \end{aligned}$$

We note the following estimates on ρ_f : firstly, since f is bounded with $\text{supp } f \subset \mathbb{T}^d \times B_{\mathbb{R}^2}(0; R(t))$,

$$|\rho_f(t, x)| = \left| \int_{v \in \mathbb{R}^2} f(t, x, v) dv \right| = \left| \int_{v \in B_{\mathbb{R}^2}(0; R(t))} f(t, x, v) dv \right| \leq C \|f\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)} R(t)^2. \quad (2.38)$$

Thus

$$\|\rho_f(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq CC_0 R(t)^2. \quad (2.39)$$

Secondly, by (2.4) we have a uniform $L^2(\mathbb{T}^2)$ bound

$$\|\rho_f(t, \cdot)\|_{L^2(\mathbb{T}^2)} \leq C, \quad (2.40)$$

where C depends on C_0 only. Substituting this into (2.38), we find that, for any $\delta \in (0, 1)$,

$$\left| \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \frac{y}{|y|^2} (\rho_f(x-y) - 1) dy \right| \leq C \left(\delta R(t)^2 + (-\log \delta)^{1/2} \right).$$

Choosing $\delta = (1 + R(t))^{-2}$, we obtain

$$\left| \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \frac{y}{|y|^2} (\rho_f(x-y) - 1) dy \right| \leq C \left(1 + (\log(1 + R(t)))^{1/2} \right). \quad (2.41)$$

Substituting (2.41) and (2.37) into (2.36), we obtain

$$|\bar{E}(t, x)| \leq C \left(1 + (\log(1 + R(t)))^{1/2} \right),$$

for C depending on C_0 only. □

2.7.1.2 Control of E

Using Lemma 2.20, we obtain an $L^\infty(\mathbb{T}^2)$ control on the whole electric field E .

Corollary 2.21. *Let $d = 2$. Let f be a solution of (2.1) satisfying, for some constant C_0 ,*

$$\|f\|_{L^\infty([0, T] \times \mathbb{T}^2 \times \mathbb{R}^2)} \leq C_0, \quad \sup_{t \in [0, T]} \mathcal{E}^{ME}[f(t)] \leq C_0,$$

and such that the support of f at any time t is contained in $\mathbb{T}^2 \times B_{\mathbb{R}^2}(0; R_t)$ for some function $t \mapsto R_t$. Then

$$\|E(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C \left(1 + (\log(1 + R_t))^{1/2} \right),$$

where C depends on C_0 only.

Proof. By the triangle inequality, $|E| \leq |\bar{E}| + |\hat{E}|$. Since $\hat{E} = -\nabla \hat{U}$, by Proposition 2.4,

$$\|\hat{E}(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C \exp \left(C \left(1 + \|\rho_f(t, \cdot)\|_{L^2(\mathbb{T}^d)} \right) \right).$$

By (2.40),

$$\|\hat{E}(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C$$

for some C depending on C_0 only. Therefore, by Lemma 2.20,

$$\|E(t, \cdot)\|_{L^\infty(\mathbb{T}^2)} \leq C \left(1 + (\log(1 + R_t))^{1/2} \right).$$

This completes the proof. \square

2.7.1.3 Control of support

Proof of Proposition 2.19. Our goal now is to control the growth of the support of ρ_f . If we know that $\rho_f \in L^\infty([0, T]; L^\infty(\mathbb{T}^d))$, then characteristic trajectories of (2.1) exist uniquely by the log-Lipschitz regularity of E . We may then control the growth of the support of f by studying the maximal possible growth of these characteristic trajectories. Let $V(t; 0, x, v)$ denote the v coordinate at time t of the characteristic trajectory that starts from phase space position (x, v) at time 0. We choose

$$R_t := \sup_{s \in [0, t], (x, v) \in \text{supp } f_0} |V(s; 0, x, v)|;$$

then the support of f at time t is contained in $\mathbb{T}^2 \times B_{\mathbb{R}^2}(0; R_t)$ for this choice of R_t . Next we use the previous estimates to perform a Gronwall type estimate on this quantity.

For any fixed trajectory $V(t) = V(t; x, v)$, observe that

$$|V(t)| \leq |V(0)| + \int_0^t \|E(s)\|_{L^\infty(\mathbb{T}^2)} ds.$$

Thus, by the uniform bound on E from Lemma 2.21,

$$R_t \leq R_0 + \int_0^t C \left(1 + (\log(1 + R_s))^{1/2} \right) ds.$$

By comparison with the function

$$z(t) = (1 + 2Ct) [R_0 + \log(1 + 2Ct)],$$

which satisfies the differential inequality

$$\dot{z} \geq C(1 + \log(1 + z))$$

(see Lemma A.1), we deduce that

$$R_t \leq (1 + 2Ct) [R_0 + \log(1 + 2Ct)].$$

Recalling (2.39), we conclude that

$$\sup_{t \in [0, T]} \|\rho_f(t)\|_{L^\infty(\mathbb{T}^2)} \leq C_T (1 + R_0)^2,$$

which completes the proof. \square

2.7.2 Three-dimensional Case

In this section, we consider the case $d = 3$. We adapt estimates by Batt and Rein [10] in order to prove a growth estimate on the mass density ρ_f .

Proposition 2.22. *Let f be a solution of (2.1) in dimension $d = 3$. Assume that there exists C_0 such that*

$$\|f\|_{L^\infty([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C_0, \quad \sup_{t \in [0,T]} \mathcal{E}^{ME}[f(t)] \leq C_0,$$

and that f is a solution with bounded density ρ_f , such that $f(0)$ has compact support contained in $\mathbb{T}^3 \times B_{\mathbb{R}^3}(0, R_0)$. Then

$$\sup_{t \in [0,T]} \|\rho_f(t)\|_{L^\infty(\mathbb{T}^3)} \leq \max\{T^{-81/8}, C(R_0^3 + T^6)\}.$$

The arguments of Batt and Rein [10] relate the growth of the mass density to the maximal possible growth of the characteristics corresponding to the equation, for the case of the classical Vlasov-Poisson equation. Our aim is to adapt these arguments to the massless electrons case.

Let $(X(t, \tau, x, v), V(t, \tau, x, v))$ denote the phase space position at time t of the characteristic trajectory that starts from phase space position (x, v) at time τ . The existence of these characteristics follows from the assumption that the solution f we consider has bounded density. The electric field E therefore has log-Lipschitz regularity, and the existence of characteristic trajectories then follows. The idea is to control how large $|V(t, 0, x, v)|$ may become as t grows, for all x, v contained in the support of the initial datum $f(0)$. The supremal value of $|V(t, 0, x, v)|$ gives the extent of the support of the solution f at time t : if

$$R_t := \sup\{|V(s, 0, x, v)| : s \in [0, t], x \in \mathbb{T}^d, v \in B_{\mathbb{R}^d}(0; R_0)\},$$

then

$$\text{supp } f(t, \cdot, \cdot) \subset \mathbb{T}^d \times B_{\mathbb{R}^d}(0; R_t).$$

In particular this is useful because, since the $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ norm of f is conserved by the VPME system, we may deduce the following bound on the mass density ρ_f :

$$\|\rho_f(t, \cdot)\|_{L^\infty(\mathbb{T}^d)} \leq \|f(t, \cdot, \cdot)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R_t^d \leq C_0 R_t^d, \quad (2.42)$$

where C_0 depends on $f(0)$ only.

The growth of $V(t, \tau, x, v)$ can be estimated using bounds on the electric field E , since

$$V(t, \tau, x, v) = v + \int_{\tau}^t E(X(s, \tau, x, v)) \, ds. \quad (2.43)$$

As a first estimate, we may write

$$\begin{aligned} |V(t, \tau, x, v)| &\leq |v| + \int_{\tau}^t \|E(s, \cdot)\|_{L^{\infty}(\mathbb{T}^d)} \, ds \\ &\leq |v| + \int_{\tau}^t \|\bar{E}(s, \cdot)\|_{L^{\infty}(\mathbb{T}^d)} + \|\hat{E}(s, \cdot)\|_{L^{\infty}(\mathbb{T}^d)} \, ds, \end{aligned} \quad (2.44)$$

which remains true in any dimension d . The term involving \hat{E} can be estimated using Proposition 2.4:

$$\|\hat{E}(s, \cdot)\|_{L^{\infty}(\mathbb{T}^d)} \leq \|\hat{U}(s, \cdot)\|_{C^1(\mathbb{T}^d)} \leq C_d \exp \left(C_d \left(1 + \|\rho_f(s, \cdot)\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right).$$

The conservation of energy implies that this estimate is in fact uniform in s . The remaining task is to estimate

$$\bar{E}(s, x) = K *_{\mathbf{x}} \rho_f(s, x).$$

In the two-dimensional case, we accomplished this by using an interpolation argument to estimate \bar{E} in terms of the $L^{\infty}(\mathbb{T}^d)$ and $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ norms on $\rho_f(s, \cdot)$ - see Lemma 2.20. In the two-dimensional case, the dependence on $\|\rho_f\|_{L^{\infty}(\mathbb{T}^d)}$ was logarithmic, resulting in a differential inequality that could be closed. In the three-dimensional case, the corresponding interpolation results in the following estimate:

$$\begin{aligned} \|\bar{E}(s, x)\|_{L^{\infty}(\mathbb{T}^3)} &\leq C \|\rho_f\|_{L^{\frac{5}{3}}(\mathbb{T}^3)}^{\frac{5}{9}} \|\rho_f\|_{L^{\infty}(\mathbb{T}^3)}^{\frac{4}{9}} \\ &\leq C_0 \|\rho_f\|_{L^{\infty}(\mathbb{T}^3)}^{\frac{4}{9}}. \end{aligned} \quad (2.45)$$

The estimate (2.42) then implies that

$$\|\bar{E}(s, x)\|_{L^{\infty}(\mathbb{T}^3)} \leq C_0 R_t^{\frac{4}{3}},$$

where C_0 depends on $f(0)$ only. Substituting this into (2.44) results in the estimate

$$R_t \leq R_0 + C_0 \int_0^t (1 + R_s^{\frac{4}{3}}) \, ds. \quad (2.46)$$

Since the exponent $4/3$ is greater than one, the integral inequality (2.46) cannot be used to obtain an estimate on R_t for large t .

The strategy of Batt and Rein [10] works by improving the exponent in the estimate (2.45) using a bootstrap argument. As in [10], we introduce the quantities

$$\begin{aligned} h_\rho(t) &:= \sup\{\|\rho_f(s, \cdot)\|_{L^\infty(\mathbb{T}^3)}; 0 \leq s \leq t\} \\ h_\eta(t, \Delta) &:= \sup\{|V(t_1, \tau, x, v) - V(t_2, \tau, x, v)|; 0 \leq t_1, t_2, \tau \leq t, |t_1 - t_2| \leq \Delta, (x, v) \in \mathbb{T}^3 \times \mathbb{R}^3\}. \end{aligned} \quad (2.47)$$

In [10], Batt and Rein prove a technical lemma, which we state below as Lemma 2.23. Roughly speaking, this shows that, if an inequality of the form

$$h_\eta(t, \Delta) \leq C_* h_\rho(t)^\beta \Delta \quad (2.48)$$

holds for some exponent $\beta > \frac{1}{6}$, then, along any characteristic trajectory, an improved estimate holds on \bar{E} :

$$|\bar{E}(X(t, \tau, x, v))| \leq C h_\rho(t)^{\beta'}, \quad (2.49)$$

for a new exponent β' satisfying $\frac{1}{6} \leq \beta' < \beta$. Notice for instance that (2.45) implies that (2.48) holds for the choice $\beta = \frac{4}{9}$. This can then be used in a bootstrap argument, because (2.43) and (2.49) together imply that an estimate of the form (2.48) holds for a smaller value of β . Once β is sufficiently small, then it is possible to prove a version of (2.46) which can be closed to deduce an estimate on R_t .

Lemma 2.23. *Let $(X(t; s, x, v), V(t; s, x, v))$ denote the solution at time t of an ODE*

$$\begin{pmatrix} \dot{X}(t) \\ \dot{V}(t) \end{pmatrix} = a(t, X(t), V(t)), \quad \begin{pmatrix} X(s) \\ V(s) \end{pmatrix} = \begin{pmatrix} x \\ v \end{pmatrix},$$

where a is of the form

$$a(t, X, V) = \begin{pmatrix} V \\ a_2(t, X, V) \end{pmatrix},$$

for some vector-field $a_2 : [0, T] \times \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

Assume that, for $t \in [0, T]$, $f = f(t, x, v)$ is the pushforward of f_0 along the associated characteristic flow; that is, for all $\phi \in C_b(\mathbb{T}^3 \times \mathbb{R}^3)$,

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, x, v) \phi(x, v) dx dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(s, x, v) \phi(X(t; s, x, v), V(t; s, x, v)) dx dv.$$

Assume that there exists $C_* > 1$ such that

$$\|f\|_{L^\infty([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C_*, \quad \sup_{t \in [0,T]} \|f_t |v|^2\|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C_*.$$

Also, suppose that

$$h_\eta(t, \Delta) \leq C_* h_\rho(t)^\beta \Delta \quad \text{for all } h_\rho(t)^{-\beta} \leq \Delta \leq t, \quad (2.50)$$

where h_ρ, h_η are defined as in (2.47). Then for all $0 \leq t_1 < t_2 \leq t$ with $t_2 - t_1 \leq \Delta$, if

$$h_\rho(t)^{-\beta/2} \leq \Delta \leq t$$

then

$$\int_{t_1}^{t_2} \int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_f(s, y) dy ds \leq C \left(h_\rho(t)^{2\beta/3} + h_\rho(t)^{1/6} \right) \Delta,$$

where C depends only on C_* .

Using this technical lemma, we now carry out the bootstrap strategy outlined above.

Proof of Proposition 2.22. Using the representation of the Coulomb kernel on the torus K_{per} discussed in Section 1.3.3, the total force E has the representation

$$E(t, x) = C \int_{\mathbb{T}^3} \frac{x - y}{|x - y|^3} \rho_f(t, y) dy + [K_0 * (\rho_f - 1)](t, x) + \widehat{E}(t, x),$$

where K_0 is a smooth function. Fix a characteristic trajectory $(X(t), V(t))$. Along the trajectory we have an estimate

$$\begin{aligned} |V(t_2) - V(t_1)| &\leq C \int_{t_1}^{t_2} \int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_f(s, y) dy ds \\ &\quad + C \int_{t_1}^{t_2} |K_0 * \rho_f(s, X(s)) - 1| ds + \int_{t_1}^{t_2} |\widehat{E}(s, X(s))| ds. \end{aligned} \quad (2.51)$$

Since K_0 is a $C^1(\mathbb{T}^3)$ function, we have

$$|K_0 * \rho_f(s, \cdot) - 1| \leq \|K_0\|_{L^\infty(\mathbb{T}^3)} \|\rho_f(s, \cdot) - 1\|_{L^1(\mathbb{T}^3)} \leq \|K_0\|_{L^\infty(\mathbb{T}^3)}, \quad (2.52)$$

where the last inequality follows from conservation of mass.

For the smooth part of the field, we use Proposition 2.4 and Lemma 1.12 to get

$$\|\widehat{E}(s, \cdot)\|_{L^\infty(\mathbb{T}^3)} \leq \|\widehat{U}(s, \cdot)\|_{C^{1,\alpha}(\mathbb{T}^3)} \leq C_d \exp \left(C_d \left(1 + \|\rho_f(s, \cdot)\|_{L^{\frac{5}{3}}(\mathbb{T}^3)} \right) \right) \leq C, \quad (2.53)$$

for some C depending on C_0 .

Now, in order to apply Lemma 2.23, we need to ensure that (2.50) holds for some $\beta > 0$. To do that, we rely on the following estimate from [50, Lemma 4.5.4]:

$$\int_{\mathbb{T}^3} |x - y|^{-2} \rho_f(s, y) dy \leq C \|\rho_f(s, \cdot)\|_{L^{5/3}}^{5/9} \|\rho_f(s, \cdot)\|_{L^\infty(\mathbb{T}^3)}^{4/9} \leq C \|\rho_f(s, \cdot)\|_{L^\infty(\mathbb{T}^3)}^{4/9}, \quad (2.54)$$

where the second inequality follows from (2.4). Recalling the definition of h_η and h_ρ , we may combine (2.54), (2.52) and (2.53) with (2.51), to deduce that (2.50) holds with $\beta = \frac{4}{9}$, provided that $h_\rho(t)^{-2/9} \leq \Delta \leq t$. This allows us to apply Lemma 2.23 to obtain a better control on the term $\int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_f(s, y) dy ds$. Using (2.51) again we get

$$h_\eta(t, \Delta) \leq C \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{4}{9}} + h_\rho(t)^{1/6} \right) \Delta + C\Delta \leq Ch_\rho(t)^{\frac{8}{27}} \Delta, \quad \text{if } h_\rho(t)^{-2/9} \leq \Delta \leq t,$$

where we used that $h_\rho \geq 1$ (since $\|\rho_f\|_{L^1(\mathbb{T}^d)} = 1$).

This implies that (2.50) holds with $\beta = \frac{8}{27}$, so we may reapply Lemma 2.23 to obtain a even better control on the term $\int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_f(s, y) dy ds$. Using (2.51) again we get

$$h_\eta(t, \Delta) \leq C \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{8}{27}} + h_\rho(t)^{1/6} \right) \Delta + C\Delta \leq Ch_\rho(t)^{\frac{16}{81}} \Delta \quad \text{if } h_\rho(t)^{-4/27} \leq \Delta \leq t.$$

After one more iteration, we get

$$h_\eta(t, \Delta) \leq C \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{16}{81}} + h_\rho(t)^{1/6} \right) \Delta + C\Delta \leq C \left(h_\rho(t)^{\frac{32}{243}} + h_\rho(t)^{1/6} \right) \Delta \leq Ch_\rho(t)^{1/6} \Delta, \quad (2.55)$$

provided that $h_\rho(t)^{-8/81} \leq \Delta \leq t$.

Thus if the support of $f(0)$ is contained in $\mathbb{T}^3 \times B_{\mathbb{R}^3}(0, R_0)$, then arguing as in (2.38) for $d = 3$ we have

$$h_\rho(t) \leq C \|f(0)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} (R_0 + h_\eta(t))^3.$$

If $t \geq h_\rho(t)^{-8/81}$, it follows by (2.55) and the Cauchy-Schwarz inequality that

$$h_\rho(t) \leq C \left(R_0 + h_\rho(t)^{1/6} t \right)^3 \leq CR_0^3 + Ch_\rho(t)^{1/2} t^3 \leq CR_0^3 + C \frac{t^6 + h_\rho(t)}{2}$$

and therefore

$$h_\rho(t) \leq C(R_0^3 + t^6).$$

On the other hand, for $t \leq h_\rho(t)^{-8/81}$ we trivially have

$$h_\rho(t) \leq t^{-81/8}.$$

Combining these two bounds together we obtain, as desired,

$$h_\rho(t) \leq \max\{t^{-81/8}, C(R_0^3 + t^6)\}.$$

□

2.8 A Regularised System

Next, we turn to the construction of solutions for the VPME system. We begin by considering a regularised system. We will be able to prove the existence of solutions for this system using a variation on the theory for Vlasov equations with smooth interaction that we discussed in Section 1.3.2.

In Section 2.9, we will then construct solutions of the VPME system by proving compactness of the solutions of the regularised system, using the mass density estimates from Section 2.7. After extracting a convergent subsequence, we will prove that the limit is a solution of VPME.

To introduce our regularisation, we define a scaled mollifier χ_r by letting

$$\chi_r(x) = r^{-d} \chi\left(\frac{x}{r}\right). \quad (2.56)$$

Here $\chi : \mathbb{R}^d \rightarrow \mathbb{R}$ is a fixed smooth, compactly supported and (therefore) bounded function. We assume further that χ is radially symmetric, non-negative and has total mass 1. We then consider the following regularised system:

$$\left\{ \begin{array}{l} \partial_t f_r + v \cdot \nabla_x f_r + E_r[f_r] \cdot \nabla_v f_r = 0, \\ E_r = -\chi_r * \nabla U_r, \\ \Delta U_r = e^{U_r} - \chi_r * \rho[f_r], \\ f_r|_{t=0} = f(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f(0) dx dv = 1. \end{array} \right. \quad (2.57)$$

We regularise the ion density but not the electron density. This is a slightly different approach from that of Bouchut [17], where both densities are regularised. The idea is that the thermalisation assumption should lead to a regularising effect. This choice is further motivated in part by our later study of the mean field limit in Chapter 4, where we will use a microscopic system of this form.

We introduce the decomposition

$$E_r = \bar{E}_r + \hat{E}_r,$$

where

$$\bar{E}_r = -\chi_r * \nabla \bar{U}_r, \quad \hat{E}_r = -\chi_r * \nabla \hat{U}_r,$$

with \bar{U}_r, \hat{U}_r satisfying

$$\Delta \bar{U}_r = 1 - \chi_r * \rho[f_r], \quad \Delta \hat{U}_r = e^{\bar{U}_r + \hat{U}_r} - 1.$$

Notice that we are using a technique of ‘double regularisation’; for instance, if we define the ‘singular’ part of the electric field to be $\bar{E}_r = -\chi_r * \nabla \bar{U}_r$, where

$$\Delta \bar{U}_r = 1 - \chi_r * \rho[f_r],$$

then \bar{E}_r can be represented in the form

$$\bar{E}_r = \chi_r * \chi_r * K_{\text{per}} * \rho[f_r].$$

This type of regularisation appeared in the work of Horst [53], and has subsequently been used in many other contexts. An advantage of this approach is that the system (2.57) has an associated conserved energy, defined by

$$\mathcal{E}_r^{\text{ME}}[f] := \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla U_r|^2 \, dx + \int_{\mathbb{T}^d} U_r e^{U_r} \, dx. \quad (2.58)$$

If f_r converges to some f sufficiently strongly as r tends to zero, then we would expect $\mathcal{E}_r^{\text{ME}}[f_r]$ to converge to $\mathcal{E}^{\text{ME}}[f]$, where \mathcal{E}^{ME} is the energy of the original VPME system, defined in (2.2).

In the regularised system, the force E_r will be smooth. It is therefore possible to construct solutions for this system using standard methods for Vlasov equations with smooth interactions, following for example [18, 27, 31, 72]. The standard results cannot be applied directly since the force is not of convolution type, but the method can be adapted to our case.

Lemma 2.24 (Existence of regularised solutions). *For every $f(0) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, there exists a unique solution $f_r \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ of (2.57). If $f(0) \in L^p(\mathbb{T}^d \times \mathbb{R}^d)$ for some $p \in [1, \infty]$, then for all $t \in [0, \infty)$,*

$$\|f_r(t)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq \|f(0)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}.$$

Proof. We sketch the proof, which is a modification of the methods described in Section 1.3.2 in order to handle the extra term in the electric field. First consider the linear problem for fixed

$\mu \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$:

$$\begin{cases} \partial_t g_r^{(\mu)} + v \cdot \nabla_x g_r^{(\mu)} + E_r[\mu] \cdot \nabla_v g_r^{(\mu)} = 0, \\ E_r^{(\mu)} = -\chi_r * \nabla U_r^{(\mu)}, \\ \Delta U_r^{(\mu)} = e^{U_r^{(\mu)}} - \chi_r * \rho[\mu], \\ g_r^{(\mu)}|_{t=0} = f(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f(0) (dx dv) = 1, \end{cases} \quad (2.59)$$

for $f(0) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$.

Observe that, for any probability measure μ , $\chi_r * \rho[\mu]$ is a function satisfying

$$|\chi_r * \rho[\mu]| \leq \|\chi_r\|_{L^\infty(\mathbb{T}^d)}.$$

This is shown in the proof of Lemma 1.1. Then by Proposition 2.4,

$$\|U_r^{(\mu)}\|_{C^1(\mathbb{T}^d)} \leq \exp \left[C_d \left(1 + \|\chi_r\|_{L^\infty(\mathbb{T}^d)} \right) \right],$$

and hence $E_r^{(\mu)} = \chi_r * \nabla U_r^{(\mu)}$ is of class $C^1(\mathbb{T}^d)$, with the uniform-in-time estimate

$$\|E_r^{(\mu)}\|_{C^1(\mathbb{T}^d)} \leq \|\chi_r\|_{C^1(\mathbb{T}^d)} \|\nabla U_r^{(\mu)}\|_{C(\mathbb{T}^d)} \leq \|\chi_r\|_{C^1(\mathbb{T}^d)} \exp \left[C_d \left(1 + \|\chi_r\|_{L^\infty(\mathbb{T}^d)} \right) \right] \leq C_{r,d}. \quad (2.60)$$

This implies the existence of a unique global-in-time C^1 characteristic flow. Using this flow we may construct a unique solution $g_r^{(\mu)} \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ to the linear problem (2.59) by the method of characteristics. Since the vector field (v, E_r) is divergence free, this solution conserves $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ norms for $p \in [1, +\infty]$.

To prove that the nonlinear equation has a unique solution in $C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$, we adapt the methods presented for smooth Vlasov equations in Section 1.3. We have discussed in Subsection 1.3.2.5 that it is enough to show that the electric field $E_r^{(\mu)}$ is Lipschitz and has a stability property in W_1 with respect to μ :

$$\|E_r^{(\mu)}\|_{\text{Lip}} \leq C_r \quad (2.61)$$

$$\|E_r^{(\mu)} - E_r^{(\nu)}\|_{L^\infty(\mathbb{T}^d)} \leq C_r W_1(\mu, \nu). \quad (2.62)$$

The Lipschitz regularity (2.61) holds by (2.60). For the stability (2.62), once again we use the decomposition $E_r^{(\mu)} = \bar{E}_r^{(\mu)} + \hat{E}_r^{(\mu)}$. First,

$$\bar{E}_r^{(\mu)} = -\chi_r * \nabla \bar{U}_r^{(\mu)} = \chi_r * K_{\text{per}} * \chi_r * \rho[\mu],$$

where K_{per} is the Coulomb kernel on the torus (recall the definitions in Subsection 1.3.3). This is a force of convolution type. The kernel is Lipschitz since $K_{\text{per}} \in L^1(\mathbb{T}^d)$ and χ_r is smooth. We can therefore use the stability estimate of Lemma 1.10. It remains to verify stability of \widehat{E}_r with respect to μ .

Consider two continuous paths of probability measures $\mu, \nu \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$. First note that by Young's inequality,

$$\|\widehat{E}_r^{(\mu)} - \widehat{E}_r^{(\nu)}\|_{L^\infty(\mathbb{T}^d)} = \|\chi_r * (\nabla \widehat{U}_r^{(\mu)} - \nabla \widehat{U}_r^{(\nu)})\|_{L^\infty(\mathbb{T}^d)} \leq \|\chi_r\|_{L^2(\mathbb{T}^d)} \|\nabla \widehat{U}_r^{(\mu)} - \nabla \widehat{U}_r^{(\nu)}\|_{L^2(\mathbb{T}^d)}.$$

By the L^2 stability estimate from Proposition 2.11,

$$\begin{aligned} \|\nabla \widehat{U}_r^{(\mu)} - \nabla \widehat{U}_r^{(\nu)}\|_{L^2(\mathbb{T}^d)} &\leq \exp \left[C \left(\max_{\gamma \in \{\mu, \nu\}} \|\bar{U}_r^{(\gamma)}\|_{L^\infty(\mathbb{T}^d)} + \max_{\gamma \in \{\mu, \nu\}} \|\widehat{U}_r^{(\gamma)}\|_{L^\infty(\mathbb{T}^d)} \right) \right] \\ &\quad \times \|\bar{U}_r^{(\mu)} - \bar{U}_r^{(\nu)}\|_{L^2(\mathbb{T}^d)}. \end{aligned}$$

By Proposition 2.4,

$$\max_{\gamma \in \{\mu, \nu\}} \|\bar{U}_r^{(\gamma)}\|_{L^\infty(\mathbb{T}^d)} + \max_{\gamma \in \{\mu, \nu\}} \|\widehat{U}_r^{(\gamma)}\|_{L^\infty(\mathbb{T}^d)} \leq \exp \left[C_d \left(1 + \|\chi_r\|_{L^\infty(\mathbb{T}^d)} \right) \right].$$

Hence

$$\begin{aligned} \|\nabla \widehat{U}_r^{(\mu)} - \nabla \widehat{U}_r^{(\nu)}\|_{L^2(\mathbb{T}^d)} &\leq C_{r,d} \|\bar{U}_r^{(\mu)} - \bar{U}_r^{(\nu)}\|_{L^2(\mathbb{T}^d)} \\ &\leq C_{r,d} \|\bar{U}_r^{(\mu)} - \bar{U}_r^{(\nu)}\|_{L^\infty(\mathbb{T}^d)} = C_{r,d} \|\chi_r *_{\chi} G *_{\chi} (\rho[\mu] - \rho[\nu])\|_{L^\infty(\mathbb{T}^d)}. \end{aligned}$$

Note that $\chi_r *_{\chi} G$ is smooth and hence Lipschitz. We can therefore apply Lemma 1.10. Explicitly, by Kantorovich duality for the W_1 distance we have

$$W_1(\rho_\mu, \rho_\nu) = \sup_{\|\phi\|_{\text{Lip}} \leq 1} \left\{ \int_{\mathbb{T}^d} \phi \, d\rho_\mu - \int_{\mathbb{T}^d} \phi \, d\rho_\nu \right\}.$$

Thus for any $x \in \mathbb{T}^d$

$$\begin{aligned} \chi_r *_{\chi} G *_{\chi} (\rho_\mu - \rho_\nu)(x) &= \int_{\mathbb{T}^d} [\chi_r *_{\chi} G](x - y) \, d(\rho_\mu - \rho_\nu)(y) \\ &\leq \|\chi_r *_{\chi} G(x - \cdot)\|_{\text{Lip}} W_1(\rho_\mu, \rho_\nu) \leq C_{r,d} W_1(\rho_\mu, \rho_\nu), \end{aligned}$$

where $C_{r,d}$ is independent of x . Hence

$$\|\chi_r * G * (\rho_\mu - \rho_\nu)\|_{L^\infty(\mathbb{T}^d)} \leq C_{r,d} W_1(\rho_\mu, \rho_\nu).$$

We conclude that

$$\|\chi_r * (\nabla \widehat{U}_r^{(\mu)} - \nabla \widehat{U}_r^{(\nu)})\|_{L^\infty(\mathbb{T}^d)} \leq C_{r,d} W_1(\rho_\mu, \rho_\nu) \leq C_{r,d} W_1(\mu, \nu),$$

which shows that (2.62) holds.

The methods of proof of Theorem 1.2 (existence of solutions) and Theorem 1.5 (uniqueness of solutions) can therefore be adapted to this case. This proves the existence of a unique solution $f_r \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ for the nonlinear regularised equation (2.57).

This solution also preserves all $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ norms, since it is the solution of the linear transport equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + E_r[f_r] \cdot \nabla_v g = 0, \\ g|_{t=0} = f(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f(0) dx dv = 1, \end{cases}$$

and $(v, E_r[f_r])$ is a divergence-free C^1 vector field.

□

2.9 Construction of Solutions

In this section, we show that the approximate solutions f_r converge to a limit as r tends to zero, and that this limit may be identified as the unique bounded density solution of (2.1) with data $f(0)$. In the following lemma, we collect together some useful uniform estimates for the approximate solutions f_r .

Lemma 2.25. *Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported. For each $r > 0$, let f_r denote the solution of (2.57) with initial datum $f(0)$. Then f_r have the following properties:*

(i) *L^p bounds: for all $p \in [1, +\infty]$,*

$$\sup_{r>0} \sup_{t \in [0, T]} \|f_r(t)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq \|f(0)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}. \quad (2.63)$$

(ii) *Moment bounds:*

$$\sup_{r>0} \sup_{t \in [0, T]} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t, x, v) dx dv \leq C[f(0)].$$

Proof. Property (i) was proved in Lemma 2.24. Property (ii) is a consequence of the conservation of the energy functional $\mathcal{E}_r^{\text{ME}}[f_r]$ defined by (2.58). First we must check that $\mathcal{E}_r^{\text{ME}}[f(0)]$ is bounded uniformly in r . Since $f(0) \in L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ with compact support, we have $\rho_0 := \rho[f(0)] \in L^\infty(\mathbb{T}^d)$. Since $\|\chi_r * \rho_0\|_{L^\infty(\mathbb{T}^d)} \leq \|\rho_0\|_{L^\infty(\mathbb{T}^d)}$, by Proposition 2.4 we have for any $\alpha \in (0, 1)$

$$\|U_r(0)\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C\left(\alpha, \|\rho_0\|_{L^\infty(\mathbb{T}^d)}\right).$$

Moreover $\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(0, x, v) \, dx \, dv$ is finite since $f(0)$ is compactly supported. Therefore there exists C_0 depending on $f(0)$ such that

$$\sup_{r>0} \sup_{t \in [0, T]} \mathcal{E}_r^{\text{ME}}[f_r(t)] \leq C_0. \quad (2.64)$$

Note that $xe^x \geq -e^{-1}$. This implies that for all r and all $t \in [0, T]$,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t, x, v) \, dx \, dv \leq C, \quad (2.65)$$

which completes the proof of Property (ii). \square

Proposition 2.26 (Density bounds). *Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported. For each $r > 0$, let f_r denote the solution of (2.57) with initial datum $f(0)$. Fix $T > 0$. Then, for all $r \in [0, 1]$ the mass density $\rho[f_r]$ satisfies the following bounds, which are uniform with respect to r :*

$$\sup_{r>0} \|\rho[f_r(t)]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C[f(0)], \quad (2.66)$$

$$\sup_{r>0} \|\rho[f_r(t)]\|_{L^\infty(\mathbb{T}^d)} \leq C[T, f(0)]. \quad (2.67)$$

Proof. The $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ bound follows from the moment bound (2.65) and the uniform $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ bound (2.63), after applying the interpolation estimate Lemma 1.12. For the $L^\infty(\mathbb{T}^d)$ bound we use the estimates on the growth of the support from Propositions 2.19 and 2.22, which are valid for solutions of the regularised equation, with the same constants. We give the details in Lemmas 2.27, 2.28 and 2.29 below. \square

To prove L^∞ bounds on the mass densities $\rho[f_r]$ that are uniform in r , we revisit the calculations in Section 2.7, and show that they can be adapted to the regularised system. We recall that we used different methods in the two cases $d = 2$ and $d = 3$. In both cases, we make

use of the decomposition

$$E_r = \bar{E}_r + \hat{E}_r.$$

This results in the estimate

$$|E_r(x)| \leq C_d \left| \left(\frac{1}{|\cdot|^{d-1}} \right) * \chi_r * \chi_r * \rho[f_r](x) \right| + |K_0 * \chi_r * \chi_r * \rho[f_r](x)| + |\hat{E}_r(x)|. \quad (2.68)$$

Note firstly that, since $K_0 \in C^\infty(\mathbb{T}^d)$, by Young's inequality,

$$\begin{aligned} \|K_0 * \chi_r * \chi_r * \rho[f_r]\|_{L^\infty(\mathbb{T}^d)} &\leq \|K_0\|_{L^\infty(\mathbb{T}^d)} \|\chi_r * \chi_r * \rho[f_r]\|_{L^1(\mathbb{T}^d)} \\ &\leq \|K_0\|_{L^\infty(\mathbb{T}^d)}, \end{aligned}$$

using the fact that $\rho[f_r]$ and χ_r have total mass one.

Secondly, by Proposition 2.4, for $\alpha \in (0, 1)$, for all $r > 0$,

$$\|\hat{U}_r\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C_{\alpha,d} \exp \left(C_{\alpha,d} \left(1 + \|\chi_r * \rho[f_r]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right).$$

Thus

$$\begin{aligned} \|\hat{E}_r\|_{L^\infty(\mathbb{T}^d)} &= \|\chi_r * \nabla \hat{U}_r\|_{L^\infty(\mathbb{T}^d)} \leq \|\nabla \hat{U}_r\|_{L^\infty(\mathbb{T}^d)} \\ &\leq C_d \exp \left(C_d \left(1 + \|\chi_r * \rho[f_r]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right) \\ &\leq C_d \exp \left(C_d \left(1 + \|\rho[f_r]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right) \leq C(d, f(0)), \end{aligned}$$

where the final inequality uses the bound from (2.66):

$$\sup_{r>0} \|\rho[f_r(t)]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C[f(0)].$$

It remains to estimate the term

$$\left| \left(\frac{1}{|\cdot|^{d-1}} \right) * \chi_r * \chi_r * \rho[f_r](x) \right|.$$

To do this, we revisit the estimates of Section 2.7 and show that they apply when the regularisation by convolution is included. Since f_r satisfies the regularised system (2.57), we will be able to show that these bounds hold rigorously rather than being a priori estimates. We begin by showing that the regularised kernel is comparable to the non-regularised one.

Lemma 2.27 (Bounds on the regularised kernel). *Let $d > 1$ and let χ_r be defined by (2.56) for some fixed $\chi \in C_c(\mathbb{R}^d)$. Then there exists a constant $C(d, \chi) > 0$, independent of r , such that, for all $x \in [-\frac{1}{2}, \frac{1}{2}]^d$,*

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{\chi_r(x-y)}{|y|^{d-1}} dy \leq \frac{C(d, \chi)}{|x|^{d-1}}.$$

Proof. For each $x \in [-\frac{1}{2}, \frac{1}{2}]^d$, consider the function

$$|x|^{d-1} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{\chi_r(x-y)}{|y|^{d-1}} dy.$$

There exists a constant $C_d > 0$ such that, for all $x, y \in \mathbb{R}^d$,

$$|x|^{d-1} \leq C_d \left(|y|^{d-1} + |x-y|^{d-1} \right).$$

Thus

$$|x|^{d-1} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{\chi_r(x-y)}{|y|^{d-1}} dy \leq \int_{[-\frac{1}{2}, \frac{1}{2}]^d} |\chi_r(x-y)| dy + \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{|x-y|^{d-1}}{|y|^{d-1}} |\chi_r(x-y)| dy. \quad (2.69)$$

Note that

$$\int_{x+[-\frac{1}{2}, \frac{1}{2}]^d} |\chi_r(y)| dy = r^{-d} \int_{x+[-\frac{1}{2}, \frac{1}{2}]^d} \left| \chi \left(\frac{y}{r} \right) \right| dy \leq \int_{\mathbb{R}^d} |\chi(y)| dy = C(\chi).$$

Split the second term of (2.69) as follows: for any $L > 0$,

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{|x-y|^{d-1}}{|y|^{d-1}} |\chi_r(x-y)| dy \leq \int_{|y| \leq L} \frac{|x-y|^{d-1}}{|y|^{d-1}} |\chi_r(x-y)| dy + \int_{2 \geq |y| > L} \frac{|x-y|^{d-1}}{|y|^{d-1}} |\chi_r(x-y)| dy.$$

The first term is estimated by

$$\int_{|y| \leq L} \frac{|x-y|^{d-1}}{|y|^{d-1}} |\chi_r(y)| dy \leq \| |\cdot|^{d-1} \chi_r \|_{L^\infty(\mathbb{R}^d)} \int_{|y| \leq L} \frac{1}{|y|^{d-1}} dy \leq C_d L \| |\cdot|^{d-1} \chi_r \|_{L^\infty(\mathbb{R}^d)}.$$

Observe that

$$|x|^{d-1} |\chi_r(x)| = r^{-1} \left| \frac{x}{r} \right|^{d-1} \chi \left(\frac{x}{r} \right) \leq r^{-1} \| |\cdot|^{d-1} \chi \|_{L^\infty(\mathbb{R}^d)} \leq C(\chi) r^{-1}.$$

The second term is estimated by

$$\int_{2 \geq |y| > L} \frac{|x-y|^{d-1}}{|y|^{d-1}} |\chi_r(x-y)| dy \leq L^{1-d} \| |\cdot|^{d-1} \chi_r \|_{L^1(\mathbb{R}^d)}.$$

For the constant, we find that

$$\begin{aligned} \| |\cdot|^{d-1} \chi_r \|_{L^1(\mathbb{R}^d)} &= r^{-d} \int_{\mathbb{R}^d} |x|^{d-1} \left| \chi\left(\frac{x}{r}\right) \right| dx \\ &= r^{d-1} \int_{\mathbb{R}^d} |x|^{d-1} \chi(x) dx \\ &\leq C(\chi) r^{d-1}. \end{aligned}$$

Altogether this gives

$$|x|^{d-1} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{\chi_r(x-y)}{|y|^{d-1}} dy \leq C(\chi) \left[1 + r^{-1} (C_d L + L^{1-d} r^d) \right].$$

Minimising over L , the optimal value is $L = C_d r$. Then

$$|x|^{d-1} \int_{[-\frac{1}{2}, \frac{1}{2}]^d} \frac{\chi_r(x-y)}{|y|^{d-1}} dy \leq C(d, \chi).$$

This completes the proof. \square

Remark 3. Lemma 2.27 also applies in the case of the double regularisation $\chi_r * \chi_r$, since

$$\chi_r * \chi_r(rx) = r^{-2d} \int_{\mathbb{R}^d} \chi\left(x - \frac{y}{r}\right) \chi\left(\frac{y}{r}\right) dy = r^{-d} \int_{\mathbb{R}^d} \chi(x-y) \chi(y) dy = r^{-d} \chi * \chi(x).$$

Lemma 2.28 (Mass bounds, $d = 2$). *Let $d = 2$. Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)$ have compact support contained in $\mathbb{T}^2 \times B_{\mathbb{R}^2}(0, R_0)$. For each $r > 0$, let f_r denote the unique solution of (2.57) in the space $C([0, +\infty); \mathcal{P}(\mathbb{T}^2 \times \mathbb{R}^2))$ with initial datum $f(0)$.*

Then $f_r(t)$ is compactly supported in $\mathbb{T}^2 \times \mathbb{R}^2$ for all $t \in [0, \infty)$. Moreover, the mass density $\rho[f_r](t)$ lies in the space $L^\infty(\mathbb{T}^2)$, with the bound

$$\sup_{r \in [0, 1]} \sup_{t \in [0, T]} \|\rho[f_r](t)\|_{L^\infty(\mathbb{T}^2)} \leq C_T (1 + R_0)^2.$$

Proof. We discussed in the proof of Lemma 4.7 that $E_r = E_r[f_r]$ is a $C^1(\mathbb{T}^2)$ function for all t , with the estimate (2.60), which is uniform in $t \geq 0$ but not in r .

The characteristic system of system (2.57) is the following ODE for paths $t \mapsto Z_r(t; s, z)$, where $Z_r(t; s, z) = (X_r(t; \tau, z), V_r(t; \tau, z))$:

$$\begin{cases} \dot{X}_r(t; \tau, z) = V_r(t; \tau, z) \\ \dot{V}_r(t; \tau, z) = E_r[t, X_r(t; \tau, z)] \\ Z_r(\tau; \tau, z) = z = (x, v). \end{cases} \quad (2.70)$$

This system therefore has a unique global-in-time solution for all $s \in \mathbb{R}$, $z \in \mathbb{T}^2 \times \mathbb{R}^2$.

For any characteristic trajectory $Z_r(t; 0, z)$, the velocity component satisfies

$$|V_r(t; 0, z)| \leq |v| + \int_0^t \|E_r(s)\|_{L^\infty(\mathbb{T}^2)} ds.$$

By estimate (2.60) and the fact that $f(0)$ is compactly supported,

$$|V_r(t; 0, z)| \leq R_0 + C_r t.$$

Thus $f_r(t)$ is compactly supported for all $t \geq 0$, with the following preliminary bound on R_t :

$$R_t \leq R_0 + C_r t.$$

Then the mass density satisfies

$$\|\rho[f_r](t)\|_{L^\infty(\mathbb{T}^2)} \leq C(R_0 + C_r t)^2.$$

We now seek a bound that is uniform in r . By (2.71) and the decomposition (2.68) of E_r ,

$$|V_r(t; 0, z)| \leq |v| + C(f_0)t + \int_0^t \left| \left(\frac{1}{|\cdot|} \right) * \chi_r * \chi_r * \rho[f_r](s, X_r(s; 0, z)) \right| ds.$$

By Lemma 2.27,

$$|V_r(t; 0, z)| \leq |v| + C(f_0)t + \int_0^t \left\| \int_{\mathbb{T}^2} \frac{\rho[f_r](s, \cdot - y)}{|y|} dy \right\|_{L^\infty(\mathbb{T}^2)} ds.$$

By Lemma 2.20,

$$|V_r(t; 0, z)| \leq |v| + Ct + C \int_0^t \left(1 + (\log(1 + R_s))^{1/2} \right) ds,$$

where the constant C depends only on $f(0)$ and is independent of r .

As in Proposition 2.19, we deduce that

$$R_t \leq (1 + 2Ct) [R_0 + \log(1 + 2Ct)],$$

and conclude that

$$\sup_{t \in [0, T]} \|\rho[f_r](t)\|_{L^\infty(\mathbb{T}^2)} \leq C_T(1 + R_0)^2.$$

□

Lemma 2.29 (Mass bounds, $d = 3$). *Let $d = 3$. Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ have compact support contained in $\mathbb{T}^3 \times B_{\mathbb{R}^3}(0, R_0)$. For each $r > 0$, let f_r denote the unique solution of (2.57) in the space $C([0, +\infty); \mathcal{P}(\mathbb{T}^3 \times \mathbb{R}^3))$ with initial datum $f(0)$.*

Then $f_r(t)$ is compactly supported in $\mathbb{T}^3 \times \mathbb{R}^3$ for all $t \in [0, +\infty)$. Moreover, the mass density $\rho[f_r](t)$ lies in the space $L^\infty(\mathbb{T}^3)$, with the bound

$$\sup_{r > 0} \sup_{t \in [0, T]} \|\rho[f_r](t)\|_{L^\infty(\mathbb{T}^3)} \leq \max\{T^{-81/8}, C(R_0^3 + T^6)\}.$$

Proof. By (2.60), the characteristic system (2.70) is well-posed for all $s \in \mathbb{R}$, $z \in \mathbb{T}^3 \times \mathbb{R}^3$. We define the quantities

$$\begin{aligned} h_\rho^{(r)}(t) &:= \sup\{\|\rho[f_r](s)\|_{L^\infty(\mathbb{T}^3)}; 0 \leq s \leq t\} \\ h_\eta^{(r)}(t, \Delta) &:= \sup\{|V_r(t_1, \tau, z) - V_r(t_2, \tau, z)|; 0 \leq t_1, t_2, \tau \leq t, |t_1 - t_2| \leq \Delta, z \in \mathbb{T}^3 \times \mathbb{R}^3\}. \end{aligned}$$

We check that these quantities are well defined. By the uniform C^1 estimate (2.60) on E_r ,

$$|V_r(t_1; \tau, z) - V_r(t_2; \tau, z)| \leq \int_{t_1}^{t_2} |E_r[s, X(s; \tau, z)]| \quad (2.71)$$

$$\leq |t_2 - t_1| \|E_r\|_{L^\infty(\mathbb{T}^3)} \leq C_r |t_2 - t_1|. \quad (2.72)$$

Thus $h_\eta^{(r)}(t, \Delta) \leq C_r \Delta$. Moreover

$$\begin{aligned} |\rho[f_r](t, x)| &\leq \int_{\text{supp } f_r(t)} f_r(t, x, v) \, dv, \\ &\leq \|f_r(t)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} (R_0 + C_r t)^3. \end{aligned}$$

By the uniform $L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)$ bounds (2.63),

$$\sup_{t \in [0, T]} \|\rho[f_r](t)\|_{L^\infty(\mathbb{T}^3)} \leq \|f(0)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)} (R_0 + C_r t)^3. \quad (2.73)$$

By estimates (2.72) and (2.73), both $h_\rho^{(r)}(t)$ and $h_\eta^{(r)}(t, \Delta)$ are finite for all $t \geq 0$ and $\Delta \in [0, t]$.

We now apply the bootstrap scheme from the proof of Proposition 2.22, using the estimates of Batt and Rein (Lemma 3.13). We use this to obtain an estimate on $h_\rho^{(r)}$ that is independent of r .

By (2.71) and the decomposition (2.68) of E_r ,

$$|V_r(t_1; \tau, z) - V_r(t_2; \tau, z)| \leq C(f_0)|t_2 - t_1| + \int_{t_1}^{t_2} \left| \left(\frac{1}{|\cdot|^2} \right) * \chi_r * \chi_r * \rho[f_r](s, X_r(s; \tau, z)) \right| ds.$$

By Lemma 2.27,

$$|V_r(t_1; \tau, z) - V_r(t_2; \tau, z)| \leq C(f_0)|t_2 - t_1| + C(\chi) \int_{t_1}^{t_2} \int_{\mathbb{T}^3} \frac{\rho[f_r](s, y)}{|X_r(s; \tau, z) - y|^2} dy ds. \quad (2.74)$$

We then apply the bootstrap argument from Proposition 2.22. The estimate (2.74) plays the role of estimate (2.51). Since

$$\int_{\mathbb{T}^3} \frac{\rho[f_r](s, y)}{|x - y|^2} dy \leq C \|\rho[f_r]\|_{L^{\frac{5}{3}}(\mathbb{T}^3)}^{\frac{5}{9}} \|\rho[f_r]\|_{L^\infty(\mathbb{T}^3)}^{\frac{4}{9}},$$

by (2.66) we have

$$h_\eta^{(r)}(t, \Delta) \leq C\Delta \left[h_\rho^{(r)} \right]^{\frac{4}{9}},$$

where C depends on $f(0)$ only. By Lemma 3.13 and (2.74), we obtain

$$h_\eta^{(r)}(t, \Delta) \leq C \left[h_\rho^{(r)}(t) \right]^{\frac{8}{27}} \Delta, \quad \text{if } \left[h_\rho^{(r)}(t) \right]^{-2/9} \leq \Delta \leq t.$$

Iterating this scheme as described in the proof of Proposition 2.22, we obtain

$$h_\eta^{(r)}(t, \Delta) \leq C \left[h_\rho^{(r)}(t) \right]^{1/6} \Delta,$$

provided that $\left[h_\rho^{(r)}(t) \right]^{-8/81} \leq \Delta \leq t$. The constant C is independent of r . As in the proof of Proposition 2.22, we conclude that

$$h_\rho(t) \leq \max\{t^{-81/8}, C(R_0^3 + t^6)\}.$$

This completes the proof. \square

From Proposition 2.26, we deduce the following regularity bounds on the regularised potential U_r . These estimates are uniform in $r > 0$.

Lemma 2.30 (Uniform regularity of the electric field). *Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported. For each $r > 0$, let f_r denote the solution of (2.57) with initial datum $f(0)$. Then, for any $\alpha \in (0, 1)$, there exist constants $C[\alpha, f(0)]$, $C[\alpha, T, f(0)]$ such that*

$$\sup_{r>0} \sup_{t \in [0, T]} \|\widehat{U}_r(t)\|_{C^{1, \alpha}(\mathbb{T}^d)} \leq C[\alpha, f(0)], \quad \sup_{r>0} \sup_{t \in [0, T]} \|\bar{U}_r(t)\|_{C^{1, \alpha}(\mathbb{T}^d)} \leq C[\alpha, T, f(0)]. \quad (2.75)$$

Proof. These estimates follow from Proposition 2.4, using the density bounds from Proposition 2.26 and the fact that $\|\chi_r * \rho[f_r(t)]\|_{L^p(\mathbb{T}^d)} \leq \|\rho[f_r(t)]\|_{L^p(\mathbb{T}^d)}$. \square

The final ingredient is a compactness property of the solutions f_r of the regularised system. We show that the solutions f_r are equicontinuous in time into the space $W^{-1, 2}(\mathbb{T}^d)$. We will use this property to extract a limit point from $\{f_r\}_r$.

Lemma 2.31 (Equicontinuity in time). *Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported. For each $r > 0$, let f_r denote the solution of (2.57) with initial datum $f(0)$. For any $t_1 < t_2$,*

$$\|f_r(t_2) - f_r(t_1)\|_{W^{-1, 2}(\mathbb{T}^d \times \mathbb{R}^d)} \leq C[f(0)] |t_2 - t_1|,$$

where $W^{-1, 2}(\mathbb{T}^d \times \mathbb{R}^d)$ denotes the dual of $W^{1, 2}(\mathbb{T}^d \times \mathbb{R}^d)$.

Proof. We use the fact that f_r satisfies the transport equation

$$\partial_t f_r = -\operatorname{div}_{x, v} \left((v, E_r[f_r(t)]) f_r \right).$$

From the weak form of this equation we have, for any function $\phi \in W^{1, 2}(\mathbb{T}^d \times \mathbb{R}^d)$,

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi f_r(t) dx dv \right| &= \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_{x, v} \phi \cdot (v, E_r[f_r(t)]) f_r(t) dx dv \right| \\ &\leq \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_{x, v} \phi|^2 f_r(t) dx dv \right)^{1/2} \\ &\quad \times \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} (|v|^2 + |E_r[f_r(t)]|^2) f_r(t) dx dv \right)^{1/2}. \end{aligned}$$

Using the uniform $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ bound on f_r (2.63), we can control this quantity by

$$\begin{aligned} \left| \frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi f_r(t) dx dv \right| &\leq \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_{x,v} \phi|^2 dx dv \right)^{1/2} \|f_r(t)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}^{1/2} \\ &\quad \times \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t) dx dv + \|f_r(t)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \int_{\mathbb{T}^d \times \mathbb{R}^d} |E_r[f_r(t)]|^2 dx dv \right)^{1/2} \\ &\leq \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla_{x,v} \phi|^2 dx dv \right)^{1/2} \|f_r(t)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \\ &\quad \times \left(\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t) dx dv + \int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla U_r|^2 dx dv \right)^{1/2}. \end{aligned}$$

We control the last factor using the regularised energy $\mathcal{E}_r^{\text{ME}}[f_r]$. From the definition of $\mathcal{E}_r^{\text{ME}}$ (2.58), we have

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t) dx dv + \int_{\mathbb{T}^d \times \mathbb{R}^d} |\nabla U_r|^2 dx dv \leq \mathcal{E}_r^{\text{ME}}[f_r] + e^{-1}.$$

Thus

$$\left| \frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi f_r(t) dx dv \right| \leq C \|f(0)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} (C + \mathcal{E}_r^{\text{ME}}[f_r(t)]) \|\nabla \phi\|_{L^2(\mathbb{T}^d \times \mathbb{R}^d)}.$$

By the conservation of the regularised energy (2.64), we conclude that

$$\left| \frac{d}{dt} \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi f_r(t) dx dv \right| \leq C[f(0)] \|\nabla \phi\|_{L^2(\mathbb{T}^d \times \mathbb{R}^d)}.$$

This estimate means that

$$\|\partial_t f_r(t)\|_{W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d)} = \sup_{\|\phi\|_{W^{1,2}(\mathbb{T}^d \times \mathbb{R}^d)} \leq 1} \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi \partial_t f_r(t) dx dv \leq C,$$

thus $\partial_t f_r \in L^\infty((0, T); W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$. Thus, for any $t_1 < t_2$,

$$\|f_r(t_2) - f_r(t_1)\|_{W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d)} \leq \int_{t_1}^{t_2} \|\partial_t f_r(t)\|_{W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d)} dt \leq C|t_2 - t_1|.$$

□

In the next lemma, we use the above bounds to extract a convergent subsequence of approximate solutions, and show that the limit is a weak solution of (2.1). This completes the proof of Theorem 2.2.

Lemma 2.32. *Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported. For each $r > 0$, let f_r denote the solution of (2.57) with initial datum $f(0)$. Then there exists a subsequence f_{r_n} converging in $L_{loc}^\infty([0, +\infty); W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$ to a limit $f \in C([0, +\infty); W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$; that is, for each time horizon $T > 0$ and for all $\phi \in W^{1,2}(\mathbb{T}^d \times \mathbb{R}^d)$,*

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi (f_{r_n}(t) - f(t)) \, dx \, dv \right| = 0.$$

Moreover, for each $t \in [0, +\infty)$, for any $p \in [1, \infty]$ and all $\phi \in L^p(\mathbb{T}^d \times \mathbb{R}^d)$,

$$\lim_{n \rightarrow \infty} \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi (f_{r_n}(t) - f(t)) \, dx \, dv \right| = 0.$$

Furthermore, f belongs to the space $C([0, +\infty); \mathcal{M}_+(\mathbb{T}^d \times \mathbb{R}^d))$ and is a weak solution of (2.1) with initial datum $f(0)$, for which $\rho_f \in L_{loc}^\infty([0, +\infty); L^\infty(\mathbb{T}^d))$ and

$$\sup_{t \in [0, \infty)} \mathcal{E}^{ME}[f(t)] \leq C.$$

Proof. To extract the convergent subsequence, we need to make careful use of the equicontinuity in time. The curves

$$t \mapsto f_r(t) \in W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d)$$

are equicontinuous in the norm topology on $W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d)$ by Lemma 2.31. They are also uniformly bounded in $W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d)$ since $f_r \in L^\infty([0, +\infty); L^2(\mathbb{T}^d \times \mathbb{R}^d))$ by (2.63). We now fix a compact time interval $[0, T]$. By an Arzelà-Ascoli type argument we may extract a subsequence r_n such that for all $\phi \in W^{1,2}(\mathbb{T}^d \times \mathbb{R}^d)$,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} (f_{r_n}(t) - f(t)) \phi \, dx \, dv \right| = 0, \quad (2.76)$$

for some $f \in C([0, T]; W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$. In particular, since $C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d) \subset W^{1,2}(\mathbb{T}^d \times \mathbb{R}^d)$,

$$\lim_{n \rightarrow \infty} \sup_{t \in [0, T]} \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} (f_{r_n}(t) - f(t)) \phi \, dx \, dv \right| = 0, \quad \text{for all } \phi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d). \quad (2.77)$$

We now want to prove that the convergence holds also in $L^p(\mathbb{T}^d \times \mathbb{R}^d) - w$, for $p \in [1, +\infty)$, and $L^\infty(\mathbb{T}^d \times \mathbb{R}^d) - w^*$. For fixed $t \in [0, T]$, we have the uniform bounds

$$\sup_{r>0} \|f_r(t)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq \|f(0)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}, \quad \sup_{r>0} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t, x, v) dx dv \leq C[f(0)]. \quad (2.78)$$

This implies that $\{f_r(t)\}_{r>0}$ is relatively compact in $L^p(\mathbb{T}^d \times \mathbb{R}^d) - w$ for $p \in [1, +\infty)$ and $L^\infty(\mathbb{T}^d \times \mathbb{R}^d) - w^*$. For each $p \in [1, +\infty)$ and $t \in [0, T]$ there is a further subsequence r_{n_k} and a limit $g \in L^p(\mathbb{T}^d \times \mathbb{R}^d)$, both depending on t and p , such that for all $\phi \in L^{p^*}(\mathbb{T}^d \times \mathbb{R}^d)$ (p^* being the Hölder conjugate of p),

$$\lim_{k \rightarrow \infty} \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(f_{r_{n_k}}(t) - g) dx dv \right| = 0.$$

In particular, this holds for $\phi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d) \subset L^{p^*}(\mathbb{T}^d \times \mathbb{R}^d)$. By (2.77), we deduce that

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f(t) \phi dx dv = \int_{\mathbb{T}^d \times \mathbb{R}^d} g \phi dx dv \quad \text{for all } \phi \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d).$$

Thus $f(t) = g$. The uniqueness of the limit implies that in fact the whole original subsequence $f_{r_n}(t)$ converges to $f(t)$ in $L^p(\mathbb{T}^d \times \mathbb{R}^d) - w$ for $p \in [1, +\infty)$ and $L^\infty(\mathbb{T}^d \times \mathbb{R}^d) - w^*$.

Note that the weak (or weak*) convergence of f_{r_n} to f implies that the bounds (2.78) pass to the limit: for all $t \in [0, T]$ and $p \in [1, +\infty]$,

$$\|f(t)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq \|f(0)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(t, x, v) dx dv \leq C[f(0)]. \quad (2.79)$$

Since $f_{r_n}(t)$ is non-negative for all n and $t \in [0, T]$, this property also holds for the limit f : note that, for fixed t , the function $\mathbb{1}_{\{f(t)<0\}} \in L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ is non-negative. Since $f_{r_n}(t)$ converges to $f(t)$ weakly in $L^1(\mathbb{T}^d \times \mathbb{R}^d)$,

$$0 \geq \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t) \mathbb{1}_{\{f(t)<0\}} dx dv = \lim_{n \rightarrow \infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} f_{r_n}(t) \mathbb{1}_{\{f(t)<0\}} dx dv \geq 0.$$

Thus

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f(t) \mathbb{1}_{\{f(t)<0\}} dx dv = 0,$$

which implies that $f(t) \geq 0$ almost everywhere.

We will now show that the path $t \mapsto f(t)$ is in fact continuous with respect to convergence of measures. This is equivalent to showing that the function

$$t \mapsto \langle f(t), \phi \rangle := \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(x, v) f(t, x, v) dx dv \quad (2.80)$$

is continuous with respect to t , for all $\phi \in C_b(\mathbb{T}^d \times \mathbb{R}^d)$. Since $f \in C([0, T]; W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$, we already know that this holds for all $\phi \in W^{1,2}(\mathbb{T}^d \times \mathbb{R}^d)$, and therefore certainly for $\phi \in C_c^1(\mathbb{T}^d \times \mathbb{R}^d)$. We will enlarge the space of admissible test functions ϕ , first to $C_c(\mathbb{T}^d \times \mathbb{R}^d)$, and then to $C_b(\mathbb{T}^d \times \mathbb{R}^d)$.

Let $\phi \in C_c(\mathbb{T}^d \times \mathbb{R}^d)$, and fix $t \in [0, T]$. We will show that the path (2.80) is continuous at t . Given $\varepsilon > 0$, there exists a function $\psi_\varepsilon \in C_c^1(\mathbb{T}^d \times \mathbb{R}^d)$ such that $\|\phi - \psi_\varepsilon\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq \varepsilon$. Then, for $s \in [0, T]$,

$$|\langle f(s), \phi \rangle - \langle f(t), \phi \rangle| \leq |\langle f(s), \phi - \psi_\varepsilon \rangle| + |\langle f(s), \psi_\varepsilon \rangle - \langle f(t), \psi_\varepsilon \rangle| + |\langle f(t), \psi_\varepsilon - \phi \rangle|.$$

By (2.79), for all $s \in [0, T]$ we have the estimate

$$\begin{aligned} |\langle f(s), \phi - \psi_\varepsilon \rangle| &\leq \|f(s)\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)} \|\phi - \psi_\varepsilon\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \\ &\leq C(f(0)) \varepsilon. \end{aligned}$$

Thus

$$|\langle f(s), \phi \rangle - \langle f(t), \phi \rangle| \leq C\varepsilon + |\langle f(s), \psi_\varepsilon \rangle - \langle f(t), \psi_\varepsilon \rangle|,$$

where $C > 0$ depends on $f(0)$ only. Since $t \mapsto \langle f(t), \psi_\varepsilon \rangle$ is continuous in t , there exists $\delta > 0$ such that, if $|s - t| < \delta$, then

$$|\langle f(s), \phi \rangle - \langle f(t), \phi \rangle| \leq 2C\varepsilon.$$

Therefore the function (2.80) is continuous for all $\phi \in C_c(\mathbb{T}^d \times \mathbb{R}^d)$.

Now let $\phi \in C_b(\mathbb{T}^d \times \mathbb{R}^d)$. For each $R > 0$, let $\eta_R \in C_c^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be a cut-off function taking values in $[0, 1]$ and such that

$$\eta_R(x) = 1 \quad \text{for } |x| \leq R, \quad \eta_R(x) = 0 \quad \text{for } |x| > 2R.$$

Let $\phi_R := \phi \eta_R$. Then $\phi_R \in C_c(\mathbb{T}^d \times \mathbb{R}^d)$. By (2.79), for all $t \in [0, T]$,

$$\begin{aligned} |\langle f(t), \phi - \phi_R \rangle| &\leq \|\phi\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \int_{x \in \mathbb{T}^d, |v| > R} f(t, x, v) \, dx \, dv \\ &\leq \|\phi\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R^{-2} \int_{x \in \mathbb{T}^d, |v| > R} |v|^2 f(t, x, v) \, dx \, dv \\ &\leq C \|\phi\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R^{-2}, \end{aligned}$$

where $C > 0$ depends only on $f(0)$. Thus

$$\begin{aligned} |\langle f(s), \phi \rangle - \langle f(t), \phi \rangle| &\leq |\langle f(s), \phi - \phi_R \rangle| + |\langle f(s), \phi_R \rangle - \langle f(t), \phi_R \rangle| + |\langle f(t), \phi - \phi_R \rangle| \\ &\leq C \|\phi\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} R^{-2} + |\langle f(s), \phi_R \rangle - \langle f(t), \phi_R \rangle|. \end{aligned}$$

Given $\varepsilon > 0$, first choose R sufficiently large so that the first term is smaller than ε . Then, since $\phi_R \in C_c(\mathbb{T}^d \times \mathbb{R}^d)$, $\langle f(t), \phi_R \rangle$ is continuous with respect to t . Thus there exists $\delta > 0$ such that, for all $s \in [0, T]$ such that $|s - t| < \delta$, the second term is smaller than ε . Thus $\langle f(t), \phi \rangle$ is continuous with respect to t . We deduce that $f \in C([0, T]; \mathcal{M}_+(\mathbb{T}^d \times \mathbb{R}^d))$.

Next we show that the convergence also holds for the mass density. Since $f_{r_n}(t)$ converges in $L^1(\mathbb{T}^d \times \mathbb{R}^d) - w$, for all $\phi \in L^\infty(\mathbb{T}^d)$ we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\mathbb{T}^d} \rho[f_{r_n}(t)] \phi(x) dx &= \lim_{n \rightarrow \infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} f_{r_n}(t, x, v) \phi(x) dx dv \\ &= \int_{\mathbb{T}^d \times \mathbb{R}^d} f(t, x, v) \phi(x) dx dv = \int_{\mathbb{T}^d} \rho_f(t, x, v) \phi(x) dx. \end{aligned}$$

In other words $\rho_{r_n}(t) \rightharpoonup \rho_f(t)$ in $L^1(\mathbb{T}^d) - w$. Since, by (2.66) and (2.67), $\rho[f_{r_n}(t)]$ are uniformly bounded in $L^p(\mathbb{T}^d)$ for all $p \in [1, +\infty]$, the convergence also holds in $L^p(\mathbb{T}^d) - w$ for $p \in [1, +\infty)$ and in $L^\infty(\mathbb{T}^d) - w^*$. In particular,

$$\sup_{t \in [0, T]} \|\rho_f(t)\|_{L^p(\mathbb{T}^d)} \leq \liminf_n \|\rho[f_{r_n}(t)]\|_{L^p(\mathbb{T}^d)}.$$

We deduce that, by Lemma 2.25,

$$\sup_{t \in [0, T]} \|\rho_f(t)\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C, \quad \sup_{t \in [0, T]} \|\rho_f(t)\|_{L^\infty(\mathbb{T}^d)} \leq C_T. \quad (2.81)$$

Next, we prove convergence of the electric field. By Lemma 2.30, for any $\alpha \in (0, 1)$,

$$\sup_r \sup_{t \in [0, T]} \|U_r(t)\|_{C^{1, \alpha}(\mathbb{T}^d)} \leq C[\alpha, T, f(0)],$$

which implies that $U_r(t), \nabla U_r(t)$ are equicontinuous on \mathbb{T}^d . Hence there exists a further subsequence for which $U_{r_{n_k}}(t), \nabla U_{r_{n_k}}(t)$ converge in $C(\mathbb{T}^d)$ to some $U(t), \nabla U(t)$.

We identify the limit $U(t)$, by showing that it is a solution of

$$\Delta U(t) = e^{U(t)} - \rho_f(t). \quad (2.82)$$

The elliptic equation for $U_r(t)$ in (2.57) in weak form tells us that for all r and all $\phi \in W^{1,2} \cap L^1(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} \nabla U_r(t) \cdot \nabla \phi + \left(e^{U_r(t)} - \chi_r * \rho[f_r(t)] \right) \phi \, dx = 0.$$

The first two terms converge by dominated convergence, since $U_r(t)$ are uniformly bounded in $C^1(\mathbb{T}^d)$. For the term involving $\chi_r * \rho[f_r(t)]$, we split

$$\int_{\mathbb{T}^d} (\chi_r * \rho[f_r(t)] - \rho_f(t)) \phi \, dx = \int_{\mathbb{T}^d} (\chi_r * \rho[f_r(t)] - \rho[f_r(t)]) \phi \, dx + \int_{\mathbb{T}^d} (\rho[f_r(t)] - \rho_f(t)) \phi \, dx. \quad (2.83)$$

For any $\phi \in L^{\frac{d+2}{2}}(\mathbb{T}^d)$, we have

$$\begin{aligned} \left| \int_{\mathbb{T}^d} (\chi_r * \rho[f_r(t)] - \rho[f_r(t)]) \phi \, dx \right| &= \left| \int_{\mathbb{T}^d} (\chi_r * \phi - \phi) \rho[f_r(t)] \, dx \right| \\ &\leq \|\chi_r * \phi - \phi\|_{L^{\frac{d+2}{2}}(\mathbb{T}^d)} \|\rho[f_r(t)]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \\ &\leq C \|\chi_r * \phi - \phi\|_{L^{\frac{d+2}{2}}(\mathbb{T}^d)}. \end{aligned}$$

The right hand side converges to zero as r tends to zero by standard results on continuity in L^p spaces. For $r = r_{n_k}$, the second term of (2.83) converges to zero as k tends to infinity, for all $\phi \in L^{\frac{d+2}{2}}(\mathbb{T}^d)$, since $\rho[f_r(t)]$ converges to $\rho_f(t)$ weakly in $L^{\frac{d+2}{d}}(\mathbb{T}^d)$. Hence, for all $\phi \in W^{1,2} \cap L^{\frac{d+2}{2}}(\mathbb{T}^d)$,

$$\int_{\mathbb{T}^d} \nabla U(t) \cdot \nabla \phi + (e^{U(t)} - \rho_f(t)) \phi \, dx = 0.$$

Since $U(t) \in C^1(\mathbb{T}^d)$ and $\rho_f(t) \in L^2(\mathbb{T}^d)$, this extends to all $\phi \in W^{1,2}(\mathbb{T}^d)$ by density of $L^{\frac{d+2}{2}}(\mathbb{T}^d)$ in $L^2(\mathbb{T}^d)$. In other words $U(t)$ is indeed a weak solution of (2.82).

Our earlier stability estimates (Proposition 2.11) imply that (2.82) has at most one solution in $L^\infty \cap W^{1,2}(\mathbb{T}^d)$. Since $U(t), \nabla U(t) \in C(\mathbb{T}^d)$ we do have $U(t) \in L^\infty \cap W^{1,2}(\mathbb{T}^d)$. Since the limit of any convergent subsequence is uniquely identified, it follows that for all t we have $U_{r_n}(t) \rightarrow U(t)$ in $C^1(\mathbb{T}^d)$, where $U(t)$ is the unique $L^\infty \cap W^{1,2}(\mathbb{T}^d)$ solution of (2.82) (that is, without passing to further subsequences).

Next we consider the convergence of the regularised electric field

$$E_{r_n}[f_{r_n}(t)] = -\chi_r * \nabla U_{r_n}(t).$$

This converges to $-\nabla U(t)$ uniformly since, for some fixed $\alpha \in (0, 1)$,

$$\begin{aligned} |\chi_r * \nabla U_r(t)(x) - \nabla U_r(t)(x)| &= \left| \int_{y \in \mathbb{T}^d: |x-y| \leq Cr} \chi_r(x-y) [\nabla U_r(t)(y) - \nabla U_r(t)(x)] dy \right| \\ &\leq \|U_r(t)\|_{C^{1,\alpha}(\mathbb{T}^d)} \int_{y \in \mathbb{T}^d: |x-y| \leq Cr} \chi_r(x-y) |x-y|^\alpha dy \\ &\leq C_{\alpha,d,T} r^\alpha \int_{\mathbb{R}^d} \chi_1(y) |y|^\alpha dy \leq C_{\alpha,d,T} r^\alpha. \end{aligned}$$

The right hand side converges to zero as r tends to zero, which proves the assertion.

Finally, we show that f is a weak solution of (2.1) on the time interval $[0, T]$. Since f_r is a solution of (2.57), for any $\phi \in C_c^\infty([0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d)$ we have

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f(0, x, v) \phi(0, x, v) dx dv + \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial_t \phi + v \cdot \nabla_x \phi + E_r(x) \cdot \nabla_v \phi) f_r dx dv dt = 0.$$

Since $\partial_t \phi + v \cdot \nabla_x \phi \in C_c^\infty([0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d)$, (2.76) implies that for all fixed t , as n tends to infinity,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial_t \phi + v \cdot \nabla_x \phi) f_{r_n} dx dv \rightarrow \int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial_t \phi + v \cdot \nabla_x \phi) f dx dv.$$

Since

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial_t \phi + v \cdot \nabla_x \phi) f_{r_n} dx dv \right| \leq \|f(0)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \int_{\mathbb{T}^d \times \mathbb{R}^d} |\partial_t \phi + v \cdot \nabla_x \phi| dx dv \in L^1([0, \infty)),$$

we deduce from the dominated convergence theorem that for all $\phi \in C_c^\infty([0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d)$, as n tends to infinity,

$$\int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial_t \phi + v \cdot \nabla_x \phi) f_{r_n} dx dv \rightarrow \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial_t \phi + v \cdot \nabla_x \phi) f dx dv.$$

For the nonlinear term we have the estimate

$$\begin{aligned} \left| \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_v \phi \cdot (E_r f_r + \nabla_x U f) dx dv dt \right| &\leq \left| \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} (E_r + \nabla_x U) \cdot \nabla_v \phi f_r dx dv dt \right| \\ &\quad + \left| \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} -\nabla_x U \cdot \nabla_v \phi (f_r - f) dx dv dt \right|. \end{aligned}$$

Now assume that $\text{supp } \phi \subset [0, T] \times \mathbb{T}^d \times \mathbb{R}^d$. We estimate the first term using the uniform $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ estimate (2.63):

$$\left| \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} (E_{r_n} + \nabla U) \cdot \nabla_v \phi f_r \, dx \, dv \, dt \right| \leq C[f(0)] \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} |(E_{r_n} + \nabla U) \cdot \nabla_v \phi| \, dx \, dv \, dt.$$

Since $E_{r_n}(t) + \nabla U(t)$ tends to zero as n tends to infinity for each t , the function $\nabla_v \phi$ belongs to $L^1([0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d)$, and the uniform bound

$$\sup_{t \in [0, T]} \|E_{r_n}(t) + \nabla U(t)\|_{L^\infty(\mathbb{T}^d)} \leq CT$$

holds (by (2.75)), it follows by the dominated convergence theorem that the right hand side converges to zero as n tends to infinity. Similarly, for the second term we use that, for each t , since $\nabla U(t) \cdot \nabla_v \phi \in L^1(\mathbb{T}^d \times \mathbb{R}^d)$,

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla U \cdot \nabla_v \phi (f_{r_n} - f) \, dx \, dv \right| \rightarrow 0.$$

Combining this with the bound

$$|\nabla U \cdot \nabla_v \phi (f_r - f)| \leq C[T, f(0)] |\nabla_v \phi| \in L^1([0, \infty) \times \mathbb{T}^d \times \mathbb{R}^d),$$

which follows from (2.63) and (2.75), we conclude that, as n tends to infinity,

$$\int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} (\nabla_v \phi \cdot E_{r_n}) f_{r_n} \, dx \, dv \, dt \rightarrow - \int_0^\infty \int_{\mathbb{T}^d \times \mathbb{R}^d} (\nabla_v \phi \cdot \nabla U) f \, dx \, dv \, dt.$$

Hence, for all $\phi \in C_c^\infty([0, T] \times \mathbb{T}^d \times \mathbb{R}^d)$ with $\phi(T, \cdot, \cdot) = 0$,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} f(0, x, v) \phi(0, x, v) \, dx \, dv + \int_0^T \int_{\mathbb{T}^d \times \mathbb{R}^d} (\partial_t \phi + v \cdot \nabla_x \phi - \nabla U(x) \cdot \nabla_v \phi) f \, dx \, dv \, dt = 0.$$

Thus f is a weak solution of the VPME system (2.1).

By (2.81), f has bounded density on $[0, T]$. Thus Theorem 2.1 implies that f is the unique bounded density solution of (2.1). Therefore the limit of any $C([0, T]; W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$ convergent subsequence f_{r_n} must be f . This implies that in fact f_r converges to f in $C([0, T]; W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$ as r tends to zero, without passing to a subsequence. Since T was arbitrary and the limit f is unique for each T , by extension a single path $f \in C([0, \infty); W^{-1,2}(\mathbb{T}^d \times \mathbb{R}^d))$ may be defined that coincides on each $[0, T]$ with the limit of f_r . This completes the proof. \square

2.9.1 Energy conservation

The solutions we have constructed have finite energy. Indeed, since each f_r satisfies a conservation of the energy $\mathcal{E}_r^{\text{ME}}$ defined in (2.58), for some constant $C[f(0)] > 0$ independent of t and r we have for all $t \in [0, +\infty)$,

$$\mathcal{E}_r^{\text{ME}}[f_r(t)] = \mathcal{E}_r^{\text{ME}}[f(0)] \leq C[f(0)].$$

Since $U_r(t)$ converges to $U(t)$ in $C^1(\mathbb{T}^d)$ and $f_r(t)$ converges to $f(t)$ weakly in $L^1(\mathbb{T}^d \times \mathbb{R}^d)$, it follows that

$$\begin{aligned} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(t) \, dx \, dv &\leq \liminf_{r \rightarrow 0} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t) \, dx \, dv \\ \int_{\mathbb{T}^d} |\nabla U(t)|^2 \, dx &= \lim_{r \rightarrow 0} \int_{\mathbb{T}^d} |\nabla U_r(t)|^2 \, dx \\ \int_{\mathbb{T}^d} U(t) e^{U(t)} \, dx &= \lim_{r \rightarrow 0} \int_{\mathbb{T}^d} U_r(t) e^{U_r(t)} \, dx. \end{aligned}$$

In particular,

$$\lim_{r \rightarrow 0} \mathcal{E}_r^{\text{ME}}[f(0)] = \mathcal{E}^{\text{ME}}[f(0)].$$

Moreover,

$$\mathcal{E}^{\text{ME}}[f(t)] \leq \liminf_{r \rightarrow 0} \mathcal{E}_r^{\text{ME}}[f_r(t)] \leq \mathcal{E}^{\text{ME}}[f(0)].$$

In other words f has uniformly bounded energy. Next we want to show that the energy is in fact conserved for solutions with compactly supported data. We will do this by controlling a moment of order strictly greater than two.

Lemma 2.33. *Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported, and let f be the unique solution of the VPME system (2.1) with initial datum $f(0)$ and locally bounded density. Then, for all $p \in [1, +\infty)$ and all $T \geq 0$,*

$$\sup_{t \in [0, T]} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^p f(t, x, v) \, dx \, dv \leq C_{p, T, f_0, M},$$

where $C_{p, T, f_0, M}$ depends on p , T , f_0 and a bound on the mass.

Proof. By assumption, there exists M such that

$$\sup_{t \in [0, T]} \|\rho_f(t)\|_{L^\infty(\mathbb{T}^d)} \leq M.$$

Then, by Proposition 2.4,

$$\|E\|_{L^\infty(\mathbb{T}^d)} \leq CM + e^{CM}.$$

Let $V(t; x, v)$ be the characteristic trajectory beginning at (x, v) at time $t = 0$. Then

$$|V(t; x, v)| \leq |v| + \int_0^t \|E\|_{L^\infty(\mathbb{T}^d)} ds \leq |v| + (CM + e^{CM})t. \quad (2.84)$$

Let R_t be a function such that f_t is supported in $\mathbb{T}^d \times B_{\mathbb{R}^d}(0; R_t)$. Then (2.84) implies that

$$R_t \leq R_0 + C_{M,t}.$$

Let $\psi_N \in C_b(\mathbb{R}^d)$ be a sequence of non-negative test functions satisfying

$$\begin{aligned} \psi_N(v) &= |v|^p, \quad |v| \leq N, \\ \psi_N(v) &\uparrow |v|^p, \quad N \rightarrow \infty. \end{aligned}$$

We take the representation of f as the pushforward of f_0 along the characteristic flow induced by f . Then

$$\begin{aligned} \int_{\mathbb{T}^d \times \mathbb{R}^d} \psi_N(v) f_t dx dv &= \int_{\mathbb{T}^d \times \mathbb{R}^d} \psi_N(V(t; x, v)) f_0 dx dv \leq \int_{\mathbb{T}^d \times \mathbb{R}^d} |V(t; x, v)|^p f_0 dx dv \\ &\leq \int_{\mathbb{T}^d \times \mathbb{R}^d} |R_0 + C_{M,t}|^p f_0 dx dv \leq C |R_0 + C_{M,t}|^p R_0^d \|f_0\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \\ &\leq C_{p,t,M,f_0}. \end{aligned}$$

By monotone convergence, we have

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^p f_t dx dv = \lim_{N \rightarrow \infty} \int_{\mathbb{T}^d \times \mathbb{R}^d} \psi_N(v) f_t dx dv \leq C_{p,M,t,f_0},$$

as required. \square

We complete the proof of the conservation of energy by showing that instead of the bound

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(t) dx dv \leq \liminf_{r \rightarrow 0} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t) dx dv$$

we in fact have the equality

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(t) dx dv = \lim_{r \rightarrow 0} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t) dx dv.$$

Lemma 2.34. *Let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported, and let f be the unique solution of the VPME system (2.1) with initial datum $f(0)$ and locally bounded density. Let f_r*

be the solution of the regularised equation (2.57) with initial datum $f(0)$. Then, for all t ,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(t) \, dx \, dv = \lim_{r \rightarrow 0} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f_r(t) \, dx \, dv.$$

Proof. Let $\psi_N \in C_b(\mathbb{R}^d)$ be a sequence of non-negative test functions satisfying

$$\begin{aligned} \psi_N(v) &= |v|^2, \quad |v| \leq N, \\ \psi_N(v) &\uparrow |v|^2, \quad N \rightarrow \infty. \end{aligned}$$

By Lemma 2.33, for any $\alpha > 0$,

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^{2+\alpha} f(t) \, dx \, dv \leq C_\alpha.$$

In fact, the same estimate holds for f_r with the same constant C_α for all r , since we showed in Lemma 2.25 that for any $T \geq 0$,

$$\sup_{t \in [0, T]} \|\rho[f_r(t)]\|_{L^\infty(\mathbb{T}^d)} \leq M,$$

where M is uniform in r . Hence we have the estimate

$$\begin{aligned} \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f(t) \, dx \, dv - \int_{\mathbb{T}^d \times \mathbb{R}^d} \psi_N(v) f(t) \, dx \, dv \right| &\leq \left| \int_{(x,v): |v| \geq N} |v|^2 f(t) \, dx \, dv \right| \\ &\leq N^{-\alpha} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^{2+\alpha} f(t) \, dx \, dv \leq C_\alpha N^{-\alpha}, \end{aligned}$$

along with the analogous one for f_r . Then

$$\left| \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 [f(t) - f_r(t)] \, dx \, dv \right| \leq 2C_\alpha N^{-\alpha} + \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \psi_N(v) [f(t) - f_r(t)] \, dx \, dv \right|.$$

The second term converges to zero as r tends to zero, since f_r converges to f weakly in $L^1(\mathbb{T}^d \times \mathbb{R}^d)$. This completes the proof. \square

Chapter 3

The Quasi-Neutral Limit for the VPME System With Rough Data

Contents

3.1	Statement of Results	136
3.1.1	Strategy of Proof	138
3.2	Estimates on the Electric Field	139
3.2.1	Regularity	140
3.2.2	Stability	144
3.3	Wasserstein Stability	146
3.4	Growth Estimates	152
3.4.1	Proof of Proposition 3.10 in the two dimensional case	152
3.4.2	Proof of Proposition 3.10 in the three dimensional case	156
3.5	Quasi-Neutral Limit	160

In this chapter, we prove a quasi-neutral limit for the VPME system. In quasi-neutral scaling, the system reads as follows:

$$(VPME)_\varepsilon := \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla U_\varepsilon, \\ \varepsilon^2 \Delta U_\varepsilon = e^{U_\varepsilon} - \int_{\mathbb{R}^d} f_\varepsilon \, dv = e^{U_\varepsilon} - \rho_\varepsilon, \\ f_\varepsilon|_{t=0} = f_\varepsilon(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon(0, x, v) \, dx \, dv = 1. \end{cases} \quad (3.1)$$

The quasi-neutral limit is the limit in which the Debye length ε tends to zero. We introduced the Debye length and the physical motivation for this limit in Subsection 1.2.3, and discussed its mathematical background in Section 1.5. The main result of this chapter is a rigorous quasi-neutral limit for the VPME system for a class of rough data.

The formal limiting system is the kinetic isothermal Euler system:

$$(KIsE) := \begin{cases} \partial_t g + v \cdot \nabla_x g + E \cdot \nabla_v g = 0, \\ E = -\nabla U, \\ U = \log \rho, \\ g(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} g(0, x, v) dx dv = 1. \end{cases} \quad (3.2)$$

The aim is to prove that, if $f_\varepsilon(0)$ converges to $g(0)$ as ε tends to zero, then the solutions f_ε of (3.1) converge to a solution g of (3.2).

Rigorous results on the quasi-neutral limit go back to the works of Brenier and Grenier [21] and Grenier [32] for the Vlasov-Poisson system for electrons (1.5). A result of particular relevance to the contents of this chapter is the work of Grenier [33], proving the limit for the classical system assuming uniformly analytic data. The works of Han-Kwan and Iacobelli [43, 44] extended this result to data that are very small, but possibly rough, perturbations of the uniformly analytic case, in dimension 1, 2 and 3.

In the massless electrons case considered here, Han-Kwan and Iacobelli [44] showed a rigorous limit in dimension one, again for rough perturbations of analytic data. The main result of this chapter extends the results of [44] to higher dimensions by showing a rigorous quasi-neutral limit for the VPME system (3.1) in dimension 2 and 3, for data that are very small, but possibly rough, perturbations of some uniformly analytic functions.

3.1 Statement of Results

We state the main result of this chapter, which is a quasi-neutral limit for rough data that are small perturbations of analytic functions. We begin by introducing some concepts and notation needed to state our results.

Energy We recall that we defined the energy associated with the VPME system in (2.2). For the quasi-neutral limit, we need to write the energy with a quasi-neutral scaling. This results in

the following functional:

$$\mathcal{E}_\varepsilon^{\text{ME}}[f] := \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv + \frac{\varepsilon^2}{2} \int_{\mathbb{T}^d} |\nabla U|^2 \, dx + \int_{\mathbb{T}^d} U e^U \, dx,$$

where U is the potential defined in (3.1).

Analytic Norms We introduced the family of analytic norms $\|\cdot\|_{B_\delta}$ in Section 1.5.4.1. Explicitly, for $\delta > 1$ the norm $\|\cdot\|_{B_\delta}$ is defined by

$$\|g\|_{B_\delta} := \sum_{k \in \mathbb{Z}^d} |\hat{g}(k)| \delta^{|k|},$$

where $\hat{g}(k)$ denotes the Fourier coefficient of g of index k .

Iterated Exponentials We will use the notation $\overline{\text{exp}}_n$ to denote the n -fold iteration of the exponential function. For example

$$\overline{\text{exp}}_3(x) := \text{expexpexp}(x).$$

Our main result is the following theorem.

Theorem 3.1 (Quasi-neutral limit). *Let $d = 2, 3$. Consider initial data $f_\varepsilon(0)$ satisfying the following conditions:*

- (Uniform bounds) $f_\varepsilon(0)$ is bounded and has bounded energy, uniformly with respect to ε :

$$\|f_\varepsilon(0)\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} \leq C_0, \quad \mathcal{E}_\varepsilon^{\text{ME}}[f_\varepsilon(0)] \leq C_0,$$

for some constant $C_0 > 0$.

- (Control of support) There exists $C_1 > 0$ such that

$$f_\varepsilon(0, x, v) = 0 \quad \text{for } |v| > \exp(C_1 \varepsilon^{-2}). \quad (3.3)$$

- (Perturbation of an analytic function) There exist $g_\varepsilon(0)$ satisfying, for some $\delta > 1$, $\eta > 0$, and $C > 0$,

$$\sup_{\varepsilon} \sup_{v \in \mathbb{R}^d} (1 + |v|^{d+1}) \|g_\varepsilon(0, \cdot, v)\|_{B_\delta} \leq C,$$

$$\sup_{\varepsilon} \left\| \int_{\mathbb{R}^d} g_\varepsilon(0, \cdot, v) dv - 1 \right\|_{B_\delta} \leq \eta,$$

as well as the support condition (3.3), such that

$$W_2(f_\varepsilon(0), g_\varepsilon(0)) \leq [\overline{\exp}_4(C\varepsilon^{-2})]^{-1} \quad (3.4)$$

for C sufficiently large with respect to C_0, C_1 .

- (Convergence of data) $g_\varepsilon(0)$ has a limit $g(0)$ in the sense of distributions as $\varepsilon \rightarrow 0$.

Let f_ε denote the unique solution of (3.1) with bounded density and initial datum $f_\varepsilon(0)$. Then there exists a time horizon $T_* > 0$, independent of ε but depending on the collection $\{g_\varepsilon(0)\}_\varepsilon$, and a solution g of (3.2) on the time interval $[0, T_*]$ with initial datum $g(0)$, such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T_*]} W_1(f_\varepsilon(t), g(t)) = 0.$$

Remark 4. The condition (3.3) should be understood as giving the fastest growth rate on the support for which the inverse quadruple exponential is an admissible rate in (3.4). In particular, this would still be the rate achievable by our methods even if the support of the data was uniform in ε .

3.1.1 Strategy of Proof

The main idea of the proof is to consider the unique bounded density solution g_ε of the VPME system (3.1) with initial datum $g_\varepsilon(0)$. Since each $g_\varepsilon(0)$ is compactly supported, the fact that a unique g_ε exists is a consequence of the results of Chapter 2. We will use g_ε as an intermediate step between f_ε and a solution g of the KIsE system (3.2).

The quasi-neutral limit for the VPME system (3.1) with uniformly analytic data can be proved using the methods of Grenier [33], with the modifications for the massless electrons case described in [44, Proposition 4.1]. We stated this result above as Theorem 1.20. Grenier's result gives an H^s convergence of a representation of the VP system as a multi-fluid pressureless Euler system. In [44, Corollary 4.2], it is shown that this implies convergence in W_1 . We can therefore make use of the following result.

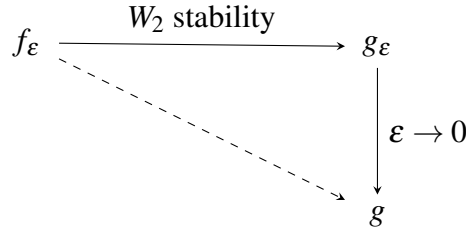


Fig. 3.1 Strategy for the proof of Theorem 3.1.

Theorem 3.2. *Let $g_\varepsilon(0)$ satisfy the technical conditions stated in Theorem 3.1, and let g_ε denote the solution of (3.1) with initial datum $g_\varepsilon(0)$. Assume that $g_\varepsilon(0)$ has a limit $g(0)$ in the sense of distributions as ε tends to zero. Then there exists a time $T_* > 0$ and a solution of (3.2) on the time interval $[0, T_*]$ such that*

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T_*]} W_1(g_\varepsilon(t), g(t)) = 0.$$

To show the convergence from f_ε to g_ε , we use a stability estimate in W_2 for solutions of the VPME system. We proved an estimate of this type in Proposition 2.12 of Chapter 2. In the setting of the quasi-neutral limit, what is important is to quantify the dependence of the constants in the estimate on ε . The quantified estimate is Proposition 3.9 stated below in Section 3.3. We use this to show that, under our assumptions on the initial data,

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T_*]} W_2(f_\varepsilon(t), g_\varepsilon(t)) = 0.$$

The combination of these two results will complete the proof.

This chapter is structured as follows. The W_2 stability estimate is proved in Section 3.3. The proof relies on regularity estimates for the electric field, which are proved in Section 3.2. The stability estimate also relies on bounds on the mass density of the solutions. We quantify the dependence of these bounds on ε in Section 3.4. The proof of Theorem 3.1 is completed in Section 3.5.

3.2 Estimates on the Electric Field

In this section we revisit the regularity estimates on the electric field that we proved in Section 2.4, with the aim of quantifying their dependence with respect to ε . Once again, we use

the decomposition of the electrostatic potential into two parts:

$$U_\varepsilon = \bar{U}_\varepsilon + \widehat{U}_\varepsilon,$$

where \bar{U}_ε and \widehat{U}_ε are solutions of the equations

$$\varepsilon^2 \Delta \bar{U}_\varepsilon = 1 - \rho[f_\varepsilon], \quad \varepsilon^2 \Delta \widehat{U}_\varepsilon = e^{\bar{U}_\varepsilon + \widehat{U}_\varepsilon} - 1.$$

We will use the notation $\bar{E}_\varepsilon = -\nabla \bar{U}_\varepsilon$ and $\widehat{E}_\varepsilon = -\nabla \widehat{U}_\varepsilon$.

3.2.1 Regularity

The following is a version of Proposition 2.4 with quasi-neutral scaling. We will give the proof of these estimates below.

Proposition 3.3 (Regularity estimates on \bar{U}_ε and \widehat{U}_ε). *Let $d = 2, 3$. Let $h \in L^\infty \cap L^{\frac{d+2}{d}}(\mathbb{T}^d)$. Then there exist unique $\bar{U}_\varepsilon \in W^{1,2}(\mathbb{T}^d)$ with zero mean and $\widehat{U}_\varepsilon \in W^{1,2} \cap L^\infty(\mathbb{T}^d)$ satisfying*

$$\varepsilon^2 \Delta \bar{U}_\varepsilon = 1 - h, \quad \varepsilon^2 \Delta \widehat{U}_\varepsilon = e^{\bar{U}_\varepsilon + \widehat{U}_\varepsilon} - 1.$$

Moreover we have the following estimates for some constant $C_{\alpha,d} > 0$:

$$\begin{aligned} \|\bar{U}_\varepsilon\|_{C^{0,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \varepsilon^{-2} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right), & \alpha \in \begin{cases} (0, 1) & \text{if } d = 2 \\ (0, \frac{1}{5}] & \text{if } d = 3 \end{cases} \\ \|\bar{U}_\varepsilon\|_{C^{1,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \varepsilon^{-2} \left(1 + \|h\|_{L^\infty(\mathbb{T}^d)} \right), & \alpha \in (0, 1) \\ \|\widehat{U}_\varepsilon\|_{C^{1,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \exp \left(C_{\alpha,d} \varepsilon^{-2} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right) & \alpha \in (0, 1) \\ \|\widehat{U}_\varepsilon\|_{C^{2,\alpha}(\mathbb{T}^d)} &\leq C_{\alpha,d} \exp_2 \left(C_{\alpha,d} \varepsilon^{-2} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right) & \alpha \in \begin{cases} (0, 1) & \text{if } d = 2 \\ (0, \frac{1}{5}] & \text{if } d = 3. \end{cases} \end{aligned}$$

3.2.1.1 Regularity of \bar{U}_ε

For the singular part of the potential \bar{U}_ε , the dependence on ε can be deduced using the fact that the Poisson equation is linear, that is, that

$$\Delta(\varepsilon^2 \bar{U}_\varepsilon) = 1 - h.$$

By applying Lemma 2.5 from Chapter 2 to $\varepsilon^2 \bar{U}_\varepsilon$, we deduce the following estimates:

- (i) If $h \in L^{\frac{d+2}{d}}(\mathbb{T}^d)$, then for all $\alpha \in (0, 1)$, if $d = 2$, or $\alpha \in (0, \frac{1}{5}]$ if $d = 3$, there exists a constant $C_{\alpha,d} > 0$ such that

$$\|\bar{U}_\varepsilon\|_{C^{0,\alpha}} \leq C_{\alpha,d} \varepsilon^{-2} \left(1 + \|h\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right)$$

- (ii) If $h \in L^\infty(\mathbb{T}^d)$, then for any $\alpha \in (0, 1)$, there exists a constant $C_{\alpha,d}$ such that

$$\|\bar{U}_\varepsilon\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C_{\alpha,d} \varepsilon^{-2} \left(1 + \|h\|_{L^\infty(\mathbb{T}^d)} \right)$$

We can also deduce the following quantified version of Lemma 2.6.

Lemma 3.4 (Log-Lipschitz regularity of \bar{E}_ε). *Let \bar{U}_ε be a solution of*

$$\varepsilon^2 \Delta \bar{U}_\varepsilon = h$$

for $h \in L^\infty(\mathbb{T}^d)$. Then

$$|\nabla \bar{U}_\varepsilon(x) - \nabla \bar{U}_\varepsilon(y)| \leq \varepsilon^{-2} C \|h\|_{L^\infty} |x - y| \left(1 + \log \left(\frac{\sqrt{d}}{|x - y|} \right) \mathbb{1}_{|x - y| \leq \sqrt{d}} \right).$$

3.2.1.2 Regularity of \hat{U}_ε

In Section 2.4.2, we proved the existence of a unique $W^{1,2}(\mathbb{T}^d)$ solution \hat{U} of the equation

$$\Delta \hat{U} = e^{\bar{U} + \hat{U}} - 1,$$

under the assumption that $\bar{U} \in L^\infty(\mathbb{T}^d) \cap W^{1,2}(\mathbb{T}^d)$. We showed that this solution in fact belongs to the Hölder space $C^{1,\alpha}(\mathbb{T}^d)$ for $\alpha \in (0, 1)$. Furthermore, if $\bar{U} \in C^{0,\alpha}(\mathbb{T}^d)$, then $\hat{U} \in C^{2,\alpha}(\mathbb{T}^d)$. The method of proof also applies for general ε . We will revisit part of the proof here in order to quantify how the constants depend on ε .

Lemma 3.5. *Consider the nonlinear Poisson equation*

$$\varepsilon^2 \Delta \hat{U}_\varepsilon = e^{\bar{U}_\varepsilon + \hat{U}_\varepsilon} - 1. \quad (3.5)$$

Assume that $\bar{U}_\varepsilon \in L^\infty(\mathbb{T}^d) \cap W^{1,2}(\mathbb{T}^d)$ with

$$\|\bar{U}_\varepsilon\|_{L^\infty(\mathbb{T}^d)} \leq M_1. \quad (3.6)$$

Then the unique $W^{1,2}(\mathbb{T}^d)$ solution \widehat{U}_ε of (3.5) belongs to $C^{1,\alpha}(\mathbb{T}^d)$ for all $\alpha \in (0, 1)$, with the estimate

$$\|\widehat{U}_\varepsilon\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C\varepsilon^{-2} (e^{2M_1} + 1).$$

If, in addition, \bar{U}_ε is Hölder regular with the estimate

$$\|\bar{U}_\varepsilon\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq M_2$$

for some $\alpha \in (0, 1)$, then $\widehat{U}_\varepsilon \in C^{2,\alpha}(\mathbb{T}^d)$ with the estimate

$$\|\widehat{U}_\varepsilon\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq [M_2 + C\varepsilon^{-2} (e^{2M_1} + 1)] \exp [C\varepsilon^{-2} (e^{2M_1} + 1)].$$

Proof. The idea is to prove $L^p(\mathbb{T}^d)$ estimates on the right hand side of (3.5), in particular on e^{U_ε} . The estimates on \widehat{U}_ε are then deduced using standard regularity estimates for the Poisson equation.

Since \bar{U}_ε is already controlled in L^∞ by assumption (3.6), it is enough to estimate $e^{\widehat{U}_\varepsilon}$. We prove an a priori estimate using the equation satisfied by \widehat{U}_ε :

$$\varepsilon^2 \Delta \widehat{U}_\varepsilon = e^{\bar{U}_\varepsilon + \widehat{U}_\varepsilon} - 1.$$

We use the function $e^{(p-1)\widehat{U}_\varepsilon}$ as a test function in the weak form of this equation. Then

$$\int_{\mathbb{T}^d} e^{(p-1)\widehat{U}_\varepsilon} dx = \varepsilon^2 (p-1) \int_{\mathbb{T}^d} |\nabla \widehat{U}_\varepsilon|^2 e^{\widehat{U}_\varepsilon} dx + \int_{\mathbb{T}^d} e^{\bar{U}_\varepsilon} \cdot e^{p\widehat{U}_\varepsilon} dx,$$

and so

$$\int_{\mathbb{T}^d} e^{\bar{U}_\varepsilon} \cdot e^{p\widehat{U}_\varepsilon} dx \leq \int_{\mathbb{T}^d} e^{(p-1)\widehat{U}_\varepsilon} dx,$$

We can make this estimate rigorous by using a truncation argument, as described in the proof of Lemma 2.9. Since

$$\|\bar{U}_\varepsilon\|_{L^\infty(\mathbb{T}^d)} \leq M_1,$$

we have

$$\|e^{\widehat{U}_\varepsilon}\|_{L^p(\mathbb{T}^d)}^p \leq e^{M_1} \|e^{\widehat{U}_\varepsilon}\|_{L^{p-1}(\mathbb{T}^d)}^{p-1}. \quad (3.7)$$

Note that the constants in this estimate do not depend on ε . We therefore deduce an estimate for all $p < \infty$ by induction, in the same way as we did in Lemma 2.9.

First, since

$$0 = \varepsilon^2 \int_{\mathbb{T}^d} \Delta \widehat{U}_\varepsilon dx = \int_{\mathbb{T}^d} (e^{U_\varepsilon} - 1) dx,$$

for any ε we still have

$$\|e^{U_\varepsilon}\|_{L^1(\mathbb{T}^d)} = 1.$$

Then

$$\|e^{\widehat{U}_\varepsilon}\|_{L^1(\mathbb{T}^d)} \leq e^{\|\bar{U}_\varepsilon\|_{L^\infty(\mathbb{T}^d)}} \leq e^{M_1}.$$

By induction, using (3.7), we conclude that for all integer p ,

$$\|e^{\widehat{U}_\varepsilon}\|_{L^p(\mathbb{T}^d)} \leq e^{M_1}. \quad (3.8)$$

By interpolation, this extends to any $p < \infty$.

Next, we use these estimates to deduce regularity for \widehat{U}_ε . By Calderón-Zygmund estimates for the Poisson equation,

$$\|\widehat{U}_\varepsilon\|_{W^{2,n}(\mathbb{T}^d)} \leq C_{n,d} \varepsilon^{-2} \|e^{\bar{U}_\varepsilon + \widehat{U}_\varepsilon}\|_{L^n(\mathbb{T}^d)} \leq C_{n,d} \varepsilon^{-2} (e^{2M_1} + 1).$$

By Sobolev embedding with n sufficiently large, for any $\alpha \in (0, 1)$,

$$\|\widehat{U}_\varepsilon\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq C_{\alpha,d} \varepsilon^{-2} (e^{2M_1} + 1).$$

Now assume that for some $\alpha \in (0, 1)$,

$$\|\bar{U}_\varepsilon\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq M_2.$$

Then $U_\varepsilon \in C^{0,\alpha}(\mathbb{T}^d)$, with

$$\|U_\varepsilon\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq \|\bar{U}_\varepsilon\|_{C^{0,\alpha}(\mathbb{T}^d)} + \|\widehat{U}_\varepsilon\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq M_2 + C \varepsilon^{-2} (e^{2M_1} + 1).$$

Since

$$\left| e^{U_\varepsilon(x)} - e^{U_\varepsilon(y)} \right| \leq e^{\max\{U_\varepsilon(x), U_\varepsilon(y)\}} |U_\varepsilon(x) - U_\varepsilon(y)|,$$

it follows that

$$\|e^{U_\varepsilon}\|_{C^{0,\alpha}(\mathbb{T}^d)} \leq \|U_\varepsilon\|_{C^{0,\alpha}(\mathbb{T}^d)} \exp [C \varepsilon^{-2} (1 + e^{2M_1})].$$

By Schauder estimates [30, Chapter 4],

$$\begin{aligned} \|\widehat{U}_\varepsilon\|_{C^{2,\alpha}(\mathbb{T}^d)} &\leq C \left(\|\widehat{U}_\varepsilon\|_{L^\infty(\mathbb{T}^d)} + \varepsilon^{-2} \|e^{U_\varepsilon} - 1\|_{C^{0,\alpha}(\mathbb{T}^d)} \right) \\ &\leq [M_2 + C \varepsilon^{-2} (1 + e^{2M_1})] \exp [C \varepsilon^{-2} (1 + e^{2M_1})], \end{aligned}$$

as required. \square

3.2.2 Stability

We are also interested in the stability of U_ε with respect to the charge density. The proposition below is a quantitative version of Proposition 2.11.

Proposition 3.6. *For each $i = 1, 2$, let $\bar{U}_\varepsilon^{(i)}$ be a zero-mean solution of*

$$\varepsilon^2 \Delta \bar{U}_\varepsilon^{(i)} = h_i - 1,$$

where $h_i \in L^\infty \cap L^{\frac{d+2}{d}}(\mathbb{T}^d)$. Then

$$\|\nabla \bar{U}_\varepsilon^{(1)} - \nabla \bar{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq \varepsilon^{-4} \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2). \quad (3.9)$$

Now, in addition, let $\widehat{U}_\varepsilon^{(i)}$ be a solution of

$$\varepsilon^2 \Delta \widehat{U}_\varepsilon^{(i)} = e^{\bar{U}_\varepsilon^{(i)} + \widehat{U}_\varepsilon^{(i)}} - 1.$$

Then

$$\begin{aligned} \|\nabla \widehat{U}_\varepsilon^{(1)} - \nabla \widehat{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 &\leq \overline{\text{exp}}_2 \left[C_d \varepsilon^{-2} \left(1 + \max_i \|h_i\|_{L^{(d+2)/d}(\mathbb{T}^d)} \right) \right] \\ &\quad \times \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2). \end{aligned} \quad (3.10)$$

For the stability of $\nabla \bar{U}_\varepsilon$, we use a result proved originally by Loeper [65] in the whole space \mathbb{R}^d , and adapted to the torus \mathbb{T}^d in [43].

Lemma 3.7 (Loeper-type estimate for Poisson's equation). *For each $i = 1, 2$, let $\bar{U}_\varepsilon^{(i)}$ be a solution of*

$$\varepsilon^2 \Delta \bar{U}_\varepsilon^{(i)} = h_i - 1,$$

where $h_i \in L^\infty(\mathbb{T}^d)$. Then

$$\|\nabla \bar{U}_\varepsilon^{(1)} - \nabla \bar{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq \varepsilon^{-4} \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2).$$

For \widehat{U}_ε we use a version of Proposition 2.11 in which we quantify the dependence of the constants on ε .

Lemma 3.8. For each $i = 1, 2$, let $\widehat{U}_\varepsilon^{(i)} \in W^{1,2} \cap L^\infty(\mathbb{T}^d)$ be a solution of

$$\varepsilon^2 \Delta \widehat{U}_\varepsilon^{(i)} = e^{\bar{U}_\varepsilon^{(i)} + \widehat{U}_\varepsilon^{(i)}} - 1, \quad (3.11)$$

for some given potentials $\bar{U}_\varepsilon^{(i)} \in L^\infty(\mathbb{T}^d)$. Then

$$\varepsilon^2 \|\nabla \widehat{U}_\varepsilon^{(1)} - \nabla \widehat{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq C_\varepsilon \|\bar{U}_\varepsilon^{(1)} - \bar{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2,$$

where C depends on the L^∞ norms of $\widehat{U}_\varepsilon^{(i)}$ and $\bar{U}_\varepsilon^{(i)}$. More precisely, C_ε can be chosen such that

$$C_\varepsilon \leq \exp \left[C \left(\max_i \|\bar{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} + \max_i \|\widehat{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} \right) \right],$$

for some sufficiently large constant C , independent of ε .

Proof. We look at the equation solved by the difference $\widehat{U}_\varepsilon^{(1)} - \widehat{U}_\varepsilon^{(2)}$. By subtracting the two equations (3.11), we find that

$$\varepsilon^2 \Delta (\widehat{U}_\varepsilon^{(1)} - \widehat{U}_\varepsilon^{(2)}) = e^{\bar{U}_\varepsilon^{(1)} + \widehat{U}_\varepsilon^{(1)}} - e^{\bar{U}_\varepsilon^{(2)} + \widehat{U}_\varepsilon^{(2)}}.$$

By taking $\widehat{U}_\varepsilon^{(1)} - \widehat{U}_\varepsilon^{(2)}$ as a test function, we find that

$$-\varepsilon^2 \int_{\mathbb{T}^d} |\nabla \widehat{U}_\varepsilon^{(1)} - \nabla \widehat{U}_\varepsilon^{(2)}|^2 dx = \int_{\mathbb{T}^d} \left(e^{\bar{U}_\varepsilon^{(1)} + \widehat{U}_\varepsilon^{(1)}} - e^{\bar{U}_\varepsilon^{(2)} + \widehat{U}_\varepsilon^{(2)}} \right) (\widehat{U}_\varepsilon^{(1)} - \widehat{U}_\varepsilon^{(2)}) dx.$$

In the proof of Proposition 2.11, we showed that for any $\bar{U}, \widehat{U}, \bar{V}, \widehat{V} \in L^\infty(\mathbb{T}^d)$,

$$-\int_{\mathbb{T}^d} \left(e^{\bar{U} + \widehat{U}} - e^{\bar{V} + \widehat{V}} \right) (\widehat{U} - \widehat{V}) dx \leq C_{U,V} \|\bar{U} - \bar{V}\|_{L^2(\mathbb{T}^d)}^2,$$

where

$$C_{U,V} \leq \exp \left\{ C \left[\|\bar{U}\|_{L^\infty(\mathbb{T}^d)} + \|\widehat{U}\|_{L^\infty(\mathbb{T}^d)} + \|\bar{V}\|_{L^\infty(\mathbb{T}^d)} + \|\widehat{V}\|_{L^\infty(\mathbb{T}^d)} \right] \right\}.$$

Thus

$$\|\nabla \widehat{U}_\varepsilon^{(1)} - \nabla \widehat{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq C_\varepsilon \varepsilon^{-2} \|\bar{U}_\varepsilon^{(1)} - \bar{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2,$$

where

$$C_\varepsilon \leq \exp \left[C \left(\max_i \|\bar{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} + \max_i \|\widehat{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} \right) \right].$$

□

Proof of Proposition 3.6. It suffices to prove (3.10), since (3.9) follows immediately from Lemma 3.7. By the Poincaré inequality for zero-mean functions,

$$\|\bar{U}_\varepsilon^{(1)} - \bar{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq C \|\nabla \bar{U}_\varepsilon^{(1)} - \nabla \bar{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2.$$

By Lemma 3.7,

$$\|\nabla \bar{U}_\varepsilon^{(1)} - \nabla \bar{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq \varepsilon^{-4} \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2).$$

Then, by Lemma 3.8,

$$\|\nabla \hat{U}_\varepsilon^{(1)} - \nabla \hat{U}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq C_\varepsilon \varepsilon^{-6} \max_i \|h_i\|_{L^\infty(\mathbb{T}^d)} W_2^2(h_1, h_2),$$

for

$$C_\varepsilon \leq \exp \left[C \left(\max_i \|\bar{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} + \max_i \|\hat{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} \right) \right].$$

For the $L^\infty(\mathbb{T}^d)$ estimates, we use Proposition 3.3:

$$\begin{aligned} \|\bar{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} &\leq C_d \varepsilon^{-2} \left(1 + \|h_i\|_{L^{\frac{d+2}{d}}} \right) \\ \|\hat{U}_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} &\leq C_d \exp \left[C_d \varepsilon^{-2} \left(1 + \|h_i\|_{L^{\frac{d+2}{d}}} \right) \right]. \end{aligned}$$

Hence

$$C_\varepsilon \leq \overline{\exp}_2 \left[C_d \varepsilon^{-2} \left(1 + \max_i \|h_i\|_{L^{\frac{d+2}{d}}} \right) \right],$$

which implies (3.10). □

3.3 Wasserstein Stability

To prove the quasi-neutral limit, we will need a quantified stability estimate between solutions of the VPME system (3.1). The following proposition is an ε -dependent version of Proposition 2.12. We revisit the proof here, keeping track of how all the constants depend on ε .

Proposition 3.9 (Stability for solutions with bounded density). *For $i = 1, 2$, let $f_\varepsilon^{(i)}$ be solutions of (3.1) satisfying for some constant M and all $t \in [0, T]$,*

$$\rho[f_\varepsilon^{(i)}(t)] \leq M.$$

Then there exists a constant C_ε such that, for all $t \in [0, T]$,

$$W_2 \left(f_\varepsilon^{(1)}(t), f_\varepsilon^{(2)}(t) \right) \leq \beta \left(t, W_2 \left(f_\varepsilon^{(1)}(0), f_\varepsilon^{(2)}(0) \right) \right),$$

where β denotes the function

$$\beta(t, x) = \begin{cases} 16de \exp \left[\left(1 + \log \frac{x}{16d} \right) e^{-C_\varepsilon t} \right] & t \leq T_0 \\ (d \vee x) e^{C_\varepsilon(1+\log 16)(t-T_0)} & t > T_0, \end{cases}$$

where

$$T_0 = \inf \left\{ t > 0 : 16de \exp \left[\left(1 + \log \frac{x}{16d} \right) e^{-C_\varepsilon t} \right] > d \right\}.$$

If in addition, for some constant C_0 ,

$$\sup_{t \in [0, T]} \|f_\varepsilon^{(i)}(t, \cdot, \cdot)\|_{L_{x,v}^\infty} \leq C_0, \quad \sup_{t \in [0, T]} \mathcal{E}_\varepsilon^{ME}[f_\varepsilon^{(i)}](t) \leq C_0, \quad (3.12)$$

then C_ε may be chosen to satisfy

$$C_\varepsilon \leq \overline{\text{exp}}_2(C\varepsilon^{-2})(M+1).$$

Proof. The W_2 distance is defined as an infimum over couplings (Definition 4). Thus it suffices to estimate the L^2 distance corresponding to a particular coupling of $f_\varepsilon^{(1)}$ and $f_\varepsilon^{(2)}$. To do this we will first represent $f_\varepsilon^{(i)}$ as the pushforward of $f_\varepsilon^{(i)}(0)$ along the characteristic flow induced by $f_\varepsilon^{(i)}$. That is, consider the following system of ODEs for $Z_z^{(i)} = (X_z^{(i)}, V_z^{(i)}) \in \mathbb{T}^d \times \mathbb{R}^d$:

$$\begin{cases} \dot{X}_z^{(i)} = V_z^{(i)}, \\ \dot{V}_z^{(i)} = E_\varepsilon^{(i)}(X_z^{(i)}), \\ (X_z^{(i)}(0), V_z^{(i)}(0)) = (x, v) = z, \end{cases} \quad (3.13)$$

where the electric field $E_\varepsilon^{(i)}$ is given by

$$\begin{aligned} E_\varepsilon^{(i)} &= -\nabla U_\varepsilon^{(i)}, \\ \varepsilon^2 \Delta U_\varepsilon^{(i)} &= e^{U_\varepsilon^{(i)}} - \rho_\varepsilon^{(i)} := e^{U_\varepsilon^{(i)}} - \rho[f_\varepsilon^{(i)}]. \end{aligned}$$

Since $f_\varepsilon^{(i)}$ has bounded mass density, Proposition 3.3 and Lemma 3.4 imply that $E_\varepsilon^{(i)}$ is a log-Lipschitz vector field, with a constant uniform on $[0, T]$. This regularity is enough to imply that there exists a unique solution of (3.13) for every initial condition z , resulting in a

well-defined characteristic flow. Since the characteristic flow is unique, by [1, Theorem 3.1] the linear Vlasov equation

$$\begin{cases} \partial_t g + v \cdot \nabla_x g + E_\varepsilon^{(i)}(x) \cdot \nabla_v g = 0, \\ g|_{t=0} = f_\varepsilon^{(i)}(0) \geq 0 \end{cases} \quad (3.14)$$

has a unique solution. Moreover, this solution can be represented, in weak form, by the relation

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} \phi g_t(dz) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi \left(Z_z^{(i)} \right) f_\varepsilon^{(i)}(0)(dz), \quad (3.15)$$

for all $\phi \in C_b(\mathbb{T}^d \times \mathbb{R}^d)$. Since $f_\varepsilon^{(i)}$ is certainly a solution of (3.14), we deduce that $g = f_\varepsilon^{(i)}$ and so $f_\varepsilon^{(i)}$ has the representation (3.15). We will use this representation to define a coupling between $f_\varepsilon^{(1)}$ and $f_\varepsilon^{(2)}$.

Fix an arbitrary initial coupling $\pi_0 \in \Pi \left[f_\varepsilon^{(1)}(0), f_\varepsilon^{(2)}(0) \right]$. We define π_t to follow the corresponding characteristic flows: for $\phi \in C_b((\mathbb{T}^d \times \mathbb{R}^d)^2)$, let

$$\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \phi(z_1, z_2) d\pi_t(z_1, z_2) = \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi \left(Z_z^{(1)}, Z_z^{(2)} \right) d\pi_0(z_1, z_2).$$

We can verify that π_t is indeed a coupling of $f_\varepsilon^{(1)}$ and $f_\varepsilon^{(2)}$ by checking the marginals:

$$\begin{aligned} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \phi(z_i) d\pi_t(z_1, z_2) &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \phi \left(Z_{z_i}^{(i)}(t) \right) d\pi_0(z_1, z_2) \\ &= \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi \left(Z_z^{(i)}(t) \right) f_\varepsilon^{(i)}(0)(dz) \\ &= \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(z) f_\varepsilon^{(i)}(t)(dz). \end{aligned} \quad (3.16)$$

Next, we define a functional which is greater than (or equal to) the squared Wasserstein distance between $f_\varepsilon^{(1)}$ and $f_\varepsilon^{(2)}$. Let

$$D(t) = \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x_1 - x_2|^2 + |v_1 - v_2|^2 d\pi_t(z_1, z_2).$$

Since the Wasserstein distance is an infimum over couplings, while π_t is a particular coupling, D must control the squared Wasserstein distance:

$$W_2^2(f_\varepsilon^{(1)}, f_\varepsilon^{(2)}) \leq D.$$

To prove our stability estimate, it therefore suffices to estimate D . We will do this using a Grönwall estimate. Taking a time derivative, we find

$$\begin{aligned} \dot{D} = 2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} & \left(X_{z_1}^{(1)}(t) - X_{z_2}^{(2)}(t) \right) \cdot \left(V_{z_1}^{(1)}(t) - V_{z_2}^{(2)}(t) \right) \\ & + \left(V_{z_1}^{(1)}(t) - V_{z_2}^{(2)}(t) \right) \cdot \left(E_{\varepsilon}^{(1)}(t, X_{z_1}^{(1)}(t)) - E_{\varepsilon}^{(2)}(t, X_{z_2}^{(2)}(t)) \right) d\pi_0(z_1, z_2). \end{aligned}$$

Using the Cauchy-Schwarz inequality, we obtain

$$\dot{D} \leq D + \sqrt{D} \left(\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \left| E_{\varepsilon}^{(1)}(t, X_{z_1}^{(1)}(t)) - E_{\varepsilon}^{(2)}(t, X_{z_2}^{(2)}(t)) \right|^2 d\pi_0(z_1, z_2) \right)^{1/2}.$$

In other words,

$$\dot{D} \leq D + \sqrt{D} \left(\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \left| E_{\varepsilon}^{(1)}(t, x_1) - E_{\varepsilon}^{(2)}(t, x_2) \right|^2 d\pi_t(z_1, z_2) \right)^{1/2}.$$

We split the electric field term into the form

$$\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \left| E_{\varepsilon}^{(1)}(t, x_1) - E_{\varepsilon}^{(2)}(t, x_2) \right|^2 d\pi_t(z_1, z_2) \leq C \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned} I_1 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\bar{E}_{\varepsilon}^{(1)}(t, x_1) - \bar{E}_{\varepsilon}^{(1)}(t, x_2)|^2 d\pi_t, & I_2 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\bar{E}_{\varepsilon}^{(1)}(t, x_2) - \bar{E}_{\varepsilon}^{(2)}(t, x_2)|^2 d\pi_t, \\ I_3 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\hat{E}_{\varepsilon}^{(1)}(t, x_1) - \hat{E}_{\varepsilon}^{(1)}(t, x_2)|^2 d\pi_t, & I_4 &:= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\hat{E}_{\varepsilon}^{(1)}(t, x_2) - \hat{E}_{\varepsilon}^{(2)}(t, x_2)|^2 d\pi_t. \end{aligned}$$

Control of I_1 : To estimate I_1 , observe that by Lemma 3.4,

$$I_1 \leq C\varepsilon^{-4} \|\rho_{\varepsilon}^{(1)}\|_{L^{\infty}(\mathbb{T}^d)}^2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x_1 - x_2|^2 \left(1 + \log \left(\frac{\sqrt{d}}{|x_1 - x_2|} \right) \mathbb{1}_{|x_1 - x_2| \leq \sqrt{d}} \right)^2 d\pi_t.$$

As in Lemma 2.15, we use the function

$$H(x) := \begin{cases} x \left(\log \frac{x}{16d} \right)^2 & \text{if } x \leq d \\ d (\log 16)^2 & \text{if } x > d, \end{cases}$$

which is concave on \mathbb{R}_+ . Then

$$I_1 \leq C\varepsilon^{-4} \|\rho_\varepsilon^{(1)}\|_{L^\infty(\mathbb{T}^d)}^2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} H(|x_1 - x_2|^2) d\pi_t.$$

By Jensen's inequality,

$$I_1 \leq C\varepsilon^{-4} \|\rho_\varepsilon^{(1)}\|_{L^\infty(\mathbb{T}^d)}^2 H\left(\int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x_1 - x_2|^2 d\pi_t\right) \leq C\varepsilon^{-4} M^2 H(D).$$

Control of I_2 : For I_2 , observe that

$$\begin{aligned} I_2 &\leq \int_{\mathbb{T}^d} |\bar{E}_\varepsilon^{(1)}(x) - \bar{E}_\varepsilon^{(2)}(x)|^2 \rho_\varepsilon^{(2)}(dx) \\ &\leq \|\rho_\varepsilon^{(2)}\|_{L^\infty(\mathbb{T}^d)} \|\bar{E}_\varepsilon^{(1)} - \bar{E}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2. \end{aligned}$$

We apply the Loeper stability estimate (Lemma 3.7) to obtain

$$I_2 \leq \varepsilon^{-4} \max_i \|\rho_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)}^2 W_2^2(\rho_\varepsilon^{(1)}, \rho_\varepsilon^{(2)}) \leq C\varepsilon^{-4} M^2 D.$$

Control of I_3 : To estimate I_3 , we recall a regularity estimate on $\widehat{U}_\varepsilon^{(1)}$ from Proposition 3.3:

$$\|\widehat{E}_\varepsilon^{(1)}\|_{C^1(\mathbb{T}^d)} \leq \|\widehat{U}_\varepsilon^{(1)}\|_{C^2(\mathbb{T}^d)} \leq C_d \overline{\exp}_2 \left(C_d \varepsilon^{-2} \left(1 + \|\rho_\varepsilon^{(1)}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \right) \right).$$

Under condition (3.12), by Lemma 1.12,

$$\|\rho_\varepsilon^{(1)}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C$$

for some C depending on C_0 only. Therefore

$$I_3 \leq \|\widehat{E}_\varepsilon^{(1)}\|_{C^1(\mathbb{T}^d)}^2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x_1 - x_2|^2 d\pi_t \leq \overline{\exp}_2(C\varepsilon^{-2}) D,$$

for some C depending on C_0 and d only. If (3.12) does not hold, we can use the fact that

$$\|\rho_\varepsilon^{(1)}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq \|\rho_\varepsilon^{(1)}\|_{L^\infty(\mathbb{T}^d)} \leq M$$

and complete the proof in the same way, to find a constant depending on M .

Control of I_4 : First, note that

$$I_4 = \int_{\mathbb{T}^d} |\widehat{E}_\varepsilon^{(1)}(x) - \widehat{E}_\varepsilon^{(2)}(x)|^2 \rho_\varepsilon^{(2)}(dx) \leq \|\rho_\varepsilon^{(2)}\|_{L^\infty(\mathbb{T}^d)} \|\widehat{E}_\varepsilon^{(1)} - \widehat{E}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2.$$

We apply the stability estimate on $\widehat{U}_\varepsilon^{(1)}$ from Proposition 3.6 to find

$$\begin{aligned} \|\widehat{E}_\varepsilon^{(1)} - \widehat{E}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 &\leq \overline{\exp}_2 \left[C_d \varepsilon^{-2} \left(1 + \max_i \|\rho_\varepsilon^{(i)}\|_{L^{(d+2)/d}(\mathbb{T}^d)} \right) \right] \\ &\quad \times \max_i \|\rho_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} W_2^2(\rho_\varepsilon^{(1)}, \rho_\varepsilon^{(2)}). \end{aligned}$$

Once again, if condition (3.12) holds then

$$\|\widehat{E}_\varepsilon^{(1)} - \widehat{E}_\varepsilon^{(2)}\|_{L^2(\mathbb{T}^d)}^2 \leq \overline{\exp}_2(C\varepsilon^{-2}) \max_i \|\rho_\varepsilon^{(i)}\|_{L^\infty(\mathbb{T}^d)} W_2^2(\rho_\varepsilon^{(1)}, \rho_\varepsilon^{(2)}),$$

for some C depending on C_0 and d only. Thus

$$I_4 \leq \overline{\exp}_2(C\varepsilon^{-2}) M^2 D.$$

Altogether we find that

$$D \leq \begin{cases} C_\varepsilon D (1 + |\log \frac{D}{16d}|) & \text{if } D < d \\ C_\varepsilon (1 + \log 16) D & \text{if } D \geq d. \end{cases}$$

If (3.12) holds, then C_ε may be chosen to satisfy

$$\begin{aligned} C_\varepsilon &\leq C\varepsilon^{-2} M + \overline{\exp}_2(C\varepsilon^{-2}) + \overline{\exp}_2(C\varepsilon^{-2}) M \\ &\leq \overline{\exp}_2(C\varepsilon^{-2}) (M + 1). \end{aligned}$$

We conclude that

$$D(t) \leq \beta(t, D(0)),$$

following the argument used in the proof of Lemma 2.14.

□

3.4 Growth Estimates

In this section we will study how the support of a solution of the VPME system grows in time. We present two different proofs for the two and three dimensional case, respectively in Sections 3.4.1 and 3.4.2. The result we obtain is the following:

Proposition 3.10 (Mass bounds). *Let f_ε be a solution of (3.1) satisfying for some constant C_0 ,*

$$\|f_\varepsilon\|_{L^\infty([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)} \leq C_0, \quad \sup_{t \in [0,T]} \mathcal{E}_\varepsilon^{ME}[f_\varepsilon(t)] \leq C_0.$$

Assume that $\rho_{f_\varepsilon} \in L^\infty([0,T]; L^\infty(\mathbb{T}^d))$. Let R_0 be any constant such that the support of $f_\varepsilon(0, \cdot, \cdot)$ is contained in $\mathbb{T}^d \times B_{\mathbb{R}^d}(0, R_0)$.

(i) *If $d = 2$, then:*

$$\sup_{t \in [0,T]} \|\rho_{f_\varepsilon}(t)\|_{L^\infty(\mathbb{T}^2)} \leq C_T e^{C\varepsilon^{-2}} [R_0 + \varepsilon^{-2}]^2.$$

The constant C depends on C_0 only, while C_T depends on C_0 and T .

(ii) *If $d = 3$, then:*

$$\sup_{t \in [0,T]} \|\rho_\varepsilon(t)\|_{L^\infty(\mathbb{T}^3)} \leq \max\{T^{-81/8}, C(R_0^3 + e^{C\varepsilon^{-2}} T^6)\}.$$

The constant C depends on C_0 only.

3.4.1 Proof of Proposition 3.10 in the two dimensional case

In this section, we prove an estimate on the mass density ρ_ε in the case $d = 2$. Observe first that if the support of f_ε is contained in the set $\mathbb{T}^2 \times B_{\mathbb{R}^2}(0; R_t)$, then

$$\|\rho_\varepsilon(t)\|_{L^\infty(\mathbb{T}^d)} \leq CR_t^2, \tag{3.17}$$

where C is a constant depending on $\|f_\varepsilon(0)\|_{L^\infty(\mathbb{T}^2 \times \mathbb{R}^2)}$. Our argument will rely on controlling the growth of the support of f_ε . To do this we will find a bound on the growth rate of the velocity component of the characteristic trajectories. Since any characteristic trajectory (X_t, V_t) satisfies

$$\dot{V}_t = E_\varepsilon(X_t),$$

we look for a uniform estimate on the electric field E_ε .

By Proposition 3.3, for the smooth part \widehat{E}_ε we have the estimate

$$\begin{aligned}\|\widehat{E}_\varepsilon\|_{L^\infty(\mathbb{T}^2)} &\leq \exp\left\{C\varepsilon^{-2}\left(1 + \|\rho_\varepsilon\|_{L^2(\mathbb{T}^2)}\right)\right\} \\ &\leq \exp(C\varepsilon^{-2}),\end{aligned}$$

where the constant C depends only on a bound on the initial energy. However, for the singular part \bar{E}_ε , Proposition 3.3 only gives us the estimate

$$\|\bar{E}_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \leq C\varepsilon^{-2}\left(1 + \|\rho_\varepsilon\|_{L^\infty(\mathbb{T}^d)}\right),$$

which depends on an L^∞ bound on the mass density. If we use this estimate in combination with (3.17), this results in a bound on the size of the support of the form

$$R_t \leq R_0 + \exp(C\varepsilon^{-2})t + C\varepsilon^{-2} \int_0^t R_s^2 ds. \quad (3.18)$$

The solution of the ODE

$$\dot{y} = C(1 + y^2)$$

blows up in finite time and so the differential inequality (3.18) is not enough to imply a bound on R_t . We need to use a more careful estimate on \bar{E}_ε . In dimension two, we can make use of the fact that the conservation of energy gives us a uniform bound on $\|\rho_\varepsilon\|_{L^2(\mathbb{T}^2)}$, by Lemma 1.12, and use an interpolation argument.

Lemma 3.11. *Let $\rho \in L^1 \cap L^\infty(\mathbb{T}^2)$ satisfy the bounds*

$$\|\rho\|_{L^1(\mathbb{T}^2)} = 1, \quad \|\rho\|_{L^2(\mathbb{T}^2)} \leq C_0, \quad \|\rho\|_{L^\infty(\mathbb{T}^2)} \leq M.$$

Let $\bar{E}_\varepsilon = -\nabla \bar{U}_\varepsilon$, where \bar{U}_ε is the unique $W^{1,2}(\mathbb{T}^2)$ solution of the Poisson equation

$$\varepsilon^2 \Delta \bar{U}_\varepsilon = 1 - \rho.$$

Then there exists a constant C depending only on C_0 such that

$$\|\bar{E}_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \leq C\varepsilon^{-2}\left(1 + |\log M|^{1/2}\right).$$

Proof. We use the representation

$$\bar{E}_\varepsilon = \varepsilon^{-2} K_{\text{per}} * (\rho - 1) = C\varepsilon^{-2} \int_{[-\frac{1}{2}, \frac{1}{2}]^2} \frac{y}{|y|^2} (\rho(x - y) - 1) dy + \varepsilon^{-2} K_0 * (\rho - 1),$$

where K_0 is a $C^1(\mathbb{T}^2)$ function. By Young's inequality,

$$\|K_0 * (\rho - 1)\|_{L^\infty(\mathbb{T}^2)} \leq \|K_0\|_{L^\infty(\mathbb{T}^2)} \|\rho - 1\|_{L^1(\mathbb{T}^2)} \leq C.$$

We split the integral term into a part where $|y|$ is small and a part where $|y|$ is large:

$$\int_{[-\frac{1}{2}, \frac{1}{2}]^2} \frac{y}{|y|^2} (\rho(x-y) - 1) dy = \int_{|y| \leq l} \frac{y}{|y|^2} (\rho(x-y) - 1) dy + \int_{2 \geq |y| \geq l} \frac{y}{|y|^2} (\rho(x-y) - 1) dy.$$

For the part where $|y|$ is large, we use Young's inequality with the L^2 control on ρ :

$$\begin{aligned} \left| \int_{2 \geq |y| \geq l} \frac{y}{|y|^2} (\rho(x-y) - 1) dy \right| &\leq \int_{2 \geq |y| \geq l} \frac{1}{|y|} |\rho(x-y) - 1| dy \\ &\leq \left\| \frac{1}{|y|} \mathbb{1}_{2 \geq |y| \geq l} * |\rho - 1| \right\|_{L^\infty(\mathbb{T}^2)} \\ &\leq \left(\int_{2 \geq |y| \geq l} \frac{1}{|y|^2} dy \right)^{1/2} \|\rho - 1\|_{L^2(\mathbb{T}^2)} \\ &\leq C(C_0 + 1) |\log l|^{1/2}. \end{aligned}$$

Where $|y|$ is small, we use Young's inequality with the L^∞ control on ρ :

$$\begin{aligned} \left| \int_{|y| \leq l} \frac{y}{|y|^2} (\rho(x-y) - 1) dy \right| &\leq \int_{|y| \leq l} \frac{1}{|y|} |\rho(x-y) - 1| dy \\ &\leq \left\| \frac{1}{|y|} \mathbb{1}_{|y| \leq l} * |\rho - 1| \right\|_{L^\infty(\mathbb{T}^2)} \\ &\leq \|\rho - 1\|_{L^\infty(\mathbb{T}^2)} \int_{|y| \leq l} \frac{1}{|y|} dy \\ &\leq CMl. \end{aligned}$$

Altogether we obtain

$$\|\bar{E}_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \leq C\varepsilon^{-2} \left[1 + C_0 |\log l|^{1/2} + Ml \right].$$

We choose $l = M^{-1}$ and conclude that

$$\|\bar{E}_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \leq C\varepsilon^{-2} \left(1 + |\log M|^{1/2} \right),$$

where C depends on C_0 only.

□

By using this estimate, we can deduce a differential inequality on R_t that can be closed.

Lemma 3.12. *Let f_ε be a solution of (3.1) with bounded energy and compact support contained in $\mathbb{T}^2 \times B_{\mathbb{R}^2}(0; R_t)$ at time t . Then R satisfies the estimate*

$$R_t \leq e^{C\varepsilon^{-2}t} (1 + R_0 + (\log t \vee 0)).$$

Proof. We consider the velocity coordinate $V_t(x, v)$ of an arbitrary characteristic trajectory starting from (x, v) at time $t = 0$. We have

$$\begin{aligned} |V_t(x, v)| &\leq |v| + \int_0^t \|E_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \, ds \\ &\leq |v| + \int_0^t \|\widehat{E}_\varepsilon\|_{L^\infty(\mathbb{T}^2)} + \|\bar{E}_\varepsilon\|_{L^\infty(\mathbb{T}^2)} \, ds \\ &\leq |v| + \int_0^t \exp(C\varepsilon^{-2}) + C\varepsilon^{-2} (1 + |\log R_s|^{1/2}) \, ds \end{aligned}$$

The size of the support is controlled by the modulus of the furthest-reaching characteristic trajectory that starts within the support of $f_\varepsilon(0)$:

$$\begin{aligned} R_t &\leq \sup_{(x,v) \in \mathbb{T}^2 \times B_{\mathbb{R}^2}(0; R_0)} |V_t(x, v)| \\ &\leq \sup_{(x,v) \in \mathbb{T}^2 \times B_{\mathbb{R}^2}(0; R_0)} \left\{ |v| + \int_0^t \exp(C\varepsilon^{-2}) + C\varepsilon^{-2} (1 + |\log R_s|^{1/2}) \, ds \right\} \\ &\leq R_0 + \int_0^t \exp(C\varepsilon^{-2}) + C\varepsilon^{-2} |\log R_s|^{1/2} \, ds. \end{aligned}$$

We compare this with the function

$$z(t) = (1 + 2Ct) [R_0 + \log(1 + 2Ct)].$$

By Lemma A.1, this satisfies

$$\dot{z} \geq C(1 + \log(1 + z)).$$

We deduce that

$$R_t \leq e^{C\varepsilon^{-2}t} (\varepsilon^{-2} + R_0 + (\log t \vee 0)).$$

□

Proof of Proposition 3.10, case $d = 2$. We combine Lemma 3.12 with the elementary estimate (3.17): for all $t \in [0, T]$,

$$\begin{aligned} \|\rho_\varepsilon(t)\|_{L^\infty(\mathbb{T}^d)} &\leq CR_t^2 \\ &\leq e^{C\varepsilon^{-2}} t \left(\varepsilon^{-2} + R_0 + (\log t \vee 0) \right)^2 \\ &\leq C_T e^{C\varepsilon^{-2}} (R_0 + \varepsilon^{-2})^2. \end{aligned}$$

□

3.4.2 Proof of Proposition 3.10 in the three dimensional case

In this section, we prove a mass bound in the case $d = 3$. In this case, the conservation of energy gives us a uniform bound on $\|\rho_\varepsilon\|_{L^{5/3}(\mathbb{T}^3)}$. This integrability is not enough to allow us to use the elementary interpolation approach that we used in the two dimensional case. Instead, we will adapt estimates devised by Batt and Rein [10] for the classical Vlasov-Poisson equation on $\mathbb{T}^3 \times \mathbb{R}^3$. We used this approach in Chapter 2 to prove the existence of solutions of the VPME system with bounded density. Here we focus on identifying how the bounds on $\|\rho_\varepsilon\|_{L^\infty(\mathbb{T}^3)}$ depend on ε .

As in the two dimensional case, Batt and Rein's argument relies on controlling the mass density using the characteristic trajectory with velocity component of greatest Euclidean norm starting within the support of $f_\varepsilon(0)$ at time zero. They prove a bootstrap estimate on the convolution integral defining the singular part of the electric field. We recall this estimate in the following technical lemma.

Lemma 3.13. *Let $(X(t; s, x, v), V(t; s, x, v))$ denote the solution at time t of an ODE*

$$\begin{pmatrix} \dot{X}(t) \\ \dot{V}(t) \end{pmatrix} = a(t, X(t), V(t)), \quad \begin{pmatrix} X(s) \\ V(s) \end{pmatrix} = \begin{pmatrix} x \\ v \end{pmatrix},$$

where a is of the form

$$a(t, X, V) = \begin{pmatrix} V \\ a_2(t, X, V) \end{pmatrix},$$

for some vector-field $a_2 : [0, T] \times \mathbb{T}^3 \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$.

Assume that, for $t \in [0, T]$, $f = f(t, x, v)$ is the pushforward of f_0 along the associated characteristic flow; that is, for all $\phi \in C_b(\mathbb{T}^3 \times \mathbb{R}^3)$,

$$\int_{\mathbb{T}^3 \times \mathbb{R}^3} f(t, x, v) \phi(x, v) \, dx \, dv = \int_{\mathbb{T}^3 \times \mathbb{R}^3} f(s, x, v) \phi(X(t; s, x, v), V(t; s, x, v)) \, dx \, dv.$$

Assume that there exists $C_* > 1$ such that

$$\|f\|_{L^\infty([0,T] \times \mathbb{T}^3 \times \mathbb{R}^3)} \leq C_*, \quad \sup_{t \in [0,T]} \|f_t |v|^2\|_{L^1(\mathbb{T}^3 \times \mathbb{R}^3)} \leq C_*.$$

Also, suppose that, for some $\beta \in (0, 1)$,

$$h_\eta(t, \Delta) \leq C_* h_\rho(t)^\beta \Delta \quad \text{for all } h_\rho(t)^{-\beta} \leq \Delta \leq t, \quad (3.19)$$

where h_ρ, h_η are defined as in (2.47). Then for all $0 \leq t_1 < t_2 \leq t$ with $t_2 - t_1 \leq \Delta$, if

$$h_\rho(t)^{-\beta/2} \leq \Delta \leq t$$

then

$$\int_{t_1}^{t_2} \int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_f(s, y) dy ds \leq C \left(h_\rho(t)^{2\beta/3} + h_\rho(t)^{1/6} \right) \Delta,$$

where C depends only on C_* .

We complete the proof of Proposition 3.10 by combining Lemma 3.13 with the estimates on \widehat{E}_ε from Proposition 3.3.

Proof of Proposition 3.10, case $d = 3$. For any characteristic trajectory (X_t, V_t) , we have for any $0 \leq t_1 < t_2 \leq T$,

$$|V_{t_1} - V_{t_2}| \leq \int_{t_2}^{t_1} |E_\varepsilon(X_s)| ds.$$

We can write the total force E_ε in the form

$$E_\varepsilon(x) = \varepsilon^{-2} [K_0 * (\rho_\varepsilon - 1)](x) + C\varepsilon^{-2} \int_{\mathbb{T}^3} \frac{x-y}{|x-y|^3} \rho_\varepsilon(y) dy + \widehat{E}_\varepsilon.$$

Since K_0 is a $C^1(\mathbb{T}^3)$ function and ρ_ε has unit mass, the first term satisfies the bound

$$\begin{aligned} |\varepsilon^{-2} [K_0 * (\rho_\varepsilon - 1)](x)| &\leq \varepsilon^{-2} \|K_0\|_{L^\infty(\mathbb{T}^3)} \|\rho_\varepsilon - 1\|_{L^1(\mathbb{T}^3)} \\ &\leq C\varepsilon^{-2}. \end{aligned}$$

For the last term, we use Proposition 3.3:

$$|\widehat{E}_\varepsilon| \leq \exp \left[C \left(1 + \|\rho_\varepsilon\|_{L^{5/3}(\mathbb{T}^3)} \right) \right] \leq \exp(C\varepsilon^{-2}).$$

Therefore,

$$|V_{t_1} - V_{t_2}| \leq \int_{t_2}^{t_1} \left[\exp(C\varepsilon^{-2}) + C\varepsilon^{-2} \int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_\varepsilon(y) dy \right] ds. \quad (3.20)$$

From [50, Lemma 4.5.4] we have the estimate

$$\int_{\mathbb{T}^3} |x - y|^{-2} \rho_\varepsilon(s, y) dy \leq C \|\rho_\varepsilon(s, \cdot)\|_{L^{5/3}}^{5/9} \|\rho_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{T}^3)}^{4/9} \leq C \|\rho_\varepsilon(s, \cdot)\|_{L^\infty(\mathbb{T}^3)}^{4/9},$$

where C depends only on $\|f_\varepsilon(0)\|_{L^\infty(\mathbb{T}^3 \times \mathbb{R}^3)}$ and the initial energy.

By (3.20), we have

$$h_\eta(t, \Delta) \leq \left(C\varepsilon^{-2} h_\rho(t)^{\frac{4}{9}} + e^{C\varepsilon^{-2}} \right) \Delta.$$

Since ρ_ε has total mass 1, $h_\rho \geq 1$. Thus

$$h_\eta(t, \Delta) \leq e^{C\varepsilon^{-2}} \Delta h_\rho(t)^{\frac{4}{9}}.$$

This means that condition (3.19) is satisfied with $C_* = e^{C\varepsilon^{-2}}$. We apply Lemma 3.13 to improve our bound on

$$\int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_\varepsilon(y) dy \leq C C_*^{4/3} \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{4}{9}} + h_\rho(t)^{1/6} \right) \Delta.$$

Feeding this new estimate into (3.20), we obtain

$$h_\eta(t, \Delta) \leq C\varepsilon^{-2} C_*^{4/3} \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{4}{9}} + h_\rho(t)^{1/6} \right) \Delta + (C\varepsilon^{-2} + C e^{C\varepsilon^{-2}}) \Delta \leq e^{C\varepsilon^{-2}} h_\rho(t)^{\frac{8}{27}} \Delta,$$

as long as

$$h_\rho(t)^{-2/9} \leq \Delta \leq t.$$

We will iterate this process until we achieve the lowest possible exponent for h_ρ , i.e. $\frac{1}{6}$. Applying Lemma 3.13 a second time, we obtain

$$\int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_\varepsilon(y) dy \leq C C_*^{4/3} \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{8}{27}} + h_\rho(t)^{1/6} \right) \Delta,$$

with $C_* = e^{C\varepsilon^{-2}}$, provided that

$$h_\rho(t)^{-4/27} \leq \Delta \leq t,$$

and therefore

$$h_\eta(t, \Delta) \leq C\varepsilon^{-2} C_*^{4/3} \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{8}{27}} + h_\rho(t)^{1/6} \right) \Delta + (C\varepsilon^{-2} + Ce^{C\varepsilon^{-2}}) \Delta \leq e^{C\varepsilon^{-2}} h_\rho(t)^{\frac{16}{81}} \Delta.$$

Applying Lemma 3.13 once more, we obtain

$$\int_{\mathbb{T}^3} |X(s) - y|^{-2} \rho_\varepsilon(y) dy \leq C C_*^{4/3} \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{16}{81}} + h_\rho(t)^{1/6} \right) \Delta,$$

with $C_* = e^{C\varepsilon^{-2}}$, provided that

$$h_\rho(t)^{-8/81} \leq \Delta \leq t,$$

and therefore

$$h_\eta(t, \Delta) \leq C\varepsilon^{-2} C_*^{4/3} \left(h_\rho(t)^{\frac{2}{3} \cdot \frac{16}{81}} + h_\rho(t)^{1/6} \right) \Delta + (C\varepsilon^{-2} + Ce^{C\varepsilon^{-2}}) \Delta \leq e^{C\varepsilon^{-2}} h_\rho(t)^{\frac{1}{6}} \Delta, \quad (3.21)$$

since $\frac{32}{243} < \frac{1}{6}$ and $h_\rho \geq 1$.

Finally, we use this new growth estimate on characteristic trajectories to control the mass density. Assuming that $f_\varepsilon(0)$ is supported in $\mathbb{T}^3 \times B_{\mathbb{R}^3}(0; R_t)$, we have

$$h_\rho \leq C \|f_\varepsilon\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)} (R_0 + h_\eta(t, t))^3$$

Since we work with $L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ solutions, we have a uniform bound on $\|f_\varepsilon\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}$ depending only on the initial data. Therefore, using (3.21), we find that if $h_\rho(t)^{-8/81} \leq t$,

$$\begin{aligned} h_\rho &\leq C (R_0 + h_\eta(t, t))^3 \leq C \left(R_0 + e^{C\varepsilon^{-2}} h_\rho(t)^{\frac{1}{6}} t \right)^3 \\ &\leq C R_0^3 + e^{C\varepsilon^{-2}} h_\rho(t)^{1/2} t^3 \leq C R_0^3 + \frac{(e^{C\varepsilon^{-2}} t^3)^2 + h_\rho(t)}{2}. \end{aligned}$$

Hence

$$h_\rho \leq C \left(R_0^3 + e^{C\varepsilon^{-2}} t^6 \right).$$

If instead $h_\rho(t)^{-8/81} \geq t$, then

$$h_\rho(t) \leq t^{-\frac{81}{8}}.$$

Therefore, we may conclude that

$$h_\rho(t) \leq \max\{t^{-81/8}, C(R_0^3 + e^{C\varepsilon^{-2}} t^6)\}.$$

□

3.5 Quasi-Neutral Limit

In this section, we complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Let g_ε denote the solution of (3.1) with data $g_\varepsilon(0)$. We will use g_ε to interpolate between the solution f_ε of (3.1) starting from $f_\varepsilon(0)$ and the solution g of (3.2) starting from $g(0)$. By the triangle inequality,

$$W_1(f_\varepsilon(t), g(t)) \leq W_1(f_\varepsilon(t), g_\varepsilon(t)) + W_1(g_\varepsilon(t), g(t)). \quad (3.22)$$

By Theorem 3.2, there exists a solution g of (3.2) such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T_*]} W_1(g_\varepsilon(t), g(t)) = 0. \quad (3.23)$$

To deal with the first term of (3.22), we use a stability estimate around g_ε for the VPME system. By the monotonicity property of Wasserstein distances (1.38) and Proposition 3.9,

$$\begin{aligned} W_1(f_\varepsilon(t), g_\varepsilon(t)) &\leq W_2(f_\varepsilon(t), g_\varepsilon(t)) \\ &\leq \begin{cases} C \exp \left[C \left(1 + \log \frac{W_2(f_\varepsilon(0), g_\varepsilon(0))}{4\sqrt{d}} \right) e^{-C_\varepsilon t} \right] & \text{if } W_2(f_\varepsilon(0), g_\varepsilon(0)) \leq d \\ W_2(f_\varepsilon(0), g_\varepsilon(0)) e^{C_\varepsilon t} & \text{if } W_2(f_\varepsilon(0), g_\varepsilon(0)) > d. \end{cases} \end{aligned} \quad (3.24)$$

where C_ε may be chosen to satisfy

$$C_\varepsilon \leq \overline{\exp}_2(C\varepsilon^{-2})(M+1).$$

for any M satisfying

$$\sup_{[0, T_*]} \|\rho_{f_\varepsilon}(t)\|_{L_x^\infty}, \quad \sup_{[0, T_*]} \|\rho_{g_\varepsilon}(t)\|_{L_x^\infty} \leq M.$$

By Proposition 3.10, we may take M such that

$$M \leq C e^{C\varepsilon^{-2}}.$$

The constant C depends on T_* , C_0 , the dimension d , and C_1 , the rate of growth of the initial support. We emphasise again that the appearance of an exponential rate here is a consequence of the form of the equation rather than because condition (3.3) allows fast growth of the initial

support. It follows that we may estimate

$$C_\varepsilon t \leq C \overline{\exp}_2(C\varepsilon^{-2})$$

for all $t \in [0, T_*]$. Hence we have convergence if

$$\frac{\left| \log \frac{W_2(f_\varepsilon(0), g_\varepsilon(0))}{4\sqrt{d}} \right|}{\overline{\exp}_3(C\varepsilon^{-2})} \rightarrow \infty$$

as ε tends to zero. This holds if

$$W_2(f_\varepsilon(0), g_\varepsilon(0)) \leq (\overline{\exp}_4(C\varepsilon^{-2}))^{-1}$$

for sufficiently large C . In this case, it follows by (3.24) that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T_*]} W_1(f_\varepsilon(t), g_\varepsilon(t)) = 0.$$

Combined with (3.23) and (3.22), this completes the proof.

□

Chapter 4

Derivation of the VPME System from a System of Extended Ions

4.1 Introduction

In this chapter, we derive the VPME system (2.1) from a microscopic particle system. The VPME system is a model for the ions in a plasma, assuming that the electrons are instantaneously thermalised. A possible choice of microscopic system would therefore be a system of N ions, modelled as point charges, in a background of thermalised electrons. On the torus, this is described by the following system of ODEs:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum_{j \neq i}^N K_{\text{per}}(X_i - X_j) - K_{\text{per}} * e^U, \end{cases} \quad (4.1)$$

where U satisfies

$$\Delta U = e^U - \frac{1}{N} \sum_{i=1}^N \delta_{X_i}.$$

The formal limit of this particle system, as N tends to infinity, is the VPME system, which reads as follows on the torus:

$$(VPME) := \begin{cases} \partial_t f + v \cdot \nabla_x f + E \cdot \nabla_v f = 0, \\ E = -\nabla_x U, \\ \Delta_x U = e^U - \int_{\mathbb{R}^d} f dv = e^U - \rho, \\ f|_{t=0} = f_0 \geq 0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_0 dx dv = 1. \end{cases} \quad (4.2)$$

In fact, we will not be able to derive VPME from the point charge system (4.1). This is due to the singularity of the Coulomb interaction, as discussed in Section 1.6. Instead, we will consider a limit from a regularised system. We use a regularisation method inspired by the work of Lazarovici [61] for the classical VPME system. The idea of the proof is to use a model consisting of ‘extended charges’. Instead of modelling the ions as point charges, we replace these with delocalised packets of charge. We define this system in detail in Subsection 4.1.1 below.

4.1.1 The Extended Ions Model

In this section, we introduce the extended ions model. We fix a smooth, radially symmetric function $\chi : \mathbb{R}^d \rightarrow [0, +\infty)$ with unit mass and compact support contained within the unit ball in \mathbb{R}^d . Define

$$\chi_r(x) = r^{-d} \chi\left(\frac{x}{r}\right). \quad (4.3)$$

We then consider a microscopic system describing the dynamics of a system of ‘delocalised ions’ of shape χ_r . For $1 \leq i \leq N$, let (X_i, V_i) denote the position and velocity of the centre of the i th delocalised ion. The system is described by the following system of ODEs:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{1}{N} \sum_{j \neq i} \chi_r * K_{\text{per}} * \chi_r(X_i - X_j) - \chi_r * K_{\text{per}} * e^U, \end{cases} \quad (4.4)$$

where U satisfies

$$\Delta U(x) = e^{U(x)} - \frac{1}{N} \sum_{i=1}^N \chi_r(x - X_i)$$

and K_{per} denotes the Coulomb kernel on \mathbb{T}^d , defined in (1.44). That is, $K_{\text{per}} = -\nabla G_{\text{per}}$, where $G_{\text{per}} = G + G_0$ for G_0 a smooth function and G the Coulomb potential on \mathbb{R}^d , defined by (1.42).

Note that we can rewrite the velocity equation as

$$\dot{V}_i = -\chi_r * \nabla U(t, X_i),$$

and the equation for U as

$$\Delta U = e^U - \chi_r * \rho_{\mu_r^N},$$

where μ_r^N denotes the empirical measure as defined in (1.58). This is valid because $\chi_r * K_{\text{per}} * \chi_r(0) = 0$, and so

$$\frac{1}{N} \sum_{j \neq i} \chi_r * K_{\text{per}} * \chi_r(X_i - X_j) = \frac{1}{N} \sum_{j=1}^N \chi_r * K_{\text{per}} * \chi_r(X_i - X_j) = \chi_r * K_{\text{per}} * \chi_r * \rho_{\mu_r^N}(X_i).$$

Indeed, since K_{per} is odd and χ_r is even, we have

$$\begin{aligned} \chi_r * K_{\text{per}} * \chi_r(0) &= \int_{\mathbb{T}^d \times \mathbb{T}^d} \chi_r(-x) K_{\text{per}}(x-y) \chi_r(y) \, dx \, dy \\ &= \int_{\mathbb{T}^d \times \mathbb{T}^d} \chi_r(-y) K_{\text{per}}(y-x) \chi_r(x) \, dx \, dy \\ &= - \int_{\mathbb{T}^d \times \mathbb{T}^d} \chi_r(y) K_{\text{per}}(x-y) \chi_r(-x) \, dx \, dy \\ &= -\chi_r * K_{\text{per}} * \chi_r(0). \end{aligned}$$

4.2 Statement of Results

The main result of this chapter is the following theorem.

Theorem 4.1 (Regularised mean field limit). *Let $d = 2, 3$, and let $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be a compactly supported choice of initial datum for (4.2). Then, for all $T_* > 0$, there exists a constant $C = C(f_0, T_*)$ such that the following holds. Assume that $r = r_N$ and the initial configurations for (4.4) are chosen such that the corresponding empirical measures satisfy,*

$$\limsup_{r \rightarrow 0} \frac{W_2^2(f(0), \mu_r^N(0))}{r^{d+2+C|\log r|^{-1/2}}} < 1. \quad (4.5)$$

Then the empirical measure associated to the particle system dynamics starting from this configuration converges to f :

$$\lim_{r \rightarrow 0} \sup_{t \in [0, T_*]} W_2(f(t), \mu_r^N(t)) = 0.$$

Remark 5. This result in fact holds for any choice of initial data for which there exists a constant $M > 0$ such that for all $r > 0$,

$$\|\rho_f\|_{L^\infty([0, T_*] \times \mathbb{T}^d)}, \sup_{r > 0} \|\rho_{f_r}\|_{L^\infty([0, T_*] \times \mathbb{T}^d)} \leq M.$$

In this case, the constant $C(f_0, T_*) = C_{M, T_*}$ in fact depends only on M and T_* .

For Theorem 4.1 to be useful, we need to check whether the condition (4.5) is reasonable. This is dealt with in the following result. We consider data chosen randomly, by drawing independent samples from f_0 . A generalised Glivenko–Cantelli theorem - a law of large numbers result for measures - implies that $\mu^N(0)$ chosen in this way will converge to f_0 almost surely as N tends to infinity. With a suitable choice of parameters, the rate of this convergence will be fast enough to satisfy (4.5).

Theorem 4.2 (Typicality). *Let $d = 2$ or 3 , and let f_0 be a choice of initial datum for (4.2) satisfying the assumptions of Theorem 4.1 and having a finite k th moment for some $k > 4$:*

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} (|x|^k + |v|^k) f_0(dx dv) < +\infty.$$

Let $r = cN^{-\gamma}$ for some γ satisfying

$$\gamma < \frac{1}{d+2} \min \left\{ \frac{1}{d}, 1 - \frac{4}{k} \right\}. \quad (4.6)$$

For each N , select initial configurations for the regularised N -particle system (4.4) by taking N independent samples from f_0 . Then with probability 1, this gives an admissible set of configurations for Theorem 4.1, i.e. the regularised mean field limit holds:

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T_*]} W_2(\mu_r^N(t), f(t)) = 0.$$

Remark 6. It is worth noticing that our assumptions on r are the same as the ones found by Lazarovici for the classical Vlasov–Poisson system in [61].

4.2.1 Strategy

The overall strategy is to apply the approach of Lazarovici [61] for the Vlasov–Poisson system, with appropriate modifications to adapt it to the VPME case. In this section, we briefly outline this approach.

The regularised mean field limit is proved in two stages. As an intermediate step between the particle system (1.63) and VPME, we introduce the following regularised version of the VPME system (4.2):

$$\begin{cases} \partial_t f_r + v \cdot \nabla_x f_r + E_r \cdot \nabla_v f_r = 0, \\ E_r = -\chi_r * \nabla U_r, \\ \Delta U_r = e^{U_r} - \chi_r * \rho[f_r], \\ f_r|_{t=0} = f(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f(0) dx dv = 1. \end{cases} \quad (4.7)$$

We studied this equation in Section 2.8 and proved that it has a unique solution f_r for any $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$. The convergence of μ_r^N to f will be proved as a two-stage limit, passing via f_r .

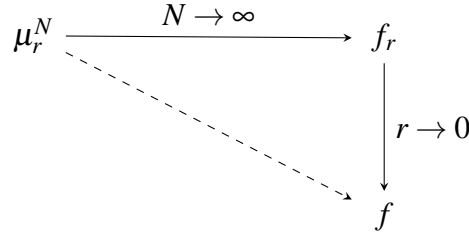


Fig. 4.1 Strategy for the proof of Theorem 4.1.

First, as N tends to infinity with r fixed, μ_r^N converges to f_r , the solution of (2.57) with initial datum $f(0)$. This holds because the force in (2.57) is regular enough that the equation has a stability property even in the class of measures, as investigated for example by Dobrushin [27]. However, the rate of this convergence will degenerate in the limit as r tends to zero, because the regularised force will converge to a singular force in this limit. The goal is therefore to quantify this convergence in W_2 , optimising the constants so as to minimise the rate of blow-up as r tends to zero. This step will be carried out in Section 4.4.

As in the article of Lazarovici [61], the technique we use to do this is to consider an anisotropic version of the W_2 distance. For some parameter $\lambda > 0$, we consider

$$W_2^{(\lambda)}(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \lambda^2 |x_1 - x_2|^2 + |v_1 - v_2|^2 d\pi(x_1, v_1, x_2, v_2) \right)^{\frac{1}{2}}.$$

For this anisotropic distance, we will prove a differential inequality of Grönwall type, with a growth constant C_λ depending on λ . We then allow λ to depend on r , and optimise with respect to r in order to achieve the smallest C_λ possible. The optimal value of λ will lead to a rate of convergence as r tends to zero that will allow us to close the estimate.

In the second step, we must show that f_r converges to f as r tends to zero. We already showed that this holds if $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ with compact support, in Section 2.9 when we constructed global solutions of the VPME system. In this chapter we will show that this convergence can be quantified, in the second order Wasserstein distance W_2 , again aiming to optimise the rate in r .

By combining these two limits, we will identify a regime for the initial data in which μ_r^N converges to f .

The main difference in adapting the proof to the VPME case is to include the additional part of the electric field $\widehat{E} = -\nabla \widehat{U}$, where

$$\Delta \widehat{U} = e^{\bar{U} + \widehat{U}} - 1, \quad -\Delta \bar{U} = \rho[f] - 1,$$

and the analogous part of the electric field for the regularised equation (4.7). This is accomplished by making use of the regularity and stability estimates on the electric field from Sections 2.4 and 2.5.

4.3 Preliminary Estimates

4.3.1 Behaviour of the Wasserstein distance under regularisation

We recall some useful results on the behaviour of Wasserstein distances under regularisation by convolution. See [83, Proposition 7.16] for proofs. The first observation is that regularising two measures cannot increase the Wasserstein distance between them.

Lemma 4.3. *Let μ, ν be probability measures, $r > 0$ any positive constant and χ_r a mollifier as defined in (4.3), where $\chi \in C_c^\infty(\mathbb{R}^d; [0, +\infty))$ has total mass one. Then*

$$W_p(\chi_r * \mu, \chi_r * \nu) \leq W_p(\mu, \nu).$$

We also have an explicit control on the Wasserstein distance between a measure and its regularisation:

Lemma 4.4. *Let μ be a probability measure and $r > 0$. Let χ_r be a mollifier as defined in (4.3), where $\chi \in C_c^\infty(\mathbb{R}^d; [0, +\infty))$ has support contained in the unit ball and total mass one. Then*

$$W_p(\chi_r * \mu, \mu) \leq r.$$

If a measure ν is close to an L^p -function in Wasserstein sense, then it is possible to estimate the L^p -norm of the regularised measure $\chi_r * \nu$ in a way that exploits this fact. We will use this in our estimates to control the regularised mass density $\chi_r * \rho_\mu$. The following estimate is shown in the whole space for $p = \infty$ in [61, Lemma 4.3] (see also [13] for similar estimates), but it is straightforward to adapt it to the case of the torus with general p . We revisit the proof here for completeness.

Lemma 4.5. *Let ν be a probability measure on \mathbb{T}^d and $h \in L^p(\mathbb{T}^d)$ a probability density function, for some $p \in [1, +\infty]$. Then there exists a constant $C_{d,\chi,p}$, depending on the dimension d , the integrability exponent p and $\|\chi\|_{L^\infty(\mathbb{T}^d)}$, such that for all $r \in (0, \frac{1}{2}]$, $q \in [1, +\infty)$,*

$$\|\chi_r * \nu\|_{L^p(\mathbb{T}^d)} \leq C_{d,\chi} \left(\|h\|_{L^p(\mathbb{T}^d)} + r^{-(q+d)} W_q^q(h, \nu) \right).$$

Proof. Throughout the proof, we identify functions and measures defined over the flat torus \mathbb{T}^d with \mathbb{Z}^d -periodic functions and measures defined on \mathbb{R}^d .

We give the proof in the case where the Wasserstein distance W_p is defined using the toroidal distance

$$d_{\mathbb{T}^d}(x, y) := \inf_{k \in \mathbb{Z}^d} |x - y + k|.$$

Note that $d_{\mathbb{T}^d}$ is less than or equal to the distance $d_{\mathcal{Q}_d}$ induced by the identification of the torus \mathbb{T}^d with the unit cube $\mathcal{Q}_d = [-\frac{1}{2}, \frac{1}{2}]^{\frac{1}{2}}$. The same result will therefore then also apply to Wasserstein distances defined using $d_{\mathcal{Q}_d}$.

Step 1: Kantorovich Duality. The proof is based on the characterisation of Wasserstein distances in terms of Kantorovich duality, which was stated in Lemma 1.6. In particular this means that for all pairs of functions $(\phi, \psi) \in L^1(\mathbb{T}^d, \nu) \times L^1(\mathbb{T}^d, h)$ such that

$$\phi(x) - \psi(y) \leq d_{\mathbb{T}^d}(x, y)^q$$

the following inequality holds:

$$W_q^q(h, \nu) \geq \int_{\mathbb{T}^d} \phi(x) \nu(dx) - \int_{\mathbb{T}^d} \psi(y) h(y) dy.$$

In particular this holds if we choose $\psi = \phi^*$, where ϕ^* is the conjugate function of ϕ with respect to the cost function $d_{\mathbb{T}^d}(x, y)^q$:

$$\phi^*(x) := \sup_{y \in \mathbb{T}^d} \{ \phi(y) - d_{\mathbb{T}^d}(x, y)^q \}.$$

Now consider the mollification $\chi_r * \nu$. We make use of the identification of the torus \mathbb{T}^d with the unit cube \mathcal{Q}_d . For a fixed $x \in \mathcal{Q}_d$, we may write

$$\chi_r * \nu(x) := \int_{x - \mathcal{Q}_d} r^{-d} \chi\left(\frac{x - y}{r}\right) \nu(dy).$$

In the expression above, the measure ν on \mathcal{Q}_d is identified with its periodic extension to \mathbb{R}^d . The function χ is only evaluated over the principal domain \mathcal{Q}_d ; note that the restriction $r \leq \frac{1}{2}$ is imposed so that the support of χ_r is entirely contained within \mathcal{Q}_d . Then

$$\chi_r * \nu(x) = r^{-(d+q)} \int_{x-\mathcal{Q}_d} r^q \chi\left(\frac{x-y}{r}\right) \nu(dy).$$

Define a function $\phi_x : x - \mathcal{Q}_d \rightarrow \mathbb{R}$ by

$$\phi_x(y) := r^q \chi\left(\frac{x-y}{r}\right).$$

Note that, for $r \leq 1/2$, $\phi_x = 0$ on the boundary of $x - \mathcal{Q}_d$, and so ϕ_x can be extended periodically to the whole of \mathbb{R}^d , thus inducing a well-defined function on \mathbb{T}^d .

Then we may write $\chi_r * \nu(x)$ in terms of ϕ_x and its conjugate:

$$\begin{aligned} \chi_r * \nu(x) &= r^{-(d+q)} \int_{x-\mathcal{Q}_d} \phi_x(y) \nu(dy) \\ &= r^{-(d+q)} \left[\int_{x-\mathcal{Q}_d} \phi_x(y) \nu(dy) - \int_{x-\mathcal{Q}_d} \phi_x^*(y) h(y) dy \right] + r^{-(d+q)} \int_{x-\mathcal{Q}_d} \phi_x^*(y) h(y) dy. \end{aligned}$$

The conjugate function $\phi_x^* : \mathbb{T}^d \rightarrow \mathbb{R}$ is defined by

$$\phi_x^*(y) := \sup_{z \in x-\mathcal{Q}_d} \{ \phi_x(z) - d_{\mathbb{T}^d}(y, z)^q \}.$$

We wish to use the bound

$$\chi_r * \nu(x) \leq r^{-(d+q)} W_q^q(h, \nu) + r^{-(d+q)} \int_{x-\mathcal{Q}_d} |\phi_x^*(y)| h(y) dy. \quad (4.8)$$

To make this rigorous, we must check that $\phi_x \in L^1(\nu)$ and $\phi_x^* \in L^1(h)$. For ϕ_x , we use the fact that

$$\|\phi_x\|_{L^\infty} \leq r^q \|\chi\|_{L^\infty}.$$

Thus, since ν is a finite measure, $\phi_x \in L^1(\nu)$. The relevant estimates on ϕ_x^* will be proved in Step 2 below.

It then remains to control the term

$$r^{-(d+q)} \int_{x-\mathcal{Q}_d} |\phi_x^*(y)| h(y) dy.$$

This is done by studying the support of ϕ_x^* .

Step 2: Support of ϕ_x^* . By definition,

$$\phi_x^*(y) = \sup_{z \in x - \mathcal{Q}_d} \alpha(z; x, y),$$

where

$$\alpha(z; x, y) = r^q \chi \left(\frac{x - z}{r} \right) - d_{\mathbb{T}^d}(y, z)^q.$$

First, we note some global lower and upper bounds. For the lower bound, let $k \in \mathbb{Z}^d$ be such that $y + k \in x - \mathcal{Q}_d$ (such a k always exists), and consider the choice $z = y + k$. We find that

$$\alpha(y; x, y) = r^q \chi \left(\frac{x - (y + k)}{r} \right) \geq 0,$$

since the function χ is non-negative. Thus $\phi_x^*(y) \geq 0$ for all $y \in \mathbb{T}^d$.

For the upper bound, since χ is a bounded function, we have the estimate

$$\alpha(z; x, y) \leq r^q \|\chi\|_{L^\infty(\mathbb{T}^d)} - d_{\mathbb{T}^d}(y, z)^q \leq r^q \|\chi\|_{L^\infty(\mathbb{T}^d)}.$$

Thus

$$\|\phi_x^*\|_{L^\infty(\mathbb{T}^d)} \leq r^q \|\chi\|_{L^\infty(\mathbb{T}^d)}. \quad (4.9)$$

Note that in particular this implies that $\phi_x^* \in L^{p'}(\mathbb{T}^d)$, where p' is the Hölder conjugate of p , and so ϕ_x^* belongs to $L^1(h)$. Thus the estimate (4.8) is rigorous.

Next we will control the support of ϕ_x^* . Firstly, since the support of χ is contained in the unit ball, if $d_{\mathbb{T}^d}(x, z) > r$ then

$$\alpha(z; x, y) = -d_{\mathbb{T}^d}(y, z)^q \leq 0.$$

Secondly, if $d_{\mathbb{T}^d}(y, z) > \eta r$ for some $\eta > 0$, then

$$\alpha(z; x, y) \leq r^q \left[\|\chi\|_{L^\infty(\mathbb{T}^d)} - \eta^q \right].$$

If we choose $\eta = \|\chi\|_{L^\infty(\mathbb{T}^d)}^{\frac{1}{q}}$, then

$$\alpha(z; x, y) \leq 0.$$

It follows that, if $\phi_x^*(y) > 0$, then the intersection

$$B_{\mathbb{T}^d}(x; r) \cap B_{\mathbb{T}^d}(y; \|\chi\|_{L^\infty(\mathbb{T}^d)}^{\frac{1}{q}} r) \quad (4.10)$$

must be non-empty. Here $B_{\mathbb{T}^d}(z; r)$ denotes the ball with radius r and centre z on \mathbb{T}^d with respect to $d_{\mathbb{T}^d}$. The set (4.10) is non-empty if and only if

$$d_{\mathbb{T}^d}(x, y) \leq r \left(1 + \|\chi\|_{L^\infty(\mathbb{T}^d)} \right).$$

We deduce that

$$\text{supp } \phi_x^* \subset B_{\mathbb{T}^d} \left(x; r \left(1 + \|\chi\|_{L^\infty(\mathbb{T}^d)} \right) \right).$$

Step 3: L^p estimate. Using the upper bound (4.9), it then follows that, for all $y \in x - \mathcal{Q}_d$,

$$\phi_x^*(y) \leq r^q \|\chi\|_{L^\infty(\mathbb{T}^d)} \mathbb{1}_{B_{\mathbb{R}^d}(0; \frac{1}{4})} \left(\frac{4(x-y)}{r(1 + \|\chi\|_{L^\infty(\mathbb{T}^d)})} \right),$$

where $B_{\mathbb{R}^d}(0; \delta)$ denotes the open ball in \mathbb{R}^d of radius δ with respect to the Euclidean metric. Define the function $\Psi_{d, \chi} : \mathcal{Q}_d \rightarrow \mathbb{R}$ by

$$\Psi_{d, \chi}(z) := r^q \|\chi\|_{L^\infty(\mathbb{T}^d)} \mathbb{1}_{B_{\mathbb{R}^d}(0; \frac{1}{4})} \left(\frac{4z}{r(1 + \|\chi\|_{L^\infty(\mathbb{T}^d)})} \right).$$

This can be extended periodically to the whole of \mathbb{R}^d and thus induces a well-defined function on \mathbb{T}^d .

We may then write

$$\int_{x - \mathcal{Q}_d} |\phi_x^c(y)| h(y) dy = \int_{x - \mathcal{Q}_d} \phi_x^c(y) h(y) dy \leq \int_{x - \mathcal{Q}_d} \Psi_{d, \chi}(x - y) h(y) dy = \Psi_{d, \chi} * h(x).$$

Thus

$$\chi_r * v(x) \leq r^{-(d+q)} W_q^q(h, v) + r^{-(d+q)} \Psi_{d, \chi} * h(x).$$

Since the torus has finite volume, we then deduce that, for all $p \in [1, +\infty]$,

$$\|\chi_r * v\|_{L^p(\mathbb{T}^d)} \leq C_d r^{-(d+q)} W_q^q(h, v) + r^{-(d+q)} \|\Psi_{d, \chi} * h\|_{L^p(\mathbb{T}^d)}.$$

The next step is to apply Young's inequality to the second term:

$$r^{-(d+q)} \|\Psi_{d, \chi} * h\|_{L^p(\mathbb{T}^d)} \leq r^{-(d+q)} \|\Psi_{d, \chi}\|_{L^1(\mathbb{T}^d)} \|h\|_{L^p(\mathbb{T}^d)}.$$

The final step is to evaluate $\|\Psi_{d,\chi}\|_{L^1(\mathbb{T}^d)}$:

$$\begin{aligned}\|\Psi_{d,\chi}\|_{L^1(\mathbb{T}^d)} &= r^q \|\chi\|_{L^\infty(\mathbb{T}^d)} \int_{\mathcal{Q}_d} \mathbb{1}_{B_{\mathbb{R}^d}(0; \frac{1}{4})} \left(\frac{4z}{r(1 + \|\chi\|_{L^\infty(\mathbb{T}^d)})} \right) dz \\ &\leq r^{d+q} \|\chi\|_{L^\infty(\mathbb{T}^d)} (1 + \|\chi\|_{L^\infty(\mathbb{T}^d)})^d |B_{\mathbb{R}^d}(0; 1)| \\ &\leq C_{d,\chi} r^{d+q}.\end{aligned}$$

We conclude that

$$\|\chi_r * v\|_{L^p(\mathbb{T}^d)} \leq C_d r^{-(d+q)} W_q^q(h, v) + C_{d,\chi} \|h\|_{L^p(\mathbb{T}^d)},$$

which completes the proof. \square

4.3.2 Regularised Kernel

We will need a Lipschitz estimate for the regularised kernel $\chi_r * K$. A suitable estimate is proved in [61, Lemma 4.2(ii)] for the whole space case. The adaptation to the torus is straightforward, since $K_{\text{per}} = K + K_0$, where K_0 is smooth. The remainder can therefore be controlled using Lemma 1.1.

Lemma 4.6. (i) *Let $h \in L^1 \cap L^\infty(\mathbb{R}^d)$. There exists $C > 0$ such that*

$$\|\chi_r * K * h\|_{Lip} \leq C |\log r| (1 + \|h\|_{L^\infty}).$$

(ii) *Let $h \in L^\infty(\mathbb{T}^d)$. There exists $C > 0$ such that*

$$\|\chi_r * K_{\text{per}} * h\|_{Lip} \leq C |\log r| (1 + \|h\|_{L^\infty}). \quad (4.11)$$

4.3.3 Estimates for the Regularised System

We make use of the existence result for the regularised system, which we proved in Section 2.8.

Lemma 4.7 (Existence of regularised solutions). *For every $f(0) \in \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d)$, there exists a unique solution $f_r \in C([0, \infty); \mathcal{P}(\mathbb{T}^d \times \mathbb{R}^d))$ of (2.57).*

We recall some basic estimates on these solutions, which are independent of $r > 0$. We have uniform-in-time $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ estimates on f_t . If $f(0) \in L^p(\mathbb{T}^d \times \mathbb{R}^d)$ for some $p \in [1, \infty]$, then for all t

$$\|f_r(t)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} \leq \|f(0)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}.$$

We also recall the definition of the regularised energy, which is conserved by the regularised VPME system:

$$\mathcal{E}_r[f] := \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv + \frac{1}{2} \int_{\mathbb{T}^d} |\nabla U_r|^2 \, dx + \int_{\mathbb{T}^d} U_r e^{U_r} \, dx. \quad (4.12)$$

By the same interpolation argument as used for VPME, we can deduce the following uniform-in-time estimate on $\rho[f_r]$:

$$\sup_{t \in [0, \infty)} \|\rho[f_r(t)]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C(\mathcal{E}_r[f(0)] + 1). \quad (4.13)$$

In Theorem 4.1, we assume that the initial datum $f(0)$ is compactly supported. Using the growth estimates in Section 2.7, we can therefore deduce a $L^\infty(\mathbb{T}^d)$ estimate on $\rho[f_r]$. That is, for any $T > 0$ there exists $M > 0$ such that

$$\sup_{r > 0} \sup_{t \in [0, T]} \|\rho[f_r(t)]\|_{L^\infty(\mathbb{T}^d)} \leq M.$$

4.4 W_2 stability for the regularised VPME system

In this section we perform the first step of the strategy outlined in Figure 4.1, showing that $\mu_r^N \rightarrow f_r$ as N tends to ∞ . To do this, we a weak-strong stability estimate for the regularised VPME system introduced in (2.57), in the Wasserstein distance W_2 . This can be applied to μ_r^N and f_r as both are weak solutions of this system. The estimate is optimised to degenerate slowly as r tends to zero. This will allow us to use solutions of (2.57) as a bridge between the particle system (4.4) and the VPME system (4.2).

Lemma 4.8 (Weak-strong stability for the regularised equation). *For each $r > 0$, let f_r, μ_r be solutions of (2.57), where the f_r have uniformly bounded density and initial energy:*

$$\sup_{r > 0} \sup_{t \in [0, T]} \|\rho_{f_r}\|_{L^\infty(\mathbb{T}^d)} \leq M, \quad (4.14)$$

$$\sup_{r > 0} \mathcal{E}_r^{ME}[f_r(0)] \leq C_0, \quad (4.15)$$

for some $C_0, M > 0$. Assume that the initial data satisfy, for some sufficiently large constant $C > 0$ depending on T, C_0, M and d ,

$$\limsup_{r \rightarrow 0} \frac{W_2^2(f_r(0), \mu_r(0))}{r^{(d+2+C|\log r|^{-1/2})}} < 1.$$

Then

$$\lim_{r \rightarrow 0} \sup_{t \in [0, T]} W_2^2(f_r(t), \mu_r(t)) = 0.$$

Proof. To lighten the notation, we drop the subscript r from f_r and μ_r . Fix an arbitrary coupling of the initial data $\pi_0 \in \Pi(\mu_0, f_0)$. As in the proof of Lemma 1.8, we define a coupling π_t that follows the characteristic flow of (2.57). We use the notation $Z^{(i)}$ for the characteristic flows, with $i = 1, 2$. They are defined by the following ODE systems:

$$\begin{cases} \dot{X}_{x,v}^{(1)} = V_{x,v}^{(1)} \\ \dot{V}_{x,v}^{(1)} = E_r^{(\mu)}(X_{x,v}^{(1)}) \\ (X_{x,v}^{(1)}(0), V_{x,v}^{(1)}(0)) = (x, v) \\ E_r^{(\mu)} = -\chi_r * \nabla U_r^{(\mu)} \\ \Delta U_r^{(\mu)} = e^{U_r^{(\mu)}} - \chi_r * \rho_\mu \end{cases} \quad \begin{cases} \dot{X}_{x,v}^{(2)} = V_{x,v}^{(2)} \\ \dot{V}_{x,v}^{(2)} = E_r^{(f)}(X_{x,v}^{(2)}) \\ (X_{x,v}^{(2)}(0), V_{x,v}^{(2)}(0)) = (x, v) \\ E_r^{(f)} = -\chi_r * \nabla U_r^{(f)} \\ \Delta U_r^{(f)} = e^{U_r^{(f)}} - \chi_r * \rho_f. \end{cases} \quad (4.16)$$

First, we check that unique global solutions exist for both systems. The same argument applies to both μ and f , so we will write it for μ only. Observe that since μ is a probability measure and χ_r is smooth, $\chi_r * \rho_\mu$ is a function with

$$\|\chi_r * \rho_\mu\|_{L^\infty(\mathbb{T}^d)} \leq \|\chi_r\|_{L^\infty(\mathbb{T}^d)}.$$

Hence by the regularity estimates in Proposition 2.4, $U_r^{(\mu)}$ is a C^1 function with

$$\|U_r^{(\mu)}\|_{C^1(\mathbb{T}^d)} \leq \exp \left[C \left(1 + \|\chi_r\|_{L^\infty(\mathbb{T}^d)} \right) \right]. \quad (4.17)$$

Then $E_r^{(\mu)} = -\chi_r * \nabla U_r^{(\mu)}$ is a smooth function with bounded derivative

$$\|E_r^{(\mu)}\|_{C^1(\mathbb{T}^d)} \leq \|\chi_r\|_{C^1(\mathbb{T}^d)} \exp \left[C \left(1 + \|\chi_r\|_{L^\infty(\mathbb{T}^d)} \right) \right].$$

Therefore there is a unique C^1 flow corresponding to (4.16).

We then define

$$\pi_t = \left(Z^{(1)}(t; 0, \cdot) \otimes Z^{(2)}(t; 0, \cdot) \right)_\# \pi_0, \quad (4.18)$$

recalling that $\#$ denotes a pushforward (Definition 2). It follows from the discussion in Subsection 1.3.2.2 that π_t is a coupling of $\mu(t)$ and $f(t)$.

Using π_t , we define an anisotropic functional D which controls the squared Wasserstein distance $W_2^2(f_t, \mu_t)$. For $\lambda > 0$, let

$$D(t) = \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \lambda^2 |x_1 - x_2|^2 + |v_1 - v_2|^2 d\pi_t(x_1, v_1, x_2, v_2). \quad (4.19)$$

We will choose λ later in order to optimise the rate obtained in our eventual estimate on D . We note some relationships between D and the Wasserstein distance. By Definition 4, since π_t is a particular coupling of μ_t and f_t , as long as we choose $\lambda^2 > 1$, we have

$$W_2^2(\mu_t, f_t) \leq D(t). \quad (4.20)$$

If we only look at the spatial variables, we can get a sharper estimate:

$$W_2^2(\rho_\mu(t), \rho_f(t)) \leq \lambda^{-2} D(t). \quad (4.21)$$

Since $\pi_0 \in \Pi(\mu_0, f_0)$ was arbitrary we may take the infimum to obtain

$$\inf_{\pi_0} D(0) \leq \lambda^2 W_2^2(\mu_0, f_0). \quad (4.22)$$

We now perform a Grönwall estimate on D . D is differentiable with respect to time, since E_r is uniformly bounded (4.17) and $\mu(0)$ and $f(0)$ have finite second moments. Taking a time derivative, we obtain

$$\dot{D} = 2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \lambda^2 (x_1 - x_2) \cdot (v_1 - v_2) + (v_1 - v_2) \cdot \left(E_r^{(\mu)}(t, x_1) - E_r^{(f)}(t, x_2) \right) d\pi_t.$$

Using a weighted Cauchy inequality, we find that for any $\alpha > 0$,

$$\begin{aligned} \dot{D} &= \lambda \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \lambda^2 |x_1 - x_2|^2 + |v_1 - v_2|^2 d\pi_t \\ &\quad + \alpha \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |v_1 - v_2|^2 d\pi_t + \frac{1}{\alpha} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \left| E_r^{(\mu)}(t, x_1) - E_r^{(f)}(t, x_2) \right|^2 d\pi_t. \end{aligned}$$

Therefore

$$\dot{D} \leq (\alpha + \lambda) D + \frac{C}{\alpha} \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned} I_1 &:= \int |\bar{E}_r^{(\mu)}(t, X_t^{(1)}) - \bar{E}_r^{(\mu)}(t, X_t^{(2)})|^2 d\pi_0, & I_2 &:= \int |\bar{E}_r^{(f)}(t, X_t^{(2)}) - \bar{E}_r^{(\mu)}(t, X_t^{(2)})|^2 d\pi_0, \\ I_3 &:= \int |\widehat{E}_r^{(\mu)}(t, X_t^{(1)}) - \widehat{E}_r^{(\mu)}(t, X_t^{(2)})|^2 d\pi_0, & I_4 &:= \int |\widehat{E}_r^{(f)}(t, X_t^{(2)}) - \widehat{E}_r^{(\mu)}(t, X_t^{(2)})|^2 d\pi_0. \end{aligned}$$

We have again used the decomposition $E_r^{(f)} = \bar{E}_r^{(f)} + \widehat{E}_r^{(f)}$, and the analogous form for $E_r^{(\mu)}$.

To estimate these quantities, we first note some basic regularity properties, which follow from Proposition 2.4. First, we wish to control the regularised mass density $\chi_r * \rho_\mu$ with an estimate that behaves well as r tends to zero. For this we use Lemma 4.5 with $q = 2$ and (4.21):

$$\begin{aligned} \|\chi_r * \rho_\mu\|_{L^p(\mathbb{T}^d)} &\leq C_d \left(\|\rho_f\|_{L^p(\mathbb{T}^d)} + r^{-(d+2)} W_2^2(\rho_\mu, \rho_f) \right) \\ &\leq C_d \left(\|\rho_f\|_{L^p(\mathbb{T}^d)} + r^{-(d+2)} \lambda^{-2} D \right). \end{aligned} \quad (4.23)$$

We will use this estimate in the cases $p = \frac{d+2}{d}$ and $p = \infty$. For $p = \infty$, by assumption (4.14) we obtain

$$\|\chi_r * \rho_\mu\|_{L^\infty(\mathbb{T}^d)} \leq C_d \left(M + r^{-(d+2)} \lambda^{-2} D \right). \quad (4.24)$$

For $p = \frac{d+2}{d}$, by the initial energy assumption (4.15) and (4.13) we have

$$\|\rho_f\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C_d,$$

where C_d depends on C_0 and d . Thus

$$\|\chi_r * \rho_\mu\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C_d \left(1 + r^{-(d+2)} \lambda^{-2} D \right) \quad (4.25)$$

for C_d depending on C_0 and d .

We also wish to control the regularity of $\widehat{U}_r^{(\mu)}$. Using (4.25) and Proposition 2.4, we obtain

$$\|\widehat{U}_r^{(\mu)}\|_{C^2(\mathbb{T}^d)} \leq C_d \overline{\exp}_2 \left(C_d \left(1 + r^{-(d+2)} \lambda^{-2} D \right) \right). \quad (4.26)$$

We estimate I_1 and I_2 in the same way as in [61]. For I_1 we use Lemma 4.6:

$$\|\chi_r * K_{\text{per}} * h\|_{\text{Lip}} \leq C |\log r| (1 + \|h\|_{L^\infty}).$$

We combine this with the mass density estimate (4.24) to obtain

$$I_1 \leq C (\log r)^2 \left(M + r^{-(d+2)} \lambda^{-2} D \right)^2 \lambda^{-2} D. \quad (4.27)$$

For I_2 , we use Proposition 2.11 and (4.24) to obtain

$$I_2 \leq CM(M + r^{-(d+2)}\lambda^{-2}D)\lambda^{-2}D. \quad (4.28)$$

For I_3 we compute:

$$\begin{aligned} I_3 &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\widehat{E}_r^{(\mu)}(t, X_t^{(1)}) - \widehat{E}_r^{(\mu)}(t, X_t^{(2)})|^2 d\pi_0 \\ &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\chi_r * (\nabla \widehat{U}_r^{(\mu)}(t, x) - \nabla \widehat{U}_r^{(\mu)}(t, y))|^2 d\pi_t \\ &\leq \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \|\chi_r * \nabla \widehat{U}_r^{(\mu)}\|_{\text{Lip}}^2 |x - y|^2 d\pi_t \\ &\leq \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \|\widehat{U}_r^{(\mu)}\|_{C^2(\mathbb{T}^d)}^2 |x - y|^2 d\pi_t. \end{aligned} \quad (4.29)$$

We apply the regularity estimate (4.26) to obtain

$$\begin{aligned} I_3 &\leq C\overline{\text{exp}}_2 \left[C_d(1 + r^{-(d+2)}\lambda^{-2}D) \right] \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x - y|^2 d\pi_t \\ &\leq C\overline{\text{exp}}_2 \left[C_d(1 + r^{-(d+2)}\lambda^{-2}D) \right] \lambda^{-2}D. \end{aligned}$$

For I_4 we compute

$$\begin{aligned} I_4 &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\widehat{E}_r^{(f)}(t, X_t^{(2)}) - \widehat{E}_r^{(\mu)}(t, X_t^{(2)})|^2 d\pi_0 \\ &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |\chi_r * (\nabla \widehat{U}_r^{(f)}(t, X_t^{(2)}) - \nabla \widehat{U}_r^{(\mu)}(t, X_t^{(2)}))|^2 d\pi_0 \\ &= \int_{\mathbb{T}^d} |\chi_r * (\nabla \widehat{U}_r^{(f)}(t, x) - \nabla \widehat{U}_r^{(\mu)}(t, x))|^2 \rho_f(t, x) dx \\ &\leq \|\rho_f\|_{L^\infty(\mathbb{T}^d)} \|\nabla \widehat{U}_r^{(f)} - \nabla \widehat{U}_r^{(\mu)}\|_{L^2(\mathbb{T}^d)}^2. \end{aligned} \quad (4.30)$$

By Proposition 2.11,

$$\begin{aligned} \|\nabla \widehat{U}_r^{(f)} - \nabla \widehat{U}_r^{(\mu)}\|_{L^2(\mathbb{T}^d)}^2 &\leq C\overline{\text{exp}}_2 \left[C_d(1 + r^{-(d+2)}\lambda^{-2}D) \right] \|\bar{U}_r^{(f)} - \bar{U}_r^{(\mu)}\|_{L^2(\mathbb{T}^d)}^2 \\ &\leq C\overline{\text{exp}}_2 \left[C_d(1 + r^{-(d+2)}\lambda^{-2}D) \right] (M + r^{-(d+2)}\lambda^{-2}D)\lambda^{-2}D, \end{aligned}$$

thus

$$I_4 \leq CM\overline{\text{exp}}_2 \left[C_d(1 + r^{-(d+2)}\lambda^{-2}D) \right] (M + r^{-(d+2)}\lambda^{-2}D)\lambda^{-2}D.$$

We summarise all these bounds as

$$\begin{aligned} \dot{D} \leq & (\lambda + \alpha)D + \frac{1}{4\alpha}C(1 + |\log r|)^2 \left(M + r^{-(d+2)}\lambda^{-2}D\right)^2 \lambda^{-2}D \\ & + C\overline{\exp}_2 \left[C_d(1 + r^{-(d+2)}\lambda^{-2}D)\right] \left(1 + M(M + r^{-(d+2)}\lambda^{-2}D)\right) \lambda^{-2}D. \end{aligned} \quad (4.31)$$

This differential inequality is nonlinear and so the estimate cannot be closed in its current form. To deal with this, we introduce a truncated functional, rescaled to be of order 1:

$$\hat{D} = 1 \wedge \left(r^{-(d+2)}\lambda^{-2}D\right)$$

Since D is differentiable, \hat{D} is at least Lipschitz. In particular, at almost all times t , \hat{D} is differentiable, with either

$$\frac{d}{dt}\hat{D}(t) = 0 \quad \text{or} \quad \frac{d}{dt}\hat{D}(t) = r^{-(d+2)}\lambda^{-2}\frac{d}{dt}D(t).$$

Thus (4.31) implies that, for almost all t ,

$$\frac{d}{dt}\hat{D} \leq (\lambda + \alpha)\hat{D} + \frac{1}{\alpha}C_d [1 + |\log r|^2] M^2 \lambda^{-2} \hat{D}.$$

We then optimise the exponent by choosing

$$\alpha_* = C_d(1 + |\log r|)M\lambda_*^{-1}, \quad \lambda_* = C_d(1 + |\log r|)^{1/2}\sqrt{M}. \quad (4.32)$$

Then

$$\frac{d}{dt}\hat{D} \leq C_d\lambda_*\hat{D}.$$

Since \hat{D} is absolutely continuous,

$$\hat{D}(t) = \hat{D}(0) + \int_0^t g(s) ds,$$

where the function $g(t) = \frac{d}{dt}\hat{D}(t)$ wherever this derivative exists, and $g(t) = 0$ otherwise. Thus

$$\hat{D}(t) \leq \hat{D}(0) + C_d\lambda_* \int_0^t \hat{D}(s) ds.$$

Then by Grönwall's inequality we deduce

$$\begin{aligned} \sup_{[0,T]} \hat{D}(t) &\leq \exp(C_d \lambda_* T) \hat{D}(0) \leq r^{-(d+2)} \lambda_*^{-2} \exp(C_d \lambda_* T) D(0) \\ &\leq C_d \exp \left[|\log r| \left((d+2) + C_d T \sqrt{M} |\log r|^{-1/2} \right) \right] \lambda_*^{-2} D(0). \end{aligned}$$

In order to use this estimate to control the Wasserstein distance, we need to ensure that for r sufficiently small,

$$\inf_{\pi_0} \sup_{t \in [0,T]} \hat{D}(t) < 1. \quad (4.33)$$

If (4.33) holds, then by (4.20), for all $t \in [0, T]$,

$$W_2^2(\mu_t, f_t) \leq r^{d+2} \lambda_*^2 \inf_{\pi_0} \sup_{t \in [0,T]} \hat{D}(t) \leq r^{d+2} \lambda_*^2 \rightarrow 0$$

as r tends to zero, since λ_*^2 only grows like $|\log r|$ by definition (4.32). Using (4.22), we obtain for any π_0 ,

$$\begin{aligned} \sup_{[0,T]} \hat{D}(t) &\leq C_d \exp \left[|\log r| \left((d+2) + C_d T \sqrt{M} |\log r|^{-1/2} \right) \right] \lambda_*^{-2} \inf_{\pi_0} D(0) \\ &\leq C_d \exp \left[|\log r| \left((d+2) + C_d T \sqrt{M} |\log r|^{-1/2} \right) \right] W_2^2(\mu_0, f_0). \end{aligned}$$

Since we assumed that the initial data satisfy

$$\limsup_{r \rightarrow 0} \frac{W_2^2(\mu_0, f_0)}{r^{d+2+C_{d,M,T} |\log r|^{-1/2}}} < 1,$$

for large $C_{d,M,T}$, (4.33) holds for sufficiently small r . This completes the proof. \square

4.5 Convergence from the Regularised System to VPME

Lemma 4.9 (Approximation of (VPME)). *Fix $f(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ satisfying a uniform bound on the energy as defined in (4.12):*

$$\sup_{r>0} \mathcal{E}_r^{ME}[f(0)] \leq C_0, \quad (4.34)$$

for some $C_0 > 0$.

For each $r > 0$, let f_r be a solution of (2.57) with initial datum $f(0)$. Let f be a solution of (4.2) with the same initial datum $f(0)$. Assume that $\{f_r\}_{r>0}, f$ have uniformly bounded

density:

$$\sup_{r>0} \sup_{t \in [0, T]} \|\rho_{f_r}\|_{L^\infty(\mathbb{T}^d)} \leq M, \quad \sup_{t \in [0, T]} \|\rho_f\|_{L^\infty(\mathbb{T}^d)} \leq M$$

for some $M > 0$. Then

$$\lim_{r \rightarrow 0} \sup_{t \in [0, T]} W_2^2(f(t), f_r(t)) = 0.$$

Proof. We fix an initial coupling $\pi_0 \in \Pi(f(0), f(0))$ and construct $\pi_t \in \Pi(f(t), f_r(t))$ as in (4.18), using the characteristic flows

$$\begin{cases} \dot{X}_{x,v}^{(1)} = V_{x,v}^{(1)} \\ \dot{V}_{x,v}^{(1)} = E(X_{x,v}^{(1)}) \\ (X_{x,v}^{(1)}(0), V_{x,v}^{(1)}(0)) = (x, v) \\ E = -\nabla U \\ \Delta U = e^U - \rho_f \end{cases} \quad \begin{cases} \dot{X}_{x,v}^{(2)} = V_{x,v}^{(2)} \\ \dot{V}_{x,v}^{(2)} = E_r(X_{x,v}^{(2)}) \\ (X_{x,v}^{(2)}(0), V_{x,v}^{(2)}(0)) = (x, v) \\ E_r = -\chi_r * \nabla U_r \\ \Delta U_r = e^{U_r} - \chi_r * \rho_{f_r}. \end{cases} \quad (4.35)$$

We define D as in (4.19). As in Lemma 4.8, we obtain for any $\alpha > 0$,

$$\dot{D} \leq (\alpha + \lambda)D + \frac{C}{\alpha} \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned} I_1 &:= \int |\bar{E}_r(t, X_t^{(1)}) - \bar{E}_r(t, X_t^{(2)})|^2 d\pi_0, & I_2 &:= \int |\bar{E}_r(t, X_t^{(1)}) - \bar{E}(t, X_t^{(1)})|^2 d\pi_0, \\ I_3 &:= \int |\hat{E}_r(t, X_t^{(1)}) - \hat{E}_r(t, X_t^{(2)})|^2 d\pi_0, & I_4 &:= \int |\hat{E}(t, X_t^{(1)}) - \hat{E}_r(t, X_t^{(1)})|^2 d\pi_0. \end{aligned}$$

For I_1 we use the regularity estimate (4.11) to deduce

$$I_1 \leq |\log r|^2 M^2 \int |X_t^{(1)} - X_t^{(2)}|^2 d\pi_0 \leq |\log r|^2 M^2 \lambda^{-2} D. \quad (4.36)$$

For I_2 we use the stability estimate from Lemma 2.10:

$$\|\nabla \bar{U} - \nabla \bar{U}_r\|_{L^2(\mathbb{T}^d)} \leq \sqrt{M} W_2(\chi_r * \rho_{f_r}, \rho_f).$$

By Lemma 4.4,

$$W_2(\chi_r * \rho_{f_r}, \rho_f) \leq r + W_2(\rho_{f_r}, \rho_f).$$

Hence

$$\|\nabla \bar{U} - \nabla \bar{U}_r\|_{L^2(\mathbb{T}^d)} \leq \sqrt{M}(r + \lambda^{-1} \sqrt{D}). \quad (4.37)$$

We account for the extra regularisation by elementary methods: first, for $g \in C^1(\mathbb{T}^d)$,

$$\begin{aligned} \|g - \chi_r * g\|_{L^2(\mathbb{T}^d)}^2 &= \int_{\mathbb{T}^d} \left| \int_{\mathbb{T}^d} \chi_r(y) [g(x) - g(x-y)] dy \right|^2 dx \\ &\leq \int_{\mathbb{T}^d} \left| \int_{\mathbb{T}^d} \int_0^1 -y \chi_r(y) \nabla g(x-hy) dh dy \right|^2 dx. \end{aligned}$$

Hence, by Jensen's inequality applied to the probability measure $\chi_r(y) dy dh$ on $\mathbb{T}^d \times [0, 1]$,

$$\begin{aligned} \|g - \chi_r * g\|_{L^2(\mathbb{T}^d)}^2 &\leq \int_0^1 \int_{\mathbb{T}^d} |y|^2 \chi_r(y) \int_{\mathbb{T}^d} |\nabla g(x-hy)|^2 dx dy dh \\ &\leq r^2 \|\nabla g\|_{L^2(\mathbb{T}^d)}^2 \int_{\mathbb{T}^d} |y|^2 \chi_1(y) dy \leq Cr^2 \|\nabla g\|_{L^2(\mathbb{T}^d)}^2. \end{aligned} \quad (4.38)$$

This estimate extends by density to $g \in W^{1,2}(\mathbb{T}^d)$.

Next, standard estimates for the Poisson equation and L^p interpolation inequalities imply that

$$\|\nabla^2 \bar{U}_r\|_{L^2(\mathbb{T}^d)} \leq C \|\rho_{f_r}\|_{L^2(\mathbb{T}^d)} \leq C \|\rho_{f_r}\|_{L^\infty(\mathbb{T}^d)}^{\frac{d-2}{2d}} \|\rho_{f_r}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}^{\frac{d+2}{2d}} \leq CM^{\frac{d-2}{2d}}, \quad (4.39)$$

since by (4.34) and (4.13) we have

$$\|\rho_{f_r}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C \quad (4.40)$$

for some C depending on d and C_0 only.

Therefore, using (4.38),

$$\|\nabla \bar{U}_r - \chi_r * \nabla \bar{U}_r\|_{L^2(\mathbb{T}^d)} \leq C \|\nabla^2 \bar{U}_r\|_{L^2(\mathbb{T}^d)} r \leq CM^{\frac{d-2}{2d}} r,$$

and we conclude that

$$I_2 \leq CM^2(r + \lambda^{-1}\sqrt{D})^2.$$

The term I_3 is estimated like I_1 , using the regularity estimate from Proposition 2.4 and (4.40):

$$\|\widehat{U}_r\|_{C^{2,\alpha}(\mathbb{T}^d)} \leq C,$$

where C depends only on the constant C_0 controlling the initial energy in (4.34). We obtain

$$I_3 \leq C \int |X_t^{(1)} - X_t^{(2)}|^2 d\pi_0 \leq C \lambda^{-2} D.$$

Finally, I_4 is estimated in the same way as I_2 , using the stability estimate Proposition 2.11, Proposition 2.4 and (4.40):

$$\|\nabla \widehat{U} - \nabla \widehat{U}_r\|_{L^2(\mathbb{T}^d)} \leq C \|\nabla \bar{U} - \nabla \bar{U}_r\|_{L^2(\mathbb{T}^d)} \leq C\sqrt{M}(r + \lambda^{-1}\sqrt{D}).$$

By Proposition 2.4 and (4.40), we have

$$\|U_r\|_{L^\infty} \leq \|\bar{U}_r\|_{C^{0,\alpha}(\mathbb{T}^d)} + \|\widehat{U}_r\|_{C^{1,\alpha}(\mathbb{T}^d)} \leq \exp\left(C\left(1 + \|\chi_r * \rho_{f_r}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}\right)\right) \leq C,$$

where C depends on C_0 and d only. Hence

$$\|e^{U_r}\|_{L^2(\mathbb{T}^d)} \leq C.$$

Since $\Delta \widehat{U}_r = e^{U_r} - 1$, by standard regularity results for the Poisson equation this implies that

$$\|D^2 \widehat{U}_r\|_{L^2(\mathbb{T}^d)} \leq C.$$

Therefore, using (4.38) again,

$$\|\nabla \widehat{U}_r - \chi_r * \nabla \widehat{U}_r\|_{L^2(\mathbb{T}^d)} \leq C \|D^2 \widehat{U}_r\|_{L^2(\mathbb{T}^d)} r \leq Cr,$$

and we conclude that

$$I_4 \leq CM^2(r + \lambda^{-1}\sqrt{D})^2.$$

Altogether we have

$$\dot{D} \leq (\alpha + \lambda)D + \frac{C}{\alpha\lambda^2} (|\log r|^2 + 1) M^2 D + CM^2 r^2.$$

Optimising the exponent, we deduce that

$$\dot{D} \leq C\lambda_* D + CM^2 r^2,$$

where

$$\lambda_* = \sqrt{M}(1 + |\log r|)^{1/4}.$$

Hence

$$\begin{aligned} D(t) &\leq (D(0) + CM^2 r^2) \exp\left[C\sqrt{M}(1 + |\log r|)^{1/4} t\right] \\ &\leq (D(0) + CM^2 r^2) e^{C\sqrt{M}|\log r|t}. \end{aligned}$$

Since D controls the squared Wasserstein distance (4.20),

$$W_2^2(f_r(t), f(t)) \leq (D(0) + CM^2 r^2) e^{C\sqrt{M|\log r|}t}.$$

Then, by (4.22), and since f and f_r share the same initial datum,

$$\begin{aligned} W_2^2(f_r(t), f(t)) &\leq (\inf_{\pi_0} D(0) + CM^2 r^2) e^{C\sqrt{M|\log r|}t} \\ &\leq (\lambda_*^2 W_2^2(f(0), f(0)) + CM^2 r^2) e^{C\sqrt{M|\log r|}t} \\ &\leq CM^2 r^2 e^{C\sqrt{M|\log r|}t}. \end{aligned}$$

Since

$$r^2 e^{C\sqrt{M|\log r|}t} = r^{2-C\sqrt{\frac{Mt^2}{|\log r|}}},$$

we conclude that for any compact time interval $[0, T]$,

$$\lim_{r \rightarrow 0} \sup_{t \in [0, T]} W_2^2(f_r(t), f(t)) = 0.$$

□

Finally, we combine these estimates to complete the proof of Theorem 4.1.

Proof of Theorem 4.1. Let f_r be the unique solution of (2.57) with initial datum f_0 . By the triangle inequality for W_2 , we have

$$\sup_{t \in [0, T]} W_2^2(\mu_r^N(t), f(t)) \leq \sup_{t \in [0, T]} W_2^2(\mu_r^N(t), f_r(t)) + \sup_{t \in [0, T]} W_2^2(f_r(t), f(t)).$$

We apply Lemma 4.8 to the first term using the assumption on the initial configurations and deduce that it converges to zero as r tends to zero. For the second term we apply Lemma 4.9, since f and f_r have the same initial datum. This completes the proof.

□

4.6 Typicality

In this last section of the chapter, we prove Theorem 4.2 concerning the relation between the choice of parameters in the mean field limit and the initial configurations. Our method follows the approach of Lazarovici [61] for the mean field case.

The idea is based on the following observation. If one constructs a collection of empirical measures $(\nu^N)_N$ by drawing N independent observation samples from a reference measure ν , then, by the Glivenko-Cantelli theorem, almost surely ν^N will converge to ν as N tends to infinity, in the sense of weak convergence of measures. The idea is to use a quantitative version of this result to find configurations for which the associated empirical measures converge to our reference data f_0 sufficiently quickly to satisfy the assumption (4.5) of Theorem 4.2. A suitable result of this type was proved by Fournier and Guillin in [29], with the distance between ν^N and ν measured in Wasserstein sense. The following result is from [29, Theorem 2].

Theorem 4.10. *Let ν be a probability measure on \mathbb{R}^m and let ν^N denote the empirical measure of N independent samples from ν . Assume that ν has a finite k th moment for some $k > 2p$:*

$$M_k(\nu) := \int_{\mathbb{R}^m} |x|^k d\nu(x) < +\infty.$$

Then there exist constants c, C depending on p, m and $M_k(\nu)$ such that for any $x > 0$,

$$\mathbb{P}(W_p^p(\nu^N, \nu) \geq x) \leq a(N, x) \mathbb{1}_{\{x \leq 1\}} + b(N, x),$$

where

$$a(N, x) = C \begin{cases} \exp(-cNx^2) & p > \frac{m}{2} \\ \exp\left(-cN \left[\frac{x}{\log(2+\frac{1}{x})}\right]^2\right) & p = \frac{m}{2} \\ \exp(-cNx^{m/p}) & p < \frac{m}{2} \end{cases} \quad (4.41)$$

and

$$b(N, x) = CN(Nx)^{-(k-\alpha)/p}$$

for any $\alpha \in (0, k)$.

Proof of Theorem 4.2. We follow the approach of [61, Theorem 3.3]. The idea of the proof is to show that, for the choice $r = cN^{-\gamma}$,

$$\mathbb{P}\left(\limsup_{N \rightarrow \infty} \frac{W_2^2(\mu_0^N, f_0)}{r^{d+2+C_{T^*, M}} |\log r|^{-1/2}} < 1\right) = 1. \quad (4.42)$$

Then we may apply Theorem 4.1 to conclude that the mean field limit holds on this full probability event. To prove (4.42), observe that

$$\bigcup_n \bigcap_{N \geq n} A_N^c \subset \left\{ \limsup_{N \rightarrow \infty} \frac{W_2^2(\mu_0^N, f_0)}{r^{d+2+C_{T^*, M}} |\log r|^{-1/2}} < 1 \right\},$$

where A_N is the event

$$A_N := \left\{ W_2^2(\mu_0^N, f_0) > \frac{1}{2} r^{d+2+C_{T_*,M}|\log r|^{-1/2}} \right\}.$$

Since $(\bigcup_n \bigcap_{N \geq n} A_N^c)^c = \bigcap_n \bigcup_{N \geq n} A_N$, by the Borel-Cantelli lemma it suffices to show that

$$\sum_N \mathbb{P}(A_N) < \infty. \quad (4.43)$$

We estimate $\mathbb{P}(A_N)$ using Theorem 4.10, with

$$x_N = \frac{1}{2} r^{d+2+C_{T_*,M}|\log r|^{-1/2}} = cN^{-\gamma(d+2)-C_{T_*,M,\gamma}|\log N|^{-1/2}}.$$

Note that $p = 2$ and $m = 2d$. The assumptions on γ in (4.6) are chosen such that

$$\sum_N a(N, x_N) + b(N, x_N) < \infty.$$

In this way (4.43) holds and the result follows. □

Chapter 5

Particle Derivations for Kinetic Euler Systems

Contents

5.1	Introduction	188
5.2	Statement of Results	191
5.2.1	From Extended Electrons to Kinetic Incompressible Euler	191
5.2.2	From Extended Ions to Kinetic Isothermal Euler	194
5.3	Strategy	195
5.3.1	General Configurations	195
5.3.2	Typicality	197
5.4	Preliminaries	197
5.4.1	Regularised Vlasov–Poisson Systems	197
5.4.2	Filtering	200
5.4.3	Quasi-neutral Limit for Classical Vlasov–Poisson	200
5.5	Proof for General Configurations	201
5.5.1	Stability for the Regularised Vlasov–Poisson Systems	201
5.5.2	Stability for Regularised VPME	204
5.5.3	Removal of the Regularisation	209
5.5.4	Conclusion of the Proof	217
5.6	Typicality	219
5.6.1	Concentration Estimates	219

5.6.2	Rescaling of Measures	220
5.6.3	Conclusion of the Proof.	221

5.1 Introduction

This chapter focuses on the derivation of kinetic Euler systems from underlying particle systems. We consider the kinetic incompressible Euler (KInE) (1.16) and kinetic isothermal Euler (KIsE) systems (1.20) introduced in Section 1.2.3.3.

We recall that the KInE system reads as follows:

$$(KInE) := \begin{cases} \partial_t g + v \cdot \nabla_x g - \nabla_x U \cdot \nabla_v g = 0, \\ \rho_g = 1, \\ g|_{t=0} = g_0, \int_{\mathbb{T}^d \times \mathbb{R}^d} g_0(x, v) dx dv = 1, \end{cases} \quad (5.1)$$

The KInE system formally describes the dynamics of the electrons in a plasma in the quasi-neutral regime. The physical situation therefore suggests that an appropriate microscopic model would be a system of interacting electrons. In this chapter, we consider this problem on the torus. Due to the confinement, we may also include a uniform background of ions. This leads to the following model:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \alpha(N) \sum_{j \neq i} K_{\text{per}}(X_i - X_j). \end{cases} \quad (5.2)$$

Recall that K_{per} was defined to be the Coulomb kernel on the torus \mathbb{T}^d .

In order to derive the KInE system from the particle system (5.2), it is necessary to identify an appropriate choice of the scaling $\alpha(N)$ that connects the two systems. With the choice $\alpha(N) = \frac{\varepsilon^{-2}}{N}$, for a fixed value of ε , the formal limit as N tends to infinity is the Vlasov–Poisson system

$$(VP)_\varepsilon := \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E_\varepsilon \cdot \nabla_v f_\varepsilon = 0, \\ E_\varepsilon = -\nabla_x U_\varepsilon, \\ -\varepsilon^2 \Delta_x U_\varepsilon = \rho_{f_\varepsilon} - 1, \\ f_\varepsilon|_{t=0} = f_\varepsilon(0) \geq 0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon(0) dx dv = 1. \end{cases} \quad (5.3)$$

The quasi-neutral limit $\varepsilon \rightarrow 0$ then connects the Vlasov–Poisson system to the KInE system.

To obtain the KInE system from the particle system (5.2), a possible strategy to identify the correct scaling is to consider a two-stage limit, combining the mean field and and quasi-neutral

limits. This amounts to choosing a scaling of the form

$$\alpha(N) = \frac{\varepsilon(N)^{-2}}{N},$$

where ε is now allowed to depend on N . We look for a choice of $\varepsilon = \varepsilon(N)$ such that both limits can be taken simultaneously. The limit should hold in the sense of convergence of the empirical measure, as outlined in Section 1.6. In this chapter, we follow a strategy along these lines.

We have discussed in previous chapters that the mean field limit from the particle system (5.2) to the Vlasov–Poisson system remains an open problem in any dimension $d > 1$. We will therefore not consider limits from the system (5.2) in our derivation of KInE. Instead, we use a regularised system.

We use the same regularisation as was discussed for the mean field limit for VPME in Chapter 4. This represents a system of ‘extended charges’; for KInE, these represent extended electrons. Fixing a smooth non-negative mollifier $\chi \in C_c^\infty(\mathbb{T}^d)$ with mass one, we define the scaled mollifier

$$\chi_r(x) = r^{-d} \chi\left(\frac{x}{r}\right), \quad r > 0.$$

We then consider the regularised system

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{\varepsilon^{-2}}{N} \sum_{j \neq i} \chi_r * \chi_r * K_{\text{per}}(X_i - X_j). \end{cases} \quad (5.4)$$

For this system, the aim is to find a choice $r = r(N)$, $\varepsilon = \varepsilon(N)$ such that the limit from (5.4) to KInE holds.

We also consider the corresponding problem for the KIsE system, which reads as follows:

$$(KIsE) := \begin{cases} \partial_t g + v \cdot \nabla_x g - \nabla_x U \cdot \nabla_v g = 0, \\ U = \log \rho_g, \\ g|_{t=0} = g_0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} g_0(x, v) \, dx \, dv = 1. \end{cases} \quad (5.5)$$

This is a model for the ions in a plasma, under the assumption of quasi-neutrality.

For KIsE, the natural microscopic model is a system of N ions, interacting with each other and with a background of thermalised electrons. This is represented by the following system of

ODEs:

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = \frac{\varepsilon^{-2}}{N} \sum_{j \neq i}^N K_{\text{per}}(X_i - X_j) - \varepsilon^{-2} K_{\text{per}} * e^U, \end{cases} \quad (5.6)$$

where U satisfies

$$\varepsilon^2 \Delta U = e^U - \frac{1}{N} \sum_{i=1}^N \delta_{X_i}.$$

An intermediate step between these systems is the VPME system

$$(VPME)_\varepsilon := \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + E \cdot \nabla_v f_\varepsilon = 0, \\ E = -\nabla_x U, \\ \varepsilon^2 \Delta U = e^U - \rho[f_\varepsilon], \\ f_\varepsilon|_{t=0} = f_\varepsilon(0). \end{cases} \quad (5.7)$$

We also replace the point charges in (5.6) by ‘extended ions’ in this case. This results in the system

$$\begin{cases} \dot{X}_i = V_i \\ \dot{V}_i = -\chi_r * \nabla_x U(X_i), \end{cases} \quad (5.8)$$

where U satisfies

$$\varepsilon^2 \Delta U = e^U - \frac{1}{N} \sum_{i=1}^N \chi_r(x - X_i).$$

As for KInE, the goal is to find a relationship $\varepsilon = \varepsilon(N), r = r(N)$ between the parameters such that the KIsE system (5.5) is obtained from (5.8) in the limit as N tends to infinity.

Choice of Initial Configurations. To prove these limits, it will be necessary to make restrictions on the initial configurations that we specify for the particle systems (5.4) and (5.8). These constraints arise due to the issues that we have discussed regarding the choice of initial data in the mean field and quasi-neutral limits. We need to avoid pathological configurations in the mean field limit as well as avoiding triggering unstable behaviour, such as the two-stream instability, that would obstruct the quasi-neutral limit.

Our strategy will be to choose initial data for the particle systems whose empirical measures approximate, as N tends to infinity, distributions $f_\varepsilon(0)$ for which the quasi-neutral limit holds. We will work in the close-to-analytic regime of the quasi-neutral limit, using the results of Han-Kwan–Iacobelli [43] for the electron models, and the results of Chapter 3 for the ion models.

5.2 Statement of Results

5.2.1 From Extended Electrons to Kinetic Incompressible Euler

5.2.1.1 General Configurations

In this section, we state a result concerning the derivation of the KInE system from the extended electrons system (5.4). First, we must make some preliminary definitions.

Empirical Measure. Throughout this chapter, we use the notation $\mu_{\varepsilon,r}^N$ to denote the empirical measure associated to the solution of the relevant particle system, either (5.4) or (5.8). It is defined by the formula

$$\mu_{\varepsilon,r}^N(t) = \frac{1}{N} \sum_{i=1}^N \delta_{(X_i(t), V_i(t))}. \quad (5.9)$$

Correctors. In the quasi-neutral limit for the classical Vlasov–Poisson system, we need to account for the effect of plasma oscillations, as explained in Section 1.5.1.2. The method for dealing with this is to introduce a corrector function \mathcal{R}_ε that describes the oscillations. The corrector \mathcal{R}_ε is defined in (1.53). The oscillations can then be removed from $\mu_{\varepsilon,r}^N$ by a ‘filtering’ process, which we define below.

Definition 5. Let μ be a probability measure on $\mathbb{T}^d \times \mathbb{R}^d$. Let $\mathcal{R} : \mathbb{T}^d \rightarrow \mathbb{R}^d$ be given. The corresponding filtered measure $\tilde{\mu}$ is defined to be the measure such that

$$\langle \tilde{\mu}, \phi \rangle = \int_{\mathbb{T}^d \times \mathbb{R}^d} \phi(x, v + \mathcal{R}(x)) \mu(dx dv)$$

for all test functions ϕ .

We will apply this definition with the choice $\mathcal{R} = \mathcal{R}_\varepsilon$.

Energy. We define the energy associated to the classical Vlasov–Poisson system (5.3) on the torus, with quasi-neutral scaling. This functional is defined as follows:

$$\mathcal{E}_\varepsilon^{\text{VP}}[f] := \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f dx dv + \frac{\varepsilon^2}{2} \int_{\mathbb{T}^d} |\nabla_x U_\varepsilon|^2 dx, \quad (5.10)$$

where $U_\varepsilon = \varepsilon^{-2} G_{\text{per}} * (\rho[f] - 1)$. This energy functional is conserved by strong solutions of (5.3).

Reference Data. We will choose the initial data for the particle system (5.4) to approximate given functions $f_\varepsilon(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$, which we interpret as possible choices of initial data for the Vlasov–Poisson system (5.3). We assume that $f_\varepsilon(0)$ satisfy the following set of assumptions. These are precisely the assumptions of the quasi-neutral limit for rough data proved in [43].

Assumption 2 (KInE, $d = 2, 3$). The data $f_\varepsilon(0)$ satisfy the following:

- (i) There exists C_0 independent of ε such that

$$\|f_\varepsilon(0)\|_{L^\infty} \leq C_0, \quad \mathcal{E}_\varepsilon^{\text{VP}}(f_\varepsilon(0)) \leq C_0.$$

- (ii) (Control of support) For some $\gamma > 0$,

$$f_\varepsilon(0, x, v) = 0 \quad \text{for } |v| > \varepsilon^{-\gamma}. \quad (5.11)$$

- (iii) (Perturbation of analytic functions) There exist functions $g_\varepsilon(0)$, satisfying (5.11) and

$$\sup_{\varepsilon \in (0,1)} \sup_{v \in \mathbb{R}^d} (1 + |v|^{d+1}) \|g_\varepsilon(0)(\cdot, v)\|_{B_{\delta_0}} \leq C,$$

such that

$$W_2(f_\varepsilon(0), g_{0,\varepsilon}) \leq \left[\overline{\text{exp}}_2 \left(C \varepsilon^{-(2+d\zeta)} \right) \right]^{-1},$$

for $C > 0$ sufficiently large. The exponent $\zeta = \zeta(\gamma)$ is defined as follows:

- For $d = 2$, we fix any $\delta > 2$ and let

$$\zeta = \max\{\gamma, \delta\}. \quad (5.12)$$

- For $d = 3$ we let

$$\zeta = \max\left\{\gamma, \frac{38}{3}\right\}. \quad (5.13)$$

Main Result. Under these assumptions, we have the following theorem, in which the KInE system is derived from the system of extended electrons.

Theorem 5.1. *Let $d = 2$ or 3 . For each $\varepsilon > 0$, let $f_\varepsilon(0)$ satisfy Assumption 2. Suppose that $g_\varepsilon(0)$ has a limit $g(0)$ in the sense of distributions.*

Fix $T > 0$ and $\eta > 0$. Then there exists a constant C_T and a weak solution $g(t)$ of (5.1) with initial datum g_0 such that the following holds:

Recall the exponent ζ depending on $f_\varepsilon(0)$ and defined in (5.12)-(5.13) and let $\varepsilon = \varepsilon_N$, $r = r_N$ be chosen such that

$$r < e^{-C_T \varepsilon^{-2-d\zeta}}.$$

Let the initial configurations $[Z_i^\varepsilon(0)]_{i=1}^N$ for the N -particle system (5.4) be chosen such that the corresponding empirical measures satisfy, for some $\eta > 0$,

$$\limsup_{N \rightarrow \infty} \frac{W_2(\mu_\varepsilon^N(0), f_\varepsilon(0))}{\varepsilon^{-\gamma} r^{1+d/2+\eta/2}} < \infty. \quad (5.14)$$

Let $[Z_i^{\varepsilon,r}(t)]_{i=1}^N$ denote the solution of (5.4) with initial datum $[Z_i^\varepsilon(0)]_{i=1}^N$. Let $\mu_{\varepsilon,r}^N(t)$ denote the corresponding empirical measure. Let $\tilde{\mu}_{\varepsilon,r}^N(t)$ denote the measures constructed by filtering $\mu_{\varepsilon,r}^N(t)$ using the corrector \mathcal{R}_ε defined from g in (1.53), according to Definition 5. Then

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} W_1(\tilde{\mu}_{\varepsilon,r}^N(t), g(t)) = 0.$$

5.2.1.2 Typicality

Theorem 5.2 (KInE, typicality). *Let $d = 2$ or 3 . For each $\varepsilon > 0$, let $f_\varepsilon(0)$ satisfy Assumption 2. Suppose that $g_\varepsilon(0)$ has a limit $g(0)$ in the sense of distributions. For fixed T , there exists a constants $C_T, A_T > 0$ such that the following holds:*

Recall the exponent ζ defined in (5.12)-(5.13) and let $\varepsilon = \varepsilon_N$, $r = r_N$ be chosen such that

$$\begin{aligned} r &\geq A_T N^{-\frac{1}{d(d+2)} + \alpha} \\ r &< e^{-C_T \varepsilon^{-2-d\zeta}} \end{aligned}$$

for some $\alpha > 0$. Then, if the initial N -particle configurations $[Z_i^\varepsilon(0)]_{i=1}^N$ are chosen by taking N independent samples from $f_\varepsilon(0)$, with probability one the following limit holds:

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} W_1(\tilde{\mu}_{\varepsilon,r}^N(t), g(t)) = 0,$$

where

- $g(t)$ is a weak solution of (5.1) with initial datum g_0 ;
- $\mu_{\varepsilon,r}^N(t)$ denotes the empirical measure corresponding to the solution of (5.4) with initial datum $[Z_i^\varepsilon(0)]_{i=1}^N$;

- $\tilde{\mu}_{\varepsilon,r}^N(t)$ is the measure constructed by filtering $\mu_{\varepsilon,r}^N(t)$ using the corrector \mathcal{R}_ε defined in (1.53), according to Definition 5, using the given choice of $f_\varepsilon(0)$ and $g(0)$.

5.2.2 From Extended Ions to Kinetic Isothermal Euler

5.2.2.1 General Configurations

For the KIIE system, we can prove the following limit.

Theorem 5.3 (From ions to kinetic isothermal Euler). *Let $d = 2$ or 3 , and let $f_\varepsilon(0), g_\varepsilon(0)$ and $g(0)$ satisfy the assumptions of Theorem 3.1. Given ε, r, N , let $(Z_{0,i}^{(\varepsilon,r)})_{i=1}^N \in (\mathbb{T}^d \times \mathbb{R}^d)^N$ be a choice of initial data for the regularised and scaled N -particle ODE system (5.8). Let $\mu_{\varepsilon,r}^N$ denote the empirical measure associated to the solution of (5.8) with this initial data as defined in (5.9).*

Let T_ be the maximal time of convergence from Theorem 3.1. There exists a constant $C > 0$ depending on $\{f_\varepsilon(0)\}_\varepsilon$ such that if the parameters $(r, \varepsilon) = (r(N), \varepsilon(N))$ and the initial data $\mu_{\varepsilon,r}^N(0)$ satisfy*

$$r \leq [\overline{\exp}_3(C\varepsilon^{-2})]^{-1}, \quad \lim_{N \rightarrow \infty} \frac{W_2(\mu_{\varepsilon,r}^N(0), f_\varepsilon(0))}{r^{(d+2+\eta)/2}} = 0, \quad (5.15)$$

for some $\eta > 0$, then

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T_*]} W_1(\mu_{\varepsilon,r}^N(t), g(t)) = 0,$$

where g is a solution of the KIE system (5.5) with initial data $g(0)$ on the time interval $[0, T_]$.*

5.2.2.2 Typicality

Theorem 5.4. *Let $d = 2$ or 3 , and $f_\varepsilon(0), g_\varepsilon(0)$ and $g(0)$ satisfy the assumptions of Theorem 3.1. Let $(r, \varepsilon) = (r(N), \varepsilon(N))$ be chosen to satisfy*

$$r \leq [\overline{\exp}_3(C\varepsilon^{-2})]^{-1}$$

where $C > 0$ is the constant from Theorem 5.3. Assume that $r = r(N)$ are chosen such that $r = cN^{-\gamma}$, where $c > 0$ is an arbitrary constant and γ satisfies

$$0 \leq \gamma < \frac{1}{d(d+2)}. \quad (5.16)$$

For each N , let the initial configurations for the particle system (5.8) be chosen by taking N independent samples from $f_\varepsilon(0)$. Then, with probability one, this procedure selects a set of

configurations for which Theorem 5.3 holds, that is,

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T_*]} W_1(\mu_{\varepsilon, r}^N(t), g(t)) = 0,$$

where g is the solution of (5.5) with initial datum $g(0)$ on the time interval $[0, T_*]$ provided by Theorem 3.1.

5.3 Strategy

5.3.1 General Configurations

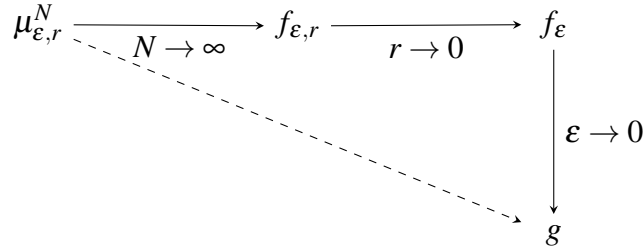


Fig. 5.1 Strategy of proof for Theorems 5.1 and 5.3.

For the derivation of the KInE system (Theorem 5.1), the overall strategy of the proof is to combine the regularised mean field limit for Vlasov–Poisson, due to Lazarovici [61], with the quasi-neutral limit from Han-Kwan–Iacobelli [43]. For the KIsE system (Theorem 5.3), the strategy is the same, using the VPME variants of these results, which are Theorems 4.1 and 3.1 of this thesis respectively.

Since we work with a regularised microscopic system, it will be convenient to introduce a regularised version of each Vlasov–Poisson with quasi-neutral scaling. For the electron models, we use the following system:

$$(VP)_{\varepsilon, r} = \begin{cases} \partial_t f_{\varepsilon, r} + v \cdot \nabla_x f_{\varepsilon, r} + E_{\varepsilon, r}[f_{\varepsilon, r}] \cdot \nabla_v f_{\varepsilon, r} = 0, \\ E_{\varepsilon, r}[f_{\varepsilon, r}] = \varepsilon^{-2} \chi_r *_{\mathbf{x}} \chi_r *_{\mathbf{x}} K_{\text{per}} *_{\mathbf{x}} \rho_f, \\ f_{\varepsilon, r}(0) = f_{\varepsilon}(0) \geq 0, \int_{\mathbb{T}^d \times \mathbb{R}^d} f_{\varepsilon}(0, \mathbf{x}, v) d\mathbf{x} dv = 1. \end{cases} \quad (5.17)$$

For the ion models, we use the following system:

$$(VPME)_{\varepsilon,r} := \begin{cases} \partial_t f_{\varepsilon,r} + v \cdot \nabla_x f_{\varepsilon,r} + E_{\varepsilon,r}[f_{\varepsilon,r}] \cdot \nabla_v f_{\varepsilon,r} = 0, \\ E_{\varepsilon,r} = -\chi_r * \nabla U_{\varepsilon,r}, \\ \varepsilon^2 \Delta U_{\varepsilon,r} = e^{U_{\varepsilon,r}} - \chi_r * \rho[f_{\varepsilon,r}], \\ f_{\varepsilon,r}(0) = f_\varepsilon(0) \geq 0, \quad \int_{\mathbb{T}^d \times \mathbb{R}^d} f_\varepsilon(0, x, v) dx dv = 1. \end{cases} \quad (5.18)$$

The strategy is illustrated in Figure 5.1. The steps of the proof are as follows:

- (i) With ε and r fixed, we prove the mean field limit for the regularised system. We show that the empirical measure $\mu_{\varepsilon,r}^N$ coming from the regularised particle system (5.4) or (5.8) converges to the solution $f_{\varepsilon,r}$ of the regularised Vlasov–Poisson system. The key point is to quantify this convergence in the Wasserstein distance W_2 .
- (ii) With ε fixed, we remove the regularisation from the Vlasov–Poisson system. We show that the solution $f_{\varepsilon,r}$ of the regularised system converges, as r tends to zero, to the solution f_ε of the Vlasov–Poisson system (5.3) or (1.19). Again we quantify the convergence in W_2 .
- (iii) Finally, we use the known results on the quasi-neutral limit ([43] or Theorem 3.1) to conclude that f_ε converges to a solution g of the relevant kinetic Euler system, as ε tends to zero.

The main goal of our estimates is to quantify the rates of convergence in steps (i) and (ii). To do this, we again use an anisotropic distance, as we did in Chapter 4. For some parameter λ , we consider

$$W_2^{(\lambda)}(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \lambda^2 |x_1 - x_2|^2 + |v_1 - v_2|^2 d\pi(x_1, v_1, x_2, v_2) \right)^{\frac{1}{2}}.$$

We then choose λ in order to optimise the rate of convergence we obtain. In this chapter, λ will be allowed to depend on both ε and r .

The estimates for steps (i) and (ii) are performed using similar methods to those used in Chapter 4. The constants in those estimates depend on $L^\infty(\mathbb{T}^d)$ bounds on the mass densities $\rho[f_{\varepsilon,r}]$ and $\rho[f_\varepsilon]$. The main difference for the estimates in this chapter is that we have to keep track of how this bound depends on ε . To do this, in the KInE case we use mass density estimates from [43]. For the KIsE case we proved the necessary estimates in Chapter 3. We recall these estimates below in Section 5.4.1.3.

5.3.2 Typicality

Our strategy for the proof of Theorems 5.2 and 5.4 is the same as used in the proof of Theorem 4.2. We check that, if we choose the initial configurations for the particle system by drawing independent samples from the reference data $f_\varepsilon(0)$, then the necessary rate of convergence (5.14) or (5.15) occurs with high probability.

To prove this, we use the concentration estimates from Fournier–Guillin [29] on the rate of convergence of the empirical measures in Wasserstein sense. In comparison to the mean field limit, in this chapter we need to deal with the fact that the reference data $f_\varepsilon(0)$ depend on ε . It is therefore necessary to keep track of how the constants in the concentration estimate depend on ε . Our strategy is to use the fact that the reference data $f_\varepsilon(0)$ are compactly supported, and to use a scaling argument.

5.4 Preliminaries

5.4.1 Regularised Vlasov–Poisson Systems

5.4.1.1 Existence of Solutions

We need to check that solutions of the regularised Vlasov–Poisson systems exist globally in time for our choice of initial data $f_\varepsilon(0)$. This follows from the fact that each $f_\varepsilon(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ is compactly supported. For the ion model (5.18), this is proved in Section 2.8 of this thesis. For the electron model, we can use the general result for Vlasov equations with Lipschitz kernels (Theorem 1.2), since the function $\chi_r * \chi_r * K_{\text{per}}$ is Lipschitz.

In both cases, since $f_{\varepsilon,r}$ solves a transport equation with a divergence free field vector field, we have the a priori estimate

$$\sup_{t \in [0, \infty)} \|f_{\varepsilon,r}(t)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)} = \|f_{\varepsilon,r}(0)\|_{L^p(\mathbb{T}^d \times \mathbb{R}^d)}. \quad (5.19)$$

5.4.1.2 Energy

Each of the regularised systems (5.17) and (5.18) has an associated energy functional. For the electron model, it is a regularisation of (5.10):

$$\mathcal{E}_{\varepsilon,r}^{\text{VP}}[f] := \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv + \frac{\varepsilon^2}{2} \int_{\mathbb{T}^d} |E_{\varepsilon,r}[f]|^2 \, dx. \quad (5.20)$$

This functional is conserved by solutions $f_{\varepsilon,r}$ of (5.17):

$$\mathcal{E}_{\varepsilon,r}^{\text{VP}}[f_{\varepsilon,r}(t)] = \mathcal{E}_{\varepsilon,r}^{\text{VP}}[f_{\varepsilon}(0)].$$

For the ion model, we have the following rescaling of (2.58):

$$\mathcal{E}_{\varepsilon,r}^{\text{ME}}[f] := \frac{1}{2} \int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv + \frac{\varepsilon^2}{2} \int_{\mathbb{T}^d} |\nabla U_{\varepsilon,r}|^2 \, dx + \int_{\mathbb{T}^d} U_{\varepsilon,r} e^{U_{\varepsilon,r}} \, dx.$$

This functional is conserved by solutions $f_{\varepsilon,r}$ of (5.18):

$$\mathcal{E}_{\varepsilon,r}^{\text{ME}}[f_{\varepsilon,r}(t)] = \mathcal{E}_{\varepsilon,r}^{\text{ME}}[f_{\varepsilon}(0)].$$

The conservation of these functionals implies a uniform-in-time a priori bound on $\|\rho[f]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}$ for a solution f of one of the regularised Vlasov–Poisson systems. For the electron model, this follows from the fact that both terms in (5.20) are non-negative. Thus if $\mathcal{E}_{\varepsilon,r}^{\text{VP}}[f] \leq C_0$, then we have an upper bound on the second moment

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv \leq C_0.$$

For the ion model, since $xe^x \geq -e^{-1}$, we have a similar estimate: if $\mathcal{E}_{\varepsilon,r}^{\text{ME}}[f] \leq C_0$, then

$$\int_{\mathbb{T}^d \times \mathbb{R}^d} |v|^2 f \, dx \, dv \leq C_0 + e^{-1}.$$

It then follows from the interpolation estimate in Lemma 1.12 that

$$\|\rho[f]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C_1, \tag{5.21}$$

where C_1 depends on C_0 and $\|f\|_{L^\infty(\mathbb{T}^d \times \mathbb{R}^d)}$.

For solutions $f_{\varepsilon,r}$ of the regularised Vlasov–Poisson systems, the uniform $L^p(\mathbb{T}^d \times \mathbb{R}^d)$ estimate (5.19) on $f_{\varepsilon,r}$ then implies that $\|\rho[f_{\varepsilon,r}]\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}$ is bounded uniformly in time for solutions of the regularised Vlasov–Poisson systems.

5.4.1.3 Mass Bounds

In our proofs, we will need uniform estimates on the mass density. That is, we use a bound of the form

$$\sup_{t \in [0, T]} \|\rho[f_\varepsilon(t)]\|_{L^\infty(\mathbb{T}^d)} \leq M_{\varepsilon, T}, \quad \sup_r \sup_{t \in [0, T]} \|\rho[f_\varepsilon(t)]\|_{L^\infty(\mathbb{T}^d)} \leq M_{\varepsilon, T}. \quad (5.22)$$

For the classical Vlasov–Poisson system, we can obtain these estimates from [43]. These results are based on controlling the growth of the size of the support of $f_\varepsilon(t)$ over time, in the style of Batt and Rein [10], quantifying the dependence on ε . We use the following result.

Proposition 5.5. *Let $d = 2, 3$. Let f_ε be a solution of the Vlasov–Poisson system (5.3), for initial data satisfying Assumption 2. Fix $T > 0$. Then there exists $C_{d, T}$ such that for all $t \in [0, T]$,*

$$\|\rho[f_\varepsilon]\|_{L^\infty(\mathbb{T}^d)} \leq C_{d, T} \varepsilon^{-d\zeta}.$$

where ζ is defined in (5.12)–(5.13).

The same estimate applies to the density $\rho[f_{\varepsilon, r}]$ for solutions of the regularised system (5.17).

For the VPME system, we use estimates from Chapter 3, in particular Proposition 3.10. Again the same estimates apply to the solution of the regularised system (5.18).

Proposition 5.6. *Let $d = 2, 3$. Let $f_\varepsilon(0) \in L^1 \cap L^\infty(\mathbb{T}^d \times \mathbb{R}^d)$ be compactly supported, with support contained in $\mathbb{T}^d \times B_R(0)$, where $R \leq e^{A\varepsilon^{-2}}$ for some constant A . Then there exists $C = C(A, d)$ depending on A and d and C_T depending on T such that*

$$\sup_{t \in [0, T]} \|\rho_{f_\varepsilon}(t)\|_{L^\infty(\mathbb{T}^d)} \leq C_T e^{C\varepsilon^{-2}}.$$

5.4.1.4 Empirical measures

Note that, if $(X_i(t), V_i(t))_{i=1}^N$ is a solution of the extended electrons systems (5.4), then the resulting empirical measure $\mu_{\varepsilon, r}^N$, defined by (5.9), is a weak solution of the regularised Vlasov–Poisson system (5.17). A similar property holds for (5.8) and the regularised VPME system (5.18).

5.4.2 Filtering

In the derivation of KInE, convergence holds for a version of the empirical measure filtered using the corrector function \mathcal{R}_ε . In our estimates, we will need to account for the effect of the filtering on the Wasserstein distance. For this we quote the following lemma from [43]:

Lemma 5.7. *Let ν_1, ν_2 be probability measures on $\mathbb{T}^d \times \mathbb{R}^d$, and let $\tilde{\nu}_i$ denote a measure obtained by filtering ν_i using a given vector field $\mathcal{R} : \mathbb{T}^d \rightarrow \mathbb{R}^d$ (see Definition 5). Then*

$$W_1(\tilde{\nu}_1, \tilde{\nu}_2) \leq (1 + \|\nabla_x \mathcal{R}\|_{L^\infty}) W_1(\nu_1, \nu_2).$$

In this chapter we will always use the corrector \mathcal{R}_ε defined by (1.53). In this case

$$|\nabla_x \mathcal{R}_\varepsilon| \leq |\nabla_x d_+| + |\nabla_x d_-|.$$

Thus there exists C_T independent of ε such that for $t \in [0, T]$,

$$\|\nabla_x \mathcal{R}_\varepsilon\|_{L^\infty(\mathbb{T}^d)} \leq C_T.$$

Thus

$$W_1(\tilde{\nu}_1, \tilde{\nu}_2) \leq C_T W_1(\nu_1, \nu_2).$$

5.4.3 Quasi-neutral Limit for Classical Vlasov–Poisson

We recall the quasi-neutral limit for the classical Vlasov–Poisson system proved in [43, Theorem 1].

Theorem 5.8. *Let γ , δ_0 , and C_0 be positive constants, with $\delta_0 > 1$. Consider a choice $\{f_\varepsilon(0)\}$ of non-negative initial data in L^1 satisfying Assumption 2. For all $\varepsilon \in (0, 1)$, consider $f_\varepsilon(t)$ a global weak solution of $(VP)_\varepsilon$ with initial condition $f_\varepsilon(0)$. Define the filtered distribution function*

$$\tilde{f}_\varepsilon(t, x, v) := f_\varepsilon\left(t, x, v - \mathcal{R}_\varepsilon\right)$$

where \mathcal{R}_ε is the corrector defined in (1.53).

There exist $T > 0$ and $g(t)$ a weak solution on $[0, T]$ of (KIE) with initial condition g_0 such that

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0, T]} W_1(\tilde{f}_\varepsilon(t), g(t)) = 0.$$

5.5 Proof for General Configurations

5.5.1 Stability for the Regularised Vlasov–Poisson Systems

In this section we prove a quantitative estimate of the rate of convergence of the empirical measure of the solution $\mu_{\varepsilon,r}^N$ of the scaled and regularised N -particle system (5.4), to the solution $f_{\varepsilon,r}$ of the mean-field regularised equation (5.17). Our approach is similar to the methods of [61].

Proposition 5.9. *Fix $T > 0$. For any small $\beta > 0$, there exists $C_{\beta,T}$ such that the following holds: Let $f_{\varepsilon,r}$ be a solution of (5.17) and $\mu_{\varepsilon,r}^N$ be defined as in (1.58). Let $\varepsilon = \varepsilon_N$, $r = r_N$ be chosen such that*

$$r < e^{-C_{\beta,T}\varepsilon^{-2-d\zeta}} \quad (5.23)$$

Assume that the initial condition for (5.4) is ‘well-placed’ in the sense that there exists $\eta > \beta$ such that

$$\lim_{N \rightarrow \infty} \frac{W_2(\mu_{\varepsilon,r}^N(0), f_{\varepsilon}(0))}{\varepsilon^{-\gamma} r^{1+d/2+\eta/2}} = 0. \quad (5.24)$$

Then for any $\eta' \in (\beta, \eta)$, there exists a constant $C = C(\beta, T, \eta, \eta', \gamma, \zeta)$ such that for all N sufficiently large, for all $t \leq T$ we have

$$W_2^2(\mu_{\varepsilon,r}^N(t), f_{\varepsilon,r}(t)) \leq C r^{d+2+\eta'-\beta}.$$

Proof. We follow the proof of Lemma 4.8, with a quasi-neutral scaling. To lighten the notation, we drop the sub- and superscripts r, ε, N from $f_{\varepsilon,r}$ and $\mu_{\varepsilon,r}^N$. We define a coupling π_t that follows the characteristic flows associated to f and μ :

$$\left\{ \begin{array}{l} \dot{X}_1 = V_1 \\ \dot{V}_1 = E_{\varepsilon,r}^{(\mu)}(X_1) \\ (Z_1(0;0,z)) = z = (x,v) \\ E_{\varepsilon,r}^{(\mu)} = -\chi_r * \nabla U_{\varepsilon,r}^{(\mu)} \\ \varepsilon^2 \Delta U_{\varepsilon,r}^{(\mu)} = e^{U_{\varepsilon,r}^{(\mu)}} - \chi_r * \rho_{\mu} \end{array} \right. \quad \left\{ \begin{array}{l} \dot{X}_2 = V_2 \\ \dot{V}_2 = E_{\varepsilon,r}^{(f)}(X_2) \\ (Z_2(0;0,z)) = z = (x,v) \\ E_{\varepsilon,r}^{(f)} = -\chi_r * \nabla U_{\varepsilon,r}^{(f)} \\ \varepsilon^2 \Delta U_{\varepsilon,r}^{(f)} = e^{U_{\varepsilon,r}^{(f)}} - \chi_r * \rho_f. \end{array} \right.$$

These systems have unique classical solutions by the same arguments as in the proof of Lemma 4.8.

Fixing an arbitrary coupling of the initial data $\pi_0 \in \Pi(\mu_0, f_0)$, we define a coupling $\pi_t \in \Pi(\mu(t), f(t))$ by

$$\pi_t = (Z_1(t;0,\cdot) \otimes Z_2(t;0,\cdot))_{\#} \pi_0. \quad (5.25)$$

Using π_t , we define an anisotropic functional D . For $\lambda > 0$, let

$$D(t) = \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \lambda^2 |x_1 - x_2|^2 + |v_1 - v_2|^2 d\pi_t(x_1, v_1, x_2, v_2). \quad (5.26)$$

We perform a Grönwall estimate on D . Following the same steps as in the proof of Lemma 4.8, we obtain

$$\dot{D} \leq (\alpha + \lambda)D + \frac{C}{\alpha} \sum_{i=1}^2 I_i,$$

where

$$I_1 := \int |E_{\varepsilon,r}^{(\mu)}(X_t^{(1)}) - E_{\varepsilon,r}^{(\mu)}(X_t^{(2)})|^2 d\pi_0, \quad I_2 := \int |E_{\varepsilon,r}^{(f)}(X_t^{(2)}) - E_{\varepsilon,r}^{(\mu)}(X_t^{(2)})|^2 d\pi_0.$$

I_1 and I_2 are estimated in the same way as in the proof of Lemma 4.8, which follows the argument of [61]. Compared to the case $\varepsilon = 1$, here we gain an extra factor of ε^{-4} due to the quasi-neutral scaling. The estimates are otherwise the same. We obtain

$$\dot{D} \leq (\lambda + \alpha)D + \frac{\varepsilon^{-4}}{4\alpha} C(1 + |\log r|)^2 \left(M + r^{-(d+2)} \lambda^{-2} D\right)^2 \lambda^{-2} D. \quad (5.27)$$

We introduce the truncated functional

$$\hat{D} = 1 \wedge \left(r^{-(d+2)} \lambda^{-2} D\right).$$

By the same argument as in the proof of Lemma 4.8, \hat{D} is Lipschitz and therefore differentiable almost everywhere and absolutely continuous, with either

$$\frac{d}{dt} \hat{D}(t) = 0 \quad \text{or} \quad \frac{d}{dt} \hat{D}(t) = r^{-(d+2)} \lambda^{-2} \frac{d}{dt} D(t).$$

Thus (5.27) implies that, for almost all t ,

$$\frac{d}{dt} \hat{D} \leq (\lambda + \alpha)D + \frac{C\varepsilon^{-4}}{\alpha} M^2 |\log r|^2 \lambda^{-2} D.$$

We then optimise the exponent by choosing

$$\alpha_* = C_d \varepsilon^{-2} (1 + |\log r|) M \lambda_*^{-1}, \quad \lambda_* = C_d \varepsilon^{-1} (1 + |\log r|)^{1/2} \sqrt{M}.$$

Then

$$\frac{d}{dt} \hat{D} \leq C_d \lambda_* \hat{D}.$$

Since \hat{D} is absolutely continuous, the integrated form of this inequality also holds. Then by Grönwall's inequality we deduce the following estimate:

$$\sup_{[0,T]} \hat{D}(t) \leq \exp(C_d \lambda_* T) \hat{D}(0). \quad (5.28)$$

Now we must choose r and ε in such a way as to prevent the factor $\exp(C_d \lambda_* T)$ from exploding too quickly.

First we check that $\inf_{\pi_0} \hat{D}(0) \rightarrow 0$. For convenience we will define a function ω by

$$\omega(N) = \frac{W_2(\mu_\varepsilon^N(0), f_\varepsilon(0))}{\varepsilon^{-\gamma} r^{1+d/2+\eta/2}};$$

thus (5.24) implies that ω is bounded. Recalling (4.22),

$$\inf_{\pi_0} D(0) \leq \frac{1}{2} \lambda^2 W_2^2(\mu_\varepsilon^N(0), f_\varepsilon(0)).$$

Thus, since by definition $\hat{D}(0) \leq \lambda^{-2} r^{-(d+2)} D(0)$, it follows that

$$\inf_{\pi_0} \hat{D}(0) \leq \frac{1}{2} r^{-(d+2)} W_2^2(\mu_\varepsilon^N(0), f_\varepsilon(0)).$$

By definition of ω we have

$$\inf_{\pi_0} \hat{D}(0) \leq \frac{1}{2} r^\eta \varepsilon^{-2\gamma} \omega(N)^2.$$

If (5.23) holds, then $\varepsilon^{-1} \leq C_{\beta,T} |\log r|^{\frac{1}{2+d\zeta}}$ and thus

$$\inf_{\pi_0} \hat{D}(0) \leq C r^\eta |\log r|^{\frac{2\gamma}{2+d\zeta}} \omega(N)^2.$$

Then for any $\eta' < \eta$ there exists $C = C_{\beta,T,\eta',\gamma,\zeta}$ such that

$$\inf_{\pi_0} \hat{D}(0) \leq C r^{\eta'}.$$

Next, we use the Gronwall estimate to get convergence at later times. We can do this by controlling the exponential growth factor in (5.28). Observe that for $r < 1$ this factor satisfies

$$e^{CM^{1/2}\varepsilon^{-1}|\log r|^{1/2}T} = r^{-CM^{1/2}\varepsilon^{-1}|\log r|^{-1/2}T}. \quad (5.29)$$

If ε, r satisfy (5.23), then by Proposition 5.5,

$$CM^{1/2}\varepsilon^{-1}|\log r|^{-1/2}T \leq CT\varepsilon^{-1-\frac{d\zeta}{2}}|\log r|^{-1/2} \leq \beta \quad (5.30)$$

and hence

$$r^{-CM^{1/2}\varepsilon^{-1}|\log r|^{-1/2}T} \leq r^{-\beta}. \quad (5.31)$$

This implies that for all $t \in [0, T]$

$$\begin{aligned} \inf_{\pi_0} \hat{D}(t) &\leq \inf_{\pi_0} \hat{D}(0) r^{-\beta} \\ &\leq Cr^{\eta' - \beta}. \end{aligned} \quad (5.32)$$

Upon choosing $\eta' > \beta$, we find that $\inf_{\pi_0} \hat{D}(T) \rightarrow 0$ as $N \rightarrow \infty$.

For N sufficiently large, we have $\inf_{\pi_0} \hat{D}(T) < 1$. Then

$$\sup_{[0, T]} W_2^2(f_{\varepsilon, r}, \mu_{\varepsilon, r}^N) \leq \lambda^2 r^{d+2} \inf_{\pi_0} \hat{D}(T). \quad (5.33)$$

Recall that we chose $\lambda = CM^{1/2}\varepsilon^{-1}|\log r|^{1/2}$. Thus by (5.30) we have $\lambda \leq C\beta T^{-1}|\log r|$. Hence

$$\lambda^2 r^{d+2} \leq C_{\beta, T} |\log r|^2 r^{d+2}.$$

Then by combining (5.33) with (5.32) and (5.33) we obtain

$$\sup_{[0, T]} W_2^2(f_{\varepsilon, r}, \mu_{\varepsilon, r}^N) \leq C |\log r|^2 r^{d+2+\eta' - \beta}.$$

By adjusting η' and C so as to absorb the logarithmic factor, we may conclude that for N sufficiently large,

$$\sup_{[0, T]} W_2^2(f_{\varepsilon, r}, \mu_{\varepsilon, r}^N) \leq Cr^{d+2+\eta' - \beta}.$$

□

5.5.2 Stability for Regularised VPME

The following lemma is a quantified version of Lemma 4.8.

Lemma 5.10 (Weak-strong stability for the regularised equation, with quasineutral scaling). *Let $f_\varepsilon(0)$ satisfy the assumptions of Theorem 3.1. Let $f_{\varepsilon, r}$ be the solution of (5.18) with data $f_\varepsilon(0)$. Fix any $T > 0$. Then there exists a constant A depending on T and $\{f_\varepsilon(0)\}$ such that the*

following holds. For each (ε, r) , let $\mu_{\varepsilon, r}$ be a measure solution of (5.18). Assume that for some $\eta > 0$, $r = r(\varepsilon)$ satisfies

$$r \leq [\overline{\exp}_3(A\varepsilon^{-2})]^{-1} \quad \text{and} \quad \lim_{r \rightarrow 0} \frac{W_2(\mu_{\varepsilon, r}(0), f_\varepsilon(0))}{r^{(d+2+\eta)/2}} = 0. \quad (5.34)$$

Then

$$\lim_{r \rightarrow 0} \sup_{t \in [0, T]} W_2(\mu_{\varepsilon, r}(t), f_{\varepsilon, r}(t)) = 0.$$

Proof. We follow the proof of Lemma 4.8, tracking the dependence of the constants on ε . For ease of notation we drop the subscripts on $f_{\varepsilon, r}$ and $\mu_{\varepsilon, r}$. Let π_t be defined as in (5.25), and let D be defined as in (5.26). Following the same steps as for Lemma 4.8, we obtain

$$\dot{D} \leq (\alpha + \lambda)D + \frac{C}{\alpha} \sum_{i=1}^4 I_i,$$

where

$$\begin{aligned} I_1 &:= \int |\bar{E}_{\varepsilon, r}^{(\mu)}(X_t^{(1)}) - \bar{E}_{\varepsilon, r}^{(\mu)}(X_t^{(2)})|^2 d\pi_0, & I_2 &:= \int |\bar{E}_{\varepsilon, r}^{(f)}(X_t^{(2)}) - \bar{E}_{\varepsilon, r}^{(\mu)}(X_t^{(2)})|^2 d\pi_0, \\ I_3 &:= \int |\widehat{E}_{\varepsilon, r}^{(\mu)}(X_t^{(1)}) - \widehat{E}_{\varepsilon, r}^{(\mu)}(X_t^{(2)})|^2 d\pi_0, & I_4 &:= \int |\widehat{E}_{\varepsilon, r}^{(f)}(X_t^{(2)}) - \widehat{E}_{\varepsilon, r}^{(\mu)}(X_t^{(2)})|^2 d\pi_0. \end{aligned}$$

To estimate these quantities, we first note some basic $L^p(\mathbb{T}^d)$ estimates on the regularised mass density $\chi_r * \rho_\mu$, using (4.23). For $p = \frac{d+2}{d}$, since $f_\varepsilon(0)$ has uniformly bounded energy (1.66), by (5.21) we obtain

$$\|\rho_f\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C,$$

where C depends on C_0 and d only. Thus

$$\|\chi_r * \rho_\mu\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)} \leq C_d \left(1 + r^{-(d+2)} \lambda^{-2} D\right), \quad (5.35)$$

for C_d depending on C_0 and d .

For $p = \infty$ we obtain

$$\|\chi_r * \rho_\mu\|_{L^\infty(\mathbb{T}^d)} \leq C_d \left(M_{\varepsilon, T} + r^{-(d+2)} \lambda^{-2} D\right). \quad (5.36)$$

where $M_{\varepsilon, T}$ is a constant such that

$$\sup_{t \in [0, T]} \|\rho_{f_t}\|_{L^\infty(\mathbb{T}^d)} \leq M_{\varepsilon, T}.$$

By Proposition 5.6, there exists a constant C depending on T such that

$$\sup_{t \in [0, T]} \|\rho_{f_t}\|_{L^\infty(\mathbb{T}^d)} \leq \exp(C\varepsilon^{-2}).$$

Therefore $M_{\varepsilon, T}$ may be chosen to satisfy

$$M_{\varepsilon, T} \leq \exp(C\varepsilon^{-2}). \quad (5.37)$$

We estimate I_1 as in (4.27). An extra factor of ε^{-4} appears due to the quasineutral scaling on the force:

$$I_1 \leq C\varepsilon^{-4}(\log r)^2 \left(M_{\varepsilon, T} + r^{-(d+2)}\lambda^{-2}D\right)^2 \lambda^{-2}D.$$

Similarly, I_2 is estimated as in (4.28) using Lemma 2.10 and (5.36) to obtain

$$I_2 \leq C\varepsilon^{-4}M_{\varepsilon, T}(M_{\varepsilon, T} + r^{-(d+2)}\lambda^{-2}D)\lambda^{-2}D.$$

For I_3 the same computation as in (4.29) implies that

$$I_3 \leq \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \|\widehat{U}_{\varepsilon, r}^{(\mu)}\|_{C^2(\mathbb{T}^d)}^2 |x - y|^2 d\pi_t.$$

Using (5.35) we apply Proposition 3.3, which is a regularity estimate on $\widehat{U}_{\varepsilon, r}^{(\mu)}$, taking account of the quasi-neutral scaling, to obtain

$$\|\widehat{U}_{\varepsilon, r}^{(\mu)}\|_{C^2(\mathbb{T}^d)} \leq C_d \overline{\exp}_2 \left(C_d \varepsilon^{-2} \left(1 + r^{-(d+2)}\lambda^{-2}D \right) \right).$$

Thus

$$\begin{aligned} I_3 &\leq C \overline{\exp}_2 \left[C_d \varepsilon^{-2} (1 + r^{-(d+2)}\lambda^{-2}D) \right] \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x - y|^2 d\pi_t \\ &\leq C \overline{\exp}_2 \left[C_d \varepsilon^{-2} (1 + r^{-(d+2)}\lambda^{-2}D) \right] \lambda^{-2}D. \end{aligned}$$

For I_4 , as in (4.30) we obtain

$$I_4 \leq \|\rho_f\|_{L^\infty(\mathbb{T}^d)} \|\nabla \widehat{U}_{\varepsilon, r}^{(f)} - \nabla \widehat{U}_{\varepsilon, r}^{(\mu)}\|_{L^2(\mathbb{T}^d)}^2.$$

By the stability estimate from Lemma 3.8 and the density bound (5.36),

$$\begin{aligned} \|\nabla \widehat{U}_{\varepsilon,r}^{(f)} - \nabla \widehat{U}_{\varepsilon,r}^{(\mu)}\|_{L^2(\mathbb{T}^d)}^2 &\leq C\varepsilon^{-2} \overline{\exp}_2 \left[C_d \varepsilon^{-2} (1 + r^{-(d+2)} \lambda^{-2} D) \right] \|\bar{U}_{\varepsilon,r}^{(f)} - \bar{U}_{\varepsilon,r}^{(\mu)}\|_{L^2(\mathbb{T}^d)}^2 \\ &\leq C\varepsilon^{-6} \overline{\exp}_2 \left[C_d \varepsilon^{-2} (1 + r^{-(d+2)} \lambda^{-2} D) \right] (M_{\varepsilon,T} + r^{-(d+2)} \lambda^{-2} D) \lambda^{-2} D. \end{aligned}$$

Thus

$$I_4 \leq C\varepsilon^{-6} M_{\varepsilon,T} \overline{\exp}_2 \left[C_d \varepsilon^{-2} (1 + r^{-(d+2)} \lambda^{-2} D) \right] (M_{\varepsilon,T} + r^{-(d+2)} \lambda^{-2} D) \lambda^{-2} D.$$

We summarise this as

$$\begin{aligned} \dot{D} &\leq (\lambda + \alpha) D + \frac{1}{\alpha} C\varepsilon^{-4} (1 + |\log r|)^2 \left(M_{\varepsilon,T} + r^{-(d+2)} \lambda^{-2} D \right)^2 \lambda^{-2} D \\ &\quad + C \overline{\exp}_2 \left[C_d \varepsilon^{-2} (1 + r^{-(d+2)} \lambda^{-2} D) \right] \left(1 + M_{\varepsilon,T} (M_{\varepsilon,T} + r^{-(d+2)} \lambda^{-2} D) \right) \lambda^{-2} D. \end{aligned} \quad (5.38)$$

We introduce the truncated functional

$$\hat{D} = 1 \wedge \left(r^{-(d+2)} \lambda^{-2} D \right),$$

which is Lipschitz and so differentiable almost everywhere. Where the derivative exists, (5.38) implies that

$$\frac{d}{dt} \hat{D} \leq (\lambda + \alpha) \hat{D} + \frac{1}{4\alpha} C_d \left[\varepsilon^{-4} (1 + |\log r|)^2 + \overline{\exp}_2(C\varepsilon^{-2}) \right] M_{\varepsilon,T}^2 \lambda^{-2} \hat{D}.$$

We optimise the exponent by choosing

$$\begin{aligned} \alpha_* &= C_d \left[\varepsilon^{-4} (1 + |\log r|)^2 + \overline{\exp}_2(C\varepsilon^{-2}) \right]^{1/2} M_{\varepsilon,T} \lambda_*^{-1} \\ \lambda_* &= C_d \left[\varepsilon^{-4} (1 + |\log r|)^2 + \overline{\exp}_2(C\varepsilon^{-2}) \right]^{1/4} \sqrt{M_{\varepsilon,T}}. \end{aligned} \quad (5.39)$$

Then

$$\frac{d}{dt} \hat{D} \leq C_d \lambda_* \hat{D}.$$

By absolute continuity of \hat{D} , it follows that

$$\hat{D}(t) \leq \hat{D}(0) + C_d \lambda_* \int_0^t \hat{D}(s) ds.$$

From this we deduce that

$$\begin{aligned}
\sup_{[0,T]} \hat{D}(t) &\leq \exp(C_d \lambda_* T) \hat{D}(0) \\
&\leq r^{-(d+2)} \lambda_*^{-2} \exp(C_d \lambda_* T) D(0) \\
&\leq r^{-(d+2)} \exp(C_d \lambda_* T) D(0) \\
&\leq C_d \exp \left[|\log r| \left((d+2) + C_d T \sqrt{M_{\varepsilon,T}} \varepsilon^{-1} |\log r|^{-1/2} \right) + \sqrt{M_{\varepsilon,T}} \overline{\exp}_2(C\varepsilon^{-2}) \right] D(0).
\end{aligned}$$

By the estimate (5.37) on $M_{\varepsilon,T}$, we obtain

$$\sup_{[0,T]} \hat{D}(t) \leq C_d \exp \left[|\log r| \left((d+2) + C_d T \exp(C\varepsilon^{-2}) |\log r|^{-1/2} \right) \right] \overline{\exp}_3(C\varepsilon^{-2}) D(0).$$

By assumption (5.34) on the relationship between ε and r ,

$$|\log r|^{-1/2} \leq \exp \left[-\frac{1}{2} \exp(A\varepsilon^{-2}) \right].$$

Thus we have

$$\sup_{[0,T]} \hat{D}(t) \leq C_d \exp \left[|\log r| \left((d+2) + \exp(C\varepsilon^{-2}) - \frac{1}{2} \exp(A\varepsilon^{-2}) \right) \right] \overline{\exp}_3(C\varepsilon^{-2}) D(0).$$

By assumption on $W_2(\mu_{\varepsilon,r}^N(0), f_{\varepsilon,r}(0))$, for all sufficiently small r there exists a choice of initial coupling $\pi_0^{(r)}$ such that

$$D(0) < r^{d+2+\eta}.$$

Then

$$\sup_{[0,T]} \hat{D}(t) \leq C_d \exp \left[|\log r| \left(\exp(C\varepsilon^{-2}) - \frac{1}{2} \exp(A\varepsilon^{-2}) - \eta \right) + \overline{\exp}_2(C\varepsilon^{-2}) \right].$$

For sufficiently small ε ,

$$\begin{aligned}
\sup_{[0,T]} \hat{D}(t) &\leq C_d \exp \left[-\frac{1}{2} \eta |\log r| + \overline{\exp}_2(C\varepsilon^{-2}) \right] \\
&\leq C_d \exp \left[-\frac{1}{2} \eta \overline{\exp}_2(A\varepsilon^{-2}) + \overline{\exp}_2(C\varepsilon^{-2}) \right].
\end{aligned}$$

Thus if $A > C$, then

$$\sup_{[0,T]} \hat{D}(t) \rightarrow 0$$

as ε tends to zero. In particular, for ε sufficiently small,

$$\inf_{\pi_0} \sup_{t \in [0,T]} \hat{D}(t) < 1.$$

Hence, for ε sufficiently small,

$$\inf_{\pi_0} \sup_{t \in [0,T]} \hat{D}(t) = \inf_{\pi_0} \sup_{t \in [0,T]} r^{-(d+2)} \lambda_*^{-2} D(t).$$

Thus

$$\sup_{t \in [0,T]} W_2^2(\mu_{\varepsilon,r}^N(t), f_{\varepsilon,r}(t)) \leq \inf_{\pi_0} \sup_{t \in [0,T]} D(t) = r^{d+2} \lambda_*^2 \inf_{\pi_0} \sup_{t \in [0,T]} \hat{D}(t) \leq r^{d+2} \lambda_*^2.$$

By (5.39),

$$\lambda_*^2 = C_d \left[\varepsilon^{-4} (1 + |\log r|)^2 + \overline{\exp}_2(C\varepsilon^{-2}) \right]^{1/2} \exp(C\varepsilon^{-2}).$$

Hence, for any $\alpha > 0$,

$$r^{d+2} \lambda_*^2 \leq C r^{d+2-\alpha} \overline{\exp}_2(C\varepsilon^{-2}) \leq C \exp \left\{ \exp(C\varepsilon^{-2}) - (d+2-\alpha) \overline{\exp}_2(A\varepsilon^{-2}) \right\}.$$

The right hand side converges to zero as ε tends to zero. Therefore

$$\lim_{\varepsilon \rightarrow 0} \sup_{t \in [0,T]} W_2^2(f_t, \mu_t) = 0.$$

□

5.5.3 Removal of the Regularisation

5.5.3.1 Classical Vlasov–Poisson

In this section, we prove the following Gronwall-type estimate between solutions of the regularised and unregularised Vlasov–Poisson systems.

Proposition 5.11. (i) Let $f_{\varepsilon,r}$ be a solution of (5.17) and f_ε a solution of (5.3), both having the same initial datum $f_\varepsilon(0)$. Let $M = M_{\varepsilon,T}$ be chosen such that (5.22) is satisfied. Then

there exists a constant C , independent of r , M and ε , such that for all $t \in [0, T]$

$$W_2(f_{\varepsilon,r}(t), f_\varepsilon(t)) \leq C\varepsilon^{-3/2}M^{3/4}r|\log r|^{-1/4}e^{C\varepsilon^{-1}M^{1/2}|\log r|^{1/2}t}.$$

(ii) Let $\{f_\varepsilon(0)\}$ be a set of initial data satisfying Assumption 2. Let ζ be defined as in (5.12)-(5.13), and let $T > 0$ be fixed. If $\varepsilon = \varepsilon_N$ and $r = r_N$ are chosen to satisfy (5.23) for some $\beta < 1$, then

$$\lim_{N \rightarrow \infty} \sup_{t \in [0, T]} W_2(f_{\varepsilon,r}(t), f_\varepsilon(t)) = 0. \quad (5.40)$$

Proof. We consider couplings of $f_{\varepsilon,r}$ and f_ε that evolve along the characteristic flows of their respective equations. Since $f_{\varepsilon,r}$ and f_ε have the same initial datum $f_\varepsilon(0)$, the choice $\pi_0(x, v, y, w) = f_0(x, v)\delta(x - y, v - w)$ is an optimal initial coupling, so that at time $t = 0$ we have

$$W_2^2(f_{\varepsilon,r}, f_\varepsilon) = \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x - y|^2 + |v - w|^2 d\pi_0(x, v, y, w)$$

We take π_t to solve

$$(\partial_t + v \cdot \nabla_x + w \cdot \nabla_y)\pi_t + (E_{\varepsilon,r}[f_{\varepsilon,r}](x) \cdot \nabla_v + E_\varepsilon[f_\varepsilon](y) \cdot \nabla_w)\pi_t = 0.$$

Then π_t is a coupling of $f_{\varepsilon,r}(t)$ and $f_\varepsilon(t)$ for all times t . As before we define an anisotropic distance D : let

$$D(t) = \frac{1}{2} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \lambda^2 |x - y|^2 + |v - w|^2 d\pi_t(x, v, y, w)$$

for some λ to be specified later. Then, as in the proof of Lemma 4.8, we estimate that

$$D'(t) \leq (\lambda + \alpha)D(t) + \frac{1}{2\alpha} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |E_{\varepsilon,r}[f_{\varepsilon,r}](x) - E_\varepsilon[f_\varepsilon](y)|^2 d\pi_t. \quad (5.41)$$

By the triangle inequality, the second term in (5.41) may be estimated as follows:

$$\begin{aligned} \frac{1}{2\alpha} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |E_{\varepsilon,r}[f_{\varepsilon,r}](x) - E_\varepsilon[f_\varepsilon](y)|^2 d\pi_t &\leq \underbrace{\frac{1}{\alpha} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |E_{\varepsilon,r}[f_{\varepsilon,r}](x) - E_{\varepsilon,r}[f_{\varepsilon,r}](y)|^2 d\pi_t}_{:=I_1} \\ &\quad + \underbrace{\frac{1}{\alpha} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |E_{\varepsilon,r}[f_{\varepsilon,r}](y) - E_\varepsilon[f_\varepsilon](y)|^2 d\pi_t}_{:=I_2}. \end{aligned}$$

The term I_1 is estimated using the Lipschitz regularity of the regularised forces (Lemma 4.6):

$$\begin{aligned}
I_1 &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |E_{\varepsilon,r}[f_{\varepsilon,r}](x) - E_{\varepsilon,r}[f_{\varepsilon,r}](y)|^2 d\pi \\
&\leq \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} \|\varepsilon^{-2} \chi_r * K_{\text{per}} * (\chi_r * \rho_{f_{\varepsilon,r}})\|_{\text{Lip}}^2 |x - y|^2 d\pi \\
&\leq C\varepsilon^{-4} |\log r|^2 (1 + \|\chi_r * \rho_{f_{\varepsilon,r}}\|_{L^\infty})^2 \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x - y|^2 d\pi \\
&\leq C\varepsilon^{-4} |\log r|^2 M^2 \lambda^{-2} D(t).
\end{aligned}$$

For I_2 , we first observe that since the y -marginal of π_t is $\rho_{f_\varepsilon}(y) dy$,

$$\begin{aligned}
I_2 &= \int_{\mathbb{T}^d} |\varepsilon^{-2} K_{\text{per}} * (\chi_r * \chi_r * \rho_{f_{\varepsilon,r}} - \rho_{f_\varepsilon})|^2(y) \rho_{f_\varepsilon}(y) dy \\
&\leq M \|\varepsilon^{-2} K_{\text{per}} * (\chi_r * \chi_r * \rho_{f_{\varepsilon,r}} - \rho_{f_\varepsilon})\|_{L^2(\mathbb{T}^d)}^2,
\end{aligned} \tag{5.42}$$

where (5.42) follows from the fact that $\|\rho_{f_\varepsilon}\|_{L^\infty} \leq M$ by definition (recall (5.22)).

We then apply the Loeper stability estimate, Theorem 1.17:

$$\|\varepsilon^{-2} K_{\text{per}} * (\chi_r * \chi_r * \rho_{f_{\varepsilon,r}} - \rho_{f_\varepsilon})\|_{L^2(\mathbb{T}^d)} \leq \varepsilon^{-2} M^{1/2} W_2(\chi_r * \chi_r * \rho_{f_{\varepsilon,r}}, \rho_{f_\varepsilon}).$$

To control the Wasserstein distance we first apply the triangle inequality:

$$W_2(\chi_r * \chi_r * \rho_{f_{\varepsilon,r}}, \rho_{f_\varepsilon}) \leq W_2(\chi_r * \chi_r * \rho_{f_{\varepsilon,r}}, \chi_r * \rho_{f_{\varepsilon,r}}) + W_2(\chi_r * \rho_{f_{\varepsilon,r}}, \rho_{f_{\varepsilon,r}}) + W_2(\rho_{f_{\varepsilon,r}}, \rho_{f_\varepsilon}).$$

The third term can be controlled by $\lambda^{-1} D^{1/2}$ due to (4.21). We apply Lemma 4.4 to each of the first two terms. This results in the estimate

$$W_2(\chi_r * \chi_r * \rho_{f_{\varepsilon,r}}, \rho_{f_\varepsilon}) \leq 2r + \lambda^{-1} D^{1/2}.$$

Thus we get the following estimate for I_2 :

$$\begin{aligned}
I_2 &\leq C\varepsilon^{-4} M^2 (2r + \lambda^{-1} D^{1/2})^2 \\
&\leq C\varepsilon^{-4} M^2 (r^2 + \lambda^{-2} D).
\end{aligned}$$

Substituting these estimates into (5.41) gives us that

$$\begin{aligned} D'(t) &\leq \left[\lambda + \alpha + \frac{1}{\alpha} C \varepsilon^{-4} M^2 (|\log r|^2 + 1) \lambda^{-2} \right] D(t) + \frac{C}{\alpha} \varepsilon^{-4} M^2 r^2 \\ &\leq \left[\lambda + \alpha + \frac{1}{\alpha} C \varepsilon^{-4} M^2 |\log r|^2 \lambda^{-2} \right] D(t) + \frac{C}{\alpha} \varepsilon^{-4} M^2 r^2. \end{aligned} \quad (5.43)$$

We will now choose the parameters α and λ so as to minimise the constant in the exponential part of our Gronwall estimate; that is, the coefficient of D in (5.43). This has a minimum for (α, λ) satisfying

$$\begin{cases} \alpha &= C \lambda^{-1} \varepsilon^{-2} M |\log r| \\ \lambda &= \left(\frac{C \varepsilon^{-4} M^2 |\log r|^2}{\alpha} \right)^{1/3}, \end{cases}$$

that is, $\alpha = \lambda = C \varepsilon^{-1} M^{1/2} |\log r|^{1/2}$. With this choice of parameters (5.43) becomes

$$D'(t) \leq C \varepsilon^{-1} M^{1/2} |\log r|^{1/2} D(t) + C \varepsilon^{-3} M^{3/2} r^2 |\log r|^{-1/2}.$$

Since $D(0) = 0$, the above inequality implies that

$$D(t) \leq C \varepsilon^{-3} M^{3/2} r^2 |\log r|^{-1/2} e^{C \varepsilon^{-1} M^{1/2} |\log r|^{1/2} t}.$$

We conclude that

$$W_2(f_{\varepsilon, r}(t), f_{\varepsilon}(t)) \leq C \varepsilon^{-3/2} M^{3/4} r |\log r|^{-1/4} e^{C \varepsilon^{-1} M^{1/2} |\log r|^{1/2} t}.$$

Finally, we use this to prove (5.40). Since $\{f_{\varepsilon}(0)\}$ are assumed to satisfy Assumption 2, we may apply Proposition 5.5 to deduce that

$$M \leq C \varepsilon^{-\zeta d},$$

for ζ defined in (5.12)-(5.13). Next, we observe that the relation (5.23) implies that

$$\varepsilon^{-(2+\zeta d)} \leq C_{T, \beta} |\log r|,$$

for some $C_{\beta, T}$. Thus

$$\varepsilon^{-1} M^{1/2} \leq C_{\beta, T} |\log r|^{1/2}.$$

Moreover, by (5.29) and (5.31), we have

$$e^{C\varepsilon^{-1}M^{1/2}|\log r|^{1/2}t} \leq r^{-\beta}.$$

Thus

$$\sup_{t \in [0, T]} W_2(f_{\varepsilon, r}(t), f_{\varepsilon}(t)) \leq C_T r^{1-\beta} |\log r|^{1/2}.$$

Since $\beta < 1$, the right hand side converges to 0 as $N \rightarrow \infty$. This completes the proof. \square

5.5.3.2 VPME

The next lemma is a quantified version of Lemma 4.9.

Lemma 5.12 (Approximation of (VPME) in quasineutral scaling). *Let $f_{\varepsilon}(0)$ satisfy the assumptions of Theorem 3.1. Let $f_{\varepsilon, r}$ be the solution of the scaled and regularised Vlasov equation (5.18) with initial datum $f_{\varepsilon}(0)$. Let f_{ε} be the unique bounded density solution of (3.1).*

Fix $T > 0$. Then there exists a constant C depending on T and on $\{f_{\varepsilon}(0)\}_{\varepsilon}$ such that the following holds. If r and ε satisfy

$$r \leq [\overline{\exp}_3(C\varepsilon^{-2})]^{-1},$$

then

$$\limsup_{r \rightarrow 0} \sup_{t \in [0, T]} W_2(f_{\varepsilon, r}(t), f_{\varepsilon}(t)) = 0.$$

Proof. By (1.66), Lemma 1.12 and (5.21) there exists a constant C depending on C_0 and d only such that

$$\|\rho_{f_{\varepsilon}^{(r)}}\|_{L^{\infty}([0, T]; L^{\frac{d+2}{d}}(\mathbb{T}^d))}, \|\rho_{f_{\varepsilon}}\|_{L^{\infty}([0, T]; L^{\frac{d+2}{d}}(\mathbb{T}^d))} \leq C. \quad (5.44)$$

By Proposition 3.10, there exists a constant C depending on the initial data and T such that

$$\|\rho_{f_{\varepsilon}^{(r)}}\|_{L^{\infty}([0, T]; L^{\infty}(\mathbb{T}^d))}, \|\rho_{f_{\varepsilon}}\|_{L^{\infty}([0, T]; L^{\infty}(\mathbb{T}^d))} \leq \exp(C\varepsilon^{-2}). \quad (5.45)$$

We will control the Wasserstein distance between $f_{\varepsilon, r}$ and f_{ε} using a particular coupling π_t . Since both solutions share the same initial datum, we take π_0 to be the trivial coupling $f_{\varepsilon}(0)(x, v)\delta((x, v) - (y, w)) dx dv dy dw$. We construct $\pi_t \in \Pi(f_{\varepsilon}(t), f_{\varepsilon, r}(t))$ as in (2.25), using

the scaled version of the characteristic systems in (4.35):

$$\begin{cases} \dot{X}_{x,v}^{(1)} = V_{x,v}^{(1)} \\ \dot{V}_{x,v}^{(1)} = E(X_{x,v}^{(1)}) \\ (X_{x,v}^{(1)}(0), V_{x,v}^{(1)}(0)) = (x, v) \\ E = -\nabla U \\ \varepsilon^2 \Delta U = e^U - \rho_{f_\varepsilon} \end{cases} \quad \begin{cases} \dot{X}_{x,v}^{(2)} = V_{x,v}^{(2)} \\ \dot{V}_{x,v}^{(2)} = E_r(X_{x,v}^{(2)}) \\ (X_{x,v}^{(2)}(0), V_{x,v}^{(2)}(0)) = (x, v) \\ E_r = -\chi_r * \nabla U_r \\ \varepsilon^2 \Delta U_r = e^{U_r} - \chi_r * \rho_{f_{\varepsilon,r}}. \end{cases}$$

We define D as in (4.19). As in Lemma 4.8, we obtain for any $\alpha > 0$,

$$\dot{D} \leq (\alpha + \lambda)D + \frac{C}{\alpha} \sum_{i=1}^5 I_i,$$

where

$$\begin{aligned} I_1 &:= \int |\nabla \bar{U}_r(X_t^{(1)}) - \nabla \bar{U}_r(X_t^{(2)})|^2 d\pi_0, & I_2 &:= \int |\nabla \bar{U}_r(X_t^{(1)}) - \nabla \bar{U}(X_t^{(1)})|^2 d\pi_0, \\ I_3 &:= \int |\nabla \hat{U}_r(X_t^{(1)}) - \nabla \hat{U}_r(X_t^{(2)})|^2 d\pi_0, & I_4 &:= \int |\nabla \hat{U}(X_t^{(1)}) - \nabla \hat{U}_r(X_t^{(1)})|^2 d\pi_0 \\ I_5 &:= \int |\chi_r * \nabla U_r(X_t^{(2)}) - \nabla U_r(X_t^{(2)})|^2 d\pi_0. \end{aligned}$$

I_1 is estimated as in (4.36). There is an extra factor of ε^{-4} due to the quasineutral scaling and the mass bound $M = M_{\varepsilon,T}$ depends on ε :

$$I_1 \leq C\varepsilon^{-4} |\log r|^2 M_{\varepsilon,T}^2 \lambda^{-2} D.$$

We estimate I_2 as in (4.37), keeping track of the dependence on ε in Lemma 2.10:

$$\|\nabla \bar{U} - \nabla \bar{U}_r\|_{L^2(\mathbb{T}^d)} \leq C\varepsilon^{-2} \sqrt{M_{\varepsilon,T}} (r + \lambda^{-1} \sqrt{D}).$$

We conclude that

$$I_2 \leq C\varepsilon^{-4} M_{\varepsilon,T}^2 (r + \lambda^{-1} \sqrt{D})^2.$$

For I_3 , by the regularity estimate from Proposition 2.4 and the uniform $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ estimate on the density (5.44) we have

$$\|\hat{U}_r\|_{C^{2,\alpha}(\mathbb{T}^d)} \leq \overline{\exp}_2(C\varepsilon^{-2}),$$

where C depends only on C_0 . We obtain

$$\begin{aligned} I_3 &\leq \overline{\text{exp}}_2(C\varepsilon^{-2}) \int_{\mathbb{T}^d \times \mathbb{R}^d} |X_t^{(1)} - X_t^{(2)}|^2 d\pi_0 \\ &\leq \overline{\text{exp}}_2(C\varepsilon^{-2}) \lambda^{-2} D. \end{aligned}$$

For I_4 we use the stability estimate from Lemma 3.6 and the $L^{\frac{d+2}{d}}(\mathbb{T}^d)$ estimate (5.44):

$$\|\nabla \widehat{U} - \nabla \widehat{U}_r\|_{L^2(\mathbb{T}^d)} \leq \overline{\text{exp}}_2(C\varepsilon^{-2}) \|\nabla \bar{U} - \nabla \bar{U}_r\|_{L^2(\mathbb{T}^d)} \leq \overline{\text{exp}}_2(C\varepsilon^{-2}) \sqrt{M_{\varepsilon,T}} (r + \lambda^{-1} \sqrt{D}).$$

Hence

$$I_4 \leq \overline{\text{exp}}_2(C\varepsilon^{-2}) M_{\varepsilon,T}^2 (r + \lambda^{-1} \sqrt{D})^2.$$

For I_5 , by (4.38) we have

$$\|\chi_r * \nabla U_r - \nabla U_r\|_{L^2(\mathbb{T}^d)}^2 \leq Cr^2 \|U_r\|_{W^{2,2}(\mathbb{T}^d)}^2.$$

Thus

$$\begin{aligned} I_5 &:= \int_{\mathbb{T}^d} |\chi_r * \nabla U_r(x) - \nabla U_r(x)|^2 \rho_{f_\varepsilon^{(r)}}(dx) \\ &\leq \|\rho_{f_\varepsilon^{(r)}}\|_{L^\infty(\mathbb{T}^d)} \|\chi_r * \nabla U_r - \nabla U_r\|_{L^2(\mathbb{T}^d)}^2 \\ &\leq Cr^2 M_{\varepsilon,T} \|U_r\|_{W^{2,2}(\mathbb{T}^d)}^2 \\ &\leq Cr^2 \varepsilon^{-4} M_{\varepsilon,T} \|e^{U_r} - \rho_{f_\varepsilon^{(r)}}\|_{L^2(\mathbb{T}^d)}^2. \end{aligned}$$

As in (4.39), we have

$$\|\rho_{f_{\varepsilon,r}}\|_{L^2(\mathbb{T}^d)} \leq \|\rho_{f_{\varepsilon,r}}\|_{L^\infty(\mathbb{T}^d)}^{\frac{d-2}{2d}} \|\rho_{f_{\varepsilon,r}}\|_{L^{\frac{d+2}{d}}(\mathbb{T}^d)}^{\frac{d+2}{2d}} \leq CM_{\varepsilon,T}^{\frac{d-2}{2d}}.$$

To estimate e^{U_r} , first note that by Proposition 2.4 and (5.44),

$$\|\bar{U}_r\|_{L^\infty(\mathbb{T}^d)} \leq C\varepsilon^{-2}.$$

By (3.8),

$$\|e^{\widehat{U}_r}\|_{L^2(\mathbb{T}^d)} \leq \exp(C\varepsilon^{-2}).$$

Thus

$$\|e^{U_r}\|_{L^2(\mathbb{T}^d)} \leq C \exp\left(\|\bar{U}_r\|_{L^\infty(\mathbb{T}^d)}\right) \|e^{\widehat{U}_r}\|_{L^2(\mathbb{T}^d)} \leq \exp(C\varepsilon^{-2}).$$

Therefore

$$I_5 \leq Cr^2 \varepsilon^{-4} M_{\varepsilon, T} \left(M_{\varepsilon, T}^{\frac{d-2}{d}} + e^{C\varepsilon^{-2}} \right).$$

Putting these five estimates together, we obtain

$$\begin{aligned} \dot{D} \leq (\alpha + \lambda)D + \frac{C}{\alpha\lambda^2} r^2 \overline{\exp}_2(C\varepsilon^{-2}) M_{\varepsilon, T}^2 \\ + \frac{C}{\alpha\lambda^2} \{ \overline{\exp}_2(C\varepsilon^{-2}) M_{\varepsilon, T}^2 + \varepsilon^{-4} |\log r|^2 M_{\varepsilon, T}^2 + \overline{\exp}_2(C\varepsilon^{-2}) \} D. \end{aligned}$$

By (5.45), we may estimate that

$$M_{\varepsilon, T} \leq \exp(C\varepsilon^{-2}).$$

From this we deduce

$$\dot{D} \leq (\alpha + \lambda)D + \frac{C}{\alpha\lambda^2} \{ \exp(C\varepsilon^{-2}) |\log r|^2 + \overline{\exp}_2(C\varepsilon^{-2}) \} D + \frac{C}{\alpha\lambda^2} \overline{\exp}_2(C\varepsilon^{-2}) r^2.$$

After choosing α and λ so as to minimise the constant in front of D we obtain

$$\dot{D} \leq C\lambda_* D + \overline{\exp}_2(C\varepsilon^{-2}) r^2,$$

where

$$\lambda_* = [\exp(C\varepsilon^{-2}) |\log r|^2 + \overline{\exp}_2(C\varepsilon^{-2})]^{1/4} \geq 1,$$

for r, ε sufficiently small.

Therefore, by a Grönwall estimate

$$D(t) \leq (D(0) + \overline{\exp}_2(C\varepsilon^{-2}) r^2) \exp[Ct\lambda_*].$$

Since π_0 was trivial, $D(0) = 0$. Since D controls the squared Wasserstein distance,

$$W_2^2(f_{\varepsilon, r}(t), f_{\varepsilon}(t)) \leq \exp(C\varepsilon^{-2}) r^2 \exp[Ct\lambda_*].$$

Then, by definition of λ_* ,

$$\sup_{t \in [0, T]} W_2^2(f_{\varepsilon, r}(t), f_{\varepsilon}(t)) \leq r^2 \exp \left[\exp(C\varepsilon^{-2}) |\log r|^{1/2} \right] \cdot \overline{\exp}_3(C\varepsilon^{-2}).$$

If $r \leq [\overline{\text{exp}}_3(A\varepsilon^{-2})]^{-1}$, then

$$e^{C\varepsilon^{-2}} |\log r|^{-1/2} \leq \exp \left[C_T \varepsilon^{-2} - \frac{1}{2} \exp(A\varepsilon^{-2}) \right] \rightarrow 0$$

as ε tends to zero, for any $A \geq 0$. Hence, for any $\eta > 0$, for ε sufficiently small,

$$\begin{aligned} r^2 \exp \left[\exp(C_T \varepsilon^{-2}) |\log r|^{1/2} \right] &\leq r^{2 - \exp(C_T \varepsilon^{-2}) |\log r|^{1/2}} \\ &\leq r^{2 - \eta}. \end{aligned}$$

Moreover,

$$r^{2 - \eta} \overline{\text{exp}}_3(C_T \varepsilon^{-2}) \leq \exp \left[\overline{\text{exp}}_2(C_T \varepsilon^{-2}) - (2 - \eta) \overline{\text{exp}}_2(A\varepsilon^{-2}) \right],$$

which converges to zero for any $\eta < 2$ as ε tends to zero, as long as $A \geq C_T$.

Therefore, if $r \leq [\overline{\text{exp}}_3(A\varepsilon^{-2})]^{-1}$ for $A \geq C_T$, then as ε tends to zero (and so r also tends to zero),

$$\sup_{t \in [0, T]} W_2(f_{\varepsilon, r}(t), f_{\varepsilon}(t)) \rightarrow 0.$$

□

5.5.4 Conclusion of the Proof

5.5.4.1 From Extended Electrons to KInE

We combine the previous results to complete the proof of Theorem 5.1. We use $\tilde{\mu}$ to denote the distribution produced by filtering μ using the correctors defined in (1.53), following Definition 5.

Proof of Theorem 5.1. In the following, $\mu_{\varepsilon, r}^N$ denotes the empirical measure associated to the solution of the particle system (5.4), while $f_{\varepsilon, r}$ denotes the solution of the regularised Vlasov–Poisson system (5.17) with initial datum $f_{\varepsilon}(0)$. f_{ε} denotes the solution of the scaled Vlasov–Poisson system (5.3) with initial datum $f_{\varepsilon}(0)$. Let g be the solution of the KInE system (5.1), on some time interval $[0, T_*]$, obtained in the quasineutral limit from f_{ε} using Theorem 5.8.

Our aim is to estimate $W_1(\tilde{\mu}_{\varepsilon, r}^N, g)$. First, we apply the triangle inequality for the Wasserstein distance to get

$$W_1(\tilde{\mu}_{\varepsilon, r}^N, g) \leq W_1(\tilde{\mu}_{\varepsilon, r}^N, \tilde{f}_{\varepsilon}) + W_1(\tilde{f}_{\varepsilon}, g)$$

We begin by estimating $W_1(\tilde{\mu}_{\varepsilon,r}^N, \tilde{f}_\varepsilon)$. By Lemma 5.7 and the fact that $W_1 \leq W_2$,

$$\begin{aligned} \sup_{[0,T]} W_1(\tilde{\mu}_{\varepsilon,r}^N, \tilde{f}_\varepsilon) &\leq C_T \sup_{[0,T]} W_1(\mu_{\varepsilon,r}^N, f_\varepsilon) \\ &\leq C_T \sup_{[0,T]} W_2(\mu_{\varepsilon,r}^N, f_\varepsilon) \\ &\leq C_T \left(\sup_{[0,T]} W_2(\mu_{\varepsilon,r}^N, f_{\varepsilon,r}) + \sup_{[0,T]} W_2(f_{\varepsilon,r}, f_\varepsilon) \right). \end{aligned}$$

Under conditions (5.24) and (5.23), the first term converges to zero by Proposition 5.9. The second term converges by Proposition 5.11. Hence

$$\lim_{N \rightarrow \infty} \sup_{[0,T]} W_1(\tilde{\mu}_{\varepsilon,r}^N, \tilde{f}_\varepsilon) = 0.$$

For $W_1(\tilde{f}_\varepsilon, g)$, we apply Theorem 5.8 to deduce that

$$\lim_{N \rightarrow \infty} \sup_{[0,T]} W_1(\tilde{f}_\varepsilon, g) = 0.$$

Therefore

$$\lim_{N \rightarrow \infty} \sup_{[0,T]} W_1(\tilde{\mu}_{\varepsilon,r}^N, g) = 0.$$

□

5.5.4.2 From Extended Ions to KIsE

Proof of Theorem 5.3. In the following, $\mu_{\varepsilon,r}^N$ denotes the empirical measure associated to the solution of the particle system (5.8), while $f_{\varepsilon,r}$ denotes the solution of the scaled version of (5.18) with initial datum $f_\varepsilon(0)$. f_ε denotes the solution of the scaled VPME system (5.7) with initial datum $f_\varepsilon(0)$. Let g be the solution of the KIE system (3.2), on some time interval $[0, T_*]$, obtained in the quasineutral limit from f_ε using Theorem 3.1.

Our aim is to estimate $W_1(\mu_{\varepsilon,r}^N(t), g(t))$. By the triangle inequality for W_1 ,

$$\begin{aligned} \sup_{t \in [0, T_*]} W_1(\mu_{\varepsilon,r}^N(t), g(t)) &\leq \sup_{t \in [0, T_*]} W_1(\mu_{\varepsilon,r}^N(t), f_{\varepsilon,r}(t)) \\ &\quad + \sup_{t \in [0, T_*]} W_1(f_{\varepsilon,r}(t), f_\varepsilon(t)) + \sup_{t \in [0, T_*]} W_1(f_\varepsilon(t), g(t)). \end{aligned} \quad (5.46)$$

The last term converges to zero as ε tends to zero, by Theorem 3.1.

For the other two terms, we first observe that

$$W_1(\mu_{\varepsilon,r}^N(t), f_{\varepsilon,r}(t)) \leq W_2(\mu_{\varepsilon,r}^N(t), f_{\varepsilon,r}(t)), \quad W_1(f_{\varepsilon,r}(t), f_{\varepsilon}(t)) \leq W_2(f_{\varepsilon,r}(t), f_{\varepsilon}(t)).$$

Then the second term of (5.46) converges to zero by Lemma 5.10 and the third term of (5.46) converges to zero by Lemma 5.12, provided that (5.34) is satisfied for C depending on T_* . \square

5.6 Typicality

5.6.1 Concentration Estimates

To prove the typicality theorems - Theorem 5.2 and Theorem 5.4, we use the same strategy as in Section 4.6. We employ concentration estimates, due to Fournier and Guillin [29], that allow us to quantify the rate of convergence of the empirical measures to the reference data $f_{\varepsilon}(0)$. For the regularised mean field limit, we used [29, Theorem 2], which we quoted as Theorem 4.10.

In this chapter, for each N we choose initial configurations by sampling from a different reference distribution $f_{\varepsilon(N)}(0)$. To use Theorem 4.10, we would need to track the dependence of the constants c, C on (the moments of) $f_{\varepsilon(N)}(0)$.

Instead, we will make use of the fact that we work with compactly supported data. We will find it more convenient to use a slightly different version of the concentration estimates from Fournier–Guillin [29]. These estimates are designed for compactly supported measures. The following result is from [29, Proposition 10].

Theorem 5.13. *Let ν be a probability measure supported on $(-1, 1]^m$. Let ν^N denote the empirical measure of N independent samples from ν . Then there exist constants c, C depending on p and m only such that for any $x > 0$,*

$$\mathbb{P}(W_p^p(\nu^N, \nu) \geq x) \leq a(N, x) \mathbb{1}_{\{x \leq 1\}},$$

where $a(N, x)$ is defined, as in (4.41), by

$$a(N, x) = C \begin{cases} \exp(-cNx^2) & p > \frac{m}{2} \\ \exp\left(-cN \left[\frac{x}{\log(2+\frac{1}{x})}\right]^2\right) & p = \frac{m}{2} \\ \exp(-cNx^{m/p}) & p < \frac{m}{2} \end{cases}$$

5.6.2 Rescaling of Measures

Our goal is to apply Theorem 5.13 in the case where $\nu = f_\varepsilon(0) dx dv$, which has support contained in $\mathbb{T}^d \times B_R$ for some $R > 0$. In order to apply the above estimate, we first rescale the velocity variable in order to work with measures supported in $[-1, 1]^{2d}$.

Definition 6. Let ν be a measure on $\mathbb{T}^d \times \mathbb{R}^d$. We define a scaled measure $\mathcal{S}_R[\nu]$ such that for any $X \in \mathcal{B}(\mathbb{T}^d)$ and $V \in \mathcal{B}(\mathbb{R}^d)$,

$$\mathcal{S}_R[\nu](X \times V) = \nu(X \times RV).$$

Similarly, let ν_1 and ν_2 be measures on $\mathbb{T}^d \times \mathbb{R}^d$ and let $\pi \in \Pi(\nu_1, \nu_2)$. Then let $\mathcal{S}_R^{(2)}[\pi]$ be defined via

$$\mathcal{S}_R^{(2)}[\pi](X_1 \times V_1 \times X_2 \times V_2) = \pi(X_1 \times RV_1 \times X_2 \times RV_2).$$

Remarks 1. (i) Note that $\mathcal{S}_R^{(2)}[\pi] \in \Pi(\mathcal{S}_R[\nu_1], \mathcal{S}_R[\nu_2])$.

(ii) $\mathcal{S}_R^{(2)}$ gives a bijection between $\Pi(\nu_1, \nu_2)$ and $\Pi(\mathcal{S}_R[\nu_1], \mathcal{S}_R[\nu_2])$.

We examine the effect of this scaling on the Wasserstein distance.

Lemma 5.14. Let ν_1, ν_2 be measures on $\mathbb{T}^d \times \mathbb{R}^d$. Then

$$W_p(\nu_1, \nu_2) \leq RW_p(\mathcal{S}_R[\nu_1], \mathcal{S}_R[\nu_2]).$$

Proof. Observe that for any $\pi \in \Pi(\nu_1, \nu_2)$,

$$\begin{aligned} \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x - y|^p + |v - w|^p d\pi &= \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x - y|^p + R^p |v - w|^p d\mathcal{S}_R^{(2)}[\pi] \\ &\leq R^p \int_{(\mathbb{T}^d \times \mathbb{R}^d)^2} |x - y|^p + |v - w|^p d\mathcal{S}_R^{(2)}[\pi] \end{aligned}$$

Since $\mathcal{S}_R^{(2)}$ is a bijection, taking infimum over π yields

$$W_p(\nu_1, \nu_2) \leq RW_p(\mathcal{S}_R[\nu_1], \mathcal{S}_R[\nu_2]).$$

□

Remark 7. Note that, if $(Z_i^{(R)})_{i=1}^N$ are N independent samples from $\mathcal{S}_R[\nu]$, then $(Z_i^{(R)})_{i=1}^N$ has the same law as $(X_i, \frac{1}{R}V_i)_{i=1}^N$, where $(Z_i)_{i=1}^N = (X_i, V_i)_{i=1}^N$ are N independent samples from ν . We will use this property in the proofs below to identify scaled empirical measures with empirical measures drawn from scaled distributions.

5.6.3 Conclusion of the Proof.

Proof of Theorem 5.2. We follow the strategy of Lazarovici [61], and consider the events where the desired rate of convergence does not hold at N :

$$A_N := \left\{ W_2^2(\mu_\varepsilon^N(0), f_\varepsilon(0)) \geq C\varepsilon^{-2\gamma} r^{d+2+\eta} \right\}$$

Our goal is to show that $\sum_N \mathbb{P}(A_N)$ is finite for some choice of C and η . By the Borel-Cantelli lemma, this will imply that the probability that these events occur infinitely often is zero. In other words, with probability one, there exists N such that A_n^c occurs for all $n \geq N$. This means precisely that the desired rate of convergence holds. We estimate $\mathbb{P}(A_N)$ using the concentration inequality from Theorem 5.13 and the scaling argument described above.

By Lemma 5.14,

$$\mathbb{P}(A_N) \leq \mathbb{P}\left(\varepsilon^{-2\gamma} W_2^2(\mathcal{S}_{\varepsilon^{-\gamma}}[\mu_\varepsilon^N(0)], \mathcal{S}_{\varepsilon^{-\gamma}}[f_\varepsilon(0)]) \geq C\varepsilon^{-2\gamma} r^{d+2+\eta}\right).$$

By Remark 7, $\mathcal{S}_{\varepsilon^{-\gamma}}[\mu_\varepsilon^N(0)]$ has the same law (as a random measure) as the empirical measure of N independent samples from $\mathcal{S}_{\varepsilon^{-\gamma}}[f_\varepsilon(0)]$. Since the scaled measures have support contained in $[-1, 1]^{2d}$, we may apply Theorem 5.13 to deduce the existence of constants κ, c, C depending only on d such that

$$\mathbb{P}\left(W_2^2(\mathcal{S}_{\varepsilon^{-\gamma}}[\mu_{0,\varepsilon}^N], \mathcal{S}_{\varepsilon^{-\gamma}}[f_\varepsilon(0)]) \geq \kappa r^{d+2+\eta}\right) \leq C \begin{cases} \exp\left(-cN\left(\frac{r^{4+\eta}}{\log(2+r^{-(4+\eta)})}\right)^2\right) & \text{for } d = 2 \\ \exp\left(-cNr^{3(5+\eta)}\right) & \text{for } d = 3. \end{cases}$$

Using this we deduce that

$$\mathbb{P}(A_N) \leq C \begin{cases} \exp\left(-cN^{8\alpha+\eta(2\alpha-\frac{1}{4})}\left(\log\left(2+N^{-\frac{1}{2}+4\alpha+\eta(\alpha-\frac{1}{8})}\right)\right)^{-2}\right) & \text{for } d = 2 \\ \exp\left(-cN^{15\alpha+\eta(3\alpha-\frac{1}{5})}\right) & \text{for } d = 3. \end{cases}$$

Thus $\sum_N \mathbb{P}(A_N)$ is finite for

$$\eta < \alpha \frac{d(d+2)^2}{1-\alpha d(d+2)},$$

which completes the proof. □

Proof of Theorem 5.4. We use the same strategy as for Theorem 5.2. Recall that we already assumed that $f_\varepsilon(0)$ were compactly supported with the support in velocity growing no faster

than $e^{C\varepsilon^{-2}}$ for some C . It is enough to show that

$$\sum_N \mathbb{P}(A_N) < \infty, \quad (5.47)$$

where A_N denotes the event

$$A_N := \left\{ W_2^2(\mathcal{S}_{e^{-C\varepsilon^{-2}}}[\mu_\varepsilon^N(0)], \mathcal{S}_{e^{-C\varepsilon^{-2}}}[f_\varepsilon(0)]) > \frac{1}{2} r^{d+2+\eta} \exp(-2C\varepsilon^{-2}) \right\}.$$

We observe that the assumption

$$r < [\overline{\exp}_3(A\varepsilon^{-2})]^{-1},$$

implies that

$$\exp(-C\varepsilon^{-2}) > c(\log \log N)^{-\zeta} > cN^{-\alpha}$$

for ζ depending on C and A , any $\alpha > 0$ and c depending on α , C and A . We then apply Theorem 5.13 with the choice

$$x_N = cN^{-\gamma(d+2+\eta)-\alpha}.$$

The assumption (5.16) on γ implies that it is possible to find $\eta > 0$ such that

$$\sum_N a(N, x_N) < \infty.$$

This yields (5.47), which completes the proof. □

References

- [1] Ambrosio, L. (2008). Transport equation and Cauchy problem for Non-smooth Vector Fields. In Dacorogna, B. and Marcellini, P., editors, *Calculus of Variations and Nonlinear Partial Differential Equations: Lectures given at the C.I.M.E. Summer School held in Cetraro, Italy, June 27 - July 2, 2005*. Springer-Verlag Berlin Heidelberg.
- [2] Arsenev, A. (1975). Existence in the large of a weak solution to the Vlasov system of equations. *Zh. Vychisl. Mat. i Mat. Fiz.*, 15:136–147.
- [3] Baradat, A. (2018). Nonlinear instability in Vlasov type equations around rough velocity profiles. arXiv:1811.01350.
- [4] Bardos, C. (2014). About a variant of the 1d Vlasov equation, dubbed “Vlasov-Dirac-Benney equation”. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2012–2013*, Exp. No. XV, Sémin. Équ. Dériv. Partielles. École Polytech., Palaiseau.
- [5] Bardos, C. and Besse, N. (2013). The Cauchy problem for the Vlasov-Dirac-Benney equation and related issues in fluid mechanics and semi-classical limits. *Kinet. Relat. Models*, 6(4):893–917.
- [6] Bardos, C. and Degond, P. (1985). Global existence for the Vlasov–Poisson equation in 3 space variables with small initial data. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2(2):101–118.
- [7] Bardos, C., Golse, F., Nguyen, T. T., and Sentis, R. (2018). The Maxwell-Boltzmann approximation for ion kinetic modeling. *Phys. D*, 376/377:94–107.
- [8] Bardos, C. and Nouri, A. (2012). A Vlasov equation with Dirac potential used in fusion plasmas. *J. Math. Phys.*, 53(11):115621.
- [9] Batt, J. (1977). Global symmetric solutions of the initial value problem of stellar dynamics. *J. Differential Equations*, 25(3):342–364.
- [10] Batt, J. and Rein, G. (1991). Global classical solutions of the periodic Vlasov-Poisson system in three dimensions. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(6):411–416.
- [11] Bellan, P. M. (2008). *Fundamentals of Plasma Physics*. Cambridge University Press.
- [12] Berk, H. L., Nielsen, C. E., and Roberts, K. V. (1970). Phase space hydrodynamics of equivalent nonlinear systems: Experimental and computational observations. *The Physics of Fluids*, 13(4):980–995.

- [13] Bolley, F., Guillin, A., and Villani, C. (2007). Quantitative concentration inequalities for empirical measures on non-compact spaces. *Probab. Theory Related Fields*, 137(3-4):541–593.
- [14] Boltzmann, L. (1872). Weitere Studien über das Wärmegleichgewicht unter Gasmolekülen. *Sitzungsberichte Akademie der Wissenschaften*, 66:275–370.
- [15] Bonhomme, G., Pierre, T., Leclert, G., and Trulsen, J. (1991). Ion phase space vortices in ion beam-plasma systems and their relation with the ion acoustic instability: numerical and experimental results. *Plasma Physics and Controlled Fusion*, 33(5):507–520.
- [16] Bossy, M., Fontbona, J., Jabin, P.-E., and Jabir, J.-F. (2013). Local existence of analytical solutions to an incompressible Lagrangian stochastic model in a periodic domain. *Comm. Partial Differential Equations*, 38(7):1141–1182.
- [17] Bouchut, F. (1991). Global weak solution of the Vlasov-Poisson system for small electrons mass. *Comm. Partial Differential Equations*, 16(8-9):1337–1365.
- [18] Braun, W. and Hepp, K. (1977). The Vlasov dynamics and its fluctuations in the $1/N$ limit of interacting classical particles. *Comm. Math. Phys.*, 56(2):101–113.
- [19] Brenier, Y. (1989). Une formulation de type Vlasov–Poisson pour les équations d’Euler des fluides parfaits incompressibles. [Rapport de recherche] RR-1070, INRIA.
- [20] Brenier, Y. (2000). Convergence of the Vlasov–Poisson system to the incompressible Euler equations. *Comm. Partial Differential Equations*, 25(3-4):737–754.
- [21] Brenier, Y. and Grenier, E. (1994). Limite singulière du système de Vlasov-Poisson dans le régime de quasi neutralité : le cas indépendant du temps. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(2):121–124.
- [22] Caflisch, R. E. (1990). A simplified version of the abstract Cauchy-Kowalewski theorem with weak singularities. *Bull. Amer. Math. Soc. (N.S.)*, 23(2):495–500.
- [23] Chen, F. F. (2016). *Introduction to Plasma Physics and Controlled Fusion*. Springer International Publishing, 3 edition.
- [24] Courant, R. and Hilbert, D. (1989). *Methods of Mathematical Physics, Vol. II: Partial Differential Equations*. Wiley-VCH.
- [25] Desvillettes, L. and Villani, C. (2005). On the trend to global equilibrium for spatially inhomogeneous kinetic systems: The Boltzmann equation. *Invent. Math.*, 159:245–316.
- [26] DiPerna, R. J. and Lions, P. L. (1989). Ordinary differential equations, transport theory and Sobolev spaces. *Inventiones mathematicae*, 98(3):511–547.
- [27] Dobrushin, R. L. (1979). Vlasov Equations. *Funktsional. Anal. i Prilozhen.*, 13(2):48–58.
- [28] Evans, L. C. (2010). *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society.
- [29] Fournier, N. and Guillin, A. (2015). On the rate of convergence in Wasserstein distance of the empirical measure. *Probab. Theory Related Fields*, 162(3-4):707–738.

- [30] Gilbarg, D. and Trudinger, N. S. (1977). *Elliptic partial differential equations of second order*, volume 224 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg.
- [31] Golse, F. (2016). On the dynamics of large particle systems in the mean field limit. In Muntean, A., Rademacher, J., and Zagaris, A., editors, *Macroscopic and Large Scale Phenomena: Coarse Graining, Mean Field Limits and Ergodicity*, volume 3 of *Lect. Notes Appl. Math. Mech.*, pages 1–144. Springer.
- [32] Grenier, E. (1995). Defect measures of the Vlasov-Poisson system in the quasineutral regime. *Comm. Partial Differential Equations*, 20(7-8):1189–1215.
- [33] Grenier, E. (1996). Oscillations in quasineutral plasmas. *Comm. Partial Differential Equations*, 21(3-4):363–394.
- [34] Grenier, E. (1999). Limite quasineutre en dimension 1. In *Journées “Équations aux Dérivées Partielles” (Saint-Jean-de-Monts, 1999)*, pages Exp. No. II, 8. Univ. Nantes, Nantes.
- [35] Griffin-Pickering, M. and Iacobelli, M. Global well-posedness in 3-dimensions for the Vlasov-Poisson system with massless electrons. arXiv:1810.06928.
- [36] Griffin-Pickering, M. and Iacobelli, M. Singular limits for plasmas with thermalised electrons. *J. Math. Pures Appl.* In press, preprint available at arXiv:1811.03693.
- [37] Griffin-Pickering, M. and Iacobelli, M. (2018). A mean field approach to the quasi-neutral limit for the Vlasov-Poisson equation. *SIAM J. Math. Anal.*, 50(5):5502–5536.
- [38] Gurevich, A., Pariiskaya, L., and Pitaevskii, L. (1966). Self-similar motion of rarefied plasma. *Soviet Phys. JETP*, 22(2):449–454.
- [39] Gurevich, A., Pariiskaya, L., and Pitaevskii, L. (1968). Self-similar motion of a low-density plasma. II. *Soviet Phys. JETP*, 27(3):476–482.
- [40] Gurevich, A. V. and Pitaevsky, L. P. (1975). Non-linear dynamics of a rarefied ionized gas. *Progress in Aerospace Sciences*, 16(3):227 – 272.
- [41] Han-Kwan, D. (2011). Quasineutral limit of the Vlasov-Poisson system with massless electrons. *Comm. Partial Differential Equations*, 36(8):1385–1425.
- [42] Han-Kwan, D. and Hauray, M. (2015). Stability issues in the quasineutral limit of the one-dimensional Vlasov-Poisson equation. *Comm. Math. Phys.*, 334(2):1101–1152.
- [43] Han-Kwan, D. and Iacobelli, M. (2017a). Quasineutral limit for Vlasov-Poisson via Wasserstein stability estimates in higher dimension. *J. Differential Equations*, 263(1):1–25.
- [44] Han-Kwan, D. and Iacobelli, M. (2017b). The quasineutral limit of the Vlasov-Poisson equation in Wasserstein metric. *Commun. Math. Sci.*, 15(2):481–509.
- [45] Han-Kwan, D. and Nguyen, T. T. (2016). Ill-posedness of the hydrostatic Euler and singular Vlasov equations. *Arch. Ration. Mech. Anal.*, 221(3):1317–1344.

- [46] Han-Kwan, D. and Rousset, F. (2016). Quasineutral limit for Vlasov-Poisson with Penrose stable data. *Ann. Sci. Éc. Norm. Supér. (4)*, 49(6):1445–1495.
- [47] Hauray, M. (2014). Mean field limit for the one dimensional Vlasov-Poisson equation. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2012–2013*, Exp. No. XXI, Sémin. Équ. Dériv. Partielles. École Polytech., Palaiseau.
- [48] Hauray, M. and Jabin, P.-E. (2007). N -particles approximation of the Vlasov equations with singular potential. *Arch. Ration. Mech. Anal.*, 183(3):489–524.
- [49] Hauray, M. and Jabin, P.-E. (2015). Particle approximation of Vlasov equations with singular forces: propagation of chaos. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(4):891–940.
- [50] Hörmander, L. (1990). *The Analysis of Linear Partial Differential Operators I*. Springer-Verlag, second edition.
- [51] Horst, E. (1981). On the classical solutions of the initial value problem for the unmodified nonlinear Vlasov equation. I. General theory. *Math. Methods Appl. Sci.*, 3(2):229–248.
- [52] Horst, E. (1982). On the classical solutions of the initial value problem for the unmodified nonlinear Vlasov equation. II. Special cases. *Math. Methods Appl. Sci.*, 4(1):19–32.
- [53] Horst, E. (1990). Global solutions of the relativistic Vlasov-Maxwell system of plasma physics. *Dissertationes Math. (Rozprawy Mat.)*, 292.
- [54] Horst, E. (1993). On the asymptotic growth of the solutions of the Vlasov-Poisson system. *Math. Methods Appl. Sci.*, 16(2):75–86.
- [55] Horst, E. and Hunze, R. (1984). Weak solutions of the initial value problem for the unmodified non-linear Vlasov equation. *Math. Methods Appl. Sci.*, 6(2):262–279.
- [56] Iordanskii, S. (1961). Cauchy’s problem for a kinetic equation of plasma. *Trudy Mat. Inst. im. VA Steklova*, 60:181–194.
- [57] Jabin, P. and Nouri, A. (2011). Analytic solutions to a strongly nonlinear Vlasov equation. *C.R. Acad. Sci. Paris, Sér. I*, 349:541–546.
- [58] Jeans, J. H. (1915). On the theory of star-streaming and the structure of the universe. *Monthly Notices of the Royal Astronomical Society*, 76:70–84.
- [59] Kurth, R. (1952). Das Anfangswertproblem der Stelldynamik. *Z. Astrophys.*, 30:213–229.
- [60] Landau, L. (1936). Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung. *Phys. Z. Sowjet.*
- [61] Lazarovici, D. (2016). The Vlasov-Poisson dynamics as the mean field limit of extended charges. *Comm. Math. Phys.*, 347(1):271–289.
- [62] Lazarovici, D. and Pickl, P. (2017). A mean field limit for the Vlasov-Poisson system. *Arch. Ration. Mech. Anal.*, 225(3):1201–1231.

- [63] Lieberman, M. A. and Lichtenberg, A. J. (2005). *Principles of Plasma Discharges and Materials Processing*. John Wiley & Sons.
- [64] Lions, P. L. and Perthame, B. (1991). Propagation of moments and regularity for the 3-dimensional Vlasov-Poisson system. *Invent. Math.*, 105(2):415–430.
- [65] Loeper, G. (2006). Uniqueness of the solution to the Vlasov-Poisson system with bounded density. *J. Math. Pures Appl. (9)*, 86(1):68–79.
- [66] Majda, A. and Bertozzi, A. (2002). *Vorticity and Incompressible Flow*, volume 27 of *Cambridge Texts in Applied Mathematics*. Cambridge University Press.
- [67] Masmoudi, N. (2001). From Vlasov–Poisson system to the incompressible Euler system. *Comm. Partial Differential Equations*, 26(9-10).
- [68] Mason, R. J. (1971). Computer simulation of ion-acoustic shocks. The diaphragm problem. *The Physics of Fluids*, 14(9):1943–1958.
- [69] Maxwell, J. C. (1867). IV. On the dynamical theory of gases. *Phil. Trans. R. Soc.*, 157.
- [70] Medvedev, Y. V. (2011). Ion front in an expanding collisionless plasma. *Plasma Physics and Controlled Fusion*, 53(12):125007.
- [71] Mouhot, C. and Villani, C. (2011). On Landau damping. *Acta Math.*, 207(1):29–201.
- [72] Neunzert, H. and Wick, J. (1974). Die Approximation der Lösung von Integro-Differentialgleichungen durch endliche Punktmengen. In *Numerische Behandlung nicht-linearer Integrodifferential-und Differentialgleichungen.*, volume 395 of *Lecture Notes in Math.*, pages 275–290, Berlin, Heidelberg. Springer.
- [73] Pallard, C. (2012). Moment propagation for weak solutions to the Vlasov-Poisson system. *Comm. Partial Differential Equations*, 37(7):1273–1285.
- [74] Pfaffelmoser, K. (1992). Global classical solutions of the Vlasov-Poisson system in three dimensions for general initial data. *J. Differential Equations*, 95(2):281–303.
- [75] Rein, G. (2007). Collisionless kinetic equations from astrophysics - the Vlasov-Poisson system. In *Handbook of Differential Equations: Evolutionary Equations*, volume III, chapter 5, pages 383–476. Elsevier/North Holland, Amsterdam.
- [76] Robert, R. (1997). Unicité de la solution faible à support compact de l'équation de Vlasov–Poisson. *C. R. Acad. Sci. Paris Sér. I Math.*, 324(8):873–877.
- [77] Sakanaka, P., Chu, C., and Marshall, T. (1971). Formation of ion-acoustic collisionless shocks. *The Physics of Fluids*, 14(611).
- [78] Schaeffer, J. (1987). Global existence for the Poisson-Vlasov system with nearly symmetric data. *J. Differential Equations*, 69(1):111–148.
- [79] Schaeffer, J. (1991). Global existence of smooth solutions to the Vlasov-Poisson system in three dimensions. *Comm. Partial Differential Equations*, 16(8-9):1313–1335.
- [80] Tonks, L. and Langmuir, I. (1929). Oscillations in ionized gases. *Phys. Rev.*, 33(195).

- [81] Ukai, S. and Okabe, T. (1978). On classical solutions in the large in time of two-dimensional Vlasov's equation. *Osaka J. Math.*, 15(2):245–261.
- [82] Van der Vaart, A. and Wellner, J. (1996). *Weak Convergence of Empirical Processes*. Springer-Verlag New York.
- [83] Villani, C. (2003). *Topics in optimal transportation*, volume 58 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI.
- [84] Vlasov, A. A. (1938). On the vibration properties of the electron gas. *Zh. Eksper. Teor. Fiz.*, 8(3):291.
- [85] Vlasov, A. A. (1945). Vlasov equation and plasma dispersion relation. *J. Phys. USSR*, 9:25.
- [86] Yudovich, V. I. (1963). Non-stationary flow of an ideal incompressible liquid. *U.S.S.R. Comput. Math. and Math. Phys.*, 3:1032–1066.

Appendix A

Differential Inequalities

Lemma A.1. *Let $K, C > 0$, and define*

$$z(t) = (1 + 2Ct) [K + \log(1 + 2Ct)].$$

Then

$$\dot{z} \geq C(1 + \log(1 + z)).$$

Proof. By direct computation,

$$\dot{z} = 2C[(K + 1) + \log(1 + 2Ct)].$$

Secondly,

$$\log(1 + z) = \log[1 + (1 + 2Ct)[K + \log(1 + 2Ct)]] \leq \log[(1 + 2Ct)[(K + 1) + \log(1 + 2Ct)]].$$

Thus, since $\log(1 + x) \leq x$,

$$\begin{aligned} \log(1 + z) &= \log(1 + 2Ct) + \log[K + 1 + \log(1 + 2Ct)] \leq \log(1 + 2Ct) + \log[K + 1 + 2Ct] \\ &\leq \log(1 + 2Ct) + \log[(K + 1)(1 + 2Ct)] = 2\log(1 + 2Ct) + \log(K + 1). \end{aligned}$$

Therefore

$$1 + \log(1 + z) \leq 2\log(1 + 2Ct) + K + 1 \leq \frac{1}{C}\dot{z},$$

which completes the proof.

□