Intersection and mixing times for reversible chains

Yuval Peres[*]   Thomas Sauerwald[†]   Perla Sousi[‡]

Alexandre Stauffer[§]

Abstract

We consider two independent Markov chains on the same finite state space, and study their *intersection time*, which is the first time that the trajectories of the two chains intersect. We denote by t_I the expectation of the intersection time, maximized over the starting states of the two chains. We show that, for any reversible and lazy chain, the total variation mixing time is $O(t_I)$. When the chain is reversible and transitive, we give an expression for t_I using the eigenvalues of the transition matrix. In this case, we also show that t_I is of order $\sqrt{n\mathbb{E}[I]}$, where I is the number of intersections of the trajectories of the two chains up to the uniform mixing time, and n is the number of states. For random walks on trees, we show that t_I and the total variation mixing time are of the same order. Finally, for random walks on regular expanders, we show that t_I is of order $\sqrt{n}$.

1 Introduction

Intersections of Markov chains have been intensively studied, partly due to their connection with loop-erased walks and spanning trees. The 1991 book of Lawler [12] focuses on intersections of random walks on lattices. In 1989, Fitzsimmons and Salisbury [6] developed techniques for analysing intersections of Brownian motions and Lévy processes. In 1996, Salisbury [20] adapted those techniques in order to bound intersection probabilities for discrete time Markov chains. In 2003, Lyons, Peres and Schramm [16] used Salisbury's result to extend certain intersection probability estimates from lattices to general Markov chains.

[*]Microsoft Research, Redmond, Washington, USA. E-mail: `peres@microsoft.com`
[†]University of Cambridge, Cambridge, UK. E-mail: `thomas.sauerwald@cl.cam.ac.uk`
[‡]University of Cambridge, Cambridge, UK. E-mail: `p.sousi@statslab.cam.ac.uk`
[§]University of Bath, Bath, UK. E-mail: `a.stauffer@bath.ac.uk`. Supported in part by a Marie Curie Career Integration Grant PCIG13-GA-2013-618588 DSRELIS.

In this paper we focus on finite Markov chains and study the *intersection time*, defined as follows. Let P denote the transition matrix of an irreducible Markov chain on a finite state space Ω, with stationary distribution π. Let X and Y be two independent Markov chains with transition matrix P. Define

$$\tau_I = \inf\{t \geq 0 : \{X_0, \ldots, X_t\} \cap \{Y_0, \ldots, Y_t\} \neq \varnothing\},$$

i.e. τ_I is the first time the trajectories of X and Y intersect. The key quantity will be the expectation of the random time defined above, maximized over starting states:

$$t_{\mathrm{I}} = \max_{x,y \in \Omega} \mathbb{E}_{x,y}[\tau_I].$$

This quantity was considered in [5], where it was estimated in many examples, in particular random walks on tori $\mathbb{Z}_\ell^d$ for $d \geq 1$.

We denote by $t_{\mathrm{mix}} = t_{\mathrm{mix}}(1/4)$ the total variation mixing time; that is,

$$t_{\mathrm{mix}} = \inf\left\{t \geq 0 \colon \|p_t(x, \cdot) - \pi\|_{\mathrm{TV}} \leq \tfrac{1}{4} \text{ for all } x \in \Omega\right\},$$

where $p_t(x, \cdot)$ is the distribution after t steps of the chain started from x, and $\|p_t(x, \cdot) - \pi\|_{\mathrm{TV}} = \frac{1}{2} \sum_{y \in \Omega} |p_t(x, y) - \pi(y)|$ is the total variation distance between two distributions in the same state space. Let $t_{\mathrm{hit}} = \max_{x,y} \mathbb{E}_x[\tau_y]$ be the maximum hitting time, where for all y

$$\tau_y = \inf\{t \geq 0 : X_t = y\}.$$

In order to avoid periodicity and near-periodicity issues, in many places we consider the lazy version of a Markov chain, i.e. the chain with transition matrix $P_L = (P + I)/2$.

For functions f, g we will write $f(n) \lesssim g(n)$ if there exists a universal constant $c > 0$ such that $f(n) \leq cg(n)$ for all n. We write $f(n) \gtrsim g(n)$ if $g(n) \lesssim f(n)$. Finally, we write $f(n) \asymp g(n)$ if both $f(n) \lesssim g(n)$ and $f(n) \gtrsim g(n)$.

We define

$$t_{\mathrm{H}} = \max_{x \in \Omega, A \subseteq \Omega : \pi(A) \geq 3/8} \mathbb{E}_x[\tau_A], \tag{1.1}$$

where τ_A stands for the first hitting time of the set A.

Our first result shows that t_{I} is an upper bound on t_{H} for all chains.

Theorem 1.1. *For all finite and irreducible Markov chains, we have*

$$t_{\mathrm{H}} \lesssim t_I.$$

In other words, there is a universal constant $c > 0$ such that for every n and every irreducible Markov chain we have $t_{\mathrm{H}} \leq c\, t_I$.

Using the equivalence between mixing times and t_{H} for lazy reversible chains (Theorem 2.1), which was proven independently by [18] and [19], we obtain the following corollary.

Corollary 1.2. *For all finite, irreducible, reversible and lazy Markov chains, we have*

$$t_{\mathrm{mix}} \lesssim t_I.$$

For lazy weighted random walks on finite trees, we have

$$t_{\mathrm{mix}} \asymp t_I.$$

We prove Theorem 1.1 and Corollary 1.2 in Section 2, where we also state the equivalence between mixing and hitting times.

Remark 1.3. We recall the definition of the Cesaro mixing time

$$t_{\text{Ces}} = \min\left\{t : \max_{x \in \Omega}\left\|\frac{1}{t}\sum_{s=0}^{t-1}p_s(x,\cdot) - \pi\right\|_{\text{TV}} \le \frac{1}{4}\right\},$$

where $p_s(x,y)$ stands for the transition probability from x to y in s steps. Since $t_{\text{Ces}} \asymp t_{\text{H}}$ for all lazy and irreducible chains without assuming reversibility (see, for instance, [19, Theorem 6.1 and Proposition 7.1]), it follows from Theorem 1.1 that $t_{\text{Ces}} \lesssim t_{\text{I}}$.

Remark 1.4. We note that $t_{\text{I}} \le 2t_{\text{hit}}$, since we can fix a state and wait until both chains hit it. So Theorem 1.1 demonstrates that the intersection time can be sandwiched between the mixing time and the maximum hitting time of the chain. Hence this double inequality can be viewed as a refinement of the basic inequality stating that the mixing time is upper bounded by the maximum hitting time, which is rather loose for many chains.

We denote by t_{unif} the uniform mixing time, i.e.

$$t_{\text{unif}} = \inf\left\{t \ge 0 : \max_{x,y \in \Omega}\left|\frac{p_t(x,y)}{\pi(y)} - 1\right| \le \frac{1}{4}\right\}.$$

Note that $t_{\text{mix}} \le t_{\text{unif}}$. Benjamini and Morris [3] related t_{unif} to intersection properties of multiple random walks. Also, we say that a chain is *transitive* if, for any two $x, y \in \Omega$, there is a bijection $\varphi : \Omega \to \Omega$ such that $\varphi(x) = y$ and $p(z,w) = p(\varphi(z),\varphi(w))$ for all $z, w \in \Omega$.

For transitive reversible chains, our next theorem gives an expression for the intersection time. We prove it in Section 3.

Theorem 1.5. *Let X be a finite, transitive, irreducible, reversible and lazy chain on n states. Let $Q = \sum_{j=2}^{n}(1 - \lambda_j)^{-2}$, where $(\lambda_j)_j$ are the non-unit eigenvalues of the chain in decreasing order. Then we have*

$$t_{\text{I}} \asymp \sqrt{Q} \quad \text{and} \quad Q \asymp n\sum_{i,j=0}^{t_{\text{unif}}}p_{i+j}(x,x)$$

for any state x.

Remark 1.6. Let X and Y be two independent copies of a finite, transitive, irreducible, reversible and lazy chain starting from $x \in \Omega$. We note that if $I = \sum_{i=0}^{t_{\text{unif}}}\sum_{j=0}^{t_{\text{unif}}}\mathbf{1}(X_i = Y_j)$, then $\mathbb{E}[I] = \sum_{i,j=0}^{t_{\text{unif}}}p_{i+j}(x,x)$. So Theorem 1.5 can be restated by saying that

$$t_{\text{I}} \asymp \sqrt{n \cdot \mathbb{E}[I]}.$$

Remark 1.7. When the Markov chains are lazy, simple random walks on $\mathbb{Z}_\ell^d$, the local central limit theorem implies that $p_t(x,x) \asymp t^{-d/2}$ for each fixed d when $t \le t_{\text{unif}} \asymp \ell^2$. Thus the above theorem gives the intersection time in $\mathbb{Z}_\ell^d$, for any $d \ge 1$. In particular, $t_I \asymp \ell^2$ for $d = 1,2,3$, while $t_I \asymp \sqrt{n\log n}$ for $d = 4$ and $t_I \asymp \sqrt{n}$ for $d \ge 5$, where $n = \ell^d$. These estimates were derived in [5] by a less systematic method.

For a finite, irreducible and reversible Markov chain let $1 = \lambda_1 \ge \lambda_2 \ge \lambda_3, \ldots$ be its eigenvalues in decreasing order. Let $\lambda_* = \max_{i \ge 2}|\lambda_i|$, and define the relaxation time $t_{\text{rel}} = (1 - \lambda_*)^{-1}$. Note that $\lambda_* = \lambda_2$ for lazy chains. We obtain the following result for all regular graphs, which we prove in Section 4.

Proposition 1.8. *Consider a lazy, simple random walk on a finite, connected, regular graph G on n vertices. Then we have that $t_{\text{I}} \lesssim \sqrt{n}(t_*)^{\frac{3}{4}}$, where $t_* = \min\{t_{\text{rel}}(\log t_{\text{rel}} + 1), t_{\text{unif}}\}$. In particular, if G is a regular expander on n vertices, we have $t_{\text{I}} \asymp \sqrt{n}$.*

Remark 1.9. For any regular graph on n vertices, we have that $t_I \gtrsim \sqrt{n}$. Indeed, the expected number of intersections by time t, which we denote by I_t, is given by

$$\mathbb{E}_{\pi,\pi}[I_t] = \mathbb{E}_{\pi,\pi}\left[\sum_{i,j=0}^{t} \mathbf{1}(X_i = Y_j)\right] = \frac{(t+1)^2}{n}.$$

Hence, taking $t = \sqrt{n}/2$ and using $\mathbb{P}_{\pi,\pi}(I_t \geq 1) \leq \mathbb{E}_{\pi,\pi}[I_t]$ prove the claim.

The intersection time is related to basic sampling questions [10], testing statistical properties of distributions [2] and testing structural properties of graphs, in particular expansion and conductance [4, 8, 9]. Many of the approaches used in these works rely on collision or intersections of random walks (or more generally, random experiments), which is quite natural if one is interested in the algorithms which work even in sublinear time (or space). In this context, it is particularly important to understand the relation between these parameters and the expansion of the underlying graph, as done in our result which relates the mixing time to the intersection time.

We further point out that there exists a seemingly related notion for single random walks, called self-intersection time. This time plays an important role in the context of finding the discrete logarithm using Markov chains [11]. However, we are not aware of any direct connection between this parameter and the intersection time of two random walks, as the self-intersection time will be just a constant for many natural classes of graphs.

In the remainder of this work, all Markov chains under consideration are assumed to be finite and irreducible.

2   Intersection time for reversible Markov chains

In this section we give the proof of Theorem 1.1. We start by stating a result proved independently by Oliveira [18], and Peres and Sousi [19] that relates the total variation mixing time to the maximum hitting time of large sets for lazy reversible Markov chains.

Theorem 2.1 ([18], [19]). *Let X be a lazy reversible Markov chain with stationary distribution π. Then we have*

$$t_{\mathrm{mix}} \asymp t_{\mathrm{H}},$$

where t_H was defined in (1.1).

For random walks on trees, mixing times are equivalent to hitting times of the so-called "central nodes".

Definition 2.2. A node v of a tree T is called *central* if each component of $T - \{v\}$ has stationary probability at most $1/2$.

Theorem 2.3 ([19]). *Let X be a lazy weighted random walk on a tree and let v be a central node* (*which always exists*). *Then*

$$t_{\mathrm{mix}} \asymp \max_x \mathbb{E}_x[\tau_v],$$

where τ_v is the first hitting time of v.

Before proving Theorem 1.1 we introduce another notion

$$t_I^* = \max_x \mathbb{E}_{x,\pi}[\tau_I].$$

Note that, instead of maximizing over all starting states, in t_I^* we start one chain from stationarity and maximize over the starting state of the other chain.

Proposition 2.4. *For all Markov chains we have*

$$t_{\mathrm{H}} \lesssim t_I^*.$$

Proof. Let X and Y be two independent Markov chains such that $X_0 = x$ and $Y_0 \sim \pi$. Let A be a set with $\pi(A) \geq 3/8$ and define

$$\tau_A = \inf\{t \geq 0 : X_t \in A\}.$$

Then we claim that for all x we have

$$\mathbb{P}_x(\tau_A \leq 20 t_I^*) \geq c > 0. \tag{2.1}$$

First of all by Markov's inequality we immediately get

$$\mathbb{P}_{x,\pi}(\tau_I \geq 10 t_I^*) \leq \frac{1}{10}. \tag{2.2}$$

Let $t = 10 t_I^*$ and for $0 \leq k \leq t$ we let

$$M_k = \mathbb{P}_\pi(Y_t \in A^c \mid Y_0, \ldots, Y_k) = \mathbb{P}_\pi(Y_t \in A^c \mid Y_k),$$

where the second equality follows by the Markov property. It follows from the definition of M that it is a martingale, and hence applying Doob's maximal inequality, we immediately obtain

$$\mathbb{P}_\pi\left(\max_{0 \leq k \leq t} M_k \geq \frac{3}{4}\right) \leq \frac{4}{3} \cdot \mathbb{E}_\pi[M_t] = \frac{4}{3} \cdot \mathbb{P}_\pi(Y_t \in A^c) \leq \frac{5}{6}, \tag{2.3}$$

since $\pi(A) \geq 3/8$. We now let

$$G = \left\{ \max_{0 \leq k \leq t} M_k \leq \frac{3}{4} \quad \text{and} \quad \tau_I \leq t \right\}.$$

By the union bound and using (2.3) and (2.2) we obtain

$$\mathbb{P}_{x,\pi}(G^c) \leq \frac{5}{6} + \frac{1}{10} = \frac{14}{15}.$$

Letting $\sigma = \min\{k : X_k \in Y[0, t]\} \wedge t$ and $B = \{z : \mathbb{P}_z(\tau_A \leq t) \geq 1/4\}$, we now get

$$\mathbb{P}_x(\tau_A \leq 2t) \geq \mathbb{P}_{x,\pi}(\tau_A \leq 2t, G) = \sum_{z \in B} \mathbb{P}_{x,\pi}(\tau_A \leq 2t, G, X_\sigma = z).$$

The last equality is justified, since on G if $X_\sigma = z \notin B$, then $\exists k$ such that $Y_k = z \notin B$, and hence on this event we have

$$\mathbb{P}_\pi(Y_t \in A \mid Y_k) < \frac{1}{4} \Rightarrow \max_{0 \leq k \leq t} M_k > \frac{3}{4} \Rightarrow G^c.$$

Therefore we deduce that

$$\mathbb{P}_x(\tau_A \leq 2t) \geq \sum_{z \in B} \mathbb{P}_{x,\pi}(\tau_A \leq 2t \mid G, X_\sigma = z) \, \mathbb{P}_{x,\pi}(X_\sigma = z \mid G) \, \mathbb{P}_{x,\pi}(G)$$

$$\geq \sum_{z \in B} \mathbb{P}_z(\tau_A \leq t) \, \mathbb{P}_{x,\pi}(X_\sigma = z \mid G) \, \mathbb{P}_{x,\pi}(G) \geq \frac{1}{4} \cdot \frac{1}{15} = \frac{1}{60},$$

where the second inequality follows by the Markov property, since the events G and $\{X_\sigma = z\}$ only depend on the paths of the chains up to time t. This concludes the proof of (2.1) and by performing independent geometric experiments, we finally get that

$$\max_x \mathbb{E}_x[\tau_A] \lesssim t_I^*.$$

Since A was an arbitrary set with $\pi(A) \geq 3/8$, we get

$$t_{\mathrm{H}} \lesssim t_I^*$$

and this finishes the proof. $\qquad\square$

Proposition 2.5. *For all Markov chains we have*

$$t_{\mathrm{I}} \asymp t_{\mathrm{I}}^*.$$

Proof. Obviously we have $t_{\mathrm{I}}^* \leq t_{\mathrm{I}}$, so we only need to prove that $t_{\mathrm{I}} \lesssim t_{\mathrm{I}}^*$. To do so, we consider three independent chains, X, Y and Z such that $X_0 = x$, $Y_0 = y$ and $Z_0 \sim \pi$. We will denote by $\tau_I^{X,Y}$ the first time that X and Y intersect and similarly for $\tau_I^{X,Z}$.

Let $t = 6t_I^*$. It suffices to show that for all x, y we have

$$\mathbb{P}_{x,y}\left(\tau_I^{X,Y} \leq 4t_I^*\right) \geq c > 0, \tag{2.4}$$

since then by performing geometric experiments, we would get that $t_{\mathrm{I}} \lesssim t_{\mathrm{I}}^*$. For all $0 \leq k \leq t$ we define

$$M_k = \mathbb{P}_{y,\pi}(Y[0,4t] \cap Z[2t,3t] = \varnothing \mid Z_0, \ldots, Z_k) = \mathbb{P}_{y,\pi}(Y[0,4t] \cap Z[2t,3t] = \varnothing \mid Z_k),$$

where the last equality follows from the Markov property. Then clearly M is a martingale. By Doob's maximal inequality we get

$$
\begin{aligned}
\mathbb{P}_{y,\pi}\left(\max_{0 \leq k \leq t} M_k \geq \frac{3}{4}\right) &\leq \frac{4}{3} \cdot \mathbb{P}_{y,\pi}(Y[0,4t] \cap Z[2t,3t] = \varnothing) \\
&\leq \frac{4}{3} \cdot \mathbb{P}_{y,\pi}(Y[2t,3t] \cap Z[2t,3t] = \varnothing) \\
&\leq \frac{4}{3} \cdot \max_x \mathbb{P}_{x,\pi}(\tau_I \geq t) \leq \frac{4}{3} \cdot \frac{\max_x \mathbb{E}_{x,\pi}[\tau_I]}{t} = \frac{4t_I^*}{3t} = \frac{2}{9},
\end{aligned}
$$

where in the final inequality we used Markov's inequality. Next we define

$$G = \left\{\max_{0 \leq k \leq t} M_k \leq \frac{3}{4} \quad \text{and} \quad \tau_I^{X,Z} \leq t\right\}.$$

By the union bound and Markov's inequality we obtain

$$\mathbb{P}_{x,y,\pi}(G^c) \leq \frac{2}{9} + \frac{1}{6} = \frac{7}{18}. \tag{2.5}$$

Writing $\sigma = \inf\{k : X_k \in Z[0,t]\} \wedge t$ and $B = \{w : \mathbb{P}_{y,w}(Y[0,4t] \cap Z[t,3t] \neq \varnothing) \geq 1/4\}$, then we have

$$\mathbb{P}_{x,y}\left(\tau_I^{X,Y} \leq 5t\right) \geq \mathbb{P}_{x,y,\pi}\left(\tau_I^{X,Y} \leq 5t, G\right) = \sum_{w \in B} \mathbb{P}_{x,y,\pi}\left(\tau_I^{X,Y} \leq 5t, G, X_\sigma = w\right).$$

For the last equality we note that on G if $X_\sigma = w \notin B$, then $\exists \ell \leq t$ such that $Z_\ell = w \notin B$, and hence on this event we have

$$
\begin{aligned}
\mathbb{P}_{y,\pi}(Y[0,4t] \cap Z[2t,3t] \neq \varnothing \mid Z_\ell) &= \mathbb{P}_{y,\pi}(Y[0,4t] \cap Z[2t,3t] \neq \varnothing \mid Z_\ell = w) \\
&= \mathbb{P}_{y,w}(Y[0,4t] \cap Z[2t - \ell, 3t - \ell] \neq \varnothing) \\
&\leq \mathbb{P}_{y,w}(Y[0,4t] \cap Z[t,3t] \neq \varnothing) < \frac{1}{4} \implies G^c.
\end{aligned}
$$

We now deduce

$$
\begin{aligned}
\mathbb{P}_{x,y}\left(\tau_I^{X,Y} \le 5t\right) &\ge \sum_{w \in B} \mathbb{P}_{x,y,\pi}\left(\tau_I^{X,Y} \le 5t \mid G, X_\sigma = w\right) \mathbb{P}_{x,y,\pi}(X_\sigma = w \mid G)\, \mathbb{P}_{x,y,\pi}(G) \\
&\ge \sum_{w \in B} \mathbb{P}_{w,y}\left(\tau_I^{X,Y} \le 4t\right) \mathbb{P}_{x,y,\pi}(X_\sigma = w \mid G)\, \mathbb{P}_{x,y,\pi}(G) \\
&= \sum_{w \in B} \mathbb{P}_{w,y}(Y[0,4t] \cap X[0,4t] \ne \varnothing)\, \mathbb{P}_{x,y,\pi}(X_\sigma = w \mid G)\, \mathbb{P}_{x,y,\pi}(G) \\
&\ge \sum_{w \in B} \mathbb{P}_{y,w}(Y[0,4t] \cap Z[t,3t] \ne \varnothing)\, \mathbb{P}_{x,y,\pi}(X_\sigma = w \mid G)\, \mathbb{P}_{x,y,\pi}(G) \\
&\ge \frac{1}{4} \cdot \frac{11}{18},
\end{aligned}
$$

The first inequality follows from the Markov property, since the events G and $\{X_\sigma = w\}$ only depend on the paths of the chains X and Z up to time t. The last inequality follows from (2.5) and the definition of the set B and this concludes the proof of (2.4). □

Proof of Theorem 1.1. Proposition 2.4 immediately gives that for all Markov chains we have

$$
t_{\mathrm{H}} \lesssim t_{\mathrm{I}}
$$

and this finishes the proof. □

Proof of Corollary 1.2. Combining Theorem 2.1 with Theorem 1.1 yields

$$
t_{\mathrm{mix}} \asymp t_{\mathrm{H}} \lesssim t_{\mathrm{I}}.
$$

It remains to prove that for trees the two quantities, t_{mix} and t_{I}, are equivalent. Since $t_{\mathrm{mix}} \lesssim t_{\mathrm{I}}$ for all reversible Markov chains, we only need to show that $t_{\mathrm{I}} \lesssim t_{\mathrm{mix}}$. Let v be a central node. Then if we wait until both chains X and Y hit v, this will give an upper bound on their intersection time, and hence

$$
\mathbb{E}_{x,y}[\tau_I] \le \mathbb{E}_x\left[\tau_v^X\right] + \mathbb{E}_y\left[\tau_v^Y\right] \le 2 \max_x \mathbb{E}_x[\tau_v]
$$

Now Theorem 2.3 finishes the proof. □

3 Intersection time for transitive chains

In this section we prove Theorem 1.5. We start by showing that for transitive chains instead of considering one or two worst starting points, both chains can start from stationarity. In particular, we have the following.

Lemma 3.1. *Let X be a transitive Markov chain. Then*

$$
t_I \asymp \mathbb{E}_{\pi,\pi}[\tau_I].
$$

Proof. From Proposition 2.5 we have that for all chains

$$
t_I \asymp \max_x \mathbb{E}_{x,\pi}[\tau_I].
$$

By transitivity it follows that $\mathbb{E}_{x,\pi}[\tau_I]$ is independent of x. Therefore, averaging over all x in the state space proves the lemma. □

Definition 3.2. Let P be a general transition matrix. We define for all $t > 0$

$$
g_t(x,z) = \sum_{j=0}^{t} p_j(x,z), \quad Q_t(x) = \sum_z g_t^2(x,z) \quad \text{and} \quad Q_t = \max_x Q_t(x).
$$

Remark 3.3. Note that if a chain is transitive, then $Q_t(x) = Q_t$ for all x.

The next lemma gives a control on the first and second moment of the number of intersections of two independent chains. It will be used in the proof of Theorem 1.5 and Proposition 1.8. In this form it appeared in [16], but the idea goes back to Le-Gall and Rosen [13, Lemma 3.1]. We include the proof here for the reader's convenience.

Lemma 3.4. *Let X and Y be two independent copies of a (not necessarily transitive) Markov chain and $I_t = \sum_{i=0}^{t} \sum_{j=0}^{t} \mathbf{1}(X_i = Y_j)$ count the number of intersections up to time t. Then for all x, y we have*

$$\mathbb{E}_{x,x}[I_t] = Q_t(x) \quad \text{and} \quad \mathbb{E}_{x,y}[I_t^2] \leq 4Q_t^2.$$

Proof. For the first moment of the number of intersections we have

$$\mathbb{E}_{x,x}[I_t] = \sum_{i=0}^{t} \sum_{j=0}^{t} \mathbb{P}_{x,x}(X_i = Y_j) = \sum_z \sum_{i=0}^{t} \sum_{j=0}^{t} \mathbb{P}_x(X_i = z)\,\mathbb{P}_x(Y_j = z) = \sum_z g_t^2(x, z) = Q_t(x).$$

For the second moment of I_t we have

$$\mathbb{E}_{x,y}[I_t^2]$$

$$= \sum_{i,j,\ell,m=0}^{t} \mathbb{P}_{x,y}(X_i = Y_j, X_\ell = Y_m) = \sum_{z,w} \sum_{i,j,\ell,m=0}^{t} \mathbb{P}_x(X_i = z, X_\ell = w)\,\mathbb{P}_y(Y_j = z, Y_m = w)$$

$$\leq \sum_{z,w} (g_t(x, z)g_t(z, w) + g_t(x, w)g_t(w, z))(g_t(y, z)g_t(z, w) + g_t(y, w)g_t(w, z))$$

$$\leq \sum_{z,w} \left((g_t^2(x, z)g_t^2(z, w) + g_t^2(x, w)g_t^2(w, z)) + (g_t^2(y, z)g_t^2(z, w) + g_t^2(y, w)g_t^2(w, z)) \right)$$

$$\leq 4Q_t^2.$$

For the second inequality we used that $ab \leq (a^2 + b^2)/2$ and $(a + b)^2 \leq 2(a^2 + b^2)$. $\qquad \square$

Lemma 3.5. *Let X be a transitive Markov chain starting from x, and let $S_t(x) = \sum_{j=0}^{t} g_t(x, X_j)$. Then*

$$\mathbb{P}_x\left(S_t(x) \geq \frac{Q_t}{2} \right) \geq \frac{1}{16}.$$

Proof. By Remark 3.3 we have that $Q_t(x) = Q_t$ for all x, since the chain is transitive. Let X and Y be two independent copies of the chain starting from x. We write

$$I_t = \sum_{j=0}^{t} \sum_{\ell=0}^{t} \mathbf{1}(X_j = Y_\ell)$$

for the total number of intersections up to time t. We now observe that

$$S_t(x) = \mathbb{E}_x[I_t \mid X_0, \ldots, X_t],$$

and hence we get

$$\mathbb{E}_x[S_t(x)] = \mathbb{E}_{x,x}[I_t] = Q_t \quad \text{and} \quad \mathbb{E}_x[S_t^2(x)] \leq \mathbb{E}_{x,x}[I_t^2].$$

From Lemma 3.4 we now obtain

$$\mathbb{E}_x[S_t^2(x)] \leq \mathbb{E}_{x,x}[I_t^2] \leq 4(\mathbb{E}_{x,x}[I_t])^2 = 4Q_t^2.$$

Applying the second moment method (Paley-Zygmund inequality) finally gives

$$\mathbb{P}_x\left(S_t(x) \geq \frac{Q_t}{2} \right) \geq \frac{1}{4} \cdot \frac{(\mathbb{E}_x[S_t(x)])^2}{\mathbb{E}_x[S_t^2(x)]} \geq \frac{1}{16}$$

and this concludes the proof. $\qquad \square$

The following proposition is the main ingredient of the proof of Theorem 1.5. We now explain the key idea behind the proof which was used in [7, Theorem 5.1]. We define a set of good points on the path of the chain X and show that conditional on X and Y intersecting before time t, then they intersect at a good point with constant probability.

Proposition 3.6. *Let X and Y be two independent copies of a transitive Markov chain on n states started from stationarity. Let I_t denote the number of intersections of X and Y up to time t. Then*

$$\frac{(t+1)^2}{4nQ_t} \leq \mathbb{P}_{\pi,\pi}(I_t > 0) \leq \frac{2^7(t+1)^2}{nQ_t}.$$

Proof. For all t using the independence between X and Y we get

$$\mathbb{E}_{\pi,\pi}[I_t] = \sum_z \sum_{i,j=0}^t \mathbb{P}_{\pi,\pi}(X_i = z, Y_j = z) = \frac{(t+1)^2}{n}. \tag{3.1}$$

For the second moment we have

$$\mathbb{E}_{\pi,\pi}\left[I_t^2\right] = \sum_{i,j,\ell,m=0}^t \sum_{z,w} \mathbb{P}_\pi(X_i = z, X_j = w)\, \mathbb{P}_\pi(Y_\ell = z, Y_m = w)$$

$$\leq \frac{(t+1)^2}{n^2} \sum_{z,w} \left(g_t(z,w) + g_t(w,z)\right)^2 \leq \frac{4(t+1)^2}{n} Q_t, \tag{3.2}$$

where for the last equality we used transitivity. Using the second moment method we obtain

$$\mathbb{P}_{\pi,\pi}(I_t > 0) \geq \frac{(t+1)^2}{4nQ_t}.$$

We now turn to prove the upper bound. For every $x = (x_0, \ldots, x_{2t})$ we define the set

$$\Gamma_t(x) = \left\{ r \leq t : \sum_{j=0}^t g_t(x_r, x_{r+j}) \geq \frac{Q_t}{2} \right\}.$$

Next we define

$$\tau = \min\{j \in [0,t] : X_j \in \{Y_0, \ldots, Y_t\}\},$$

and $\tau = \infty$ if the above set is empty. Conditioned on $(Y_s)_{s \leq t}$, we see that τ is a stopping time for X. Thus using Lemma 3.5 and the strong Markov property we get that τ satisfies

$$\mathbb{P}_{\pi,\pi}(\tau \in \Gamma_t(X) \mid \tau < \infty) \geq \frac{1}{16},$$

where to simplify notation we write $\Gamma_t(X)$ for the random set $\Gamma_t((X_s)_{s \leq 2t})$. Therefore

$$\mathbb{P}_{\pi,\pi}(I_t > 0) = \mathbb{P}_{\pi,\pi}(\tau < \infty) \leq 16 \cdot \mathbb{P}_{\pi,\pi}(\tau \in \Gamma_t(X)). \tag{3.3}$$

It now remains to bound $\mathbb{P}_{\pi,\pi}(\tau \in \Gamma_t(X))$. We define $\sigma = \min\{\ell \in [0,t] : Y_\ell \in \cup_{r \in \Gamma_t(X)} X_r\}$ with $\sigma = \infty$ if the above set is empty. We note that

$$\mathbb{P}_{\pi,\pi}(\tau \in \Gamma_t(X)) \leq \mathbb{P}_{\pi,\pi}(\sigma \in [0,t]). \tag{3.4}$$

Writing $A_k = \{Y_\sigma = X_k, k \in \Gamma_t, k \text{ is minimal}, \sigma \in [0,t]\}$ for all $k \leq t$ we now have

$$\mathbb{E}_{\pi,\pi}[I_{2t} \mid \sigma \in [0,t]] = \sum_{k=0}^t \mathbb{E}_{\pi,\pi}[I_{2t} \mid A_k]\, \mathbb{P}_{\pi,\pi}(A_k \mid \sigma \in [0,t]). \tag{3.5}$$

For every $k \leq t$ we obtain

$$\mathbb{E}_{\pi,\pi}[I_{2t} \mid A_k]$$

$$\geq \sum_{\substack{x=(x_0,\ldots,x_{2t}) \\ \text{s.t. } k \in \Gamma_t(x)}} \mathbb{E}_{\pi,\pi}\left[\sum_{i,j=0}^{t} \mathbf{1}(Y_{\sigma+i} = X_{k+j}) \,\middle|\, (X_s)_{s \leq 2t} = x, A_k\right] \mathbb{P}_{\pi,\pi}((X_s)_{s \leq 2t} = x \mid A_k)$$

$$= \sum_{\substack{x=(x_0,\ldots,x_{2t}) \\ \text{s.t. } k \in \Gamma_t(x)}} \sum_{j=0}^{t} g_t(x_k, x_{k+j}) \mathbb{P}_{\pi,\pi}((X_s)_{s \leq 2t} = x \mid A_k) \geq \frac{Q_t}{2}.$$

Substituting the above lower bound into (3.5) we deduce

$$\mathbb{E}_{\pi,\pi}[I_{2t} \mid \sigma \in [0,t]] \geq \frac{Q_t}{2}.$$

Using (3.1) and the above bound we finally get

$$\mathbb{P}_{\pi,\pi}(\sigma \in [0,t]) \leq \frac{\mathbb{E}_{\pi,\pi}[I_{2t}]}{\mathbb{E}_{\pi,\pi}[I_{2t} \mid \sigma \in [0,t]]} \leq \frac{(2t+1)^2/n}{Q_t/2} \leq \frac{2^3(t+1)^2}{nQ_t}.$$

This in conjunction with (3.3) and (3.4) gives

$$\mathbb{P}_{\pi,\pi}(I_t > 0) \leq \frac{2^7(t+1)^2}{nQ_t},$$

and this concludes the proof of the upper bound. $\qquad \square$

The following lemma follows by the spectral theorem and will be used for the upper bound in the proof of Theorem 1.5. Combined with the statement of Theorem 1.5 it gives that for transitive and reversible chains $t_{\mathrm{unif}} \lesssim t_{\mathrm{I}}$, which is an improvement over Corollary 1.2 which gives $t_{\mathrm{mix}} \lesssim t_{\mathrm{I}}$. Note that this is not true in general, if the chain is not transitive. Take for instance two cliques of sizes $\sqrt{n}$ and n connected by a single edge.

Lemma 3.7. *Let X be a reversible, transitive and lazy Markov chain on n states and let $(\lambda_j)_j$ be the corresponding non-unit eigenvalues. Then*

$$t_{\mathrm{unif}} \leq 2\sqrt{Q},$$

where $Q = \sum_{k=2}^{n}(1-\lambda_k)^{-2}$.

Proof. We start by noting that for a transitive, reversible and lazy chain the uniform mixing time is given by

$$t_{\mathrm{unif}} = \min\left\{t \geq 0 : p_t(x,x) \leq \frac{5}{4n}\right\}.$$

See for instance [17, equation (16)] or [15, Proposition A.1]. By the spectral theorem and using transitivity of X we have

$$p_t(x,x) = \frac{1}{n} \cdot \sum_{k=1}^{t} \lambda_k^t = \frac{1}{n} + \frac{1}{n} \cdot \sum_{k=2}^{n} \lambda_k^t.$$

Therefore $t_{\mathrm{unif}} = \min\{t : \sum_{k=2}^{n} \lambda_k^t \leq 1/4\}$. We now set $\varepsilon_j = 1 - \lambda_j$ for all j. Since the chain is lazy, it follows that $\varepsilon_j \in [0,1]$ for all j. So we now need to show

$$\sum_{k=2}^{n}(1-\varepsilon_k)^{2\sqrt{\sum_{j=2}^{n} \varepsilon_j^{-2}}} \leq \frac{1}{4}. \tag{3.6}$$

In order to prove (3.6) it suffices to show

$$\sum_{k=2}^{n} \exp\left(-2\varepsilon_k \cdot \sqrt{\sum_{j=2}^{n} \varepsilon_j^{-2}}\right) \leq \frac{1}{4}.$$

Writing $r_k = \varepsilon_k \cdot \sqrt{\sum_{j=2}^{n} \varepsilon_j^{-2}}$, we get $r_k \geq 1$ and $\sum_{k=2}^{n} r_k^{-2} = 1$. Since $e^r \geq r^2$ for all $r \geq 0$, we finally deduce

$$\sum_{k=2}^{n} e^{-2r_k} \leq \frac{1}{4} \cdot \sum_{k=2}^{n} r_k^{-2} = \frac{1}{4}$$

and this finishes the proof. $\qquad\square$

We are now ready to give the proof of Theorem 1.5.

Proof of Theorem 1.5. Since the chain is reversible and transitive, it follows that for any state x we have

$$Q_t = \sum_{i=0}^{t} \sum_{j=0}^{t} p_{i+j}(x,x).$$

Using the spectral theorem together with transitivity, we obtain

$$Q_t = \frac{1}{n} \cdot \sum_{k=1}^{n} \sum_{i,j=0}^{t} \lambda_k^{i+j} = \frac{(t+1)^2}{n} + \frac{1}{n} \cdot \sum_{k=2}^{n} \frac{(1 - \lambda_k^{t+1})^2}{(1 - \lambda_k)^2}. \tag{3.7}$$

For $t \geq t_{\mathrm{rel}} = (1 - \lambda_2)^{-1} \geq -\frac{1}{\log(\lambda_2)}$ we get

$$\left(1 - \lambda_2^{t+1}\right)^2 \geq 1 - 2\lambda_2^{t+1} \geq 1 - 2\lambda_2^t \geq 1 - \frac{2}{e}.$$

Since for all $j \geq 2$ we have $\lambda_j \leq \lambda_2$ using the above inequality we obtain for all $j \geq 2$ and $t \geq t_{\mathrm{rel}}$

$$\left(1 - \lambda_j^{t+1}\right)^2 \geq 1 - \frac{2}{e}.$$

Therefore for all $t \geq t_{\mathrm{rel}}$ we deduce

$$Q_t \geq \frac{(t+1)^2}{n} + \left(1 - \frac{2}{e}\right) \cdot \frac{Q}{n}. \tag{3.8}$$

Using (3.8) together with Proposition 3.6 now gives for $t \geq t_{\mathrm{rel}}$

$$\mathbb{P}_{\pi,\pi}(\tau_I \leq t) \leq \frac{2^7 (t+1)^2}{(t+1)^2 + \left(1 - \frac{2}{e}\right)Q}. \tag{3.9}$$

We now claim that $t_{\mathrm{I}} \gtrsim \sqrt{Q}$. Let C_1 be a large constant to be specified later. If $\sqrt{Q} \leq C_1 t_{\mathrm{rel}}$, then since $t_{\mathrm{mix}} \gtrsim t_{\mathrm{rel}}$ (cf. [14, Theorem 12.4]) the claim follows from Corollary 1.2. So we may assume that $\sqrt{Q} \geq C_1 t_{\mathrm{rel}}$. Setting $t + 1 = C\sqrt{Q} \geq t_{\mathrm{rel}}$ for a constant $C \geq 1/C_1$ to be determined we get

$$\mathbb{P}_{\pi,\pi}(\tau_I \leq t) \leq 2^7 \cdot \frac{C^2 Q}{C^2 Q + \left(1 - \frac{2}{e}\right)Q}.$$

If we take C so that $C^2 = (1 - 2/e)/2^8$ and we choose $C_1 = (1 - 2/e)^{-1/2} \cdot 2^4$, then from the above we obtain

$$\mathbb{P}_{\pi,\pi}(\tau_I \leq t) \leq \frac{1}{2}$$

and this proves the claim that $t_I \gtrsim \sqrt{Q}$. It remains to show that $t_I \lesssim \sqrt{Q}$. It suffices to show that there are positive constants c_1 and c_2 such that for all x, y we have

$$\mathbb{P}_{x,y}\left(\tau_I \leq c_1\sqrt{Q}\right) \geq c_2. \tag{3.10}$$

Indeed, by then performing independent experiments, we would get that $t_I \lesssim \sqrt{Q}$. From (3.7) we immediately get

$$Q_t \leq \frac{(t+1)^2}{n} + \frac{Q}{n}. \tag{3.11}$$

This together with Proposition 3.6 gives that for all t we have

$$\mathbb{P}_{\pi,\pi}(\tau_I \leq t) \geq \frac{1}{4} \cdot \frac{(t+1)^2}{(t+1)^2 + Q}. \tag{3.12}$$

Taking $t = \sqrt{Q}$ in (3.12) gives

$$\mathbb{P}_{\pi,\pi}\left(\tau_I \leq \sqrt{Q}\right) \geq \frac{1}{8}. \tag{3.13}$$

From Lemma 3.7 we have $t_{\text{unif}} \leq 2\sqrt{Q}$. Setting $s = 2\sqrt{Q}$ we now have for all x, y

$$\begin{aligned}
\mathbb{P}_{x,y}\left(\tau_I \leq s + \sqrt{Q}\right) &\geq \mathbb{P}_{x,y}\left(X[s, s+\sqrt{Q}] \cap Y[s, s+\sqrt{Q}] \neq \varnothing\right) \\
&= \sum_{x',y'} p_s(x, x') p_s(y, y') \mathbb{P}_{x',y'}\left(\tau_I \leq \sqrt{Q}\right) \\
&\geq \frac{9}{16} \cdot \sum_{x',y'} \pi(x')\pi(y')\mathbb{P}_{x',y'}\left(\tau_I \leq \sqrt{Q}\right) \\
&\geq \frac{9}{16}\mathbb{P}_{\pi,\pi}\left(\tau_I \leq \sqrt{Q}\right) \geq \frac{9}{128},
\end{aligned}$$

where for the last inequality we used (3.13). This proves (3.10). Finally, from (3.8), (3.11) and since $t_{\text{unif}} \leq 2\sqrt{Q}$ by Lemma 3.7 we obtain

$$Q_{t_{\text{unif}}} = \sum_{i,j=0}^{t_{\text{unif}}} p_{i+j}(x, x) \asymp \frac{Q}{n}.$$

and this concludes the proof of the theorem. $\qquad\square$

Corollary 3.8. *Let X be a transitive, reversible, lazy Markov chain. Then $t_I \gtrsim \sqrt{t_{\text{hit}}}$.*

Proof. From [1, Proposition 3.13] for all reversible chains we have

$$\mathbb{E}_\pi[\tau_\pi] = \sum_{i=2}^{n} \frac{1}{1 - \lambda_i},$$

where τ_π is the first time X hits a state chosen according to the stationary distribution π. Using that X is transitive we get

$$\mathbb{E}_\pi[\tau_\pi] \leq t_{\text{hit}} \leq 2\mathbb{E}_\pi[\tau_\pi],$$

where the second inequality holds since for all states x, y we have $\mathbb{E}_x[\tau_y] \leq \mathbb{E}_x[\tau_\pi] + \mathbb{E}_\pi[\tau_y]$. From Theorem 1.5 the intersection time is given by

$$t_I^2 \asymp \sum_{i=2}^n \frac{1}{(1-\lambda_i)^2} \geq \sum_{i=2}^n \frac{1}{1-\lambda_i}$$

and this concludes the proof. $\qquad\square$

Remark 3.9. To see that the inequality of Corollary 3.8 does not hold for all regular graphs start with a 3-regular expander with k^4 vertices, remove 2 edges, say e_1 and e_2 and replace them by a ladder of length k as in Figure 1. In this graph $t_I \asymp k^2$ and $t_{\text{hit}} \asymp k^5$.
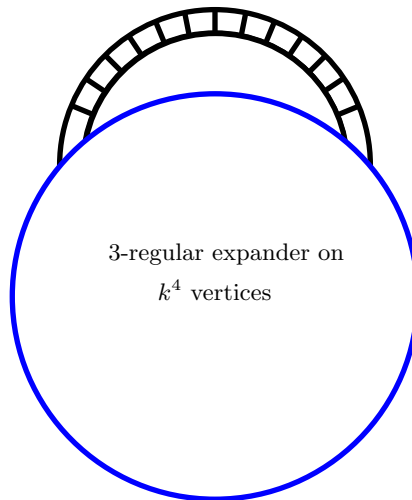

Figure 1: 3-regular expander on k^4 vertices and ladder on k vertices

Remark 3.10. For transitive, reversible, lazy chains we know

$$t_I \asymp \sqrt{\sum_{i=2}^n \frac{1}{(1-\lambda_i)^2}}.$$

In what generality does this equivalence hold?

4 Intersection time for regular graphs

In this section we prove Proposition 1.8 which gives a bound on the intersection time for random walks on regular graphs. In this section all random walks are assumed to be simple and lazy. We start by stating some standard results about return probabilities for random walks on regular graphs.

Lemma 4.1. *Let G be a regular graph on n vertices and $t \leq n^2$. Then for all vertices x the return probability to x satisfies*

$$p_t(x, x) \lesssim \frac{1}{\sqrt{t}}.$$

The proof of the above lemma follows for instance from [1, Proposition 6.16, Chapter 6].

Lemma 4.2. *Let G be a regular graph on n vertices and λ_2 the second eigenvalue of the lazy simple random walk on G. Then for all vertices x and all t the return probability in t steps satisfies*

$$p_t(x,x) \lesssim \frac{1}{n} + \lambda_2^t.$$

The statement of Lemma 4.2 follows from [14, inequality 12.11]. Using [14, inequality (19.8)] gives the following claim.

Claim 4.3. *For all $t \geq 2t_{\mathrm{mix}}$ we have*

$$p_t(x,y) \geq \frac{\pi(y)}{4}.$$

Recall the definition of $t_* = \min\{t_{\mathrm{rel}}(\log t_{\mathrm{rel}} + 1), t_{\mathrm{unif}}\}$.

Claim 4.4. *For a lazy random walk on a regular graphs with n vertices we have $t_{\mathrm{mix}} \lesssim \sqrt{n}(t_*)^{3/4}$.*

Proof. Since for all regular graphs $t_{\mathrm{unif}} \lesssim n^2$ we get

$$t_{\mathrm{mix}} \leq t_{\mathrm{unif}} \lesssim \sqrt{n}(t_{\mathrm{unif}})^{3/4}.$$

To prove that $t_{\mathrm{mix}} \lesssim \sqrt{n}(t_{\mathrm{rel}}(\log t_{\mathrm{rel}} + 1))^{3/4}$ we will consider cases depending on whether $t_{\mathrm{rel}} \geq n$ or $t_{\mathrm{rel}} < n$. If $t_{\mathrm{rel}} \geq n$, then using that $t_{\mathrm{mix}} \lesssim t_{\mathrm{rel}} \log n$ (cf. [14, Theorem 12.3]) we get

$$\sqrt{n}(t_{\mathrm{rel}} \log t_{\mathrm{rel}})^{3/4} \gtrsim (t_{\mathrm{mix}})^{1/4}(t_{\mathrm{rel}} \log n)^{3/4} \gtrsim t_{\mathrm{mix}}.$$

If $t_{\mathrm{rel}} < n$, then using again $t_{\mathrm{mix}} \lesssim t_{\mathrm{rel}} \log n$, it suffices to prove

$$(t_{\mathrm{rel}})^{1/4} \lesssim \frac{\sqrt{n}}{\log n}(\log t_{\mathrm{rel}} + 1)^{3/4}.$$

But this clearly holds by the assumption that $t_{\mathrm{rel}} < n$. $\square$

Recall Definition 3.2 of $Q_t(x)$ and Q_t from Section 3. The following lemma gives an upper bound on Q_t for all regular graphs.

Lemma 4.5. *For all regular graphs on n vertices, for all times t we have*

$$Q_t \lesssim (t_*)^{3/2} + \frac{t^2}{n}.$$

Proof. For any x we have

$$Q_t(x) = \sum_z g_t^2(x,z) = \sum_{i,j=0}^t \sum_z p_i(x,z)p_j(x,z) = \sum_{i,j=0}^t \sum_z p_i(x,z)p_j(z,x) = \sum_{i,j=0}^t p_{i+j}(x,x),$$

where the third equality follows from the fact that G is regular. We thus obtain that

$$Q_t(x) \lesssim \sum_{k=0}^{2t} k p_k(x,x).$$

We now divide time into four intervals, $A_1 = [0, n^2]$, $A_2 = (n^2, t_*]$, $A_3 = (t_*, t_{\mathrm{unif}}]$ and $A_4 = (t_{\mathrm{unif}}, t]$. Note that A_2, A_3 and A_4 could also be empty. For $k \in A_1$ we use Lemma 4.1

and for $k \in A_3$ we use Lemma 4.2. Since $t_* \lesssim n^2$ and $p_t(x, x)$ is decreasing as a function of t, for $k \in A_2$ we use that $p_k(x, x) \lesssim 1/n$. We thus get

$$Q_t(x) \lesssim (t_*)^{3/2} + \sum_{k=t_*}^{t_{\text{unif}}} k\lambda_2^k + \frac{t^2}{n}, \qquad (4.1)$$

where we used that for $k \geq t_{\text{unif}}$ the return probability to x satisfies $p_k(x, x) \lesssim 1/n$. In the case when $t_* = t_{\text{rel}}(\log t_{\text{rel}} + 1)$, we get

$$\sum_{k=t_*}^{t_{\text{unif}}} k\lambda_2^k \lesssim t_{\text{rel}}^2(\log t_{\text{rel}})\lambda_2^{t_*} \leq t_{\text{rel}}^2(\log t_{\text{rel}}) \exp\left(-\frac{1}{t_{\text{rel}}} \cdot t_{\text{rel}}(\log t_{\text{rel}})\right) \lesssim t_{\text{rel}}(\log t_{\text{rel}}) \leq t_*.$$

For the case $t_* = t_{\text{unif}}$, the second-term in the right-hand side of (4.1) is zero since A_3 is empty. Therefore, we deduce

$$Q_t(x) \lesssim (t_*)^{3/2} + \frac{t^2}{n}.$$

Since this holds for all x the statement of the lemma follows. $\qquad \square$

Proof of Proposition 1.8. Let X and Y be two independent lazy simple random walks on G. Recall I_t stands for the number of intersections up to time t. We define I_t' to be the number of intersections between $2t_{\text{mix}}$ and $2t_{\text{mix}} + t$, i.e.

$$I_t' = \sum_{i,j=2t_{\text{mix}}}^{2t_{\text{mix}}+t} \mathbf{1}(X_i = Y_j).$$

Using Claim 4.3 we obtain that

$$\mathbb{E}_{x,y}[I_t'] \geq \frac{1}{16}\mathbb{E}_{\pi,\pi}[I_t] = \frac{(t+1)^2}{16n}.$$

From Claim 4.3 again we get that the second moment of I_t' satisfies

$$\mathbb{E}_{x,y}\big[(I_t')^2\big] \asymp \mathbb{E}_{\pi,\pi}\big[I_t^2\big] \lesssim Q_t^2,$$

where the last inequality follows from Lemma 3.4.

Therefore, we obtain

$$\mathbb{P}_{x,y}(I_t' > 0) \geq \frac{(\mathbb{E}_{x,y}[I_t'])^2}{\mathbb{E}_{x,y}[(I_t')^2]} \gtrsim \frac{\left(\frac{t^2}{n}\right)^2}{Q_t^2}.$$

Choosing t such that $t^2/n = (t_*)^{3/2}$ or equivalently $t = \sqrt{n}(t_*)^{3/4}$ gives us $Q_t \lesssim t^2/n$ by Lemma 4.5, and hence the ratio above becomes of order 1. Since this bound holds uniformly for all x and y we can perform independent experiments to finally conclude that for regular graphs $t_I \lesssim \sqrt{n}(t_*)^{3/4} + t_{\text{mix}}$. This together with Claim 4.4 concludes the proof.

In the case when G is a regular expander graph, $t_{\text{rel}} = O(1)$, and hence $t_I \asymp \sqrt{n}$, using Remark 1.9 for the lower bound. $\qquad \square$

References

[1] David Aldous and James Fill. *Reversible Markov Chains and Random Walks on Graphs.* In preparation, http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[2] Tugkan Batu, Lance Fortnow, Ronitt Rubinfeld, Warren D. Smith, and Patrick White. Testing closeness of discrete distributions. *J. ACM*, 60(1):4, 2013.

[3] Itai Benjamini and Ben Morris. The birthday problem and Markov chain Monte Carlo, 2007. arXiv:math/0701390.

[4] Artur Czumaj and Christian Sohler. Testing expansion in bounded-degree graphs. *Combinatorics, Probability & Computing*, 19(5-6):693–709, 2010. MR-2726075

[5] Quentin de Mourgues and Thomas Sauerwald. Intersection time, 2013. preprint.

[6] P. J. Fitzsimmons and Thomas S. Salisbury. Capacity and energy for multiparameter Markov processes. *Ann. Inst. H. Poincaré Probab. Statist.*, 25(3):325–350, 1989. MR-1023955

[7] Olle Häggström, Yuval Peres, and Jeffrey E. Steif. Dynamical percolation. *Ann. Inst. H. Poincaré Probab. Statist.*, 33(4):497–528, 1997.

[8] Satyen Kale and C. Seshadhri. Combinatorial approximation algorithms for maxcut using random walks. In *Innovations in Computer Science - ICS 2010, Tsinghua University, Beijing, China, January 7-9, 2011. Proceedings*, pages 367–388, 2011.

[9] Satyen Kale and C. Seshadhri. An expansion tester for bounded degree graphs. *SIAM J. Comput.*, 40(3):709–720, 2011. MR-2810912

[10] Liran Katzir, Edo Liberty, Oren Somekh, and Ioana A. Cosma. Estimating sizes of social networks via biased sampling. *Internet Mathematics*, 10(3-4):335–359, 2014. MR-3259270

[11] Jeong Han Kim, Ravi Montenegro, Yuval Peres, and Prasad Tetali. A birthday paradox for Markov chains with an optimal bound for collision in the Pollard rho algorithm for discrete logarithm. *Ann. Appl. Probab.*, 20(2):495–521, 2010.

[12] Gregory F. Lawler. *Intersections of random walks*. Probability and its Applications. Birkhäuser Boston, Inc., Boston, MA, 1991.

[13] Jean-François Le Gall and Jay Rosen. The range of stable random walks. *Ann. Probab.*, 19(2):650–705, 1991.

[14] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov chains and mixing times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson.

[15] Russell Lyons and Shayan Oveis Gharan. Sharp Bounds on Random Walk Eigenvalues via Spectral Embedding, 2012. arXiv:1211.0589.

[16] Russell Lyons, Yuval Peres, and Oded Schramm. Markov chain intersections and the loop-erased walk. *Ann. Inst. H. Poincaré Probab. Statist.*, 39(5):779–791, 2003. MR-1997212

[17] Ben Morris and Yuval Peres. Evolving sets and mixing. In *Proceedings of the Thirty-Fifth Annual ACM Symposium on Theory of Computing*, pages 279–286 (electronic). ACM, New York, 2003.

[18] Roberto Imbuzeiro Oliveira. Mixing and hitting times for finite Markov chains. *Electron. J. Probab.*, 17:no. 70, 12, 2012. MR-2968677

[19] Yuval Peres and Perla Sousi. Mixing times are hitting times of large sets. *Journal of Theoretical Probability*, pages 1–32, 2013.

[20] Thomas S. Salisbury. Energy, and intersections of Markov chains. In *Random discrete structures (Minneapolis, MN, 1993)*, volume 76 of *IMA Vol. Math. Appl.*, pages 213–225. Springer, New York, 1996.