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The stability analysis of a system with two delays

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Abstract

This paper presents new results of stability analysis for a linear system with two delays. We attempt to determine the asymptotic stability regions of the system in a parameter space by using D-partition method. Moreover, some stability and instability conditions in terms of coefficient inequalities have been obtained for the system.

Keywords: Stability analysis; Delay differential equations; Multiple delays

1 Introduction

While modeling by using ordinary differential equations, the delay in the system is always ignored. However, even a small amount of delay may cause large changes in the system solution. Therefore, the use of delay differential equations is more realistic when any encountered problems are modeled.

For a long time, many problems in the fields of engineering [1–4], biology [5–8], chemistry [9], physics [10, 11], economy [12], psychology [13, 14], etc. have been modeled by delay differential equations.

In this paper, we consider the problem of stability of zero stationary solution of the following system:

$$x'(t) = \alpha_1 x(t) + \beta_1 y(t) + \theta_1 y(t - r_1), \tag{1}$$

$$y'(t) = \alpha_2 x(t) + \beta_2 y(t) + \theta_2 y(t - r_2), \tag{2}$$

where the coefficients α_1, α_2, β_1, β_2, θ_1, and θ_2 are real and r_1, r_2 are positive real delays. To shorten the notation, we will write stability/instability of the system instead of stability/instability of zero solution of the system. Some similar systems were investigated by many researchers, see, for example, the work by Nussbaum [15]. Another interesting study of a similar system was conducted by Ruan and Wei [16]. Hale and Huang [17] gave a very thorough characterization of the boundary of the stability region in the delay parameter space. Gu et al. [18] provided a detailed study on the stability crossing curves for more general systems. Mohammad and Mohammad [19] presented a novel method to study the stability map of linear fractional order systems with multiple delays. Josef and Zdeněk [20] utilized the method of complexification and the method of Lyapunov–Krasovskii functional to study asymptotic behavior of a differential system with a finite number of non-constant delays. Additionally, the existence of positive periodic solutions

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for a fourth-order nonlinear neutral differential equation with variable delay was studied by Ardjouni et al. [21]. Grace [22] established the results for oscillation of a third-order nonlinear delay differential equation. Öztürk and Akın [23] investigated nonoscillatory solutions of two-dimensional systems of first-order delay dynamic equations.

We attempt to determine the stability regions of the system in a parameter space by using D-partition method [24] which is explained in Sect. 2.

2 D-partition method

The method originated from paper [24]. This method consists in obtaining a "partition" of the parameter space in several regions, so that each region is bounded by a hyper surface which corresponds to the case when at least one root lies on the imaginary axis. Furthermore, for all the parameters lying in a given region, the corresponding characteristic equation has the same number of roots with positive real parts [25]. Following theorems and definitions and more details on them can be found in references [26–28] and [29].

In order to analyze the stability of the system, the characteristic equation of the system is obtained. The characteristic roots λ_j, $j = 1, 2, \dots$, of equations (1)–(2) are obtained by solving the characteristic equation

$$g(\lambda) = \lambda^2 - (\alpha_1 + \beta_2)\lambda + (\alpha_1 - \lambda)\theta_2 e^{-\lambda r_2} - \alpha_2\theta_1 e^{-\lambda r_1} + \alpha_1\beta_2 - \alpha_2\beta_1 = 0,$$

where λ_j is a complex number. If the characteristic roots have negative real parts, i.e., $\operatorname{Re}(\lambda_j) < 0$ for all $j = 1, 2, \dots$, then the solution of the system is asymptotically stable; and if at least one of the characteristic roots has positive real parts, i.e., $\operatorname{Re}(\lambda_j) > 0$ for some $j = 1, 2, \dots$, then the solution of the system is unstable.

The characteristic equation above is a special case of the general characteristic equation

$$g(\lambda, k_1, k_2) = 0, \tag{3}$$

where g depends linearly on k_1 and k_2 which could be any two of the parameters α_1, α_2, β_1, β_2, θ_1, θ_2.

The roots $\lambda = ib$ are called critical roots of the characteristic equation since stability regions of the system are determined by the analysis of the system at critical roots.

Substitute a pure imaginary number $\lambda = ib$ in (3) of a system linearly depending on two parameters k_1 and k_2. Equating the real and imaginary parts to zero, we have

$$U(b, k_1, k_2) := \operatorname{Re} g(ib, k_1, k_2) = k_1 P_1(b) + k_2 Q_1(b) + R_1(b) = 0, \tag{4}$$

$$V(b, k_1, k_2) := \operatorname{Im} g(ib, k_1, k_2) = k_1 P_2(b) + k_2 Q_2(b) + R_2(b) = 0, \tag{5}$$

where P_1, P_2, R_1, R_2, Q_1, and Q_2 are differentiable functions. Note that, for some values of b, the function $g(\lambda, k_1, k_2)$ becomes a real-valued function. Then, for these values of b, the solutions of characteristic equation $g(ib, k_1, k_2)$=0 are called singular solutions.

It is well known that the solution of system (4)–(5) exists under the assumption $P_1(b)Q_2(b) - P_2(b)Q_2(b) \neq 0$.

Definition 1 The parametric curves

$$k_1 = k_1(b), \qquad k_2 = k_2(b),$$

which are obtained by solving equations (4)–(5) and singular solutions, are called D-curves [29].

Theorem 1 *The D-curves divide the complex plane up into a finite number of regions* [29].

Theorem 2 *The characteristic equation* $g(\lambda)$ *has a root on the imaginary axis if and only if* (k_1, k_2) *is on the D-curves* [29].

Theorem 3 *In each region, determined by the D-curves in the* (k_1, k_2) *plane,* $g(\lambda)$ *has the same number of roots with positive real parts* [29].

For every region u_k of the D-partition, bounded by D-curves, it is possible to assign a number k which is the number of roots with positive real parts of the characteristic equation defined by the points of this region. Among the regions of this decomposition are also found regions u_0 (if they exist) on which the characteristic equation does not have any root with positive real part. On these regions, the solutions are asymptotically stable. The determination of these numbers for the individual domains is not an easy task. One technique is analysis of sign of partial derivative along the D-curves. Alternatively, without calculating partial derivatives, the following Stepan's formulas [30] can also be used to determine the number of roots with positive real parts.

Theorem 4 *Assume that the characteristic equation* $g(\lambda)$ *of the* n *dimensional system has no zeros on the imaginary axis. If* n *is even, i.e.,* $n = 2m$ *with* m *being an integer, then the number of unstable exponents is*

$$N = m + (-1)^m \sum_{k=1}^{r} (-1)^{k+1} \operatorname{sgn} V(\rho_k), \tag{6}$$

where $\rho_1 \geq \cdots \geq \rho_r > 0$ *are the positive real zeros of* $U(b)$. *If the* n *is odd, i.e.,* $n = 2m + 1$ *with* m *being an integer, then the number of unstable exponents is*

$$N = m + \frac{1}{2} + (-1)^m \left[\frac{1}{2} (-1)^s \operatorname{sgn} U(0) + \sum_{k=1}^{s-1} (-1)^k \operatorname{sgn} U(\sigma_k) \right], \tag{7}$$

where $\sigma_1 \geq \cdots \geq \sigma_s = 0$ *are the nonnegative real zeros of* $V(b)$ [30].

Since the delay terms have a direct effect on the solution of the characteristic equation, the delay differential equations with the same coefficients but different delay terms $r_1, r_2, \ldots, r_n$ may have different stability regions.

The following definitions are given for the delay differential equations with delay terms $r_1, r_2, \ldots, r_n$.

Definition 2 The system the stability of which depends on the delay terms is called delay-dependent stable system.

Let the delay-dependent stable region be defined as follows:

$$S_r = \{(k_1, k_2) \mid \text{the system is asymptotically stable for } r = (r_1, r_2, \ldots, r_n)\},$$

where $r_1, r_2, \ldots, r_n$ denote the values of the delays of the delay differential equation.

Definition 3 The system for which the stability is preserved for every value of delay terms is called delay independent stable.

Let the delay independent stable region be defined in the following form:

$$S_\infty = \left\{(k_1, k_2) \mid \text{the system is asymptotically stable for } \forall r_i \in IR^+\right\},$$

where r_i, $i = 1, 2, \ldots, n$, denote the values of the delays of the equation.

Methodology The method evolves as follows:

(i) Find a parametric equation of D-curves of the system.

(ii) Construct the graph of the D-curves. (In this paper, D-curves are obtained by means of MATLAB.)

(iii) Select specific points in the regions whose boundaries are the D-curves.

(iv) Determine the number of roots with positive real parts for the specific points by using Theorem 4 and generalize them to the relevant regions.

(v) Denote the region, on which the number of roots with positive real parts is k for a chosen specific point, by u_k.

3 Stability regions and main results

In this section, system (1)–(2) is studied for two different cases in two different spaces. One of these cases is $r_1 = r_2 = r$ for which the characteristic equation becomes much simpler. In other case, $r_1 \neq r_2$ state, which constitutes the main part of our analysis.

3.1 $(\alpha_1 - \alpha_2)$ parameter space

In this subsection, we determine the conditions under which the system is unstable. Moreover, the regions on which the characteristic equation has roots with the same number of positive real parts are shown for fixed delay values.

In order to find the stability region of system (1)–(2) in the parameter space, the D-partition method is applied. If the characteristic equation

$$g(\lambda) = \lambda^2 - (\alpha_1 + \beta_2)\lambda + (\alpha_1 - \lambda)\theta_2 e^{-\lambda r_2} - \alpha_2\theta_1 e^{-\lambda r_1} + \alpha_1\beta_2 - \alpha_2\beta_1 = 0, \tag{8}$$

corresponding to system (1)–(2), has a zero root, then we have

$$(\theta_2 + \beta_2)\alpha_1 = (\theta_1 + \beta_1)\alpha_2. \tag{9}$$

This straight line is one of the lines forming the boundary of the D-partition. Substituting $\lambda = ib$ in characteristic equation (8) and equating the real and imaginary parts to zero, we have

$$U = -b^2 - \theta_2 b\sin(br_2) + \alpha_1\theta_2\cos(br_2) - \alpha_2\theta_1\cos(br_1) + \alpha_1\beta_2 - \alpha_2\beta_1 = 0, \tag{10}$$

$$V = -\alpha_1 b - \beta_2 b - \theta_2 b\cos(br_2) - \alpha_1\theta_2\sin(br_2) + \alpha_2\theta_1\sin(br_1) = 0. \tag{11}$$

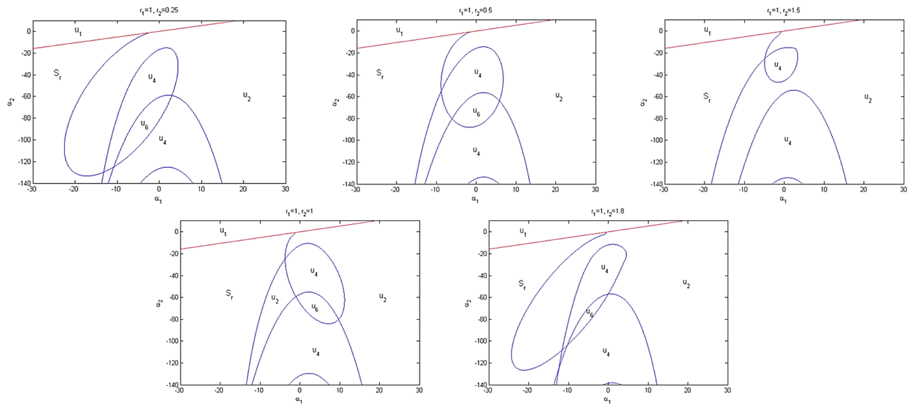

Figure 1 The D-curves and stability region of system (1)–(2) for $\beta_1 = 1.4$, $\beta_2 = 0.4$, $\theta_1 = 1.25$, $\theta_2 = 1$, $r_1 = 1$, and $r_2 = 0.25, 0.5, 1, 1.5, 1.8$

From equations (10)–(11), D-curves are obtained as follows:

$$\alpha_1(b) = \frac{\theta_1\theta_2 b\cos(b(r_1 - r_2)) + \theta_1 b^2 \sin(br_1) + \theta_1\beta_2 b\cos(br_1) + \theta_2\beta_1 b\cos(br_2) + \beta_1\beta_2 b}{\theta_1\theta_2 \sin(b(r_1 - r_2)) + \theta_1\beta_2 \sin(br_1) - \theta_2\beta_1 \sin(br_2) - \theta_1 b\cos(br_1) - \beta_1 b}, \tag{12}$$

$$\alpha_2(b) = \frac{b^3 + (\theta_2^2 + \beta_2^2)b + 2\theta_2\beta_2 b\cos(br_2) + 2\theta_2 b^2 \sin(br_2)}{\theta_1\theta_2 \sin(b(r_1 - r_2)) + \theta_1\beta_2 \sin(br_1) - \theta_2\beta_1 \sin(br_2) - \theta_1 b\cos(br_1) - \beta_1 b} \tag{13}$$

as b tends to 0, we obtain the cusp point

$$p_1 := \lim_{b\to 0} \alpha_1(b) = -\frac{(\theta_1 + \beta_1)(\theta_2 + \beta_2)}{\theta_1 + \beta_1 - \theta_1 r_1(\theta_2 + \beta_2) + \theta_2 r_2(\theta_1 + \beta_1)},$$

$$p_2 := \lim_{b\to 0} \alpha_2(b) = -\frac{(\theta_2 + \beta_2)^2}{\theta_1 + \beta_1 - \theta_1 r_1(\theta_2 + \beta_2) + \theta_2 r_2(\theta_1 + \beta_1)}.$$

As the next step, these results are illustrated for various values of parameters. The curves (12)–(13) and the straight line (9) form the D-partition shown in Fig. 1 for $\beta_1 = 1.4$, $\beta_2 = 0.4$, $\theta_1 = 1.25$, $\theta_2 = 1$, $r_1 = 1$ and $r_2 = 0.25, 0.5, 1, 1.5, 1.8$.

Lemma 1 *If* $0 \le 2A \le \frac{1}{r\mu}$, *then* $x^2 + 2Ax\sin(xr) \ge 0$ *for* $\forall x \in IR$, *where* r *is a positive real number and* $\mu = \sup(\frac{-\sin x}{x}) \approx 0.218$.

Proof Since $f(x) = x^2 + 2Ax\sin(xr)$ is an even function, it is sufficient to show only for $\forall x \ge 0$. When $\sin(xr) \ge 0$, we obtain

$$x^2 + 2Ax\sin(xr) \ge x^2 \ge 0.$$

On the other hand, when $\sin(xr) < 0$, we obtain the following inequality:

$$x^2 + 2Ax\sin(xr) \ge x^2 + \frac{1}{r\mu}x\sin(xr). \tag{14}$$

Suppose that $x^2 + \frac{1}{r\mu} x \sin(xr) < 0$ for $\forall x \in IR^+$ when $\sin(xr) < 0$. By taking $x = \frac{y}{r}$ on the left-hand side of inequality (14), we have

$$\mu < -\frac{\sin(y)}{y},$$

which contradicts the definition of μ.

Consequently, the desired inequality

$$x^2 + 2Ax\sin(xr) \geq 0 \quad \text{for } \forall x \geq 0$$

is obtained. □

Lemma 2 *Let μ be defined as in Lemma 1. If $r_1 = r_2 = r$, $\beta_2\theta_1 = \theta_2\beta_1$, $|\theta_1| < \beta_1$, and $0 \leq 2\theta_2 \leq \frac{1}{r\mu}$, then $\alpha_2(b) \leq 0$ for $\forall b \in IR$.*

Proof Under the conditions of the theorem, for $\forall b \in IR - \{0\}$, equality (13) can be rewritten as follows:

$$\alpha_2(b) = -\frac{b^2 + \theta_2^2 + \beta_2^2 + 2\theta_2\beta_2\cos(br) + 2\theta_2 b\sin(br)}{\theta_1\cos(br) + \beta_1}. \tag{15}$$

(i) If $|\theta_1| < \beta_1$ for $\forall b \in IR$, then inequality $\theta_1\cos(br) + \beta_1 > 0$ holds.

(ii) If $\theta_2\beta_2 \geq 0$, then

$$\theta_2^2 + \beta_2^2 + 2\theta_2\beta_2\cos(br) \geq \theta_2^2 + \beta_2^2 - 2\theta_2\beta_2 = (\theta_2 - \beta_2)^2 \geq 0$$

holds and if $\theta_2\beta_2 < 0$, then

$$\theta_2^2 + \beta_2^2 + 2\theta_2\beta_2\cos(br) > \theta_2^2 + \beta_2^2 + 2\theta_2\beta_2 = (\theta_2 + \beta_2)^2 > 0$$

holds. As a result, $\theta_2^2 + \beta_2^2 + 2\theta_2\beta_2\cos(br) > 0$ holds for $\forall b \in IR$.

(iii) If $0 \leq \theta_2 \leq \frac{1}{r\mu}$, then it follows from Lemma 1 that $b^2 + 2\theta_2 b\sin(br) \geq 0$ holds for $\forall b \in IR$.

It follows from (i), (ii), and (iii) that $\alpha_2(b) \leq 0$ for all $\forall b \in \Re - \{0\}$. Moreover, we have

$$\lim_{b\to 0}\alpha_2(b) = -\frac{(\theta_2 + \beta_2)^2}{\theta_1 + \beta_1},$$

which is negative when $|\theta_1| < \beta_1$. □

Lemma 3 *If $r_1 = r_2 = r$, $\beta_2\theta_1 = \theta_2\beta_1$, $|\theta_1| < \beta_1$, and $2|\theta_2| \leq |\beta_2|$, then $\alpha_2(b) \leq 0$ for $\forall b \in IR$.*

Proof It follows from the proof of Lemma 2 that the denominator of equality (15) is positive. Now let us investigate the following cases for the numerator of equality (15). Since the numerator of equality (15) is an even function of b, it is sufficient to show for $\forall b \geq 0$.

(i) If $0 \leq 2\theta_2 \leq \beta_2$, then

$$\begin{aligned} b^2+\theta_2^2+\beta_2^2+2\theta_2\beta_2\cos(br)+2\theta_2 b\sin(br) &\geq b^2+\theta_2^2+\beta_2^2-2\theta_2\beta_2-2\theta_2 b \\ &= (b+\theta)^2+\beta_2^2-2\theta_2\beta_2 \geq 0 \end{aligned}$$

holds.

(ii) If $0 \leq 2\theta_2 \leq -\beta_2$, then

$$\begin{aligned} b^2+\theta_2^2+\beta_2^2+2\theta_2\beta_2\cos(br)+2\theta_2 b\sin(br) &\geq b^2+\theta_2^2+\beta_2^2+2\theta_2\beta_2-2\theta_2 b \\ &= (b-\theta)^2+\beta_2^2+2\theta_2\beta_2 \geq 0 \end{aligned}$$

holds.

(iii) If $0 \leq -2\theta_2 \leq \beta_2$, then

$$\begin{aligned} b^2+\theta_2^2+\beta_2^2+2\theta_2\beta_2\cos(br)+2\theta_2 b\sin(br) &\geq b^2+\theta_2^2+\beta_2^2+2\theta_2\beta_2+2\theta_2 b \\ &= (b+\theta)^2+\beta_2^2+2\theta_2\beta_2 \geq 0 \end{aligned}$$

holds.

(iv) If $\beta_2 \leq 2\theta_2 \leq 0$, then

$$\begin{aligned} b^2+\theta_2^2+\beta_2^2+2\theta_2\beta_2\cos(br)+2\theta_2 b\sin(br) &\geq b^2+\theta_2^2+\beta_2^2-2\theta_2\beta_2+2\theta_2 b \\ &= (b+\theta)^2+\beta_2^2-2\theta_2\beta_2 \geq 0 \end{aligned}$$

holds. □

Theorem 5 *We suppose that the conditions of Lemma* 2 *or Lemma* 3 *hold and if*

$$\left.\begin{array}{l} (\theta_2+\beta_2)\alpha_1 < (\theta_1+\beta_1)\alpha_2, \\ \alpha_2 > 0 \end{array}\right\} \tag{16}$$

are satisfied, then the characteristic equation of system (1)–(2) *has only one root with positive real part.*

Proof It follows from Lemma 2 or Lemma 3 that D-curves of system (1)–(2) are located outside of the region which is determined by inequality system (16) in $(\alpha_1 - \alpha_2)$ parameter space. Suppose that there were two points within the region (16) with different numbers of roots with positive real parts. Then along any arc within that region connecting the points there must be a point where some of the roots of the characteristic equation lie on the imaginary axis. This point must lie on the D-curves, giving a contradiction. As a result, the number of roots of the characteristic equation with positive real parts does not change in the region (16). In other words, if the system is unstable for specific values of the parameters which satisfy the conditions of the theorem, the instability of the system has been shown. Using Stepan's formula (6) with $\beta_1 = \beta_2 = 1.2$, $\theta_1 = \theta_2 = 1$, $r_1 = r_2 = 1$, $\alpha_1 = 0.1$, $\alpha_2 = 1$ values, it is obtained that the characteristic equation of the system has one root with a positive real part. □

Theorem 6 *We suppose that the conditions of Lemma 2 or Lemma 3 hold and if*

$$\left.\begin{array}{l}(\theta_2+\beta_2)\alpha_1 > (\theta_1+\beta_1)\alpha_2,\\ \alpha_2 > 0\end{array}\right\} \tag{17}$$

are satisfied, then the characteristic equation of system (1)–(2) *has two roots with positive real parts.*

Proof The proof follows the lines of the proof of Theorem 5. □

It follows from Theorems 5 and 6 that since the region on which $\alpha_2 > 0$ has no curves but straight line (9), the stability of the system does not change. Moreover, the straight line (9) splits this region into two subregions on which the characteristic equation of the system has either one or two roots with positive real parts. As a result, the system is unstable under the assumption of Theorems 5 and 6.

3.2 ($\beta_1 - \beta_2$) parameter space

From equation (10)–(11), D-curves are obtained in ($\beta_1 - \beta_2$) parameter space as follows:

$$\beta_1(b) = -\frac{b^3+\theta_2 b^2\sin(br_2)+\alpha_2\theta_1(b\cos(br_1)-\alpha_1\sin(br_1))+\alpha_1^2(b+\theta_2\sin(br_2))}{b\alpha_2}, \tag{18}$$

$$\beta_2(b) = -\frac{\alpha_1 b+\theta_2 b\cos(br_2)+\alpha_1\theta_2\sin(br_2)-\alpha_2\theta_1\sin(br_1)}{b} \tag{19}$$

as b tends to 0, we obtain the cusp point

$$p_1 := \lim_{b\to 0}\beta_1(b) = -\frac{\alpha_1^2(1+\theta_2 r_2)+\alpha_2\theta_1(1-\alpha_1 r_1)}{\alpha_2},$$

$$p_2 := \lim_{b\to 0}\beta_2(b) = -\alpha_1(1+\theta_2 r_2)-\theta_2+\alpha_2\theta_1 r_1.$$

The curves (18)–(19) and the straight line (9) that form the D-partition are shown in Fig. 2 for $\alpha_1 = 0.2$, $\alpha_2 = 0.2$, $\theta_1 = 1.25$, $\theta_2 = 1$, $r_1 = 1$, $r_2 = 0.25, 0.5, 1, 1.5, 2$ and for $r_1 = 0.25, 1, 1.5, 2, 3$, $r_2 = 1$.

Proposition 1 *There exist real numbers m and M such that $m < \beta_2(b) < M$ for $\forall b \in IR$.*

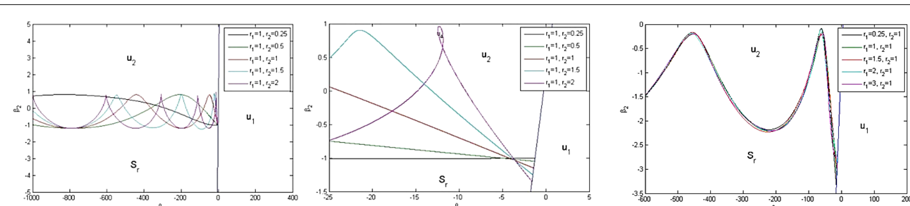

Figure 2 The D-curves and stability region of system (1)–(2) for $\alpha_1 = 0.2$, $\alpha_2 = 0.2$, $\theta_1 = 1.25$, $\theta_2 = 1$, $r_1 = 1$, $r_2 = 0.25, 0.5, 1, 1.5, 2$ and for $r_1 = 0.25, 1, 1.5, 2, 3$, $r_2 = 1$

Proof Let $\mu = \sup(\frac{-\sin x}{x})$. If m and M are defined as follows:

$$m = \begin{cases} -\alpha_1 - |\theta_2| + \mu\alpha_1\theta_2 r_2 + \alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 < 0, \alpha_2\theta_1 < 0, \\ -\alpha_1 - |\theta_2| + \mu\alpha_1\theta_2 r_2 - \mu\alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 < 0, \alpha_2\theta_1 > 0, \\ -\alpha_1 - |\theta_2| - \alpha_1\theta_2 r_2 + \alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 > 0, \alpha_2\theta_1 < 0, \\ -\alpha_1 - |\theta_2| - \alpha_1\theta_2 r_2 - \mu\alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 > 0, \alpha_2\theta_1 > 0, \end{cases} \tag{20}$$

$$M = \begin{cases} -\alpha_1 + |\theta_2| - \alpha_1\theta_2 r_2 - \mu\alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 < 0, \alpha_2\theta_1 < 0, \\ -\alpha_1 + |\theta_2| - \alpha_1\theta_2 r_2 + \alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 < 0, \alpha_2\theta_1 > 0, \\ -\alpha_1 + |\theta_2| + \mu\alpha_1\theta_2 r_2 - \mu\alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 > 0, \alpha_2\theta_1 < 0, \\ -\alpha_1 + |\theta_2| + \mu\alpha_1\theta_2 r_2 + \alpha_2\theta_1 r_1 & \text{if } \alpha_1\theta_2 > 0, \alpha_2\theta_1 > 0. \end{cases} \tag{21}$$

Then it is clearly obtained that $m < \beta_2(b) < M$ for $\forall b \in IR$. □

Theorem 7 *If* $\frac{\alpha_2\theta_1 - \alpha_1\theta_2 + \alpha_2\beta_1}{\alpha_1} < \beta_2 < m$ *is satisfied, then system* (1)–(2) *is stable.*

Proof From equation (9) and Proposition 1, the region which is determined by $\frac{\alpha_2\theta_1 - \alpha_1\theta_2 + \alpha_2\beta_1}{\alpha_1} < \beta_2 < m$ does not include D-curves in $(\beta_1 - \beta_2)$ parameter space.

In order to prove that on each region the characteristic equation of the system has the same number of roots with positive real parts, let us suppose that in a region there were two points for which the numbers of roots with positive real parts are different. Then along any arc within that region connecting these points there must be a point where some of the roots of the characteristic equation lie on the imaginary axis. This point must lie on the D-curves, giving a contradiction. As a result, the number of roots of the characteristic equation with positive real part does not change in the region. Thus, it is sufficient to show that it is stable for specific values which satisfy the condition of the theorem. Using Stepan's formula (6) with $\alpha_1 = 0.2, \alpha_2 = 0.2, \beta_1 = -5, \beta_2 = -2, \theta_1 = 1.25, \theta_2 = 1, r_1 = 1, r_2 = 2$, it is shown that the characteristic equation of the system has no root with apositive real part, which implies the stability of the system (1)–(2). □

Theorem 8 *Suppose that one of the following conditions holds*:

(A1) $M < \beta_2 < \dfrac{\alpha_2\theta_1 - \alpha_1\theta_2 + \alpha_2\beta_1}{\alpha_1}$;

(A2) $M < \beta_2$ *and* $\dfrac{\alpha_2\theta_1 - \alpha_1\theta_2 + \alpha_2\beta_1}{\alpha_1} < \beta_2$;

(A3) $\beta_2 < m$ *and* $\beta_2 < \dfrac{\alpha_2\theta_1 - \alpha_1\theta_2 + \alpha_2\beta_1}{\alpha_1}$;

then system (1)–(2) *is unstable.*

Proof The proof is similar to the proof of Theorem 7. Figure 2 can be used to find the specific values that satisfy the conditions. □

Proposition 2 *Let* $\mu = \sup(\frac{-\sin x}{x})$. *If* $0 \leq \theta_2 \leq \frac{1}{r_2\mu}$ *and* $\alpha_2 > 0$, *then there exists a real number* N *such that* $\beta_1(b) < N$ *for* $\forall b \in IR$.

Proof It follows from Lemma 1 and equation (18) that

$$\beta_1(b) < -\frac{\alpha_2\theta_1\cos(br_1) + \alpha_1^2}{\alpha_2} - \frac{\alpha_1^2\theta_2\sin(br_2)}{b\alpha_2} + \frac{\theta_1\alpha_1\sin(br_1)}{b},$$

holds for $\forall b \in IR$. If N is defined as follows:

$$N = \begin{cases} \frac{\alpha_2|\theta_1|+\alpha_1^2+\mu\alpha_1^2\theta_2 r_2}{\alpha_2} + \theta_1\alpha_1 r_1 & \text{if } \theta_1\alpha_1 > 0, \\ \frac{\alpha_2|\theta_1|+\alpha_1^2+\mu\alpha_1^2\theta_2 r_2}{\alpha_2} - \mu\theta_1\alpha_1 r_1 & \text{if } \theta_1\alpha_1 < 0, \end{cases}$$

then it is obtained that $\beta_1(b) < N$ for $\forall b \in IR$. □

Theorem 9 *System* (1)–(2) *is unstable if* $N < \beta_1$ *holds*.

Proof The proof is similar to the proof of Theorem 7. Figure 2 can be used to find the specific values that satisfy the conditions. □

4 Conclusion

It shown that the delay plays an important role on the stability of the system for both cases. The stability region gets either expanded or contracted in one direction as the delay increases. Having examined the stability locally, we found that a certain range of delays gain stability. However, this is not a common result for all values of coefficients of the system. In Fig. 2, increasing delay r_1 does not change stability considerably.

In Theorems 5–9, new conditions in terms of coefficients are obtained for stability and instability of the system. These conditions are derived by exploiting D-partition method.

In the future work, we will develop the conditions of theorems as figures imply.

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