

Rizanesander's *Recknekonsten* or “The art of arithmetic” – the oldest known textbook of mathematics in Swedish

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The present paper considers an important cultural treasure in the early Swedish history of mathematics education. After the reformation in the 16th century it became possible to study mathematics in Sweden. The first printed textbook in Swedish on arithmetic appeared in 1614, but already in 1601, the oldest known manuscript in Swedish on arithmetic was written by Hans Larsson Rizanesander. In this paper we investigate Rizanesander's manuscript in its historical context.

Keywords: Mathematics history, history of education, arithmetic, mathematics education.

Introduction

Swedish school mathematics has a history of about four centuries. During recent years there has been an increased research interest in the history of Swedish mathematics education (see, for example, Hatami (2007), Lundin (2008), Pejlare (2017), and Prytz (2007)). However, not much research has been conducted on the early history of Swedish mathematics education. The first printed textbook on arithmetic in Swedish was written by Aegidius Aurelius (c.1580–1648) in 1614 (Aurelius & Johansson, 1994). However, the oldest known manuscript in Swedish of a mathematics textbook was written in 1601 by Hans Larsson Rizanesander (1574–1646); this manuscript is the focus of the present paper. The manuscript, entitled *Recknekonsten* (*The art of arithmetic*), constitutes a significant part of the history of Swedish mathematics education. There seems to be a pedagogical idea behind the structure of the text, and it contains numerous examples with solutions. Even though it was never printed, and it is not known to what extent it was used, it can give us insights into the early history of Swedish mathematics education. Hatami and Schéele have interpreted the handwritten manuscript, and in 2018 it was, after more than 400 years, finally printed (Rizanesander, 2018). Hultman (1868–1871, 1874) wrote on the Swedish history of arithmetic, but he was probably not aware of Rizanesander's manuscript, since it was not mentioned. In his dissertation on the Swedish history of mathematics up to 1679, Dahlin (1875) summarized parts of Rizanesander's manuscript, but he did not consider it in the context of the Swedish history of mathematics education. The present paper is part of the outcome from a larger project aiming to contribute to the understanding of the early history of mathematics education in Sweden. The aim of this paper is to investigate Rizanesander's manuscript as well as possible influence on him by other authors. We conduct a content analysis on relevant parts of the manuscript and perform comparisons to other textbooks on arithmetic that he may have had access to. In order to better understand the context in which Rizanesander wrote his manuscript, we will first give an overview of the Swedish history of mathematics education (13th to 17th century).

Swedish history of mathematics education

To better understand the cultural value of Rizanesander's manuscript and its importance for mathematics education, we will first give a summary of the history of Swedish mathematics education until the 17th century. The Swedish education has its origin in the 13th century, when education was committed at cathedral schools, convent schools and provincial schools. Through the Fourth Council of the Lateran, convoked by Pope Innocentius III in 1215, each cathedral was committed to have a school where future priests could get free education. In 1237 the Dominican Order established the first Swedish convent school in Åbo. The convent schools were oriented towards theology, but gave a higher education than the cathedral schools. In particular, those belonging to the higher states had the possibility to get educated. However, many nobles on the continent could not read or write and possibly it was the same in Sweden. During the reformation in the 16th century the convent schools were closed and the opportunity of higher education disappeared. King Gustav Vasa (1496–1560) took over the management of the cathedral schools and provincial schools, but there was no particular interest of education among the people. Not until 1842 there was a Royal decision to implement a public-school system in Sweden (Lundgren, 2015).

The oldest university in the Nordic countries is Uppsala University, which was founded in 1477, and it was the only university in Sweden until King Gustav II Adolf in 1632 founded the university of Dorpat, which today is the University of Tartu in Estonia. During the Reformation there was very little activity at Uppsala University, but during the synod of the Lutheran Church of Sweden in 1593 the Duke Charles (later King Charles IX) gave new privileges to the university, which reopened in 1595 (Pejlare & Rodhe, 2016). Until the reformation the traces of mathematical knowledge in Sweden are few; only at the end of the 13th century the Arabic numerals became known (Dahlin, 1875). At the beginning of the 14th century there probably were a few that were skilled in using the Arabic numerals in simple calculations, but at the beginning of the 16th century there were most likely very few Swedes who could perform calculations except with finger calculations or with an abacus.

The first Swede mentioned to own mathematical literature is the canon Hemming from Uppsala, who testamentated his mathematics books to a relative when he died in 1299. Among these books we find those written by two of the most well-known mathematicians of that time: Campanus of Novara (c. 1220–1296) and Johannes de Sacrobosco (John of Halifax, c.1195–1256) (Dahlin, 1875). The oldest Swedish mathematical work known is a short overview of the calendar with the title *Tabula cerei paschalis*, written in Uppsala in 1344. The first Swedish mathematician mentioned to have been relatively knowledgeable of mathematics and astronomy is King Charles VIII (c. 1408–1470), but some details of what kind of knowledge he had is not known.

When Uppsala University was given new privileges in 1593, Ericus Jacobi Skinnerus (deceased 1597), became the first Swedish professor of mathematics (Rodhe, 2002). No mathematical works by Skinnerus have been encountered. Nevertheless, he was an avid supporter of the French philosopher Petrus Ramus' (1515–1572) controversial ideas (Rodhe, 2002): Ramus questioned the Aristotelian theories that were then dominating in the academic world. Even if we cannot see any clear traces of mathematical activity during the time of Skinnerus, his interest of ramism should have provided a basis for a positive development of education.

The first Swedish professor of mathematics whose activity we know somewhat well is Laurentius Paulinus Gothus (1565–1646). He was a professor at Uppsala University from 1594 to 1601 where

he lectured on arithmetic, algebra, logic, geometry and philosophy (Dahlin, 1875). He also studied the mathematical works by Ramus, and in his spirit Paulinus struggled for the Aristotelian theories to be removed from school. Ramus preached on the usefulness of science, which for mathematics implies the search for applications to other subjects. The followers of Ramism in particular fought the mysterious and superstitious features found in the Aristotelian scholastics (Rodhe, 2002). In 1637 Paulinus became arch bishop and promulgated a program for students of theology: the demand for becoming a priest would be to have good knowledge of arithmetic, Euclid's *Elements*, Ramus' physics and in astronomy, in particular the *doctrina sphaerica* and the calendar (Dahlin, 1875).

During the end of the 16th century it became possible to teach mathematics at the cathedral schools as long as the teaching did not have a negative influence on other subjects. In the curricula from 1611, *Skolordningen 1611*, we can read that in school the children should be educated in Buscherus' arithmetic and in "*Sphaera Johannis de Sacro busto*" with the condition that no other subjects would be neglected (Dahlin, 1875). Buscherus, or Heizo Buscher (1564–1598), was a German philosopher. His book on arithmetic, *Arithmeticae logica methodo conscriptae libri duo* (1590), was an important textbook in Sweden during the early 17th century (Vanäs, 1955) and was later reedited by the Swedish bishop Johannes Bothvidi (1575–1635). *Johannis de Sacro busto* most likely refers to Johannes de Sacrobosco, who wrote a book on astronomy, *Libellus de sphaera*, that was widely used at universities during the middle ages.

The first known mathematics book on arithmetic in Sweden is a handwritten manuscript in Latin on the rule of three, by Peder Månsson (c.1465–1534), written in the early 16th century (Hultman 1870). In 1609 Olof Bures' (1578–1655) book on arithmetic, *Arithmetica instrumentalis abacus*, was printed. The first printed textbook on arithmetic in Swedish is Aegidius Aurelius' *Arithmetica eller Räknebook, medh heele och brutne Taal (Arithmetica or arithmetic textbook, with integers and fractions)*, which was published in eleven editions between 1614 and 1705. But already in 1601, the first Swedish textbook on arithmetic was written: *Recknekonsten* by Hans Larsson Rizanesander. It is written in Swedish at a time when education was reserved for privileged boys with knowledge of Latin, and when mathematics education still was not generally in question.

Rizanesander and *Recknekonsten*

Not much is known about Hans Larsson Rizanesander (1574–1646). He studied at Vasa Akademi in Gävle – a school that had been founded by King Gustav Vasa in 1557 – and became a judge in Gästrikland, a province north of Uppsala, in 1605. Originally, he came from Rotskär in Älvkarleby. The linguistic origin of the name Rizanesander is a Greek translation of the Swedish word Rot-skärman (in English: Root-skerry-man), i.e., *Rhiza-nes-ander*.

Rizanesander wrote his manuscript in Tallinn, which during this time was a dominion of Sweden called Räfte or Reval. Today the manuscript is kept at the university library *Carolina Rediviva* at Uppsala University; to our knowledge it only exists in one copy. The manuscript is dated the 21st of August 1601. It does not have a title, but since the word *Recknekonsten* (*The art of arithmetic*) is used in the dedication this is how we will refer to the book. The manuscript consists of 147 pages, including twelve pages that may have been written by Rizanesander's son Lars Hansson. It is divided into 20 chapters: The first two chapters contain definitions and explanations of the hindu-arabic positional notation (4 pages), as well as a description of the abacus (4 pages). The following

four chapters deal with addition (11 pages), subtraction (7 pages), multiplication (10 pages) and division (12 pages), both on the abacus and with numerals. Chapters VII and VIII deal with the greatest common divisor (1 page) and the least common multiple (4 pages). Chapters IX to XIII deal with fractions (3 pages) and the four basic mathematical operations with fractions (5, 4, 3, and 4 pages, respectively). Chapter XIV deals with arithmetic and geometric series (10 pages), and chapters XV and XVI deal with the rule of three (14 pages) and reversed rule of three (2 pages). The last four chapters deal with general counting (3 pages), square roots and cubic roots (19 pages), *regula cecis* (also called *regula virginum*, 7 pages), and the rule of false position (8 pages).

The manuscript is dedicated (4 pages) to Duke John (1589–1618), son of King John III of Sweden. At this time Duke John was 12 years old. In the introduction Rizanesander claims that the art of arithmetic is indeed a glorious and useful art, since God herself used arithmetic. He also refers to Plato, who demanded that kings and princes should have the acquirement of the art of arithmetic. At the end of the introduction Rizanesander asks Duke John for financial support in order to have the manuscript printed. It is not known if there was any contact between Rizanesander and Duke John, but apparently Rizanesander never got any funding, since the manuscript was not printed. It is not known if Duke John ever mastered the art of arithmetic. However, he never became king; twice he gave up the throne and he died only 28 years old.

Recknekonsten or The Art of Arithmetic

Rizanesander begins the first chapter of *Recknekonsten* by stating that “The art of arithmetic is a knowledge of calculating well”¹. He uses nine *significant* digits (1–9) and one *insignificant* (0) called *Nulla* or *Ziphra*.² The insignificant is explained to “do nothing by itself”³ but it fills up the space where there is no significant digit. This means that the zero only has a meaning as an empty place indicator in the place value number system; the zero does not yet have a meaning in itself. These ten significant and insignificant digits are explained to be “the wood of the art of arithmetic of which it is built in the same way as a house is built by timber, stones, lime, clay and sand”⁴. It is further explained how numbers should be pronounced. Rizanesander does not have a word for numbers greater than 1000. To pronounce a great number, he uses a dot on each thousand to indicate how many multiples of thousand the digits represent. The example he gives, 13̇45̇6̇79̇8654̇321̇06̇2489, should be pronounced as follows:

Thirteen thousand, thousand thousand, thousand thousand times thousand
 Four hundred fiftysix thousand thousand thousand, thousand times thousand
 [...]
 four hundred eighty nine⁵

¹ “Recknekonsten ähr een lärdom till att wäll Reckna” (Rizanesander, 1601, p. 3 r).

² In modern Swedish the word for *zero* is *noll* and the word for *numerical digit* is *siffra*.

³ “förmå inthett vthi sig sielfff” (ibid, p. 3 r).

⁴ “wirkin till Reckne konsten af hwilken hon warder vpbygd medh Lijka såsom till itt hus att vpbyggia hörer Stocker, Stenar, Kalck, Leer och Sandh” (ibid, p. 3 r).

⁵ Thretton tusendh, tusend tusend, tusend tusend gånger tusend fyre hundrade femtiyosex tusend tusend tusend, tusend gånger tusend [...] fyre hundrade ottotiye niyo (ibid, p. 4 v).

This method is not very efficient, but any great number can be pronounced with it. In Christopher Clavius' (1537–1612) *Epitome arithmeticae practicae* from 1583 great numbers were also marked with dots and described to be pronounced in the same way.

After presenting numbers and how to pronounce them, Rizanesander continues to give a description of the abacus (see Figure 1) and the meaning of the lines and their spacing on it. A coin on the lines represents, from below, a unit, a tenth, a hundredth and so on. A coin in the spaces between the lines represents five, fifty, five hundred, and so on. His abacus resembles a traditional abacus but with the supplement that a coin below the bottom line represents half a unit. This is interesting, considering that Rizanesander has not yet introduced fractions.

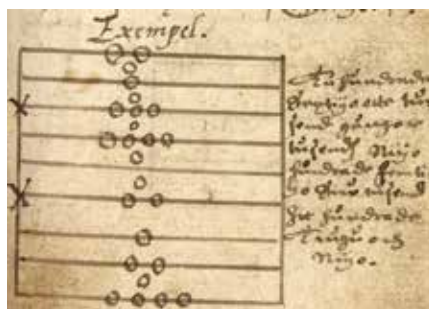


Figure 1: Rizanesander's abacus representing the number 268,957,129 (Rizanesander, 1601, p. 6 r).

Rizanesander proceeds through the following four chapters by explaining the four basic operations addition, subtraction, multiplication and division, both with the abacus, and with algorithms. The abacus will only be used in these chapters; in later chapters numerical methods, only, will be considered. The chapters on the four basic operations with numbers include many examples. Most of the examples concern money, but there are also some numerical examples without a context where it is explained how to carry through the algorithms.

The algorithm for the addition of numbers is the same as the standard addition algorithm that we use today, starting from the right. However, when Rizanesander describes subtraction with numbers, he starts the algorithm from the left, which forces him at each step to look at the position to the right in order to decide whether he has to decompose and recompose the numbers or not. For example:

One asks how many years there has been since we wrote 1574 and now write 1601? *Reliquus* 27.

Put $\begin{array}{r} 1601 \\ 1574 \end{array}$. Say 1 from 1 is nothing left. Draw a line through them both. Say again 5 from 6

Reliquus is 1, which should be written above. But the next number, 7 could not be taken from 0.

Therefore, write under :| less than one is 0 |: and keep this. Draw a line through 6 and 5.

Say again from 10 :| Since you had it in memory to do so |: *Reliquus* is 3. But write less that is 2

above, the same reason as before, draw a line through 0 and 7, and again keep one in memory;

Thereafter take four from 11 *Reliquus* is 7 which can complete be written above. Draw a line

27

through 1 and 4 and now the example is thus: $\begin{array}{r} 1601 \\ 1574 \end{array}$.⁶

$\begin{array}{r} 1601 \\ 1574 \end{array}$

⁶ Een frågar huru många år, ähre sedan thz skreffz 1574 och nu skriffwes 1601? *Reliquus* 27. Sätt såledhes. $\begin{array}{r} 1601 \\ 1574 \end{array}$
Sågh 1 ifrå 1 år inthz öffwer. Dragh så en linie igenom them bådhen. Sågh åther 5 ifrå 6 *Reliquus* år 1, hwilkett skulle

Ramus used the same algorithm for subtraction in his *Arithmeticae libri* from 1555, but Clavius in 1583 used the same subtraction algorithm as we do today.

Chapter V deals with multiplication, and this chapter Ryzanesander begins by presenting the Pythagorean table (see Figure 2), which he states has to be learned by heart.

Figure 2: Ryzanesander's table of multiplication (Ryzanesander, 1601, p. 16 r).

Interesting is that Ryzanesander uses a triangular table, where the commutative law implicitly has to be considered. This means that from the table of multiplication with 2 to the table of multiplication with 9 the length of the tables successively gets shorter. Also, other mathematicians active during the 17th and first half of the 18th century, such as for example Aurelius, but also Nils Buddaeus (1595–1653), Nicolaus Petri Agrelius (c. 1625–1681) and Anders Celsius (1701–1744), have taken the commutative law into account when they designed their multiplication tables, which makes the table triangular and compact. However, Swedish mathematics textbook authors during the end of the 18th century, such as Nils Petter Beckmarck (1753–1815), Olof Hansson Forsell (1762–1853) and Per Anton Zweigbergk (1811–1862) used quadratic multiplication tables and do not focus on the commutative law. Clavius (1583) also used a quadratic multiplication table, but Ramus (1555) only presented a multiplication algorithm and not a multiplication table.

Ryzanesander refers to the Pythagorean table when he describes division with numbers. The division algorithm with numbers he uses is a scratch method, or *divisione per galea*, which was a common method in the middle ages (Vanäs, 1955). Characteristic for this method is that the divisor is written under the dividend and is moved one position for every new quotient digit, and the quotient is written to the right. As the computation is performed, the digits belonging to the same number does not have to be written next to each other on the same line and the remainder is written above the dividend. Also, the partial products are computed from the left to the right and are subtracted as they are computed.

After the chapters on the four basic mathematical operations, Ryzanesander proceeds with finding the greatest common divisor, and the least common multiple, of two numbers. He needs this in his rendering of fractions and the four basic mathematical operations with fractions in the following chapters. Compared to the presentation of the four basic operations with numbers, the presentation of the four basic operations with fractions is different. Both presentations contain numerous

skriffwas offwanföre. Men her till nest föliande taal, icke kunna 7 tages vthaf 0. Derföre skriff ret vnnder :| mindre ähn eett ähr 0 |: och beholtt thz. Dragh een linie igenom 6 och 5. Sägh åther ifrå 10 :| Ty war thu i sinnett hadhe gör thz så |: *Reliquus* ähr 3. Men skriff i mindre som ähr 2 offwanföre, för förne orsackz skuldh; drag een linie igenom 0. Och 7, och beholtt åther eet i sinnett; Tagh sidhan fyra ifrå 11 *Reliquus* ähr 7 hwilken kan fulkommeligen schriffwas offwan före.

2 7

Dragh och een Linie igenom 1 och 4 och ståår nu hele Exemplett såledhes. $\frac{1601}{1574}$ (Ryzanesander, 1601, p. 14 v–15 r).

$\frac{1601}{1574}$

examples, but when the examples with numbers often are in a context there is, with only one exception (unit weights), no context in the examples with fractions. Also, there is no explanation of how the fractions should be interpreted: focus lies instead on how the algorithm should be carried through. However, in the following chapter XIV on arithmetic and geometric series we find examples where fractions are used in a context. One example of an arithmetic series is the following:

I would like to know how many times the bell rang since it rang at one during the night until it rang at 12 during the day.⁷

Many of the examples use a context with money. The money used in Sweden at this time was thaler, mark, öre and penningar (pg). The thaler was an international silver coin used throughout Europe. One thaler is four mark, one mark is eight öre, and one öre is 24 penningar. One example of a geometric series in the context of money, which resembles an example we also find in Buscherus' *Arithmetica* (1590), is the following:

One buys a horse that is shod with 32 nails and give 1 pg for the first nail, 2 pg for the second nail, 4 pg for the third nail and 8 for the fourth and so on doubling. How expensive is the horse? Answer: 4294967295 pg, that is 5592405 thaler 1 mark 6 öre 19 pg. That was an expensive horse.⁸

This was indeed an expensive horse. But Rizanesander has actually made an arithmetical mistake when he changes his money: the correct answer should be 5,592,405 thaler, 1 mark, 2 öre and 15 penningar.

Concluding remarks

Being the oldest mathematics textbook in Swedish, Rizanesander's manuscript is an important source through which we can learn more on the early Swedish history of mathematics education. The manuscript includes a rich repertoire of examples, giving students the possibility to master the art of arithmetic. However, explanations to, for example, why certain algorithms work, are not given. Regarding the algorithms, we see some similarities with both Ramus' and Clavius' books, indicating that Rizanesander was influenced by them. Also, it is indicated that he was influenced by Buscherus' book. We suggest that further investigations of Rizanesander's manuscript should be done; in particular it would be valuable to consider not only the algorithms, but also the contextualization of the examples in order to further investigate in what way he was influenced by Ramus, Clavius and Buscherus.

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⁷ Jagh will gerna weeta huru många slagh Klockan haffwer slagitt ifrå thz hon slogh 1 om natten och in till hon slogh 12 om daghen: Summa 78 slagh (Rizanesander, 1601, p. 41 r).

⁸ Een köper een hest som ähr skodd med 32 söm och giffwer för then förste sö 1 pg^r för then andre 2 pg^r för then tridie 4 pg^r för then fierde 8. Och så aitt dubbelt vp huru dyr ähr thå samme hest Facit. 4294967295 pg^r. Thz ähr 5592405 daler 1 mk 6 öre 19 pg^r thz war een dyr heest (Rizanesander, 1601, p. 42 v).

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