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THE BOURGAIN-TZAFRIRI CONJECTURE AND CONCRETE CONSTRUCTIONS OF NON-PAVABLE PROJECTIONS

PETER G. CASAZZA, MATTHEW FICKUS, DUSTIN G. MIXON AND JANET C. TREMAIN

ABSTRACT. It is known that the Kadison-Singer Problem (KS) and the Paving Conjecture (PC) are equivalent to the Bourgain-Tzafriri Conjecture (BT). Also, it is known that (PC) fails for 2-paving projections with constant diagonal 1/2. But the proofs of this fact are existence proofs. We will use variations of the discrete Fourier Transform matrices to construct concrete examples of these projections and projections with constant diagonal $1/r$ which are not r-pavable in a very strong sense.

In 1989, Bourgain and Tzafriri showed that the class of zero diagonal matrices with small entries (on the order of $\leq 1/log^{1+\epsilon}n$, for an n-dimensional Hilbert space) are *pavable*. It has always been assumed that this result also holds for the BT-Conjecture - although no one formally checked it. We will show that this is not the case. We will show that if the BT-Conjecture is true for vectors with small coefficients (on the order of $\leq C/\sqrt{n}$) then the BT-Conjecture is true and hence KS and PC are true.

KEYWORDS. Kadison-Singer Problem, Anderson Paving Problem Discrete Fourier Transform.
AMS MSC (2000). 42C15, 46C05, 46C07.

1. INTRODUCTION

It is now known that the 1959 Kadison-Singer Problem is equivalent to fundamental unsolved problems in a dozen areas of research in pure mathematics, applied mathematics and engineering [7, 8]. In 1979, Anderson [1] showed that the Kadison-Singer Problem is equivalent to the *Paving Conjecture.*

Paving Conjecture (PC). *For $\epsilon > 0$, there is a natural number r so that for every natural number n and every linear operator T on l_2^n whose matrix has zero diagonal, we can find a partition (i.e. a paving) $\{A_j\}_{j=1}^r$ of*

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$\{1,\ldots,n\}$, such that

$$\|Q_{A_j}TQ_{A_j}\| \le \epsilon\|T\| \quad \textit{for all } j=1,2,\ldots,r,$$

where Q_{A_j} is the natural projection onto the A_j coordinates of a vector.

Operators satisfying the Paving Conjecture are called **pavable operators**. A projection P on $\mathcal{H}_n$ is (ϵ, r)**-pavable** if there is a partition $\{A_j\}_{j=1}^r$ of $\{1,2,\ldots,n\}$ satisfying

$$\|Q_{A_j}PQ_{A_j}\| \le \epsilon, \quad \text{for all } j=1,2,\ldots,r.$$

It was shown in [5] that projections with constant diagonal $1/r$ are not (r,ϵ)-pavable for any $\epsilon > 0$. But the argument in [5] is an existence proof and the actual matrices failing paving were not known. In this note we will construct concrete examples of these projections. As a consequence, we will obtain a stronger result than that of [5]. The main question now is whether this construction can be generalized to produce a counter-example to KS.

Notation 1.1. Throughout this paper, if $\mathcal{H}_n$ is an n-dimensional Hilbert space, then $\{e_i\}_{i=1}^n$ denotes a fixed orthonormal basis for $\mathcal{H}$.

It was shown [7] that BT is equivalent to PC. Our construction of non-2-pavable projections starts with a construction of non-2-Riesable sequences (See Section 2 for the definitions). The vectors we will produce have very small coefficients, on the order of $1/\sqrt{n}$ for an n-dimensional Hilbert space. However, conventional wisdom indicates that we cannot construct a counter-example to PC out of vectors with small coefficients. So next, we will show that conventional-wisdon has been wrong for the last 20 years and a counter-example to BT exists in general if and only if it exists for matrices with coefficients on the order of $1/\sqrt{n}$. Conventional wisdom came from a result of Bourgain and Tzafriri [2, 3] where they showed that the Paving Conjecture has a positive solution for the class of zero diagonal matrix operators $A = (a_{ij})_{i,j=1}^n$ on $\mathbb{H}_n$ with small coefficients. In particular, a matrix is pavable if the coefficients satisfy for some $\epsilon > 0$,

$$|a_{ij}| \le \frac{C}{log^{1+\epsilon}n}.$$

It has always been assumed that the corresponding result holds for BT if

$$|Te_i(j)| \le \frac{C}{log^{1+\epsilon}n}, \quad \text{for all } i,j=1,2,\ldots,n.$$

We will show that this is not the case. This is the second main theorem of this paper (See Section 2 for the definitions).

Theorem 1.2. *The following are equivalent:*

(1) The Bourgain-Tzafriri Conjecture is true.

(2) There are constants δ and $r \in \mathbb{N}$ so that for every $C > 0$ there is an N_0 so that for every $N \ge N_0$ if $\{f_i\}_{i=1}^{2N}$ is a unit norm 2-tight frame for $\mathcal{H}_N$

satisfying

$$|f_i(j)| \leq \frac{C}{\sqrt{2N}},$$

then $\{f_i\}_{i=1}^{2N}$ *is* (δ, r)*-Rieszable.*

2. Preliminaries

We will actually work with an equivalent form of the Paving Conjecture for projections with constant diagonal. In 1989, Bourgain and Tzafriri proved one of the most celebrated theorems in analysis: The *Bourgain-Tzafriri Restricted Invertibility Theorem* [2]. This gave rise to a major open problem in analysis.

Bourgain-Tzafriri Conjecture (BT)**.** *There is a universal constant* $A > 0$ *so that for every* $B > 1$ *there is a natural number* $r = r(B)$ *satisfying: For any natural number* n*, if* $T : \ell_2^n \to \ell_2^n$ *is a linear operator with* $\|T\| \leq B$ *and* $\|Te_i\| = 1$ *for all* $i = 1, 2, \ldots, n$*, then there is a partition* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \ldots, n\}$ *so that for all* $j = 1, 2, \ldots, r$ *and all choices of scalars* $\{a_i\}_{i \in A_j}$ *we have:*

$$\|\sum_{i \in A_j} a_i Te_i\|^2 \geq A \sum_{i \in A_j} |a_i|^2.$$

It was shown in [7] that BT is equivalent to the Paving Conjecture.

Definition 2.1. *A family of vectors* $\{f_i\}_{i=1}^M$ *for an* n*-dimensional Hilbert space* $\mathcal{H}_n$ *is* (δ, r)-**Rieszable** *if there is a partition* $\{A_j\}_{j=1}^r$ *of* $\{1, 2, \ldots, M\}$ *so that for all* $j = 1, 2, \ldots, r$ *and all scalars* $\{a_i\}_{i \in A_j}$ *we have*

$$\|\sum_{i \in A_j} a_i f_i\|^2 \geq \delta \sum_{i \in A_j} |a_i|^2.$$

A projection P *on* $\mathcal{H}_n$ *is* (δ, r)-**Rieszable** *if* $\{Pe_i\}_{i=1}^n$ *is* (δ, r)*-Rieszable.*

Recall that a family of vectors $\{f_i\}_{i \in I}$ is a **frame** for a Hilbert space $\mathcal{H}$ if there are constants $0 < A, B < \infty$, called the **lower (upper) frame bounds)** respectively satisfying for all $f \in \mathcal{H}$:

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2.$$

If $\|f_i\| = \|f_j\|$ for all i, j, we call this an **equal norm** frame and if $\|f_i\| = 1$ for all i, it is a **unit norm frame**. If $A = B$ this is an A**-tight frame** and if $A = B = 1$, it is a **Parseval frame**. It is known [4, 6, 9] that $\{f_i\}_{i \in I}$ is an A-tight frame if and only if the matrix with the $f_i's$ as rows has orthogonal columns and the square sums of the column coefficients equal A. It is also known [4, 9] that $\{f_i\}_{i=1}^M$ is a Parseval frame for $\mathcal{H}_n$ if and only if there is an othogonal projection $P : \ell_2^M \to \mathcal{H}_n$ with

$$Pe_i = f_i, \quad \text{for all } i = 1, 2, \ldots, M,$$

where $\{e_i\}_{i=1}^M$ is the unit vector basis of ℓ_2^M.

The following result can be found in [5, 10].

Proposition 2.2. *Fix a natural number $r \in \mathbb{N}$. The following are equivalent:*

(1) The class of projections with constant diagonal $1/r$ are pavable.
(2) The class of projections with constant diagonal $1/r$ are Rieszable.
(3) The class of unit norm r-tight frames $\{f_m\}_{m=1}^{nr}$ for $\mathcal{H}_n$ are Rieszable.

Moreover, the Paving Conjecture is equivalent to (1)-(3) holding for some $r \in \mathbb{N}$.

We will construct concrete counterexamples for (4) of Proposition 2.2 for the case $r = 2$. These will give concrete counterexamples to 1-3 in the proposition by the following result which can be found in [5]. The point here is that the proof of this proposition gives an explicit representation of each of the equivalences in the proposition in terms of all the others.

Proposition 2.3. *Let P be an orthogonal projection on $\mathcal{H}_n$ with matrix $B = (a_{ij})_{i,j=1}^n$. The following are equivalent:*

(1) The vectors $\{Pe_i\}_{i=1}^n$ is (δ, r)-Rieszable.
(2) There is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, n\}$ so that for all $j = 1, 2, \ldots, r$ and all scalars $\{a_i\}_{i \in A_j}$ we have

$$\| \sum_{i \in A_j} a_i (I - P) e_i \|^2 \leq (1 - \delta) \sum_{i \in I} |a_i|^2.$$

(3) The matrix of $I - P$ is (δ, r)-pavable.

As a fundamental tool in our work, we will work with the $n \times n$ discrete Fourier transform matrices which we will just call DFT matrices or $DFT_{n\times n}$. For these, we fix $n \in \mathbb{N}$ and let ω be a primative n^{th} root of unity and define

$$DFT_{n\times n} = \left(\frac{1}{\sqrt{n}} \omega^{ij} \right)_{i,j=1}^n .$$

The main point of these $DFT_{n\times n}$ matrices is that they are unitary matrices for which the modulus of all of the entries of the matrix are equal to 1. We will use on the following simple observation.

Proposition 2.4. *If $A = (a_{ij}\}_{i,j=1}^n$ is a matrix with $|a_{ij}|^2 = a$ for all i, j and orthogonal columns and we multiply the j^{th}-column of A by a constant C_j to get a new matrix B, then*

(1) The columns of B are orthogonal.
(2) The square sums of the coefficients of any row of B all equal

$$a \sum_{j=1}^n C_j^2.$$

(3) The square sum of the coefficients of the j^{th} column of B equal aC_j^2.

3. The Bourgain-Tzafriri Conjecture for r=2

Let us first outline our construction. For any natural number n, we will alter two $2n \times 2n$ DFT matrices along the lines of Proposition 2.4 and then stack them on top of one another to get a $4n \times 2n$ matrix with the following properties:

(1) Each altered DFT has the square sums of the coefficients of any row equal to 1.

(2) The top altered DFT will have the square sums of the coefficients of each column j with $1 \leq j \leq n-1$ equal to 2, and the square sums of the coefficients of the remaining columns will all equal $2/(n+1)$.

(3) The combined matrix will have the square sums of the coefficients of each column equal to 2.

(4) The columns of the combined matrix are orthogonal.

It follows that this is the matrix of a unit norm 2-tight frame and hence multiplying the matrix by $1/\sqrt{2}$ will turn it into an equal norm Parseval frame, creating the matrix of a rank $2n$ projection on $\mathcal{C}^{4n}$ with constant diagonal 1/2. We will then show that the rows of this class of matrices are not uniformly 2-Rieszable to complete the example.

So we start with a $2n \times 2n$ DFT and multiply the first $n-1$ columns by $\sqrt{2}$ and the remaining columns by $\sqrt{\frac{2}{n+1}}$ to get a new matrix B_1. Now, we take the second $2n \times 2n$ DFT matrix and multiply the first $n-1$ columns by 0 and the remaining columns by $\sqrt{\frac{2n}{n+1}}$ to get a matrix B_2. We form the matrix B by stacking the matrices B_1 and B_2 on top of one another to get the matrix B given below.

(n-1)-Colmns	(n+1)-Colmns.
$\sqrt{2}$	$\sqrt{\frac{2}{n+1}}$
0	$\sqrt{\frac{2n}{n+1}}$

Now we can prove:

Proposition 3.1. *The matrix B satisfies:*

(1) The columns are orthogonal and the square sum of the coefficients of every column equals 2.

(2) The square sum of the coefficients of every row equals 1.

The row vectors of the matrix B are not $(\delta, 2)$-Rieszable, for any δ independent of n.

Proof. Clearly the columns of B are orthogonal. To check the square sums of the column coefficients, recall that for columns $1 \leq \ell \leq n-1$ the modulus of all the coefficients of B_1 are $\frac{1}{\sqrt{n}}$, the the coefficients of B_2 are 0. So the square sum of the coefficients in column ℓ are:

$$\frac{1}{n} \cdot 2n + 0 = 2.$$

For the columns $n \leq \ell \leq 2n$, the modulus of the coefficients of B_1 are $\frac{1}{\sqrt{n(n+1)}}$ and the coefficients of B_2 are $\frac{1}{\sqrt{n+1}}$. So the square sum of the coefficients of B in column ℓ are:

$$2n \cdot \frac{1}{n(n+1)} + 2n \cdot \frac{1}{n+1} = \frac{2}{n+1} + \frac{2n}{n+1} = 2.$$

Now we check the row sums. For any row of B_1, the first $n-1$ column coefficients have modulus $\frac{1}{\sqrt{n}}$, and the modulus of the coefficients of the last $n+1$ columns of B_1 have modulus $\frac{1}{\sqrt{n(n+1)}}$. So the square sum of the coefficients of any row of B_1 are:

$$(n-1)\frac{1}{n} + (n+1)\frac{1}{n(n+1)} = 1.$$

For any row of B_2, the first $n-1$ column coefficients are equal to 0 and the remaining $n+1$ column coefficients have modulus $\frac{1}{\sqrt{n+1}}$. So the square sum of the row coefficients of B_2 are

$$(n+1)\frac{1}{n+1} + 0 = 1.$$

We will now show that the row vectors of B are not two Rieszable. So let $\{A_1, A_2\}$ be a partition of $\{1, 2, \ldots, 4n\}$. Without loss of generality, we may assume that $|A_1 \cap \{1, 2, \ldots, 2n\}| \geq n$. Let the row vectors of the matrix B be $\{f_i\}_{i=1}^{4n}$ as elements of $\mathcal{C}^{2n}$. Let P_{n-1} be the orthogonal projection of $\mathcal{C}^{2n}$ onto the first $n-1$ coordinates. Since $|A_1| \geq n$, there are scalars $\{a_i\}_{i \in A_1}$ so that $\sum_{i \in A_1} |a_i|^2 = 1$ and

$$P_{n-1}\left(\sum_{i \in A_1} a_i f_i\right) = 0.$$

Also, let $\{g_j\}_{j=1}^{2n}$ be the orthonormal basis consisting of the original rows of the $DFT_{2n \times 2n}$. We now have:

$$\begin{aligned}
\|\sum_{i \in A_1} a_i f_i\|^2 &= \|(I - P_{n-1})\left(\sum_{i \in A_1} a_i f_i\right)\|^2 \\
&= \frac{2}{n+1}\|(I - P_{n-1})\left(\sum_{i \in A_1} a_i g_i\right)\|^2 \\
&\leq \frac{2}{n+1}\|\sum_{i \in A_1} a_i g_i\|^2 \\
&= \frac{2}{n+1}\sum_{i \in A_1} |a_i|^2 \\
&= \frac{2}{n+1}.
\end{aligned}$$

Letting $n \to \infty$, we have that this class of matrices is not $(\delta, 2)$-pavable for any $\delta > 0$. $\square$

4. The Bourgain-Tzafriri Conjecture for general r

In this section we will extend our construction to projections with constant diagonal $1/r$ and actually prove a stronger result.

Proposition 4.1. *For every natural number $r \geq 2$, there is a $r^2n \times rn$ projection matrix with constant diagonal $1/r$ so that whenever we partition $\{1, 2, \ldots, r^2n\}$ into sets $\{A_j\}_{j=1}^r$, and for all $k = 1, 2, \ldots, r$, if $D_k = \{(k-1)rn+1, (k-1)rn+2, \ldots, krn\}$, then for every $k = 1, 2, \ldots, r-1$, there is a j so that the vectors $\{f_i\}_{i \in A_j \cap D_k}$ are not uniformly 2-Rieszable.*

This time, we will take r DFT matrices of size $rn \times rn$ and alter their columns by certain amounts so that when we stack them on top of one another to get a matrix B of size $r^2n \times rn$ satifying:

1. The columns of B are orthogonal and the sums of the squares of the coefficients of each row of B equals 1.

2. The sums of the squares of the coefficients of each column of B equals r.

3. B satisfies the requirements of the proposition.

For the first matrix B_1 we take the $rn \times rn$ DFT and multiply the first $n-1$ columns by $\sqrt{r}$ and the remaining columns by $\sqrt{\delta_1}$ (to be chosen later). For B_2 we take the $rn \times rn$ DFT and multiply the first $n-1$ columns by 0, multiply the columns $n-1+j$, $j = 1, 2, \ldots, n-1$ by $\sqrt{r-\delta_1}$, and multiply the remaining columns by $\sqrt{\delta_2}$ (to be chosen later). And for $k = 3, \ldots, r-1$ we construct the matrix B_k by taking the $rn \times rn$ DFT and multiplying the first $(k-1)(n-1)$ columns by 0, multiply the columns $(k-1)(n-1)+j$ for $j = 1, 2, \ldots, n-1$ by

$$\sqrt{r - \sum_{i=1}^{k-1} \delta_{k-1}},$$

and multiplying the remaining columns by $\sqrt{\delta_k}$ (to be chosen later). Finally, for B_r we take the $rn \times rn$ DFT and multiplying the first $(r-1)(n-1)$ columns by 0 and the remaining columns by $\sqrt{\delta_r}$ (to be chosen later).

We then stack these r, $rn \times rn$ matrices $\{B_k\}_{k=1}^r$ on top of each other to produce the matrix B for which the moduli of the coefficients of B are given in figure 2 below. Now we must show that the matrix B has all of the properties of Proposition 4.1.

It is clear that the columns of B are orthogonal. To show that the square sums of the row coefficients of the matrix B are all equal to 1, we need a lemma.

Lemma 4.2. *To get the rows of the matrix B to square sum to 1, we need*

$$\delta_k = \frac{r^2 n}{[(r-k+1)n+k-1][(r-k)n+k]}. \tag{1}$$

Proof. We will proceed by induction on k to show Equation 1 for all $k = 1, 2, \ldots, r$. For $k = 1$, we observe that the coefficients of the first $n-1$ columns of B_1 have modulus equal to $1/n$, while the coefficients of the remaining $rn-(n-1)$ columns of B_1 have modulus $\sqrt{\frac{\delta_1}{rn}}$. So the sum of the squares of the coefficients of any row of B_1 equals

$$\frac{1}{rn}\left[r(n-1)+\delta_1(rn-(n-1))\right] = 1.$$

Hence,

$$\delta_1(rn-(n-1)) = rn - r(n-1) = r.$$

So,

$$\delta_1 = \frac{r}{(r-1)n+1} = \frac{r^2 n}{[(r-1+1)n+1-1][(r-1)n+1]]}.$$

For $k = 2$, our matrix B_2 has coefficients of the first $n-1$ columns equal to 0, coefficients of the columns $(n-1)+j$, $j = 1, 2, \ldots, n-1$ have modulus equal to

$$\sqrt{\frac{r-\delta_1}{rn}}.$$

and the remaining $rn - 2(n-1)$ columns have modulus equal to $\sqrt{\frac{\delta_2}{rn}}$. So the square sums of the coefficients of any row of B_2 equals

$$\frac{1}{rn}\left[(n-1)(r-\delta_1) + (rn-2(n-1))\delta_2\right] = 1.$$

Since

$$r - \delta_1 = r - \frac{r}{(r-1)(n+1)} = \frac{r(r-1)n}{(r-1)n+1},$$

we can solve the equation to get

$$\delta_2 = \frac{r^2 n}{[(r-1)n+1][(r-2)n+2]}.$$

Now assume our formula holds for any $k \le r-1$ and we check it for $k+1$. The matrix B_{k+1} has coefficients of the first $k(n-1)$ columns equal to 0, coefficients of the columns $k(n-1)+j$, $j = 1, 2, \ldots, n-1$ of modulus

$$\left(\frac{r - \sum_{j=1}^{k}\delta_k}{rn}\right)^{1/2},$$

and the coefficients of the remaining columns have modulus $\sqrt{\frac{\delta_{k+1}}{rn}}$. It follows that the square sums of the row coefficients of the matrix B_{k+1} must satisfy

$$\left(r-\sum_{j=1}^{k}\delta_j\right)(n-1)+\delta_{k+1}[rn-(k+1)(n-1)]=rn. \tag{2}$$

Hence, letting $a=rn/(n-1)$ we have

$$\begin{aligned}
\sum_{j=1}^{k}\delta_j &= r^2n\sum_{j=1}^{k}\frac{1}{[r-j+1)n+j-1][(r-j)n+j]}\\
&= \frac{r^2n}{(n-1)^2}\sum_{j=1}^{k}\frac{1}{(a+1-j)(a-j)}\\
&= \frac{r^2n}{(n-1)^2}\sum_{j=1}^{k}\left(\frac{1}{a-j}-\frac{1}{a-(j-1)}\right)\\
&= \frac{r^2n}{(n-1)^2}\frac{1}{a-k}-\frac{1}{a-0}\\
&= \frac{r^2n}{(n-1)^2}\frac{k}{a(a-k)}\\
&= \frac{r^2kn}{rn(rn-k(n-1))}\\
&= \frac{rk}{(r-k)n+k}.
\end{aligned}$$

Combining this with Equation 2 we have

$$\begin{aligned}
\delta_{k+1} &= \frac{r+(n-1)\sum_{j=1}^{k}\delta_j}{rn-(k+1)(n-1)}\\
&= \frac{r+(n-1)\left(\frac{rk}{(r-k)n+k}\right)}{(r-k+1)n+k-1}\\
&= \frac{r[(r-k)n+k]+(n-1)rk}{[(r-k+1)n+k-1][(r-k)n+k]}\\
&= \frac{r^2n-rkn+rk+rnk-kr}{[(r-k+1)n+k-1][(r-k)n+k]}\\
&= \frac{r^2n}{[(r-k+1)n+k-1][(r-k)n+k]}.
\end{aligned}$$

□

By Lemma 4.2, we know that the rows of the matrix B square sum to 1. Now we need to check the column sums. Most of this is true by our definitions. We check two cases:

Case 1: For a column $\ell = k(n-1)+j$, $k = 1, 2, \ldots, r-1$, the column coefficients for $1 \le j \le k-1$ and $i = jrn + m$, $m = 1, 2, \ldots, rn$, have modulus $\sqrt{\frac{\delta_j}{rn}}$, and for $i = krn + m$, $m = 1, 2, \ldots, rm$ the modulus of the coefficients are $\sqrt{\frac{r - \sum_{j=1}^{k-1}\delta_j}{rn}}$, and all other coefficients are 0. Hence, the square sum of the column coefficients is

$$rn \sum_{j=1}^{k-1} \frac{\delta_j}{rn} + rn \left(\frac{r - \sum_{j=1}^{k-1}\delta_j}{rn} \right) = r.$$

Case 2: For a column $\ell = (r-1)(n-1)+j$, with $j = 1, 2, \ldots rn - (r-1)(n-1) = r+n-1$, the square sum of the coefficients of column ℓ are (using our formula for the sum of the δ_k above):

$$\sum_{k=1}^{r} \delta_k = \frac{r^2}{(r-r)n + r} = r.$$

Finally, we need to show that our matrix B is not pavable (with paving constants independent of n) in the strong sense given in the proposition. This follows similarly to the $DFT_{2n\times 2n}$ case. Let $\{f_i\}_{i=1}^{r^2 n}$ be the rows of the matrix B and let $\{g_i\}_{i=1}^{rn}$ be the rows of the DFT matrix. Also, let P_k be the orthogonal projection of $\mathcal{C}_2^{rn}$ onto the first $k(n-1)$ coordinates. Now let $\{A_j\}_{j=1}^{r}$ be a partition of $\{1, 2, \ldots, r^2 n\}$ and fix $1 \le k \le r-1$. Then there is a j so that $|A_j \cap D_k| \ge n$. Since the vectors $\{f_i\}_{i \in A_j \cap D_k}$ have zero coordinates for all $j = 1, 2, \ldots, (k-1)(n-1)$, and there are scalars $\{a_i\}_{i \in A_j \cap D_k}$ satisfying

1. $\sum_{i \in A_j \cap D_k} |a_i|^2 = 1$.
2. We have

$$P_k \left(\sum_{i \in A_j \cap D_k} a_i f_i \right) = 0.$$

It follows from our construction that

$$\begin{aligned}
\|\sum_{i\in A_j\cap D_k} a_i f_i\|^2 &= \|(I-P_k)\left(\sum_{i\in A_j\cap D_k} a_i f_i\right)\|^2 \\
&= \delta_k \|(I-P_k)\left(\sum_{i\in A_j\cap D_k} a_i g_i\right)\|^2 \\
&\le \delta_k \|\sum_{i\in A_j\cap D_k} a_j g_j\|^2 \\
&= \delta_k \sum_{i\in A_j\cap D_k} |a_i|^2 \\
&= \delta_k
\end{aligned}$$

Since $\lim_{n\to\infty}\delta_k = 0$, it follows that our family of matrices are not 2-Rieszable in the strong sense of the Proposition. This argument looks pictorially as:

Each square is a $rn \times (n-1)$ submatrix

$\sqrt{\frac{r}{rn}}$	$\sqrt{\frac{\delta_1}{rn}}$	$\sqrt{\frac{\delta_1}{rn}}$	$\cdots$	$\sqrt{\frac{\delta_1}{rn}}$
0	$\sqrt{\frac{r}{rn}}$	$\sqrt{\frac{\delta_2}{rn}}$	$\cdots$	$\sqrt{\frac{\delta_2}{rn}}$
0	0	$\sqrt{\frac{r}{rn}}$	$\cdots$	$\sqrt{\frac{\delta_3}{rn}}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
0	0	0	$\cdots$	$\sqrt{\frac{\delta_r}{rn}}$

The main question is whether it is possible to take the concrete constructions in this paper and generalize them to give a complete counterexample to the Paving Conjecture.

5. The Proof of Theorem 1.2

Proof. (1) $\Rightarrow$ (2): This is from Proposition 2.2.

(2) $\Rightarrow$ (1): Let P be a projection with constant diagonal 1/2 on $\mathcal{H}_{2N}$. So $\{\sqrt{2}Pe_i\}_{i=1}^{2N}$ is a unit norm 2-tight frame for $\mathcal{H}_{2N}$. Let A be the $N\times N$ matrix with row vectors $\{\sqrt{2}Pe_i\}_{i=1}^{2N}$. Define recursively,

$$A_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} A & A \\ A & -A \end{bmatrix}$$

and

$$A_{K+1} = \frac{1}{\sqrt{2}}\begin{bmatrix} A_K & A_K \\ A_K & -A_K \end{bmatrix}$$

Note: Each A_K (their rows) is a unit norm 2-tight frame for $\mathcal{H}_{2^K N}$. Since the columns of A_K are orthogonal, this implies that the columns of A_{K+1} are orthogonal. Also, clearly the sums of the squares of the row elements are still one and the sums of the squares of the column elements are still one.

Also, the entries $(a_{i,j})$ of A_K satisfy

$$|a_{i,j}| \leq \frac{1}{\sqrt{2^K}} = \frac{\sqrt{N}}{\sqrt{2^K N}}. \tag{3}$$

Letting $C = \sqrt{N}$ in (2) of the theorem, there is some N_0 such that for every $L \geq N_0$, if $\{f_i\}_{i=1}^{2L}$ is a unit norm 2-tight frame for $\mathcal{H}_L$ with

$$|f_{i,j}| \leq \frac{C}{\sqrt{2L}},$$

then $\{f_i\}_{i=1}^{2L}$ is (δ, r)-Rieszable. Hence, for K large enough, Equation 3 has this inequality. So, A_K is (δ, r)-Rieszable. That is, there is a partition $\{A_j\}_{j=1}^r$ of $\{1, 2, \ldots, 2^K N\}$ so that for every $j = 1, 2, \ldots, r$ and all scalars $\{a_i\}_{i \in A_j}$ we have

$$\|\sum_{i \in A_j} a_i f_i\|^2 \geq \delta \sum_{i \in A_j} |a_i|^2,$$

where $\{f_i\}$ are the row vectors of A_K. Let

$$B_j = A_j \cap \{1, 2, \ldots, N\}.$$

Then $\{B_j\}_{j=1}^r$ is a partition of $\{1, 2, \ldots, N\}$. Now we compute,

$$\begin{aligned}
\delta &\leq \|\sum_{i \in B_j} a_i f_i\|^2 \\
&= \frac{1}{2^K} \sum_{\ell=1}^{2^K} \|\sum_{i \in B_j} a_i \sum_{j=1}^{N} f_{i,\ell+j}\|^2 \\
&= \frac{1}{2^K} \cdot 2^K \|\sum_{i \in B_j} a_i \sqrt{2} P e_i\|^2 \\
&= \|\sum_{i \in B_j} a_i \sqrt{2} P e_i\|^2.
\end{aligned}$$

Hence, A is (δ, r)-Rieszable and hence KS holds by Proposition 2.2. $\square$

Remark 5.1. The above points out that there really is a major difference between "paving" and "Rieszing". Recall that if $\{f_i\}_{i=1}^M$ is a set of vectors, the *Grammian* of this family is the $M \times M$ matrix $(\langle f_i, f_j \rangle)$. In the above construction, if G_A is the Grammian of the row vectors A then the

Grammian of of the row vectors of A_K is

$$\begin{bmatrix} G_A & 0 & 0 & \cdots \\ 0 & G_A & 0 & \cdots \\ 0 & 0 & G_A & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

That is, the coefficients of the Grammian do not get smaller in this construction while the coefficients of the matrix do get smaller.

Remark 5.2. This result also says that passing results on paving from the Grammian back to the matrix and the other way do not hold in general.

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