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STUDENTS' REVERSIBLE REASONING ON FUNCTION COMPOSITION PROBLEM: REVERSIBLE ON FUNCTION AND SUBSTITUTION

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Abstract

In this paper we report students' reversible reasoning types on function composition problem. Reversible reasoning can be observed according to operation and structural correlation among input, process, and result. There are seven students participate in doing function composition related to structural correlation in (f∘g)(x)=h(x), The researchers assume that there are four types of reversible reasoning in composition problem, namely: (1) reversible on composition; (2) reversible on function; (3) reversible on substitution; (4) reversible on variables. However, there are only two appearing reversible reasoning, they are: reversible on function and reversible on substitution. Each type were selected a subject to be interviewed for 25 minutes and asked to do the Function Composition Task (FCT). Subject with reversible on function type identifies structural correlation among input, source and result as well as involving inverse in input with permissibility (f(y)) to produce basic function (f(x)). Meanwhile, subject with reversible on substitution type, constructs result based on the input, identifies structural similarity and generalizes from the structural similarity to produce basic function

Keywords: Reversible Reasoning, Function Composition, Reversible on Function, Reversible on Substitution

INTRODUCTION

Reasoning is studied in different terms(Beckmann & Izsák, 2015; Blanton, Brizuela, Gardiner, Sawrey, & Newman-Owens, 2015; Hackenberg & Lee, 2015; Lobato, Hohense, & Rhodehamel, 2014; Whitacre & Nickerson, 2016).Beckmann & Izsák(2015) used the term of proportional reasoning on the problem of multiplication of $M \times N = P$ involving proportion, Maria Blanton et al(2015)also used the term of proportional, but on the object of algebra. Hackenberg & Lee, (2015)used the term of quantitative reasoning related to fractional knowledge in writing equations with unknown quantity $(ax = b)$, besides Whitacre & Nickerson (2016) also used the term of quantitative reasoning especially in multidigit multiplication with the teacher candidates as the subjects. Lobato et al. (2014)used the term of algebraic reasoning on algebraic issues. So, the terms of reasoning are interpreted based on the object which is studied. For example, examining algebraic objects will bring up the term of algebraic reasoning.

Understanding mathematical objects is important for students in learning mathematics(Hackenberg & Lee, 2015; Pierson et al., 2014; Raven McCrory, Robert Floden, Joan Ferrini-Mundy, Mark D. Reckase, & Sharon L. Senk, 2012; Whitacre & Nickerson, 2016). This is because interpreting an object can develop the students' way of thinking and minimize their barriers of thinking in solving a problem(Pierson et al., 2014). Some other studies(Bofferding, 2014; Ian Jones & Dave Pratt, 2012; Percival Matthews, Bethany Rittle-Johnson, Katherine McEldoon, & Roger Taylor, 2012; Ramful, 2014; Simon, Kara, Placa, & Sandir, 2016) view objects as forms of operation (addition, subtraction, multiplication, division) and equal sign. For example, equal sign (Jones & Pratt, 2012) can be interpreted as: (1) computation result ($2 + 2 = 4$); (2) identity ($(x - y)(x + y) = x^2 - y^2$); (3) function ($f(x) = \sin x$); and (4) substitution ($x = 1/2$). Students who do not understand objects as forms of operation and equal sign, especially in writing other forms, for example those unable to explain other form of equation $y = 5x (x = \frac{1}{5}y)$ due to the lack of reversible reasoning and reciprocal reasoning (Hackenberg & Lee, 2015; Steffe & Olive, 2009).

Before explaining reversible reasoning in detail, some previous studies conducted studies with the term of reversibility (Ramful, 2014; Ramful & Olive, 2008; Vilkomir & O'Donoghue, 2009). Inhelder & Piaget(1958) characterize reversibility into two forms, namely negation or inverse and compensation or reciprocity. Negation is related to the operational structure which states that each operation has an inverse that cannot be canceled or negated, for example addition is negated by subtraction. Compensation is related to the relationship of relational structures, for example in concluding $a < b$ (the relationship between a and b) then $b > a$ (relationship between a and b) or if $A = B$ then the opposite is $B = A$.

The concept of reversibility was developed by Piaget. It is important for students in solving problems (Adi, 1978; Krutetskii, 1976; Linchevski & Herscovics, 1996; Steffe & Olive, 2009; Tzur, 2004). Suppose that the equation $14 - \frac{15}{7-x} = 9$(Adi, 1978) can be solved by inverse (Number which is subtracted from 14 is 9) or compensation (multiplying both segments by $(7 - x)$). Reversibility is also related to discovering basic form or initial form which is unknown (starting unknown), e.g., in addition and subtraction of $a \pm x = b$, in which a and b are known and x is unknown the form of equation $ax + b = c$ (Linchevski & Herscovics, 1996), and equation $ax = b$ involves fractional knowledge (partitive and iterative) and write other forms of equation (Hackenberg & Lee, 2015; Ramful, 2014; Simon et al., 2016; Steffe & Olive, 2009).

The results of several other studies (Hackenberg & Lee, 2015; Ramful, 2014; Simon et al., 2016; Steffe & Olive, 2009) suggest that reversibility is not only on the object of operation and finds other forms of an equation. Reversibility also can be seen from the objects of function and inverse, derivatives and anti-derivatives, the rules of exponent and the rules of logarithm. Reversibility can be explained based on the structure, e.g. input (I), process (P), and result (R) in the form of equation (Simon, Kara, & Placa, 2016.) Based on the structure, the reversibility in this article examines the form of function composition $f(g(x)) = h(x)$, in which $h(x)$ (result) and $g(x)$ (input) are known, while $f(x)$ (process) is unknown.The reversible reasoning for the function composition can be seen from the direction of the result ($h(x)$) which is known to source ($f(g(x))$) by utilizing the function of inverse or other ways in determining one of unknown functions.

Several studies have examined the conceptions of students in understanding the concept of function(Bagley, Rasmussen, & Zandieh, 2015; Dubinsky & Wilson, 2013; Leron & Paz, 2014;

Proulx, 2015; Tabach & Nachlieli, 2015; Weber & Thompson, 2014). Implicitly, some of these studies talk about reversible reasoning, but it is not the primary purpose. For example, describing the graph of function by involving translations (the graph of $f(x) \rightarrow f(x+k)$ or otherwise)(Proulx, 2015; Weber & Thompson, 2014), searching for the function composition $f(g(x)) = h(x)$(Arnon, Cottrill, & Dubinsky, 2014; Bowling, 2014), searching for the derivative of function composition of $f(g(x))$ with chain rule(Jojo, Maharaj, & Brijlall, 2013; Kontorovich, 2017; Park, 2015). Some researchers explain the function composition based on mental structures and mental mechanisms (Arnon et al., 2014) and the conception of students on the function composition related to the area of circle ($A = \pi r^2$as $f(r)$) and diameter ($d = 2r$ as $g(r)$)(Bowling, 2014). Especially for the function composition, there has been no explanation on reversible reasoning based on structural relationships involving input, process, and result. The researchers added a change of variable (transformation) to unknown functions, e.g. $f(x)$ as basic function and found $f(x+k)$.

The researcher uses reversible reasoning as the mental process of students in reconstructing the source (input, basic form) using the result (output, final form) based on the structural relationship between source ($f(g(x))$) and result ($h(x)$). Therefore, students become the subjects in this research because they have various experience in solving the problems of function composition, so that it can answer the research question: how is the description of the types of students' reversible reasoning on the problem of function composition? The type is based on the Function Composition Task (FCT) and based on theoretical studies, as well as interview with several students in solving the problems of function composition. The type of reversible reasoning on the problem of function composition can be utilized by educators (teachers or lecturers) in the learning process.

Prior Research about Reversibility

Some researchers are inspired by the reversibility concept of Piaget as thinking in the opposite direction or from the results to the source (Krutetskii, 1976; Nathan & Koedinger, 2000) in the process of reversing, one does not need to do the same way from the source to the result. For example $a + x = b$ or $ax = b$ where b is as the known result, for students, it is more difficult to solve the problem. This is also because they only learn the direct process in finding results rather than finding the basic form of the known results.

Several previous studies have discussed reversibility by involving multiplication reasoning, fractional reasoning, and reasoning with unknown quantities in equation $ax = b$(Hackenberg & Lee, 2015; Ramful, 2014; Simon et al., 2016) For example, if "$2/3$ of the total number of cars in the parking lot is 30, how many cars are in the parking lot?" The problem involves part-to-whole to find whole-to-part(Anderson Norton & Jesse L. M. Wilkins, 2012; Hackenberg & Lee, 2015; Ramful, 2014; Simon et al., 2016; Steffe & Olive, 2009; Tzur, 2004).Reversibility can be explained based on the inherent structure in mathematics (operation, transformation, and relations) (Ramful, 2014), such as addition and subtraction, multiplication and division, doubling and halving. In addition, the reciprocal relationship on the quantity can also be regarded as reversibility (Hackenberg, 2013) e.g. if A is $2/3$ of B, then B is $3/2$ from A.

In general, those studies explain reversibility structurally from the source and result with multiplication, fraction, and unknown quantity. This article focuses more on the explanation of Simon et al (2016) about Input (*I*), Process (*P*), and Result(R) on equation form of $I(P) = R$, in addition, some other studies (Hackenberg & Lee, 2015; Ramful, 2014; Ramful & Olive, 2008; Steffe & Olive, 2009) are limited to the form of equation $ax = b$ related to multiplication, fraction, and unknown quantities and suggest the development of reversibility in other aspects, e.g. in function. Based on the form of equation $I(P) = R$, researchers see that there is a relationship with the form of function composition. Researchers suspect that there are 2 other functions (table 1.1) that allow students to do reversible reasoning, but this article only examines the form of $f(g(x)) = h(x)$, with $g(x)$as the input, $f(x)$as the process, and h(x)as the result.

Table 1.1 Composition Form of Function $f \circ g(x) = h(x)$

Form	$f(g(x)) = h(x)$	$f(x) = h(g^{-1}(x))$	$f^{-1}(x) = g(h^{-1}(x))$
Input	[illegible]	[illegible]	[illegible]
Result	[illegible]	[illegible]	[illegible]
Process	[illegible]	[illegible]	[illegible]

Reversible Reasoning Involves Function Composition Problems

Function composition is a combination of the operation of two functions orderly to produce a new function. Some researchers have examined the problem of function composition differently(Arnon et al., 2014; Bagley et al., 2015; Bowling, 2014; Dubinsky & Wilson, 2013; Kontorovich, 2017), but in this article, the researchers develop the research results of Dubinskyby giving 3 types of problems of function composition:

1. F and G are known, determine H so that $H = F \circ G$
2. G and H are known, determine F so that $H = F \circ G$
3. F and H are known, determine G so that $H = F \circ G$

Three types of problems described by Dubinsky are associated with reversible reasoning, type (1) cannot involve reversible reasoning because the input and process are known and also include the problems with direct process (Sangwin & Jones, 2017), while type (2) and (3) involve reversible reasoning because the result is known, and from the results, it can be constructed to obtain input or process. In this article, the researchers chose type (2) (with the form $f(g(x)) = h(x)$) and involve variable change (transformation) in finding the solution (e.g. if it is asked $f(x) \to f(x + k)$)). The first suspect of the researcher for type (2) is that there are 4 types of reversible reasoning done by students in solving that problem, namely: (1) reversible on function composition; (2) reversible on function; (3) reversible on substitution; and (4) reversible of variable. This type of reversible reasoning (table 1.2) is based on the theory and the work pattern of some students by giving the following problem "If it is known that $f(g(x)) = h(x)$, with $h(x) = \frac{x-5}{x+1}$ and $g(x) = 2x - 1$, then determine $f(x + 5)$!"

Table 1.2 Types of Students' *Reversible* Reasoning

Type	Characteristic	Behavior
Reversible on composition	• Identify the meaning of "=" as a structural relationship between input, process and result • Interpret "∘" to have the characteristic of function composition • Identify input, process, and result by involving the inverse of functions on input $(g^{-1}(x))$ • Determine the basic function (f(x)) of the result of composition $h(g^{-1}(x))$	Convert $(f \circ g)(x) = h(x)$ into $f(x) = h \circ g^{-1}(x)$
Reversible on function	• Identify the structural relationship between input, process, and result • Convert the known function composition into $f(g(x)) = h(x)$ • Identify input, process, and result by involving the inverse on input ($f(g(x))$is converted into f(y) with premise of$y = 2x - 1$) • Convert the inverse of f(y) into f(x)as a basic function to determine $f(x + 5)$	Convert $f(2x - 1)$ into f(y) with premise of $y = 2x - 1$
Reversible on substitution	• Identify structural relationship between input, process, and result • Convert the known function composition into $f(g(x)) = h(x)$ • Identify the result based on input. For example, if $f(2x + 1) = x + 1$, then the result is converted into $\frac{2x+2}{2}$. • Identify the similarities between the source and the result. For example, $$f(2x + 1) = \frac{(2x + 1) + 1}{2}$$ • Generalize the relationship of similarities between source and result to determine its basic function (f(x)). For example, $f(2x + 1) = \frac{(2x+1)+1}{2}$ to be $f(x) = \frac{x+1}{2}$	Construct the result based on input $f(2x - 1) = \frac{x + 5}{x + 1}$ into $f(2x - 1) = \frac{(2x - 1) - 9}{(2x - 1) + 3}$
Reversible on variable	• Identify structural relationship between input, process, and result. • Convert the known function composition into $f(g(x)) = h(x)$ • Identify xon input by doing transformation without changing to the basic function (f(x))	Convert variable $f(2x + 1)$ into $f(x + 5)$ with substitution

METHOD

Type of The Study and Subject

The present study employs explorative descriptive approach and seven students participate in the instrument test of the present study. The students are selected since they have learned the concept of function and inverse in intermediate level and higher education, they are able to solve composition function problem in class as well. The subject selection through observation in class and based on Composition Function Problem (FCT) (Appendix 1 and 2) given to assess the type of students' reversible reasoning. According to FCT, there are three students have wrong in solving the composition function problem, while four out of seven students solve the problem in a different way. Three of them have reversible reasoning on function type and 1 student has reversible reasoning on substitution type.

Data Collection

1. Interview

In the present study there are two students being interviewed in order to provide information related to their result of their work in solving FCT. One (1)student with reversible reasoning on function type and one (1) student with reversible reasoning on substitution type. The interview is conducted for 25 minutes and recorded by using mobile phone. During the interview, the researchers follow interview manual previously developed including asking the students to explain the problem, select strategy in solving problem, explanation in finding the solution, and other strategies in solving the problem. The researchers also provide other FCT to perceive subjects' persistent in solving FCT. For example, in an interview the subject was given "if$f \circ g(x) = h(x)$, with $g(x) = 2x + 1$and $h(x) = 3x - 2$, find $f(x + 5)$?"

2. Composition Function Problem (FCT**)**

FCT refers to problem type developed by Dubinskyand the result study byRamful(2014) and Simon et al (2016)related to sources (input and process) and the result of the equation $I(P) = R$, in present studythat equation form is identical to $f\big(g(x)\big) = h(x)$with $h(x)$and $g(x)$is known and find the transformation in $f(x)$ (or$f(x + k)$). The result of FCT which then become a reference in typing the students' reversible reasoning in composition function problem explained above.

Data and Analysis

Analysis is conducted into two stages. First, making initial hypothesis of students' reversible reasoning types theoretically and the result of FCT. Then, selecting the result of students' works (2 students) and conducting interview related to their explanation during solving FCT. The result of the interview are transcribed both gesture and verbal during the interview and scanning the answers. Afterwards, reading the whole transcribed data interview, giving certain notes or idea from the data obtained and data coding for the appearing reversible reasoning types during the interview. Second, the researchers present students' reversible reasoning based on formulated type previously, namely: students' with reversible on function and substitution types.

RESULTS AND DISCUSSIONS

In this section, the researcher presents the findings of the types of students' reversible reasoning on the problem of function composition, i.e. the type of reversible on function (the subject of Ith) and the type of reversible on substitution (subject Zul).

Students' Reversible Reasoning On Function

The type of reversible on function is characterized by: (1) identifying the structural relationship between input, process, and result ($RF1$); (2) changing the known function composition ($RF2$); (3) identifying the information of input, process, and result by involving inverse on input ($RF3$); and (4) converting the result of inverse $f(y)$ to $f(x)$ as the basic function ($RF4$). In this article, 3 students are identified to have the type of reversible reasoning on function based on FCT work result. Subsequently, the subject (Ith) was selected to provide information about FCT work result. He explained while working on FCT by identifying $f(x), g(x),$ and $h(x)$ $RF1$, and converting the function composition $(f \circ g)(x) = h(x)$ to $f(g(x)) = \frac{x-5}{x+1}$ $(RF2)$.

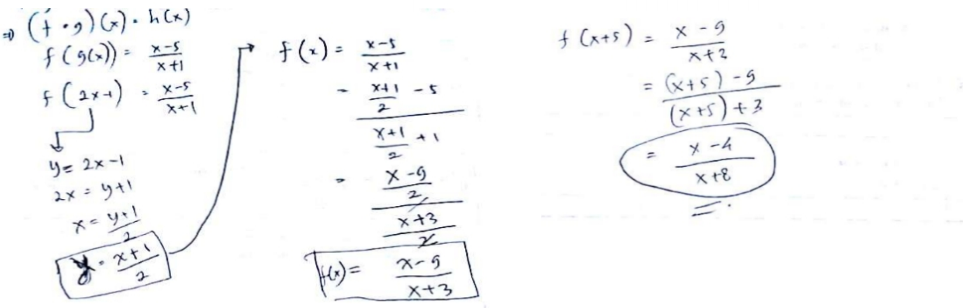

Figure 4.1 Result CFT Ith

Furthermore, in an interview with Ith, she determines the basic function $f(x)$ first and then function $g(x)$ is converted by premising $y = 2x + 1$ on $g(x)$ $(RF3)$.

Int	Why do you determine f(x) first if you want to find f(x + 5)?	
Ith	⋯ Because here, in the question, it is known that (f ∘ g)(x) = h(x)and one of the functions which is known isg(x) which is 2x − 1. So, to get function f(x + 5) we need to find function f(x) which is known in the question first, from function f(x) we can get function f(x + 5)	Determining basic function (f(x))
Int	What is the meaning of inversing g(x) first?	
Ith	⋯ g(x) is converted because we want to find the function of f(x) itself. In which 2x − 1 is not in the form of x yet, it is still in g(x), to find it, we premise y = 2x − 1, so that the result of inverse is substituted to h(x) , substitute the value of x	Involving inverse on input (RF3)
Int	After $x = \frac{x+1}{2}$ is obtained, then it is converted to $y = \frac{x+1}{2}$, why	

Ith	It is known that $(f \circ g)(x) = h(x)$, with $h(x) = \frac{x-5}{x+1}$ and $g(x) = 2x - 1$, then determine function $f(x+5)$, to solve function $(f \circ g)(x) = h(x)$ the form is converted into $f(g(x)) = \frac{x-5}{x+1}$, after that $g(x)$ is converted *to* $2x - 1$ so that $f(2x-1) = \frac{x-5}{x+1}$. To get $f(x+5)$ then at first, find function $f(x)$ by inversing the value of function $g(x)$. is it so?	Identifying input, process, and result (RF1) and converting the known function composition (RF2)
Ith	The value of $x = \frac{y+1}{2}$ is converted to the original form to its function x. therefore, the value of y premised is substituted by x because in the equation, there is variable x which will be changed with the result of inverse of $g(x)$	Converting $y = g^{-1}(x)$

Ith explains another way of solving the problem by changing the form of composition, but there is a mistake in changing $(f \circ g)(x) = h(x)$ it by writing $(f \circ g)(x) \circ g^{-1}(x) = h(x) \times g^{-1}(x)$ and interpreting that composition ($\circ$) is not commutative. She implicitly interprets that function which is composed by its inverse generates identity, $(g \circ g^{-1}(x) = x)$. Therefore, by revealing that way, Ith also has the type of reversible reasoning on composition.

Int	Is there another way?	
Ith	Yes, but the pattern is different	Determining another way
Int	How is the pattern?	
Ith	How, like this, $(f \circ g)(x) = h(x)$ is converted into $(f \circ g)(x) \circ g^{-1}(x) = h(x) \times g^{-1}(x)$ so that the result of $g(x)$ and $g^{-1}(x)$ will be used up, so that the remainder is $f(x) = \frac{x+5}{x-1}$ with the value of x is substituted with the value of $g^{-1}(x)$	Interpret that "$\circ$" has the character of function composition (not commutative) and involving inverse on input ($g^{-1}(x)$)

The type of reversible on function is more dominantly used by Ith in solving the problem based on his experience in solving the problem of function composition. She revealed that the way she did (with premise $y = 2x - 1$) was more systematic rather than using the composition of the function ($g^{-1}(x)$). She also implicitly revealed that composition characteristic is not widely used in solving the problem of function composition. However, she understood that composition had no commutative ($(f \circ g)(x) \neq (g \circ f)(x)$) and saw operation of composition between source and result required an inverse function ($((f \circ g) \circ g^{-1})(x) = (h \circ g^{-1})(x)$).

Students' reversible reasoning on substitution

The type of reversible on substitution is indicated by: (1) Identifying the structural relationship

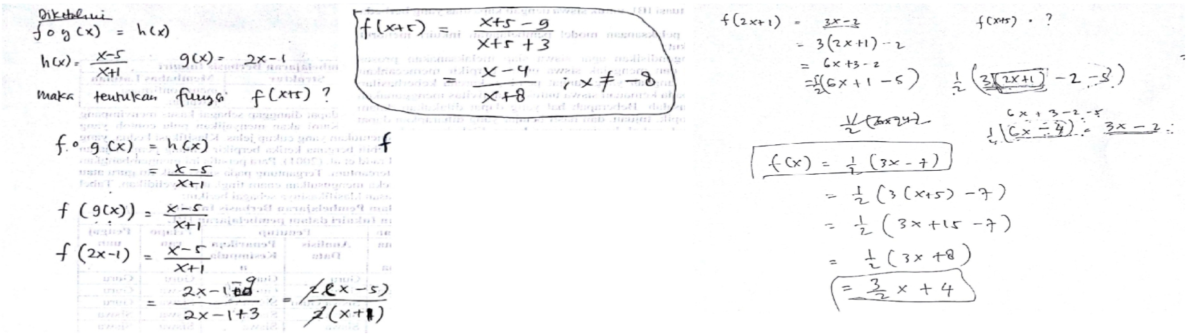

Figure 4.2 Result CFT Zul

between input, process, and result ($RS1$); (2) Converting the known function composition into $f(g(x)) = h(x)$ ($RS2$); (3) Identifying result based on input ($RS3$); (4) Identifying the similarity between source and result ($RS4$); and (5) Generalizing the similarity of functions to determine its basic function ($f(x)$) ($RS5$). From the work result of FCT with 7 subjects, there is only 1 subject (Zul) that has that type. Similar to that have been done by Itriah[8], Zul identified the structural relationship between input, process, and result ($RS1$), and converted the function composition into $f(g(x)) = h(x)$ ($RS2$).

Zul	This is the one being questioned right? (pointing $f(x+5)$), it is known that $h(x)$ (pointing $h(x) = \frac{x-5}{x+1}$) and $g(x)$ (pointing $g(x) = 2x-1$), the next is $(f \circ g)(x) = \frac{x-5}{x+1}$, ini $(f \circ g)(x) = f(g(x))$ after that $h(x)$ is substituted	Identifying input, process, and result (RS1) and converting function composition which is known (RS2)

Through the interview, Zul explained how FCT works by linking the relationship between $f(g(x))$ and $h(x)$. Zul began by converting the result (($h(x)$) into $\frac{2x-1-9}{2x-1+3}$ which is adjusted to $f(2x-1)$). He identifies the similarity between the source and the result, in the source there is $f(2x-1)$ and in the result there is $h(x) = \frac{2x-1-9}{2x-1+3}$, he operationally proved that $\frac{2x-1-9}{2x-1+3}$ equals $\frac{x-5}{x+1}$ is true. Then, through the generalization of the similarity of form $f(2x-1) = \frac{2x-1-9}{2x-1+3}$, he concluded that $f(x) = \frac{x-9}{x+3}$.

Int	How to solve that question?	
Zul	⋯ because $f \circ g(x) = f(g(x))$, automatically this is equal to this (pointing $h(x) = \frac{x-5}{x+1}$ then into $f(g(x)) = \frac{x-5}{x+1}$), $g(x)$ is known	Identifying result based on

> and substituted here (pointing f(g(x))). Oh it appears that f(2x − 1) equals $\frac{x-5}{x+1}$. The one being questioned is f(x + 5), to make it easier, find the function f(x) first. I get $\frac{2x-1-9}{2x-1+3}$ because it is equal to this (pointing h(x)) or $\frac{2x-10}{2x+2}$ is equal to $\frac{x-5}{x+1}$). It will be converted in f(x), this (pointing 2x − 1) is converted into x, oh it appears that in this f(x) the x is this (pointing 2x − 1 subtracted by 9) subtracted by 9 this is the x (pointing 2x − 1 is subtracted by 3) subtracted by 3. Therefore, f(x) = $\frac{x-9}{x+3}$. The one being questioned is f(x + 5), so it is substituted to x + 5 (RS3)

The researchers provided additional question to look at the consistency of Zul in solving function composition problems, as researchers suspected from the previous work result of FCT that the constructed results has variable coefficient of 1 or $h(x) = \frac{x-5}{x+1}$. So, it was easily solved by Zul. Then the problem given is "if $g(x) = 2x + 1$ and $h(x) = 3x - 2$, with $(f \circ g)(x) = h(x)$, then determine $f(x + 5)$?". He was working on the problem for 20 minutes, he did the same way with how he had done in FCT before. He wrote down $f((2x + 1) = 3x - 2$ and started from the result $3x - 2$. Zul converted the result into $3(2x + 1) - 2$ by using the distributive property of multiplication on subtraction by writing $6x + 3 - 2$, then it was subtracted by 5 in order to obtain $6x - 4$, but it revealed the error in solving the problem as it produced $f(x) = x$.

Zul concluded that the way in solving the problem in FCT previously could not be applied to that problem (because $h(x) = 3x - 2$, with x variable is not constant), but he was still trying to find $f(x)$ from that function composition. Zul revealed that in the result ($h(x) = 3x - 2$), there must be $2x + 1$ in that function ($RS3$). Therefore, he concluded that the solution of that problem involves fractions. He wrote $\frac{1}{2}(6x + 1 - 5)$ then he linked the input and the result in $6x + 1 - 5$, involved $2x + 1$ and obtained $\frac{1}{2}(3(2x + 1) - 2 - 5)$. Zul concluded that there is similarity between source and result ($RS4$), $f(2x + 1) = \frac{1}{2}(3(2x + 1) - 7)$, thus $f(x) = \frac{1}{2}(3x - 7)$. Next, with he substituted $f(x)$ into $f(x + 5) = \frac{1}{2}(3(x + 5) - 7)$.

The researchers also explored another way that Zul understood in solving the problem. However, he did not know other ways to solve the problem. He simply identified the result ($h(x)$), constructed it to obtain process ($f(x)$) based on input ($g(x)$), then identified the similarity between the source and the result. Thus, the researchers concluded that Zul only focused on the type of reversible on substitution on the problem of function composition

Discussion

The results show that the students' reversible reasoning can be investigated on the function composition even though the object of the study is different from the previous studies(Hackenberg & Lee, 2015; Ramful, 2015; Simon et al., 2016; Steffe & Olive, 2009). Reversible reasoning on the problem of function composition $(f \circ g)(x) = h(x)$ is seen based on the structural relationship between source and outcome (Ramful & Olive, 2008; Simon et al., 2016). If the source (input and process) is known to obtain results, it cannot express reversible reasoning because it is a direct process (Sangwin & Jones, 2017), but moving from the opposite direction can reveal reversible reasoning (Sangwin & Jones, 2017; Steffe & Olive, 2009). For example, in function composition $(f \circ g)(x) = h(x)$, if $f(x)$ and $g(x)$ are known, it includes direct process, while if $h(x)$ and $g(x)$ are known, it includes inverse process in determining $f(x)$.

In contrast to the previous research instruments (Hackenberg & Lee, 2015; Ramful, 2014; Simon et al., 2016) by giving two problems, namely problems requiring direct and inverse process, e.g. giving major problems and double problems(Ramful, 2014), problems of fraction and equation(Hackenberg & Lee, 2015), also joint and separate terms (Simon et al., 2016). Researchers only provided problems with the inverse process on the function composition. Dubinsky describes functions composition through three types of problems, the type of problem with direct process and the type of problem with inverse process of a function ($f^{-1}(x)$or $g^{-1}(x)$). However, in this study, the researchers took one part of the type of problem developed by Dubinsky (1991) and added a change of variable (transformation) from basic function (e.g. $f(x)$) to the function being questioned ($f(x + k)$).

Inverse process can also be said to be a reverse process from the direction of the result to the source(Vilkomir & O'Donoghue, 2009). Thus, reversible reasoning for function composition problems occurs when students perform inverse process involving the inverse function from the result to the source. Preliminary findings obtained based on theoretical and instrument test, the researchers found four types of reversible reasoning for subjects of student: (1) reversible on composition; (2) reversible on function; (3) reversible on substitution; and (4) reversible on variable. Those types refer to the concept of reversibility developed by Piaget (Inhelder & Piaget, 1958) on negation and compensation and (Ramful, 2014)describing the situation of reversibility based on operation, relationship and transformation.

Based on the results of the research, there are two types of reversible reasoning found, namely: the type of reversible on function (Ith) and reversible on substitution (Zul). The reversible function is characterized by the behavior of the subjects using premise ($f(y)$) in determining the inverse of a function, while reversible on substitution is characterized by the behavior of subjects constructing the result based on its input by identifying the structural similarity and generalizing from the relationship of the similarity. If associated with the situation of reversibility based on research conducted by(Ramful, 2014), the type of reversible on function is more dominant over operations because it involves reversal operations orderly $y = 2x - 1$ into $x = \frac{y+1}{2}$). While the type of reversible on substitution is more dominant in relation and transformation because it involves the structural relationship between the source and the result ($f(2x-1) = \frac{x-5}{x+1}$), the transformation is seen from the construct to the result ($\frac{2x-10}{2x+2} = h(x)$), and generalizing in determining the basic functions based on the similarity of structure ($f(2x-1) = \frac{(2x-1)-9}{(2x-1)+3}$into $f(x) = \frac{x-9}{x+3}$).

The explanation of the two subjects (Ith and Zul) on other ways of solving the problem of function composition is influenced by their previous experiences in the learning process. In addition to having a type of reversible on function, Ith also has a type of reversible on composition. She understands the nature of function composition, namely: not commutative and functions composed with the inverse generate identity (implicitly, $(f \circ g \circ g^{-1})(x) = (h \circ g^{-1})(x)$), but she prefers the way of premise based on structural relationship ($f(2x-1) = \frac{x-5}{x+1}$, to find$f(x)$, it must be converted with$y = 2x - 1$). While Zul, more focused on starting the construction in the results and using structural relationship in finding basic function. He only knew that there is a way of premise ($f(y)$) in solving the problem. Here is the picture of the reversible reasoning type of both subjects:

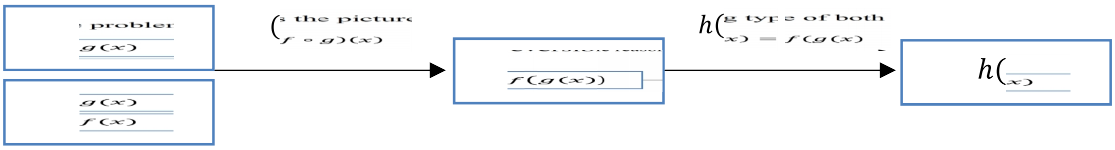

Figure 3.1 Direct processess on composition of function

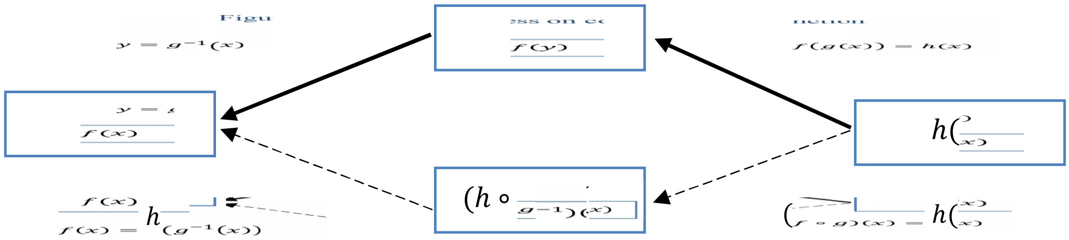

Figure 3.2 reversible on function (Ith)

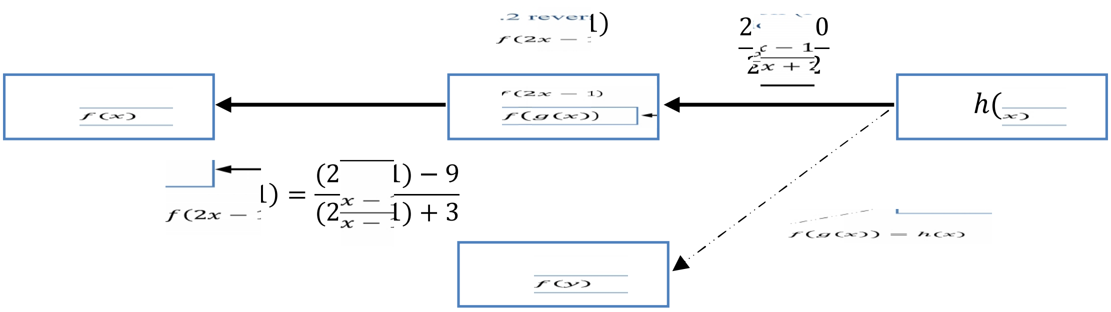

Figure 3.3 Reversible on subtitution (Zul)

Based on figure 3.2 and 3.3, further research of reversible reasoning can be studied based on the reverse direction of the type of reversible reasoning owned. For example, for Ith, it can be claimed as reversible reasoning with divergent conditions because it has two types of reversible reasoning although they are weak (assume that $f(x) = (h \circ g^{-1})(x)$ is rarely used way), while Zul is as reversible reasoning with convergent conditions because he focused on the type of reversible on substitution. The role of educators (teachers and lecturers) in the learning process affects the experience of both subjects, so they have different types of reversible reasoning. In general, in learning process, teaching staff encourages students to do reversible reasoning for the type of reversible on function or by converting the functions (Bagley et al., 2015; Bowling, 2014; Kontorovich, 2017; Tabach & Nachlieli, 2015), but they do not provide other ways to solve the problem of function composition.

CONCLUSION

The present study proceeds the suggestions of the former studies related to development of reversible reasoning is not only occurred on addition and substraction, multiplication and division squaring and factoring(Hackenberg & Lee, 2015; Ramful, 2014; Simon et al., 2016; Steffe & Olive, 2009). Reversible reasoning can also be viewed by the structural correlation among input, process, and result with the following form $I[P] = R$. Based on the structural correlation above, the researchers perceives the relationship between $I[P] = R$ and $f(g(x)) = h(x)$. Therefore, the function composition is become object of the study to describe students' reversible reasoning.

The study formulates four types of reversible reasoning on function composition problem, reversible on function, reversible on substitution, and reversible on variables. There are seven students participate in the present study. It is found out that there are two reversible reasoning types namely: reversible on function and reversible on substitution. The researchers assume that there are only two types of reasoning which appear since there are only few students participate in the study. So that, the advanced study will be conducted with more subjects to obtain four types of reversible reasoning.

Reversible on function type (Ith) is marked by the using of permissibility $f(y)$ by the subjectin determining inverse from a function, while reversible on substitution (Zul) is marked by the behavior of the subject that construct result based on the input by identifying structural similarity and generalizing from those structural similarity. However, the researchers is limited on mental structure and mechanism of students' mental in both types (Arnon et al., 2014). Furthermore, for the development of advanced study, every subject does not only have one reverse process on function composition problem, for instance, Itria$_8$beside having reversible on function type also having reversible on composition type, while Zulonly focus on reversible on composition type. Therefore, it is required to review the reversible process on divergent and convergent conditions which is related to reversible reasoning types.

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