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On some coarse geometric notions inspired by topology and category theory

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To Debbie and Jane

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Abstract

Coarse geometry is the study of the large scale properties of spaces. The interest in large scale properties is mainly motivated by applications to geometric group theory and index theory, as well as to important open problems such as the Novikov Conjecture. In this thesis, we introduce and study coarse versions of the following classical topological notions: connectedness, monotone-light factorizations, extension theorems, and quotients by properly discontinuous group actions. We will draw on the analogy between large scale geometry and topology as well as on the perspective of category theory using Roe's coarse category. In the first of four research chapters, we look at a large scale connectedness condition arising from the coarse category and show that it coincides with the topological connectedness of the Higson corona. In the second, we introduce coarse versions of monotone and light maps (calling them coarsely monotone and coarsely light maps respectively) and show that these maps constitute a factorization system on the coarse category. We also show that coarsely light maps preserve some important large scale properties. In the third research chapter, we unify the proof of three extension theorems: the classical Tietze Extension Theorem from topology, Katetov's extension theorem for uniform spaces, and an extension theorem for slowly oscillating functions (an important class of functions in coarse geometry). The unification is achieved via a general extension theorem for neighbourhood operators. In the final research chapter, we study warped spaces associated to group actions on metric spaces, focussing in particular on coarsely discontinuous actions which we introduce as large scale analogues of properly discontinuous actions in topology. For such actions, we relate the (maximal) Roe algebra of the warped space with the crossed product of the (maximal) Roe algebra of the original space and the group, and prove a deck transformation result.

Table of Contents

1 Introduction **1**

1.1 When are two spaces the same? 1

1.2 A brief survey of some coarse invariants 7

1.2.1 The Roe algebra 7

1.2.2 Dimension-type invariants 10

1.2.3 Amenability-type invariants 11

1.3 Category theory 13

1.4 Abstract contexts for coarse geometry 14

1.5 Themes of the thesis and guide to the text 15

2 A coarse version of connectedness **17**

2.1 Introduction 17

2.2 Preliminaries 20

2.3 Coarse coproducts 21

2.4 Connectedness of the Higson corona 24

2.5 ω-Excisive decompositions and coarse pushouts 28

2.6 Cohomological characterisation 31

2.7 Geodesic spaces and finitely generated groups 33

2.8 Abstract coarse spaces 36

3 A coarse Tietze Extension theorem **41**

3.1 Introduction 41

3.2 Hybrid large scale spaces 43

3.3 Proper hls-spaces 47
3.4 Neighbourhood operators 49
3.5 Hybrid large scale Urysohn's Lemma 54
3.6 Hls-normal spaces 56
3.7 Non-normal spaces 58
3.8 The Tietze Extension Theorem 60
3.9 The Higson compactification and corona 65
3.10 Hybrid structures induced by compactifications 67
3.11 Topological groups as hls-spaces 70
3.12 Coarse neighbourhoods and ls-structures 73

4 A coarse version of monotone-light factorizations **77**
4.1 Introduction 77
4.2 Preliminaries 79
4.3 Coarsely light maps 81
4.4 Coarsely monotone maps and monotone-light factorizations 85
4.5 Pullbacks in the coarse category 88
4.6 Coarsely monotone maps arising from a reflection 90
4.7 Pullback-stability of coarsely monotone maps 96
4.8 Maps extended to the Higson corona 98
4.9 Asymptotic dimension and exactness 103
4.10 Groups 105

5 A coarse version of quotients by group actions **109**
5.1 Introduction 109
5.2 Preliminaries 112
5.2.1 Large scale spaces 112
5.2.2 Large scale continuous maps 115
5.2.3 The coarse category 116
5.3 Group actions and X_G 117
5.4 Coarsely discontinuous actions 123

5.5 The maximal Roe algebra 128
5.6 Weak coarse quotient maps 140
5.7 Metrization of quotient large scale structures 146

6 Conclusion **151**
6.1 Concluding remarks 151
6.2 Future avenues for research 151
6.2.1 Coarse covering spaces 152
6.2.2 Neighbourhood operators in coarse geometry 152
6.2.3 Warped spaces and warped cones 152

Bibliography **154**

Vita **162**

Chapter 1

Introduction

1.1 When are two spaces the same?

The main object of study in this thesis will be that of a **metric space**. A metric space consists of a set X and a "distance" function d from pairs of points in X to the real numbers satisfying the following conditions

- $d(x, y) \geq 0$ for all $x, y \in X$,
- $d(x, y) = d(y, x)$ for all $x, y \in X$,
- $d(x, y) = 0 \Leftrightarrow x = y$,
- $d(x, y) + d(y, z) \leq d(x, z)$ for all $x, y, z \in X$.

Metric spaces appear in many places in mathematics. For example, $\mathbb{R}^n$ can be equipped with a variety of different metrics – the most common probably being the Euclidean distance function

$$d((x_1, x_2, \ldots),\ (y_1, y_2, \ldots)) = \sqrt{\sum_i (x_i - y_i)^2}.$$

which we will assume in this thesis unless stated otherwise. Here are some of the main classes of metric spaces of interest in this thesis.

Example 1. *Let G be a finitely generated group and S a finite generating set for G. We will assume that S is symmetric (i.e. if $x \in S$ then $x^{-1} \in S$). Define a metric d on the* *elements of G as follows: $d(g,h)$ is the least number of elements of S required to write $g^{-1}h$ as a product of elements of S (where we identify the identity element of G with an empty product, so that $d(x,x) = 0$). This is called a* ***word-length metric*** *and it depends on the choice of S.*

Example 2. *Let M be a smooth Riemannian manifold of dimension n. Roughly speaking, this is a space which looks locally like $\mathbb{R}^n$ and for which a length has been assigned to every tangent vector at every point in a smooth way. For a smooth curve $\gamma : [0,1] \to M$, one can define its length by*

$$L(\gamma) = \int_0^1 ||\gamma'(t)||dt.$$

We can then define the distance $d(x,y)$ to be the infimum

$$\inf\{L(\gamma) \mid \gamma(0) = x,\ \gamma(1) = y\}$$

and this defines a metric.

We should warn the reader already that we will often refer to a metric space simply by a single letter X and use d as the distance function for all metric spaces in a particular statement. Here is an obvious but important construction for putting metrics on spaces.

Example 3. *Let (X, d_X) be a metric space and $A \subseteq X$ a subset of X. Then there is a natural subspace metric on A, given by $d_A(x,y) = d_X(x,y)$.*

Metric spaces appear outside pure mathematics as well. For example, in many applied sciences data is often represented as a point cloud in some high dimensional space, with a natural metric inherited from that higher dimensional space, or one may want to consider vectors of features with a metric capturing the similarity between them.

A key question when dealing with metric spaces is to ask when two metric spaces should be considered "the same". A very strong condition for being the same is being isometric. Two metric spaces (X, d_X) and (Y, d_Y) are (globally) **isometric** if there is a bijection f from X to Y such that for all $x, x' \in X$,

$$d_X(x,x') = d_Y(f(x), f(x')).$$

Such a map f is called a (global) **isometry**. For example, the x-axis and the y-axis in $\mathbb{R}^2$ are isometric, where each is equipped with the subspace metric coming from $\mathbb{R}^2$, with the isometry given by $(x, 0) \mapsto (0, x)$.

Being isometric is a very strong condition, and many fields of mathematics prefer to use a weaker notion of similarity. In topology, a common condition is that of homeomorphism. A map $f : (X, d_X) \to (Y, d_Y)$ between metric spaces is called **continuous** if for all $x \in X$ and $\varepsilon > 0$ there is a $\delta > 0$ such that

$$d_X(x, x') < \delta \implies d_Y(f(x), f(x')) < \varepsilon$$

Two metric spaces (X, d_X) and (Y, d_Y) are **homeomorphic** if there is a bijection $f : X \to Y$ which is continuous and which has continuous inverse $f^{-1} : Y \to X$. Such a map f is called a **homeomorphism**. It is easy to check that any isometry is a homeomorphism. On the other hand, the map $f(x) = \tan(x)$ is a homeomorphism from $(-\pi/2, \pi/2)$ to $\mathbb{R}$ which is not an isometry. Indeed, $(-\pi/2, \pi/2)$ and $\mathbb{R}$ cannot be isometric since any two points in the former are at most π apart, while the latter has unbounded distances between points.

Homeomorphism is a good notion for studying "smooth" spaces like manifolds, but the group example above (Example 1) illustrates that it is not useful for every space. Indeed, if G is any infinite finitely generated group given the metric in Example 1, then it is homeomorphic to the integers with their usual distance metric $d(a, b) = |a - b|$. Indeed, G must be countable and we can pick a bijection between the elements of G and $\mathbb{Z}$. One easily checks that this bijection is automatically continuous and has continuous inverse (simply choose $\delta = 1/2$ for any ε). What is behind this fact is that both G and $\mathbb{Z}$ have the discrete topology under their respective metrics.

The fact that the finite diameter space $(-\pi/2, \pi/2)$ is homeomorphic to the infinite diameter space $\mathbb{R}$ should convince us that homeomorphisms do not care that much about the large scale structure of a space, i.e. the information contained in large distances. To put it even more starkly, one may observe that given any metric space (X, d_X), the metric d' defined by

$$d'(x, y) = \mathsf{min}(d(x, y), 1)$$

makes the identity set map a homeomorphism from (X, d_X) to (X, d'). Notice how d' "ignores large distances". There is another notion of similarity of spaces which cares *only* about the large scale structure of a space, and this notion turns out to be very useful when studying spaces like groups with word metrics, whose topology is trivial. Here are two fundamental definitions for this thesis, which together will help us formulate this notion.

Definition 1. *Let (X, d_X) and (Y, d_Y) be metric spaces. A set map $f : X \to Y$ is called* ***bornologous*** *(or* ***uniformly expansive*** *or* ***large scale continuous****) if for every $R > 0$ there is an $S > 0$ such that*

$$d_X(x, x') \leq R \implies d_Y(f(x), f(x')) \leq S.$$

Definition 2. *Let (X, d_X) and (Y, d_Y) be metric spaces. Two set maps $f, g : X \to Y$ are called* ***close*** *if there is an $R > 0$ such that for all $x \in X$.*

$$d_Y(f(x), g(x)) \leq R.$$

At large scale, we should think of close maps as being more or less the same. Our notion of being the same at large scale is thus going to be based on a bornologous map which is invertible up to closeness.

Definition 3. *Let (X, d_X) and (Y, d_Y) be metric spaces. Then X and Y are* ***coarsely equivalent*** *(or* ***large scale equivalent****) if there exist bornologous maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f$ and $f \circ g$ are close to the identity maps on X and Y respectively.*

For the sake of completeness, we prove a classical example of a coarse equivalence. The idea is that $\mathbb{Z}$ is coarsely equivalent to $\mathbb{R}$ because as you "zoom out" they resemble one another.

Proposition 1. *$\mathbb{Z}$ is coarsely equivalent to $\mathbb{R}$ (with the usual metrics).*

Proof. We begin by defining our maps. Let $f : \mathbb{Z} \to \mathbb{R}$ be the inclusion, and let $g : \mathbb{R} \to \mathbb{Z}$ be the floor map, i.e. $g(x) = \max\{r \in \mathbb{Z} \mid r \leq x\}$. We have

$$|f(x) - f(y)| = |x - y|$$

and

$$|g(x) - g(y)| \leq |x - y| + 2$$

which shows that f and g are bornologous. Finally, $g \circ f$ is the identity, and for $x \in \mathbb{R}$

$$|f(g(x)) - x| \leq 1.$$

which shows that $f \circ g$ is close to the identity. □

Note that if two spaces are isometric, then they are coarsely equivalent. Thus coarse equivalence is a weaker condition than isometry which captures a very different similarity between metric spaces. Coarse geometry, the area in which the results of this thesis naturally lie, can be defined as the *study of those properties of metric spaces which are invariant under coarse equivalence.* More informally, it is the study of the large scale properties of spaces. Before proceeding on, we should briefly mention another very useful and well-known characterisation of coarse equivalences.

Definition 4. *Let $f : X \to Y$ be a map between metric spaces. The map f is called a* ***coarse embedding*** *if for all $S > 0$, there is an $R > 0$ such that for all $x, x' \in X$,*

$$d(f(x), f(x')) \leq S \implies d(x, x') \leq R.$$

The map f is called ***coarsely surjective*** *if there is some $C > 0$ such that for all $y \in Y$, $d(f(x), y) \leq C$ for some $x \in X$.*

Proposition 2. *Let $f : X \to Y$ be a bornologous map between metric spaces. Then f is a coarse equivalence if and only if f is a coarsely surjective coarse embedding.*

Proof. ($\Rightarrow$) Let $g : Y \to X$ be a bornologous map such that $d(f(g(y)), y) \leq C$ and $d(g(f(x)), x) \leq C$ for all $x \in X$, $y \in Y$. The former inequality already shows that f is coarsely surjective. Suppose now that $d(f(x), f(x')) \leq S$. There is an $R > 0$ depending only on S so that $d(g(f(x)), g(f(x'))) \leq R$ because g is bornologous. Thus

$$d(x, x') \leq d(x, g(f(x))) + d(g(f(x)), g(f(x')) + d(g(f(x')), x') \leq 2C + R.$$

which shows that f is a coarse embedding.

($\Leftarrow$) For every $y \in Y$, there is an $x \in X$ such that $d(f(x), y) \leq C$. For each y, choose such an x and define $g(y) = x$. Clearly $f \circ g$ is close to the identity. We claim that g is bornologous. If $d(y, y') \leq R$ for $y, y' \in Y$, then

$$d(f(g(y)), f(g(y'))) \leq d(y, f(g(y))) + d(y, y') + d(y', f(g(y'))) \leq R + 2C,$$

so since f is a coarse embedding, $d(g(y), g(y')) \leq S$ for some S depending on R and C. Finally, if $x \in X$, then since $d(f(x), f(g(f(x)))) \leq C$, we have that $d(x, g(f(x))) \leq D$ for some D depending on C since f is a coarse embedding. □

Having looked at a number of different senses in which metric spaces can be the same, a natural question is to ask what techniques exist for determining when two metric spaces are the same in a given sense. For example, the circle S^1 is not homeomorphic to the real line, but how can we prove this? A common technique in topology and related areas is to use invariants. The fundamental group π_1 is a classical example of a topological invariant – that is, if X and Y are homeomorphic, then $\pi_1(X)$ is isomorphic to $\pi_1(Y)$ in a natural way. In particular, it can be shown that $\pi_1(S^1)$ is isomorphic to the group of integers, while $\pi_1(\mathbb{R})$ is trivial, so that we can be sure that S^1 is not homeomorphic to $\mathbb{R}$.

As the above example illustrates, topological invariants can be readily used to tell when two spaces are not homeomorphic. The reverse question: whether two spaces are the same given isomorphic topological invariants, is usually very hard and often the subject of long-standing open questions. For example, the Borel Conjecture posits that any two aspherical topological manifolds are homeomorphic if their fundamental groups are isomorphic; it is still open over half a century after it was stated though it has been shown in some cases [27]. The Borel Conjecture is an example of a so-called rigidity conjecture – an assertion that some weaker form of "sameness" implies a stronger form.

At the heart of coarse geometry is the study of **coarse invariants** - objects or properties associated to spaces which are invariant under coarse equivalence. In the next section we will briefly survey some of these invariants in order to give context for the research chapters that follow while also giving some idea of applications of coarse geometry to other fields.

1.2 A brief survey of some coarse invariants

1.2.1 The Roe algebra

In this thesis, the Roe algebra will feature only in the final research chapter, where its maximal counterpart (introduced in [31]) will also make an appearance. Nonetheless, it is an important object of study in applications of coarse geometry, and the fact that it is a coarse invariant illustrates how coarse geometry is motivated by questions from manifold topology.

The Roe algebra was introduced by John Roe in his work on index theory for non-compact Riemannian manifolds. This work represents one of the main motivations for the study of abstract coarse geometric concepts. Roughly speaking, one is interested in studying smooth Riemannian manifolds via (elliptic) differential operators on them and their associated indices. For the case of compact manifolds, these indices are computed by the celebrated Atiyah-Singer Index Theorem, but to move beyond the compact case (as we should expect if we are to reach large scale theory, since all compact manifolds have finite diameter), one needs a more general notion of index based on something called the Roe algebra. For details, we refer the reader to [56].

We will define the Roe algebra here for certain kinds of metric spaces – this is no great restriction as many spaces of interest turn out to be coarsely equivalent to a space of this kind. Firstly, our spaces will be discrete, so that they are topologically trivial. They will also have bounded geometry. A discrete space X is said to have **bounded geometry** if for every $R > 0$ there is an $N > 0$ such that for every $x \in X$, the ball $B_R(x)$ of radius R around x has at most N points in it. Let H be a separable infinite dimensional Hilbert space, for example $\ell^2(\mathbb{N})$, the set of all sequences $(a_i)_{i \in \mathbb{N}}$ of complex numbers such that $\sum |a_i|^2 < \infty$. We let $\ell^2(X, H)$ be the set of all functions f from X to H such that

$$\sum_{x \in X} ||f(x)||^2 < \infty$$

If we have a linear transformation H from $\ell^2(X,H)$ to $\ell^2(X,H)$, we can define an X-by-X matrix T whose entries are linear transformations from H to H as follows:

$$T_{x,y}(h) = H(e_{y,h})(x)$$

where $e_{y,h}$ is the function which sends y to h and every other point to 0. This gives a bijection between the set of bounded operators on $\ell^2(X,H)$ and a subset of X-by-X matrices with entries in $B(H)$, the algebra of bounded operators from H to H. Recall that an operator S is **bounded** if

$$\sup\{||S(v)|| \mid ||v|| = 1\} < \infty.$$

Denote by $C[X]$ the algebra of all bounded operators T on $\ell^2(X,H)$ such that, when T is written as a matrix $(T_{x,y})_{x,y\in X}$ of operators,

- $T_{x,y}$ is a compact operator for all x and y (that is, T is **locally compact**), and
- there exists an $R > 0$ such that for all $x, y \in X$ with $d(x,y) > R$, $T_{x,y} = 0$.

The second condition is referred to as **finite propagation**, and it is where the large scale structure of X is finally coming into play. Associated to bounded operators on a Hilbert space one always has the **operator norm** $\sup\{||S(v)|| \mid ||v|| = 1\} < \infty$ and the corresponding metric $d(S_1, S_2) = ||S_1 - S_2||$. The **Roe algebra** $C^*(X)$ of X is the closure of $C[X]$ in $B(\ell^2(X,H))$ under this norm.

Proposition 3. *The Roe algebra is a coarse invariant.*

Proof. This proof will be based on the proof of Lemma 2 in [36]. Let X and Y be discrete metric spaces of bounded geometry, and let $f : X \to Y$ be a coarse equivalence. We will assume that f is surjective. For every $y \in Y$, the inverse image $f^{-1}(y)$ is a finite non-empty set since it is bounded and X has bounded geometry. For every y, we can find an isometry V_y from $\ell^2(\{y\},H) \cong H$ to $\ell^2(f^{-1}(y),H) \cong \oplus_{x\in f^{-1}(y)}\ell^2(\{x\},H) \subseteq \ell^2(X,H)$. This is because H is isometric to any finite direct sum of copies of H. Putting these isometries together gives an isometry V from $\ell^2(Y,H)$ to $\ell^2(X,H)$. We now define a map $\mathsf{Ad}V$ from bounded operators on $\ell^2(X, H)$ to bounded operators on $\ell^2(Y, H)$ via

$$\mathsf{Ad}V(T) = V^*TV.$$

It is straightforward to check that $\mathsf{Ad}V$ is an isometric isomorphism with inverse $\mathsf{Ad}V^*$ given by $T \mapsto VTV^*$. If $T \in C[X]$ with propagation R (i.e. $T_{x,y} = 0$ for $d(x, y) > R$), then $\mathsf{Ad}V(T)$ has propagation given by

$$\max\{d(f(x), f(x')) \mid d(x, x') \leq R\}$$

which is bounded by some S depending only on R because f is bornologous. Similarly, $\mathsf{Ad}V^*$ maps $C[Y]$ to $C[X]$ because f is a coarse embedding. Thus $C[X]$ and $C[Y]$ are isometrically isomorphic, and so $C^*(X)$ and $C^*(Y)$ are too.

The only remaining obstacle is to remove the assumption that f be surjective. Note that f factorizes as two maps: the surjective map $f' : X \to \mathsf{Im}(f)$ and the inclusion $i : \mathsf{Im}(f) \to Y$. Since f is coarsely surjective, there is a $C > 0$ such that for all $y \in Y$, $d(f(x), y) \leq C$ for some $x \in X$. Define $g : Y \to \mathsf{Im}(f)$ by choosing for each y such an x in such a way that $g \circ i$ is the identity. One can check that g is a surjective coarse equivalence, and so we can apply our earlier result to it. Finally, we conclude that $C^*(X) \cong C^*(\mathsf{Im}(f)) \cong C^*(Y)$ as required. □

Note that the isometry V and hence the isometry $\mathsf{Ad}V$ above is far from canonical. However, it turns out that the induced map on K-theory *is* canonical (see [36]). This is fortunate, since the K-theory of the Roe algebra is in fact the codomain of the "coarse index map" [56] featuring in the applications to index theory mentioned earlier.

The K-theory of the Roe algebra is also one side of the coarse Baum-Connes Conjecture, a conjecture which is false in general but has implications for the Novikov Conjecture via the principle of descent when it holds [56]. In the next two sections we will encounter some coarse properties which imply this Conjecture, and which are consequently the subject of a great deal of interest.

1.2.2 Dimension-type invariants

Many different notions of dimension have been introduced in topology – see [48] for one treatment of this topic. In coarse geometry, the most important notion of dimension is the concept of asymptotic dimension introduced by Gromov in his seminal monograph on the large scale geometry of groups [32].

There are a number of equivalent definitions of asymptotic dimension, so we will choose one which is most relevant for this thesis. Let $\mathcal{U}$ be a cover of a metric space X, that is, a family of subsets of X whose union is all of X. We say that $\mathcal{U}$ has **Lebesgue number at least** R if every subset of diameter less than R is contained in some element of $\mathcal{U}$. We say that $\mathcal{U}$ has **point multiplicity at most** M if every point in X is contained in at most M elements of $\mathcal{U}$.

Definition 5. *Let X be a metric space. We say that X has asymptotic dimension less than n, writing* $\mathrm{asdim} X \leq n$*, if for every $R > 0$, there is a cover $\mathcal{U}$ of X and a number $S > 0$ such that $\mathcal{U}$ has Lebesgue number at least R and point multiplicity at most $n+1$ and the diameter of any $U \in \mathcal{U}$ is at most S. The asymptotic dimension of X is then the least n such that* $\mathrm{asdim} X \leq n$.

Another way to state this above is as follows. Call a cover $\mathcal{U}$ **uniformly bounded** if there is an $S > 0$ such that the diameter of any $U \in \mathcal{U}$ is at most S. Also, say that a cover $\mathcal{U}$ is a **refinement** of a cover $\mathcal{V}$ if every element of $\mathcal{U}$ is contained in some element of $\mathcal{V}$. Then $\mathrm{asdim} X \leq n$ if and only if every uniformly bounded cover refines a uniformly bounded cover of point multiplicity $n+1$. The advantage of such a statement is that it can be easily compared with an important topological notion of dimension called covering dimension – indeed, recall that a metric space has **Lebesgue covering dimension at most** n if every open cover is refined by an open cover of point multiplicity at most $n+1$.

Example 4. *The space $\mathbb{Z}^n$ has asymptotic dimension n, although this is not easy to prove without relying on results from topological dimension theory (see Chapter 2 of [50]).*

Often one is not interested in the asymptotic dimension of a space, but just whether it is finite. For example, hyperbolic groups in the sense of Gromov has finite asymptotic dimension (see Theorem 9.25 in [57]). The interest in finite asymptotic dimension is readily explained by the following famous result of Guoliang Yu.

Theorem 1 (Yu [73])**.** *Let G be a finite generated group of finite homotopy type which (as a metric space with the word metric) has finite asymptotic dimension. Then the Novikov Conjecture holds for G.*

Apart from the conceptual link between asymptotic dimension and covering dimension, there is also a direct link due to Dranishnikov via the Higson corona: for proper metric spaces, the asymptotic dimension coincides with the (topological) covering dimension of the Higson corona whenever the former is finite [18]. The Higson corona is a kind of boundary of a space which captures large scale information. It plays an important role in the Chapters 2, 3 and 4, so we will defer the definition and further discussion until then.

1.2.3 Amenability-type invariants

Amenability is a classical property of groups introduced by Von Neumann in the 1920s motivated by the Banach-Tarski paradox. There are many equivalent definitions, but for us the following will be the most natural.

Definition 6. *A discrete finitely generated group G is amenable if for every finite subset $F \subseteq G$ and every $\varepsilon < 0$, there is a finite subset $E \subseteq G$ such that for all $f \in F$,*

$$\frac{|fE\Delta E|}{|E|} \leq \varepsilon$$

where $A\Delta B = (A \cup B) \setminus (A \cap B)$ is the symmetric difference, and $|\cdot|$ denotes cardinality.

Amenability has long been known to be a coarse invariant of groups, i.e. if G is amenable and H is coarsely equivalent to it, then H is also amenable (see Theorem 3.1.5 of [50]). In [74], Yu introduced the following "non-equivariant" generalization of amenability for general metric spaces.

Definition 7. *A discrete metric space X has* ***Property A*** *if for any $R > 0$ and $\varepsilon > 0$, there exist a family of finite subsets $\{A_x\}$ of $X \times \mathbb{N}$ indexed by points in X such that*

1. $(x, 1) \in A_x$ *for all* $x \in X$*,*

2. $\frac{|A_x \Delta A_y|}{|A_x \cap A_y|} \leq \varepsilon$ *whenever* $d(x, y) \leq R$*,*

3. there exists $S > 0$ *such that if* $(y, m) \in A_x$*, then* $d(x, y) \leq S$*.*

To see the comparison with amenability, one should think of the sets A_x as being the translates of the finite set E, and of the R being related to the maximum word length of elements of the finite set F. In particular, any discrete finitely generated amenable group has Property A when considered as a metric space (see e.g. [50]). Other equivalent definitions of Property A are often used in the literature, and we will have occasion in this thesis to use them when convenient. The interest in Property A is again explained by its applications to the Novikov Conjecture. In the same paper [74] where Yu introduced Property A, Yu also proved the following two results.

Proposition 4 (Yu [74])**.** *If a discrete metric space* X *has Property A, then there is a coarse embedding from* X *to a Hilbert space.*

Theorem 2 (Yu [74])**.** *Let* X *be a discrete metric space of bounded geometry. If* X *admits a coarse embedding into Hilbert space, then the coarse Baum-Connes Conjecture holds for* X*.*

The coarse Baum-Connes Conjecture was already mentioned in this introduction in connection with the Roe algebra. By the principle of descent, the coarse Baum-Connes Conjecture implies the Novikov Conjecture for finitely generated groups of finite homotopy type [56], which allowed Yu to prove the following corollary.

Theorem 3 (Yu [74])**.** *Let* G *be a finite generated group of finite homotopy type which (as a metric space with the word metric) coarsely embeds into Hilbert space. Then the Novikov Conjecture holds for* G*.*

Note that any discrete metric space of finite asymptotic dimension has Property A (see Section 4.3 of [50] for a proof), so that this result is an improvement on Yu's earlier result in [73].

This result, along with other applications of coarse invariants, has given rise to an industry of introducing and studying coarse properties which are weaker than finite asymptotic dimension but which imply coarse embeddability in Hilbert space. These concepts will not feature very prominently in this thesis and so we will not recall their definitions here. They include, for example, Asymptotic Property C [18] and finite decomposition complexity and its variants [33].

1.3 Category theory

One of the main tools which will be used in this thesis is category theory. The use of category theory is motivated by the discussion in Section 1.1 of this introduction since a category is the most natural abstract framework in which one can talk about spaces being "the same". One of the main uses of category theory is to unify concepts and proofs from different areas of mathematics – in this thesis these areas will usually be topology and coarse geometry.

Very little understanding of category theory will be necessary to understand the work in this thesis, however for the sake of completeness we should collect some basic definitions here just in case. A good reference for basic category theory is [43].

Definition 8. *A category $\mathbb{C}$ consists of objects and arrows between objects (called morphisms) together with*

- *the choice, for each object X, of an identity morphism 1_X, and*
- *the choice, for any two morphisms $f : A \to B$, $g : B \to C$, of a composite $gf : A \to C$,*

such that

- *for any morphism $f : X \to Y$, $f \circ 1_X = 1_Y \circ f = f$, and*
- *composition of morphisms is associative.*

For example, there is a category whose objects are all metric spaces and whose morphisms are all bornologous maps, with the usual composition and identity maps. To prove that this is a category is straightforward once one checks that the composition of two bornologous maps is again a bornologous map. Within the framework of a category, one can define certain kinds of morphisms.

Definition 9. *Let $\mathbb{C}$ be a category and let $f : X \to Y$ be a morphism. Then f is said to be*

- *a* ***monomorphism*** *if for any two morphisms $g, h : W \to X$, $fg = fh \implies g = h$,*
- *an* ***epimorphism*** *if for any two morphisms $g, h : Y \to Z$, $gf = hf \implies g = h$,*
- *an* ***isomorphism*** *if there is a morphism $f^{-1} : Y \to X$ such that $ff^{-1} = 1_Y$ and $f^{-1}f = 1_X$.*

For example, in the category whose objects are all sets and whose morphisms are all functions between sets, monomorphisms are precisely the injections, epimorphisms precisely the surjections and isomorphisms precisely the bijections. One often wants isomorphisms to capture a certain notion of being the same. In the category of metric spaces and bornologous maps mentioned earlier, the isomorphisms are the *bijective* coarse equivalences. If we want to capture all coarse equivalences, then we need to change the category – see Chapter 2 of this thesis.

We may also need the notion of a functor. A **functor** F from a category $\mathbb{C}$ to a category $\mathbb{D}$ assigns to each object (resp. morphism) in $\mathbb{C}$ an object (resp. morphism) in $\mathbb{D}$ such that identity morphisms are sent to identity morphisms and composition is respected, i.e. $F(gf) = F(g)F(f)$ whenever the composite gf makes sense. A basic exercise is to show that functors send isomorphisms to isomorphisms – this explains their usefulness in constructing invariants of one type or another. Important functors in coarse geometry are the Higson corona functor and the functor which sends a space to the K-theory of its Roe algebra.

1.4 Abstract contexts for coarse geometry

During the development of topology, it became clear that it was important to work in a context more general than that of metric spaces. What followed was the introduction of the notion of topological space. The advantage of such a generalization is firstly to be able to examine more general kinds of spaces (such as non-metrizable compactifications), but also to allow for more intuitive statements and proofs of results by illuminating the abstract ideas behind them rather than focussing on details such as the particular metric. In coarse geometry, it also makes sense to move beyond metric spaces into more general contexts. This thesis will make use of two such contexts – Chapter 2 uses Roe's notion of coarse space, probably the earliest abstract context for coarse geometry, while the other three research chapters make use of the notion of large scale space introduced by Dydak and Hoffland in [21]. The definitions will be recalled when needed, but we also give them here.

Definition 10 (Roe [57])**.** *A **coarse space** is a pair $(X, \mathcal{X})$ where X is a set and $\mathcal{X}$ is a family of binary relations on X which contains the diagonal Δ and which is closed under taking subrelations, inverses, products (i.e. composition of relations) and finite unions.*

Definition 11 (Dydak-Hoffland [21])**.** *A **large scale structure** $\mathcal{L}$ on a set X is a nonempty set of families $\mathcal{B}$ of subsets of X (which we call the **uniformly bounded families** in X) satisfying the following conditions:*

(1) $\mathcal{B}_1 \in \mathcal{L}$ implies $\mathcal{B}_2 \in \mathcal{L}$ if each element of $\mathcal{B}_2$ consisting of more than one point is contained in some element of $\mathcal{B}_1$.

(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}$ implies $\mathsf{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{L}$.

Those readers familiar with the theory of uniform spaces (see e.g. [38]) will notice the similarity between these definitions and the two main definitions of uniform space (in terms of entourages and covers respectively). This analogy is useful to keep in mind, since coarse geometry is often viewed as "dual" to the theory of uniform spaces.

1.5 Themes of the thesis and guide to the text

This aim of this thesis is to develop new theory and prove new results in coarse geometry by drawing on the following two themes:

- looking for concepts and results which are analogous to existing ones in classical and algebraic topology,
- using category theory to give new theory a natural and rigorous underpinning.

These themes are expressed in the research chapters in the following ways:

In Chapter 2, a basic topological notion (connectedness) is transported to coarse geometry using the language of category theory.

In Chapter 3, a coarse geometric version of a foundational result in classical topology (the Tietze Extension Theorem) is proved using the abstract notion of neighbourhood operator.

In Chapter 4, a classical notion in topology (monotone-light factorizations) has been transported to coarse geometry using dimension-theoretic ideas. The language of category theory is then used to verify the naturalness of the new definitions and to compare them in an abstract way to the topological versions.

In Chapter 5, we look at group actions in coarse geometry and study a coarse version of properly discontinuous group actions (an important class of actions in algebraic topology). Category theory mainly plays a role in this chapter in allowing us to view Roe's warped spaces as coarse quotients.

The chapters are each based on a published paper by the author (sometimes with a coauthor). The chapters have been designed to be self-contained for readers familiar with basic notions in topology and geometry. The result is that some definitions are repeated, however we believe that this cost is worth it to ensure that the reader be able to pick up any individual chapter and understand it, rather than have to carry definitions in their heads for four chapters.

Chapter 2

A coarse version of connectedness

This chapter is based on the accepted manuscript of the following paper: T. Weighill, On spaces with connected Higson coronas, Topology and its Applications 209, 2016, 301–315. *One round of revisions took place after comments by the anonymous referee. The introduction has been adapted, otherwise the manuscript has remained more or less unchanged.*

2.1 Introduction

In this chapter we will examine a coarse version of the notion of connectedness in topology. Connectedness and its variants such as local connectedness play a central role in general topology – for example in the study of covering spaces. A topological space X is said to be connected if there are disjoint non-empty open subsets A and B in X such that $X = A \cup B$. Note that by this definition, the empty space is presumed to be disconnected, while some authors prefer to call it connected.

In fact, our goal in this chapter will be to show that two very natural notions of connectedness for coarse geometry are one and the same. These two notions come from two different important ways to study coarse properties, namely:

1. studying metric spaces as objects in the appropriate category, for example the coarse category (defined in the next section), and

2. associating to each metric space a topological space or algebra which captures coarse properties of the space, for example the Higson corona [34] or the uniform Roe algebra [54, 55] (see also [56, 57]).

In this chapter, we will exhibit a connection between a condition stated in the language of the coarse category and a topological condition on the Higson corona. For a metric space X, the *Higson corona* of X, denoted by νX, is a compact topological space which captures coarse properties of X. It was introduced in [34], motivated by considerations in index theory, and is defined for proper metric spaces as the complement of X in the so-called *Higson compactification* of X. While the Higson compactification is only defined for proper metric spaces (or for proper coarse spaces in the sense of [57]), the Higson corona can be defined for arbitrary metric spaces (and arbitrary coarse spaces). We recall this definition here, following [57].

Definition 2.1. *Let X be a metric space. Given a bounded map $f : X \to \mathbb{C}$ (not necessarily continuous) to the complex numbers, f is said to be* slowly oscillating *if for every $\varepsilon > 0$ and $R > 0$, there is a bounded set B such that*

$$d(x, x') \leq R \Rightarrow d(f(x), f(x')) \leq \varepsilon$$

for $x, x' \in X \setminus B$.

We say that a bounded map $f : X \to \mathbb{C}$ *tends to zero at infinity* if for every $\varepsilon > 0$, there is a bounded set B in X such that $|f(x)| \leq \varepsilon$ whenever $x \notin B$. Let $B_h(X)$ be the set of bounded slowly oscillating functions from X to $\mathbb{C}$, and let $B_0(X)$ be the set of bounded functions from X to $\mathbb{C}$ which tend to zero at infinity. It is easy to check that $B_h(X)$ is a unital C^*-algebra with the sup-norm and pointwise operations, and that $B_0(X)$ is a closed ideal of $B_h(X)$.

Definition 2.2. *The* Higson corona *of X, denoted by νX, is the spectrum of the C^*-algebra $B_h(X)/B_0(X)$.*

In other words, νX is the unique (up to homeomorphism) compact Hausdorff space whose algebra $C(\nu X)$ of continuous complex-valued functions is $*$-isomorphic to $B_h(X)/B_0(X)$

(for more on the theory of C^*-algebras, see for example [2]). For proper metric spaces this definition coincides with the original definition in terms of the Higson compactification (see Lemma 2.40 in [57]). The study of coarse properties of metric spaces can be viewed as the study of properties which are invariant under *coarse equivalence* of metric spaces (defined in the next section). It turns out that coarsely equivalent metric spaces have homeomorphic Higson coronas [57], so that the Higson corona is indeed a coarse invariant of a space. A good example of the relationship between coarse properties of a space and topological properties of its Higson corona is Theorem 7.2 of [18], where it was shown that if the asymptotic dimension of a proper metric space is finite, then the asymptotic dimension coincides with the covering dimension of its Higson corona.

Topological connectedness of the Higson corona can be easily stated very easily in terms of the algebra $B_h(X)/B_0(X)$. Note that any topological space X is disconnected if and only if it admits a continuous non-constant map to the discrete space $\{0, 1\} \subset \mathbb{C}$. As an immediate consequence, we obtain the following result:

Lemma 2.3. *Let X be a compact Hausdorff space and $C(X)$ its C^*-algebra of continuous complex-valued functions. Then X is disconnected if and only if $C(X)$ contains a non-trivial idempotent element. In particular, the Higson corona of a space X is connected if and only if $B_h(X)/B_0(X)$ contains no non-trivial idempotent elements.*

The main result of this chapter will be to show that connectedness of the Higson corona can be characterised by a categorical condition (condition (C) in Section 2.4) which is a natural generalization of the notion of connectedness in topological spaces, stated in the language of the coarse category. This result motivates the study of categorical conditions in the coarse category. Note that, in general, any condition stated in categorical language in the coarse category is automatically invariant under coarse equivalence (since coarse equivalences are isomorphisms in this category). We will also give one further connection between a categorical notion and a well-known coarse condition in this chapter, namely, we give a categorical characterisation of the notion of ω-excisive decomposition introduced in [36]. It would be interesting in the future to investigate what other categorical conditions stated in terms of the coarse category turn out to correspond to well-known or interesting coarse properties of spaces.

We should note that since the Higson corona can be defined for arbitrary metric spaces via the algebra $B_h(X)/B_0(X)$ (see Definition 2.2 above), metric spaces will not be assumed to be proper in this chapter, unless otherwise stated.

2.2 Preliminaries

Throughout this chapter, we will often deal with the disjoint union $X+Y$ of two sets X and Y. For convenience, we will not distinguish between (relations on) the set X and (relations on) the image of X under the inclusion $\iota_X : X \to X+Y$. For a subset A of a metric space X and $R > 0$, we denote the set $\{x \in X \mid d(x, A) < R\}$ by $B(A, R)$. Throughout the chapter we will make use of elementary category theoretic notions, most importantly the notion of coproduct and pushout, although we give explicit descriptions of the universal property in question whenever possible. For an introduction to category theory, we direct the reader to [43].

Let X and Y be metric spaces, and let $f : X \to Y$ be a map. We call the map f *ρ-bornologous*, where ρ is a function $\rho : [0, \infty) \to [0, \infty)$, if for any points x, x' in X,

$$d_X(x, x') \leq R \Rightarrow d_Y(f(x), f(x')) \leq \rho(R).$$

The map f is called *bornologous* if it is ρ-bornologous for some ρ. In other words, f is bornologous if and only if for every $R > 0$ there is an $S > 0$ such that $d_X(x, x') \leq R \Rightarrow d_Y(f(x), f(x')) \leq S$. It is easy to check that the composite of two bornologous maps is again bornologous. Consequently, metric spaces together with bornologous maps form a category, which we denote throughout by $\mathbf{Met_{Born}}$. A map f between metric spaces is called *proper* if the inverse image of any bounded set under f is bounded. A map is called *coarse* if it is both bornologous and proper. Proper maps are closed under composition, so coarse maps form a subcategory of $\mathbf{Met_{Born}}$.

Two maps f and g from a metric space X to a metric space Y are *close* if there is a $R > 0$ such that $d_Y(f(x), g(x)) \leq R$ for all $x \in X$. A map f from X to Y is a *coarse equivalence* if there exists a bornologous map $f^* : Y \to X$ such that ff^* and f^*f are close to the respective identities. Note that coarse equivalences are always proper. Coarse equivalences are also coarsely surjective: if $u : X \to Y$ is a coarse equivalence, then there is an $R > 0$ such that $Y \subseteq B(\mathsf{Im}(u), R)$.

In $\mathbf{Met_{Born}}$ the isomorphisms are precisely the bijective coarse equivalences. This is inconvenient in practice, since one would like any coarse equivalence to be an isomorphism. Thus it is often convenient to consider the category whose objects are metric spaces and whose morphisms are equivalence classes of coarse maps under the closeness relation (note that in in this category, coarse equivalences represent isomorphisms). In this chapter, we will call this category the *coarse category of metric spaces*, by analogy with the coarse category as defined in [56] (where objects are abstract coarse spaces). Note that composition is well-defined in this category, since if f is close to g and h is close to k, where f,g, h and k are bornologous maps, then hf is close to kg whenever these composites are defined.

2.3 Coarse coproducts

In this section we introduce coarse coproducts of metric spaces and prove some basic results about them.

Definition 2.4. *Let X and Y be two metric spaces, and let $x_0 \in X$ and $y_0 \in Y$ be two arbitrary points (which we will call the* base points *for the coproduct). Then the* coarse coproduct *of (X, d_X) and (Y, d_Y) is the space $(X + Y, d_{X+Y})$ whose underlying set is the disjoint union of X and Y and where the distance d_{X+Y} is defined as follows:*

$$d_{X+Y}(a,b) = \begin{cases} d_X(a,b) & a, b \in X \\ d_Y(a,b) & a, b \in Y \\ d_X(a,x_0) + 1 + d_Y(y_0,b) & a \in X, b \in Y. \end{cases}$$

Proposition 2.5. *Let X and Y be metric spaces with coarse coproduct $X + Y$ and let $\iota_X : X \to X + Y$ and $\iota_Y : Y \to X + Y$ be the evident isometric embeddings. Then*

(1) $X + Y$ (together with ι_X and ι_Y) is the coproduct in $\mathbf{Met_{Born}}$ of X and Y, i.e. if there are bornologous maps $f : X \to Z$ and $g : Y \to Z$, then there exists a unique bornologous map $h : X + Y \to Z$ such that $h\iota_X = f$ and $h\iota_Y = g$;

(2) in the notation of (1) above, both f and g are proper if and only if h is proper;

(3) in the notation of (1) above, if $h' : X + Y \to Z$ is a map such that $h'\iota_X$ is close to f and $h'\iota_Y$ is close to g, then h' is close to h.

Proof. For (1), define h to coincide with f on X and g on Y. It remains to show that h is bornologous, since then h is clearly unique with the desired property. Let $r = d_Z(f(x_0), g(y_0))$, where $x_0 \in X$ and $y_0 \in Y$ are the chosen base points, and suppose f and g are ρ- and σ-bornologous respectively. Since h is clearly bornologous on X and Y, it remains to consider points $a \in X$ and $b \in Y$ with $d(a, b) \leq R$. We have

$$d(h(a), h(b)) = d(f(a), g(b)) \leq d(f(a), f(x_0)) + r + d(g(y_0), g(b))$$

Since $d(a, x_0) \leq d(a, b) \leq R$ and $d(y_0, b) \leq d(a, b) \leq R$ in $X + Y$, we have

$$d(h(a), h(b)) \leq \rho(R) + r + \sigma(R)$$

which gives the required result. For (2), note that a subset of $X + Y$ is bounded if and only if its restrictions to both X and Y are bounded. (3) is easy to check. □

It follows from the proposition above that the coarse coproduct of X and Y is defined up to bijective coarse equivalence – that is, if different base points $x_1 \in X$ and $y_1 \in Y$ are chosen for the construction, then the resulting coarse coproduct is coarsely equivalent to the one with base points x_0, y_0 via the identity set map.

The proposition above also shows that not only is $X + Y$ the coproduct in $\mathbf{Met_{Born}}$, but it also gives the coproduct in the subcategory of coarse maps, as well as in the coarse category of metric spaces. As usual, the existence of binary coproducts gives the existence of finite coproducts in all these categories. It is easy to show that arbitrary coproducts (for example, an uncountable coproduct of singleton spaces) do not exist in $\mathbf{Met_{Born}}$. Countable coproducts do exist in $\mathbf{Met_{Born}}$, however, as the following proposition shows.

Proposition 2.6. *Let $X_1, X_2, \ldots$ be a countable family of metric spaces and $x_1 \in X_1, x_2 \in X_2, \ldots$ chosen base points in each space. Let $\sum_i X_i$ be the metric space whose underlying set is the disjoint union of $X_1, X_2, \ldots$ and whose distance d is defined as follows:*

$$d(a,b) = \begin{cases} 0 & a = b \\ d_{X_i}(a,b) + i & a, b \in X_i,\ a \neq b \\ d_{X_i}(a,x_i) + i + j + d_{X_j}(b,x_j) & a \in X_i, b \in X_j, i \neq j \end{cases}$$

Then $\sum_i X_i$, together with the obvious injections $(\iota_i)_{i \geq 1}$, is the coproduct of the X_i in $\mathbf{Met_{Born}}$.

Proof. Suppose Z is a metric space and $f_i : X_i \to Z$ a family of bornologous maps. Let $f : \sum_i X_i \to Z$ be the induced set map. Then for any $R > 0$, there is a $k \in \mathbb{N}$ such that $d(a,b) \leq R \Rightarrow a = b$ whenever $a \in X_i$ and $b \in X_j$ with $\mathsf{max}(i,j) \geq k$. For f to be bornologous it is thus enough for it to be bornologous on subspaces of the form

$$\bigcup_{i \leq k} X_i \subseteq \sum_i X_i$$

which is easy to show using similar arguments to Proposition 2.5. □

Note that it is not true in general that, in the notation of the proof, the map $f : \sum_i X_i \to Z$ is proper whenever the f_i are. Moreover, if $f'\iota_i$ is close to f_i for every i for some bornologous map f', then f' need not be close to f.

If one changes the definition of the metric d defined in Proposition 2.6 to

$$d(a,b) = \begin{cases} d_{X_i}(a,b) & a, b \in X_i \\ d_{X_i}(a,x_i) + i + j + d_{X_j}(b,x_j) & a \in X_i, b \in X_j, i \neq j \end{cases}$$

then one obtains a space $\square_i X_i$ with the following universal property: for any family $f_i : X_i \to Z$ of ρ-bornologous maps, there is a unique bornologous map $f : \square_i X_i \to Z$ such that $f\iota_i = f_i$ for each i. Note that in this case the f_i must share the same function ρ.

This construction is well known in the case when $X = G$ is a finitely generated group with the word length metric and $X_i = G/G_i$ is a sequence of finite quotients of G such that every finite index normal subgroup of G contains some G_i. The space $\square_i X_i$ is then (up to bijective coarse equivalence) the *box space* of G introduced in [57]. An important result about box spaces is as follows: a residually finite group G (i.e. one which admits such a sequence G_i) is amenable if and only if the box space satisfies Yu's Property A (see [50]).

2.4 Connectedness of the Higson corona

Given any category with coproducts, there are a number of conditions on an object X which in the category of topological spaces and continuous maps all reduce to the usual notion of topological connectedness. Some of these conditions are listed and compared in [39]. In this chapter, we will use the following condition from this list:

- any morphism f from X to a coproduct $Y + Z$ factors through a coproduct injection, i.e. there exists either a map $g : X \to Y$ such that $\iota_Y g = f$ or a map $h : X \to Z$ such that $\iota_Z h = f$.

It is easy to check that this captures the notion of connectedness in the case of topological spaces when applied in the category of topological spaces and continuous maps. In the coarse category of metric spaces, this condition becomes the following condition on a metric space X:

(C) for every coarse map $f : X \to Y + Z$, there exists either a coarse map $g : X \to Y$ such that $\iota_Y g$ is close to f or a coarse map $h : X \to Z$ such that $\iota_Z h$ is close to f.

Theorem 2.7. *For a metric space X, the following are equivalent:*

(a) X doesn't satisfy (C)*;*

(b) there are two unbounded subsets A and B of X such that

– $X = A \cup B$, *and*

– *for any* $R > 0$, *there is a bounded set* $C_R \subseteq X$ *such that*

$$a \in (A \setminus C_R) \wedge b \in (B \setminus C_R) \Rightarrow d_X(a,b) \geq R.$$

(c) X *is (bijectively) coarsely equivalent to a coarse coproduct* $Y + Z$ *where neither* Y *nor* Z *is bounded;*

(d) there exists a coarse map $f : X \to \mathbb{Z}$ *such that the image of* f *has no maximum or minimum.*

Proof. (a) $\Rightarrow$ (b): Suppose $f : X \to Y + Z$ is a coarse map such that f does not factor, up to closeness, through either ι_Y or ι_Z, and suppose that f is ρ-bornologous. Then in particular, neither $\mathsf{Im}(f) \cap Y$ nor $\mathsf{Im}(f) \cap Z$ are bounded. Let $A = f^{-1}(Y)$, $B = f^{-1}(Z)$; since f is bornologous, neither A nor B are bounded subspaces of X. For any $R > 0$, let $K = B(y_0, \rho(R)) \cup B(z_0, \rho(R))$ where $y_0 \in Y$ and $z_0 \in Z$ are the base points of the coproduct. Since f is proper, $C_R = f^{-1}(K)$ is bounded. If $a \in (A \setminus C_R)$ and $b \in (B \setminus C_R)$, then using the definition of $Y + Z$, we have that

$$d(f(a), f(b)) \geq 2\rho(R) + 1 > \rho(R),$$

so that $d(a,b) \geq R$ as required.

(b) $\Rightarrow$ (c): It is easy to see that $A \cap B$ has to be bounded, so that $B \setminus A$ must be non-empty. Choose points $a_0 \in A$ and $b_0 \in B \setminus A$. Note that we may choose the C_R such that for all $R > 0$, $\{a_0, b_0\} \subseteq C_R$. Let $W = A + (B \setminus A)$, choosing a_0 and b_0 as base points and where the metric on A and $B \setminus A$ is induced by X. The identity set map $W \to X$ is clearly bornologous, so it remains to prove that it has bornologous inverse. The inverse is clearly bornologous on A and $B \setminus A$, so let $a \in A$ and $b \in B \setminus A$, and choose a bounded subset C_R corresponding to the value $R = d_X(a,b)$. Let D be the diameter of C_R. Then one of a and b must be in C_R, so

$$d_W(a,b) = d_X(a,a_0) + d_X(b,b_0) + 1 \leq d_X(a,b) + 1 + 2D$$

since $\{a_0, b_0\} \subseteq C_R$. This shows that the inverse of i is bornologous, since D depends only on $d_X(a, b)$.

(c) ⇒ (d): Suppose X is coarsely equivalent to $Y + Z$ with chosen base points y_0, z_0. Define a map $g : Y \to \mathbb{Z}$ by $g(y) = k$, where $k \leq d_Y(y, y_0) < k + 1$. Define a map $h : Z \to \mathbb{Z}$ by $h(z) = -k$, where $k \leq d_Z(z, z_0) < k + 1$. Clearly both g and h are coarse, so the induced map $f : Y + Z \to \mathbb{Z}$ is also coarse. Moreover, the image of f has no maximum or minimum since Y and Z are unbounded. Finally, composing with the coarse equivalence from X to $Y + Z$ gives the required map.

(d) ⇒ (a): This follows from the fact that $\mathbb{Z}$ is (bijectively coarse equivalent to) the coarse coproduct of $\mathbb{N}$ with itself. □

Example 1. *The metric spaces $\mathbb{Z}$ and $\mathbb{R}$ do not satisfy* (C) *(indeed, they are both coarsely equivalent to the coarse coproduct $\mathbb{N} + \mathbb{N}$). The metric space $\{n^2 \mid n \in \mathbb{N}\}$ also does not satisfy* (C) *as can be seen from (b) in the above theorem (take A to be the even numbers and B the odd ones). It is easy to show using condition (b) in Theorem 2.7, however, that the metric space $\mathbb{N}$ does satisfy* (C)*. In particular, there are no surjective coarse maps from $\mathbb{N}$ to $\mathbb{Z}$.*

In topological spaces, a space X is disconnected if and only if it admits a non-trivial map to the two element discrete space. Thus the space $\mathbb{Z}$ in some sense plays the role of the two element discrete space for condition (C).

Corollary 1. *If $f : X \to Y$ is a surjective coarse map and X satisfies* (C)*, then Y satisfies* (C).

Proof. This follows from condition (d) in Theorem 2.7. □

Recall from [57] that the map which takes a metric space X to its Higson corona νX extends to a functor ν from the coarse category of metric spaces to the category of compact Hausdorff spaces and continuous maps (in fact, the result in [57] is stated only for the case of proper metric/coarse spaces, but the proof works for arbitrary metric/coarse spaces). In particular, coarsely equivalent metric spaces have homeomorphic Higson coronas. It turns out that the functor ν preserves coproducts, as we will now show.

Lemma 2.8. *Let X and Y be metric spaces with coarse coproduct $X+Y$. Let $f : X \to \mathbb{C}$ and $g : Y \to \mathbb{C}$ be maps to the complex numbers. Then the map $h : X+Y \to \mathbb{C}$ which agrees with f on X and with g on Y is slowly oscillating/tends to zero at infinity if and only if both f and g are slowly oscillating/tend to zero at infinity.*

Proof. The equivalence for tending to zero at infinity is clear, as is the fact that if h is slowly oscillating, then f and g both are. Suppose then that f and g are slowly oscillating. Let $\varepsilon > 0$ and $R > 0$. Since f and g are slowly oscillating, there are bounded sets B_1 in X and B_2 in Y such that, outside of $B_1 \cup B_2 \subseteq X+Y$, $d(a,b) \leq R \Rightarrow d(h(a),h(b)) \leq \varepsilon$ whenever a and b are either both in X or both in Y. Let $x_0 \in X$ and $y_0 \in Y$ be the base points chosen for $X+Y$, and let $C = B(x_0,R) \cup B(y_0,R)$. Then for $x, x' \in (X+Y) \setminus (B_1 \cup B_2 \cup C)$, we have that x and x' are either both in X or both in Y. Consequently, $d(h(x),h(x')) \leq \varepsilon$ as required. □

Proposition 2.9. *The Higson corona preserves binary coarse coproducts. That is, if X and Y are metric spaces, then $\nu X + \nu Y$ and $\nu(X+Y)$ are homeomorphic.*

Proof. Consider the algebras $C(\nu(X)+\nu(Y)) \cong C(\nu X) \times C(\nu Y)$ and $C(\nu(X+Y))$. There is a canonical $*$-homomorphism $F : C(\nu(X+Y)) \to C(\nu X) \times C(\nu Y)$ which sends an equivalence class of maps $[f]$ to the pair $([f\iota_X],[f\iota_Y])$. It follows from Lemma 2.8 that this map has trivial kernel and is surjective, so F is an isomorphism. Thus we obtain that $\nu X + \nu Y$ and $\nu(X+Y)$ are homeomorphic. □

We are now ready to state the main result of this chapter.

Theorem 2.10. *The following are equivalent for a metric space X:*

(a) X satisfies (C)*;*

(b) the Higson corona of X is (topologically) connected;

(c) $B_h(X)/B_0(X)$ does not contain a non-trivial idempotent element.

Proof. (b) $\Rightarrow$ (a): This follows from Proposition 2.9 and the fact that for an unbounded metric space, the Higson corona is non-empty.

(b) $\Leftrightarrow$ (c): This follows from Lemma 2.3.

(a) $\Rightarrow$ (c): Suppose that (c) doesn't hold. Then there is a slowly oscillating map $f : X \to \mathbb{C}$ such that $[f^2 - f] = [0]$, with $[f] \neq [0]$, $[f] \neq [1]$. This means that for any $\varepsilon > 0$, there is a bounded set C such that the image of $X \setminus C$ under f is contained in $B(0, \varepsilon) \cup B(1, \varepsilon)$. In particular, one can choose C such that the image of $X \setminus C$ under f is contained in $B(0, 1/4) \cup B(1, 1/4)$. Let $A = f^{-1}(B(0, 1/4))$ and $B = X \setminus A$. The non-triviality of $[f]$ ensures that neither A nor B are bounded. It follows that for any $R > 0$, we can choose a bounded set C' such that

$$d(x, x') \leq R \Rightarrow d(f(x), f(x')) \leq 1/4$$

for any $x, x' \notin C'$, and such that

$$f(X \setminus C') \subseteq B(0, 1/4) \cup B(1, 1/4).$$

In particular, if $d(x, x') \leq R$ and $x, x' \notin C'$, then x and x' must either both be in A or both be in B. Thus A and B satisfy the conditions in (b) in Theorem 2.7, so we obtain the required contradiction. □

2.5 ω-Excisive decompositions and coarse pushouts

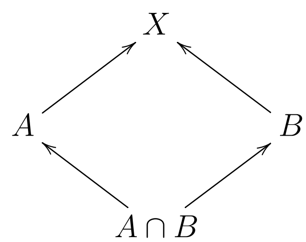
In this section we make some connections between the work done so far and the notion of ω-excisive decomposition found in [36]. We do so via a result (Theorem 2.12 below) which in its own right further motivates the study of categorical conditions in the coarse category. Recall the following definition from [36].

Definition 2.11. *Let X be a metric space and let A and B be closed subspaces with $X = A \cup B$. Then $X = A \cup B$ is an* ω-excisive decomposition *if for each $R > 0$ there exists a $S > 0$ such that $B(A, R) \cap B(B, R) \subseteq B(A \cap B, S)$.*

Such decompositions are important because they give rise to Mayer-Vietoris sequences at the level of coarse cohomology as well as at the level of K-theory of uniform Roe algebras [36].

In particular, this allows ω-excisive decompositions to be used to prove the coarse Baum-Connes conjecture for certain spaces [72]. We now show that such decompositions amount to pushouts in the coarse category.

Theorem 2.12. *Let X be a metric space and let A and B be closed subspaces with $X = A \cup B$. Then $X = A \cup B$ is an ω-excisive decomposition if and only if $A \cap B$ is non-empty and the diagram of inclusions*



(2.1)

is a pushout in the coarse category of metric spaces, i.e. for any coarse maps $f : A \to Y$ and $g : B \to Y$ which are close on $A \cap B$, there is a unique-up-to-closeness map $h : X \to Y$ such that h is close to f on A and close to g on B.

Proof. ($\Rightarrow$): Suppose $X = A \cup B$ is a ω-excisive decomposition, and that $f : A \to C$ and $g : B \to C$ are two coarse maps such that f and g are close on $A \cap B$, with f ρ-bornologous and g σ-bornologous. Define $h : X \to C$ to be f on A and g on $X \setminus A$. It remains to show that h is coarse. It is clearly coarse on A and B, so it remains to consider $a \in A$, $b \in X \setminus A$. Suppose $d(a, b) \leq R$. Then $a, b \in B(A, R) \cap B(B, R)$, so by hypothesis, there is an S such that $a, b \in B(A \cap B, S)$. Suppose $d(a, c_1) \leq S + 1$ and $d(b, c_2) \leq S + 1$ for $c_1, c_2 \in A \cap B$. The distance $d(h(a), h(b)) = d(f(a), g(b))$ is bounded above by

$$d(f(a), f(c_1)) + d(f(c_1), f(c_2)) + d(f(c_2), g(c_2)) + d(g(c_2), g(b)).$$

The first and last terms are bounded by $\rho(S + 1)$ and $\sigma(S + 1)$ respectively. The third term is bounded by a constant since f and g are close on $A \cap B$. Finally, the second term is bounded by $\rho(2S + 2 + R)$ since

$$d(c_1, c_2) \leq d(a, c_1) + d(a, b) + d(b, c_2).$$

This shows that h is bornologous (since S depends only on R), and properness of h is easy to check.

($\Leftarrow$): Define a new metric d' on X as follows:

$$d'(a,b) = \begin{cases} d(a,b) & a,b \in A \setminus B \\ d(a,b) & a,b \in B \setminus A \\ \mathsf{inf}\{d(a,c)+d(c,b) \mid c \in A \cap B\} & a \in A, b \in B \end{cases}$$

One checks that this is a metric. Consider the inclusions $i : A \to (X, d')$, $j : B \to (X, d')$. They are actually isometric embeddings, and hence coarse. The maps i and j agree on $A \cap B$, so by the universal property of the pushout, there must be a coarse map $h : X \to (X, d')$ which is close to the identity. Since maps which are close to bornologous maps are bornologous, we may assume that h *is* the identity. Suppose that h is ρ-bornologous. Let $R > 0$, and let $x \in B(A, R) \cap B(B, R)$. Without loss of generality, suppose that $x \in A$, and that $d(x, b) \leq 2R$ for $b \in B$. Since h is bornologous, we have (by definition of the metric d') that x must be at most $\rho(2R) + 1$ away from $A \cap B$. Thus we can set $S = \rho(2R) + 1$. □

Note that in the coarse category of metric spaces, there is at most one morphism from a bounded space K to a metric space X (since any two coarse maps $K \to X$ are close). Thus, by general category theoretic arguments, if $A \cap B$ is bounded then the diagram (2.1) is a pushout if and only if X (together with the inclusions) is the coproduct of A and B. Furthermore, note that in condition (b) of Theorem 2.7, one may choose the sets A and B to have non-empty intersection and to be closed (simply take all points which are at most R distance from A and B respectively, for a suitable R). This leads to the following corollary, which can also be verified directly using condition (b) in Theorem 2.7.

Corollary 2. *A metric space X satisfies* (C) *(or equivalently, has a connected Higson corona) if and only if in every ω-excisive decomposition $X = A \cup B$ of X with $A \cap B$ bounded, one of A and B is bounded.*

The fact that proper metric spaces which do not satisfy (C) have disconnected Higson coronas can now be seen as a direct consequence of Proposition 1 in [36], while Lemma 2.8

in this chapter can be seen as a consequence of Proposition 2 in [36] for the proper case (noting that every coarse map is slowly oscillating on a bounded subset). Theorem 2.10 of the present chapter states in part that disconnectedness of the Higson corona νX ensures the existence of a ω-excisive decomposition $X = A \cup B$ of X with A and B unbounded and $A \cap B$ bounded.

2.6 Cohomological characterisation

In this section, we prove the following:

Theorem 2.13. *A metric space M has a connected Higson corona if and only if its first coarse cohomology group $HX^1(M)$ is trivial.*

We briefly recall the definition of coarse cohomology, following [57]. For M a metric space and q a natural number, a subset $E \subseteq M^{q+1}$ is called *controlled* if all the product projections $\pi_1, \ldots, \pi_{q+1}$ are close on E. The subset E is called *bounded* if every product projection is close to a constant map. Note that if $q = 0$, then every subset is controlled, while the bounded sets are precisely the bounded sets in the usual sense.

Definition 2.14. *Let M be a metric space. A subset $D \subseteq M^{q+1}$ is called* cocontrolled *if its intersection with every controlled set $E \subseteq M^{q+1}$ is bounded.*

In the case of $q = 0$, the cocontrolled subsets are precisely the bounded ones. Given an abelian group G, the *coarse complex of M with coefficients in G*, denoted by $CX^*(M; G)$, is defined as the space of functions $\phi : M^{q+1} \to G$ with cocontrolled support. The complex can be equipped with coboundary maps $\delta : CX^{q+1}(M; G) \to CX^{q+2}(M; G)$ defined as follows:

$$\delta\phi(x_0, \ldots, x_{q+1}) = \sum_{i=0}^{q+1} (-1)^i \phi(x_0, \ldots, \hat{x}_i, \ldots, x_{q+1}),$$

where the ‘hat’ denotes omission of a specific term. One checks that this defines a cochain complex, and the *coarse cohomology* $HX^*(M; G)$ is defined to be the cohomology of this complex. When $G = \mathbb{Z}$, we denote the cohomology simply by $HX^*(M)$.

Proof of Theorem 2.13. ($\Rightarrow$): Suppose $f : M^2 \to \mathbb{Z}$ represents a non-trivial cohomology class in $HX^1(M)$. In other words, f satisfies $f(b,c) - f(a,c) + f(a,b) = 0$ (because $\delta f = 0$) but cannot be written as $f(a,b) = g(a) - g(b)$ for any function $g : M \to \mathbb{Z}$ with cocontrolled (equivalently, bounded) support. Define a relation R on M as follows:

$$aRb \Leftrightarrow f(a,b) = 0.$$

It follows from the conditions on f that R is an equivalence relation. We claim that any equivalence class of R has an unbounded complement. Suppose not; let A be an equivalence class with bounded complement and pick $a \in A$. Define a function $g : M \to \mathbb{Z}$ as follows:

$$g(x) = f(x,a).$$

Note that g has bounded support, since the complement of A is assumed to be bounded, and that $\delta g = f$, a contradiction. Thus we conclude that we can divide M into two unbounded sets A and B, each a union of equivalence classes. Let $S > 0$. Then $a \in A$, $b \in B$ and $d(a,b) \leq S$ implies

$$(a,b) \in \mathsf{supp}(f) \cap \{(x,y) \mid d(x,y) \leq S\}$$

where the intersection is bounded because f has cocontrolled support. It follows that A and B satisfy the conditions in (b) of Theorem 2.7, which gives the required result.

($\Leftarrow$): Let $M = A \cup B$ with A and B satisfying the conditions in (b) of Theorem 2.7. We may suppose that A and B are disjoint (simply take $B = M \setminus A$ in the event this is not the case). Let $f : M^2 \to \mathbb{Z}$ be the map

$$f(x,y) = \begin{cases} 1 & x \in A, y \in B \\ -1 & x \in B, y \in A \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that f represents a non-trivial cohomology class in $HX^1(M)$. Indeed, the conditions on A and B force f to have cocontrolled support, while the unboundedness of A and B ensure that f is non-trivial in cohomology. □

Theorem 2.13 shows that coarse coproducts are not preserved (i.e. taken to direct sums) when taking first cohomology groups, since a space may be the coproduct of two unbounded spaces each having a connected Higson corona. For proper metric spaces, one part of the proof above (namely that having a disconnected Higson corona implies non-trivial first cohomology), follows from Corollary 2 in the previous section and the result in [36] that there is a long exact Mayer-Vietoris sequence

$$\dots \longrightarrow HX^0(A) \oplus HX^0(B) \longrightarrow HX^0(A \cap B) \longrightarrow HX^1(M) \longrightarrow \dots$$

for any ω-excisive decomposition $M = A \cup B$. Indeed, if A and B are unbounded and $A \cap B$ is bounded, then $HX^0(A)$ and $HX^0(B)$ are trivial, while $HX^0(A \cap B)$ is isomorphic to $\mathbb{Z}$ (see Section 5.1 of [57]).

2.7 Geodesic spaces and finitely generated groups

We expect that connectedness of the Higson corona should be the same as being "connected at infinity" for certain spaces. In this section we show such a result for the case of geodesic spaces. Recall that a metric space X is said to be *geodesic* (see for example [50]) if for any two points $x, y \in X$ there is an isometric embedding γ of the interval $[0, d(x, y)]$ into X with $\gamma(0) = x$, $\gamma(d(x, y)) = y$. We refer to the image of γ as the geodesic from x to y.

Theorem 2.15. *The following are equivalent for a geodesic metric space X:*

(a) the Higson corona of X is (topologically) disconnected;

(b) there exists a bounded set $X_0 \subseteq X$ such that for any bounded set C containing X_0, $X \setminus C$ is topologically disconnected.

Proof. (a) ⇒ (b): Suppose $X = A \cup B$ with A and B satisfying the conditions in (b) in Theorem 2.7. Let X_0 be the bounded set such that $a \in A \setminus X_0$, $b \in B \setminus X_0 \Rightarrow d(a, b) \geq 1$. Then the result follows easily from the fact that A and B are unbounded.

(b) ⇒ (a): Let X_0 be as in (b). By the assumption on X_0, every connected component of $X \setminus X_0$ must have an unbounded complement in $X \setminus X_0$. It follows that we can divide $X \setminus X_0$ into two sets A and B', each a union of connected components. By connectedness, the geodesic from a point $a \in A \setminus X_0$ to a point $b \in B' \setminus X_0$ must pass through X_0. Thus, for any $R > 0$, the distance between points $a \in A \setminus B(X_0, R)$ and $b \in B' \setminus B(X_0, R)$ is at least R, so A and $B = B' \cup X_0$ satisfy the conditions in (b) in Theorem 2.7. □

Example 2. *The above theorem shows that $\mathbb{R}^n$ has a connected Higson corona for $n \neq 1$, and a disconnected Higson corona when $n = 1$. Note, however, that by Theorem 5 of [42], the Higson corona of $\mathbb{R}^n$ is neither locally connected nor arcwise connected for $n > 1$.*

We now consider the special case of finitely generated groups, seen as metric spaces. Let G be a finitely generated group generated by a finite set S which is closed under taking inverses. We can construct two coarsely equivalent metric spaces associated to the pair (G, S):

- the group G equipped with the *word length metric*, i.e. where $d(g, h)$ is the length of the minimal representation of gh^{-1} using elements of S, and
- the *Cayley graph* $\Gamma(G, S)$ of G, i.e. the graph with vertices the elements of G and an edge from g to gs for every $g \in G$, $s \in S$, viewed as a 1-complex and equipped with the path length metric.

It turns out that both of these metric spaces do not depend, up to coarse equivalence, on the choice of finite generating set S (see for example [50]). In particular, the Higson corona of a group G with the word length metric is invariant under choice of generating set, and is moreover homeomorphic to the Higson corona of any Cayley graph associated to G. The *number of ends* of G is defined to be the number of (topological) ends of its Cayley graph. Note that the number of ends of the Cayley graph also does not depend on the finite generating set S (see for example Section 13 of [30] for more on ends of groups).

Corollary 3. *A finitely generated group G with the word length metric has a connected Higson corona if and only if it has at most one end.*

Proof. By the above remarks, a group G with the word length metric has a connected Higson corona if and only if its Cayley graph $\Gamma(G,S)$ does. Note that $\Gamma(G,S)$ is a geodesic space. Fix a vertex g in $\Gamma(G,S)$ and consider the cover of $\Gamma(G,S)$ by compact sets

$$\overline{B(g,1)} \subseteq \overline{B(g,2)} \subseteq \overline{B(g,3)} \subseteq \cdots$$

Here we use the fact that $\Gamma(G,S)$ is locally finite, i.e. every vertex is an endpoint of finitely many edges. An end of $\Gamma(G,S)$ is then a sequence

$$U_1 \supseteq U_2 \supseteq U_3 \supseteq \cdots$$

where for each i, U_i is a connected component of $\Gamma(G,S) \setminus \overline{B(g,i)}$. If $\Gamma(G,S)$ has more than one end, then there must be an n such that $\Gamma(G,S) \setminus \overline{B(g,n)}$ has two unbounded connected components, in which case $\Gamma(G,S)$ has a disconnected Higson corona by Theorem 2.15. Conversely, if $\Gamma(G,S)$ has a disconnected Higson corona, then by Theorem 2.15 there is some n such that for any bounded set K containing $\overline{B(g,n)}$, $\Gamma(G,S) \setminus K$ has at least two connected components. Notice that since $\Gamma(G,S)$ is locally finite, $\Gamma(G,S) \setminus \overline{B(g,n)}$ has finitely many connected components. It follows that two of its connected components must be unbounded, so that $\Gamma(G,S)$ has more than one end as required. □

Note that Corollary 3 also follows from Theorem 13.5.5 in [30] and the fact that the usual group cohomology of a finitely generated group G (with coefficients in the group ring $\mathbb{Z}G$) coincides with the coarse cohomology of the group as a metric space (see Example 5.21 of [57]).

Remark 1. *A complete characterisation of finitely generated groups with more than one end is already known. A finitely generated group can have either 0, 1, 2 or infinitely many ends; a finitely generated group G has two ends if and only if G has an infinite cyclic subgroup of finite index. The characterisation for infinitely many ends is given by a theorem of Stallings [64, 65]. For more details and proofs of these facts, we refer the reader to [30].*

2.8 Abstract coarse spaces

In this section we consider the more general setting of coarse spaces. Most of the results of the previous sections generalise to this context, so long as one works with coarse spaces which are "connected" in the sense of [57], i.e. in which finite sets are bounded.

Recall from [57] that a *coarse space* is a pair $(X, \mathcal{X})$ where X is a set and $\mathcal{X}$ is a family of binary relations on X which contains the diagonal Δ and which is closed under taking subrelations, inverses, products (i.e. composition of relations) and finite unions. A map f between (the underlying sets of) coarse spaces $(A, \mathcal{A})$ and $(B, \mathcal{B})$ is called *bornologous* if $(f \times f)(R) \in \mathcal{B}$ for every $R \in \mathcal{A}$. Given a coarse space $(A, \mathcal{A})$, a subset B of A is called *bounded* if it is contained in $\{a \in A \mid aRx\}$ for some $R \in \mathcal{A}$ and $x \in A$. The notion of proper map can thus be defined for coarse spaces. Two maps $f, g : (A, \mathcal{A}) \to (B, \mathcal{B})$ are said to be *close* if $\{(f(a), g(a)) \mid a \in A\} \in \mathcal{B}$.

Every metric d on a set A induces a *bounded coarse structure*, consisting of all those relations R for which the set $\{d(a, b) \mid aRb\}$ is bounded. A map $f : A \to B$ between metric spaces is bornologous with respect to the metrics if and only if it is bornologous with respect to the respective bounded coarse structures. Thus $\mathbf{Met_{Born}}$ can be viewed as a full subcategory of the category of coarse spaces and bornologous maps. A coarse space $(A, \mathcal{A})$ is called *connected* [57] if every finite subset of $A \times A$ is in $\mathcal{A}$. Bounded coarse structures associated to metrics (which are not allowed to take the value ∞) are always connected. We say that a coarse structure on a set A is *metrizable* if it is the bounded coarse structure associated to a metric on A. We recall the following result.

Theorem 2.16 ([57])**.** *A connected coarse structure $\mathcal{A}$ on A is metrizable if and only if it is countably generated.*

Let $\mathcal{A}_0$ be a collection of relations on a set A. Then it is easy to show that there exists a smallest coarse structure on A containing $\mathcal{A}_0$, which we denote by $\overline{\mathcal{A}_0}$. By "countably generated" in Theorem 2.16, we mean there is a countable set $\mathcal{A}_0$ of relations such that $\overline{\mathcal{A}_0} = \mathcal{A}$.

Lemma 2.17. *Let $\mathcal{A}$ be a set of relations on a set A and let $f : A \to B$ be a map. Then we have*

$$(f \times f)(\overline{\mathcal{A}}) \subseteq \overline{(f \times f)(\mathcal{A})}.$$

Proof. This follows from the fact that for two relations R and S on A,

$$(f \times f)(R \circ S) \subseteq (f \times f)(R) \circ (f \times f)(S).$$

□

Proposition 2.18. *The category of connected coarse spaces and bornologous maps admits arbitrary coproducts. Moreover, the countable (or finite) coproduct of metrizable coarse spaces is metrizable.*

Proof. Suppose $(A_\alpha, \mathcal{A}_\alpha)_{\alpha \in I}$ is a family of connected coarse spaces. For each A_α, pick a base point $a_\alpha \in \iota_\alpha(A_\alpha)$. Define a coarse structure $\sum_\alpha \mathcal{A}_\alpha$ on the disjoint union $\sum_\alpha A_\alpha$ of the A_α as follows:

$$\sum_\alpha \mathcal{A}_\alpha = \overline{\bigcup_\alpha \mathcal{A}_\alpha \cup \bigcup_{\alpha \in I} \bigcup_{\alpha' \in I} \{(a_\alpha, a_{\alpha'})\}},$$

Note that since each $(A_\alpha, \mathcal{A}_\alpha)$ is connected, $(\sum_\alpha A_\alpha, \sum_\alpha \mathcal{A}_\alpha)$ is also connected. Moreover, it is clearly the smallest connected coarse structure containing all the $\mathcal{A}_\alpha$, so using Lemma 2.17 it is easy to check that this gives the required coproduct.

Suppose now that $I = \mathbb{N}$ and that each $(A_i, \mathcal{A}_i)$ is metrizable, with each $\mathcal{A}_i$ generated by relations $\{\mathcal{A}_{i,1}, \mathcal{A}_{i,2}, \ldots\}$. Note that the set

$$\bigcup_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \{(a_i, a_j)\},$$

is countable, as is the set

$$\{\bigcup_{i=1}^{k-1} \mathcal{A}_{i,k-i} \mid k \in \{2, 3 \ldots\}\}$$

and together they generate $\sum_\alpha \mathcal{A}_\alpha$. Thus by Theorem 2.16, $\sum_\alpha \mathcal{A}_\alpha$ is metrizable. □

It is easy to check that the construction of binary coproducts given in the proposition above gives the binary coproduct in the subcategory of coarse maps as well as in the *coarse* *category of connected coarse spaces*, i.e. the category whose objects are connected coarse spaces and whose morphisms are equivalence classes of coarse maps under the closeness relation.

Throughout the rest of this section, by a *coarse coproduct* of two connected coarse spaces, we mean the coproduct of the two spaces in the category of connected coarse spaces and bornologous maps. By the proof of Proposition 2.18, the coarse coproduct of $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$ is given by

$$(X + Y, \overline{\mathcal{X} \cup \mathcal{Y} \cup \{(x_0, y_0)\}}),$$

where $x_0 \in X$ and $y_0 \in Y$ are arbitrary chosen base points.

Proposition 2.18 gives an alternative proof that the category of metric spaces and bornologous maps admits finite and countable coproducts. The explicit construction of the metric d in Proposition 2.6 can be derived from the proof of Theorem 2.16 (see [57]) and the second half of the proof of Proposition 2.18 above. In the case of binary coproducts (see Proposition 2.5), the description of the metric follows naturally from the following lemma. In order to state the lemma, we introduce the following notation. If X and Y are sets and R is a relation on X, we denote by R_r the smallest reflexive relation on the disjoint union $X+Y$ whose restriction to X is R (it is nothing but the union of R with the diagonal relation on $X + Y$).

Lemma 2.19. *Let $(X + Y, \mathcal{X} + \mathcal{Y})$ be the coarse coproduct of $(X, \mathcal{X})$ and $(Y, \mathcal{Y})$, and let $x_0 \in X$, $y_0 \in Y$. Then every relation in $\mathcal{X} + \mathcal{Y}$ is contained in a relation of the form*

$$(R_r \circ U \circ S_r) \cup (S_r \circ U \circ R_r),$$

where R and S are in $\mathcal{X}$ and $\mathcal{Y}$ respectively, and $U = \{(x_0, y_0), (y_0, x_0)\} \cup \Delta$.

Proof. Let $\mathcal{Z}_0$ be the set of all relations of the form

$$(R_r \circ U \circ S_r) \cup (S_r \circ U \circ R_r),$$

for $R \in \mathcal{X}$ and $S \in \mathcal{Y}$, and let $\mathcal{Z}_1$ be its closure under taking subrelations. Since $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{Z}_1 \subseteq \mathcal{X} + \mathcal{Y}$, it is enough to show that $\mathcal{Z}_1$ is a coarse structure by minimality of $\mathcal{X} + \mathcal{Y}$.

Some straightforward computations show that $\mathcal{Z}_0$ is closed under composition of relations, and the result follows. □

Since condition (C) in Section 2.4 was stated in the language of coarse coproducts, coarse maps and closeness of maps, the definition extends to the case of general coarse spaces. Theorem 2.7 generalizes partially to this setting, as shown in the following theorem. The proof of the theorem is a straightforward adaptation of the proof of Theorem 2.7 now that we have Lemma 2.19, so we omit it.

Theorem 2.20. *For a connected coarse space $(X, \mathcal{X})$, the following are equivalent:*

(a) X doesn't satisfy (C)*;*

(b) there are two unbounded subsets A and B of X such that

- *$X = A \cup B$, and*
- *for any $R \in \mathcal{X}$, there is a bounded set C_R such that if xRx' for $x, x' \in X \setminus C_R$, then $\{x, x'\}$ intersects at most one of A and B,*

(c) is (bijectively) coarsely equivalent to a coarse coproduct $(Y + Z, \mathcal{Y} + \mathcal{Z})$ where neither Y nor Z is bounded.

Example 3. *Let X be any infinite set and let $\mathcal{X}$ be the smallest connected coarse structure on X. In other words, $\mathcal{X}$ consists of all finite subsets of $X \times X$ together with all relations of the form $R \cup \Delta$, where R is a finite subset of $X \times X$. It is easy to show that $(X, \mathcal{X})$ does not satisfy* (C) *(using for example (b) from Theorem 2.20 and noting that finite sets are always bounded in a connected coarse space), and thus has a disconnected Higson corona.*

The above example shows that we cannot hope for a version of (d) from Theorem 2.7 to appear in the above theorem. Indeed, take X to be any uncountable set and $\mathcal{X}$ the smallest connected coarse structure on X. Then there are no proper maps from $(X, \mathcal{X})$ to $\mathbb{Z}$.

A map $f : (X, \mathcal{X}) \to Y$ from a coarse space to a metric space Y is called *slowly oscillating* if for every $R \in \mathcal{X}$ and $\varepsilon > 0$, there exists a bounded set $K \subseteq X$ such that for any $a, b \in X$, $a, b \notin K$,

$$aRb \Rightarrow d(f(a), f(b)) \leq \varepsilon.$$

This, together with a similar definition for tending to 0 at infinity, allows one to define the C^*-algebra $B_h(X)/B_0(X)$ for any coarse space, and consequently also the Higson corona [57]. We have the following generalization of the main theorem.

Theorem 2.21. *The following are equivalent for a connected coarse space $(X, \mathcal{X})$.*

(a) $(X, \mathcal{X})$ *satisfies* (C)*;*

(b) the Higson corona of X is (topologically) connected;

(c) $B_h(X)/B_0(X)$ *does not contain a non-trivial idempotent element.*

Proof. This is an easy adaptation of the metric case now that we have Theorem 2.20 (and in particular, part (b)). Note that for connected coarse spaces, the finite union of bounded sets is bounded. □

Chapter 3

A coarse Tietze Extension theorem

This chapter is based on the accepted manuscript of the following paper: J. Dydak and T. Weighill, Extension theorems for large scale spaces via coarse neighbourhoods, Mediterranean Journal of Mathematics 15, 2018, 59. *The contributions of each author of the above manuscript may be considered roughly equal. Two rounds of revisions took place after comments by the anonymous referees. The introduction has been adapted, otherwise the manuscript has remained more or less unchanged.*

3.1 Introduction

In this chapter, we prove an analogue of a foundational result in general topology: the Tietze Extension Theorem.

Theorem 4 (Tietze Extension Theorem)**.** *Let X be a normal topological space and let A be a closed subset of X. Then any continuous function $f : A \to [0, 1]$ extends to a continuous function $g : X \to [0, 1]$.*

When studying the large scale properties of spaces, one typically replaces continuous functions to subsets of $\mathbb{R}^n$ with slowly oscillating functions to subsets of $\mathbb{R}^n$. A good example of this analogy in action can be found in [22], where asymptotic dimension is approached by considering extensions of slowly oscillating functions to spheres, based on the approach to covering dimension via extensions of continuous maps to spheres. In the same paper, the authors also prove the following result:

Theorem 5 (Dydak-Mitra [22]). *Given a metric space X, any slowly oscillating function on a subset of X to $[0, 1]$ extends to a slowly oscillating function on the whole of X to $[0, 1]$.*

This may be seen as a large scale Tietze Extension Theorem for metric spaces. However, as can be seen, in the context of metric spaces no additional criteria are needed to prove the extension theorem. This is not surprising as the topological Tietze theorem holds true for all metric spaces. Thus just as the topological theorem makes more sense in the broader context of general topological spaces, we will prove a result in a more general framework, namely that of large scale spaces. In fact, we will work in the even more general context of hybrid large scale spaces introduced in [4] – that is, sets equipped with a large scale structure and a topology satisfying a compatibility axiom – so that we are interested in maps which are both slowly oscillating and continuous. Results for large scale spaces (with no topology) can be recovered as special cases of the hybrid results by endowing the large scale space with the discrete topology.

Our purely large scale result (in other words, Theorem 10 with the discrete topology) is very close to a result of Protasov in [53]. The differences between the two results are twofold: firstly, Protasov works within the abstract framework of balleans, a framework which is known to be equivalent to coarse spaces and large scale spaces but which has different axioms. Secondly, his "normality" condition (the condition required for the extension theorem to hold) is different from ours, although the equivalence follows fairly readily from Lemma 2.2 in [53], modulo the translation between balleans and large scale spaces.

The main contribution of this chapter, though, is not so much the result itself but the proof, in that we unite the proofs of three different extension theorems: the classical Tietze theorem in topology, Katetov's theorem for extension of uniformly continuous functions, and the (hybrid) large scale result. This unification is achieved via the general notion of a neighbourhood operator, which is already present in the topology literature. This approach via abstract neighbourhood operators may be of independent interest to readers outside of coarse geometry, and suggests possible further investigation into the use of relational structures such as neighbourhood operators in uniting coarse geometry and topology.

3.2 Hybrid large scale spaces

The main context for the results in this chapter is that of a hybrid large scale space, which is a set equipped with a topology (representing the small scale) and a large scale structure (or ls-structure) which are compatible in a suitable sense. The idea to consider a space equipped with a topology and ls-structure which are compatible goes back to Roe (see Chapter 2 of [57]). Note that the notion of ls-structure is equivalent to the notion of coarse structure in the sense of [57] (see [21]).

Let us recall the definition of ls-structure from [21]. Let X be a set. Recall that the **star** $\mathsf{st}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B. More generally, for two families $\mathcal{B}$ and $\mathcal{U}$ of subsets of X, $\mathsf{st}(\mathcal{B},\mathcal{U})$ is the family $\{\mathsf{st}(B,\mathcal{U}) \mid B \in \mathcal{B}\}$.

Definition 12. *A* ***large scale structure*** $\mathcal{L}$ *on a set* X *is a nonempty collection of families of subsets of* X *(which we call the* ***uniformly bounded families****) satisfying the following conditions:*

(1) $\mathcal{B}_1 \in \mathcal{L}$ *implies* $\mathcal{B}_2 \in \mathcal{L}$ *if each element of* $\mathcal{B}_2$ *consisting of more than one point is contained in some element of* $\mathcal{B}_1$*.*

(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}$ *implies* $\mathsf{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{L}$*.*

A set equipped with a large scale structure will be called an ***ls-space****. A uniformly bounded family which is a cover is also called a* ***scale****.*

Definition 13. *A* ***hybrid large scale space*** *(or* ***hls-space*** *for short) is a set* X *equipped with both a large scale structure and a topology (which together we call the hybrid large scale structure on* X*) such that there is a uniformly bounded cover of* X *which consists of open sets. We call a uniformly bounded open cover an* ***open scale****.*

Note that in an hls-space, any scale can be coarsened to an open scale (we say that $\mathcal{U}$ **coarsens** $\mathcal{V}$ in case $\mathcal{V}$ refines $\mathcal{U}$).

Lemma 1. *Let X be a set equipped with an ls-structure and a topology. Then the following are equivalent:*

(1) X, together with the two structures, gives an hls-space.

(2) there is a uniformly bounded cover $\mathcal{U}$ relative to the ls-structure such that for every subset A in X, $\mathsf{cl}(A) \subseteq \mathsf{st}(A,\mathcal{U})$.

Moreover, if X is an hls-space, then for any open scale $\mathcal{U}$ and any subset A of X, we have $\mathsf{cl}(A) \subseteq \mathsf{st}(A,\mathcal{U})$.

Proof. To prove the last statement, notice that for any open scale $\mathcal{U}$ and subset A of an hls-space X, if $x \in \mathsf{cl}(A)$ and $x \in U \in \mathcal{U}$, then U intersects A. This also gives (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1): Let $\mathcal{U}$ be as in (2). Then the interiors of the elements of the cover $\mathsf{st}(\mathcal{U},\mathcal{U})$ form an uniformly bounded open family $\mathcal{V}$ of subsets. Moreover, if $U \in \mathcal{U}$, then $\mathsf{cl}(X \setminus \mathsf{st}(U,\mathcal{U})) \subseteq \mathsf{st}(X \setminus \mathsf{st}(U,\mathcal{U}),\mathcal{U}) \subseteq X \setminus U$, so $\mathcal{V}$ coarsens $\mathcal{U}$, and thus is a cover as required. □

Example 5. *The canonical example of a large scale space is a metric space (X, d) equipped with the ls-structure consisting of all families $\mathcal{U}$ of subsets which refine $\{B(x, R) \mid x \in X\}$ for some $R > 0$. In fact, this ls-structure together with the metric topology gives an hls-space.*

Every result in this chapter which is proved for hls-spaces provides a version of the result for ls-spaces as a special case. This is because any ls-space can be viewed as an hls-space by equipping it with the discrete topology. On the other hand, every topological space can be viewed as an hls-space by equipping it with the ls-structure consisting of all families of subsets.

Let us now consider the notion of connectedness in the context of (hybrid) large scale spaces. In any scale category (see [4]) an important issue is connectedness at some scale, that is, the existence of a scale such that any two points in X are connected by a chain in that scale.

Definition 14. *Given a cover $\mathcal{U}$ of a set X and two points x and y in X, we say that x and y are $\mathcal{U}$-connected and write $x \sim_{\mathcal{U}} y$ if there is a finite sequence U_i, $1 \le i \le n$, of elements of $\mathcal{U}$ such that $U_i \cap U_{i+1} \neq \emptyset$ for all $i < n$, $x \in U_1$ and $y \in U_n$. A **$\mathcal{U}$-component** of X is an equivalence class of the equivalence relation $\sim_{\mathcal{U}}$. We say that X is **$\mathcal{U}$-connected**, or **connected at the scale $\mathcal{U}$**, if it has at most one $\mathcal{U}$-component.*

In the case of ls-spaces one is often interested in spaces that are $\mathcal{U}$-connected for some uniformly bounded cover $\mathcal{U}$ (for example, every geodesic metric space, as an ls-space, is such). This is not to be confused with the weaker condition, called **coarse connectedness** by Roe [57], which, in terms of uniformly bounded covers, can be translated as saying that any two points are $\mathcal{U}$-connected for some uniformly bounded cover $\mathcal{U}$. Clearly this is the same as to say that all finite subsets of X are bounded (a subset of an ls-space is called **bounded** if it is an element of some uniformly bounded cover).

Definition 15. *A **coarse component** of a point x in an ls-space is the union of all the bounded sets containing x.*

A non-empty ls-space X is thus coarsely connected in the sense of Roe if it has only one coarse component. To distinguish Roe's version of connectivity from the stronger one we introduce the following concept:

Definition 16. *An ls-space is **scale connected** if it is $\mathcal{U}$-connected for some uniformly bounded cover $\mathcal{U}$.*

For example, the subspace $\{x^2 \mid x \in \mathbb{N}\}$ with the ls-structure induced by the usual metric, is coarsely connected but not scale connected. In an hls-space, the topology of X dictates large scale connectivity:

Proposition 5. *Let X be an hls-space and let $\mathcal{U}$ be an open scale. Then*

(1) the $\mathcal{U}$-components of X are open-closed,

(2) the coarse components of X are open-closed,

(3) if the topology of X is connected, then X is connected at all open scales.

Proof. Given an open scale $\mathcal{U}$, the $\mathcal{U}$-components of X are clearly open and they partition X, which proves (1). Part (2) follows from the fact that every bounded set is contained in an open bounded set. Finally, (3) follows from (1). □

Lemma 2. *If X is a hybrid large scale space and $\mathcal{U} = \{U_s\}_{s\in S}$ is an open scale of X, then each $\mathcal{U}$-component of X can be expressed as a union $\bigcup_{n=1}^{\infty} A_n$, where the sequence $\{A_n\}$ satisfies the following properties:*

1. *each A_n is closed and bounded,*

2. *A_n is contained in the interior of A_{n+1} for each $n \geq 1$.*

Proof. Let A be a $\mathcal{U}$-component of X, and let $U_s \in \mathcal{U}$ be contained in A. Consider the sequence B_n defined as follows:

1. $B_1 = U_s$,

2. $B_{n+1} = st(B_n, \mathcal{U})$.

By the definition of $\mathcal{U}$-component, the union $\bigcup_{n=1}^{\infty} B_n$ is the whole of A. Define $A_n = \mathrm{cl}(B_n)$ for each n. Then the sequence A_n satisfies the conditions by Lemma 1. □

Lemma 3. *If X is a hybrid large scale space that is coarsely connected, then all precompact subsets (that is, subsets whose closure is compact) of X are bounded.*

Proof. This is an easy consequence of Lemma 2 and the fact that in any coarsely connected ls-space, the finite union of bounded sets is bounded. □

Definition 17. *A subset K of an ls-space X is called **weakly bounded** if its intersection with any coarse component is bounded.*

We now recall the definition of a slowly oscillating map. For a coarsely connected ls-space X, a slowly oscillating map is usually defined as a map to a metric space M such that for every uniformly bounded family $\mathcal{U}$ in X and every $\varepsilon > 0$ there is a bounded set K in X such that $(U \in \mathcal{U}) \wedge (U \cap K = \varnothing) \implies \mathsf{diam}(f(U)) < \varepsilon$. If X is not coarsely connected, then this definition is too restrictive. Indeed, it is easy to check that, under this definition, a slowly oscillating map must be constant on all but one of the coarse components of X. Thus we use the following definition taken from [4], which reduces to the usual definition when X is coarsely connected, and agrees with the classical definition of Higson function in [57] for proper metric spaces (or more generally for proper hls-spaces introduced in the next section).

Definition 18. *Let X be an ls-space, M a metric space and $f : X \to M$ a map. Then f is* ***slowly oscillating*** *if for every uniformly bounded family $\mathcal{U}$ in X and every $\varepsilon > 0$ there is a weakly bounded set K in X such that $(U \in \mathcal{U}) \wedge (U \cap K = \varnothing) \implies \mathsf{diam}(f(U)) < \varepsilon$.*

Note that under Definition 18, a map from an ls-space X is slowly oscillating if and only if its restriction to each coarse component is slowly oscillating.

3.3 Proper hls-spaces

By Lemma 3, every precompact subset of a coarsely connected hls-space is bounded. On the other hand, even for metric spaces with the induced hls-structure, it may not be the case that all bounded sets are precompact.

Definition 19. *A hybrid large scale space X is called* ***proper*** *if its topology is Hausdorff, and its family of bounded sets is identical with the family of all precompact subsets of X. In particular, X is (topologically) locally compact and coarsely connected.*

For example, any proper metric space (i.e. in which bounded sets are precompact) together with the induced hls-structure is a proper hls-space. It might initially appear that the notion of a proper hybrid large scale space is a generalization of the notion of coarsely connected **proper coarse space** introduced by Roe (see [57], Definition 2.35), since the assumption of paracompactness is missing in our definition. However, Corollary 4 below shows that a proper hls-space X must be paracompact, so the two notions are, in fact, identical.

Corollary 4. *The topology of any proper hybrid large scale space is paracompact.*

Proof. Pick an open scale $\mathcal{U}$ and express each $\mathcal{U}$-component of X as in Lemma 2. Suppose $\mathcal{V}$ is an open cover of X. Since each A_n is closed and compact, we may suppose that each A_n intersects only finitely many elements of $\mathcal{V}$. Moreover, each A_n is paracompact since it is compact Hausdorff. Pick a partition of unity on A_1 subordinate to the cover $\{V \cap A_1 \mid V \in \mathcal{V}\}$. By Theorem 1.5 in [20], we can extend this to a partition of unity on A_2 which is subordinate to the cover $\{V \cap A_2 \mid V \in \mathcal{V}\}$. Inductively we obtain a partition of unity on the whole of X subordinate to the cover $\mathcal{V}$, where the continuity follows from the fact that each A_n is contained in the interior of A_{n+1}. □

Corollary 5. *There is no proper hybrid large scale structure on the space of all countable ordinals S_Ω whose topology is the order topology.*

Proof. S_Ω with the order topology is the basic example of a normal space that is not paracompact (see [47]). □

We also have the following corollary of Lemma 2.

Corollary 6. *If X is a proper hls-space and $\mathcal{U}$ is an open scale, then each $\mathcal{U}$-component admits a countable basis of bounded sets, that is, a countable set $\mathcal{B}$ of bounded sets such that every bounded set is contained in some element of $\mathcal{B}$. In particular, each $\mathcal{U}$-component of X is σ-compact.*

If $f : X \to Y$ is a map from an ls-space X to an ls-space Y, we say that f is **large-scale continuous** or **ls-continuous** if for every uniformly bounded family $\mathcal{U}$ in X, the family

$$f(\mathcal{U}) = \{f(U) \mid U \in \mathcal{U}\}$$

is uniformly bounded in Y. A map $f : X \to Y$ between hls-spaces is called **hls-continuous** if it is continuous with respect to the topologies and ls-continuous with respect to the ls-structures. Two ls-continuous maps $f, g : X \to Y$ are said to be **close** if the family $\{\{f(x), g(x)\} \mid x \in X\}$ is uniformly bounded.

Recall that an ls-continuous map $f : X \to Y$ between ls-spaces is called a **large scale equivalence** (or coarse equivalence) if there is an ls-continuous map f' in the other direction such that ff' and $f'f$ are both close to the identity, i.e. such that both families

$$\{\{ff'(y), y\} \mid y \in Y\},\ \{\{f'f(x), x\} \mid x \in X\}$$

are uniformly bounded. It is easy to check that an ls-continuous map $f : X \to Y$ is a large scale equivalence if and only if both of the following hold:

- f is **coarsely surjective**, i.e. there is a uniformly bounded family $\mathcal{U}$ in Y such that $Y \subseteq \mathsf{st}(f(X), \mathcal{U})$;
- f is a a **coarse embedding**, i.e. for every uniformly bounded family $\mathcal{U}$ in Y, $f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is uniformly bounded in X.

Proposition 6. *If X is a hybrid large scale space, then it contains a topologically discrete subset Y such that the inclusion $i : Y \to X$, with the induced ls-structure on Y, is a large scale equivalence. If X is proper, then Y can be chosen such that the bounded subsets of Y are finite.*

Proof. Pick an open scale $\mathcal{U}$ of X. Let Y be a maximal subset of X with respect to the following property: $Y \cap U$ contains at most one point for all $U \in \mathcal{U}$. The inclusion $i : Y \to X$ is clearly coarsely surjective, and it is a coarse embedding since the ls-structure on Y is induced by X. □

Proposition 6 is well known in the case where X is a metric space. Indeed, the discrete subset Y can be realised as a maximal 1-separated subset of X, that is, a maximal subset Y with respect to the property that any two points of Y are at least distance 1 apart (see for example Section 1 of [32]).

3.4 Neighbourhood operators

In order to make clear the connection between the results for (hybrid) large scale spaces contained in this chapter and the classical topological results for topological spaces, we prove some results in the context of a set equipped with a neighbourhood operator satisfying certain axioms. By a **neighbourhood operator** on a set X we mean a binary relation $\prec$ on the power set $\mathcal{P}(X)$ of X such that $A \prec B \implies A \subseteq B$. If $A \prec B$, we say that B is a neighbourhood of A with respect to $\prec$. Neighbourhood operators appear in many places in the literature: see for example [13] for applications to topology, or [37] for a more categorical approach. For our purposes, we will be interested in neighbourhood operators $\prec$ on a set X satisfying the following conditions:

(N0) $A \prec X$ for all $A \subseteq X$.

(N1) if $A \prec B$ then $X \setminus B \prec X \setminus A$.

(N2) if $A \prec B \subseteq C$, then $A \prec C$.

(N3) if $A \prec N$ and $A' \prec N'$ then $A \cup A' \prec N \cup N'$.

It is easy to see that, together, axioms (N0) – (N3) imply:

(N0′) $\varnothing \prec A$ for all $A \subseteq X$.

(N2′) if $A \subseteq B \prec C$ then $A \prec C$.

(N3′) if $A \prec N$ and $A' \prec N'$ then $A \cap A' \prec N \cap N'$.

We now introduce some examples of neighbourhood operators, the first three of which are the most important for our purposes.

- the **topological neighbourhood operator** on a topological space X: define $A \prec B$ if and only if B contains an open set containing $\mathsf{cl}(A)$.

- the **coarse neighbourhood operator** on an ls-space X: define $A \prec B$ if and only if B is a **coarse neighbourhood** of A, that is, $A \subseteq B$ and for every uniformly bounded cover $\mathcal{U}$ of X, $\mathsf{st}(A, \mathcal{U})$ is contained in $B \cup K$ for some weakly bounded set K.

- the **hybrid neighbourhood operator** on an hls-space X: define $A \prec B$ if and only if B is a neighbourhood of A with respect to the topological neighbourhood operator and the coarse neighbourhood operator on X.

- the **uniform neighbourhood operator** on a uniform space X (see for example [38]): define $A \prec B$ if there is a uniform cover $\mathcal{U}$ such that such that $\mathsf{st}(A,\mathcal{U}) \subseteq B$.

If B is a neighbourhood of A with respect to the topological neighbourhood operator, then we say that B is a topological neighbourhood of A, and similarly for the coarse and hybrid neighbourhood operators. For proper metric spaces the notion of coarse neighbourhood is closely related to the notion of asymptotic neighbourhood in [6]. Indeed, a coarse neighbourhood of a subset A of a proper metric space is nothing but an asymptotic neighbourhood of A which contains A. Note that what we in this chapter call the topological neighbourhood operator does not capture the neighbourhood relation in the usual sense (that is, where B is a neighbourhood of A if and only if B contains an open set which contains A). Indeed, the usual neighbourhood relation does not satisfy (N1), while one can check that the topological neighbourhood operator above does.

Observation 1. *Conditions* (N0)-(N3) *are satisfied by all four examples given above.*

Remark 2. *Clearly every neighbourhood operator $\prec$ on a set X which satisfies* (N0) *and* (N3′) *induces a topology on X wherein a set $U \subseteq X$ is open if and only if for every $x \in U$, $\{x\} \prec U$. For a T_1 topological space and the topological neighbourhood operator, this recovers the original topology. For a uniform space and the uniform neighbourhood operator, this recovers the topology induced by the uniform structure in the usual sense. For a large scale space and the coarse neighbourhood operator, every superset of a singleton set is a coarse neighbourhood of that set, so the induced topology is the discrete topology.*

Definition 20. *Let X and Y be sets equipped with neighbourhood operators $\prec_X$ and $\prec_Y$ respectively. A set map $f : X \to Y$ is called* ***neighbourhood continuous with respect to $\prec_X$ and $\prec_Y$*** *if $A \prec_Y B \implies f^{-1}(A) \prec_X f^{-1}(B)$ for any subsets A and B of Y.*

We now show that neighbourhood continuity generalises both topological continuity and being slowly oscillating for maps into $[0, 1]$.

Proposition 7. *Let X and Y be topological spaces and let $f : X \to Y$ be a set map. If f is topologically continuous, then it is neighbourhood continuous with respect to the topological neighbourhood operators on X and Y. If Y is a T_1-space then the converse also holds.*

Proof. It is easy to check that if f is topologically continuous then it is also neighbourhood continuous. Suppose then that f is neighbourhood continuous, Y is a T_1 space and let A be an open set with $f(x) \in A$. Since the point $f(x)$ is closed, we have $\{f(x)\} \prec A$. Thus we have $f^{-1}(f(x)) \prec f^{-1}(A)$, which gives us continuity at x. □

For convenience, when we are referring to a map f from a set X equipped with a neighbourhood operator $\prec$ to a subset of $\mathbb{R}$, we say that f is neighbourhood continuous to mean that it is neighbourhood continuous with respect to the neighbourhood operator $\prec$ and the topological neighbourhood operator on the codomain.

Lemma 4. *Let X be a set and $\prec$ a neighbourhood operator on X satisfying* (N0) – (N3)*. A map $f : X \to [0,1]$ is neighbourhood continuous if and only if for every $a < b$ in $[0,1]$ we have $f^{-1}([0,a]) \prec f^{-1}([0,b))$.*

Proof. ($\Rightarrow$) is obvious.

($\Leftarrow$) Suppose A has neighbourhood N in $[0,1]$ relative to the topological neighbourhood operator. We must show that $f^{-1}(A) \prec f^{-1}(N)$. We may suppose that N is open and A is closed since the neighbourhood operator $\prec$ satisfies (N2) and (N2′). Since $[0,1]$ is compact, there is an $\varepsilon > 0$ such that $B(A,\varepsilon) \cap [0,1] \subseteq N$ (indeed, the function $x \mapsto d(x, X \setminus N)$ achieves a minimum on A which cannot be 0). The connected components of $A' = \{x \in [0,1] \mid \exists_{a\in A}\, d(a,x) \leq \varepsilon/2\}$ have diameter at least ε, so A' is a finite union of closed intervals. Moreover, A' contains A and is contained in $N' = B(A',\varepsilon/2) \cap [0,1]$, which is in turn contained in N. Using (N2) and (N2′) again together with (N3), we may thus reduce to the case where $A = [a,b]$ and $N = (a',b') \cap [0,1]$ for $a' < a$ and $b < b'$, and we can choose $b' < 1$ if $b < 1$. If $0 < a' < b' < 1$, then noticing that $[a,b] = [0,b] \cap ([0,1] \setminus [0,a))$ and $(a',b') = [0,b') \cap ([0,1] \setminus [0,a'])$ and using condition (N2′) and (N1), we can reduce to the case of $A = [0,x]$ and $N = [0,y)$ for $x < y$. If $a = 0$ and $b < 1$, then we are already reduced to the desired case. Finally, if $A = [0,1]$, then by (N0), we have that $f^{-1}(A) = X \prec X = f^{-1}(N)$, so we can discard this case. □

Proposition 8. *Let X be an ls-space and $f : X \to [0,1]$ a set map. Then f is slowly oscillating if and only if f is neighbourhood continuous with respect to the coarse neighbourhood operator on X and the topological neighbourhood operator on $[0,1]$.*

Proof. It is enough to consider the case when X is coarsely connected.

($\Rightarrow$) Let $a < b$, with $b-a = \varepsilon$. If $\mathcal{U}$ is a uniformly bounded cover, then there is a bounded set K in X such that $f(U)$ has diameter less than $\varepsilon/2$ for every U in $\mathcal{U}$ not contained in K. Thus $\mathsf{st}(f^{-1}([0,a]),\mathcal{U})$ is contained in $f^{-1}([0,a+\varepsilon)) \cup K$, which gives the result by Lemma 4.

($\Leftarrow$) Suppose that f is not slowly oscillating. Then there is an $\varepsilon > 0$ and a uniformly bounded cover $\mathcal{U}$ of X such that $Y = \bigcup\{U \in \mathcal{U} \mid \mathsf{diam}(f(U)) > \varepsilon\}$ is unbounded. Divide $[0,1]$ into consecutive closed intervals $I_1, \ldots, I_k$ of length less than $\varepsilon/2$ with non-empty interior, and let $I_0 = I_{k+1} = \varnothing$ for convenience. Then there exists a $1 \leq m \leq k$ such that $f^{-1}(I_m) \cap Y$ is unbounded (otherwise Y is a finite union of bounded sets). The subset $N = I_{m-1} \cup I_m \cup I_{m+1}$ is a topological neighbourhood of I_m, but $\mathsf{st}(f^{-1}(I_m),\mathcal{U}) \setminus f^{-1}(N)$ is not bounded, so $f^{-1}(N)$ is not a coarse neighbourhood of $f^{-1}(I_m)$. □

Proposition 9. *Let X be an hls-space and $f : X \to [0,1]$ a set map. Then f is continuous and slowly oscillating if and only if f is neighbourhood continuous with respect to the hybrid neighbourhood operator on X and the topological neighbourhood operator on $[0,1]$.*

Proof. This follows from Proposition 7 and 8 above. □

Now that we have motivated the notion of neighbourhood continuity, we are ready to prove some general results about neighbourhood operators. Before we do, we introduce a "normality" condition on a neighbourhood operator $\prec$.

(N4) for every pair of subsets $A \prec C$, there is a subset B with $A \prec B \prec C$.

Lemma 5. *Let X be a set equipped with a neighbourhood operator $\prec$ satisfying* (N0)–(N3) *and let $\{A_s\}_{s \in S}$ be a family of subsets of X indexed by a dense subset S of $[0,1]$. If, for each $s < t \in S$, we have $A_s \prec A_t$, then the function $f : X \to [0,1]$ defined by*

$$f(x) = \inf\{t \mid x \in A_t\}$$

is neighbourhood continuous.

Proof. Let $[0,a] \subseteq [0,b)$ be subsets of $[0,1]$. Pick $s, s' \in S$ such that $a < s < s' < b$. Then

$$f^{-1}([0,a]) \subseteq A_s \prec A_{s'} \subseteq f^{-1}([0,b)).$$

Thus by Lemma 4 we obtain the result. □

Theorem 6 (Urysohn's Lemma for neighbourhood operators)**.** *Let X be a set and $\prec$ a neighbourhood operator satisfying* (N0)–(N3)*. Then the following are equivalent:*

(1) $\prec$ satisfies (N4)*,*

(2) for any subsets A and B of X such that $A \prec X \setminus B$, there is a neighbourhood continuous function $f : X \to [0,1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.

Proof. (1) $\implies$ (2): By Lemma 5, it is enough to produce a family of subsets A_s indexed by a dense subset S of $[0,1]$ such that $A_s \prec A_t$ whenever $s < t$. Using (N4) we can define such subsets indexed by the dyadic fractions, starting with $A_0 = A$ and $A_1 = X \setminus B$.

(2) $\implies$ (1): Given $A \prec N$, construct a neighbourhood continuous map f taking A to 0 and $X \setminus N$ to 1. Then $f^{-1}([0,1/2))$ is the required intermediate neighbourhood. □

Notice that the proof of Theorem 6 is a straightforward adaptation of the standard proof of Urysohn's Lemma from topology. We can recover the classical Urysohn's Lemma from Theorem 6 above.

Lemma 6. *Let X be a topological space. Then X is normal if and only if the topological neighbourhood operator on X satisfies* (N4)*.*

Corollary 7 (Urysohn's Lemma)**.** *Let X be a normal topological space. Then for any closed disjoint subsets A and B of X there is a continuous map $f : X \to [0,1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.*

3.5 Hybrid large scale Urysohn's Lemma

In this section we apply the results of the previous section to prove results for hybrid large scale spaces.

Definition 21. *Let X be an ls-space and A, B be subsets of X. We say that A and B are* ***coarsely separated*** *for every uniformly bounded family $\mathcal{U}$ in X, $\mathsf{st}(A,\mathcal{U}) \cap \mathsf{st}(B,\mathcal{U})$ is weakly bounded.*

Note that in the case of metric spaces, this is the same as saying that A and B **diverge** in the sense of [17]. Clearly if A and B are disjoint subsets of an ls-space X, then A and B are coarsely separated if and only if $X \setminus B$ is a coarse neighbourhood of A.

Definition 22. *Let X be a hybrid large scale space. We say that X is* ***hybrid large scale normal*** *(or hls-normal) if for every closed subset A and every hybrid neighbourhood N of A, there is a closed subset V of X such that V is a hybrid neighbourhood of A and N is a hybrid neighbourhood of V. We say that an ls-space is* ***ls-normal*** *if it is hybrid large scale normal when equipped with the discrete topology.*

Lemma 7. *An hls-space is hls-normal if and only if the hybrid neighbourhood operator satisfies* ($\mathsf{N4}$)*.*

Proof. ($\Rightarrow$) Suppose A has a hybrid neighbourhood N. In particular then, $\mathsf{cl}(A)$ is contained in the interior of N. But since $\mathsf{cl}(A) \subseteq \mathsf{st}(A,\mathcal{U})$ for any open scale $\mathcal{U}$, N is a coarse neighbourhood of $\mathsf{cl}(A)$. Thus N is a hybrid neighbourhood of $\mathsf{cl}(A)$, and we can obtain an intermediate hybrid neighbourhood as required.

($\Leftarrow$) Consider a closed subset A and a hybrid neighbourhood N of A. Condition ($\mathsf{N4}$) gives us an intermediate hybrid neighbourhood V. Taking $\mathsf{cl}(V)$, by similar arguments to the previous direction, produces the required closed intermediate hybrid neighbourhood. □

Combining Lemma 7 and Lemma 6 we obtain:

Corollary 8 (Urysohn's Lemma for hybrid large scale spaces)**.** *Let X be an hls-space. Then the following are equivalent:*

(1) X is hls-normal,

(2) if A has hybrid neighbourhood N, then there is a continuous slowly oscillating map $f : X \to [0, 1]$ such that $f(A) \subseteq \{0\}$ and $f(X \setminus N) \subseteq \{1\}$,

(3) for any closed coarsely separated disjoint subsets A and B of X there is a continuous slowly oscillating map $f : X \to [0,1]$ such that $f(A) \subseteq \{0\}$ and $f(B) \subseteq \{1\}$.

Proof. The only part which needs proving is the equivalence of (2) and (3). Clearly (2) implies (3). To show (3) implies (2), notice that if A has hybrid neighbourhood N, then $\mathsf{cl}(A)$ and $\mathsf{cl}(X \setminus N)$ are closed, coarsely separated and disjoint. □

3.6 Hls-normal spaces

In this section we look at some more properties of hls-normal spaces, as well as some classes of examples of hls-normal hls-spaces.

Lemma 8. *The topology of any hls-normal hls-space is normal.*

Proof. It is enough to consider the case where X is $\mathcal{U}$-connected for some open scale $\mathcal{U}$ (since the $\mathcal{U}$-components are open-closed). Notice that the topology induced on any closed and bounded subset Y of X is normal due to the fact that any two disjoint, closed subsets of Y are coarsely separated in X. Express X as in Lemma 2. Since each A_i is topologically normal, so is their directed union. Indeed, for any closed subsets A and B of X, define a continuous function f_1 taking $A \cap A_1$ to 0 and $B \cap A_1$ to 1. Then use the Tietze Extension Theorem to extend the function which agrees with f_1 on A_1 and which sends $A \cap A_2$ to 0 and $B \cap A_2$ to 1 to all of A_2. Continuing this process one defines a function f inductively which sends A to 0 and B to 1, and which is continuous by the conditions on the A_n. □

Theorem 7. *If X is a hybrid large scale space, then the following conditions are equivalent:*

(1) X is hls-normal,

(2) X is ls-normal as an ls-space and the topology of X is normal.

Proof. (1) $\Rightarrow$ (2) : In view of Lemma 8, the topology of X is normal. Suppose A and B are two disjoint and coarsely separated subsets of X. Then the closures of A and B are also coarsely separated and disjoint outside of a bounded set K by Lemma 1. Thus there is a a slowly oscillating continuous function $f : X \to [0,1]$ sending $cl(A) \setminus K$ to 0 and sending

$cl(B) \setminus K$ to 1 by hls-normality. Redefine f on K such that it sends $A \cap K$ to 0 and $B \cap K$ to 1, and notice that the new f is slowly oscillating.

(2) $\Rightarrow$ (1) : Suppose A is a closed subset of X and U is its hybrid neighbourhood. We need to find a closed coarse neighbourhood V of A such that U is a coarse neighbourhood of V. Pick an open scale $\mathcal{U}$ of X and pick a coarse neighbourhood W of A so that U is a coarse neighbourhood of W. The set $V = \mathsf{cl}(\mathsf{st}(W, \mathcal{U}))$ is closed and is a coarse neighbourhood of A. However, it may not be contained in U. Nonetheless, since V is contained in $\mathsf{st}(\mathsf{st}(W, \mathcal{U}), \mathcal{U})$, it is contained in $U \cup K$ for some bounded set K. Using the topological normality of X, we may find a closed topological neighbourhood V' of A which is contained in U. The required intermediate closed hybrid neighbourhood between A and U is then $(V \setminus \mathsf{st}(K, \mathcal{U})) \cup (V' \cap \mathsf{cl}(\mathsf{st}(K, \mathcal{U})))$. □

Thus we may say that

$$\textbf{hls-normality} = \textbf{topological normality} + \textbf{ls-normality}.$$

We should note that the compatibility axiom played a crucial role in the proof of this fact. We now present some examples of hls-normal spaces. In particular, we show that metric spaces, both with the usual ls-structure and with the C_0 structure introduced by Wright, are hls-normal, as is any set equipped with the maximal uniformly locally finite ls-structure.

Definition 23 (Wright [71])**.** *Let (X, d) be a metric space. Let $\mathcal{L}$ be the collection of all families $\mathcal{U}$ of subsets of X such that for every $\varepsilon > 0$, there is a bounded set $B \subseteq X$ such that $\mathsf{diam}(U) \le \varepsilon$ for all $U \in \mathcal{U}$ not intersecting B. Then $\mathcal{L}$ is an ls-structure, called the C_0* ***ls-structure*** *associated to the metric d.*

Proposition 10. *Let X be a metric space equipped with the metric topology. Then X equipped with either the metric or C_0 ls-structure is hls-normal.*

Proof. The same construction works for both the metric and C_0 ls-structures. Let A be a closed subset and U a hybrid neighbourhood of A. Let V be the set of all points $x \in X$ such that $d(x, A) \le d(x, X \setminus U)$. Clearly V is closed and contains a neighbourhood of A. We claim that V is an intermediate coarse neighbourhood between A and U. Indeed, let $\mathcal{U}$ be a cover of X by balls of bounded radii. If $\mathsf{st}(A,\mathcal{U})$ intersects $X \setminus V$ in an unbounded set, then it is easy to check that $\mathsf{st}(A,\mathcal{U}')$ intersects $X \setminus U$ in an unbounded set, where $\mathcal{U}'$ is the set formed from $\mathcal{U}$ by replacing every ball $B(x,R)$ by $B(x,2R)$. This is a contradiction since $\mathcal{U}'$ is uniformly bounded whenever $\mathcal{U}$ is for both ls-structures. A similar argument shows that U is a hybrid neighbourhood of V. □

A family $\mathcal{U}$ of subsets of a set X is **uniformly locally finite** if there is a natural number m so that $\text{card}(st(x,\mathcal{U})) \leq m$ for all $x \in X$

Definition 24 (Sako [59]). *A large scale space X is **uniformly locally finite** if every uniformly bounded cover $\mathcal{U}$ of X is uniformly locally finite.*

On any set X the collection of all uniformly locally finite families forms an ls-structure (called the **maximal uniformly locally finite ls-structure**), which is the largest uniformly locally finite ls-structure on X. Viewed as a coarse structure in the sense of Roe, the maximal uniformly locally finite structure is nothing but the universal bounded geometry structure in the sense of [57].

Proposition 11. *Let X be a set equipped with the maximal uniformly locally finite ls-structure. Then X is an ls-normal space.*

Proof. Note that given any two coarsely separated subsets A and B of X relative to this structure, one of them is finite. The result follows easily from this observation. □

3.7 Non-normal spaces

At this point, one might ask if there are any hls-spaces which are not hls-normal. An example of an ls-space which is not ls-normal is described below in Proposition 12. In Section 3.11 we will also see a class of topological groups which are not hls-normal as hls-spaces.

Proposition 12. *Let X be the subset of the upper half-plane of $\mathbb{Z}^2$ given by $-y \leq x \leq y$, $y > 0$. Let $A = \{(x,x) \mid x \in \mathbb{Z}, x > 0\}$ and $B = \{(-x,x) \mid x \in \mathbb{Z}, x > 0\}$. Define an ls-structure on this space as follows: let $\mathscr{L}$ be the set of all uniformly locally finite families $\mathcal{V}$ such that for any scale $\mathcal{U}$ in the metric ls-structure on X the set of $V \in \mathcal{V}$ intersecting $\mathsf{st}(A \cup B,\mathcal{U})$ is uniformly bounded in the metric ls-structure on X. Then:*

(1) The collection $\mathscr{L}$ is a uniformly locally finite ls-structure on X. The uniformly bounded families with respect to the metric ls-structure are members of $\mathcal{L}$.

(2) A and B are coarsely separated in $(X, \mathscr{L})$.

(3) there is no slowly oscillating (with respect to $\mathscr{L}$) function $f : X \to [0,1]$ such that $f(A) = 0$, $f(B) = 1$.

so that, in particular, X is not hls-normal.

Proof. (1) Suppose $\mathcal{V}_1, \mathcal{V}_2 \in \mathscr{L}$ are covers. We will show that $\mathsf{st}(\mathcal{V}_1, \mathcal{V}_2)$ is in $\mathscr{L}$. Clearly $\mathsf{st}(\mathcal{V}_1, \mathcal{V}_2)$ is uniformly locally finite. Suppose then that $\mathcal{U}$ is a scale in the metric ls-structure on X, and let $U = \mathsf{st}(A \cup B, \mathcal{U})$. Let $\mathcal{V}_2'$ be the family of elements of $\mathcal{V}_2$ intersecting U, $\mathcal{V}_1'$ the family of elements of $\mathcal{V}_1$ intersecting $\mathsf{st}(U, \mathcal{V}_2')$ and $\mathcal{V}_2''$ the family of elements of $\mathcal{V}_2$ intersecting $\mathsf{st}(\mathsf{st}(U, \mathcal{V}_2'), \mathcal{V}_1')$. Each of these families is uniformly bounded in the metric ls-structure, and the family of elements of $\mathsf{st}(\mathcal{V}_1, \mathcal{V}_2)$ intersecting U clearly refines $\mathsf{st}(\mathcal{V}_1', \mathcal{V}_2' \cup \mathcal{V}_2'')$, so it is uniformly bounded in the metric ls-structure.

(2) Suppose $\mathcal{V} \in \mathscr{L}$. The set $\mathsf{st}(A, \mathcal{V}) \cap \mathsf{st}(B, \mathcal{V})$ is contained in the union of all sets $V_1 \cap V_2$, where $V_1 \in \mathcal{V}$ intersects A and $V_2 \in \mathcal{V}$ intersects B, so the family of those sets forms a uniformly bounded family $\mathcal{U}$ in the metric ls-structure on X. Therefore $\mathsf{st}(A, \mathcal{V}) \cap \mathsf{st}(B, \mathcal{V}) \subset \mathsf{st}(A, \mathcal{U}) \cap \mathsf{st}(B, \mathcal{U})$ which is finite.

(3) Suppose such an f exists. Let X_i be the set $\{(x, i) \mid x \in \mathbb{Z}\} \cap X$. Consider the cover of X by 2-balls in the l_1-metric structure on X. Since f is in particular slowly oscillating with respect to the metric ls-structure, there is some $M > 0$ such that outside of the M-ball at $(1, 1)$ one has $|f(z_1) - f(z_2)| < 1/6$ if z_1, z_2 are on the same horizontal line and their distance is 1. Therefore $f^{-1}([1/6, 1/3])$ and $f^{-1}([2/3, 5/6])$ both intersect X_i for $i > M + 2$. Take $z_i \in f^{-1}([1/6, 1/3]) \cap X_i$ and $w_i \in f^{-1}([2/3, 5/6]) \cap X_i$ for $i > M+2$. Notice $\mathcal{Z} = \{z_i, w_i\}_{i > M+2}$ belongs to $\mathscr{L}$. Indeed, since f is slowly oscillating with respect to the metric structure, the union of $\mathcal{Z}$ must be coarsely separated from A and B in the metric ls-structure. Thus the family $\mathcal{Z}$ is an element of $\mathscr{L}$. However, $|f(z_i) - f(w_i)| \geq 1/3$ for all $i > M + 2$, which contradicts the fact that f is slowly oscillating with respect to $\mathscr{L}$. □

3.8 The Tietze Extension Theorem

As with Urysohn's Lemma, the proof of the Tietze Extension Theorem for neighbourhood operators is a straightforward adaptation of the classical proof, and gives us the result for (hybrid) large scale spaces as a corollary.

Lemma 9. *Let $\prec$ be a neighbourhood operator on a set X satisfying* $(\mathsf{N0}) - (\mathsf{N3})$ *and let $f, g : X \to [-M, M]$ be two neighbourhood continuous maps. Then $f + g$ is neighbourhood continuous.*

Proof. By Lemma 4 (since $[-2M, 2M]$ is homeomorphic to $[0, 1]$), it is enough to show that for any interval $[-2M, b]$ in $\mathbb{R}$ and $\varepsilon > 0$, that $(f + g)^{-1}([-2M, b])$ has neighbourhood $(f + g)^{-1}([-2M, b + \varepsilon))$ relative to $\prec$. Cover $[-M, M]$ by finitely many intervals $I_n = [-M,\ n\varepsilon/4 + \varepsilon/4]$ and $J_n = [-M,\ b - n\varepsilon/4]$, $n \in \mathbb{Z}$. It follows that

$$\begin{aligned}(f+g)^{-1}([-2M,b]) &\subseteq \bigcup_n f^{-1}(I_n) \cap g^{-1}(J_n) \\ &\subseteq \bigcup_n f^{-1}(B(I_n,\varepsilon/4)) \cap g^{-1}(B(J_n,\varepsilon/4)) \\ &\subseteq (f+g)^{-1}([-2M, b+\varepsilon)).\end{aligned}$$

Since f and g are neighbourhood continuous and $\prec$ satisfies $(\mathsf{N0}) - (\mathsf{N3})$, we have

$$\bigcup_n f^{-1}(I_n) \cap g^{-1}(J_n) \prec \bigcup_n f^{-1}(B(I_n,\varepsilon/4)) \cap g^{-1}(B(J_n,\varepsilon/4))$$

which completes the proof. □

Lemma 10. *Let $\prec$ be a neighbourhood operator on a set X satisfying* $(\mathsf{N0}) - (\mathsf{N3})$. *Suppose $g_n : X \to [-m_n, m_n]$ is a sequence of neighbourhood continuous maps such that*

$$\sum_{i=1}^{\infty} m_n = m < \infty.$$

Then $f = \sum_{i=1}^{\infty} g_n : X \to \mathbb{R}$ is neighbourhood continuous.

Proof. By Lemma 4 (since $[-m, m]$ is homeomorphic to $[0, 1]$), it is enough to show that for any interval $[-m, b]$ in $\mathbb{R}$ and $\varepsilon > 0$, that $f^{-1}([-m, b])$ has neighbourhood $f^{-1}([-m, b + \varepsilon))$ relative to $\prec$. Pick M such that $\sum_{n=M}^{\infty} m_n < \varepsilon/4$ and let $f' = \sum_{n=1}^{M-1} g_n$. Then f' is neighbourhood continuous by Lemma 9, so $f'^{-1}([-m, b + \varepsilon/4])$ has neighbourhood $f'^{-1}([-m, b + \varepsilon/2))$. But

$$f^{-1}([-m, b]) \subseteq f'^{-1}([-m, b + \varepsilon/4])$$

and

$$f'^{-1}([-m, b + \varepsilon/2)) \subseteq f^{-1}([-m, b + \varepsilon))$$

from which we obtain the required result. □

Definition 25. *Let X be a set and $\prec$ a neighbourhood operator. If A is a subset of X, then the* ***induced neighbourhood operator*** *$\prec_A$ on subsets of A is defined as follows: $S \prec_A T$ precisely when there exists a subset T' of X such that $S \prec T'$ as subsets of X and $T = T' \cap A$.*

Observation 2. *If a neighbourhood operator $\prec$ on X satisfies* (N0) – (N4)*, then so does the induced neighbourhood operator on any subset.*

Theorem 8 (Tietze Extension Theorem for neighbourhood operators)**.** *Let X be a set and $\prec$ a neighbourhood operator satisfying* (N0) – (N3)*. Then $\prec$ satisfies* (N4) *if and only if for any function $f : A \to [-2, 2]$ from a subset A of X which is neighbourhood continuous with respect to the operator induced by $\prec$ on A and the topological neighbourhood operator on $[-2, 2]$, there is a neighbourhood continuous function $g : X \to [-2, 2]$ which extends f.*

Proof. The proof follows the classical topological proof closely. Suppose $\prec$ satisfies (N4).

Claim: Given a neighbourhood continuous map $f : A \to [-3m, 3m]$, $m > 0$, there is a neighbourhood continuous map $g : X \to [-m, m]$ such that $|f(x) - g(x)| \leq 2m$ for all $a \in A$.

Proof of Claim: Let $S = f^{-1}([-3m, -m])$ and $T = f^{-1}([m, 3m])$. Since f is neighbourhood continuous, we have $S \prec_A A \setminus T$. It follows from the definition of $\prec_A$ and condition (N2) that $S \prec X \setminus T$, so by Theorem 6, there is a neighbourhood continuous map $g' : X \to [0, 1]$ such that $g'(S) \subseteq \{0\}$ and $g'(T) \subseteq \{1\}$. Composing with the appropriate linear map $[0, 1] \to [-m, m]$, we obtain the required map g. This proves the claim.

Now, define $m(n) = 2^{n+1}/3^n$ for $n \geq 0$. Using the Claim, inductively construct a sequence of functions $g_n : X \to [-m(n), m(n)]$ which are neighbourhood continuous and such that for all $a \in A$,

$$|f(a) - \sum_{i=1}^{n+1} g_i(a)| \leq 2m(n).$$

Then the map $g = \sum_{i=1}^{\infty} g_i$ is neighbourhood continuous by Lemma 10 and agrees with f on A.

For the other direction, note that if $A \prec N$, then the function which sends A to 0 and $X \setminus N$ to 1 is neighbourhood continuous on $A \cup (X \setminus N)$. Thus by Theorem 6 we have the result.

□

Corollary 9 (Tietze Extension Theorem)**.** *Let X be a normal topological space and let A be a closed subset of X. Then any continuous function $f : A \to [0, 1]$ extends to a continuous function $g : X \to [0, 1]$.*

Proof. In order to apply Theorem 8, we have only to show that the function f is continuous with respect to the neighbourhood operator $\prec_A$ induced by the topological neighbourhood operator on X. Suppose S and T are subsets of A, and that the closure of S in A is contained in a subset $V \subseteq T$ which is open in A. Since A is closed, the closure of S in A coincides with the closure of S in X. Let V' be an open set in X such that $V' \cap A = V$. Thus the closure of S (in X) is contained in V' which is contained in $T \cup X \setminus A$, so $S \prec_A T$. This shows that the topological neighbourhood operator associated to the subspace topology is contained (as a relation) in $\prec_A$, which gives the required result. □

We also obtain a result for hybrid large scale spaces. Note that if A is a subset of an ls-space X, then the coarse neighbourhood operator induced by the subspace ls-structure on A coincides with the neighbourhood operator induced on A by the coarse neighbourhood operator on X.

Corollary 10 (Tietze Extension Theorem for hybrid large scale spaces)**.** *Let X be an hls-space. Then X is hls-normal if and only if for any closed subset A of X, any continuous* *slowly oscillating function $f : A \to [0,1]$ extends to a continuous slowly oscillating function $g : X \to [0,1]$.*

Corollary 11. *Given a metric space X, any bounded continuous slowly oscillating function on a closed subset of X to $\mathbb{R}$ extends to a bounded continuous slowly oscillating function on the whole of X to $\mathbb{R}$.*

Proof. We have already seen that metric spaces are hls-normal as hls-spaces, so the result follows from Corollary 10. □

The purely large scale version of the above result is just Theorem 5.

Neighbourhood operators can also be applied to obtain results for small scale/uniform spaces. We will use the definition of uniform space in terms of covers introduced by Tukey [66] (see also [38]), which is equivalent to the original definition in terms of entourages introduced by Weil [69] and used in Chapter 2 of Bourbaki's book on general topology [8]. A **uniform space** is a set X equipped with a collection $\mathcal{S}$ of covers of X (which we call "uniform covers") satisfying the following axioms:

- $\{X\}$ is in $\mathcal{S}$,
- If $\mathsf{st}(\mathcal{U},\mathcal{U})$ refines $\mathcal{V}$ and $\mathcal{U}$ is in $\mathcal{S}$, then $\mathcal{V}$ is also in $\mathcal{S}$,
- if $\mathcal{U}$ and $\mathcal{V}$ are elements of $\mathcal{S}$, then there exists an element $\mathcal{W}$ of $\mathcal{S}$ such that $\mathsf{st}(\mathcal{W},\mathcal{W})$ refines both $\mathcal{U}$ and $\mathcal{V}$.

Note the apparent duality with the notion of large scale space. Indeed, uniform spaces are also called **small scale spaces** in the literature. For a more formal investigation of the connections and duality between large and small scale structures, we refer the reader to [3] and [4]. A map $f : X \to Y$ from a uniform space X to a uniform space Y is called **uniformly continuous** if for every uniform cover $\mathcal{V}$ of Y, $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ is a uniform cover of X. Metric spaces such as $\mathbb{R}$ carry a natural uniform structure consisting of all covers which have positive Lebesgue number. For compact metric spaces, this is just the set of all covers which are refined by an open cover.

Lemma 11. *Let X be a uniform space and $f : X \to [0,1]$ a function. Then f is uniformly continuous if and only if it is neighbourhood continuous with respect to the uniform neighbourhood operator on X and the topological neighbourhood operator on $[0,1]$.*

Proof. ($\Rightarrow$) is easy to check using Lemma 4.

($\Leftarrow$) Let $\varepsilon > 0$. Choose a finite number of points $t_1, \dots, t_n$ in $[0,1]$ such that $0 < t_{k+1} - t_k < \varepsilon/2$ for all $1 \leq k \leq n-1$. By neighbourhood continuity, for each of the intervals $[0, t_i]$, we have a uniform cover $\mathcal{U}_n$ of X such that $\mathsf{st}(f^{-1}([0,t_n]), \mathcal{U}_n) \subseteq f^{-1}([0, t_{n+1}))$. Taking a common refinement $\mathcal{V}$ of the $\mathcal{U}_n$, we have that $\mathsf{diam}(V) \leq \varepsilon$ for every $V \in \mathcal{V}$ as required. $\square$

One can check that for a subset A of a uniform space X, the uniform neighbourhood operator on A induced by the subspace uniform structure on A coincides with the neighbourhood operator induced by the uniform neighbourhood operator on X. Thus we recover the result of Katetov below.

Corollary 12 (Katetov [41])**.** *Let X be a uniform space and $A \subseteq X$ a subspace. Then any uniformly continuous function $f : X \to [0,1]$ extends to a uniformly continuous function g on the whole of X.*

Proof. It is enough to show that the uniform neighbourhood operator always satisfies the axiom (N4). Suppose we have $N, M \subseteq X$ with $N \prec M$ with respect to the uniform neighbourhood operator. Then there is a uniform cover $\mathcal{U}$ in X such that $\mathsf{st}(N, \mathcal{U}) \subseteq M$. By the definition of uniform structure, there is a uniform cover $\mathcal{V}$ such that $\mathsf{st}(\mathcal{V}, \mathcal{V})$ refines $\mathcal{U}$. Then $N \prec \mathsf{st}(N, \mathcal{V}) \prec \mathsf{st}(\mathsf{st}(N, \mathcal{V}), \mathcal{V}) \subseteq M$ as required. $\square$

We should mention some similarities between the work in [41] and the approach to extension theorems via neighbourhood operators in this chapter (which was developed independently with large scale spaces in mind). In [41], Katetov proves a version of his insertion theorem for abstract relations on sets and functions preserving them, with two key examples of such relations being what we call the topological and uniform neighbourhood operators in this chapter. From this he is able to obtain the (topological) Katetov-Tong Theorem (Theorem 1 in [41]), as well as the result for uniform spaces given above. In the corrections to [41], the author notes that some axioms are needed (on the relations) in order to prove the insertion theorem for relations. These axioms (found in Lemma 1 of the corrections) closely resemble axioms (N3) and (N4) given in this chapter.

3.9 The Higson compactification and corona

The concept of Higson compactification really belongs to hybrid large scale geometry. For completeness, let's prove the following result:

Proposition 13. *Given a hybrid large scale space X the following conditions are equivalent:*

(1) There is a Hausdorff compactification $h(X)$ of X with the property that every continuous slowly oscillating function $f : X \to [0, 1]$ extends uniquely over $h(X)$,

(2) X is Tychonoff as a topological space.

Proof. The implication (1) $\implies$ (2) holds for any space X that admits a Hausdorff compactification. To show (2) $\implies$ (1) first observe that, given any $x_0 \in X$ and given a bounded open neighbourhood U of x_0 in X, any continuous function $f : X \to [0, 1]$ that vanishes outside of U is slowly oscillating. Thus the set of continuous slowly oscillating functions on X separates closed sets from points. It is easy to check that the collection of continuous slowly oscillating functions from X to $[0, 1]$ is a subring of the ring of continuous functions from X to $[0, 1]$ that is complete with respect to the sup-norm and contains the constant functions. Thus (1) follows from well-known results in compactification theory (see for example Theorem (m) in Section 4.5 of [52]). □

In [57] (pp. 30–31) the Higson corona of a coarse space X is defined abstractly as a compact space νX satisfying

$$C(\nu X) = \frac{B_h(X)}{B_0(X)}.$$

Here $B_h(X)$ is the C*-algebra of all bounded slowly oscillating (not necessarily continuous) complex-valued functions and $B_0(X)$ is the closed two-sided ideal of functions that "approach 0 at infinity", i.e. all $f \in B_h(X)$ such that for every $\varepsilon > 0$ there is a bounded set K such that $|f(x)| < \varepsilon$ for all $x \notin K$. It is shown that the geometric realization of the Higson corona, in the case of a (paracompact) proper coarse space, can be obtained as $h(X) \setminus X$, where $h(X)$

is the Higson compactification of X, i.e. the compactification corresponding to the algebra of all continuous bounded slowly oscillating functions $X \to [0,1]$.

In case of arbitrary hybrid large scale spaces we can talk about two ways of defining the Higson corona: one as above (using $B_h(X)/B_0(X)$) and the other using continuous slowly oscillating functions, that is via the formula

$$C(\nu X) = \frac{B_h^c(X)}{B_0^c(X)}.$$

where B_h^c and B_0^c are the subalgebras of continuous functions in B_h and B_0 respectively. One purpose of this section is to show that for normal hls-spaces these definitions are equivalent. There is a natural homomorphism $\frac{B_h^c(X)}{B_0^c(X)} \to \frac{B_h(X)}{B_0(X)}$ induced by the inclusion of B_h^c into B_h; what we are interested in is when that homomorphism is an isomorphism.

Theorem 9. *If X is hls-normal as a hybrid ls-space, then the natural homomorphism α : $\frac{B_h^c(X)}{B_0^c(X)} \to \frac{B_h(X)}{B_0(X)}$ is an isomorphism.*

Proof. Since α has trivial kernel, it is enough to show that $B_h = B_0 + B_h^c$. Let $f \in B_h$ and let $\mathcal{U}$ be an open scale. Let A be a subset of X which is maximal with the property that no two elements of A are in the same element of $\mathsf{st}(\mathcal{U},\mathcal{U})$. Then A is a discrete subset of X, and no element of x belongs to the closure of more than one element of A. Define a map f' from $\mathsf{cl}(A)$ to $[0,1]$ which sends $a' \in A$ to $f(a)$, where a' is in closure of $\{a\}$. Then one checks that f' is slowly oscillating and continuous. By Theorem 10, we can extend f' to a continuous slowly oscillating function g on all of X. It remains to show that $g - f$ is in B_0. Indeed, let $\varepsilon > 0$. Then for some bounded set K, $\{a,b\} \in U \in \mathsf{st}(\mathcal{U},\mathcal{U})$ implies that $|g(a) - g(b)| < \varepsilon/2$ and $|f(a) - f(b)| < \varepsilon/2$. Since every element of X is in the same element of $\mathsf{st}(\mathcal{U},\mathcal{U})$ as some element of A, we have that $|f(x) - g(x)| < \varepsilon$ for every $x \notin K$. □

Proposition 14. *Suppose X is a hybrid large scale space whose topology is Tychonoff. Then X is hls-normal if and only if, for each closed subset Y of X, its closure $\overline{Y}$ in the Higson compactification $h(X)$ is the Higson compactification of Y.*

Proof. The Higson compactification of a closed subset Y of X is completely characterized by the fact that any continuous slowly oscillating complex-valued function on Y extends uniquely to hY. If X is hls-normal, then any continuous slowly oscillating function on Y extends to the whole of X by Corollary 10, and hence to hX and in particular, to $\overline{Y}$, the closure of Y in hX. Uniqueness is easy to check. Conversely, if $\overline{Y}$ is the Higson compactification of Y then any continuous bounded slowly oscillating complex-valued function f on Y extends to a continuous function on $\overline{Y} = hY$. By the classical Tietze Extension Theorem, this extends to a continuous function on hX, which when restricted to X is a continuous bounded slowly oscillating function extending f. □

Proposition 15. *Let X be an hls-space whose topology is Tychonoff. Then the following are equivalent, where for $Y \subseteq X$, $\overline{Y}$ denotes the closure of Y in hX:*

(1) X is hls-normal,

(2) two disjoint closed subsets A and B of X are coarsely separated if and only if $\overline{A} \cap \overline{B} = \varnothing$.

Proof. (1) $\implies$ (2): Suppose X is hls-normal. If $\overline{A} \cap \overline{B} = \varnothing$ then we can define a continuous map f from hX to $[0,1]$ which sends $\overline{A}$ to 0 and $\overline{B}$ to 1. The restriction of f to X is a slowly oscillating function sending A to 0 and B to 1. It follows that A and B are coarsely separated. If A and B are coarsely separated, then by Corollary 8, we can define a slowly oscillating function sending A to 0 and B to 1. Extending this to hX we see that we must have $\overline{A} \cap \overline{B} = \varnothing$ as required.

(2) $\implies$ (1): By Corollary 8 it is enough to produce, for closed subsets A and B of X such that $\overline{A} \cap \overline{B} = \varnothing$, a slowly oscillating continuous map f sending A to 0 and B to 1. This can be accomplished by constructing a continuous map sending $\overline{A}$ to 0 and $\overline{B}$ to 1 and restricting to X. □

Note that for proper metric spaces, condition (2) in the above proposition follows from Proposition 2.3 in [17] and plays a crucial role in relating properties of a proper metric space with its Higson corona in various places in the literature (see for example [5] or [17]),

3.10 Hybrid structures induced by compactifications

In this section we discuss hybrid ls-structures related to the work of Mine, Yamashita, and Yamauchi (see [44], [45]) who studied properties of the C_0-structure on a locally compact metric space relative to a compact metric compactification. Our next definition generalises that concept.

Definition 26. *Given a closed subset A of a topological space X with empty interior define the large scale structure $LS(X,A)$ on $X \setminus A$ as follows: a family $\mathcal{U}$ of subsets of $X \setminus A$ is in $LS(X,A)$ if and only if for each open neighbourhood U of any $a \in A$ in X there is an open neighbourhood V of a in U such that $W \in \mathcal{U}$ and $W \cap V \neq \emptyset$ implies $W \subset U$.*

It is easy to check that this indeed defines an ls-structure. Note that the bounded sets in $X \setminus A$ equipped with the ls-structure $LS(X,A)$ are precisely the subsets of $X \setminus A$ whose closure does not intersect A.

Proposition 16. *Given a closed subset A of a topological space X with empty interior and given a continuous function $f : X \setminus A \to Y$ to a complete metric space Y, consider the following statements:*

1. *f extends continuously over X,*

2. *f is slowly oscillating with respect to the large scale structure $LS(X,A)$ on $X \setminus A$.*

It is always the case that (1) $\Rightarrow$ (2) and, if each point of A has a countable basis of neighbourhoods and X is Hausdorff, then (2) $\Rightarrow$ (1).

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{U}$ be an element of $LS(X,A)$ and let $\varepsilon > 0$. For each point $a \in A$, pick a open neighbourhood V_a of a such that $f(V_a)$ has diameter less than ε, and choose an open neighbourhood W_a of a inside V_a such that for all $U \in \mathcal{U}$, $U \cap W_a \neq \varnothing \Rightarrow U \subseteq V_a$. Consider the union W of all the W_a. Its complement is a closed subset of $X \setminus A$, hence bounded. Any set $U \in \mathcal{U}$ which intersects $(X \setminus A) \setminus W$ must be contained in an element of V_a, so that f is slowly oscillating as required.

(2) $\Rightarrow$ (1) : Suppose $f : X \setminus A \to Y$ is continuous and slowly oscillating. The first issue is to construct an extension $g : X \to Y$ of f and then to show its continuity. The natural way to define $g(a)$ for $a \in A$ is as the only point belonging to the intersection of all sets $cl(f(U))$, U being a neighbourhood of a in X. Choose a decreasing sequence $\{U_n\}$ of neighbourhoods of a in X. The intersection of all sets $cl(f(U_n))$, $n \geq 1$, consists of exactly

one point if for each $\epsilon > 0$ there is $M > 0$ such that the diameter of $f(U_n)$ for $n > M$ is smaller than ϵ. Suppose for contradiction that there is a sequence $x_n, y_n \in U_n$ so that $dist(f(x_n), f(y_n)) \geq \epsilon$. Since every bounded set in $X \setminus A$ is contained in some $X \setminus U_n$, the family $\{x_n, y_n\}_{n\geq 1}$ cannot be uniformly bounded because f is slowly oscillating. By definition of the ls-structure $LS(X, A)$, there must exist a $b \in A$ and a neighbourhood V of b such that for every neighbourhood $V' \subseteq V$ of b, there is an n for which $\{x_n, y_n\} \cap (X \setminus V)$ and $\{x_n, y_n\} \cap V'$ each have exactly one point. We claim that $a = b$. Indeed, if not, then since X is Hausdorff, we can choose a neighbourhood of b which contains none of the x_n, y_n, a contradiction. Suppose then that $U_k \subseteq V$. We can choose a neighbourhood $V' \subseteq U_k$ of a such that V' does not contain x_i or y_i for $i \leq k$. It follows that if $\{x_n, y_n\} \cap V' \neq \varnothing$, then $\{x_n, y_n\} \subseteq U_k \subseteq V$. This is a contradiction. Thus f is well-defined, and its continuity is easy to show. □

Corollary 13. *If X is compact Hausdorff, A is a closed subset of X with empty interior whose every point has a countable basis of neighbourhoods in X, and $LS(X, A)$ is a hybrid large scale space when equipped with the topology induced from X, then the Higson compactification of $X \setminus A$ equipped with the ls-structure $LS(X, A)$ is exactly X.*

Proposition 17. *If X is a compact metric space and A is a closed subset of X with empty interior, then $LS(X, A)$ is a hybrid large scale space when equipped with the topology induced from X.*

Proof. Consider the family $\{B(x, d(x))\}_{x\in X\setminus A}$, where $d(x)$ is half the distance from x to A. It is a scale in $LS(X, A)$. □

Proposition 18. *Suppose X is a Hausdorff topological space, A is a closed subset of X with empty interior, and each point of A has a countable basis of neighbourhoods in X. If $LS(X, A)$ is a hybrid large scale space when equipped with the topology induced from X, then two closed subsets B and C of $X \setminus A$ are coarsely disjoint if and only if their closures in X are disjoint.*

Proof. Suppose B and C are coarsely disjoint but $a \in A$ belongs to $cl(B) \cap cl(C)$. Pick sequences $b_n \in B$ and $c_n \in C$, both converging to a. We claim that $\mathcal{F} := \{b_n, c_n\}_{n=1}^{\infty}$ is a uniformly bounded family in $LS(X, A)$. Indeed, let $d \in A$ have open neighbourhood U in X. If $d = a$, then we can choose $N > 0$ such that $b_n \in U$ and $a_n \in U$ for all $n > N$. Using the fact that X is Hausdorff, we can choose a smaller neighbourhood $V \subseteq U$ of d which does not contain a_i or b_i for $i \leq N$. Thus if $F \in \mathcal{F}$ intersects V, it must be contained in U. If $a \neq d$, then we can use the fact that X is Hausdorff to choose an open neighbourhood $V \subseteq U$ of d which contains none of the a_n or b_n, so that no element of $\mathcal{F}$ intersects V. Thus $\mathcal{F}$ is uniformly bounded. On the other hand, $st(B, \mathcal{F}) \cap st(C, \mathcal{F})$ is not bounded because its closure contains a. This is a contradiction.

Suppose B and C are closed in $X \setminus A$ and $cl(B) \cap cl(C) = \emptyset$. Since any scale of $LS(X, A)$ can be coarsened to an open scale, it suffices to show that $st(B,\mathcal{U}) \cap st(C,\mathcal{U})$ is bounded for any open scale $\mathcal{U}$ of $LS(X, A)$. Suppose, on the contrary, that $a \in A$ belongs to the closure of $st(B,\mathcal{U}) \cap st(C,\mathcal{U})$. Without loss of generality, we may assume $a \notin cl(B)$. Pick a neighbourhood V of a in $X \setminus cl(B)$ such that $U \cap V \neq \emptyset$, $U \in \mathcal{U}$, implies $U \subset X \setminus cl(B)$. Since $V \cap st(B,\mathcal{U}) \neq \emptyset$, there is $U \in \mathcal{U}$ intersecting both B and V. But then $U \subset X \setminus cl(B)$, a contradiction. □

Corollary 14. *Suppose X is a normal topological space, A is a closed subset of X with empty interior, and each point of A has a countable basis of neighbourhoods in X. If $LS(X, A)$ is a hybrid large scale space when equipped with the topology induced from X, then it is hls-normal.*

Proof. Suppose B and C are disjoint, closed, coarsely separated subsets of $X \setminus A$. By Proposition 18, the closures of B and C in X are disjoint. Thus the function f from $cl(B) \cup cl(C) \subseteq X$ to $[0, 1]$ which sends $cl(B)$ to 0 and $cl(C)$ to 1 is well-defined. Since it is continuous, it can be extended to the whole of X by topological normality. Thus the restriction of f to $X \setminus A$ is a slowly oscillating function, and sends B to 0 and C to 1. □

3.11 Topological groups as hls-spaces

Let G be a group. Then G admits a natural ls-structure given by all families of subsets $\mathcal{U}$ which refine a family of the form $\{g \cdot F \mid g \in G\}$ for some finite subset F [21]. If the group is finitely generated, then this ls-structure coincides with the one given by the word-length metric associated to any finite generating set (see for example [50] for a definition of this metric). If the group is countable, then this ls-stucture coincides with the unique ls-structure which is induced by a proper left-invariant metric on the group [62]. For a subset F of G, we denote the cover $\{g \cdot F \mid g \in G\}$ by $G(F)$. The following lemma gives an explicit formula for starring such covers.

Lemma 12. *Let E and F be subsets of G. Then we have*

$$\mathsf{st}(E, G(F)) = E \cdot F^{-1} \cdot F$$

$$\mathsf{st}(G(E), G(F)) = G(E \cdot F^{-1} \cdot F)$$

Proof. If $x \in \mathsf{st}(E, G(F))$, then there is an $e \in E$ and $g \in G$ such that $e = gf_1$ and $x = gf_2$ for some $f_1,\ f_2 \in F$. Thus $g = ef_1^{-1}$ so $x \in E \cdot F^{-1} \cdot F$ as required. On the other hand, if $x = ef_1^{-1}f_2 \in E \cdot F^{-1} \cdot F$, then $e \in E$ and $\{e = ef_1^{-1}f_1,\ x = ef_1^{-1}f_2\} \subseteq ef^{-1} \cdot F$, so $x \in \mathsf{st}(E, G(F))$ as required. Since $\mathsf{st}(G(E), G(F))$ is the collection of all sets $\mathsf{st}(g \cdot E, G(F))$, to prove the second equation it suffices to note that $g \cdot (E \cdot F^{-1} \cdot F) = (g \cdot E) \cdot F \cdot F^{-1}$. □

More generally, if G is a locally compact topological group, then G admits an ls-structure given by all families of subsets which refine $G(K)$ for some compact set K. Since the product of two compact subsets in a topological group is again compact, Lemma 12 shows that this is indeed an ls-structure. Moreover, G together with this structure and the topology given form a hybrid large scale space (the uniformly bounded open cover is just $G(V)$, where V is a precompact neighbourhood of the identity element). We now describe coarse neighbourhoods in the case of a locally compact topological group.

Lemma 13. *Suppose V is a precompact, symmetric neighbourhood of the identity element in a locally compact topological group G. Then the following conditions are equivalent:*

(1) N is a coarse neighbourhood of U,

(2) $U \cdot V \cdot F \cdot V \setminus N$ is precompact for each finite subset F of G,

(3) $U \cdot V \cdot x \cdot V \setminus N$ is precompact for each point x of G.

Proof. (1) $\Longrightarrow$ (2). Suppose N is a coarse neighbourhood of U and F is a finite subset of G. Enlarge F, if necessary, to contain the neutral element 1_G of G and be symmetric. Consider the uniformly bounded family $\mathcal{U} = G(F \cdot V)$. If N is a coarse neighbourhood of U, then there is a precompact set C such that $st(U, \mathcal{U}) \subset N \cup C$. By 12 that implies $U \cdot V \cdot F \cdot F \cdot V \setminus N \subset C$ is precompact. In particular, $U \cdot V \cdot F \cdot V \setminus N$ is precompact.

(2) $\Longleftrightarrow$ (3) is obvious.

(2) $\Longrightarrow$ (1) Given a precompact $C \subset G$, find a symmetric finite subset F of G satisfying $C \subset F \cdot V$. The uniformly bounded family $\mathcal{W} = G(F \cdot V)$ coarsens the cover $\mathcal{C} = G(C)$. Since (using 12) $st(U, \mathcal{W}) \setminus N$ is precompact, so is $st(U, \mathcal{C}) \setminus N$. □

Theorem 10. *Let G be a locally compact abelian group. Then the following conditions are equivalent:*

(1) G is hybrid large scale normal as an hls-space,

(2) G is σ-compact,

(3) the ls-structure on G is metrizable, that is, induced by a metric.

Proof. (1) $\Rightarrow$ (2) Suppose G is not σ-compact. By local compactness, we can pick a countably infinite discrete subset B of G. Let V be a precompact symmetric neighbourhood of the identity element. Notice that for any countable set C in G, G cannot be generated by $V \cup C$. Thus we can construct an uncountable set $A = \{a_t\}_{t<\omega_1}$ of elements of G indexed by countable ordinals such that for any $t < \omega_1$, a_t is not in the subgroup generated by $B \cup V \cup \{a_r \mid r < t\}$. Note that B is discrete, as is A (since any subset $g \cdot V$ intersects at most one element of A), so that every precompact subset of either A or B must be finite. We claim that A has coarse neighbourhood $G \setminus B$. First note that $\mathsf{cl}(A) \subseteq \mathsf{st}(A, G(V))$ and $\mathsf{cl}(B) \subseteq \mathsf{st}(B, G(V))$ are disjoint, so that $G \setminus B$ contains an open set which contains the closure of A. We now show that $G \setminus B$ is a coarse neighbourhood of A. Let x be an element of G, and consider the set $A \cdot V \cdot x \cdot V \cap B$. If $b_1 = a_1 v_1 x v_1' \in B$ and $b_2 = a_2 v_2 x v_2' \in B$ with $a_1, a_2 \in A$ and $v_1, v_1', v_2, v_2' \in V$, then a_1 (resp. a_2) is in the subgroup generated by $B \cup V$ and a_2 (resp. a_1) so we must have $a_1 = a_2$. Thus $A \cdot V \cdot x \cdot V \cap B$ contains only a single point, so by Lemma 13 we have that $X \setminus B$ is a coarse neighbourhood of A. Suppose for contradiction there is an intermediate coarse neighbourhood N between A and $G \setminus B$. For $b \in B$, let $Z(b) = \{a \in A \mid a \cdot b \notin N\}$. Since N is a coarse neighbourhood of A, each $Z(b) \cdot b$, and thus each $Z(b)$, must be precompact, hence finite. Thus the union of all the $Z(b)$ is countable, so there is an $a \in A$ such that $a \cdot b \in N$ for all $b \in B$. But then $a^{-1} \cdot N \subseteq N \cdot V \cdot a^{-1} \cdot V$ intersects B in an infinite (in particular, not precompact) set, which by Lemma 13 contradicts the fact that $G \setminus B$ is a coarse neighbourhood of N.

(2) $\Rightarrow$ (3) Suppose $G = \cup_{i=1}^{\infty} K_i$, where the K_i are compact subspaces. If $\mathcal{V}$ is an uniformly bounded open cover, then G is the union of the countable set $\{\mathsf{st}(K_i, \mathcal{V})\}_{i=1}^{\infty}$ of precompact open sets. Every compact set is contained in a union of finitely many of the $\mathsf{st}(K_i, \mathcal{V})$. It follows that there is a countable set C of precompact subsets such that each precompact subset is contained in an element of C, and that consequently, the ls-structure on G is countably generated, i.e. metrizable (see Theorem 2.55 of [57]).

(3) $\Rightarrow$ (1) Since the ls-structure is metric, G is ls-normal as an ls-space. It is well-known that locally compact Hausdorff groups are topologically normal, so by Theorem 7 we have the result.

□

Corollary 15. *Let X be the set $\mathbb{R}$ be equipped with the ls-structure coming from the group structure and the discrete topology. Then X is not hls-normal.*

Question 1. *The proof of Proposition 10 holds more generally for any group where the "left-translation structure" (that is, the ls-structure used here) and the "right-translation structure" (that is, the ls-structure generated by families of the form $\{K \cdot g \mid g \in G\}$ for K compact) coincide: only the last sentence of (1) $\Rightarrow$ (2) needs to be changed. Does Proposition 10 hold for all groups?*

3.12 Coarse neighbourhoods and ls-structures

Since the coarse neighbourhood operator completely determines which maps to $[0, 1]$ from an ls-space are slowly oscillating, one might ask to what extend coarse neighbourhoods determine the ls-structure on a set.

Definition 27. *Given any ls-space X with ls-structure $\mathcal{X}$, we define $\mathcal{X}^{cn}$ to be the collection of all families of subsets $\mathcal{U}$ such that for every coarse neighbourhood N of A in X, $\mathsf{st}(A,\mathcal{U}) \subseteq N \cup K$ for some bounded set K (that is, bounded in the sense of the original structure $\mathcal{X}$).*

Clearly we always have $\mathcal{X} \subseteq \mathcal{X}^{cn}$, and if $\mathcal{X}_1 \subseteq \mathcal{X}_2$ are ls-structures on X which induce the same bounded sets, then $\mathcal{X}_1^{cn} \subseteq \mathcal{X}_2^{cn}$. The collection $\mathcal{X}^{cn}$ need not coincide with $\mathcal{X}$ in general, as the following example shows.

Example 6. *Let X be an infinite set and let $\mathcal{X}$ be the maximal uniformly locally finite ls-structure. Then N is a coarse neighbourhood of $A \subseteq N$ if and only if either A is finite or N is cofinite. Consider a cover of X by infinitely many disjoint finite subsets of unbounded cardinality. One checks that this cover is in $\mathcal{X}^{cn}$, but it is clearly not in $\mathcal{X}$.*

For metric spaces, however, $\mathcal{X}^{cn}$ turns out to be equal to the original (metric) ls-structure.

Proposition 19. *Let X be a metric space and $\mathcal{X}$ the associated ls-structure. Then $\mathcal{X} = \mathcal{X}^{cn}$.*

Proof. We already have $\mathcal{X} \subseteq \mathcal{X}^{cn}$, so suppose for contradiction that there is a family $\mathcal{U}$ in $\mathcal{X}^{cn}$ which is not in $\mathcal{X}$. Since $\mathcal{U}$ is not uniformly bounded, we may choose a sequence U_n of elements of $\mathcal{U}$ and pairs $\{a_n, b_n\} \subseteq U_n$ of points in X such that the b_n are unbounded, and for each n, $d(a_n, b_n) > n$. Consider the subset $N = \bigcup_{n=0}^{\infty} B(a_n, n)$. Clearly it is a coarse neighbourhood of $A = \{a_n \mid n \in \mathbb{N}\}$, but $\mathsf{st}(A,\mathcal{U}) \cap X \setminus N$ contains the b_ns, so it is unbounded. □

More generally, one may ask when $\mathcal{X}^{cn}$ is an ls-structure. As shown below, for ls-normal ls-spaces $\mathcal{X}^{cn}$ turns out to be an ls-structure. Note that the maximal uniformly locally finite structure is ls-normal, so even if $\mathcal{X}^{cn}$ is an ls-structure, it need not coincide with the original structure $\mathcal{X}$ as Example 6 above shows.

Proposition 20. *Let X be an ls-space which is ls-normal. Then $\mathcal{X}^{cn}$ is an ls-structure.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be elements of $\mathcal{X}^{cn}$ and let A have coarse neighbourhood N. By normality, there are intermediate coarse neighbourhoods $A \prec L \prec M \prec N$. Then $A_1 = \mathsf{st}(A, \mathcal{V})$ is contained in $L \cup K$ for some bounded subset K. In particular, M is a coarse neighbourhood of $\mathsf{st}(A, \mathcal{V})$. Similarly, $A_2 = \mathsf{st}(A_1, \mathcal{U})$ has coarse neighbourhood N.

Finally, $\mathsf{st}(A_2, \mathcal{V}) = \mathsf{st}(A, \mathsf{st}(\mathcal{U}, \mathcal{V}))$ is contained in $N \cup K'$ for some bounded set K', which gives the result. □

Proposition 21. *Let X and Y be metric spaces and let $f : X \to Y$ be a map which sends bounded sets in X to bounded sets in Y and which is proper (that is, the inverse image of a bounded set is bounded). Then f is ls-continuous if and only if for every subset A of Y and every coarse neighbourhood N of A, $f^{-1}(N)$ is a coarse neighbourhood of $f^{-1}(A)$.*

Proof. ($\Rightarrow$) Suppose N is a coarse neighbourhood of $A \subseteq Y$. Then for every uniformly bounded cover $\mathcal{U}$ of X, the image of $\mathsf{st}(f^{-1}(A), \mathcal{U})$ is contained in $\mathsf{st}(A, f(\mathcal{U}))$, which in turn is contained in $N \cup K$ for some bounded set K. It follows, since f is coarse, that $f^{-1}(N)$ is a coarse neighbourhood of $f^{-1}(A)$.

($\Leftarrow$) Suppose that f is not ls-continuous. Then there is some $R > 0$ such that the set

$$\{d(f(x), f(x')) \mid (x, x') \in X \times X,\ d(x, x') < R\}$$

is unbounded. In particular, we may choose a sequence of pairs of points $(a_n, b_n)_{n \in \mathbb{N}}$ in X such that $d(a_n, b_n) < R$ and $d(f(a_n), f(b_n)) > n$ for every n. Because f sends bounded sets to bounded sets, the a_n and b_n cannot all be contained in a single bounded set K, since otherwise all the $f(a_n)$ and $f(b_n)$ would be contained in the bounded set $f(K)$. Since each a_n is distance at most R from b_n, we can moreover say that neither of the sequences $(a_n)_{n \in \mathbb{N}}$ and $(b_n)_{n \in \mathbb{N}}$ are bounded. Thus neither of the sequences $(f(a_n))_{n \in \mathbb{N}}$ and $(f(b_n))_{n \in \mathbb{N}}$ are bounded, because f is a proper map.

We now choose a subsequence of (a_n, b_n) for which the images of the points are sufficiently spread out. To do so, we pick a base point $y_0 \in Y$ and set $\phi(0) = 0$. The induction proceeds as follows: for $i \in \mathbb{N}$, let

$$p(i) = \max\big\{\{d(y_0, f(a_{\phi(k)})) \mid k \leq i\} \cup \{d(y_0, f(b_{\phi(k)})) \mid k \leq i\}\big\} + i + 1$$

and choose $\phi(i+1) \geq i+1$ so that $f(a_{\phi(i+1)})$ and $f(b_{\phi(i+1)})$ are both outside the bounded set $B(y_0, p(i))$. Thus we obtain a subsequence $(a_{\phi(n)}, b_{\phi(n)})_{n \in \mathbb{N}}$ with the property that for every $k, i \in \mathbb{N}$, $f(b_{\phi(k)})$ is distance at least $i - 1$ from $f(a_{\phi(i)})$.

Clearly $N = \bigcup_{i=0}^{\infty} B(f(a_{\phi(n)}), n-1)$ is a coarse neighbourhood of $A = \{f(a_\phi(n)) \mid n \in \mathbb{N}\}$. We claim, however, that $f^{-1}(N)$ is not a coarse neighbourhood of $f^{-1}(A)$. Indeed, if $\mathcal{U}$ is the uniformly bounded cover of X by balls of radius R, then $\mathsf{st}(f^{-1}(A), \mathcal{U})$ contains all the $b_{\phi(n)}$. On the other hand, $X \setminus f^{-1}(N)$ also contains each $b_{\phi(n)}$ by construction. Since the set $\{b_{\phi(n)} \mid n \in \mathbb{N}\}$ is unbounded, we have that $\mathsf{st}(f^{-1}(A), \mathcal{U}) \cap (X \setminus f^{-1}(N))$ is unbounded, so $f^{-1}(N)$ is not a coarse neighbourhood of $f^{-1}(A)$. $\square$

Chapter 4

A coarse version of monotone-light factorizations

This chapter is based on the accepted manuscript of the following paper: J. Dydak and T. Weighill, Monotone-light factorizations in coarse geometry, Topology and its Applications 239, 2018, 160–180. *The contributions of each author of the above manuscript may be considered roughly equal. No revisions were recommended by the referees. The introduction has been adapted, otherwise the manuscript has remained more or less unchanged.*

4.1 Introduction

In this chapter we are interested in a coarse version of a particular factorization of continuous maps between compact Hausdorff spaces, namely the so-called monotone-light factorization. Recall that continuous map from a compact Hausdorff space X to a compact Hausdorff space Y is called **monotone** if it is surjective and each of its fibres is connected, and is called **light** if each of its fibres is totally disconnected (see for example [70]). Eilenberg showed in [26] that every continuous map f between compact metric spaces factorizes as $f = me$, where m is light and e is monotone (in fact, the result holds more generally for compact Hausdorff spaces, see [12]). Moreover, this factorization satisfies a universal property, namely that for any commutative diagram

$$
\begin{array}{ccccc}
\bullet & \xrightarrow{e} & \bullet & \xrightarrow{m} & \bullet \\
{\scriptstyle u}\downarrow & & & & \downarrow{\scriptstyle v} \\
\bullet & \xrightarrow[e']{} & \bullet & \xrightarrow[m']{} & \bullet
\end{array}
$$

where the arrows are continuous maps and the objects are compact Hausdorff spaces, with e' monotone and m' light, there is a unique continuous map h making the diagram commute:

$$
\begin{array}{ccccc}
\bullet & \xrightarrow{e} & \bullet & \xrightarrow{m} & \bullet \\
{\scriptstyle u}\downarrow & & \vdots{\scriptstyle h} & & \downarrow{\scriptstyle v} \\
\bullet & \xrightarrow[e']{} & \bullet & \xrightarrow[m']{} & \bullet
\end{array}
$$

In the language of category theory, this is to say that the classes of monotone maps and light maps constitute a **factorization system** [29] on the category of compact Hausdorff spaces and continuous maps.

In this chapter, we introduce large scale analogues of the topological monotone and light maps mentioned above, to which we give the names coarsely monotone and coarsely light maps respectively. These classes of maps will turn out to constitute a factorization system on the coarse category (defined in the next section). A large part of the chapter is devoted to making some connections between the topological and large scale notions of monotone and light. We do so in two ways. Firstly, we examine these classes of maps from a categorical perspective inspired by the results in [12]. Secondly, we make some connections using the Higson corona in the case when the large scale spaces involved are proper metric spaces. Coarsely light maps generalize both coarse embeddings and coarsely n-to-1 maps; we prove that coarsely light maps preserve certain coarse properties such as finite asymptotic dimension and Yu's Property A in a similar way to these classes of maps. In the final section of the chapter, we make some remarks on coarsely monotone and light maps between groups.

The main goal of this chapter is to introduce two interesting classes of maps between large scale spaces and study some of their properties. Along the way, however, we also investigate some of the structure of the coarse category and apply some basic categorical arguments to large scale spaces and maps between them. It would be interesting to see what other categorical notions turn out to be useful in the study of large scale spaces.

4.2 Preliminaries

We recall some basic terminology from [21]. Let X be a set. Recall that the **star** $\mathsf{st}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B. More generally, for two families $\mathcal{B}$ and $\mathcal{U}$ of subsets of X, $\mathsf{st}(\mathcal{B},\mathcal{U})$ is the family $\{\mathsf{st}(B,\mathcal{U}) \mid B \in \mathcal{B}\}$.

Definition 28. *A* ***large scale structure*** $\mathcal{L}$ *on a set* X *is a nonempty set of families* $\mathcal{B}$ *of subsets of* X *(which we call the* ***uniformly bounded families*** *in* X*) satisfying the following conditions:*

(1) $\mathcal{B}_1 \in \mathcal{L}$ *implies* $\mathcal{B}_2 \in \mathcal{L}$ *if each element of* $\mathcal{B}_2$ *consisting of more than one point is contained in some element of* $\mathcal{B}_1$*.*

(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}$ *implies* $\mathsf{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{L}$*.*

Remark 3. *Note that any uniformly bounded family* $\mathcal{U}$ *can be extended to a cover which is also uniformly bounded by adding singleton sets to the family (we call this cover the* ***trivial extension*** *of* $\mathcal{U}$*), so we will often assume that a given family is in fact a cover for convenience. Note that a family* $\mathcal{U}$ *of subsets of* X *refines* $\mathsf{st}(\mathcal{U}, \mathcal{V})$ *for any cover* $\mathcal{V}$ *of* X*.*

By a large scale space (or ls-space for short), we mean a set equipped with a large scale structure. A subset of a large scale space X is called **bounded** if it is an element of some uniformly bounded family in X. A classical example of a large scale space is as follows. Let (X, d) be an ∞-metric space. Define the uniformly bounded families in X to be all those families $\mathcal{U}$ for which there is a $M > 0$ such that every element of $\mathcal{U}$ has diameter at most M. In fact, a large scale structure on X arises in this way from an ∞-metric on X if and only if it is countably generated [21]. We call such a large scale structure **metrizable**.

Given a set map $f : X \to Y$ from an ls-space X to an ls-space Y, we say that f is **large scale continuous** or **ls-continuous** if for every uniformly bounded family $\mathcal{U}$ in X, the family

$$f(\mathcal{U}) = \{f(U) \mid U \in \mathcal{U}\}$$

is uniformly bounded in Y. Given two set maps $f, g : X \to Y$, we say that f and g are **close** and write $f \sim g$ if the family

$$\{\{f(x), g(x)\} \mid x \in X\}$$

is uniformly bounded, in which case we say that this family (or any uniformly bounded family which it refines) **witnesses** the closeness of f and g.

Certain types of ls-continuous maps are worth mentioning. Recall that a map $f : X \to Y$ between ls-spaces is called **coarsely surjective** if there is a uniformly bounded family $\mathcal{U}$ in Y such that $Y \subseteq \mathsf{st}(f(X), \mathcal{U})$. An ls-continuous map f is called a **coarse equivalence** if there is an ls-continuous map f' in the other direction such that ff' and $f'f$ are both close to the identity. An ls-continuous map $f : X \to Y$ is called a **coarse embedding** if for every uniformly bounded family $\mathcal{U}$ in Y, $f^{-1}(\mathcal{U}) = \{f^{-1}(U) \mid U \in \mathcal{U}\}$ is uniformly bounded in X. It is easy to check that an ls-continuous map is a coarse equivalence if and only if it is coarsely surjective and a coarse embedding.

We are now ready to introduce the category on which we will construct a factorization system. In the present chapter, by the **coarse category** we mean the category whose objects are large scale spaces and whose morphisms are equivalence classes of ls-continuous maps under the closeness relation. Note that this differs from Roe's coarse category in [56], where the maps are further required to be proper (an ls-continuous map is **proper** if the inverse image of every bounded set is bounded), although similar results will hold for this category as well (see Remark 5). It is easy to check that composition is well-defined in the coarse category, that is, if $f \sim g$ and $h \sim j$, then $hf \sim jg$ whenever these composites are defined. Note that the isomorphisms in the coarse category are represented by coarse equivalences.

For a set X, a family $\mathcal{U}$ of subsets of X and $x, x' \in X$, we write $x\mathcal{U}x'$ to mean that there is an element of $\mathcal{U}$ containing both x and x'. We say that x and x' are **$\mathcal{U}$-connected** if there is a finite sequence $x = x_1,\ x_2,\ \ldots,\ x_k = x'$ of elements of X with $x_i\mathcal{U}x_{i+1}$. Equivalence classes under the relation "x is $\mathcal{U}$-connected to x'" will be called **$\mathcal{U}$-components**. For $\mathcal{U}$ and $\mathcal{V}$ two families of subsets of X, we write $\mathcal{U} \leq \mathcal{V}$ in case $\mathcal{U}$ refines $\mathcal{V}$, in which case we also say that $\mathcal{V}$ **coarsens** $\mathcal{U}$.

4.3 Coarsely light maps

In this section we introduce the large scale analogue of topological light maps.

Definition 29. *Let $f : X \to Y$ be an ls-continuous map between ls-spaces. For every pair of uniformly bounded families $\mathcal{U}$ in X and $\mathcal{V}$ in Y, denote by $c(\mathcal{U}, f, \mathcal{V})$ the family of subsets consisting of all $\mathcal{U}$-components of elements of $f^{-1}(\mathcal{V})$. The closure under refinement of the set of all such $c(\mathcal{U}, f, \mathcal{V})$ is called the **light structure on X with respect to f**.*

Proposition 22. *Let $f : X \to Y$ be an ls-continuous map between ls-spaces. Then the light structure on X with respect to f is an ls-structure which contains the ls-structure on X.*

Proof. Let $c(\mathcal{U}, f, \mathcal{V})$ and $c(\mathcal{U}', f, \mathcal{V}')$ be two elements of the light structure. We may suppose that $\mathcal{U}$, $\mathcal{U}'$, $\mathcal{V}$ and $\mathcal{V}'$ are covers and that $\mathcal{V}$ and $\mathcal{V}'$ coarsen $f(\mathcal{U})$ and $f(\mathcal{U}')$ respectively. It is easy to check that

$$\mathsf{st}(c(\mathcal{U}, f, \mathcal{V}), c(\mathcal{U}', f, \mathcal{V}')) \leq c(\mathsf{st}(\mathcal{U}, \mathcal{U}'), f, \mathsf{st}(\mathcal{V}, \mathcal{V}')).$$

It follows that the light structure is an ls-structure. To see that it contains the ls-structure on X, note that for a uniformly bounded cover $\mathcal{U}$ of X, we have $\mathcal{U} \leq c(\mathcal{U}, f, f(\mathcal{U}))$. □

Proposition 23. *If $f : X \to Y$ is a map between metrizable coarse spaces, then the light structure on X with respect to f is also metrizable.*

Proof. Since the ls-structures on X and Y are generated by metrics, we may assume that there are countable families $(\mathcal{U}_i)_{i\in\mathbb{N}}$ and $(\mathcal{V}_i)_{i\in\mathbb{N}}$ of uniformly bounded covers of X and Y respectively such that any uniformly bounded cover of X refines some $\mathcal{U}_i$ and any uniformly bounded cover of Y refines some $\mathcal{V}_i$. It follows that the light structure on X with respect to f is generated by the countable family $(c(\mathcal{U}_i, f, \mathcal{V}_j))_{i,j\in\mathbb{N}}$. □

Definition 30. *We say that an ls-continuous map $f : X \to Y$ is **coarsely light** if the light structure on X with respect to f coincides with the ls-structure on X.*

If X is an ls-space, and $A \subseteq X$ is a subset, then there is a natural ls-structure on A induced by X, namely, all those families in A which are uniformly bounded as families in

X. We will call such an A together with the induced structure a **subspace** of X. Given a collection $(A_\alpha)_{\alpha\in I}$ of subspaces of an ls-space X, let $\bigsqcup A_\alpha$ be the disjoint union of the A_α, with the ls-structure given by all families $\mathcal{U}$ which satisfy the following conditions:

(1) the image of $\mathcal{U}$ under the obvious map $p : \bigsqcup A_\alpha \to X$ is a uniformly bounded family;

(2) each member of $\mathcal{U}$ intersects at most one of the A_α.

Using the above construction, we can formulate a generalization of the notion of uniform asymptotic dimension of subspaces of a metric space given in [7]. A family $(A_\alpha)_{\alpha\in I}$ of subspaces of an ls-space X **satisfies the inequality** asdim $\leq n$ **uniformly** if $\operatorname{asdim} \bigsqcup A_\alpha \leq n$, that is, for every uniformly bounded cover $\mathcal{U}$ of $\bigsqcup A_\alpha$, there is a uniformly bounded cover $\mathcal{V}$ of $\bigsqcup A_\alpha$ which coarsens $\mathcal{U}$ and which has point multiplicity at most $n+1$. In particular, a space X is of asymptotic dimension less than n iff $\{X\}$ satisfies asdim $\leq n$ uniformly. It is easy to see that an ls-space is of asymptotic dimension 0 if and only if for every uniformly bounded family $\mathcal{U}$, the $\mathcal{U}$-components of X form a uniformly bounded family.

Proposition 24. *Let $f : X \to Y$ be an ls-continuous map. Then the following are equivalent:*

(a) f is coarsely light,

(b) for any uniformly bounded cover $\mathcal{V}$ of Y, the family of subspaces $f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$ satisfies the inequality asdim ≤ 0 *uniformly.*

If Y is a metric space with the induced ls-structure, then the above are further equivalent to

(c) $\operatorname{asdim} f = 0$ *in the sense of [9], that is, for every subspace $B \subseteq Y$ with* $\operatorname{asdim}(B) = 0$, $\operatorname{asdim} f^{-1}(B) = 0$.

Proof. (a) $\Leftrightarrow$ (b): To say that f is coarsely light is precisely to say that the $\mathcal{U}$-components of the elements of $f^{-1}(\mathcal{V})$ form a uniformly bounded family for any uniformly bounded families $\mathcal{U}$ in X and $\mathcal{V}$ of Y. This is clearly equivalent to (b).

(a) $\Rightarrow$ (c): Suppose $B \subseteq Y$ has asymptotic dimension 0, and let $\mathcal{U}$ be a uniformly bounded cover of $f^{-1}(B)$. Consider the uniformly bounded cover $f(\mathcal{U})$ of B. By hypothesis, the $f(\mathcal{U})$-components of B form a uniformly bounded family $\mathcal{V}$. Thus the $\mathcal{U}$-components of the family $f^{-1}(\mathcal{V})$ form a uniformly bounded cover of $f^{-1}(B)$. But this cover is precisely the set of $\mathcal{U}$-components of $f^{-1}(B)$.

(c) $\Rightarrow$ (a): If f is not coarsely light, then there is a family $c(\mathcal{U}, f, \mathcal{V})$ which is not uniformly bounded, with $\mathcal{U}$ and $\mathcal{V}$ uniformly bounded families in X and Y respectively. In particular, there is a sequence $V_1, V_2, \ldots$ of elements of $\mathcal{V}$ such that $f^{-1}(V_i)$ has a $\mathcal{U}$-component of diameter greater than i. If there is a bounded set K in Y containing infinitely many of the V_i, then asdim$K = 0$, but $f^{-1}(K)$ has an unbounded $\mathcal{U}$-component, which contradicts (c). Thus every bounded set K in Y contains only finitely many of the V_i. Since the V_i are uniformly bounded, we may choose a subsequence W_i such that $d(W_i, W_{i+1}) > i$. The union $W = \bigcup W_i$ is clearly of asympotic dimension 0, but its inverse image has an unbounded $\mathcal{U}$-component, which once again contradicts (c). □

Example 7. *The following are examples of coarsely light maps:*

- *any ls-continuous map $f : X \to Y$ where* asdim$X = 0$.
- *any ls-continuous map $f : X \to Y$ where Y has bounded geometry (i.e. where the elements of any uniformly bounded cover have bounded cardinality) and*

$$\mathsf{sup}\{|f^{-1}(y)| \mid y \in Y\} < \infty.$$

- *any coarse embedding, and in particular any coarse equivalence.*

Recall that an ls-continuous map $f : X \to Y$ is called **coarsely n-to-1** [3, 46] if for every uniformly bounded cover $\mathcal{V}$ of Y there is a uniformly bounded cover $\mathcal{U}$ of X such that each element of $f^{-1}(\mathcal{V})$ is contained in the union of n elements of $\mathcal{U}$.

Proposition 25. *If $f : X \to Y$ is coarsely n-to-1, then it is coarsely light.*

Proof. Let $\mathcal{U}$ and $\mathcal{V}$ be uniformly bounded covers of X and Y respectively. We may assume that $\mathcal{U}$ is large enough so that every element of $f^{-1}(\mathcal{V})$ is contained in a union of n elements of $\mathcal{U}$. It follows that $c(\mathcal{U}, f, \mathcal{V})$ refines $\mathcal{U}^{n-1}$, the star of $\mathcal{U}$ with itself $n - 1$ times, so it is uniformly bounded. □

Lemma 14. *If $f : X \to Y$ is coarsely light and is close to g, then g is coarsely light.*

Proof. This follows from the fact that for any uniformly bounded cover $\mathcal{V}$ of Y, there is a uniformly bounded cover $\mathcal{U}$ of Y such that $g^{-1}(\mathcal{V}) \leq f^{-1}(\mathcal{U})$. □

By the lemma above, it makes sense to speak of the class of coarsely light maps in the coarse category: a morphism in the coarse category is coarsely light if and only if one (and hence all) of its representatives are coarsely light. The following lemma shows that coarsely light maps form a subcategory of the coarse category, i.e. that they are closed under composition.

Lemma 15. *If $f : X \to Y$ and $g : Y \to Z$ are coarsely light maps, then gf is a coarsely light map.*

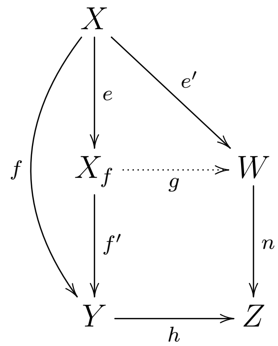
Proof. Let $\mathcal{V}$ be a uniformly bounded cover of Z and $\mathcal{U}$ a uniformly bounded cover of X. Since f and g are coarsely light, $c(f(\mathcal{U}), g, \mathcal{V})$ is uniformly bounded, as is $c(\mathcal{U}, f, c(f(\mathcal{U}), g, \mathcal{V}))$. Since $c(\mathcal{U}, gf, \mathcal{V})$ is refinement of this cover, it is uniformly bounded. □

Given any ls-continuous map $f : X \to Y$ of coarse spaces, let X_f denote X with the light structure with respect to f. Then f factorises as

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ & {\scriptstyle e}\searrow \quad \nearrow{\scriptstyle f'} & \\ & X_f & \end{array} \tag{4.1}$$

where e is the identity set map. One checks that f' is ls-continuous and coarsely light; we will call f' the **light-part of** f. This factorization satisfies a universal property, as the following lemma shows.

Lemma 16. *Let $f = f'e$ be the factorization as above. Given any diagram of solid arrows below consisting of ls-continuous maps which commutes up to closeness and in which n is a coarsely light map, there is a unique-up-to-closeness ls-continuous map g making the diagram* *commute up to closeness.*



(4.2)

Proof. Since e is the identity as a set map, the map g, if it exists, is clearly unique-up-to-closeness. Define a $g : X_f \to W$ to be the same as e' at the level of underlying sets. It that remains to show that g so defined is ls-continuous. Consider a uniformly bounded family in X_f, which we may suppose to be of the form $c(\mathcal{U}, f, \mathcal{V})$. If x and x' are in an element U of $\mathcal{U}$, then $g(x)$ and $g(x')$ are in the subset $g(U) = e'(U) \in e'(\mathcal{U})$. Moreover, if $f'(x) = f(x)$ and $f'(x') = f'(x')$ are both in an element $V \in \mathcal{V}$, then $hf'(x)$ and $hf'(x')$ are both in $h(V) \in h(\mathcal{V})$. Let $\mathcal{T}$ be the uniformly bounded cover which witnesses the closeness of hf and ne'. Then under the assumptions on x and x', $ng(x)$ and $ng(x')$ are in some $V' \in \mathcal{V}' = \mathsf{st}(h(\mathcal{V}), \mathcal{T})$. In other words, $g(c(\mathcal{U}, f, \mathcal{V})) \leq c(e'(\mathcal{U}), n, \mathcal{V}')$. Since n is coarsely light, this second cover is uniformly bounded, so g is ls-continuous as required. □

4.4 Coarsely monotone maps and monotone-light factorizations

We are now ready to define coarsely monotone maps. Let $f : X \to Y$ be an ls-continuous map and consider its factorization $f = f'e$ as in (4.1). We say that f is **coarsely monotone** if f' (i.e. the light-part of f) is a coarse equivalence. Since coarse equivalences are always light, it is easy to see that any coarse equivalence is also monotone. The following lemma is easy to show.

Lemma 17. *An ls-continuous map $f : X \to Y$ is coarsely monotone if and only if it is coarsely surjective and for every uniformly bounded cover $\mathcal{V}$ of Y, there is a uniformly* *bounded family $\mathcal{U}$ in X and a family $\mathcal{T}$ of $\mathcal{U}$-connected subsets of X which coarsens $f^{-1}(\mathcal{V})$ such that $f(\mathcal{T})$ is a uniformly bounded family in Y.*

The following lemma shows that it makes sense to speak of coarsely monotone maps in the coarse category.

Lemma 18. *If $f \sim g$ and f is coarsely monotone, then g is coarsely monotone.*

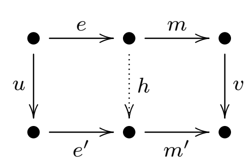
Proof. The light structures induced by g and f are the same, so the light-parts of g and f are close. The result follows. □

Let $\mathbb{C}$ be a general category and let $(\mathcal{E}, \mathcal{M})$ be a pair of classes of morphisms in $\mathbb{C}$. Recall that the pair $(\mathcal{E}, \mathcal{M})$ is said to constitute a factorization system [29] if the following conditions are satisfied:

(1) each of $\mathcal{E}$ and $\mathcal{M}$ contains the isomorphisms and is closed under composition;

(2) every morphism f in $\mathbb{C}$ can be written as $f = me$ with $m \in \mathcal{M}$ and $e \in \mathcal{E}$;

(3) given any commutative diagram of solid arrows below, with $e, e' \in \mathcal{E}$, $m, m' \in \mathcal{M}$, there is a unique morphism h making the diagram commute:

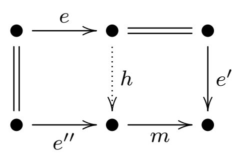


Note that in (3), if u and v are isomorphisms, then so is h. Let CMon be the class of (equivalence classes of) coarsely monotone maps and CLight be the class of (equivalence classes of) coarsely light maps in the coarse category.

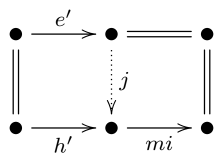
Theorem 11. *The pair $(\mathsf{CMon}, \mathsf{CLight})$ constitutes a factorization system on the coarse category. In particular, every ls-continuous map f factorizes as $f = f'e$ where f' is coarsely light and e is coarsely monotone.*

Proof. Let f be an ls-continuous map, and let $f = f'e$ be the factorization of f as in (4.1). Clearly e is coarsely monotone, so this proves (2) in the definition of a factorization system.

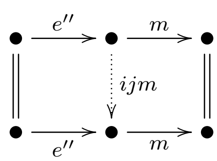
Condition (3) is an easy consequence of Lemma 16. Thus, after Lemma 15, all that remains to be shown is that coarsely monotone maps are closed under composition. This in fact follows from (2), (3) and Lemma 15. Let $e : X \to Y$ and $e' : Y \to Z$ be coarsely monotone maps, thought of as morphisms in the coarse category, and factorize $e'e$ as $e'e = me''$ with m coarsely light and e'' coarsely monotone. Then by (3) we have the morphism h in the following commutative diagram:



We also have the morphism j in the diagram



where $h = ih'$ is the $(\mathsf{CMon}, \mathsf{CLight})$-factorization of h. Thus ij is a right inverse (in the coarse category) to m. To show that it is a two-sided inverse, we apply the uniqueness part of (3) to the commutative diagram



□

Remark 4. *In the language of category theory, Lemma 16 states that the coarse category admits a **right** CLight-**factorization system** in the sense of [15] (see also [25]). It is well known that for a category $\mathbb{C}$ and a class $\mathcal{M}$ of morphisms in $\mathbb{C}$, if $\mathcal{M}$ contains the isomorphisms and is closed under composition and $\mathbb{C}$ admits a right $\mathcal{M}$-factorization system, then $\mathcal{M}$ is part of a unique factorization system $(\mathcal{E}, \mathcal{M})$ on the category. Thus the above theorem can be proved using categorical arguments once we have Lemmas 15 and 16.*

We will call a factorization $f \sim f'e$ with f' coarsely light and e coarsely monotone a **coarse monotone-light factorization of** f. Note that, by condition (3) in the definition of a factorization system, such a factorization is unique up to a coarse equivalence which makes the obvious diagram commute up to closeness. A large number of useful properties of coarsely monotone and light maps follow from the above theorem and general facts about factorization systems. For example, if fg and f are coarsely light, then so is g; dually, if fg and g are coarsely monotone, then so is f. We conclude this section with the following easy observations, which show how coarsely light and coarsely monotone maps can be used to characterise certain types of ls-spaces.

Proposition 26. *An ls-space X has asymptotic dimension 0 if and only if every ls-continuous map $f : X \to Y$ is coarsely light. An ls-space X is $\mathcal{U}$-connected for some uniformly bounded cover $\mathcal{U}$ if and only if every ls-continuous map $f : X \to B$ to a bounded space B is coarsely monotone.*

Remark 5. *If f, g and h are ls-continuous maps such that $f = gh$ and h is surjective, then g and h are both proper whenever f is proper. From this and Theorem 11 it follows easily that the classes of coarsely monotone coarse maps and coarsely light coarse maps form a factorization system on Roe's coarse category (i.e. the subcategory of the coarse category consisting of the coarse maps).*

4.5 Pullbacks in the coarse category

In this section we collect some basic facts about pullbacks in the coarse category which we will need in the next section. Suppose the following diagram of ls-continuous maps represents a pullback in the coarse category:

$$\begin{array}{ccc} P & \xrightarrow{j} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[h]{} & B \end{array} \tag{4.3}$$

Explicitly, this means that the above square commutes up to closeness, and for any ls-space X and any ls-continuous maps $u : X \to A$ and $v : X \to C$ such that $hu \sim fv$, there is a unique-up-to-closeness map $w : X \to P$ such that $gw \sim u$ and $jw \sim v$. Consider the product $A \times C$, with the ls-structure given by all families $\mathcal{U}$ such that $\pi_A(\mathcal{U})$ and $\pi_C(\mathcal{U})$ are uniformly bounded, where $\pi_A : A \times C \to A$ and $\pi_C : A \times C \to C$ are the evident projections. There is a canonical ls-continuous map $k = (g, j) : P \to A \times C$.

Lemma 19. *The map $k = (g, j)$ given above is a coarse embedding.*

Proof. Suppose not. Let $\mathcal{U}$ be a uniformly bounded family in $A \times C$ such that $k^{-1}(\mathcal{U})$ is not uniformly bounded. Let W be the set $\{(p, p') \in P \times P \mid k(p)\mathcal{U}k(p')\}$ equipped with the ls-structure consisting only of families of singleton sets. The projections $\alpha_1, \alpha_2 : W \to P$ are ls-continuous, and by construction, $k\alpha_1$ and $k\alpha_2$ are close. It follows that $g\alpha_1 \sim g\alpha_2$ and $j\alpha_1 \sim j\alpha_2$. But then the uniqueness part of the pullback property forces α_1 and α_2 to be close, which cannot be the case since $k^{-1}(\mathcal{U})$ is not uniformly bounded. □

Remark 6. *The above lemma also follows from the observation that the monomorphisms in the coarse category are precisely the coarse embeddings.*

From Lemma 19 above it follows that the pullback, when it exists, can be canonically embedded into the product. Thus, we can replace P by the image of the map k and replace j and g by the obvious projections and still obtain a pullback. The fact that the resulting diagram must commute up to closeness means that we may always assume that P is a subset of

$$A \times_{\mathcal{S}} C = \{(a, c) \in A \times C \mid h(a)\mathcal{S}f(c)\}$$

for some $\mathcal{S}$ a uniformly bounded family in B. We are now ready to present an example to show that not all pullbacks exist in the coarse category.

Example 8. *Let A be the one point set, B be the natural numbers with the ls-structure arising from the usual metric, and C the subspace*

$$\{(a_0, a_1 \ldots) \mid (i > a_0) \Rightarrow (a_i = 0)\} \subseteq \bigoplus_{i=1}^{\infty} \mathbb{N}$$

where the metric on $\bigoplus_{i=1}^{\infty} \mathbb{N}$ is Euclidean distance. Let $h : A \to B$ be the inclusion of 0 and let $f : C \to B$ be the projection $(a_0, a_1, \ldots) \mapsto a_0$. Then f and h are ls-continuous, but no pullback of f along h exists.

Proof. Suppose a pullback does exist, given by the diagram (4.3). By the above arguments, we may set $P \subseteq A \times_{\mathcal{S}} C$ for some uniformly bounded family $\mathcal{S}$ in B. Since h is the inclusion of 0, this means that P can be viewed as a subspace of $f^{-1}(B(0, N))$ for some $N > 0$. Let Q be the subspace $f^{-1}(B(0, N+1))$, with $l : Q \to C$ the obvious inclusion and m the unique map to A. Since $hm \sim fl$, by the property of the pullback, there must be a unique ls-continuous map $k : Q \to P$ such that $jk \sim l$. In particular, everything in the image of l should be a bounded distance from the image of j, but while the $(N+1)^{\text{th}}$ coordinate in P has to be 0, in Q it can be arbitrary, giving the required contradiction. □

Note that products do exist in the coarse category (as defined in the present chapter): the product of a ls-space A and B is given by the set $A \times B$ with the ls-structure consisting of all those families $\mathcal{U}$ such that $\pi_A(\mathcal{U})$ and $\pi_B(\mathcal{U})$ are both uniformly bounded, where π_A and π_B are the projections.

4.6 Coarsely monotone maps arising from a reflection

Recall that a subcategory $\mathbb{X}$ of a category $\mathbb{C}$ is **reflective** if for every object C of $\mathbb{C}$, there is an object $I(C)$ of $\mathbb{X}$ and morphism $\eta_C : C \to I(C)$ which is universal in the following sense: for any X an object of $\mathbb{X}$ and $f : C \to X$ a morphism in $\mathbb{C}$, there is a unique morphism $g : I(C) \to X$ in $\mathbb{X}$ such that $g\eta_C = f$. The assignment $C \mapsto I(C)$ extends to a functor $I : \mathbb{C} \to \mathbb{X}$ in the obvious way, which we call the **reflection** of $\mathbb{C}$ onto $\mathbb{X}$.

Let **CHaus** be the category of compact Hausdorff spaces and continuous maps (the category on which the classical monotone and lights maps constitute a factorization system), and let $\mathbf{CHaus}_0$ be the full subcategory consisting of the totally disconnected spaces. Then $\mathbf{CHaus}_0$ is reflective in **CHaus**. Indeed, for X a compact Hausdorff space, let $I(X)$ be the set of connected components of X with the quotient topology, and let $\eta_X : X \to I(X)$ be the quotient map. Then $I(X)$ is a totally disconnected compact Hausdorff space, and η_X is the universal continuous map from X to a totally disconnected compact Hausdorff space (see for example [8]).

The monotone maps between compact Hausdorff spaces can be recovered from this reflection in the following way. For a general reflection $I : \mathbb{C} \to \mathbb{X}$, let $\mathcal{E}_I$ be the class of morphisms f such that $I(f)$ is an isomorphism. Thus in the case currently being considered, $\mathcal{E}_I$ is the class of continuous maps which induce a bijection on connected components. Every monotone map is in $\mathcal{E}_I$, but not every map in $\mathcal{E}_I$ is monotone. It is easy to check that a continuous map f is monotone if and only if for every pullback

$$\begin{array}{ccc} P & \longrightarrow & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ 1 & \longrightarrow & Y \end{array}$$

where 1 is the one point space, the map g is in $\mathcal{E}_I$.

Remark 7. *In fact, a continuous map f is monotone if and only if* every *pullback of f is in $\mathcal{E}_I$. The construction of the class of monotone maps from the reflection I is a special case of a much more general process outlined in [12]. The context in [12], however, is that of a category admitting pullbacks, of which the coarse category is not an example as we have seen.*

Since a compact Hausdorff space is totally disconnected if and only if it has inductive dimension 0 (see for example [1]), one might wonder if a similar process can be applied using the class of ls-spaces of asymptotic dimension zero to arrive at the class of coarsely monotone maps. We must first describe the reflection. If X is an ls-space, then let $I(X)$ be the ls-space whose underlying set is the same as X, but whose ls-structure consists of all families which refine the set of $\mathcal{U}$-components of X for some uniformly bounded family $\mathcal{U}$ in X. Clearly $I(X)$ has asymptotic dimension zero, and the identity set map $\eta_X : X \to I(X)$ is ls-continuous. The following lemma is easy to prove.

Lemma 20. *Let $f : X \to Y$ be an ls-continuous map, where* $\text{asdim} Y = 0$*. Then f factors (up to closeness) uniquely (up to closeness) through $\eta_X : X \to I(X)$.*

It follows that the assignment $X \mapsto I(X)$ gives a reflection from the coarse category to the full subcategory of ls-spaces of asymptotic dimension zero. In particular, the assignment $X \mapsto I(X)$ extends to a functor I, which (in terms of representatives) assigns to each ls-continuous map $f : X \to Y$ an ls-continuous map $I(f) : I(X) \to I(Y)$ (which is the same as f at the level of underlying sets). As in the classical case, we let $\mathcal{E}_I$ be the class of all (equivalence classes of) ls-continuous maps f such that $I(f)$ is a isomorphism (i.e. is represented by a coarse equivalence). Clearly:

Lemma 21. *A coarsely surjective ls-continuous map $f : X \to Y$ is in $\mathcal{E}_I$ if and only if for every $\mathcal{V}$ a uniformly bounded cover in Y there is a uniformly bounded family $\mathcal{U}$ in X such that the inverse image of each $\mathcal{V}$-component of Y is contained in a $\mathcal{U}$-component of X.*

We will need the following easy categorical lemma.

Lemma 22. *Let $I : \mathbb{C} \to \mathbb{X}$ be any reflection. Then, for any pair of morphisms $f : A \to B$, $g : B \to C$, such that f has a right inverse, we have*

$$gf \in \mathcal{E}_I \Rightarrow g \in \mathcal{E}_I.$$

Proof. Let h be the two-sided inverse of $I(gf) = I(g)I(f)$. Then $I(f)h$ is a right inverse of $I(g)$. We also have

$$I(f)hI(g)I(f) = I(f)$$

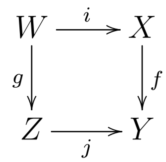
so applying the right inverse of $I(f)$ to both sides, we have that $I(f)h$ is a two-sided inverse of $I(g)$ as required. □

We can already observe from Lemma 21 that coarsely monotone maps are always in $\mathcal{E}_I$. The converse, however, does not hold. By analogy with the classical case, one might ask for a characterisation of coarsely monotone maps in terms of stability under certain pullbacks. However, as we have seen, not all pullbacks exist in the coarse category. This motivates us to instead consider an alternative condition.

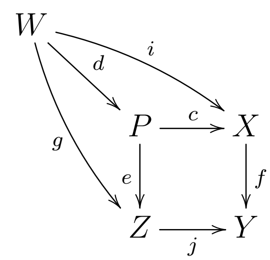
Lemma 23. *Let $\mathbb{C}$ be a category which admits all pullbacks, and let $\mathcal{E}_I$ be the class of morphisms inverted by a reflection $I : \mathbb{C} \to \mathbb{X}$. Then the following are equivalent for morphisms $f : X \to Y$ and $j : Z \to Y$ in $\mathbb{C}$:*

(P_1) *the pullback of f along j is in $\mathcal{E}_I$,*

(P_2) *for every commutative diagram*



there is a commutative diagram with $e \in \mathcal{E}_I$



(4.4)

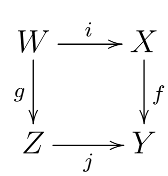
Proof. ($\mathsf{P}_1 \Rightarrow$ (P_2) is obvious – simply let e be the pullback of f along j.

($\mathsf{P}_2 \Rightarrow$ (P_1): In (4.4) above, let W be the pullback of f along j and i and g the projections. It follows from the property of the pullback that d has a left inverse s such that $gs = e \in \mathcal{E}_I$. Thus s has a right inverse, so applying Lemma 22, we obtain $g \in \mathcal{E}_I$ as required. □

Let $\mathcal{F}$ be a class of morphisms in a category $\mathbb{C}$. We say that a morphism $f : X \to Y$ in $\mathcal{E}_I$ is **stably in $\mathcal{E}_I$ with respect to $\mathcal{F}$** if for every $j : Z \to Y$ in $\mathcal{F}$, the condition (P_2) in the above lemma is satisfied. The classical monotone maps are thus precisely the continuous maps which are stably in $\mathcal{E}_I$ with respect to all maps j whose domain is the one point space (where I is the reflection onto the totally disconnected spaces). We are now ready to state an analogue for coarsely monotone maps. Recall that a ls-space is called *monogenic* if its ls-structure is generated by a single family of subsets [57].

Theorem 12. *Let $\mathcal{F}$ be the set of all (equivalence classes of) ls-continuous maps whose domain is monogenic. Then a map $f : X \to Y$ is coarsely monotone if and only if it is stably in $\mathcal{E}_I$ with respect to $\mathcal{F}$, where I is the reflection onto ls-spaces of asymptotic dimension* 0*.*

Proof. ($\Rightarrow$) Consider a diagram



of ls-continuous maps which commutes up to closeness as witnessed by the uniformly bounded cover $\mathcal{T}$ in Y, where f is coarsely monotone and $\mathcal{U}$ is a cover that generates the ls-structure on Z. Since f is coarsely surjective, there is some $\mathcal{R}$ a uniformly bounded cover in Y such that $Y \subseteq \mathsf{st}(\mathsf{Im}(f), \mathcal{R})$. We now construct the P in diagram (4.4). For every $y \in Y$, select an $s(y) \in X$ such that $y\mathcal{R}fs(y)$. Define the uniformly bounded family

$$\mathcal{S} = \mathsf{st}(\mathsf{st}(j(\mathcal{U}), \mathcal{T}), \mathcal{R}).$$

Since f is coarsely monotone, there is a uniformly bounded family $\mathcal{W}$ in X and a family $\mathcal{Q}$ of $\mathcal{W}$-conncted subsets in X such that $f(\mathcal{Q})$ is uniformly bounded. Define P to be the subspace of $Z \times X$ consisting of all pairs (z, x) such that $js(z)\mathcal{Q}x$, and let $e : P \to Z$ and $c : P \to X$ be the projections. Note that e is surjective, and that je and fc are close as witnessed by $\mathsf{st}(f(\mathcal{Q}), \mathcal{R})$. If $f(x)\mathcal{T}j(z)$, then by construction of $\mathcal{Q}$, (z, x) is in P, so there is a canonical map $d : W \to P$ making diagram (4.4) commute. It remains to show that e is in $\mathcal{E}_I$. Suppose (z, x) and (z', x') are points in P such that $z\mathcal{U}z'$. We claim that (z, x) and (z', x') are $\mathcal{U} \times \mathcal{W}$-connected in P. Indeed, (z, x) is clearly $\Delta \times \mathcal{W}$-connected to $(z, sj(z))$ in P (where Δ is the trivial cover by singletons), and similarly, (z', x') is $\Delta \times \mathcal{W}$-connected to $(z', sj(z'))$ in P. Since $fsj(z)$ and $fsj(z')$ are both in some element of $\mathcal{S}$, there is a $\Delta \times \mathcal{W}$ chain from $(z, sj(z))$ to $(z, sj(z'))$. Finally, $(z, sj(z'))$ and $(z', sj(z'))$ are $\mathcal{U} \times \Delta$-connected. Composing the chains, we obtain a proof of the claim. This shows that the inverse image of a $\mathcal{U}$-component of Z is $\mathcal{U} \times \mathcal{W}$-connected. Since the ls-structure on Z is generated by $\mathcal{U}$, this is enough to show that e is in $\mathcal{E}_I$.

($\Leftarrow$) We first claim that f is coarsely surjective. In the diagram

$$\begin{array}{ccc} W & \xrightarrow{i} & X \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \xrightarrow[j]{} & Y \end{array}$$

let Z be the underlying set of Y equipped with the smallest ls-structure, i.e. such that the only uniformly bounded families are families of singletons. Let j be the identity set map and let W be the empty ls-space with the empty maps g and i. Then there is a diagram

(4.4) with $e \in \mathcal{E}_I$. In particular, e must be surjective, and it follows by commutativity of (4.4) that f must be coarsely surjective. Now let $\mathcal{U}$ be a uniformly bounded cover of Y. For each element U of $\mathcal{U}$, pick a point $s_U \in U$. Let Z be the set of all such s_U equipped with the smallest ls-structure, and let j be the map induced by the inclusion of each point s_U. Clearly Z is monogenic as an ls-space. Let W be the set of all pairs (x, U) such that $U \in \mathcal{U}$ and $x \in f^{-1}(U)$, let $i : W \to X$ be the map $(x, U) \mapsto x$ and $g : W \to Z$ the map $(x, U) \to s_U$. Put the largest ls-structure on W for which i and g are ls-continuous. In particular, $\{(x, U), (y, V)\}$ is never bounded if $U \neq V$. Since fi and jg commute up to closeness (as witnessed by $\mathcal{U}$), we must have a diagram of the form of (4.4) with $e \in \mathcal{E}_I$. Since e is in $\mathcal{E}_I$, each $e^{-1}(s_U)$ must be $\mathcal{W}$-connected for some fixed uniformly bounded family $\mathcal{W}$ in P. Thus for every $U \in \mathcal{U}$, $f^{-1}(U) = i(g^{-1}(s_U))$ must be contained in $\mathsf{st}(c(e^{-1}(s_U)), \mathcal{T})$, a $\mathsf{st}(c(\mathcal{W}), \mathcal{T})$-connected subset of X, where $\mathcal{T}$ witnesses the closeness of cd and i. Moreover, the family $f(\mathsf{st}(c(e^{-1}(s_U)), \mathcal{T}))$ is uniformly bounded because $je \sim fc$. Since $\mathcal{U}$ was arbitrary, f is coarsely monotone as required. □

Recall from [57] that an ls-space is monogenic if and only if it is coarsely equivalent to a geodesic ∞-metric space (that is, in which points which are finite distance apart are connected by a geodesic). Thus we have the following corollary of the above result.

Corollary 16. *Let $\mathcal{F}$ be the set of all (equivalence classes of) ls-continuous maps whose domain is a geodesic ∞-metric space. Then a map $f : X \to Y$ is coarsely monotone if and only if it is stably in $\mathcal{E}_I$ with respect to $\mathcal{F}$, where I is the reflection onto ls-spaces of asymptotic dimension* 0.

A result which more closely resembles the topological situation (which involves pullbacks along maps from the singleton space) is as follows, where the singleton set is replaced by a disjoint union of singleton ls-spaces.

Corollary 17. *Let $\mathcal{F}$ be the set of all (equivalence classes of) ls-continuous maps whose domain is a set with the trivial ls-structure (i.e. the ls-structure consisting of families of singleton sets). Then a map $f : X \to Y$ is coarsely monotone if and only if it is stably in $\mathcal{E}_I$ with respect to $\mathcal{F}$, where I is the reflection onto ls-spaces of asymptotic dimension* 0.

Proof. (⇒): This follows from Theorem 12 and the fact that the trivial ls-structure is monogenic.

(⇐): This follows from the proof of Theorem 12. □

Remark 8. *One can easily show that a continuous map $f : A \to B$ between compact Hausdorff spaces is (classically) monotone if and only if for every open set $V \in B$, the restriction $f|_{f^{-1}(V)}$ induces a bijection between connected components. We can thus think of classical monotone maps as those that induce bijections on connected components "on any fixed small scale", i.e. open neighbourhood. We can view Theorem 12 as saying that coarsely monotone maps are those that induce bijections between $\mathcal{U}$-components "on any fixed large scale".*

Remark 9. *The absence of pullbacks in the coarse category presents significant obstacles to applying basic category theory arguments in coarse geometry. The above result shows that it can sometimes be useful to consider weaker notions along the lines of condition* (P_2) *in the coarse category, which in categories that do admit pullbacks reduce to familiar notions.*

4.7 Pullback-stability of coarsely monotone maps

Given any factorization system $(\mathcal{E}, \mathcal{M})$ on a category $\mathbb{C}$, it is always the case that the class $\mathcal{M}$ is stable under all pullbacks that exist in the category (that is, every pullback of an element of $\mathcal{M}$ is in $\mathcal{M}$). The same is not true in general for the class $\mathcal{E}$. Many important factorization systems do have this property, though, including the classical monotone-light factorization system. We now show that this also holds for the coarse monotone-light factorization system.

Proposition 27. *If the diagram below is a pullback in the coarse category and f is coarsely monotone, then g is also monotone.*

$$\begin{array}{ccc} P & \xrightarrow{j} & C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[h]{} & B \end{array}$$

Proof. By the remarks in Section 4.5, we may assume that P is a subspace of $A \times C$ with g and j the projections. Let $\mathcal{P}$ be the uniformly bounded family in B which witnesses the closeness of fj and hg, extended to a cover. We first show that g is coarsely surjective. We know that f is coarsely surjective, so suppose that $B \subseteq \mathsf{st}(\mathsf{Im}(f), \mathcal{R})$ for some uniformly bounded cover $\mathcal{R}$ of B. Consider the subspace of $A \times C$

$$R = \{(a, c) \mid a \in A,\ c \in C,\ h(a)\mathcal{R}f(c)\}$$

and the obvious projections ρ_1 and ρ_2 to A and C respectively. Note that ρ_1 is surjective. By the property of the pullback, there is a map $l : R \to P$ such that $gl \sim \rho_1$. Since ρ_1 is surjective, it follows that gl is coarsely surjective, and consequently that g is coarsely surjective as required.

Now, let $\mathcal{V}$ be a uniformly bounded family in A, and consider the uniformly bounded family $h(\mathcal{V})$ in B. Let $\mathcal{V}' = \mathsf{st}(h(\mathcal{V}), \mathcal{P})$. Since f is coarsely monotone, there is a uniformly bounded cover $\mathcal{V}''$ of B, which we may suppose to be coarser than $\mathcal{V}'$, and a uniformly bounded cover $\mathcal{W}$ in C such that each element of $f^{-1}(\mathcal{V}')$ is contained in a $\mathcal{W}$-component of an element of $f^{-1}(\mathcal{V}'')$. Let Q be the subspace

$$\{(a, c) \mid a \in A, c \in C, h(a)\mathcal{V}''f(c)\}$$

and let $\pi_1 : Q \to A$, $\pi_2 : Q \to C$ be the obvious projections. Since $h\pi_1$ is close to $f\pi_2$, by the property of the pullback, there must be an ls-continuous map $k : Q \to P$ such that $gk \sim \pi_1$ and $jk \sim \pi_2$. Since a map which is close to an ls-continuous map is ls-continuous, we may suppose that k is the identity on those elements which are also in P. Consider the uniformly bounded cover

$$\mathcal{V} \times \mathcal{W} = \{V \times W \mid V \in \mathcal{V},\ W \in \mathcal{W}\}$$

of $A \times C$ restricted to Q, and let $\mathcal{T}$ be its image under k. We claim that $g^{-1}(\mathcal{V})$ refines the set of $\mathcal{T}$-components of $g^{-1}(\mathsf{st}(\mathcal{V}, \mathcal{X}))$, where $\mathcal{X}$ is the cover of A which witnesses the closeness of kg and π_1. Indeed, suppose (a, c) and (a', c') are elements of P such that $\{a, a'\} \subseteq V \in \mathcal{V}$.

Then there is a $V' \in \mathcal{V}'$ containing all of $f(c)$, $f(c')$, $h(a)$ and $h(a')$, and a $V'' \in \mathcal{V}''$ containing V' such that c and c' are $\mathcal{W}$-connected inside $f^{-1}(V'')$. Consider the chain

$$(a, c) \text{ ——— } (a', c) \text{ ——— } (a', c_1) \text{ ——— } \cdots \text{ ——— } (a', c')$$

where $c, c_1, \ldots, c'$ is a chain of $\mathcal{W}$-related elements in $f^{-1}(V'')$. First note that every element in this chain is in Q, and in particular, in $\pi_1^{-1}(V) \subseteq Q$. Moreover, each pair of consecutive elements is related by $\mathcal{V} \times \mathcal{W}$. Taking the image of the chain under k, it follows that (a, c) and (a', c') are connected by a $\mathcal{T}$-chain inside an element of $g^{-1}(\mathsf{st}(\mathcal{V}, \mathcal{X}))$. □

4.8 Maps extended to the Higson corona

We will now deal with the special case of a map $f : X \to Y$ where X and Y are proper metric spaces (with the induced ls-structures) and f is a coarse map (i.e. an ls-continuous map such that the inverse image of a bounded set in Y is bounded in X). Such a map induces a continuous map $\nu f : \nu X \to \nu Y$ between the **Higson coronas** of X and Y. We briefly recall the relevant definitions from [57].

Definition 31. *Let X be a metric space, and $g : X \to \mathbb{C}$ a bounded function to the complex numbers. Then g is said to be **slowly oscillating** if for every $R > 0$ and $\varepsilon > 0$ there is a bounded set B in X such that $d(x, x') \leq R \Rightarrow |g(x) - g(x')| \leq \varepsilon$ for $x, x' \in X \setminus B$.*

The **Higson compactification** hX of a proper metric space X is the compactification of X characterised by the fact that a bounded continuous complex-valued function $g : X \to \mathbb{C}$ extends continuously to $\tilde{g} : hX \to \mathbb{C}$ if and only if it is slowly oscillating. In particular, X is dense in hX. The complement $\nu X = hX \setminus X$ is called the **Higson corona** of X.

In terms of C^* algebras, hX is the compact Hausdorff space which corresponds (under Gelfand duality) to the algebra $C_h(X)$, while νX corresponds to $C_h(X)/C_0(X)$, where $C_h(X)$ is the algebra of continuous bounded slowly-oscillating complex-valued functions on X and $C_0(X)$ is the ideal of $C_h(X)$ consisting of those functions which tend to zero at infinity. Let $B_h(X)$ be the algebra of (not necessarily continuous) bounded slowly oscillating complex-valued functions on X and $B_0(X)$ be the ideal of $B_h(X)$ consisting of those functions which tend to zero at infinity. Recall from [57] that $C_h(X)/C_0(X)$ is canonically isomorphic to $B_h(X)/B_0(X)$ via the map induced by the inclusion $C_h(X) \to B_h(X)$. Thus the corona can be defined in a way which is independent of the topology of X, which is expected since the corona captures large-scale behaviour of X.

Given a coarse map $f : X \to Y$ between metric spaces, the map $[g] \mapsto [gf]$ defines a $*$-homomorphism $f^* : B_h(Y)/B_0(Y) \to B_h(X)/B_0(X)$, which corresponds under Gelfand duality to a continuous map $\nu f : \nu X \to \nu Y$. This defines a functor ν from the category of proper metric spaces and closeness classes of coarse maps to the category of compact Hausdorff spaces [57]. In particular, νf is a homeomorphism whenever f is a coarse equivalence.

Lemma 24. *Let $f : X \to Y$ be a coarse map between proper metric spaces and $\nu f : \nu X \to \nu Y$ the induced map. If f is continuous, then νf is (up to homeomorphism) the restriction to νX of the unique continuous extension $hf : hX \to hY$ of f.*

Proof. One only needs to check that the diagram

$$\begin{array}{ccc} C_h(Y)/C_0(Y) & \xrightarrow{[g]\mapsto[gf]} & C_h(X)/C_0(X) \\ \cong\downarrow & & \downarrow\cong \\ B_h(Y)/B_0(Y) & \xrightarrow{f^*} & B_h(X)/B_0(X) \end{array}$$

commutes, which is easy. □

We will need the following lemma, which is taken from [22].

Lemma 25. *Any slowly oscillating function from a subset A of a proper metric space X to $[0, 1]$ extends to a slowly oscillating function on the whole of X to $[0, 1]$.*

Corollary 18. *Any bounded slowly oscillating function f from a subset A of a proper metric space X to $\mathbb{C}$ extends to a bounded slowly oscillating function on the whole of X to $\mathbb{C}$.*

Proof. Rescaling by a constant and translating preserves slowly oscillating functions, so we may assume that f has image $[0, 1] \times [0, 1]$. The projections $\pi_1 f$ and $\pi_2 f$ are slowly oscillating functions from A to $[0, 1]$, so by Lemma 25 we can extend each of them to slowly oscillating functions on X. Taking the induced map from X to $[0,1] \times [0,1]$ we obtain the required extension. □

Proposition 28. *Let $f : X \to Y$ be an ls-continuous map and $\nu f : \nu X \to \nu Y$ the induced continuous map between Higson coronas. Then*

(1) νf is injective if and only if f is a coarse embedding;

(2) νf is surjective if and only if f is coarsely surjective;

(3) νf is a homeomorphism if and only if f is a coarse equivalence.

Proof. (1): Suppose νf is injective; under duality, this is the same as to say that

$$f^* : B_h(Y)/B_0(Y) \to B_h(X)/B_0(X)$$

is surjective. Suppose that f is not a coarse embedding. Pick a sequence of points (a_n, b_n) in X such that $d(a_n, b_n)$ tends to infinity, but $d(f(a_n), f(b_n))$ is bounded. Since f is proper, a_n and b_n cannot be bounded, so we may choose the a_n and b_n such that each a_i (resp. b_i) is at least i away from all the b_j (resp. a_j) for $j < i$. Define a map on the union of the a_n and the b_n to $[0,1]$ which sends every a_n to 0 and every b_n to 1. This map is slowly oscillating, so we can extend it to a slowly-oscillating map on the whole of X. However, this map cannot be written as $gf + b$ for $g \in B_h(Y)$ and $b \in B_0(X)$, which contradicts the surjectivity of $f^* : B_h(Y)/B_0(Y) \to B_h(X)/B_0(X)$. Now suppose that f is a coarse embedding. Then f is, up to coarse equivalence, the inclusion of its image into Y. By Lemma 18, such inclusions give rise to surjective maps $B_h(Y)/B_0(Y) \to B_h(X)/B_0(X)$, which gives the required result.

(2): Suppose νf is surjective, i.e. that $f^* : B_h(Y)/B_0(Y) \to B_h(X)/B_0(X)$ is injective, and that f is not coarsely surjective. For every n, pick a point y_n in Y such that $d(f(X), Y) \geq n$. Define a function on $\{y_n\}$ to $\mathbb{C}$ which sends every y_n to 1. This map is clearly slowly oscillating, so it extends by Lemma 25 to a slowly oscillating function $g : Y \to \mathbb{C}$. Then g is not in $B_0(Y)$ but gf is in $B_0(X)$, contradicting the fact that $f^* : B_h(Y)/B_0(Y) \to B_h(X)/B_0(X)$ is injective. The other direction is to say that if f is coarsely surjective, and gf is in $B_0(X)$, then g is in $B_0(Y)$, which is easy to check.

(3) is a consequence of (1) and (2). □

If we are given a coarse map $f : X \to Y$ between proper metric spaces, then we can consider a 1-net X_1 in X (i.e. a maximal 1-separated subset of X). The space X_1 is a proper metric space, the inclusion $i : X_1 \to X$ is a coarse equivalence, and the composite fi is a continuous coarse map. As a result, the induced map $\nu(fi) : X_1 \to Y$ is the unique extension of fi to $hX_1 \to hY$ restricted to νX_1. But νi is a homeomorphism, so νf and $\nu(fi)$ are the same up to homeomorphism. Thus in the rest of the section we will often assume that a given coarse map f is actually continuous, and that the map νf is its extension restricted to the corona. The following lemma is a special case of Proposition 2.3 from [17].

Lemma 26. *For A and B two subsets of a proper metric space X, $\overline{A} \cap \overline{B} \cap \nu X$ is non-empty if and only if there is a $S > 0$ such that $B(A, S) \cap B(B, S)$ is unbounded.*

The proof of the following theorem was inspired in part by the techniques used in [5].

Theorem 13. *Let $f : X \to Y$ be a coarse map between proper metric spaces. Then f is coarsely monotone if and only if the induced map $\nu f : \nu X \to \nu Y$ is (classically) monotone.*

Proof. By the above remarks, we may assume that X is topologically discrete, and that νf is the restriction of a continuous extension hf.

($\Rightarrow$) By Proposition 28, νf is surjective. Suppose for contradiction that there is a $y \in \nu Y$ whose fibre under νf is disconnected, i.e. $hf^{-1}(y) = A \cup B$ with A and B disjoint closed subsets. Find a continuous function g from hX to $\mathbb{C}$ which sends A to 0 and B to 1 using the Tietze Extension Theorem. Then, $g|_X : X \to \mathbb{C}$ must be slowly oscillating. Fix $0 < \varepsilon < 1/8$. Since hf is a closed map, and hY is compact, we may choose a open neighbourhood D of y in hY such that

$$hf^{-1}(\overline{D}) \subseteq g^{-1}(B(0, \varepsilon) \cup B(1, \varepsilon))$$

Let $A' = g'^{-1}(B(0, \varepsilon)) \cap hf^{-1}(D) \cap X$ and $B' = g'^{-1}(B(1, \varepsilon)) \cap hf^{-1}(D) \cap X$. Since $y \in \overline{hf(A')} \cap \overline{hf(B')} \cap \nu Y$, by Lemma 26, there is an $S > 0$ such that, for every bounded set K in X, there is a pair a_K, b_K of points in X with $a_K \in A' \setminus K$, $b_K \in B' \setminus K$ and $d(f(a_K), f(b_K)) \leq S$. We now use the coarse monotone property of f to get that each a_K is $\mathcal{W}$-connected to b_K inside an element of $f^{-1}(\mathcal{V})$ for some uniformly bounded families $\mathcal{W}$ in X and $\mathcal{V}$ in Y. Since $g|_X$ is slowly oscillating, there is some bounded set K' in X such that, for $K' \subseteq K$, there must be an element of the $\mathcal{W}$-chain from a_K to b_K, say c_K, such that $g(c_K) \in \mathbb{C} \setminus (B(0, 2\varepsilon) \cup B(1, 2\varepsilon))$. It follows that the closure of $C = \{c_K \mid K' \subseteq K\}$ in X intersected with νX does not intersect $g^{-1}(B(0, \varepsilon))$, and thus that $\overline{hf(C)}$ does not intersect $\overline{hf(A')}$. But the set $\{f(c_K), f(a_K)\}$ is uniformly bounded, so the closure of the $f(c_K)$ and the $f(a_K)$ intersect on νY by Lemma 26, which gives the required contradiction.

($\Leftarrow$) By Proposition 28, f is coarsely surjective. Suppose f is not coarsely monotone. This means that there is a uniformly bounded cover $\mathcal{U}$ of Y such that for every integer $m > 0$, there is an element $U_m \in \mathcal{U}$ such that $f^{-1}(U_m)$ is not contained in an m-component of $f^{-1}(B(U_m, m))$. Moreover, we may assume that the distance from each $f^{-1}(B(U_m, m))$ to all the previous $f^{-1}(B(U_i, i))$ tends to infinity as $m \to \infty$. Define a map g on the union of the $f^{-1}(B(U_m, m))$ to $\{0, 1\}$ such that each m-component of $f^{-1}(B(U_m, m))$ is mapped to a single value, and such that g is surjective on each $f^{-1}(U_m)$. It is easy to see that this is a slowly oscillating function, so that it extends to a slowly oscillating function $g' : X \to [0, 1] \subseteq \mathbb{C}$. The map g' extends to a continuous map $g'' : hX \to \mathbb{C}$. Let $A = g''^{-1}(0) \cap X$ and $B = g''^{-1}(1) \cap X$ and let A' and B' be the intersection of the union of the $f^{-1}(U_m)$ with A and B respectively. By Lemma 26, the closures of $f(A')$ and $f(B')$ intersect on νY. Suppose $y \in \overline{f(A')} \cap \overline{f(B')}$. We claim that $\overline{A}$ and $\overline{B}$ cover $hf^{-1}(y) \subseteq \nu X$. Indeed, suppose that $hf(x) = y$ with $g''(x) \notin \{0, 1\}$. Pick a subset S of X such that $g''(S) \subseteq B(g'(x), \varepsilon)$ for $\varepsilon > 0$ small, and $x \in \overline{S}$. Since $\overline{f(A')}$ and $\overline{f(S)}$ intersect, by Lemma 26, there must be a $R > 0$ such that $B(f(A'), R) \cap B(f(S), R)$ is unbounded. But then S must intersect some $f^{-1}(B(U_m, m))$, which is a contradiction, since g'' was defined to be 0 or 1 on these sets. Thus the fibre of y under hf is covered by the disjoint closed sets $\overline{A}$ and $\overline{B}$, so it cannot be connected. □

As was previously mentioned, the Higson corona can also be defined for arbitrary ls-spaces. Unfortunately, Theorem 13 no longer holds in this more general context, as the following example shows.

Example 9. *Let X be $\mathbb{N}$ with the usual metric ls-structure, and let Y be the set $\mathbb{N}$ with the* ***universal bounded geometry structure*** *[57], i.e. wherein the uniformly bounded families are those families $\mathcal{U}$ such that $\{|U| \mid U \in \mathcal{U}\}$ is bounded and $\mathcal{U}$ has finite point* *multiplicity. Recall from [57] that the Higson corona of Y is the one-point space, and recall from [68] that $\mathbb{N}$ has a connected Higson corona. Let $f : X \to Y$ be the identity set map. It is clearly coarse, but is not coarsely monotone. Indeed, consider the uniformly bounded family $\mathcal{V} = \{\{n^2, (n+1)^2\} \mid n \in \mathbb{N}\}$ in Y. The family $f^{-1}(\mathcal{V})$ does not refine an M-connected family of subsets with bounded cardinality for any M, so f is not monotone. Nonetheless, the induced map on Higson coronas sends νX to a single point, and is consequently (classically) monotone.*

In fact, it is easy to see that in the above example, f is coarsely light, but the induced map νf is not classically light. Thus we cannot expect an equivalence of the form of Theorem 13 for coarsely/classically light maps for general ls-spaces.

Proposition 29. *Let $f : X \to Y$ be a coarse map between proper metric spaces. If νf is light, then f is coarsely light.*

Proof. Factorize $f = f'e$ with f' coarsely light and e coarsely monotone. By Theorem 13, νe is monotone. Since νf is light and factors through νe, νe must be a homeomorphism. But by Proposition 28, this implies that e is a coarse equivalence, so that f is light as required. □

Question 2. *Suppose $f : X \to Y$ is a coarse map between proper metric spaces. Is νf light if f is coarsely light?*

4.9 Asymptotic dimension and exactness

In this section we investigate the permanence of some coarse properties under coarsely light maps. Since we have already seen that coarsely n-to-1 maps are coarsely light, these results generalize some results of Dydak-Virk [23] obtained for coarsely n-to-1 maps.

Proposition 30. *Suppose $f : X \to Y$ is an ls-continuous map between ls-spaces. If f is coarsely light, then the asymptotic dimension of X is at most the asymptotic dimension of Y.*

Proof. Suppose $\operatorname{asdim}(Y) \leq k < \infty$ and $\mathcal{U}$ is a uniformly bounded cover of X. Pick a uniformly bounded cover $\mathcal{V}$ of Y coarsening $f(\mathcal{U})$ that has multiplicity at most $k+1$. Consider the family $\mathcal{W}$ consisting of the $\mathcal{U}$-components of elements of $f^{-1}(\mathcal{V})$. Since f is light, $\mathcal{W}$ is uniformly bounded. Moreover, it coarsens $\mathcal{U}$, and its multiplicity is at most $k+1$. □

The following definition generalizes the concept of exactness from metric spaces (as introduced by Dadarlat-Guentner [14]) to arbitrary ls-spaces. For an index set S, let $\Delta(S)$ denote the set of formal linear combinations

$$\sum_{s\in S} a_s \cdot s$$

such that $a_s \in [0,1]$ for each s, $a_s = 0$ for all but finitely many s, and $\sum a_s = 1$. We will equip $\Delta(S)$ with the l^1 metric. The **star** of a vertex $s \in S$ is the set of all elements of $\Delta(S)$ with $a_s \neq 0$. By a **partition of unity** on a set X, we mean a map $\phi : X \to \Delta(S)$ for some set S. Recall that the **mesh** of a family $\mathcal{U}$ of subsets of a metric space X is defined as follows

$$\mathsf{mesh}(\mathcal{U}) = \mathsf{sup}\{\mathsf{diam}(U) \mid U \in \mathcal{U}\}.$$

In particular, the family $\mathcal{U}$ is uniformly bounded if and only if it has finite mesh.

Definition 32. *A large scale space X is **exact** if for each uniformly bounded cover $\mathcal{U}$ of X and each $\epsilon > 0$ there is a partition of unity $\phi : X \to \Delta(S)$ such that point-inverses of stars of vertices form a uniformly bounded cover of X and the mesh of $\phi(\mathcal{U})$ is smaller than ϵ.*

Theorem 14. *Suppose $f : X \to Y$ is a large scale continuous map between ls-spaces. If f is coarsely light and Y is exact, then X is exact.*

Proof. Suppose $\mathcal{U}$ is a uniformly bounded cover of X and $\epsilon > 0$. Choose a partition of unity $\phi : Y \to \Delta(S)$ such that point-inverses of stars of vertices form a uniformly bounded cover of Y and the mesh of $\phi(f(\mathcal{U}))$ is smaller than ϵ. Consider the family J of $\mathcal{U}$-components of point-inverses of stars of vertices of the partition of unity $\phi \circ f : X \to \Delta(S)$. Create a new partition of unity $\psi : X \to \Delta(J)$ as follows:

$$\psi(x) = \sum_{j\in J} a_j \cdot j,$$

where $a_j \neq 0$ only if x belongs to j, in which case it equals the coefficient of $\phi(f(x))$ at the corresponding $s \in S$. If x, y belong to $U \in \mathcal{U}$, then they always belong to the same $\mathcal{U}$-component, so the distance from $\psi(x)$ to $\psi(y)$ is less than ϵ. Since f was coarsely light, the family of point-inverses of stars of vertices (that is, the family J) is uniformly bounded. □

Corollary 19. *Suppose $f : X \to Y$ is a large scale continuous map between metric spaces of bounded geometry. If f is coarsely light and Y has Property A [74], then X has Property A.*

Proof. As shown in [14], a metric space of bounded geometry has Property A if and only if it is exact. □

4.10 Groups

In this section we make some remarks on the case when the ls-spaces involved are groups and the maps are group homomorphisms. Let X be a (discrete) group. Following [21] we equip X with the ls-structure consisting of all refinements of covers of the form

$$\{x \cdot F \mid x \in X\}$$

where F is a finite subset of X. If X is countable, then this ls-structure coincides with the ls-structure arising from any proper left-invariant metric on X (see [62]). In particular, if X is finitely generated, then this ls-structure coincides with that induced by the word-length metric associated to any choice of finite generating set (see for example [50]). Clearly any group homomorphism $f : X \to Y$ is ls-continuous with respect to the ls-structures on X and Y.

Lemma 27. *Let X be a group. Then X has asymptotic dimension zero if and only if it is locally finite (i.e. every finitely generated subgroup is finite).*

Proof. ($\Rightarrow$): Let F be a finite subset, and $\mathcal{U} = \{x \cdot F \mid x \in X\}$ the corresponding cover. Note that any element of $\langle F \rangle$ is $\mathcal{U}$-connected to the identity element e, so that $\langle F \rangle$ is contained in the $\mathcal{U}$-component of e, which by hypothesis is bounded and hence finite.

($\Leftarrow$): Let $\mathcal{U} = \{x \cdot F \mid x \in X\}$ be a cover where F is a finite subset. Notice that $x\mathcal{U}y$ for elements $x, y \in X$ if and only if $x^{-1}y \in F \cdot F$, from which it follows inductively that if x and y are $\mathcal{U}$-connected, then $x^{-1}y \in \langle F \rangle$. By assumption, $\langle F \rangle$ is finite, so the $\mathcal{U}$-component of e is finite. Every other $\mathcal{U}$-component is a left translation of this component, so the family of $\mathcal{U}$-components is uniformly bounded, as required. □

Note that the above lemma was proved for countable groups in [62]; the proof given above is a straightforward adaptation of the proof found there. We will need the following lemma, based on the Finite Union Theorem in [7]. For a subspace $A \subseteq X$ of an ls-space X and a family $\mathcal{U}$ of subsets of X, we write $\mathcal{U}|_A$ to mean the family $\{A \cap U \mid U \in \mathcal{U}\}$.

Lemma 28. *Let X be an ls-space. If $X = A \cup B$ for subsets A and B of X, and A and B each have asymptotic dimension zero as a subspace, then X has asymptotic dimension zero.*

Proof. Let $\mathcal{U}$ be a uniformly bounded cover of X. Let $\mathcal{V}_A$ and $\mathcal{V}_B$ be the families of $\mathcal{U}|_A$-components and $\mathcal{U}|_B$-components of A and B respectively. Then $\mathcal{V}_A$ and $\mathcal{V}_B$ are uniformly bounded by hypothesis. If $a \in A$ is $\mathcal{U}$-connected to $a' \in A$, then it is easy to see that a is in the same $\mathsf{st}(\mathsf{st}(\mathcal{V}_B, \mathcal{U})|_A, \mathcal{V}_A)$-component of A as a'. Note that the family $\mathcal{W}_1$ of $\mathsf{st}(\mathsf{st}(\mathcal{V}_B, \mathcal{U})|_A, \mathcal{V}_A)$-components of A is uniformly bounded as a family in X. Using this and similar arguments one can construct a uniformly bounded family $\mathcal{W}$ in X which coarsens the family of $\mathcal{U}$-components of X, which gives the required result. □

We will also need the following generalization of Corollary 1.19 in [57].

Lemma 29. *Let $h : A \to X$ be an inclusion of a subgroup A into a group X. Then f is a coarse embedding.*

Proof. Let F be a finite subset in X. Pick a set of representatives S for the left cosets of A in X, and let T be the subset of S consisting of all those $s \in S$ such that $F \cap sA \neq \varnothing$. Clearly T is finite. Let F be the finite set $F' = (\bigcup_{t \in T} t^{-1} \cdot F) \cap A$. If a is an element of A with $a = xf \in x \cdot F$ for some $x \in X$, $f \in F$, then $f = x^{-1}a \in F$, so we may pick a $t \in T$ in the same left coset of A as x^{-1}. Then $a = xtt^{-1}f \in xt \cdot F'$ with $xt \in A$, and hence also $t^{-1}f \in A$. Thus we have that $\{x \cdot F \cap A \mid x \in X\}$ refines the uniformly bounded family $\{a \cdot F' \mid a \in A\}$ as required. □

We are now ready to present a characterisation of those group homomorphisms which are coarsely light as maps between ls-spaces.

Proposition 31. *Let $f : X \to Y$ be a group homomorphism. Then f is coarsely light if and only if $\mathsf{ker}(f)$ has asymptotic dimension zero, or equivalently, if $\mathsf{ker}(f)$ is locally finite.*

Proof. Note that by Lemma 29, we can either consider $\mathsf{ker}(f)$ as a group itself or as a subspace of X since the ls-structure is the same in each case.

($\Rightarrow$) This follows from Proposition 24 since the set consisting only of the identity in Y has asymptotic dimension zero.

($\Leftarrow$) By Lemma 29, the inclusion of the image of f into Y is a coarse embedding, and hence coarsely light. Since coarsely light maps are closed under composition, it is sufficient to consider the case when f is surjective. Let F be a finite set in Y. Then $f^{-1}(F)$ is a union of $|F|$ copies of the kernel of f, so by Lemma 28, $f^{-1}(F)$ has asymptotic dimension zero. Since

$$\{x \cdot f^{-1}(F) \mid x \in X\} = f^{-1}(\{y \cdot F \mid y \in Y\})$$

when f is surjective, the family of inverse images of the family $\{y \cdot F \mid y \in Y\}$ is a family of left translates of $f^{-1}(F)$, and so satisfies asdim $= 0$ uniformly. Thus by Lemma 24, f is coarsely light. □

Corollary 20. *Let $f : X \to Y$ be a group homomorphism whose kernel is locally finite. Then X has finite asymptotic dimension if Y does. If both X and Y are countable, then X has Property A if Y does.*

Lemma 30. *Let X be a group. Then X is finitely generated if and only if X is $\mathcal{U}$-connected for some uniformly bounded family $\mathcal{U}$.*

Proof. If X is finitely generated, then its ls-structure is generated by a word-length metric, under which X is clearly 1-connected. Conversely, suppose X is $\mathcal{U}$-connected, where $\mathcal{U} = \{x \cdot F \mid x \in X\}$. We claim that F generates X. Indeed, as in the proof of Lemma 27, x and y are $\mathcal{U}$-connected if and only if $x^{-1}y \in \langle F \rangle$, so that in particular, every $x \in X$, being in the $\mathcal{U}$-component of the identity, is in $\langle F \rangle$. □

Corollary 21. *A group X is finitely generated if and only if the unique map from X to the trivial group is coarsely monotone, and locally finite if and only if the unique map from X to the trivial group is coarsely light.*

Chapter 5

A coarse version of quotients by group actions

This chapter is based on the accepted manuscript of the following paper: L. Higginbotham and T. Weighill, Coarse quotients by group actions and the maximal Roe algebra, Journal of Topology and Analysis, online ready at https://doi.org/10.1142/S1793525319500341. *The contributions of each author of the above manuscript may be considered roughly equal. One round of revision took place after comments by the anonymous referee. The introduction has been adapted, otherwise the manuscript has remained more or less unchanged.*

5.1 Introduction

In this chapter, we are interested in studying a coarse version of the quotient of a space by a group action. In topology, given an action of a group G on a space X one can form the orbit space X/G with the quotient topology and study some of its properties. One may want to know the covering dimension of X/G or its fundamental group. The interest in such spaces is often motivated by the fact that any "nice space", say a manifold or CW-complex, is the quotient of a simply-connected space X (i.e. its universal cover) by a group action. Moreover, this group action has good properties such as being properly discontinuous.

If we are interested in large scale properties and are given a group G acting on a metric space X, then there is in fact more than one way to construct a kind of "quotient" by the action. One could take the orbit space X/G and put a structure on it which is natural from the large scale point of view, and this situation has been considered in the literature (see for example [40]). However, keeping in mind that close maps are often viewed as identical in coarse geometry, one could also construct the universal space for which the action of each $g \in G$ is close to the identity map. This is the so-called warped space X_G introduced by Roe.

If X is the cone over a compact metric space M and the action of G on X is induced by an action of G on M, then the resulting warped space X_G is called a **warped cone**. Warped cones were introduced along with warped spaces by Roe in [58] to construct examples of spaces with exotic large scale behaviour (such as not being coarsely embeddable into Hilbert space). Very recently there has been a considerable increase in interest in these warped cones; see for example [19, 60, 49] as well as a large number of preprints, including two very recent preprints by different authors which make essential use of the coarse fundamental group of a warped cone [28, 67]. It remains to be seen what possible applications exist for the results in this chapter in the study of warped cones (whose actions often satisfy the central conditions studied in this chapter, see Example 12 below).

In this chapter, we begin by generalizing the construction of the warped space to the setting of arbitrary groups acting on large scale spaces in the sense of Dydak and Hoffland [21]. This has the effect of making the universal property of the construction more apparent. After studying some general properties of the space X_G, we restrict our attention to a particular class of group actions, which we call coarsely discontinuous actions. These are the analogues of properly discontinuous actions for topological spaces. We show that when the action of G is coarsely discontinuous and X is an unbounded space which is coarsely one-ended (see Definition 38), then the group G can be recovered from X_G as an appropriately defined automorphism group $\mathsf{Aut}(X/X_G)$.

An important C^*-algebra in the index theory of non-compact complete Riemannian manifolds is the Roe algebra. The Roe algebra is a coarse invariant (that is, invariant under coarse equivalences up to isomorphism) and is functorial with respect to proper large scale continuous maps at the level of K-theory (see for example Lemma 3.5 in [56]). Thus the Roe algebra is naturally an object of study in coarse geometry. The coarse Baum-Connes conjecture (see for example [72]) concerns an index map from the K-homology of Rips complexes on X to the K-theory of the Roe algebra of X (the conjecture is that this map is an isomorphism; it is false in general [35]). A famous result of Yu [74] states that the conjecture is true for spaces which admit a coarse embedding into Hilbert space.

Gong, Wang and Yu introduced the related notion of maximal Roe algebra in [31] and formulated a version of the coarse Baum-Connes conjecture for this algebra in [51]. In this chapter, we obtain some results relating the maximal Roe algebras of X and X_G for coarsely discontinuous actions. In particular, for such actions we obtain a short exact sequence

$$0 \to \mathcal{K} \to C^*_{\max}(X_G) \to (C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G \to 0.$$

where $\mathcal{K}$ is the algebra of compact operators. Note that the crossed product in the above sequence is the full crossed product. It is impossible in general to replace the full crossed product by the reduced crossed product and the maximal Roe algebra by the usual one in the sequence above (see Corollary 24). However, we show that when G is amenable and X has Property A, then X_G also has Property A (recovering a result of Roe in [58]) and we have a short exact sequence

$$0 \to \mathcal{K} \to C^*(X_G) \to (C^*(X)/\mathcal{K}) \rtimes_{r,\alpha} G \to 0.$$

In Section 5.6, we ask the following question: to what extent can the space X_G can be considered a type of "coarse quotient"? To answer this question, we first introduce and study what we call weak coarse quotient maps between large scale spaces. We show that the class of weak coarse quotient maps is closed under closeness and composition with coarse equivalences, and that the weak coarse quotient maps are precisely the maps which correspond to regular epimorphisms in the coarse category. Note that a notion of coarse quotient map has already been introduced by Zhang [75], so we use the term "weak" here to avoid conflicting with that definition. When G is finitely generated, the canonical map $X \to X_G$ will turn out to be a weak coarse quotient map. The final section is devoted to explicitly constructing metrics which induce the various large scale structures considered in the chapter, including the one on X_G.

5.2 Preliminaries

5.2.1 Large scale spaces

The notion of large scale space (introduced in [21]) provides a general context for large scale geometry in the same way that uniform spaces provide a general context for questions of uniform continuity or convergence. A large scale space is a space equipped with a collection of families of subsets which are declared to be "uniformly bounded". To continue the analogy with uniform spaces above, the notion of large scale space is equivalent to the notion of coarse space in the sense of Roe [57] in roughly the same way that the uniform covers definition of uniform space is equivalent to the entourage definition of uniform space (see [21]). Since the reader may not be familiar with the terminology of large scale structures we recall all the necessary definitions in this section, based mostly on [21].

Let X be a set. Recall that the **star** $\mathsf{st}(B,\mathcal{U})$ of a subset B of X with respect to a family $\mathcal{U}$ of subsets of X is the union of those elements of $\mathcal{U}$ that intersect B. More generally, for two families $\mathcal{B}$ and $\mathcal{U}$ of subsets of X, $\mathsf{st}(\mathcal{B},\mathcal{U})$ is the family $\{\mathsf{st}(B,\mathcal{U}) \mid B \in \mathcal{B}\}$.

Definition 33. *A* ***large scale structure*** $\mathcal{L}$ *on a set* X *is a nonempty set of families* $\mathcal{B}$ *of subsets of* X *(which we call the* ***uniformly bounded families*** *in* X*) satisfying the following conditions:*

(1) $\mathcal{B}_1 \in \mathcal{L}$ *implies* $\mathcal{B}_2 \in \mathcal{L}$ *if each element of* $\mathcal{B}_2$ *consisting of more than one point is contained in some element of* $\mathcal{B}_1$*.*

(2) $\mathcal{B}_1, \mathcal{B}_2 \in \mathcal{L}$ *implies* $\mathsf{st}(\mathcal{B}_1, \mathcal{B}_2) \in \mathcal{L}$*.*

Note that any uniformly bounded family can be extended to a uniformly bounded cover by adding all the singleton subsets to it, so we will often assume that a particular uniformly bounded family is in fact a cover. Also, note that if $\mathcal{U}$ and $\mathcal{V}$ are elements of a large scale structure $\mathcal{B}$, then so is $\mathcal{U} \cup \mathcal{V}$. By a **large scale space** (or **ls-space** for short), we mean a set equipped with a large scale structure. To be precise, we should write large scale spaces as pairs $(X, \mathcal{X})$, where X is a set and $\mathcal{X}$ is a large scale structure. However, when we are dealing with only one large scale structure on each set, we will often write simply X to mean the set equipped with its large scale structure.

Example 10. *The canonical example of a large scale space is as follows. Let (X, d) be an ∞-metric space (that is, a metric space where the metric is also allowed to assume the value ∞). Define the uniformly bounded families in X to be all those families $\mathcal{U}$ for which*

$$\mathsf{mesh}(\mathcal{U}) = \mathsf{sup}\{\mathsf{diam}(U) \mid U \in \mathcal{U}\} < \infty$$

*This is the large scale structure **induced** by the metric d.*

A subset of a large scale space X is called **bounded** if it is an element of some uniformly bounded family in X. A large scale space is called **coarsely connected** if every finite set is bounded (for example, every metric space is such). Every set admits a smallest coarsely connected large scale structure, namely those families of finite subsets which have only finitely many non-singleton sets.

Example 11. *Another important class of a large scale structures comes from group structures. If G is a group, we can put a large scale structure on the underlying set of G consisting of all refinements of covers of the form*

$$\{g \cdot F \mid g \in G\}$$

for some finite subset $F \subseteq G$. If G is finitely generated, then this is the same large scale structure as the one induced by any word metric (see for example Chapter 1 of [50]) on G. If G is countable, this is the large scale structure induced by any discrete proper left-invariant metric on G (see [62]).

When dealing with quotients of metric spaces, it will often be easiest to first define (usually via a generating set) the large scale structure on the quotient, and then find a metric on the quotient space that induces this large scale structure. For this purpose, we need the following result, which was originally stated for coarse spaces by Roe in Section 2.4 of [57] and later for large scale spaces in [21]. For completeness we include a proof. We say that a collection of families of subsets $\mathcal{A}$ **generates** a large scale structure $\mathcal{X}$ if $\mathcal{X}$ is the smallest large scale structure containing $\mathcal{A}$.

Theorem 15 (Theorem 1.8 in [21])**.** *Let X be a large scale space. Then there exists an ∞-metric on X which induces the large scale structure on X if and only if the large scale structure is countably generated.*

Proof. Clearly if the large scale structure is induced by a metric d then the countable set $(\mathcal{U}_i)_{i\in\mathbb{N}}$, where

$$\mathcal{U}_i = \{B(x,i) \mid x \in X\},$$

generates the large scale structure. Suppose then that the large scale structure is generated by the countable set $(\mathcal{U}_i)_{i\in\mathbb{N}}$ of uniformly bounded families. We may assume that for every i, $\mathsf{st}(\mathcal{U}_i,\mathcal{U}_i)$ refines $\mathcal{U}_{i+1}$, and that $\mathcal{U}_0$ is actually a cover. For points x and y in X, define $d(x,y)$ to be the smallest i for which there is an element of $\mathcal{U}_i$ containing both x and y. One checks that this defines an ∞-metric that induces the large scale structure. $\square$

In fact, every collection $\mathcal{X}$ of families of subsets of some set X generates a large scale structure. Two possible constructions exist for this large scale structure, which we denote by $\overline{\mathcal{X}}$. One can either

- take the intersection of all large scale structures containing $\mathcal{X}$ (this is based on Proposition 2.12 in [57]),
- add the cover by singletons to $\mathcal{X}$ if necessary, close $\mathcal{X}$ under the star operation and then close the resulting collection under refinement.

Lemma 31. *Let $f : X \to Y$ be a map between sets and let $\mathcal{X}$ be a collection of families of subsets of X. Then*

$$f(\overline{\mathcal{X}}) \subseteq \overline{f(\mathcal{X})}$$

where $f(\mathcal{X}) = \{f(\mathcal{U}) \mid \mathcal{U} \in \mathcal{X}\}$.

Proof. Using the second construction above, this follows from the fact that for any two families $\mathcal{U}$ and $\mathcal{V}$ of subsets of X, the family $f(\mathsf{st}(\mathcal{U}, \mathcal{V}))$ refines $\mathsf{st}(f(\mathcal{U}), f(\mathcal{V}))$. □

Finally, we establish some notation. For two families of subsets $\mathcal{U}$ and $\mathcal{V}$, we write $\mathcal{U} \leq \mathcal{V}$ if $\mathcal{U}$ refines $\mathcal{V}$, in which case we also say that $\mathcal{V}$ **coarsens** $\mathcal{U}$. For two points x and y, we write $x\mathcal{U}y$, if $\mathcal{U}$ coarsens $\{\{x, y\}\}$.

5.2.2 Large scale continuous maps

We now recall the right notion of "continuous map" between large scale spaces. Given a set map $f : X \to Y$ from an large scale space X to an large scale space Y, we say that f is **large scale continuous** or **ls-continuous** if for every uniformly bounded family $\mathcal{U}$ in X, the family

$$f(\mathcal{U}) = \{f(U) \mid U \in \mathcal{U}\}$$

is uniformly bounded in Y. These maps are the equivalent of bornologous maps for coarse spaces. In particular, a map $f : X \to Y$ from a metric space X to a metric space Y is large scale continuous if and only if for every $R > 0$ there exists an $S > 0$ such that the following holds for every $x_1, x_2 \in X$:

$$d(x_1, x_2) \leq R \implies d(f(x_1), f(x_2)) \leq S.$$

Let $f, g : X \to Y$ be two set maps between large scale spaces, not necessarily large scale continuous. We say that f and g are **close** and write $f \sim g$ if the family of subsets

$$\{\{f(x), g(x)\} \mid x \in X\}$$

is uniformly bounded. Notice that any map that is close to a large scale continuous map is large scale continuous. Any uniformly bounded family which coarsens the family $\{\{f(x), g(x)\} \mid x \in X\}$ is said to **witness** the closeness of f and g.

If A is a subset of a large scale space X, then there is a natural large scale structure which makes A a **subspace** of X, namely the restriction of all uniformly bounded families in X to A. Now let $f : X \to Y$ be a large scale continuous map. We say that f is

- **coarsely surjective** if there is a uniformly bounded family $\mathcal{V}$ in Y such that $Y \subseteq \mathsf{st}(f(X), \mathcal{V})$,
- **a coarse embedding** if for every uniformly bounded family $\mathcal{V}$ in Y, the family

$$f^{-1}(\mathcal{V}) = \{f^{-1}(V) \mid V \in \mathcal{V}\}$$

  is uniformly bounded in X.
- **a coarse equivalence** if it is both coarsely surjective and a coarse embedding.

Clearly the inclusion of a subspace into a large scale space is a coarse embedding. A large scale continuous map $f : X \to Y$ being coarsely surjective is clearly the same as requiring that the subspace inclusion $f(X) \to Y$ is a coarse equivalence. One can easily check that a large scale continuous map $f : X \to Y$ is a coarse equivalence if and only if there exists a large scale continuous map $g : Y \to X$ such that fg and gf are both close to the respective identity map. This suggests that coarse equivalences should be isomorphisms in an appropriate category.

5.2.3 The coarse category

Various definitions exist in the literature for the coarse category, which was originally introduced by Roe in [56]. For example, Roe requires that all maps in this category be **proper** (the inverse image of a bounded set is bounded), while the authors of [16] prefer to exclude this requirement since otherwise the category does not admit products. One common requirement however is that close maps are identified in the category. We will use the following definition for this chapter.

Definition 34. *The category* **Coarse**$/\sim$*, called the* ***coarse category****, is the category whose objects are large scale spaces and whose morphisms are equivalence classes of large scale continuous maps under the closeness relation* $\sim$*.*

Composition in this category is defined in terms of representatives: $[\alpha] \circ [\beta] = [\alpha \circ \beta]$. One can check that this is well-defined. We recall the following lemma, which for coarse spaces is proved in [16]. The proof of the first two facts is an easy adaptation of the proof presented there. The third fact follows from the remarks in the previous subsection.

Lemma 32. *Let* f *represent a morphism* $[f]$ *in* **Coarse**$/\sim$*. Then*

- $[f]$ *is an epimorphism if and only if* f *is coarsely surjective;*
- $[f]$ *is a monomorphism if and only if* f *is a coarse embedding;*
- $[f]$ *is an isomorphism if and only if* f *is a coarse equivalence.*

The coarse category (as defined in this chapter) admits binary products. For two large scale spaces X and Y, their product is the set $X \times Y$ equipped with the large scale structure consisting of all refinements of families of the form

$$\mathcal{U} \times \mathcal{V} = \{U \times V \mid U \in \mathcal{U},\ V \in \mathcal{V}\}$$

for uniformly bounded families $\mathcal{U}$ in X and $\mathcal{V}$ in Y.

5.3 Group actions and X_G

Definition 35. *Let* X *be a large scale space with large scale structure* $\mathcal{X}$ *and let* G *be a group acting on the underlying set of* X*. Let* $\mathcal{X}_G$ *be the large scale structure on* X *generated by* $\mathcal{X}$ *together with all families of the form*

$$\{\{x, gx\} \mid x \in X\}$$

for some $g \in G$*. We denote the set* X *together with the large scale structure* $\mathcal{X}_G$ *by* X_G *and call it the* ***warped space****. We denote by* $p_G : X \to X_G$ *the identity set map.*

One way to view X_G is that it is the underlying set of X equipped with the smallest large scale structure which makes the action of each $g \in G$ close to the identity.

Remark 10. *It is easy to check that for any finite subset $F \subseteq G$, the family*

$$\{F \cdot x \mid x \in X\} = \{\{f \cdot x \mid f \in F\} \mid x \in X\}$$

is uniformly bounded in X_G. In particular, let G be a group and $|G|$ the underlying set of G with either the large scale structure consisting only of families of singletons or the smallest coarsely connected large scale structure. Let G act on $|G|$ by right translation, that is, $g \cdot h = hg^{-1}$. Then $|G|_G$ is G with the large scale structure coming from the group structure (see Example 11).

Let X be a large scale space. By an **action of a group G on X by coarse equivalences** we mean an action of G on the underlying set of X such that every $g \in G$ acts as a large scale continuous map.

Lemma 33. *Let $(X, \mathcal{X})$ be a large scale space and let G be a group acting on X by coarse equivalences. Then the large scale structure on X_G is precisely the collection $\mathcal{X}'$ of refinements of families of the form $\mathsf{st}(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ is a uniformly bounded family in X and $\mathcal{F}$ is of the form*

$$\{F \cdot x \mid x \in X\}.$$

for some finite subset $F \subseteq G$.

Proof. Since $\mathcal{X}'$ contains $\mathcal{X}$ and all families of the form $\{\{x, gx\} \mid x \in X\}$, it is enough to show that $\mathcal{X}'$ is a large scale structure, that is, closed under stars. Since

$$\mathsf{st}(\mathsf{st}(\mathcal{U}_1, \mathcal{F}_1), \mathsf{st}(\mathcal{U}_2, \mathcal{F}_2)) \leq \mathsf{st}(\mathsf{st}(\mathsf{st}(\mathsf{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2), \mathcal{U}_2), \mathcal{F}_2)$$

it is enough to prove that both $\mathsf{st}(\mathsf{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2)$ and $\mathsf{st}(\mathsf{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{U}_2)$ are in $\mathcal{X}'$ for any $\mathcal{U}_1, \mathcal{U}_2 \in \mathcal{X}$ and any two families

$$\mathcal{F}_1 = \{F_1 \cdot x \mid x \in X\}, \ \mathcal{F}_2 = \{F_2 \cdot x \mid x \in X\}.$$

where F_1 and F_2 are finite subsets of G. For the first, we have

$$\mathsf{st}(\mathsf{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{F}_2) \leq \mathsf{st}(\mathcal{U}_1, \mathcal{F}_3)$$

where

$$\mathcal{F}_3 = \{F_1 \cdot F_1^{-1} \cdot F_2 \cdot x \mid x \in X\}.$$

For the second one, let $\mathcal{V}$ be the element of $\mathcal{X}$ (using the fact that F_1 is finite)

$$\{g \cdot U \mid U \in \mathcal{U}_2 \wedge g \in F_1 \cdot F_1^{-1}\}.$$

Then

$$\mathsf{st}(\mathsf{st}(\mathcal{U}_1, \mathcal{F}_1), \mathcal{U}_2) \leq \mathsf{st}(\mathsf{st}(\mathcal{U}_1, \mathcal{V}), \mathcal{F}_1)$$

which gives the required result. □

Some readers may prefer to use Lemma 33 as the definition of the large scale structure on X_G. Definition 35 emphasises the universal property of X_G, whereas in practice the one provided by the lemma is most useful. We now briefly consider the case when G is finite. In particular, we show that X_G is the same, up to coarse equivalence, as X/G with an appropriate large scale structure.

Lemma 34. *Let $(X, \mathcal{X})$ be a large scale space and let G be a finite group acting on X by coarse equivalences. Let $q : X \to X/G$ be the quotient map onto the orbit space, and let $q(\mathcal{X})$ be the collection of all images of elements of $\mathcal{X}$ under q. Then $q(\mathcal{X})$ is a large scale structure.*

Proof. Let $\mathcal{U}, \mathcal{V} \in \mathcal{X}$. Let $\mathcal{U}' = \bigcup_{g \in G} g(\mathcal{U})$ and $\mathcal{V}' = \bigcup_{g \in G} g(\mathcal{V})$ respectively, and note that $\mathcal{U}'$ and $\mathcal{V}'$ are each in $\mathcal{X}$ because G is finite. Then $\mathcal{U}$ and $\mathcal{V}$ refine $\mathcal{U}'$ and $\mathcal{V}'$ respectively, and one can check that

$$\mathsf{st}(q(\mathcal{U}'), q(\mathcal{V}')) = q(\mathsf{st}(\mathcal{U}', \mathcal{V}')) \in q(\mathcal{X}).$$

□

Proposition 32. *Let $(X, \mathcal{X})$ be a large scale space and let G be a finite group acting on X by coarse equivalences. Let X/G be the orbit space with the large scale structure $q(\mathcal{X})$, where q is the quotient map $q : X \to X/G$. Then the natural map $p : X_G \to X/G$ is a coarse equivalence.*

Proof. Since the image of every family $\{\{x, gx\} \mid x \in X\}$ under q is a family of singletons, the map p is a large scale continuous map by Lemma 31, which is moreover surjective. It remains to show it is a coarse embedding. By Proposition 34, every uniformly bounded family $\mathcal{U}$ in X/G is the image under q of a uniformly bounded family $\mathcal{U}'$ in X. Then

$$q^{-1}(\mathcal{U}) \leq \mathsf{st}(\mathcal{U}', \mathcal{G})$$

where $\mathcal{G} = \{\{x, gx\} \mid g \in G,\ x \in X\}$. The family $\mathsf{st}(\mathcal{U}', \mathcal{G})$ is uniformly bounded in X_G because G is finite, which completes the proof. □

Proposition 33. *Let X be a metric space and G a finite group acting on X by coarse equivalences. Then the large scale structure $q(\mathcal{X})$ on the orbit space X/G, where q is the quotient map $q : X \to X/G$ and $\mathcal{X}$ is the large scale structure on X, is induced by the Hausdorff metric on orbits, that is,*

$$d_{\mathsf{Haus}}([x], [y]) = \min\{\sup_{x' \in [x]} \inf_{y' \in [y]} d(x', y'), \sup_{y' \in [y]} \inf_{x' \in [x]} d(x', y')\}.$$

Suppose further than G acts by isometries. Then $q(\mathcal{X})$ is also induced by

$$d_{\mathsf{min}}([x], [y]) = \min\{d_X(x', y') \mid x' \in [x], y' \in [y]\} = \min\{d_X(x, g \cdot y) \mid g \in G\}.$$

Proof. If $d_{\mathsf{Haus}}([x], [y]) \leq R$, then there are $x' \in [x], y' \in [y]$ such that $d_X(x', y') \leq R$, so clearly every cover which is uniformly bounded with respect to d_{Haus} is contained in $q(\mathcal{X})$. On the other hand, if $d_X(x, y) \leq R$ for $x, y \in X$, then there is an $S > 0$ depending only on R such that $d_X(g \cdot x, g \cdot y) \leq S$ for all $g \in G$. It follows that $d_{\mathsf{Haus}}([x], [y]) \leq S$, so every element of $q(\mathcal{X})$ is uniformly bounded with respect to d_{Haus}. If G acts by isometries, then $d_{\mathsf{min}}([x], [y])$ is indeed a metric, and it is easy to see that it induces $q(\mathcal{X})$. □

We recall the definition of coarsely light map from [24]. For A a subset of a large scale space X and $\mathcal{U}$ a cover of X, let $A_{\mathcal{U}}$ be the set of equivalence classes of A under the equivalence relation: $x \sim x'$ if and only if there exists a finite sequence $x_0, \ldots, x_n$ of elements of A with $x\mathcal{U}x_0$, $x_n\mathcal{U}x'$ and $x_i\mathcal{U}x_{i+1}$ for all $0 \leq i < n$. A large scale continuous map $f : X \to Y$ is called **coarsely light** if for every pair of uniformly bounded covers $\mathcal{V}$ of Y and $\mathcal{U}$ of X, the family of subsets

$$\bigcup_{V \in \mathcal{V}} f^{-1}(V)_{\mathcal{U}}$$

is uniformly bounded in X. Equivalently, f is coarsely light if for every uniformly bounded family $\mathcal{V}$ in Y, the family of subsets $f^{-1}(\mathcal{V})$ has asymptotic dimension zero uniformly (see [24] for details).

Proposition 34. *Let $(X, \mathcal{X})$ be a large scale space and let G be a group acting on X by coarse equivalences. Then the identity set map $p_G : X \to X_G$ is coarsely light.*

Proof. Let $\mathsf{st}(\mathcal{V}, \mathcal{F})$ be a uniformly bounded cover of X_G, where $\mathcal{V}$ is a uniformly bounded cover of X and $\mathcal{F} = \{F \cdot x \mid x \in X\}$ for a finite subset $F \subseteq G$ containing the identity (see Lemma 33). Since for any $\mathsf{st}(V, \mathcal{F}) \in \mathsf{st}(\mathcal{V}, \mathcal{F})$,

$$\mathsf{st}(V, \mathcal{F}) \subseteq \bigcup_{g \in F \cdot F^{-1}} g \cdot V,$$

it follows that every element of $\mathsf{st}(\mathcal{V}, \mathcal{F})$ is contained in a union of $n = |F \cdot F^{-1}|$ elements of the family $\mathcal{V}' = \bigcup_{g \in F \cdot F^{-1}} g \cdot \mathcal{V}$, which is uniformly bounded in X. Let $\mathcal{U}$ be a uniformly bounded cover in X, which we may assume coarsens $\mathcal{V}'$. Then for any $W \in \mathsf{st}(\mathcal{V}, \mathcal{F})$, each element of $W_{\mathcal{U}}$ is contained in an element of the uniformly bounded cover

$$\underbrace{\mathsf{st}(\mathsf{st}(\cdots \mathsf{st}(}_{n-1 \text{ times}}\mathcal{U}, \mathcal{U}), \mathcal{U}) \cdots, \mathcal{U}), \mathcal{U}).$$

□

Corollary 22. *Let $(X, \mathcal{X})$ be a large scale space and let G be a group acting on X by coarse equivalences. Then* asdim $X \leq$ asdim X_G.

Proof. This follows from Proposition 9.1 in [24]. □

In the case when G is finite, we can apply a result of Kasprowski [40] to get a better result. Kasprowski proves that for a proper metric space X and a finite group G acting on X by isometries, X/G with the metric

$$d([x],[x']) = \min_{g \in G} d(x, gx') \tag{5.1}$$

has asymptotic dimension equal to that of X. We get the following corollary for X_G.

Corollary 23 (of Theorem 1.1 in [40]). *Let X be a proper metric space and let G be a finite group acting on X by coarse equivalences. Then* $\text{asdim} X_G = \text{asdim} X$*.*

Proof. As was already observed in [23], the metric

$$d'_X(x, x') = \sum_{g \in G} d(g \cdot x, g \cdot x')$$

on X induces the same large scale structure as the original metric on X, and the group G acts on X by isometries with respect to the metric d'. Thus we may reduce to the case when G acts by isometries (since asymptotic dimension is invariant under coarse equivalence). Let d be the metric on X/G defined in (5.1), which by Proposition 33 induces the large scale structure $q(\mathcal{X})$ on X/G, where $q : X \to X/G$ is the quotient map. Applying Kaprowski's result, we have that asdim X/G = asdim X, and since asymptotic dimension is invariant under coarse equivalence, we obtain the result by Proposition 32. □

When X is a metric space and G is a countable group then the large scale structure on X_G is countably generated, hence metrizable. If G is finitely generated, then the space X_G is the same (up to coarse equivalence) as the warped space introduced by Roe in [58], as shown in Proposition 35 below.

Definition 36 ([58]). *Let (X, d) be a proper metric space and let G be a group acting on X, provided with a finite generating set S. The* ***warped metric*** *d_G on X is the greatest metric* *that satisfies the inequalities*

$$d_G(x, x') \leq d(x, x'), \quad d_G(x, s \cdot x) \leq 1 \ \forall s \in S.$$

We call X with the metric d_G the ***(metric) warped space****.*

Proposition 35. *The warped metric d_G induces the large scale structure on X_G.*

Proof. Let $\mathcal{L}$ be the large scale structure induced by d_G, and let $\mathcal{X}_G$ be the large scale structure on X_G. Clearly $\mathcal{L}$ contains the generating families of $\mathcal{X}_G$, hence $\mathcal{X}_G \subseteq \mathcal{L}$. On the other hand, if $d_G(x, y) \leq k$, then by Proposition 1.3 in [58], there is a sequence $x = x_0, x_1, \ldots, x_{k+1} = y$ in X and elements $\gamma_i \in G$ with

$$d(\gamma_i x_i, x_{i+1}) \leq k \quad \text{and} \quad |\gamma_i| \leq k$$

for all i, where $|\gamma|$ is the word-length of γ. This implies that $x_i \mathcal{U} x_{i+1}$ for all i, where $\mathcal{U}$ is the star of the family

$$\bigcup_{|\gamma| \leq k} \{\{x, \gamma x\} \mid x \in X\}$$

against the family of k-balls with respect to the metric d. Both of these families are in $\mathcal{X}_G$ so, since the length of the sequence depends only on k and not on x or y, it follows that the set of all k-balls with respect to d_G is in $\mathcal{X}_G$, and thus $\mathcal{L} \subseteq \mathcal{X}_G$ as required. □

Note that if the group G acts by isometries then it follows from Proposition 1.3 in [58] that the warped metric d_G has a simpler description, namely

$$d_G(x, x') = \inf_{g \in G} \{d(x, gx') + |g|\}$$

where $|g|$ is the word-length of g and d is the metric on X.

5.4 Coarsely discontinuous actions

We now restrict our attention to the analogue of properly discontinuous actions on topological spaces, which we call coarsely discontinuous actions.

Definition 37. *Let X be a large scale space and let G be a group acting on X by coarse equivalences. We say that the action of G is **coarsely discontinuous** if for every uniformly bounded family $\mathcal{U}$ and every element $g \in G \setminus \{e\}$, there is a bounded set K such that for every $U \in \mathcal{U}$ with $U \cap K = \varnothing$, we have $U \cap g \cdot U = \varnothing$.*

If X is a metric space, then to say that an action of G on X is coarsely discontinuous is clearly the same as to say that for each $g \in G \setminus \{e\}$ and each $R > 0$ there is a bounded set K such that $d(x, g \cdot x) \geq R$ for all $x \notin K$. Or, more succintly, $d(x, g \cdot x) \to \infty$ as $x \to \infty$ for every $g \neq e$.

Example 12. *We recall the construction of the warped cone given in [58] in detail. Let X be a compact metric space. By the **cone** CX on X we mean the quotient $X \times [0, \infty)/\sim$, where $\sim$ is the equivalence relation generated by the pairs $\{((x, 0), (x', 0)) \mid x, x' \in X\}$. We can turn CX into a metric space by choosing a continuous weight function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(t) = 0 \Leftrightarrow t = 0$ and defining*

$$d([(x,t)], [(x',t')]) = \inf \left\{ \sum_{j=0}^{n-1} |t_j - t_{j+1}| + \max\{\Phi(t_j), \Phi(t_{j+1})\} d_X(x_j, x_{j+1}) \right\}$$

*where the infimum is taken over all finite sequences $(x_j, t_j)_{0 \leq j \leq n}$ of points in CX with $(x_0, t_0) \sim (x, t)$ and $(x_n, t_n) \sim (x', t')$. One checks that this is a metric. If X is a path metric space, then this is the same metric defined in (3.46) of [55]. If G is a group which acts on X, then there is a natural action of G on CX given by $g \cdot (x, t) = (g \cdot x, t)$. The warped space $(CX)_G$ is called the **warped cone**.*

If the action of G on X is by isometries, then the action of G on CX is also by isometries, hence by coarse equivalences. If $\Phi(t) \to \infty$ as $t \to \infty$ and the group G acts freely on X, then the action is also coarsely discontinuous. Indeed, let $R > 0$ and $g \in G \setminus \{e\}$. Define

$$k_g = \min\{d_X(x, g \cdot x) \mid x \in X\} > 0.$$

Pick $t_0 > R$ such that $\Phi(t) > R/k_g$ for all $t > t_0 - R$. If $(x, t) \in CX$ with $t > t_0$, then for any sequence $(x_j, t_j)_{0 \leq j \leq n}$ with $(x, t) = (x_0, t_0)$ and $(g \cdot x, t) = (x_n, t_n)$, if $t_j \geq t_0$ for all j

then

$$\sum_{j=0}^{n-1} |t_j - t_{j+1}| + \max\{\Phi(t_j), \Phi(t_{j+1})\} d_X(x_j, x_{j+1}) \geq \sum_{j=0}^{n-1} R/k_g \cdot d_X(x_j, x_{j+1}) \geq R$$

otherwise $t_m < t_0$ *for some* m *and thus*

$$\sum_{j=0}^{n-1} |t_j - t_{j+1}| + \max\{\Phi(t_j), \Phi(t_{j+1})\} d_X(x_j, x_{j+1}) \geq \sum_{j=0}^{m-1} |t_j - t_{j+1}| \geq R$$

by the triangle inequality. Thus outside of the bounded set $X \times [0, t_0]/\sim$, $d((x,t), g\cdot(x,t)) \geq R$ *as required.*

Example 13. *Let* G *be a group equipped with a discrete proper left-invariant metric (such as a word-length metric). Denote* $d(e, g)$ *by* $|g|$ *so that* $d(g, h) = |g^{-1}h|$*. Consider the action of* G *on itself as a metric space by left translation:* $g \cdot x = gx$*. This action is an action by isometries, hence in particular by coarse equivalences. Since* $d(x, gx) = |x^{-1}gx|$*, this action will be coarsely discontinuous if for every* $R > 0$ *and* $g \neq e$, $\{x \mid |x^{-1}gx| \leq R\}$ *is finite. The ball of radius* R *around the identity element is itself finite, so this is clearly equivalent to requiring that for each* $x \in G$, $\{y \in G \mid y^{-1}gy = x^{-1}gx\}$ *is finite. Since* $x^{-1}gx = y^{-1}gy$ *if and only if* xy^{-1} *is in the centralizer of* g*, we have that the action of* G *on itself by left translation is coarsely discontinuous if the centralizer of every* $g \in G \setminus \{e\}$ *is finite. Conversely, if the centralizer of some* $g \in G \setminus \{e\}$ *is infinite, then* $d(x, gx) = |g|$ *for infinitely many* $x \in G$*, so the action can't be coarsely discontinuous. Thus the action of* G *on itself by left translation is coarsely discontinuous if and only if the centralizer of every non-identity element of* G *is finite.*

Nonexample 1. *Again consider a finitely generated group equipped with discrete proper left-invariant metric, but this time consider the action by right translation:* $g \cdot x = xg^{-1}$*. The action of each* $g \in G$ *by right translation is close to the identity since* $d(x, xg^{-1}) \leq |g^{-1}|$*, so it follows that this action is by coarse equivalences, but is never coarsely discontinuous if* G *is infinite.*

If X is a large scale space and G is a group acting on X by coarse equivalences, then we can consider the set of all coarse equivalences $X \to X$ which are close to the identity when considered as maps on X_G. If we identify in this set those maps which are close as maps on X, then we obtain a group under composition which we denote by $\mathsf{Aut}(X/X_G)$. In the language of category theory, this is the automorphism group of $p_G : X \to X_G$ in the slice category $(\mathbf{Coarse}/\sim)/X_G$. If X satisfies a connectedness condition and G acts coarsely discontinuously, then we will see that $\mathsf{Aut}(X/X_G)$ is naturally isomorphic to G.

If $\mathcal{U}$ is a cover of a set X, then the **$\mathcal{U}$-component of** $x \in X$ is the set of all $x' \in X$ for which there is a finite sequence $(U_i)_{0 \le i \le n}$ of elements of $\mathcal{U}$ with $x \in U_0$, $x' \in U_n$ and $U_i \cap U_{i+1} \neq \varnothing$ for $0 \le i < n$. A set is called **$\mathcal{U}$-connected** if it contains at most one $\mathcal{U}$-component.

Definition 38. *Let X be a large scale space and let $\mathcal{U}$ be a uniformly bounded cover of X. We say that X is* ***coarsely one-ended at scale*** *$\mathcal{U}$ if for every bounded set K in X there is a bounded set K' containing K such that $X \setminus K'$ is $\mathcal{U}$-connected. We say that X is coarsely one-ended if it is coarsely one ended at some scale.*

Proposition 36. *Let X be a proper geodesic metric space. Then X is coarsely one-ended if and only if it is topologically one-ended, that is, for every bounded set $K \subseteq X$ there is a bounded set $K' \subseteq X$ so that $K \subseteq K'$ and $X \setminus K'$ is topologically connected.*

Proof. $(\Rightarrow)$: Suppose X is coarsely one-ended at scale $\mathcal{U}$. Without loss of generality, let $\mathcal{U}$ be the cover by R-balls, $R > 0$. Let K be a bounded set and consider $N = \mathsf{st}(K, \mathcal{U})$. Then there is a L containing N so that $N \subseteq L$ and $X \setminus L$ is $\mathcal{U}$-connected. If $X \setminus L$ is topologically connected, then we're done. Otherwise, let $\{C_i\}_{i \in I}$ be the connected components of $X \setminus L$. Since $X \setminus L$ has one $\mathcal{U}$-component, for each connected component C_i there is a distinct connected component C_j and points $x_i \in C_i$ and $x_j \in C_j$ so that $x_i \mathcal{U} x_j$. In particular, $d(x_i, x_j) < 2R$. Let $\gamma_{i,j}$ be a geodesic from x_i to x_j. Then $\gamma_{i,j}$ must intersect L (lest C_i and C_j not be distinct connected components) but $\gamma_{i,j}$ does not intersect K since N is an $2R$-ball about K and the length of $\gamma_{i,j}$ is at most $2R$. In the new set $\gamma_{i,j} \cup X \setminus L$, x_i and x_j are in the same connected component. It follows from a Zorn's lemma argument that we can add a union of geodesics, none of which intersect K, to $X \setminus L$ to obtain a connected subspace.

($\Leftarrow$) : This follows from the observation that if a space is topologically connected, then it is $\mathcal{U}$-connected for any open cover $\mathcal{U}$. $\square$

Lemma 35. *Let $(X,\mathcal{X})$ be an unbounded large scale space, and let G be a group that acts on X by coarse equivalences. Then the action of G is coarsely discontinuous if and only if for every finite subset $F \subseteq G$ and every pair of uniformly bounded families $\mathcal{U}$ and $\mathcal{V}$ in X, there is a bounded subset K such that for any $x, y \in X \setminus K$ with $x\mathcal{U}y$, and any $g_1 \neq g_2$ in F, $\{\{g_1 \cdot x, g_2 \cdot y\}\}$ does not refine $\mathcal{V}$.*

Proof. Suppose the action is coarsely discontinuous, and let $F \subseteq G$ be a finite subset and $\mathcal{U}$ be a uniformly bounded family. For each $g \in F \cdot F^{-1} \setminus \{e\}$ choose a bounded subset K_g such that if $x \notin K_g$ then $\{\{x, g \cdot x\}\}$ does not refine $\mathsf{st}(\mathcal{V},\mathcal{U}')$, where

$$\mathcal{U}' = \bigcup_{g \in F} g \cdot \mathcal{U}.$$

Define the bounded set

$$K = \bigcup_{h \in F^{-1}} \bigcup_{g \in F \cdot F^{-1}} h \cdot K_g.$$

If $x\mathcal{U}y$, $x, y \notin K$ and $\{g_1 \cdot x, g_2 \cdot y\} \subseteq V \in \mathcal{V}$ with $g_1 \neq g_2$, then $(g_2 \cdot y)\mathcal{U}'(g_2 \cdot x)$ so $(g_1 \cdot x)\mathsf{st}(\mathcal{V},\mathcal{U}')(g_2 \cdot x)$. But $g_2 x = g_2 g_1^{-1} g_1 \cdot x$ and $g_1 \cdot x \notin K_{g_2 g_1^{-1}}$, a contradiction. The converse is easy to check. $\square$

Theorem 16. *Let $(X,\mathcal{X})$ be an unbounded large scale space, and let G be a group that acts on X coarsely discontinuously by coarse equivalences. If $(X,\mathcal{X})$ is coarsely one-ended, then there is a canonical group isomorphism $G \cong \mathsf{Aut}(X/X_G)$.*

Proof. Suppose $(X,\mathcal{X})$ is coarsely one-ended at scale $\mathcal{U}$. Define a map Φ from G to $\mathsf{Aut}(X/X_G)$ by sending g to its action on X. This is clearly a group homomorphism. We claim it is surjective. Let $f \in \mathsf{Aut}(X/X_G)$ and suppose (by use of Lemma 33) that f is close to the identity on X_G as witnessed by $\mathsf{st}(\mathcal{V},\mathcal{F})$ with $\mathcal{V}$ a cover in $\mathcal{X}$ (which we may assume coarsens $\mathcal{U}$) and $\mathcal{F} = \{\{F \cdot x\}\}_{x \in X}$ for $F \subseteq G$ finite. Then for every $x \in X$ there is a $f_1 f_2^{-1} \in F \cdot F^{-1}$ and an x' such that $f(x) = f_1 f_2^{-1} \cdot x'$ and $x\mathcal{V}x'$. Hence for every x there is a $f_1 f_2^{-1} \in F \cdot F^{-1}$ such that $(f_1 f_2^{-1} \cdot x)\mathcal{V}' f(x)$, where $\mathcal{V}'$ is the bounded family

$$\mathcal{V}' = \bigcup_{h \in F \cdot F^{-1}} h \cdot \mathcal{V}.$$

For each x, pick such an $f_1 f_2^{-1} \in F \cdot F^{-1}$ and call it $\alpha(x)$. We claim that outside of a bounded set, $\alpha(x)$ is uniquely defined. Using Lemma 35, choose a bounded set K such that for any $x, y \notin K$ with $x\mathcal{V}y$ and any $g_1 \neq g_2$ in $F \cdot F^{-1}$, $\{g_1 \cdot x, g_2 \cdot y\}$ does not refine $\mathsf{st}(f(\mathcal{U}), \mathcal{V}')$. If $x\mathcal{U}y$ with $x, y \notin K$, then $(\alpha(x) \cdot x)\mathcal{V}' f(x) f(\mathcal{U}) f(y) \mathcal{V}'(\alpha(y) \cdot y)$, so we must have $\alpha(x) = \alpha(y)$. Choose a bounded set K' containing K such that $X \setminus K'$ is $\mathcal{U}$-connected. Then for any two points $x, y \notin K'$, x and y are connected by a chain of elements of $\mathcal{U}$ and so $\alpha(x) = \alpha(y)$ by the above. Thus, f is close to the action of some $h \in F \cdot F^{-1}$ outside of a bounded set, as witnessed by $\mathcal{V}'$. Since f is large scale continuous, it must be close to the action of h on all of X, which shows that Φ is surjective. By coarse discontinuity, Φ is also injective, since no action of g is close to the identity on $(X, \mathcal{X})$ so long as X is unbounded. □

Both coarse one-endedness and coarse discontinuity are necessary for the above theorem to hold true. To see the former, take X to be the subspace of $\mathbb{R}^3$ consisting of the positive x, y and z-axes. Let $G = \mathbb{Z}/3\mathbb{Z}$ act on this space via $1 \cdot (x, y, z) = (z, x, y)$. The space X_G is then coarsely equivalent to the positive real axis. But any permutation of x,y and z gives rise to an element of $\mathsf{Aut}(X/X_G)$. For coarse discontinuity, take for example the action of a finitely generated group G on itself by right translation (see Example 11). If we denote the underlying large scale space of G by $|G|$ then $|G|_G$, with respect to this action, is just $|G|$ (see Nonexample 1), and so $\mathsf{Aut}(|G|/|G|_G)$ is trivial regardless of what G is.

5.5 The maximal Roe algebra

We recall the definition of the maximal Roe algebra from [31] (see also [51]). For the remainder of this section, X denotes a discrete bounded geometry metric space (for example, a finitely generated group with a word metric). Recall that a metric space has **bounded geometry** if for every $R > 0$ there is an integer N such that every R-ball in X has at most N elements. Our goal is to relate the Roe algebra of X with that of X_G for a coarsely discontinuous action of G, where G is a countable group.

Fix a separable infinite-dimensional Hilbert space H and consider the algebra of bounded operators on $\ell^2(X)\otimes H$. We can view an operator T on $\ell^2(X)\otimes H$ as a matrix $(T_{x,y})_{(x,y)\in X\times X}$ of operators on H. We say that T has **propagation less than** R if $T_{x,y} = 0$ for $d(x, y) \geq R$. The **support** of T is the subset of $X \times X$ for which $T_{x,y} \neq 0$. Denote by $C[X]$ the algebra of all bounded operators T on $\ell^2(X) \otimes H$ such that, when T is written as a matrix $(T_{x,y})_{x,y\in X}$ of operators,

- $T_{x,y}$ is compact for all x and y (that is, T is **locally compact**), and
- there exists an $R > 0$ such that T has propagation less than R (that is, T has **finite propagation**).

The (usual) **Roe algebra** $C^*(X)$ of X is the operator norm closure of $C[X]$ in $B(\ell^2(X)\otimes H)$. However, for our purposes we will need a different C^*-algebra: the maximal Roe algebra.

Definition 39 ([31])**.** *The* ***maximal Roe algebra*** $C^*_{\mathsf{max}}(X)$ *of* $C[X]$ *is the completion of* X *with respect to the the* $*$*-norm*

$$||T|| = \mathsf{sup}_{(\phi,H_\phi)}||\phi(T)||_{B(H_\phi)}$$

where (ϕ, H_ϕ) *runs through representations* ϕ *of* $C[X]$ *on a Hilbert space* H_ϕ*.*

Note that it follows from a "partial translation decomposition" argument (see [31]) that this norm is well-defined. If G is a countable group and X is a metric space then the large scale structure on X_G is induced by a metric (which we may assume is discrete) since it is countably generated. It is easy to see that $C[X_G]$ (and thus $C^*(X_G)$ and $C^*_{\max}(X_G)$) is the same for any two metrics inducing the same large scale structure, so from now on we will assume that some metric on X_G has been chosen which induces the large scale structure. It follows from Lemma 33 that if X has bounded geometry, then so does X_G.

Clearly $C[X]$ contains all the rank one operators

$$e_{(x,v),(y,w)} : \delta_z \otimes u \quad\mapsto\quad \langle\delta_y \otimes w, \delta_z \otimes u\rangle\delta_x \otimes v.$$

It follows that the maximal Roe algebra contains a closed two-sided ideal canonically isomorphic to $\mathcal{K}$, the compact operators on $\ell^2(X) \otimes H$ (this is because $\mathcal{K}$ is the universal C^*-algebra generated by a system of matrix units). We will want to work with the quotient $C^*_{\max}(X)/\mathcal{K}$.

Suppose a group G acts on X by coarse equivalences. For an element $g \in G$, let M_g be the operator on $\ell^2(X) \otimes H$ given by

$$(M_g)_{x,y} = \begin{cases} 1_H & \text{if } gy = x \\ 0 & \text{otherwise} \end{cases}$$

Note that $M_g M_h = M_{gh}$ and $M_g^* = M_{g^{-1}}$. Thus G has an induced action on $B(\ell^2(X) \otimes H)$ via

$$g \cdot T = M_g T M_g^*.$$

This action restricts to an action on $C[X]$ which extends to an action on $C^*_{\max}(X)$. This action preserves the ideal $\mathcal{K}$, so we obtain an action on $C^*_{\max}(X)/\mathcal{K}$. Note also that M_g, while not an element of $C[X_G]$ (it is not locally compact), is a multiplier of $C[X_G]$ because it has finite propagation.

Theorem 17. *Let G be a countable group which acts coarsely discontinuously by coarse equivalences on a discrete bounded geometry metric space X . Let α be the action of G on $C^*_{\max}(X)/\mathcal{K}$ given by $g \cdot [T] = [M_g T M_g^*]$. Then there is a canonical $*$-isomorphism*

$$(C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G \xrightarrow{\cong} C^*_{\max}(X_G)/\mathcal{K}$$

where $\rtimes_\alpha$ denotes the full crossed product.

Proof. Since for any $T \in C[X]$, $||TM_g||^2 = ||TM_g(TM_g)^*|| = ||TT^*|| = ||T||^2$ in $C^*_{\max}(X)$, the map $T \mapsto TM_g$ extends to a multiplier of $C^*_{\max}(X_G)$. Define a map Φ from $C_c[G, C^*_{\max}(X)]$ to $C^*_{\max}(X_G)$ by $T\delta_g \mapsto TM_g$. One checks that this actually defines a $*$-homomorphism, where the product on $C_c[G, C^*_{\max}(X)]$ is the convolution product with respect to the action

α. This gives rise to a $*$-homomorphism

$$\Phi : C^*_{\max}(X) \rtimes_\alpha G \to C^*_{\max}(X_G)$$

Since the image of $\mathcal{K} \rtimes_\alpha G$ under this map is contained in $\mathcal{K}(\ell^2(X) \otimes H)$, we get a map

$$\Phi'' : (C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G \to C^*_{\max}(X_G)/\mathcal{K}$$

Here we are implictly using the fact that the full crossed product functor is exact, so that

$$(C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G \cong (C^*_{\max}(X) \rtimes_\alpha G) \,/\, (\mathcal{K} \rtimes_\alpha G)\,.$$

We claim that Φ'' is a $*$-isomorphism. We will prove this by constructing an inverse Ψ'' to it.

Let $T \in C[X_G]$. We claim that T can be written as

$$T = \sum_{g \in G} T_g M_g$$

where each of the T_g are in $C[X]$ and only finitely many of the T_g are non-zero. Moreover, we claim that if $T = \sum_{g\in G} S_g M_g$ is some other decomposition of T with each $S_g \in C[X]$, then $S_g - T_g \in \mathcal{K}(\ell^2(X) \otimes H)$ for every $g \in G$. To prove the existence of such a decomposition, note that by Lemma 33, there exists an $R > 0$ and a finite set $F = \{g_0, \ldots, g_k\}$ of elements of G such that the support of T can be written as a disjoint union $\sqcup_{i=0}^k R_{g_i}$ where for any $(x, y) \in R_{g_i}$, there exists an x' such that $g_i x' = y$ and $d_X(x, x') \leq R$ (note that this distance is in X, not X_G). Write the corresponding decomposition of T (thinking of T as a map from $X \times X$ to $\mathcal{K}(H)$) as

$$T = \sum_{i=0}^{k} T|_{R_{g_i}}$$

It follows from the definition of R_{g_i} that $M_{g_i}^* T|_{R_{g_i}}$ (and thus also $M_{g_i} M_{g_i}^* (T|_{R_{g_i}}) M_{g_i}^* = (T|_{R_{g_i}}) M_{g_i}^*$) is an element of $C[X]$. Thus we have our decomposition

$$T = \sum_{i=0}^{k} T_i M_{g_i},$$

where $T_i = T|_{R_{g_i}} M_{g_i}^*$.

Now suppose there is another such decomposition $T = \sum_{i=0}^{l} S_i M_{g_i}$ with each $S_i \in C[X]$. Then $\sum_{i=0}^{l} (S_i - T_i) M_{g_i} = 0$ with $T_i = 0$ for $i > k$ for convenience. Pick an $R' > 0$ such that for every i, $S_i - T_i$ has propagation less than R'. Using Lemma 35, choose a bounded set K such that for every pair $g_i \neq g_j$ in $\{g_0, \dots, g_l\}$, $d(g_i \cdot x, g_j \cdot x) > 2R'$. Notice that for any i, $0 \leq i \leq l$, $(S_i - T_i) M_{g_i}$ can only have a nonzero (x, y) entry if $d(g_i \cdot x, y) \leq R'$. If $d(g_j \cdot x, y) \leq R'$ for some other $0 \leq j \leq l$, then

$$d(g_i \cdot x, g_j \cdot x) \leq 2R'$$

implies $x \in K$. Thus for $x \notin K$, the (x, y) entry of the sum $\sum_{i=0}^{l} (S_i - T_i) M_{g_i}$ is contributed to by exactly one $(S_i - T_i) M_{g_i}$. It follows that the support of every $(S_i - T_i)$ is a finite set, and thus each $(S_i - T_i)$ is compact.

We are now ready to define our inverse map. For any $T \in C[X_G]$, decompose T as $T = \sum_{g \in G} T_g M_g$ with each $T_g \in C[X]$ and define

$$\Psi(T) = \sum [T_g] \delta_g \in (C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G.$$

Our previous calculations show that this is well-defined. One easily checks that this defines a $*$-homomorphism, and so it extends to a map

$$\Psi' : C^*_{\max}(X_G) \to (C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G$$

The final step is to note that the image of $\mathcal{K}$ under this map is 0, so we have an induced map

$$\Psi'' : C^*_{\max}(X_G)/\mathcal{K} \to (C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G$$

which by construction is a two-sided inverse for Φ''. $\square$

Another way to write the conclusion of Theorem 17 above is that there is a short exact sequence

$$0 \to \mathcal{K} \to C^*_{\max}(X_G) \to (C^*_{\max}(X)/\mathcal{K}) \rtimes_\alpha G \to 0. \tag{5.2}$$

The proof of Theorem 17 was inspired by the proof of Proposition 2.8 in [51]. We now show that we recover this result as a special case. Let Γ be a residually finite finitely generated group and let

$$\Gamma_0 \supset \Gamma_1 \supset \dots$$

be a sequence of normal finite index subgroups such that $\cap_{i\in\mathbb{N}}\Gamma_i = \{e\}$. Let d_Γ be the left-invariant metric associated to some generating set in Γ and give Γ/Γ_i the metric d_i defined by $d_i(a\Gamma_i, b\Gamma_i) = \min\{d_\Gamma(a\gamma_1, b\gamma_2) \mid \gamma_1, \gamma_2 \in \Gamma_i\}$. We define the **box space** to be the set $X(\Gamma) = \sqcup_{i\in\mathbb{N}}\Gamma/\Gamma_i$ equipped with a metric d such that

- d agrees with the metric d_i defined above on Γ/Γ_i,
- $d(\Gamma/\Gamma_i, \Gamma/\Gamma_j) > i + j$ if $i \neq j$, and
- the action of Γ on $X(\Gamma)$ induced by left translation is an action by isometries.

Note that this last point is really a matter of appropriately defining $d(x, y)$ for $x \in \Gamma_i$ and $y \in \Gamma_j$ with $i \neq j$.

Corollary 24 (Proposition 2.8 in [51])**.** *In the situation above, there is a short exact sequence*

$$0 \to \mathcal{K} \to C^*_{\max}(X(\Gamma)) \to A_\Gamma \rtimes_\alpha \Gamma \to 0.$$

where $A_\Gamma = \ell^\infty(X(\Gamma), \mathcal{K}(H))/C_0(X(\Gamma), \mathcal{K}(H))$ and the action of Γ on A_Γ is induced by the action of Γ on $X(\Gamma)$ given by right translation, that is $g \cdot f(h\Gamma_i) = f(hg\Gamma_i)$.

Proof. Let X' be the underlying set of $X(\Gamma)$ equipped with the smallest coarsely connected large scale structure, that is, wherein uniformly bounded families are precisely those which contain finitely many non-singleton sets (since $X(\Gamma)$ is countable, this large scale structure is metrizable). Then Γ acts on X' by coarse equivalences. Moreover, this action is coarsely discontinuous. Indeed, since $\cap_{i\in\mathbb{N}}\Gamma_i = \{e\}$, every $\gamma \in \Gamma$ only fixes a finite number of points in X'. It is easy to check, by similar arguments as in Example 11, that $X(\Gamma)$ is coarsely equivalent to X'_Γ in the sense of Definition 35. Thus we can assume that $X'_\Gamma = X(\Gamma)$ as metric spaces. From Theorem 17, we have an exact sequence

$$0 \to \mathcal{K} \to C^*_{\max}(X(\Gamma)) \to (C^*_{\max}(X')/\mathcal{K}) \rtimes_\alpha \Gamma \to 0.$$

It is enough then to show that $C^*_{\max}(X')/\mathcal{K}$ is $*$-isomorphic to A_Γ in a way that preserves the action of Γ. There is an obvious $*$-homomorphism

$$\Theta : A_\Gamma \to C^*_{\max}(X')/\mathcal{K}$$

given by sending an element $f \in \ell^\infty(X(\Gamma), \mathcal{K}(H))$ to the (zero-propagation) diagonal matrix with entries $(f(\gamma))_{\gamma\in\Gamma}$. This map is clearly injective since such an f represents a compact operator if and only if it is in $C_0(X(\Gamma), \mathcal{K}(H))$. It remains to show it is surjective. Let $T \in C[X']$. Then by definition of the large scale structure on X', T can be written as $T' + T''$ where T' has finitely many entries and T'' is a diagonal matrix of compact operators. Thus T'' is in the image of the map Θ above. Thus the image of Θ is dense and so Θ is surjective. □

Remark 11. *In fact, the action of Γ on $X(\Gamma)$ in Proposition 2.8 in [51] is implied to be by* left *translation. However, the authors believe that this is an error in [51]. The key observation is that if $d(e, g) \leq R$ for some $g \in \Gamma$ with d being a left-invariant metric, then left translation by g on $\ell^2(\Gamma)$ is not in general a finite propagation operator because $d(a, ga) = |a^{-1}ga|$. On the other hand* right *translation by g* is *an operator of propagation less than R since $d(a, ag) = |a^{-1}ag| \leq R$. A similar argument shows the same fact for the action of g on $X(\Gamma)$. Thus in the proof of Proposition 2.8 in [51], the operators $L_{\gamma_i\Gamma}$ should really be right translation operators and not left translation operators.*

This corollary together with Remark 2.12 in [51] also shows that there is no hope for an exact sequence of the form of (5.2) where the maximal Roe algebra is replaced by the usual Roe algebra and the full crossed product is replaced by the reduced crossed product. We will show, however, that in case X has Yu's Property A (first introduced in [74]) and the group G acting on it is amenable, we can make the replacement.

Many equivalent definitions of Property A exist in the literature. We will use a definition (which is equivalent to Property A for bounded geometry discrete metric spaces) due to Dadarlat-Guentner. For an index set S, let $\Delta(S)$ denote the set of formal linear combinations

$$\sum_{s \in S} a_s \cdot s$$

such that $a_s \in [0,1]$ for each s, $a_s = 0$ for all but finitely many s, and $\sum a_s = 1$. We will equip $\Delta(S)$ with the l^1 metric. The **star** of a vertex $s \in S$ is the set of all elements of $\Delta(S)$ with $a_s \neq 0$. By a **partition of unity** on a set X, we mean a map $\phi : X \to \Delta(S)$ for some set S.

Definition 40. *[14] A large scale space X is **exact** if for each uniformly bounded cover $\mathcal{U}$ of X and each $\varepsilon > 0$ there is a partition of unity $\phi : X \to \Delta(S)$ such that point-inverses of stars of vertices form a uniformly bounded cover of X and the mesh of $\phi(\mathcal{U})$ is smaller than ε.*

Theorem 18. *Let G be a countable group which acts on a large scale space X by coarse equivalences. If G is amenable and X is exact, then X_G is exact.*

Proof. Let $\mathcal{X}$ be the large scale structure on X, let $\mathsf{st}(\mathcal{U}, \mathcal{F})$ be a uniformly bounded family in X_G, with $\mathcal{F} = \{F \cdot x \mid x \in X\}$ and $\mathcal{U} \in \mathcal{X}$, and let $\varepsilon > 0$. By the amenability of G, we have that there is a finite $E \subseteq G$ so that for all $g \in F \cdot F^{-1}$,

$$\frac{|E \Delta E \cdot g|}{|E|} < \varepsilon/3.$$

Since G acts by coarse equivalences, we have that $g \cdot \mathcal{U} = \{g \cdot U\}_{U \in \mathcal{U}}$ is in $\mathcal{X}$ for all g and hence

$$\mathcal{U}^E = \bigcup_{k \in E} k \cdot \mathcal{U}$$

is also in $\mathcal{X}$.

Since X is exact, we have that there is a partition of unity $(\phi_i)_{i\in I}$ such that the family $\mathcal{V} = (V_i)_{i\in I}$ is in $\mathcal{X}$, where $V_i = \{x \in X \mid \phi_i(x) \neq 0\}$, and for every $x, y \in U$ with $U \in \mathcal{U}^E$, we have

$$\sum_{i\in I} |\phi_i(x) - \phi_i(y)| < \frac{\varepsilon}{3}.$$

Define a new partition of unity $(\psi_i)_{i\in I}$ on X via

$$\psi_i(x) = \frac{1}{|E|}\sum_{k\in E} \phi_i(k\cdot x),$$

and let $\mathcal{W} = (W_i)_{i\in I}$ be the cover of X given by $W_i = \{x \in X \mid \psi_i(x) \neq 0\}$. We claim that $\mathcal{W}$ is uniformly bounded in X_G. Indeed, $x \in W_i$ implies $\psi_i(x) \neq 0$ so there is a $k \in E$ so that $k \cdot x \in V_i$. It follows that $\mathcal{W}$ refines the cover $\mathsf{st}(\mathcal{V}, \mathcal{E})$, where $\mathcal{E} = \bigcup_{k\in E}\{\{x, k\cdot x\} \mid x \in X\}$.

It remains to show that for any $x, y \in \mathsf{st}(U, \mathcal{F})$ with $U \in \mathcal{U}$, we have $\sum_{i\in I} |\psi_i(x)-\psi_i(y)| < \varepsilon$. It is enough to show that (1) for any $x, y \in U$ we have $\sum_{i\in I} |\psi_i(x) - \psi_i(y)| < \varepsilon/3$ and (2) for $x \in X$ and $g, h \in F$ we have $\sum_{i\in I} |\psi_i(g\cdot x) - \psi_i(h\cdot x)| < \varepsilon/3$.

We first show inequality (1). Let $x, y \in U$ for some $U \in \bigcup\limits_{k\in E} k\cdot\mathcal{U}$. Then

$$\sum_{i\in I} |\psi_i(x) - \psi_i(y)| \leq \frac{1}{|E|}\sum_{k\in E}\sum_{i\in I} |\phi_i(k\cdot x) - \phi_i(k\cdot y)|.$$

For any $k \in E$, $x, y \in U$ implies $k\cdot x, k\cdot y \in k\cdot U \in \mathcal{U}^E$, so by the construction of the ϕ_i,

$$\sum_{i\in I} |\psi_i(x) - \psi_i(y)| \leq \frac{1}{|E|}\cdot |E| \cdot \frac{\varepsilon}{3} = \frac{\varepsilon}{3}.$$

We now show (2). Let $x \in X$ and $g, h \in F$. Then

$$\sum_{i\in I} |\psi_i(g\cdot x) - \psi_i(h\cdot x)| = \frac{1}{|E|}\sum_{i\in I}|\sum_{k\in E} \phi_i(k\cdot g\cdot x) - \phi_i(k\cdot h\cdot x)|.$$

Notice that if $k \cdot g \in E \cdot g \cap E \cdot h$, then the term $\phi_i(k \cdot g \cdot x)$ is cancelled out. So from the above we get

$$\begin{aligned}
\sum_{i\in I} |\psi_i(g \cdot x) - \psi_i(h \cdot x)| &= \frac{1}{|E|} \sum_{i\in I} | \sum_{l \in E\cdot g \setminus E\cdot h} \phi_i(l \cdot x) \quad - \sum_{m \in E\cdot h \setminus E \cdot g} \phi_i(m \cdot x)| \\
&\leq \frac{1}{|E|} \sum_{l \in E\cdot g \Delta E \cdot h} \sum_{i \in I} |\phi_i(l \cdot x)| \\
&= \frac{|E \cdot g \Delta E \cdot h|}{|E|}.
\end{aligned}$$

since each ϕ_i is a partition of unity. But $|E \cdot g \Delta E \cdot h| = |E \Delta E \cdot hg^{-1}|$, so by the condition on E, we have

$$\frac{|E \cdot g \Delta E \cdot h|}{|E|} \leq \varepsilon/3$$

as required. □

Recall from [14] that a bounded geometry discrete metric space is exact if and only if it has Property A. Thus from the above theorem we recover the following slight generalization of Proposition 3.1 from [58]. Note that even in the metric case the proof of Theorem 18 differs from the proof in [58] since we use exactness instead of the definition of Property A using probability measures.

Corollary 25. *Let G be a countable group which acts on a discrete bounded geometry metric space X by coarse equivalences. If G is amenable and X has Property A, then X_G has Property A.*

Note that by Proposition 34 in this chapter and Corollary 9.4 in [24], X has Property A if X_G does, for any countable group G. The following corollary was already proved in [23].

Corollary 26. *Let G be a finite group which acts on a discrete bounded geometry metric space X by coarse equivalences. If X has Property A, then X/G has Property A when endowed with the Hausdorff metric.*

Proof. Since Property A is invariant under coarse equivalence, this follow from Propositions 32 and 33 and the fact that any finite group is amenable. □

We now recall the following result from [63]. In fact, the result in [63] requires the space to be uniformly discrete, but since every metric space is bijectively coarsely equivalent to a uniformly discrete one (simply increase the distance between distinct points by some fixed $\varepsilon > 0$), we can drop this assumption.

Theorem 19 (Proposition 1.3 in [63])**.** *If X is a bounded geometry discrete metric space with Yu's Property A, then the canonical quotient $\lambda : C^*_{\mathsf{max}}(X) \to C^*(X)$ is a $*$-isomorphism.*

Recalling that the full crossed product and reduced crossed product agree for amenable groups, we have the following corollary.

Corollary 27. *Let X be a bounded geometry discrete metric space with Yu's Property A, and let G be a countable amenable group acting on X coarsely discontinuously by coarse equivalences. Then we have an exact sequence*

$$0 \to \mathcal{K} \to C^*(X_G) \to (C^*(X)/\mathcal{K}) \rtimes_{r,\alpha} G \to 0.$$

By Proposition 32, if G is finite, X_G is coarsely equivalent to X/G. This coarse equivalence gives rise to a (non-canonical) $*$-isomorphism $\phi : C^*_{\max}(X_G) \to C^*_{\max}(X/G)$. The map ϕ is constructed as follows: let $H = \oplus_{g\in G} H_g$ be an orthogonal decomposition of H into infinite dimensional subspaces, and let $\psi_g : H \to H_g$ be a unitary isometry for every $g \in G$. For each equivalence class $[x] \in X/G$, choose a representative $s([x]) \in [x]$. Then we can define a unitary operator

$$U_p : \ell^2(X_G) \otimes H \to \ell^2(X/G) \otimes H$$

by $U_p(\delta_x \otimes h) = \delta_{[x]} \otimes \psi_g(h)$ where $x = g \cdot s([x])$. We can then define an isometry

$$\mathsf{Ad}U_p : C^*[X_G] \to C^*[X/G]$$

by $\mathsf{Ad}U_p(T) = U_p T U_p^*$, which extends to an isometry $\phi : C^*_{\max}(X_G) \to C^*_{\max}(X/G)$ as required. Since ϕ preserves all the rank-one projections, it also preserves the compact operators. Thus we obtain the following corollary of Theorem 17, Propositions 45 and 32.

Corollary 28. *Let G be a finite group acting coarsely discontinuously by coarse equivalences on a discrete bounded geometry metric space X, and let X/G be the orbit space with the Hausdorff metric. Then there is a $*$-isomorphism*

$$(C^*_{\max}(X)/\mathcal{K}) \rtimes_{r,\alpha} G \xrightarrow{\cong} C^*_{\max}(X/G)/\mathcal{K}$$

Moreover, if X has Property A, then the maximal Roe algebra above can be replaced by the usual Roe algebra.

Example 14. *Let $G = \{1, \gamma\} \cong \mathbb{Z}/2\mathbb{Z}$ and let G act on the metric space $X = \mathbb{Z}$ by $\gamma(x) = -x$. This action is coarsely discontinuous, and the quotient X/G is coarsely equivalent to $\mathbb{N}$. Since $\mathbb{Z}$ has Property A and $\mathbb{Z}/2\mathbb{Z}$ is amenable, we have a $*$-isomorphism*

$$(C^*(\mathbb{Z})/\mathcal{K}) \rtimes_r \mathbb{Z}/2\mathbb{Z} \xrightarrow{\cong} C^*(\mathbb{N})/\mathcal{K}$$

Example 15. *Recall from Remark 10 that if G is a finitely generated group and X is the underlying set of G equipped with the smallest coarsely connected large scale structure, then X_G is coarsely equivalent to G, where the action of G is by right translation. By similar arguments to the proof of Corollary 24, one can check that $C^*_{\max}(X)/\mathcal{K}$ is naturally isomorphic to $\ell^\infty(|G|, \mathcal{K}(H))/C_0(|G|, \mathcal{K}(H))$ where $|G|$ is the underlying set of G. It follows that there is a natural isomorphism*

$$C^*_{\max}(G)/\mathcal{K} \cong \ell^\infty(|G|, \mathcal{K}(H))/C_0(|G|, \mathcal{K}(H)) \rtimes G.$$

where the action of G on $\ell^\infty(|G|, \mathcal{K}(H))/C_0(|G|, \mathcal{K}(H))$ is given by right translation: $g \cdot f(h) = f(hg^{-1})$. Compare this result to Theorem 4.28 in [57] which states that

$$C^*_u(G) \cong \ell^\infty(|G|) \rtimes_r G$$

where $C_u^*(-)$ denotes the uniform Roe algebra. There is also a well-known isomorphism

$$C^*_{\max}(G) \cong \ell^\infty(G, \mathcal{K}(H)) \rtimes G$$

where $C^*_{\max}(G)$ is the maximal Roe algebra of G (the proof is an easy adaptation of the proof of Proposition 5.1.3 in [10]).

5.6 Weak coarse quotient maps

In this section, we introduce the notion of weak coarse quotient map, of which $p_G : X \to X_G$ is an example when G is finitely generated, and motivate the definition using the coarse category.

Recall that given a topological space X and a surjective set map f from the underlying set of X to a set Y, the quotient topology on Y is the finest topology that makes f continuous. We now introduce an analogous notion for large scale spaces, based on the definition of quotient coarse structure in [16].

Definition 41. *Let X be a large scale space and let f be a surjective set map from the underlying set of X to a set Y. Then the* ***quotient large scale structure on*** *Y is defined to be $\overline{f(\mathcal{X})}$ where $\mathcal{X}$ is the large scale structure on X and $f(\mathcal{X})$ is the collection $\{f(\mathcal{U}) \mid \mathcal{U} \in \mathcal{X}\}$.*

Clearly the quotient large scale structure is the smallest large scale structure which makes the map f large scale continuous. The quotient large scale structure also has a universal property. In fact, the existence of the quotient large scale structure and the universal property follow from general categorical considerations in [16]. For completeness, we present a direct proof here.

Proposition 37. *Let $f : X \to Y$ be a surjective large scale continuous map. Then Y has the quotient large scale structure with respect to f if and only f satisfies the following universal property:*

(Q1) *for any large scale continuous map $g : X \to Z$ which is constant on the fibres of f, there is a unique large scale continuous map h from Y to Z such that $hf = g$.*

Proof. ($\Rightarrow$) Let $\mathcal{X}$ be the large scale structure on X. The map h is uniquely defined: $h(f(x)) = g(x)$. The fact that h is large scale continuous follows from Lemma 31:

$$h(\overline{f(\mathcal{X})}) \subseteq \overline{hf(\mathcal{X})} = \overline{g(\mathcal{X})} \subseteq \mathcal{Z}$$

where $\mathcal{Z}$ is the large scale structure on Z.

($\Leftarrow$) Suppose (Q1) holds. Consider the map $f' : X \to Y'$, where Y' is the underlying set of Y equipped with the quotient large scale structure and f' is the same as f at the level of underlying sets. It follows that the identity set map $Y \to Y'$ must be large scale continuous, and the result follows from this. $\square$

We may be tempted to define a weak coarse quotient map as a surjective large scale continuous map $f : X \to Y$ such that Y has the quotient large scale structure with respect to f. The problem with this idea is that such a definition is not very "coarse". Indeed, we should expect a class of maps $\mathcal{E}$ defined by a large scale property to satisfy the following conditions:

(LS1) if f is close to g, and f is in $\mathcal{E}$, then so is g;

(LS2) if f is a large scale continuous map, and ϕ and ψ are coarse equivalences such that the composite $\phi f \psi$ is defined, then f is in $\mathcal{E}$ if and only if $\phi f \psi$ is in $\mathcal{E}$.

In fact, as the following proposition shows, (LS2) implies (LS1).

Proposition 38. *If a class $\mathcal{E}$ of large scale continuous maps satisfies* (LS2) *then it also satisfies* (LS1).

Proof. Let $f : X \to Y$ be in $\mathcal{E}$, and suppose g is a map whose closeness to f is witnessed by the uniformly bounded cover $\mathcal{U}$. Let X' be the subspace of the product $X \times Y$ given by

$$\{(x, y) \mid f(x)\mathcal{U}y\}.$$

The map $i : X \to X'$ given by $x \mapsto (x, f(x))$ is a coarse equivalence, and we have $\pi_2 \circ i = f$, where $\pi_2 : X' \to Y$ is the projection onto the second coordinate. It follows that π_2 is in $\mathcal{E}$ by

($\mathsf{LS2}$). We have a map $j : X \to X'$ given by $x \mapsto (x, g(x))$, which is also a coarse equivalence, such that $\pi_2 \circ j = g$. Applying condition ($\mathsf{LS2}$), we have that g is in $\mathcal{E}$ as well. □

The class of all surjective maps f whose codomain carries the quotient large scale structure with respect to f satisfies neither ($\mathsf{LS1}$) nor ($\mathsf{LS2}$). We thus introduce the following definition of weak coarse quotient map instead.

Definition 42. *Let $f : X \to Y$ be a large scale continuous map. Then f is a* ***weak coarse quotient map*** *if it is coarsely surjective and there exists a uniformly bounded cover $\mathcal{V}$ of Y such that the large scale structure on Y is generated by $f(\mathcal{X}) \cup \{\mathcal{V}\}$, where $\mathcal{X}$ is the large scale structure on X. A cover $\mathcal{V}$ satisfying this property is called a* ***quotient scale*** *of f.*

Observation 3. *Consider a group action G on a large scale space X. If the group G is generated by a finite set S, then the collection of all families*

$$\{\{x, gx\} \mid x \in X\}$$

is contained in the large scale structure generated by the single family

$$\mathcal{S} = \{S \cdot x \mid x \in X\},$$

so that the identity set map $X \to X_G$ is a weak coarse quotient map with quotient scale $\mathcal{S}$.

Note that if $\mathcal{V}$ is a quotient scale for the weak coarse quotient map $f : X \to Y$, then so is any uniformly bounded coarsening of $\mathcal{V}$. In particular, we can always pick a quotient scale $\mathcal{V}$ such that $Y \subseteq \mathsf{st}(f(X), \mathcal{V})$. A weak coarse quotient map satisfies a universal property.

Proposition 39. *Let $f : X \to Y$ be a large scale continuous map and let $\mathcal{V}$ be a uniformly bounded cover of Y. Then the following are equivalent.*

(a) f is a weak coarse quotient map with quotient scale $\mathcal{V}$,

(b) for any large scale continuous map $g : X \to Z$ such that $g(f^{-1}(\mathcal{V}))$ is uniformly bounded, there exists a unique-up-to-closeness map $h : Y \to Z$ such that hf is close to g.

Proof. (a) ⇒ (b): Without loss of generality, choose the weak coarse quotient scale $\mathcal{V}$ so that $Y \subseteq \mathsf{st}(f(X), \mathcal{V})$. Suppose there is a large scale continuous $g : X \to Z$ so that $g(f^{-1}(\mathcal{V})) \in \mathcal{Z}$ where $\mathcal{Z}$ is the large scale structure on Z. We define a map $h : Y \to Z$. Let $y \in Y$. Then we can pick a $V \in \mathcal{V}$ and an $x \in X$ so that $y \in V$ and $f(x) \in V$. Define $h(y) = g(x)$. We claim that $h \circ f$ and g are close. Indeed, let $x \in X$. Then $hf(x) = g(x')$ for some x' such that $f(x')\mathcal{V}f(x)$. It follows that $g(f^{-1}(\mathcal{V}))$ witnesses the closeness of g and $h \circ f$. The uniqueness up to closeness follows from the fact that f is an epimorphism in the coarse category. Finally, if $\mathcal{U}$ is a uniformly bounded family in X, then $hf(\mathcal{U})$ refines $\mathsf{st}(g(\mathcal{U}), g(f^{-1}(\mathcal{V})))$. This, together with the fact that $\mathcal{V}$ and $f(\mathcal{X})$ together generate the large scale structure on Y, give that h is large scale continuous.

(b) ⇒ (a): It is easy to check that f must be an epimorphism, and hence coarsely surjective. Let Y' be the underlying set of Y with the large scale structure generated by $\mathcal{V}$ and $f(\mathcal{X})$. By hypothesis, there is a large scale continuous map $Y \to Y'$, which must be close to the identity on $f(X)$ and thus also on all of Y. Since the identity map is close to a large scale continuous map, it is itself large scale continuous, and it follows easily that Y' and Y have the same large scale structure. □

Recall that for a category $\mathbb{C}$ and two morphisms $f, g : X \to Y$, a **coequalizer** of f and g is a morphism $h : Y \to Z$ such that $hf = hg$ and such that if $h' : Y \to Z'$ is another morphism such that $h'f = h'g$, then there exists a unique morphism $i : Z \to Z'$ such that $ih = h'$. A **regular epimorphism** is a coequalizer of a pair of morphisms. In the category of topological spaces and continuous maps, the epimorphisms are the surjective continuous maps, which are not quotient maps in general. On the other hand, the regular epimorphisms are precisely the quotient maps.

Proposition 40. *Let $f : X \to Y$ be a large scale continuous map. Then $[f]$ is a regular epimorphism in* **Coarse**$/\sim$ *if and only if f is a weak coarse quotient map.*

Proof. (⇒): Suppose f is the coequalizer of $[a], [b] : W \to X$, and let

$$\mathcal{V} = \{\{f(a(w)), f(b(w))\} \mid w \in W\}.$$

Then since $fa \sim fb$, $\mathcal{V}$ is uniformly bounded. We claim that f is a weak coarse quotient map with quotient scale $\mathcal{V}$. Indeed, if $g : X \to Z$ is some other map that sends $f^{-1}(\mathcal{V})$ to a uniformly bounded family, then $ga \sim gb$ and so $[g]$ factors uniquely through $[f]$ in the coarse category. By Proposition 39, f is a weak coarse quotient map.

($\Leftarrow$): Suppose f is a weak coarse quotient map with quotient scale $\mathcal{V}$. Let W be the subspace of $X \times X$ given by $\{(x, x') \mid f(x)\mathcal{V}f(x')\}$, and let π_1, π_2 be the projections $W \to X$, which are clearly large scale continuous. Then $f\pi_1 \sim f\pi_2$, and if $g\pi_1 \sim g\pi_2$, then g sends $f^{-1}(\mathcal{V})$ to a uniformly bounded family, so that $[g]$ factors uniquely through $[f]$ in the coarse category by Proposition 39. It follows that $[f]$ is the coequalizer of $[\pi_1]$ and $[\pi_2]$. □

Corollary 29. *Any coarse equivalence is a weak coarse quotient map. Moreover, the class of weak coarse quotient maps satisfies* (LS1) *and* (LS2)*.*

Proof. Any isomorphism is a regular epimorphism, and regular epimorphisms are closed under composition with isomorphisms, which gives (LS2). □

Clearly if $f : X \to Y$ is a surjective large scale continuous map and Y has the quotient large scale structure, then f is a weak coarse quotient map with any uniformly bounded cover of Y as a quotient scale. Much more general situations are possible, however, as the proposition below shows.

Proposition 41. *Let Y be a large scale space. Then the following are equivalent:*

(1) Y is monogenic, that is, the large scale structure on Y is generated by a single uniformly bounded family $\mathcal{V}$;

(2) every coarsely surjective large scale continuous map $f : X \to Y$ is a weak coarse quotient map.

Proof. (1)$\Rightarrow$(2): Pick the generating family $\mathcal{V}$ as the quotient scale.

(2)$\Rightarrow$(1): Let Y' be the underlying set of Y with the smallest large scale structure, and consider the identity set map $Y' \to Y$. The large scale structure on Y' must be generated solely by a quotient scale $\mathcal{V}$ of this map, and the result follows. □

In particular, any coarsely surjective large scale continuous map whose codomain is a geodesic metric space is a weak coarse quotient map. Thus at least in the setting of geodesic metric spaces (or more generally, spaces which are coarsely equivalent to geodesic metric spaces – see Proposition 2.57 in [57]), weak coarse quotients are nothing but coarsely surjective maps. In light of this observation, one may ask whether we can find a smaller class of large scale continuous maps which includes the class of all surjective maps $f : X \to Y$ such that Y has the quotient large scale structure and still satisfies (LS1) and (LS2). The following proposition shows that this is impossible.

Proposition 42. *Let $\mathcal{E}$ be the class of all surjective large scale continuous maps $f : X \to Y$ such that Y has the quotient large scale structure. Then the class of weak coarse quotients maps is the smallest class of large scale continuous maps satisfying* (LS1) *and* (LS2) *and containing $\mathcal{E}$.*

Proof. Suppose $\mathcal{E}'$ is a class of large scale continuous maps satisfying (LS1) and (LS2) and containing $\mathcal{E}$. We claim it contains all the weak coarse quotient maps. Let $f : X \to Y$ be a surjective large scale continuous map where the large scale structure on Y is generated by $f(\mathcal{X})$ and the cover of subsets $\mathcal{V}$, where $\mathcal{X}$ is the large scale structure on X. Define X' to be the subspace of the product $X \times Y$ given by

$$\{(x, y) \mid f(x)\mathcal{V}y\}$$

and let $i : X \to X'$ be the map $x \mapsto (x, f(x))$. One can check that i is a coarse equivalence. The projection onto the second coordinate $\pi_2 : X' \to Y$ is such that $\pi_2 \circ i = f$. The large scale structure on Y is the quotient large scale structure with respect to π_2. Indeed, if $\mathcal{X}'$ is the large scale structure on X', then $\pi_2(\mathcal{X}')$ clearly contains $f(\mathcal{X})$ by $\pi_2 \circ i = f$, as well as the cover $\mathcal{V}$ (take the image of the family $\Delta \times \mathcal{V}$ in $X \times Y$ restricted to X', where Δ is the cover by singletons). Thus π_2 is in $\mathcal{E}$, and so f is as well. Finally, we can weaken the requirement that f be surjective to coarsely surjective by applying the above argument to the restriction $X \to f(X)$ of f and then using (LS2) to compose with the inclusion $f(X) \to Y$, which is a coarse equivalence. □

We now make the connection with the notion of coarse quotient mapping in [75]. Recall from [75] that a map $f : X \to Y$ between metric spaces is called a **coarse quotient mapping with constant** K if it is large scale continuous and for every ε there exists a $\delta = \delta(\varepsilon)$ such that for every $x \in X$

$$B(f(x), \varepsilon) \subseteq f(B(x, \delta))^K$$

where for $A \subseteq Y$, $A^L = \{y \in Y \mid \exists_{a \in A} d(a, y) \leq L\}$. If $f : X \to Y$ is a coarse quotient mapping, then every uniformly bounded family $\mathcal{U}$ in $f(X)$ refines the image of $\mathsf{st}(f(\mathcal{V}), \mathcal{B}_K)$ for some uniformly bounded family $\mathcal{V}$ in X, where $\mathcal{B}_K$ is the cover of Y by R-balls. Thus the restriction $f : X \to f(X)$ is a weak coarse quotient map with quotient scale $\mathcal{B}_K$. As noted in [75], every coarse quotient mapping is coarsely surjective, so it follows that every coarse quotient mapping is a weak coarse quotient map. The converse is not true: simply take any large scale continuous and coarsely surjective map into a geodesic metric space which is not a coarse quotient mapping.

Proposition 43. *Let X be a metric space and let G be a finite group acting on X by coarse equivalences. Then the identity set map $p_G : X \to X_G$ is a coarse quotient mapping in the sense of [75] for any metric inducing the large scale structure on X_G.*

Proof. Let $K = \max\{d(x, g \cdot x) \mid x \in X, \ g \in G\}$, which is finite by the definition of the large scale structure on X_G. Let $\varepsilon > 0$. From Lemma 33, and using the fact that G is finite, there is a uniformly bounded family $\mathcal{U}$ such that any ball $B(f(x), \varepsilon)$ in X_G is contained in $\bigcup_{g \in G} g \cdot U$ for some $U \in \mathcal{U}$. It follows that $B(f(x), \varepsilon)$ is contained in $B(f(x), \delta)^K$ where $\delta = \mathsf{mesh}(\mathcal{U})$. □

5.7 Metrization of quotient large scale structures

If $f : X \to Y$ is a weak coarse quotient map, and X is a metric space, then the large scale structure on Y is countably generated, hence metrizable. The following proposition gives an explicit construction of a metric on Y which induces the large scale structure.

Proposition 44. *Let $f : X \to Y$ be a weak coarse quotient map with quotient scale $\mathcal{V}$. Let $\mathcal{V}' = \mathsf{st}(\mathcal{V}, \mathcal{V})$. If X is a metric space with metric d_X, then the large scale structure on Y is induced by the metric d_Y defined by $d_Y(y, y) = 0$ and for $y \neq y'$,*

$$d_Y(y, y') = \mathsf{inf}\{n + \sum_{i=1}^{n} d_X(a_i, b_i) \mid f(a_1)\mathcal{V}y,\ f(b_n)\mathcal{V}y',\ f(b_i)\mathcal{V}'f(a_{i+1}),\ n \in \mathbb{Z}_+\}.$$

Proof. Let $\mathcal{Y}$ denote the large scale structure on Y. It is easy to check that d_Y is a metric. Since $a\mathcal{V}b \implies d_Y(a, b) = 1$, the cover $\mathcal{V} \in \mathcal{Y}$ is uniformly bounded with respect to $\mathcal{V}$. The image under f of any uniformly bounded family in X is also clearly uniformly bounded with respect to d_Y. Thus since $\mathcal{Y}$ is generated by $f(\mathcal{X}) \cup \{\mathcal{V}\}$, we have the containment $\mathcal{Y} \subseteq \mathcal{L}(d_Y)$, where $\mathcal{L}(d_Y)$ is the large scale structure induced by d_Y. It remains to show that every element of $\mathcal{L}(d_Y)$ is an element of $\mathcal{Y}$. Let $\mathcal{U} \in \mathcal{L}(d_Y)$, and pick $M \in \mathbb{Z}$ such that $d_Y(y, y') < M$ for any $y\mathcal{U}y'$. By the definition of d_Y we have that for every $y\mathcal{U}y'$ there is a sequence $(a_i, b_i)_{1 \leq i \leq k}$ of pairs of elements of X such that

- $k \leq M$ and $d(a_i, b_i) \leq M$ for all i,
- $f(a_1)\mathcal{V}y,\ f(b_n)\mathcal{V}f(b)$ and $f(b_i)\mathcal{V}'f(a_{i+1})$ for all i.

If $\mathcal{W} \in \mathcal{Y}$ is a common coarsening of $\mathcal{V}'$ and $f(\mathcal{B}_M)$, where $\mathcal{B}_M$ is the cover of X by M-balls, it follows that y is connected to y' by a chain of at most $2M + 1$ elements of $\mathcal{W}$, which shows that $\mathcal{U}$ is an element of $\mathcal{Y}$. □

Corollary 30. *Let X be a metrice space, and let G be a finitely generated group that acts on X by coarse equivalences, with G generated by the finite symmetric set S containing the identity. Then the large scale structure on X_G is induced by the metric defined by $d_{X_G}(x, x) = 0$ and for $x \neq x'$,*

$$d_{X_G}(x, x') = \mathsf{inf}\{n + \sum_{i=1}^{n} d_X(a_i, b_i) \mid x \in S^2 \cdot a_1,\ x' \in S^2 \cdot b_n,\ b_i \in S^4 \cdot a_{i+1},\ n \in \mathbb{Z}_+\}.$$

where $S^n = \{s_1 s_2 \cdots s_n \mid \forall_i s_i \in S\}$.

Proof. This follows from Observation 3. □

Corollary 31. *Let X be a metric space and let f be a surjective set map from the underlying set of X to a set Y. Then the quotient large scale structure on Y is induced by the metric d'_f defined by $d'_f(y,y) = 0$ and for $y \neq y'$,*

$$d'_f(y,y') = \mathsf{inf}\{n + \sum_{i=1}^{n} d_X(a_i,b_i) \mid f(a_1) = y,\ f(b_n) = y',\ f(b_i) = f(a_{i+1}),\ n \in \mathbb{Z}_+\}$$

Proof. The quotient large scale structure on Y is the unique large scale structure for which f is a weak coarse quotient map with quotient scale the cover by singletons. $\square$

The metric d'_f in Corollary 31 may seem unfamiliar, but when X is uniformly discrete, it coincides with the classical quotient metric. We briefly recall the definition of the quotient (pseudo)metric (see for example Definition 3.1.12 in [11]). Let X be a metric space and let $f : X \to Y$ be a map from the underlying set of X to a set Y. Then the **quotient pseudometric** on Y with respect to f is defined to be

$$d_f(y,y') = \mathsf{inf}\{\sum_{i=1}^{n} d_X(a_i,b_i) \mid f(a_1) = y,\ f(b_n) = y',\ f(b_i) = f(a_{i+1}),\ n \in \mathbb{Z}_+\}$$

(in fact the definition is usually stated for an equivalence relation E on X, but this is clearly the same thing as a surjective set map $X \to Y$). Note that this may not be a metric since distinct points may be distance 0 apart. Recall the following definition (see for example [50]).

Definition 43. *A metric space X is called* uniformly discrete *if there is a constant $C > 0$ such that for any $x \neq x'$, $d(x,x') > C$.*

Proposition 45. *Let X be a uniformly discrete metric space and let f be a surjective set map from the underlying set of X to a set Y. Then the (classical) quotient pseudometric on Y is a metric and induces the quotient large scale structure on Y with respect to f.*

Proof. It is easy to see that the quotient pseudometric is a metric. Let $C > 0$ be such that for any $x \neq x'$, $d(x,x') > C$. Let d'_f be defined as in Corollary 31 and let d_f be the quotient metric on Y with respect to f. Suppose $d_f(y,y') < R$. Then there is a sequence of pairs of points $(a_i, b_i)_{1\le i\le k}$ such that $f(a_1) = y,\ f(b_n) = y'$ and $f(b_i) = f(a_{i+1})$, and such that

$$\sum_{i=1}^{k} d_X(a_i, b_i) \le R.$$

Since X is uniformly discrete, this means that $kC \le R$, which implies that

$$k + \sum_{i=1}^{k} d_X(a_i, b_i) \le R + R/C$$

from which it follows that $d'_f(y, y') \le R + R/C$. On the other hand, if $d'_f(y, y') < R$ then it is easy to check that $d_f(y, y') < R$, so we have that d_f and d'_f induce the same large scale structure. □

Finally, we have the following characterization of weak coarse quotient maps between metric spaces.

Proposition 46. *Let $f : X \to Y$ be a large scale continuous map between non-empty metric spaces. Then the following are equivalent:*

(a) f is a weak coarse quotient map;

(b) there exists a $T > 0$ such that for every $R > 0$ there is an $S(R) > 0$ and an integer $n(R)$ such that if $d_Y(y, y') \le R$ for $y, y' \in Y$ then there is a sequence of pairs of points $(a_i, b_i)_{1\le i\le n(R)}$ in X such that $d_Y(f(a_1), y) \le T,\ d_Y(f(b_n), y') \le T$, and $d_Y(f(b_i), f(a_{i+1})) \le T$ and $d_X(a_i, b_i) \le S(R)$ for all i.

Proof. (a) ⇒ (b): Suppose f is a weak coarse quotient map with quotient scale $\mathcal{V}$. We claim that $T = \mathsf{mesh}(\mathsf{st}(\mathcal{V}, \mathcal{V}))$ works. Let d'_Y be the metric on Y constructed in Proposition 44 which we know induces the large scale structure on Y. Thus for every $R > 0$ there is an $R' > 0$ such that $d_Y(y, y') \le R \implies d'_Y(y, y') \le R'$ for all $y, y \in Y$ where d_Y is the original metric on Y. By construction of the metric d'_Y, if $d'_Y(y, y') \le R'$ then there must be a sequence of pairs of points $(a_i, b_i)_{1\le i\le n}$ in X with $n \le R+1$ such that $f(a_1)\mathcal{V}y,\ f(b_n)\mathcal{V}y'$, $f(b_i)\mathsf{st}(\mathcal{V}, \mathcal{V})f(a_{i+1})$ for all i and $d(a_i, b_i) \le R + 1$ for all i. Thus setting $n(R) > R + 1$, $S(R) = R + 1$ we have the result.

(b) $\Rightarrow$ (a): Clearly f must be coarsely surjective with $Y \subseteq B(f(X), T)$. Let $R > 0$ and pick $S(R)$ and $n(R)$ as in (b). Let $\mathcal{B}_{S(R)}$ and $\mathcal{B}_T$ be covers of X and Y by $S(R)$-balls and T-balls respectively. If $d(y, y') \leq R$ for $y, y' \in Y$ then (b) implies that y and y' are connected by a chain of at most $n(R)$ elements of $\mathsf{st}(f(\mathcal{B}_{S(R)}), \mathcal{B}_T)$. Since $n(R)$ and $S(R)$ depend only on R, the cover of Y by R-balls is an element of the large scale structure generated by $f(\mathcal{X})$ and $\mathcal{B}_T$ where $\mathcal{X}$ is the large scale structure on X. It follows that f is a weak coarse quotient map with quotient scale $\mathcal{B}_T$. $\square$

Note that we can obviously choose $n(R) = S(R)$ for every R in (b) of Proposition 46, but it is more intuitively clear to keep the two quantities separate.

Chapter 6

Conclusion

6.1 Concluding remarks

This thesis makes contributions to the field of coarse geometry by introducing and studying new large scale notions. These notions arise through an analogy between coarse geometry and topology, and are analogues of notions which have already played a significant role in the field of topology and beyond. It is hoped that some of these new coarse geometric ideas will be useful in future work in the area of coarse geometry and other areas in a similar way. A guiding force in much of the study of these new notions is category theory – a powerful theoretical framework for drawing parallels between different fields. The main aim of using category theory is to ensure that the "right" definitions are chosen as far as possible, i.e. those that are natural and behave in the way one would expect.

6.2 Future avenues for research

In this section, we briefly survey some future research directions suggested by the work in this thesis.

6.2.1 Coarse covering spaces

A motivation for a lot of the theory developed in this thesis is to move towards a theory of covering spaces for coarse geometry. Covering spaces play a crucial role in foundational results in algebraic topology, and are in particular closely related to the fundamental group and other homotopy invariants of topological spaces. Of particular importance in studying covering spaces in topology are connectedness properties, both for individual spaces and for maps between spaces; connectedness was the main theme of Chapter 2. Also, topological covering maps are always topologically light – here is the connection to the work of Chapter 4. The most striking connection is to Chapter 5: covering spaces in topology are intimately related to properly discontinuous actions, whose coarse analogues are the main objects of study in that chapter. In particular, Theorem 16 resembles a classical topological result about deck transformation groups (see Chapter 13 of Munkres for example [47]).

6.2.2 Neighbourhood operators in coarse geometry

One lesson that can be learned from Chapter 3 is that neighbourhood operators can provide a bridge from topology and uniform spaces to coarse geometry. The final section of that chapter makes it clear that for metric spaces, the neighbourhood relation completely determines the coarse structure. However, it is not at all clear how to usefully formulate important coarse concepts like Property A or asymptotic dimension in terms of coarse neighbourhoods. Beyond metric spaces, the coarse neighbourhoods need no longer determine the coarse structure in a straightforward way (as was shown in that section as well), so there is also the potential to investigate using neighbourhood operators as the axiomatic framework for coarse geometry, perhaps instead of coarse structures in some cases.

6.2.3 Warped spaces and warped cones

As was mentioned in the introduction to Chapter 5, warped cones have recently become the subject of a great deal of interest in the coarse geometry community since they provide examples of spaces with exotic large scale behaviour. It would be interesting to investigate applications of the work in Chapter 5 to warped cones – the Roe algebra is involved in the coarse Baum-Connes Conjecture, after all, and Sawicki has recently showed that warped cones can violate this conjecture [61]. Looking beyond warped cones, there seems to be a vast potential for new kinds of spaces in the warped space construction. If there is already so much of interest just in warped cones, one wonders what other exotic behaviour can be exhibited by general warped spaces.

Bibliography

[1] A. Arhangel'skii, M. Tkachenko, *Topological Groups and Related Structures*, Atlantis Studies in Mathematics Vol. 1, Atlantis Press, 2008. 91

[2] W. Arveson, An invitation to C^*-algebras, Graduate Texts in Mathematics, No. 39, Springer-Verlag, New York-Heidelberg, 1976. 19

[3] K. Austin, Geometry of scales, PhD Thesis, University of Tennessee, 2015. 63, 83

[4] Kyle Austin, Jerzy Dydak, Michael Holloway, *Scale Structures and C^*-algebras*, (preprint) arXiv:1602.07301. 42, 44, 47, 63

[5] K. Austin and Ž. Virk, *Higson compactification and dimension raising*, Topology and its Applications 215, 2017, 45–57. 67, 101

[6] G. Bell and A. Dranishnikov, *Asymptotic dimension*, Topology and its Applications 155.12, 2008, 1265–1296. 51

[7] G. Bell and A. Dranishnikov, *On asymptotic dimension of groups*, Algebraic and Geometric Topology 1, 2001, 57–71. 82, 106

[8] N. Bourbaki, *General Topology*, Hermann, Paris, 1966. 63, 90

[9] N. Brodskiy, J. Dydak, M. Levin and A. Mitra, *A Hurewicz theorem for the Assouad-Nagata dimension*, Journal of the London Mathematical Society 77.3, 2008, 741–756. 82

[10] N. Brown and N. Ozawa, *C^*-algebras and finite-dimensional approximations*, Grad. Stud. Math. 88, American Mathematical Society, 2008. 140

[11] D. Burago, Y. Burago and S. Ivanov, *A course in metric geometry*, Grad. Stud. in Math. Volume 33, American Mathematical Society, Providence, 2001. 148

[12] A. Carboni, G. Janelidze, G. M. Kelly, R. Paré, *On localization and stabilization for factorization systems*, Appl. Categ. Struct. 5, 1997, 1–58. 77, 78, 91

[13] Á. Czászár, General Topology, Akadémiai Kiadó, Budapest, 1978. 50

[14] M. Dadarlat, E. Guentner, *Uniform embeddability of relatively hyperbolic groups*, J. Reine Angew. Math. 612, 2007, 1–15. 104, 105, 135, 137

[15] D. Dikranjan and W. Tholen, *Categorical structure of closure operators: with applications to topology, algebra and discrete mathematics*, Springer Science & Business Media, 2013. 87

[16] D. Dikranjan and N. Zava, *Some categorical aspects of coarse spaces and balleans*, Topology and its Applications 225, 2017, 164–194. 116, 117, 140

[17] A. N. Dranishnikov, J. Keesling , V. V. Upenskij, *On the Higson corona of uniformly contractible spaces*, Topology 37 No. 4, 1998, 791–803. 55, 67, 101

[18] A. N. Dranishnikov, *Asymptotic topology*, Russian Mathematical Surveys 55.6, 2000, 1085. 11, 13, 19

[19] C. Druţu and P. Nowak, *Kazhdan projections, random walks and ergodic theorems*, J. Reine Angew. Math. (to appear). 110

[20] J. Dydak, *Partitions of unity*, Topology Proceedings 27, 2003, 125–171. 48

[21] J. Dydak and C.S. Hoffland, *An Alternative Definition of Coarse Structures*, Topology and its Applications 155 (9), 2008, 1013-1021. 15, 43, 70, 79, 105, 110, 112, 114

[22] J. Dydak, A. Mitra, *Large scale absolute extensors*, Large scale absolute extensors, Topology and its Applications 214, 2016, 51–65. 41, 42, 99

[23] J. Dydak and Ž. Virk, *Preserving coarse properties*, Revista Matematica Complutense 29, 2016, 191–206. 103, 122, 137

[24] J. Dydak and T. Weighill, *Monotone-light factorizations in coarse geometry*, Topology and its Applications (to appear), arXiv preprint available at arXiv:1607.02403. 121, 122, 137

[25] H. Ehrbar, O. Wyler, *On subobjects and images in categories*, Technical Report, Department of Mathematical Sciences, Carnegie Mellon University, 1968. 87

[26] S. Eilenberg, *Sur les transformations continues d'espaces mêtriques compacts*, Fund. Math. 22, 1954, 292–296. 77

[27] F.T. Farrell, *The Borel conjecture*, High dimensional manifold theory 9, 2001, 225–298. 6

[28] D. Fisher, T. Nguyen, W. van Limbeek, *Rigidity of warped cones and coarse geometry of expanders* (preprint, arXiv:1710.03085). 110

[29] P. J. Freyd, G. M. Kelly, *Categories of continuous functors I*, J. Pure Appl. Algebra 2, 1972, 169–191. 78, 86

[30] R. Geoghegan, Topological methods in group theory, Springer Science & Business Media, 2008. 34, 35

[31] G. Gong, Q. Wang, and G. Yu, *Geometrization of the strong Novikov conjecture for residually finite groups*, J. Reine Angew. Math., 621:159–189, 2008. 7, 111, 128, 129

[32] M. Gromov, Asymptotic invariants of infinite groups, Geometric group theory vol. 2 (Sussex, 1991), London Math. Soc. Lecture Note Ser. vol. 182, Cambridge University Press, Cambridge, 1993, 1–295. 10, 49

[33] E. Guentner, R. Tessera and G. Yu, A notion of geometric complexity and its application to topological rigidity, Invent. Math. 189, 2012, 315–357. 13

[34] N. Higson, On the relative K-homology theory of Baum and Douglas (unpublished preprint), 1990. 18

[35] N. Higson, V. Lafforgue and G. Skandalis *Counterexamples to the Baum-Connes conjecture*, Geometric and Functional Analysis 12.2, 2002, 330–354. 111

[36] N. Higson, J. Roe and G. Yu, *A coarse Mayer-Vietoris principle*, Mathematical Proceedings of the Cambridge Philosophical Society Volume 114 no. 01, Cambridge University Press, 1993. 8, 9, 19, 28, 30, 31, 33

[37] D. Holgate, J. Šlapal, *Categorical neighborhood operators*, Topology and its Applications 158.17, 2011, 2356–2365. 50

[38] J. R. Isbell, *Uniform Spaces*, Mathematical Surveys, vol. 12, American Mathematical Society, Providence, RI, 1964. 15, 51, 63

[39] G. Janelidze, Categorical Galois theory: revision and some recent developments, Galois connections and applications, Springer Netherlands, 2004, 139–171. 24

[40] D. Kasprowski, *The asymptotic dimension of quotients by finite groups*, Proc. Amer. Math. Soc. 145, 2017, 2383–2389. 110, 122

[41] M. Katetov, *On real-valued functions in topological spaces*, Fund. Math., 38, 1951, 85–91; Correction to "On real-valued functions in topological spaces ", Fund. Math., 40, 203–205, 1953. 64

[42] J. Keesling, The one-dimensional Čech cohomology of the Higson compactification, Topology Proceedings, Volume 19, 1994. 34

[43] S. Mac Lane, Categories for the working mathematician, Springer Science & Business Media, 1978. 13, 20

[44] K. Mine and A. Yamashita, *Metric compactifications and coarse structures*, (preprint) arXiv:1106.1672. 67

[45] K. Mine, A. Yamashita, and T. Yamauchi, C_0 *Coarse Structures on Uniform Spaces*, Houston Journal of Mathematics 41, 2015, 1351–1358. 67

[46] T. Miyata, Z. Virk, *Dimension raising maps in a large scale*, Fundamenta Mathematicae 223, 2013, 83-97. 83

[47] J. Munkres, *Topology*, Prentice Hall, 2000, 2nd edition. 48, 152

[48] K. Nagami and Y. Kodama, *Dimension theory*, Academic Press, 1970. 10

[49] P. Nowak, D. Sawicki, *Warped cones and spectral gaps*, Proceedings of the American Mathematical Society 145, 2017, no. 2, 817–823. 110

[50] P.W. Nowak and G. Yu, *Large scale geometry*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zurich, 2012. 10, 11, 12, 24, 33, 34, 71, 105, 113, 148

[51] H. Oyono-Oyono and G. Yu, *K-theory for the maximal Roe algebra of certain expanders*, Journal of Functional Analysis 257.10, 2009, 3239–3292. 111, 128, 133, 134

[52] J. R. Porter and R. G. Woods, *Extensions and absolutes of Hausdorff spaces*, Springer Science & Business Media, 2012. 65

[53] I.V. Protasov, *Normal ball structures*, Mat. Stud 20, 2003, 3–16. 42

[54] J. Roe, An index theorem on open manifolds, I. J. Differential Geometry, 27:87–113, 1988. 18

[55] J. Roe, Coarse cohomology and index theory on complete Riemannian manifolds, Memoirs of the American Mathematical Society 497, 1993. 18, 124

[56] J. Roe, Index theory, coarse geometry, and topology of manifolds, CBMS Regional Conference Series in Mathematics Volume 90, Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. 7, 9, 12, 18, 21, 80, 110, 116

[57] J. Roe, *Lectures in Coarse Geometry*, University Lecture Series 31, American Mathematical Society, Providence, RI, 2003. 11, 15, 18, 19, 24, 26, 31, 33, 35, 36, 38, 40, 43, 45, 47, 58, 65, 73, 93, 95, 98, 99, 102, 103, 106, 112, 114, 139, 145

[58] J. Roe, *Warped cones and property A*, Geometry & Topology 9.1, 2005, 163–178. 110, 111, 122, 123, 124, 137

[59] H. Sako, *Property A for coarse spaces*, (preprint) arXiv:1303.7027. 58

[60] D. Sawicki, *Warped cones over profinite completions*, Journal of Topology and Analysis 145, 2015, 1–22. 110

[61] D. Sawicki, *Warped cones violating the coarse Baum-Connes conjecture*, preprint. 153

[62] J. Smith, *On asymptotic dimension of countable abelian groups* (preprint, arXiv:math/0504447). 71, 105, 106, 113

[63] J. Špakula and R. Willett, *Maximal and reduced Roe algebras of coarsely embeddable spaces*, J. Reine Angew. Math. 678, 2013, 35–68. 138

[64] J. Stallings, On torsion-free groups with infinitely many ends, Annals of Mathematics (2) Volume 88, 1968, 312–334. 35

[65] J. Stallings, Group theory and three-dimensional manifolds, Yale University Press, New Haven, Conn., 1971. 35

[66] J. W. Tukey, *Convergence and uniformity in topology*, Ann. Math. Studies no. 1, Princeton University Press, 1940. 63

[67] F. Vigolo, *Discrete fundamental groups of Warped Cones and expanders* (preprint, arXiv:1710.04955). 110

[68] T. Weighill, On spaces with connected Higson coronas, arXiv preprint arXiv:1602.06991, 2016. 103

[69] A. Weil, *Sur les espaces à structure uniforme et sur la topologie générale*, Actualités Sci. Ind. 551, Paris, 1937. 63

[70] G. T. Whyburn, *Open mappings on locally compact spaces*, Memoirs Amer. Math. Soc. 1, 1950. 77

[71] N. Wright, *C_0 coarse geometry*, PhD thesis, Penn State, 2002. 57

[72] G. Yu, *The coarse Baum-Connes conjecture*, K-theory 9, 1995. 29, 111

[73] G. Yu, *The Novikov conjecture for groups with finite asymptotic dimension*, Annals of Mathematics 147, 1998, 325–355. 11, 12

[74] G. Yu, *The coarse Baum-Connes conjecture for spaces which admit a uniform embedding into Hilbert space*, Inventiones mathematicae 139.1, 2000, 201-240. 11, 12, 105, 111, 135

[75] S. Zhang, *Coarse quotient mappings between metric spaces*, Israel Journal of Mathematics 207, 2015, 961–979. 111, 146

Vita

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