Asymptotic Theory for Robust Autocorrelation Test under Stochastic Volatility*

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Abstract: Wooldridge (1991) suggest a robust test for autocorrelations of the disturbances of regression models, under misspecified conditional heteroskedastic model. Although stochastic volatility (SV) models allow unconditional time-varying variance, the Monte Carlo results of Asai (2000) indicate that the test of Wooldridge (1991) is robust under the SV process. This paper shows that the test statistic has asymptotic χ^2 distribution under the null hypothesis of no serial correlation, even when the underlying process has stochastic volatility.

Keywords: Asymptotic theory; Autocorrelation; Misspecification; Robust Test; Stochastic Volatility.
JEL Classification: C12, C22.

1 Introduction

Wooldridge (1990, 1991) developed a general framework for robust, regression-based diagnostics to models with conditional means and conditional variances. As an application, Wooldridge (1991) proposed a test for autocorrelations of the disturbances of regression models, which is robust to the misspecification of conditional heteroskedastic models. Monte Carlo experiments of Asai (2000) show that the robust autocorrelation test of Wooldridge (1991) has satisfactory size and power in finite sample. The purpose of this paper is to give a formal proof for the asymptotic property of the test statistic.

The organization of this paper is as follows, Section 2 introduce the testing procedure in the presence of stochastic volatility. Section 3 shows that the robust test follows the χ^2 distribution under the null of no serial correlation, and Section 4 gives some concluding remarks.

The matrix (Euclidean) norm of the matrix, or vector A, is dened as $||A|| = \sqrt{\mathrm{tr}(A'A)}$. We denote a strictly positive constant by K.

2 Stochastic Volatility Model and Robust Autocorrelation Test

Consider the regression model with autoregressive disturbance:

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$$y_t = x_t\beta + u_t, \quad (1)$$

$$u_t = \gamma_1 u_{t-1} + \cdots + \gamma_p u_{t-p} + e_t \ (t = 1, 2, \ldots, T), \quad (2)$$

where y_t is a dependent variable, x_t is a $1 \times k$ vector of variables which may include exogenous variables and predetermined variables, β is a $k \times 1$ vector of parameters, $\gamma = (\gamma_1, \ldots, \gamma_p)'$ is a $p \times 1$ vector of parameters, and e_t follows a stochastic volatility (SV) process:

$$e_t = z_t \exp(\alpha_t/2) \quad (3)$$

$$\alpha_{t+1} = \omega + \phi\alpha_t + \eta_t, \quad (4)$$

with $z_t \sim iid(0,1)$ and $\eta_t \sim N(0, \sigma_\eta^2)$.

We assume $|\phi| < 1$ for the strict and covariance stationarity of α_t. By the denitions (3) and (4), Theorem 3.5.8 of Stout (1974) shows that e_t is strict stationary and ergodic. The structure 21 of the SV model (3) and (4) and property of the log-normal distribution indicate:

$$E(e_t) = 0, \ V(e_t) = \sigma_e^2 \quad E(e_t e_s) = 0 \text{ for } t \neq s, \quad (5)$$

where

$$\sigma_e^2 = \exp\left(\frac{\omega}{1-\phi} + \frac{\sigma_\eta^2}{2(1-\phi^2)}\right),$$

(see Andersen and Sørensen (1996) for the moments of the SV model). Hence, e_t is covariance stationary if $|\phi| < 1$. The autocovariance function of e_t^2 is given by:

$$E[e_t^2 e_{t-s}^2] = \exp\left(\frac{2\omega}{1-\phi} + \frac{\sigma_\eta^2}{1-\phi^2} + \frac{\phi^s\sigma_\eta^2}{1-\phi^2}\right), \quad s = 0, 1, 2, \ldots, \quad (6)$$

indicating the dependence of the second moment.

We assume that γ satisfy the stationary condition.

Assumption 1. *The roots of the characteristic polynomial,* $1 - \gamma_1 m - \cdots \gamma_p m^p = 0$, *are greater than one in absolute value.*

Remark 2.1. Since e_t is strict stationary and ergodic, Theorem 3.5.8 of Stout (1974) and equation (2) with Assumption 1 imply that u_t is strict stationary and ergodic. Hence u_t has an MA(∞) representation:

$$u_t = \sum_{i=0}^{\infty} \varrho_i e_{t-1}, \quad \varrho_0 = 1, \quad (7)$$

with unconditional moments, $E(u_t) = 0$ and $V(u_t) = \sigma_u^2$, where $\sigma_u^2 = \sigma_e^2 \sum_{i=0}^{\infty} \varrho_i^2 < \infty$.

For the model defined in (1)-(4), consider testing autocorrelations via the null hypothesis:

$$H_0 : \gamma_1 = \cdots = \gamma_p = 0. \quad (8)$$

For this purpose, we use the robust Lagrange multiplier (LM) test introduced by Wooldridge (1991). Following Wooldridge (1991), dene the 'misspecification indicator' as:

$$\lambda_t(\beta) = (y_{t-1} - x_{t-1}\beta, \ldots, y_{t-p} - x_{t-p}\beta). \quad (9)$$

Corresponding to the OLS estimate, $\hat{\beta}$, define $\hat{\lambda}_t = \lambda_t(\hat{\beta}) = (\hat{u}_{t-1}, \ldots, \hat{u}_{t-p})$ with the OLS residual defined by $\hat{u}_t = y_t - x_t\hat{\beta}$. Wooldridge (1991) considers a kind of standardization of the misspecification indicator using an approximated heteroskedastic model. For the underlying SV process, we use the ARCH(q) specication for the approximated heteroskedastic model. Note that the test statistic of Wooldridge (1990) is robust to the misspecification of heteroskedastic function, if the regularity conditions are satisfied.

The construction of the robust LM statistic involves the following steps:

1. Obtain the fitted values, $\hat{h}_t$ $(t = 1,\ldots,T)$ from the regression of $\hat{u}_t^2$ on $(1, \hat{u}_{t-1}^2,\ldots, \hat{u}_{t-q}^2)$.
2. Define $\tilde{x}_t = \hat{h}_t^{-1/2} x_t$ and $\tilde{u}_t = \hat{h}_t^{-1/2}\hat{u}_t$ $(t = 1,\ldots,T)$.
3. Save the $1 \times p$ vector of residuals, say $\tilde{r}_t$, from the regression of each of $\tilde{\lambda}_t$ on $\tilde{x}_t$, where $\tilde{\lambda}_t = (\tilde{u}_{t-1},\ldots, \tilde{u}_{t-p})$.
4. Compute $T -$ SSR, where SSR is the sum of the squared residuals from the regression of 1 on $\tilde{u}_t\tilde{r}_t$.

In the following, we show that $T -$ SSR has the asymptotic $\chi^2(p)$ distribution under H_0.

3 Asymptotic Property

In the asymptotic analysis, we use the following notations to explain quantities used in the procedure in the previous section.

In addition to the misspecification indicator (9), define the error term $\psi_t(\beta) = y_t - x_t\beta$. For the OLS estimator $\hat{\beta} = [\sum_{t=1}^T x_t' x_t]^{-1}\sum_{t=1}^T x_t' y_t$, the OLS residuals are given by $\hat{u}_t = \psi_t(\hat{\beta}) = u_t - x_t(\hat{\beta} - \beta^o)$, where β^o is the vector of true parameters. For the first step in the above procedure, we formally state the approximating ARCH(q) model as:

$$h_t(\theta) = \delta_0 + \delta_1(y_{t-1} - x_{t-1}\beta)^2 + \cdots + \delta_q(y_{t-q} - x_{t-q}\beta)^2, \tag{10}$$

where $\theta = (\delta', \beta')'$ and $\delta = (\delta_0, \delta_1,\ldots, \delta_q)'$. The OLS estimator of δ is obtained by:

$$\hat{\delta} = \left[\sum_{t=1}^T \kappa_t(\hat{\beta})'\kappa_t(\hat{\beta})\right]^{-1} \sum_{t=1}^T \kappa_t(\hat{\beta})'\varphi_t(\hat{\beta}) \tag{11}$$

where

$$\kappa_t(\beta) = [1 \ (y_{t-1} - x_{t-1}\beta)^2 \ \cdots \ (y_{t-q} - x_{t-q}\beta)^2], \quad \varphi_t(\beta) = (y_t - x_t\beta)^2. \tag{12}$$

By the definition of $h_t(\theta)$, we can write $\hat{h}_t$ in the first step as $\hat{h}_t = h_t(\hat{\theta})$ with $\hat{\theta} = (\hat{\delta}', \hat{\beta}')'$.

Based on $\tilde{x}_t$ and $\tilde{u}_t$ in the second step, the residual in the third step is given by:

$$\tilde{r}_t = \tilde{\lambda}_t - \tilde{x}_t\left[\sum_{t=1}^T \tilde{x}_t'\tilde{x}_t\right]^{-1}\sum_{t=1}^T \tilde{x}_t'\tilde{\lambda}_t = \left[h_t(\hat{\theta})\right]^{-1/2}\left[\lambda_t(\hat{\beta}) - x_t\hat{B}_T\right],$$

where

$$\hat{B}_T = \left[\sum_{t=1}^T [h_t(\hat{\theta})]^{-1}x_t'x_t\right]^{-1}\sum_{t=1}^T [h_t(\hat{\theta})]^{-1}x_t'\lambda_t(\hat{\beta}). \tag{13}$$

By regressing 1 on $\tilde{u}_t\,\tilde{r}_t$ in the fourth step, we obtain:

$$T - \mathrm{SSR} = \ddot{\zeta}_T' \ddot{\Omega}_T \ddot{\zeta}_T \tag{14}$$

where

$$\ddot{\zeta}_T = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \tilde{u}_t \tilde{r}_t' = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \frac{\psi_t(\hat{\beta})}{h_t(\hat{\theta})} \left[\lambda_t(\hat{\beta}) - x_t \hat{B}_T\right]', \tag{15}$$

$$\ddot{\Omega}_T = \frac{1}{T} \sum_{t=1}^{T} \tilde{u}_t^2 \tilde{r}_t' \tilde{r}_t = \frac{1}{T} \sum_{t=1}^{T} \left[\frac{\psi_t(\hat{\beta})}{h_t(\hat{\theta})}\right]^2 \left[\lambda_t(\hat{\beta}) - x_t \hat{B}_T\right]' \left[\lambda_t(\hat{\beta}) - x_t \hat{B}_T\right], \tag{16}$$

and SSR is the sum of the squared residuals. Note $T - SSR = TR_u^2$, where R_u^2 is the uncentered r-squared from the regression of 1 on $\tilde{u}_t \tilde{r}_t$.

Corresponding to θ, denote the parameter space as $\Theta = \Theta_\beta \times \Theta_\delta$ where $\Theta_\beta \subset \Re^k$ and $\Theta_\delta \subset \Re^{q+1}$. We make the following assumptions.

Assumption 2. *The vector process x_t is strict stationary and ergodic. For any t and s, x_t and u_s are independent. For the second moments of x_t and u_t, $V_x = E[x_t' x_t]$ is finite and positive definite, and σ_u^2 defined by equation (7) is finite, respectively. For the fourth moment of x_t, $E(|x_{it} x_{jt} x_{lt} x_{rt}|)$ is finite for all i, j, l, and r (i, j, l, r = 1,...,k).*

Assumption 3. *For the approximating ARCH(q) model (10), $\delta_0 > 0$ and $\delta_i \geq 0$ ($i = 1,\ldots, q$). The roots of the characteristic polynomial, $1 - \delta_1 m - \cdots - \delta_q m^q = 0$, are greater than one in absolute value.*

Remark 3.1. The parameter vector, δ, is determined by the property of u_t with the structure (2)-(4). The true value of δ is given by the following assumption.

Assumption 4. *Θ is compact. For the vectors of the true parameters, $\beta^o \in \Theta_\beta$ and $\delta^o \in \Theta_\delta$, where*

$$\delta^o = \left[E[\kappa_t(\beta^o)' \kappa_t(\beta^o)]\right]^{-1} E[\kappa_t(\beta^o)' \varphi_t(\beta^o)].$$

Assumption 5. *The distribution of z_t is symmetric and $E(z_t^4) < \infty$.*

Proposition 1. *Under Assumptions 1-5,*

$$\sqrt{T}(\hat{\delta} - \delta^o) = O_p(1),$$

where $\hat{\delta}$ is defined by (11).

Proposition 2. *Under Assumptions 1-5 and H_0*

$$T - SSR \xrightarrow{d} \chi^2(p).$$

where $T - SSR$ is defined in equation (14).

4 Conclusion

Wooldridge (1991) developed a serial correlation test which is robust to the misspecification of conditional variance. The paper shows that the test statistic suggested by Wooldridge (1991) has the asymptotic χ^2 distribution under the null hypothesis of no autocorrelation, when the underlying process follows the stochastic volatility (SV) model. The sufficient conditions for the result are existence of the fourth order moment and the assumption of a symmetric distribution.

We can consider several extensions of the paper. Regarding the underlying process, the approach used in this paper applicable to symmetric ARCH class model and symmetric type SV models. We may also examine asymptotic properties of various tests under misspecified heteroskedastic models. These are important directions of future researches.

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Appendix

A.1 Proof of Proposition 1

For a matrix A, $\{A\}_{ij}$ denotes the (i, j)th element of A. We introduce Lemma A.1 of Wooldridge (1990) which is repeatedly used in the following proofs.

Lemma 1. *Assume that the sequence of random functions* $\{Q_T(w_T, \theta) : \theta \in \Theta, T = 1, 2, \ldots\}$*, where* $Q_T(w_T, \cdot)$ *is continuous on* Θ *and* Θ *is a compact subset of* $\Re^P$*, and the sequence of non-random functions* $\{\bar{Q}_T(\theta) : \theta \in \Theta, T = 1, 2, \ldots\}$ *satisfy the following conditions:*

(i) $\sup_{\theta \in \Theta} \left| Q_T(w_T, \theta) - \bar{Q}_T(\theta) \right| \xrightarrow{p} 0$;

(ii) $\{Q_T(w_T, \theta) : \theta \in \Theta, T = 1, 2, \ldots\}$ *is continuous on* Θ *uniformly in T. Let* $\ddot{\theta}_T$ *be a sequence of random vectors such that* $\ddot{\theta}_T - \theta_T^o \xrightarrow{p} 0$ *where* $\{\theta_T^o\} \subset \Theta$.

Then $Q_T(w_T, \ddot{\theta}_T) - \bar{Q}_T(\theta_T^o) \xrightarrow{p} 0$.

Proof. See Lemma A.1 of Wooldridge (1990). □

Lemma 2. *Under Assumptions 1 and 2,* $\hat{\beta} \xrightarrow{a.s.} \beta^o$.

Proof. Noting that $y_t = x_t \beta^o + u_t$,

$$\hat{\beta} = \beta^o + \left[\frac{1}{T} \sum_{t=1}^{T} x_t' x_t \right]^{-1} \frac{1}{T} \sum_{t=1}^{T} x_t' u_t.$$

Since x_t is strict stationary and ergodic, the uniform law of large numbers (ULLN) for stationary ergodic processes (see Lemma A.2.2 of White (1994)) indicates:

$$\left| \frac{1}{T} \sum_{t=1}^{T} x_{it} x_{jt} - \{V_x\}_{ij} \right| \xrightarrow{a.s.} 0,$$

for all i and j $(i, j = 1, \ldots, k)$. By Assumption 2, V_x is positive definite, and the continuity of the matrix inverse indicates that $\frac{1}{T} \sum_{t=1}^{T} x_t' x_t$ is nonsingular almost surely for T sufficiently large. As the elements of V_x^{-1} are uniformly bounded,

$$\left| \left\{ \left(\frac{1}{T} \sum_{t=1}^{T} x_t' x_t \right)^{-1} \right\}_{ij} - \{V_x^{-1}\}_{ij} \right| \xrightarrow{a.s.} 0, \qquad \text{(A.1)}$$

for all i and j. Since $E(x_{it}^2) = \{V_x\}_{ii} < K$ and $E(u_t^2) = \sigma_u^2 < K$ by Assumptions 1 and 2,

$$E|x_{it} u_t| \leq \sqrt{E(x_{it}^2) E(u_t^2)} < K,$$

by Hölder's inequality. Since $(x_t', u_t)'$ is strict stationary and ergodic, $x_t' u_t$ is strict stationary and ergodic. By the ULLN for stationary ergodic processes (Lemma A.2.2 of White (1994)), we obtain:

$$\left| \frac{1}{T} \sum_{t=1}^{T} x_{it} u_{jt} - E(x_{it} u_t) \right| \xrightarrow{a.s.} 0,$$

Since V_x^{-1} has uniformly bounded elements, uniform continuity implies,

$$\left|\left\{\left[\frac{1}{T}\sum_{t=1}^{T}x_t'x_t\right]^{-1}\frac{1}{T}\sum_{t=1}^{T}x_t'u_t\right\}_i - \{V_x^{-1}E(x_t'u_t)\}_i\right| \xrightarrow{a.s.} 0,$$

for all i (i = 1,..., k). Since $E(x_t'u_t) = 0$ by Assumption 2,

$$\left[\frac{1}{T}\sum_{t=1}^{T}x_t'x_t\right]^{-1}\frac{1}{T}\sum_{t=1}^{T}x_t'u_t \xrightarrow{a.s.} 0,$$

implying that Lemma 2 holds. □

Lemma 3. *Under Assumptions 1-4,* $\sqrt{T}(\hat{\beta}-\beta^o) \xrightarrow{d} N(0,\sigma^2 V_x^{-1})$.

Proof. Consider the quantity $\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_t'u_t$. As $(x_t', u_t)'$ is strict stationary and ergodic, $x_t'u_t$ is strict stationary ergodic. Assumptions 1 and 2 indicate that $x_t'u_t$ is strict stationary ergodic martingale difference with $E(u_t^2x_tx_t') = \sigma_u^2V_x$, which is finite and positive definite. By the ULLN for stationary ergodic process (Lemma A.2.2 of White (1994)),

$$\left|\frac{1}{T}\sum_{t=1}^{T}u_t^2x_{it}x_{jt} - \sigma_u^2\{V_x\}_{ij}\right| \xrightarrow{a.s.} 0,$$

for all i and j (i, j = 1,..., k). Since V_x is finite and positive definite by Assumptions 2, we can define the symmetric positive definite matrix, $\sigma_u^{-1}V_x^{-1/2}$ such that $(\sigma_u^{-1}V_x^{-1/2})^2 = \sigma_u^{-2}V_x^{-1}$. Assumptions 2-4 imply that the elements of $V_x^{-1/2}$ and σ_u^{-1} are uniformly bounded. By Lemma 3.2 of White (1980a),

$$\left|\sigma_u^{-2}\left\{V_x^{-1/2}\left[\frac{1}{T}\sum_{t=1}^{T}u_t^2x_t'x_t\right]V_x^{-1/2}\right\}_{ij} - \{I_k\}_{ij}\right| \xrightarrow{p} 0, \tag{A.2}$$

for all i and j (i, j = 1,..., k). Also, by Chebyshev's inequality,

$$P\left(\left|\frac{x_{it}u_t}{\sqrt{T}}\right| > \varepsilon\right) \leq \frac{V(x_{it}u_t)}{T\varepsilon^2} \to 0,$$

as $T \to \infty$. Hence,

$$\max_{1\leq t\leq T}\left|\frac{x_{it}u_t}{\sqrt{T}}\right| \xrightarrow{p} 0. \tag{A.3}$$

As equations (A.2) and (A.3) satisfy the regularity conditions for the central limit theorem (CLT) for strict stationary ergodic martingale differences (Theorem 24.3 of Davidson (1994)), we obtain:

$$\sigma_u^{-1}V_x^{-1/2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_t'u_t \xrightarrow{d} N(0, I_k). \tag{A.4}$$

Now

$$\sqrt{T}\sigma_u^{-1}V_x^{1/2}(\hat{\beta}-\beta^o) = V_x^{1/2}\left[\frac{1}{T}\sum_{t=1}^{T}x_t'x_t\right]^{-1}V_x^{1/2}\sigma_u^{-1}V_x^{-1/2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}x_t'u_t.$$

By (A.1), (A.2), and Lemma 3.2 of White (1980a),

$$\left|\left\{V_x^{1/2}\left[\frac{1}{T}\sum_{t=1}^{T}x_t'x_t\right]^{-1}V_x^{1/2}\right\}_{ij} - \{I_k\}_{ij}\right| \xrightarrow{p} 0,$$

and hence,

$$\left| \sqrt{T}\sigma_u^{-1} V_x^{1/2}(\hat{\beta} - \beta^o) - \sigma_u^{-1} V_x^{-1/2} \frac{1}{\sqrt{T}} \sum_{t=1}^{T} x_t' u_t \right| \xrightarrow{p} 0. \tag{A.5}$$

Lemma 3.3 of White (1980a) with (A.4) and (A.5) indicates:

$$\sqrt{T}\sigma_u^{-1} V_x^{1/2}(\hat{\beta} - \beta^o) \xrightarrow{d} N(0, I_k). \quad \square$$

Lemma 4. *Define*

$$\Xi_{0t} = \kappa_t(\beta^o)' \kappa_t(\beta^o).$$

Under Assumptions 1-5,

(i) $\left| \frac{1}{T}\sum_{t=1}^{T} \xi_{0,ijt} - E(\xi_{0,ijt}) \right| \xrightarrow{a.s.} 0$ *for all i and j* $(i, j = 1,..., q+1)$, *where* $\xi_{0,ijt}$ *is the* (i, j)*th element of* Ξ_{0t};

(ii) $\Gamma_0 = E[\kappa_t(\beta^o)'\kappa_t(\beta^o)]$ *is positive definite.*

Proof. We can write the (i, j)th element of Ξ_{0t} as:

$$\xi_{0,ijt} = \begin{cases} 1 & (i = j = 1) \\ u_{t-j}^2 & (i = 1,\ j = 2, \ldots, q+1) \\ u_{t-i}^2 & (j = 1,\ i = 2, \ldots, q+1) \\ u_{t-i}^2 u_{t-j}^2 & (i, j = 2, \ldots, q+1). \end{cases}$$

By Remark 2.1, $E|\xi_{0,1jt}| < \infty$ and $E|\xi_{0,i1t}| < \infty$. For $i, j = 2,..., q+1$,

$$E|\xi_{0,ijt}| = E[u_{t-i}^2 u_{t-j}^2] \le \left[E[u_{t-i}^4]\right]^{1/2} \left[E[u_{t-j}^4]\right]^{1/2} = E[u_t^4] < \infty,$$

by Hölder's inequality and the finite fourth moment by Assumption 5. Hence $E|\xi_{0,ijt}|$ exists and bounded. Since u_t is strict stationary and ergodic, Theorem 3.5.8 of Stout (1974) with the structure Ξ_{0t} implies that all elements of Ξ_{0t} except for (1, 1) are strict stationary and ergodic. Note that $\xi_{0,11t} = 1$. By the ULLN for stationary and ergodic process (Lemma A.2.2 of White (1994)),

$$\left| \frac{1}{T} \sum_{t=1}^{T} \xi_{0,ijt} - E(\xi_{0,ijt}) \right| \xrightarrow{a.s.} 0, \tag{A.6}$$

for all i and j $(i, j = 1,..., q+1)$, which gives Lemma 4(i).

By the structure, $T^{-1}\sum_{t=1}^{T} \Xi_{0t}$ is the sample mean of the outer product of random vector $\kappa_t(\beta^0)$, thus its determinant is non-negative. Since $\kappa_t(\beta^0)$ is linearly independent by Assumption 3, the rank of $T^{-1}\sum_{t=1}^{T} \Xi_{0t}$ is $q+1$, which guarantees that the inverse of the matrix exists almost surely when $T > q+1$. Combined with (A.6), we obtain Lemma 4(ii). $\square$

Proof of Proposition 1 Since u_t and x_t are strictly stationary and ergodic, Theorem 3.5.8 of Stout (1974) with the structure (12) implies that elements of $\kappa_t(\beta^0)'\kappa_t(\beta^0)$ and $\kappa_t(\hat{\beta})'\kappa_t(\hat{\beta})$ are strict stationary and ergodic. Combined with Lemma 4 and the consistency of $\hat{\beta}$ by Lemma 2, Lemma 1 indicates that:

$$\left|\left\{\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\hat{\beta})'\kappa_t(\hat{\beta})\right\}_{ij} - \{\Gamma_0\}_{ij}\right| \xrightarrow{a.s.} 0,$$

for all i and j (i,j = 1,..., q + 1), where Γ^0 is stated in Lemma 4. By the proof of Lemma 4, $\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\hat{\beta})'\kappa_t(\hat{\beta})$ is nonsingular almost surely for T sufficiently large. As the elements of Γ_0^{-1} are uniformly bounded,

$$\left|\left\{\left(\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\hat{\beta})'\kappa_t(\hat{\beta})\right)^{-1}\right\}_{ij} - \{\Gamma_0^{-1}\}_{ij}\right| \xrightarrow{a.s.} 0, \quad \text{(A.7)}$$

for all i and j (i, j = 1,..., q +1). Since u_t and x_t are strictly stationary and ergodic, the elements of $\kappa_t(\beta^0)'\varphi_t(\beta^0)$ and $\kappa_t(\hat{\beta})'\varphi_t(\hat{\beta})$ defied by equation (12) are strictly stationary and ergodic. Since $E\left[[\{\kappa_t(\beta^0)\}_i]^2\right] = \{\Gamma^0\}_{ii} < K$ by Lemma 3 and $E\left[[\varphi_t(\beta^0)]^2\right] = E(u_t^4) < K$ by Assumption 5,

$$E\left|\{\kappa_t(\beta^o)\}_i\varphi_t(\beta^o)\right| \le \sqrt{E[[\{\kappa_t(\beta^o)\}_i]^2]E[[\varphi_t(\beta^o)]^2]} < K \quad (i = 1,\dots,q+1), \quad \text{(A.8)}$$

by Hölder's inequality. By the ULLN for stationary ad ergodic process indicates, we obtain:

$$\left|\left\{\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\beta^o)'\varphi_t(\beta^o)\right\}_i - \{E[\kappa_t(\beta^o)'\varphi_t(\beta^o)]\}_i\right| \xrightarrow{a.s.} 0,$$

for all i (i = 1,..., q +1). Since Γ_0^{-1} has uniformly bounded elements, uniform continuity implies

$$\left|\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\beta^o)'\kappa_t(\beta^o)\right]^{-1}\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\beta^o)'\varphi_t(\beta^o)\right\}_i - \{\Gamma_0^{-1}E[\kappa_t(\beta^o)'\varphi_t(\beta^o)]\}_i\right| \xrightarrow{a.s.} 0,$$

for all i (i = 1,..., q +1). By Assumption 4,

$$\left|\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\beta^o)'\kappa_t(\beta^o)\right]^{-1}\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\beta^o)'\varphi_t(\beta^o)\right\}_i - \delta_i^o\right| \xrightarrow{a.s.} 0,$$

for all i (i = 1,..., q +1). By (A.7) and the consistency of $\hat{\beta}$ by Lemma 2, Lemma 1 indicates:

$$\left|\left\{\left[\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\hat{\beta})'\kappa_t(\hat{\beta})\right]^{-1}\frac{1}{T}\sum_{t=1}^{T}\kappa_t(\hat{\beta})'\varphi_t(\hat{\beta})\right\}_i - \delta_i^o\right| \xrightarrow{a.s.} 0,$$

for all i (i = 1,..., q +1), showing that $\hat{\delta} \xrightarrow{a.s.} \delta^o$. The covariance matrix of $\sqrt{T}(\hat{\delta} - \delta^o)$ is given by:

$$V\left(\sqrt{T}(\hat{\delta} - \delta^o)\right) = \Gamma_0^{-1}E\left[\left(u_t^2\kappa_t(\beta^o)' - E[u_t^2\kappa_t(\beta^o)']\right)'\left(u_t^2\kappa_t(\beta^o)' - E[u_t^2\kappa_t(\beta^o)']\right)\right]\Gamma_0^{-1}.$$

Since the elements of Γ_0^{-1} are bounded and those of $E\left[u_t^4\kappa_t(\beta^0)'\kappa_t(\beta^0)\right]$ are bounded by (A.8), the elements of $V(\sqrt{T}(\hat{\delta} - \delta^o))$ are bounded. By Chebyshev's inequality,

$$P\left(\sqrt{T}\left|\hat{\delta}_i - \delta_i^o\right| < \epsilon\right) \ge 1 - \frac{V(\sqrt{T}(\hat{\delta}_i - \delta_i^o))}{\epsilon^2},$$

for all i (i = 1,..., q + 1). The result establishes $\sqrt{T}(\hat{\delta} - \delta^o) = O_p(1)$. □

A.2 Proof of Proposition 2

Define the information set up to t as $\mathfrak{I}_t = \{y_t, x_t, y_{t-1}, x_{t-1}, \dots\}$.

Lemma 5. *Under Assumption 1 and 2, $h_t(\theta)$ is strict stationary and ergodic with:*

$$E[h_t(\theta)] = \delta_0 + \left[\sigma_u^2 + (\beta - \beta_0)' V_x (\beta - \beta_0)\right] \sum_{i=1}^{q} \delta_i, \tag{A.9}$$

where

$$\sigma_u^2 = \frac{\sigma_e^2}{1 - \gamma_1^2 - \cdots - \gamma_q^2},$$

and σ_e^2 *is the variance of* e_t *defined by (5).*

Proof. Noting that $y_{t-i} - x_{t-i}\beta = u_{t-i} - x_{t-i}(\beta - \beta_0)$, we obatin:

$$h_t(\theta) = \delta_0 + \sum_{i=1}^{q} \delta_i \left[u_{t-i} - x_{t-i}(\beta - \beta_0)\right]^2, \tag{A.10}$$

where β_0 is the true value of β. Since u_t and x_t are stationary and ergodic by Assumptions 1 and 2, Theorem 3.5.8 of Stout (1974) with the structure (A.10) implies that $h_t(\theta)$ is stationary and ergodic. For obtaining $E[h_t(\theta)]$, the variance of u_t is obtained by the conventional approach. Since u_t is uncorrelated with x_t by Assumption 2, we obtain $E[\{u_{t-i} - x_{t-i}(\beta - \beta_0)\}^2] = \sigma_u^2 + (\beta - \beta_0)' V_x (\beta - \beta_0)$. Then we obtain (A.9). □

Lemma 6. Let $\Xi_{1t}(\theta) = [h_t(\theta)]^{-1} x_t' x_t$. *Under Assumptions 1-4,*

(i) $\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T} \xi_{1,ijt}(\theta) - E[\xi_{1,ijt}(\theta)]\right| \xrightarrow{p} 0$ *for all i and j* $(i, j = 1,\ldots, k)$, *where* $\xi_{1,ijt}(\theta)$ *is the* (i, j)*th element of* $\Xi_{1t}(\theta)$;

(ii) $\left\{T^{-1}\sum_{t=1}^{T} E[\Xi_{1t}(\theta)] : \theta \in \Theta, T = 1, 2,\ldots\right\}$ *is* $O(1)$ *and continuous on* Θ *uniformly in* T;

(iii) $E[\Xi_{1t}(\theta^o)]$ *is positive definite.*

Proof. By definition, we obtain:

$$\sup_{\theta\in\Theta} ||\Xi_{1t}(\theta)|| = \sup_{\theta\in\Theta} [h_t(\delta, \beta)]^{-1} ||x_t' x_t||,$$

and

$$||x_t x_t'|| = \sqrt{\mathrm{tr}\left((x_t' x_t)'(x_t' x_t)\right)} = \sqrt{\mathrm{tr}\left(x_t' x_t x_t' x_t\right)} = \sqrt{\mathrm{tr}\left((x_t x_t')^2\right)}$$
$$= x_t x_t' = \left\{\sqrt{\mathrm{tr}(x_t x_t')}\right\}^2 = ||x_t||^2.$$

Noting that $h_t(\delta, \beta) \geq \delta_0 > 0$ by Assumption 3, we obtain:

$$E\left[\sup_{\theta\in\Theta} ||\Xi_{1t}(\theta)||\right] = E\left[\sup_{\theta\in\Theta} [h_t(\theta)]^{-1} ||x_t' x_t||\right] \leq KE\left[\sup_{\theta\in\Theta} ||x_t||^2\right] = KE\left[||x_t||^2\right] < \infty. \tag{A.11}$$

The first inequality comes from Assumption 4. Since $1/h_t(\delta, \beta)$ is strict stationary and ergodic by Lemma 5, the uniform law of large numbers (ULLN) for stationary ergodic process (see Theorem A.2.2 of White (1994)) with the result $E\left[\sup_{\theta\in\Theta} ||\Xi_{1t}(\theta)||\right] < \infty$ indicate that:

$$\sup_{\theta\in\Theta} \left|T^{-1} \sum_{t=1}^{T} \xi_{1,ijt}(\theta) - E[\xi_{1,ijt}(\theta)]\right| \xrightarrow{a.s.} 0,$$

for all i and j (i, j = 1,..., k). By the almost sure convergence, we obtain the weak convergence in Lemma 6(i).

By (A.11), $E[\Xi_{1t}(\theta)]$ exists, and it does not depend on t and continuous on Θ by the structure. Thus Lemma 6(ii) holds.

By equation (10), $h_t(\theta^o)$ is independent of x_t. Hence $E[\Xi_{1t}(\theta^o)] = E[1/h_t(\theta^o)]V_x$. Since $E[1/h_t(\theta^o)] > 0$, we obtain Lemma 6(iii) by Assumption 2. □

Lemma 7. *Define*

$$\Xi_{2t}(\theta) = -[h_t(\theta)]^{-1}x_t'[(y_{t-1} - x_{t-1}\beta) \;\cdots\; (y_{t-p} - x_{t-p}\beta)]. \tag{A.12}$$

Under Assumptions 1-4,

(i) $\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T}\xi_{2,ijt}(\theta) - E[\xi_{2,ijt}(\theta)]\right| \xrightarrow{p} 0$ *for all* i (i = 1,..., k) *and* j (j = 1,..., p), *where* $\xi_{2,ijt}(\theta)$ *is the* (i,j)*th element of* $\Xi_{2t}(\theta)$;

(ii) $\left\{T^{-1}\sum_{t=1}^{T} E[\Xi_{2t}(\theta)] : \theta \in \Theta, T = 1, 2,... \right\}$ *is* $O(1)$ *and continuous on* Θ *uniformly in* T.

Proof. By (A.12), we obtain an alternative expression of $\Xi_{2t}(\theta)$ as:

$$\Xi_{2t}(\theta) = -[h_t(\theta)]^{-1}x_t'[(u_{t-1} - x_{t-1}(\beta - \beta^o)) \;\cdots\; (u_{t-p} - x_{t-p}(\beta - \beta^o))].$$

We can write the (i,j)th element of $\Xi_{2t}(\theta)$ as:

$$\xi_{2,ijt}(\theta) = -[h_t(\delta,\beta)]^{-1}x_{it}\,(u_{t-j} - x_{t-j}(\beta - \beta^o))\,.$$

To prove Lemma 7(i), we will show that $E[\sup_{\beta\in\Theta\beta}|\xi_{2,ijt}|]$ is finite. By Assumptions 3 and 4 and $h_t(\theta) \geq \delta_0 > 0$, we obtain:

$$|\xi_{2,ijt}(\theta)| \leq K|x_{it}\,(u_{t-j} - x_{t-j}(\beta - \beta^o))| \leq K\left[|x_{it}u_{t-j}| + \sum_{l=1}^{k}|x_{it}x_{l,t-j}||\beta_l - \beta_l^o|\right].$$

For the upper bound of $|\beta_l - \beta_l^o|$ (l = 1,..., k), we follow the approach of the proof of Theorem 1 of White (1980b). Since β^o is finite, there exists a compact neighborhood of ν of β^o such that $(\beta_l - \beta_l^o)$ is finite. There also exists a finite vector $\tilde{\beta}$ (not necessarily in ν) with element $\tilde{\beta}_l$ such that $|\beta_l - \beta_l^o| \leq |\tilde{\beta}_l - \beta_l^o|$ for all β in ν, so that for all β in ν:

$$\sum_{l=1}^{k}|x_{it}x_{l,t-j}||\beta_l - \beta_l^o| \leq \sum_{l=1}^{k}|x_{it}x_{l,t-j}||\tilde{\beta}_l - \beta_l^o|.$$

Hence we obtain $E[\sup_{\theta\in\Theta}|\xi_{2,ijt}(\theta)|] < \infty$. Since $1/h_t(\delta,\beta)$ is strict stationary and ergodic by Lemma 5, Theorem 3.5.8 of Stout (1974) with the structure (A.12) implies that $\xi_{2,ijt}(\theta)$ is strict stationary and ergodic. The ULLN for stationary ergodic processes (Theorem A.2.2 of White (1994)) with the result $E[\sup_{\beta\in\Theta_\beta}|\xi_{2,ijt}(\theta)|] < \infty$ indicates that:

$$\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T}\xi_{2,ijt}(\theta) - E[\xi_{2,ijt}(\theta)]\right| \xrightarrow{a.s.} 0,$$

for all i (i = 1,..., k) and j (j = 1,..., p). By the almost sure convergence, we obtain the weak convergence in Lemma 7(i). By the proof of Lemma 7(i), $E[\Xi_{2t}(\theta)]$ exists, and it does not depend on t and continuous on Θ by the structure. Thus Lemma 7(ii) holds. □

Lemma 8. *Define*

$$B_T^o = \left[E\left[[h_t(\theta^o)]^{-1}x_t'x_t\right]\right]^{-1} E\left[[h_t(\theta^o)]^{-1}x_t'\lambda_t(\beta^o)\right].$$

Under Assumptions 1-4, B_T^o exists and

$$\hat{B}_T - B_T^o = o_p(1), \tag{A.13}$$

where $\hat{B}_T$ is defined by equation (13).

Proof. Noting that $B_T^o = \left[E[\Xi_{1t}(\theta^o)]\right]^{-1} E[\Xi_{2t}(\theta^o)]$, Lemmas 6 and 7 indicate that $B_T(\theta^o)$ exists. Since $\hat{\beta} - \beta^o \xrightarrow{p} 0$ by Lemma 3 and $\hat{\delta} - \delta^o \xrightarrow{p} 0$ by Proposition 1, Lemmas 6 and 7 satisfy the conditions of Lemma 1, which establishes (A.13). □

Lemma 9. *Define*

$$\Xi_{3t}(\theta) = -\frac{\psi_t(\beta)}{h_t(\theta)}x_t'$$

Under Assumptions 1-4,

(i) $sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^T \xi_{3,it}(\theta) - E[\xi_{3,it}(\theta)]\right| \xrightarrow{p} 0$ for all i (i = 1,..., k), where $\xi_{3,it}$ (θ) is the ith element of $\Xi_{3t}(\theta)$;

(ii) $\left\{T^{-1}\sum_{t=1}^T E[\Xi_{3t}(\theta)] : \theta \in \Theta, T = 1, 2,... \right\}$ is $O(1)$ and continuous on Θ uniformly in T;

(iii) $\frac{1}{\sqrt{T}}\sum_{t=1}^T \Xi_{3t}(\theta^O) = O_p(1)$.

Proof. Assumptions 3 and 4 indicates:

$$|\xi_{3,it}(\theta)| \leq \frac{1}{h_t(\theta)}\left|x_{it}\left(u_t - x_t(\beta - \beta^o)\right)\right| \leq K|x_{it}u_t| + K\sum_{l=1}^k |x_{it}x_{lt}||\beta_l - \beta_l^o|.$$

By discussions similar to the proof of Lemma 7, we obtain $E[\sup_{\theta\in\Theta}|\xi_{3,it}(\theta)|] < \infty$, and we can show that $\xi_{3,it}(\theta)$ is strict stationary ergodic process by Theorem 3.5.8 of Stout (1974). By applying the ULLN for stationary ergodic process (Theorem A.2.2 of White (1994)) with $E[\sup_{\theta\in\Theta}|\xi_{3,it}(\theta)|] < \infty$ indicates that:

$$\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^T \xi_{3,it}(\theta) - E[\xi_{3,it}(\theta)]\right| \xrightarrow{a.s.} 0,$$

for all i (i,= 1,..., k). By the almost sure convergence, we obtain the weak convergence in Lemma 9(i).

Since $E[\sup_{\theta\in\Theta}|\xi_{3,it}(\theta)|] < \infty$, $E[\Xi_{3t}(\theta)]$ exists, and it does not depend on t and continuous on Θ by the structure. Thus Lemma 9(ii) holds.

When $\beta = \beta^o$, conditional on the information set up to $t-1$, we obtain $E(\xi_{3,it}(\theta^o)|\Im_{t-1}) = 0$ and:

$$V(\xi_{3,it}(\theta^o)|\Im_{t-1}) = \frac{E(u_t^2)E(x_{it}^2)}{[h_t(\theta^o)]^2} \leq K\sigma_u^2\{V_x\}_{ii} < \infty,$$

for all i (i,= 1,..., k), by Assumptions 2-4. Hence $V(\xi_{3,it}(\theta))$ is also bounded. By Chebyshev's inequality,

$$P\left(\left|T^{-1/2}\sum_{t=1}^{T}\Xi_{3t}(\theta^o)\right|<\epsilon\right)\geq 1-\frac{V(\xi_{3,it}(\theta))}{\epsilon^2}$$

for any $\epsilon > 0$ and all i (i,= 1,..., k), indicating that Lemma 9(iii) holds. □

Lemma 10. *Define*

$$\Xi_{4t}(\theta)=\frac{\psi_t(\beta)}{[h_t(\theta)]^3}x_t'\kappa_t(\beta) \tag{A.14}$$

Under Assumptions 1-4,

(i) $sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T}\xi_{4,ijt}(\theta)-E[\xi_{4,ijt}(\theta)]\right|\xrightarrow{p}0$ *for all* i (i = 1,..., k) *and* j (j = 1,..., q+1), *where* $\xi_{4,ijt}(\theta)$ *is the* (i,j)*th element of* $\Xi_{4t}(\theta)$;

(ii) $\left\{T^{-1}\sum_{t=1}^{T}E[\Xi_{4t}(\theta)] : \theta\in\Theta, T=1,2,...\right\}$ *is* $O(1)$ *and continuous on* Θ *uniformly in T.*

Proof. We can write the (i,j)th element of $\Xi_{4t}(\theta)$ as:

$$\xi_{4,ijt}(\theta)=\begin{cases}-[h_t(\theta)]^{-3}\psi_t(\beta)x_{it} & \text{for } j=1,\\ -[h_t(\theta)]^{-3}\psi_t(\beta)x_{it}\left(u_{t-j+1}-x_{t-j+1}(\beta-\beta^o)\right)^2 & \text{otherwise,}\end{cases}$$

for i (i = 1,..., k) and j (j = 1,..., q + 1). For j = 1, noting that $h_t(\delta,\beta)\geq\delta_0>0$, we just need to replace $h_t(\delta,\beta)$ by $[h_t(\delta,\beta)]^3$ in the proof of Lemma 9 to obtain the result of Lemma 10.

Hence, we concentrate on the case j = 2,..., q + 1.

By Assumptions 3 and 4 and $h_t(\theta)\geq\delta_0>0$, we obtain:

$$\begin{aligned}
|\xi_{4,ijt}(\theta)| &\leq [h_t(\delta,\beta)]^{-3}|x_{it}\left(u_t-x_t(\beta-\beta^o)\right)\left(u_{t-j+1}-x_{t-j+1}(\beta-\beta^o)\right)^2| \\
&\leq K\Bigg[|u_tu_{t-j}x_{it}|+\left|u_{t-j+1}^2\sum_{l=1}^{k}x_{it}x_{lt}(\beta_l-\beta_l^o)\right| \\
&\quad+2\left|u_tu_{t-j+1}\sum_{l=1}^{k}x_{it}x_{l,t-j+1}(\beta_l-\beta_l^o)\right| \\
&\quad+\left|u_t\sum_{l=1}^{k}\sum_{r=1}^{k}x_{it}x_{l,t-j+1}x_{r,t-j+1}(\beta_l-\beta_l^o)(\beta_r-\beta_r^o)\right| \\
&\quad+2\left|u_{t-j+1}\sum_{l=1}^{k}\sum_{r=1}^{k}x_{it}x_{lt}x_{r,t-j+1}(\beta_l-\beta_l^o)(\beta_r-\beta_r^o)\right| \\
&\quad+\left|\sum_{m=1}^{k}\sum_{l=1}^{k}\sum_{r=1}^{k}x_{it}x_{mt}x_{l,t-j+1}x_{r,t-j+1}(\beta_m-\beta_m^o)(\beta_l-\beta_l^o)(\beta_r-\beta_r^o)\right|\Bigg] \\
&\leq K\Bigg[|u_tu_{t-j}||x_{it}|+|u_{t-j+1}^2|\sum_{l=1}^{k}|x_{it}x_{lt}||\beta_l-\beta_l^o| \\
&\quad+2|u_tu_{t-j+1}|\sum_{l=1}^{k}|x_{it}x_{l,t-j+1}||\beta_l-\beta_l^o| \\
&\quad+|u_t|\sum_{l=1}^{k}\sum_{r=1}^{k}|x_{it}x_{l,t-j+1}x_{r,t-j+1}||\beta_l-\beta_l^o||\beta_r-\beta_r^o|
\end{aligned}$$

$$+ 2|u_{t-j+1}| \sum_{l=1}^{k}\sum_{r=1}^{k} |x_{it}x_{lt}x_{r,t-j+1}||\beta_l - \beta_l^o||\beta_r - \beta_r^o|$$

$$+ \sum_{m=1}^{k}\sum_{l=1}^{k}\sum_{r=1}^{k} |x_{it}x_{mt}x_{l,t-j+1}x_{r,t-j+1}||\beta_m - \beta_m^o||\beta_l - \beta_l^o||\beta_r - \beta_r^o| \Bigg],$$

for all i (i = 1,..., k) and j (j = 2,..., q +1). Since $|\beta_l - \beta_l^o|$ is bounded by the discussion of the proof of Lemma 7,

$$\sup_{\theta\in\Theta} |\xi_{4,ijt}(\theta)| \le K_1|u_t u_{t-j}||x_{it}| + K_2|u_{t-j+1}^2| \sum_{l=1}^{k} |x_{it}x_{lt}| + K_3|u_t u_{t-j+1}| \sum_{l=1}^{k} |x_{it}x_{l,t-j+1}|$$

$$+ K_4|u_t| \sum_{l=1}^{k}\sum_{r=1}^{k} |x_{it}x_{l,t-j+1}x_{r,t-j+1}| + K_5|u_{t-j+1}| \sum_{l=1}^{k}\sum_{r=1}^{k} |x_{it}x_{lt}x_{r,t-j+1}|$$

$$+ K_6 \sum_{m=1}^{k}\sum_{l=1}^{k}\sum_{r=1}^{k} |x_{it}x_{mt}x_{l,t-j+1}x_{r,t-j+1}|,$$

for all i (i = 1,..., k) and j (j = 2,..., q + 1). By Assumption 2, we obtain $E[\sup_{\theta\in\Theta}|\xi_{4,ijt}(\theta)|] < \infty$. Since $h_t(\theta)$, $\psi_t(\beta)$, x_t are strict stationary ergodic processes, Theorem 3.5.8 of Stout (1974) with the structure (A.14) implies that $\xi_{4,ijt}(\theta)$ is strict stationary and ergodic. The ULLN for stationary ergodic processes (Theorem A.2.2 of White (1994)) with the result $E[\sup_{\theta\in\Theta}|\xi_{4,ijt}(\theta)|] < \infty$ indicates that:

$$\sup_{\theta\in\Theta} \left| T^{-1}\sum_{t=1}^{T} \xi_{4,ijt}(\theta) - E[\xi_{4,ijt}(\theta)] \right| \xrightarrow{a.s.} 0,$$

for all i (i = 1,..., k) and j (j = 2,..., q + 1). By the almost sure convergence, we obtain the weak convergence in Lemma 10(i). By the proof of Lemma 10(i), $E[\Xi_{4t}(\theta)]$ exists, and it does not depend on t and continuous on Θ by the structure. Thus Lemma 10(ii) holds. □

Lemma 11. *Define*

$$\Xi_{5t}(\theta) = \frac{-2\psi_t(\beta)}{[h_t(\theta)]^3} x_t' \sum_{l=1}^{q} \delta_l(y_{t-l} - x_{t-l}\beta)x_{t-l} \quad . \tag{A.15}$$

Under Assumption 1-4,

(i) $\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T}\xi_{5,ijt}(\theta) - E[\xi_{5,ijt}(\theta)]\right| \xrightarrow{p} 0$ *for all* i *and* j (i, j = 1,..., k), *where* $\xi_{5,ijt}(\theta)$ *is the* (i,j)*th element of* $\Xi_{5t}(\theta)$;

(ii) $\{T^{-1}\sum_{t=1}^{T} E[\Xi_{5t}(\theta)] : \theta \in \Theta, T = 1, 2, \ldots\}$ *is* $O(1)$ *and continuous on* Θ *uniformly in* T.

Proof. We can write the (i,j)th element of $\Xi_{5t}(\theta)$ as:

$$\xi_{5,ijt}(\theta) = -2[h_t(\theta)]^{-3}\psi_t(\beta)x_{it} \sum_{l=1}^{q} \delta_l(u_{t-l} - x_{t-l}(\beta - \beta^o))x_{j,t-l}$$

for i and j (i, j = 1,..., k). Noting that $h_t(\theta) \ge \delta_0 > 0$, Assumptions 3 and 4, we obtain:

$$|\xi_{5,ijt}(\theta)| \le [h_t(\theta)]^{-3} \left| (u_t - x_t(\beta - \beta^o))\, x_{it} \sum_{l=1}^{q} \delta_l(u_{t-l} - x_{t-l}(\beta - \beta^o))x_{j,t-l} \right|$$

$$\leq K\sum_{l=1}^{q}\delta_l\left|u_tu_{t-l}x_{it}-u_tx_{it}x_{j,t-l}\sum_{r=1}^{k}x_{r,t-l}(\beta_r-\beta_r^o)\right.$$
$$-u_{t-l}x_{j,t-l}\sum_{r=1}^{k}x_{rt}(\beta_r-\beta_r^o)$$
$$\left.+x_{j,t-l}\sum_{r=1}^{k}\sum_{m=1}^{k}x_{rt}x_{m,t-l}(\beta_r-\beta_r^o)(\beta_m-\beta_m^o)\right|$$
$$\leq K\sum_{l=1}^{q}\delta_l\left[|u_tu_{t-l}||x_{it}|+|u_t|\sum_{r=1}^{k}|x_{it}x_{j,t-l}x_{r,t-l}||\beta_r-\beta_r^o|\right.$$
$$+|u_{t-l}|\sum_{r=1}^{k}|x_{j,t-l}x_{rt}||\beta_r-\beta_r^o|$$
$$\left.+\sum_{r=1}^{k}\sum_{m=1}^{k}|x_{j,t-l}x_{rt}x_{m,t-l}||\beta_r-\beta_r^o||\beta_m-\beta_m^o|\right],$$

for all i and j (i, j = 1,..., k). Since $|\beta_l-\beta_l^o|$ is bounded by the discussion of the proof of Lemma 7 and δ_l is bounded by Assumption 4,

$$\sup_{\theta\in\Theta}|\xi_{5,ijt}(\theta)|\leq K_1\sum_{l=1}^{q}\left[|u_tu_{t-l}||x_{it}|+K_2|u_t|\sum_{r=1}^{k}|x_{it}x_{j,t-l}x_{r,t-l}|\right.$$
$$\left.+K_2|u_{t-l}|\sum_{r=1}^{k}|x_{j,t-l}x_{rt}|+K_3\sum_{r=1}^{k}\sum_{m=1}^{k}|x_{j,t-l}x_{rt}x_{m,t-l}|\right],$$

for all i and j (i, j = 1,..., k). By Assumption 2, we obtain $E[\sup_{\theta\in\Theta}|\xi_{5,ijt}(\theta)|]<\infty$. Since $h_t(\theta)$, $\psi_t(\beta)$, and x_t are strict stationary ergodic processes, Theorem 3.5.8 of Stout (1974) with the structure (A.15) implies that $\xi_{5,ijt}(\theta)$ is strict stationary and ergodic. The ULLN for stationary ergodic processes (Theorem A.2.2 of White (1994)) with the result $E[\sup_{\theta\in\Theta}|\xi_{5,ijt}(\theta)|]<\infty$ indicates that:

$$\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T}\xi_{5,ijt}(\theta)-E[\xi_{5,ijt}(\theta)]\right|\xrightarrow{a.s.}0,$$

for all i and j (i, j = 1,..., k). By the almost sure convergence, we obtain the weak convergence in Lemma 11(i). By the proof of Lemma 11(i), $E[\Xi_{5t}(\theta)]$ exists, and it does not depend on t and continuous on Θ by the structure. Thus Lemma 11(ii) holds. □

Lemma 12. *Under Assumptions 1-5,* $B_T^o=0$.

Proof. *Define*

$$\Xi_{6t}^o=\frac{1}{h_t(\theta^o)}x_t'\lambda_t(\beta^o).$$

We can write the (i,j)th element of Ξ_{6t}^o as:

$$\xi_{6,ijt}^o=\frac{1}{h_t(\theta^o)}x_{it}u_{t-j},$$

for all i (i = 1,..., k) and j (j = 1,..., p). By the structure, Ξ_{6t}^o is an odd function of u_{t-j}. Since u_{t-j} has a symmetric distribution by Assumption 5, $E[\xi^o_{6,ijt}|x_t, u_t^{(-j)}]$, where $u_t^{(-j)} = \{u_{t-1},..., u_{t-j+1}, u_{t-j-1},..., u_{t-p}\} \cap \{u_{t-1},..., u_{t-q}\}$, is the integral of an odd function with respect to u_{t-j} from $-\infty$ to ∞, and thus $E[\xi^o_{6,ijt}|x_t, u_t^{(-j)}] = 0$. By the law of iterated expectation, $E[\xi^o_{6,ijt}] = E\left[E[\xi^o_{6,ijt}|x_t, u_t^{(-j)}]\right] = 0$ for all i (i = 1,..., k) and j (j = 1,..., p). Thus we obtain, $B^o_T = \left(E\left[[h_t(\theta^o)]^{-1}x_t' x_t\right]\right)^{-1} E(\Xi^o_{6t}) = 0$. □

Lemma 13. *Define*

$$\Xi^o_{7t} = \frac{1}{h_t(\theta^o)}\left[\lambda_t(\beta^o) - x_t B^o_T\right]' \frac{\partial \psi_t(\beta^o)}{\partial \beta}.$$

Under Assumptions 1-5, $\frac{1}{T}\sum_{t=1}^T \Xi_{7t} = o_p(1)$.

Proof. Noting that $\frac{\partial \psi_t(\beta)}{\partial \beta} = -x_t$ and $B^o_T = 0$ by Lemma 12, we can write the (i, j)th element of Ξ^o_{7t} as:

$$\xi^o_{7,ijt} = -\frac{1}{h_t(\theta^o)} u_{t-i} x_{jt} = \xi_{2,jit}(\theta^o),$$

for all i (i = 1,..., p) and j (j = 1,..., k). By the structure, Ξ^o_{7t} is an odd function of u_{t-i}. Since u_{t-i} has a symmetric distribution by Assumption 5, $E[\xi^o_{7,ijt}|x_t, u_t^{(-i)}]$ is the integral of an odd function with respect to u_{t-i} from $-\infty$ to ∞, and thus $E[\xi^o_{7,ijt}|x_t, u_t^{(-i)}] = 0$. By the law of iterated expectation, $E[\xi^o_{7,ijt}] = E\left[E[\xi^o_{7,ijt}|x_t, u_t^{(-i)}]\right] = 0$ for all i (i = 1,..., p) and j (j = 1,..., k). By Lemma 7 with $E[\xi_{7,ijt}(\theta^o)] = 0$ for all i and j, we obtain $\frac{1}{T}\sum_{t=1}^T \Xi_{7t}(\theta^o)| \xrightarrow{a.s.} 0$, which indicates $\frac{1}{T}\sum_{t=1}^T \Xi_{7t}(\theta^o) = o_p(1)$. □

Lemma 14. *Define*

$$\Xi^o_{8t} = \frac{\psi_t(\beta^o)}{h_t(\theta^o)}\left[\frac{\partial \lambda_t(\beta^o)}{\partial \theta}\right]'.$$

Under Assumptions 1-5, $\frac{1}{T}\sum_{t=1}^T \Xi^o_{8t}(\theta^o) = o_p(1)$.

Proof. Since $\frac{\partial \lambda_t(\beta)}{\partial \delta} = 0$, we can concentrate on the part including

$$\frac{\partial \lambda_t(\beta)}{\partial \beta} = -\left[x'_{t-1} \ \cdots \ x'_{t-p}\right].$$

We can write the (i, j)th element of Ξ^o_{8t} as:

$$\xi^o_{8,ijt} = -\frac{1}{h_t(\theta^o)} u_t x_{i,t-j},$$

for i (i = 1,..., p) and j (j = 1,..., k). With a minor change of the discussion of the proof of Lemma 9, we can show that:

$$\left|\frac{1}{T}\sum_{t=1}^T \xi^o_{8,ijt} - E[\xi^o_{8,ijt}]\right| \xrightarrow{a.s.} 0, \tag{A.16}$$

for all i (i = 1,..., p) and j (j = 1,..., k). Since $E[\xi^0_{8,ijt} \,|\, \mathfrak{I}_{t-1}] = -\frac{x_{i,t-j}}{h_t(\theta^o)} E[u_t \,|\, \mathfrak{I}_{t-1}] = 0$, the law of iterated expectation indicates $E[\xi^0_{8,ijt}] = 0$. Equation (A.16) with $E[\xi^0_{8,ijt}] = 0$ establishes Lemma 14. □

Lemma 15. *Define*

$$\Xi_{9t}^{o} = [\lambda_t(\beta^o) - x_t B_T^o]' \frac{\psi_t(\beta^o)}{[h_t(\theta^o)]^2} \frac{\partial h_t(\theta^o)}{\partial \beta'}. \tag{A.17}$$

Under Assumptions 1-5 and H_0, $\frac{1}{T}\sum_{t=1}^{T} \Xi_{9t}(\theta^o) = o_p(1)$.

Proof. Noting that $B_T^o = 0$ by Lemma 11 and

$$\frac{\partial h_t(\theta^o)}{\partial \beta'} = -2\sum_{l=1}^{q} \delta_l^o u_{t-l} x_{t-l},$$

we can write the (i, j)th element of Ξ_{9t}^o under H_0 as:

$$\xi_{9,ijt}^{o} = -\frac{2e_t e_{t-i}}{[h_t(\theta^o)]^2} \sum_{l=1}^{q} \delta_l^o e_{t-l} x_{j,t-l},$$

for i ($i = 1,..., p$) and j ($j = 1,..., k$). By Assumption 3

$$\sup_{\theta\in\Theta} \left|\xi_{9,ijt}^{o}\right| \le K \left| e_t e_{t-i} \sum_{l=1}^{q} \delta_l^o e_{t-l} x_{j,t-l} \right| \le K \sum_{l=1}^{q} \left| e_t e_{t-i} e_{t-l}\right| \left| x_{j,t-l}\right|,$$

and hence

$$E\left[\sup_{\theta\in\Theta} \left|\xi_{9,ijt}^{o}\right|\right] \le K \sum_{l=1}^{q} E\left| e_t e_{t-i} e_{t-l}\right| E\left| x_{j,t-l}\right| < \infty,$$

by Assumptions 2 and 5. Thus, $E[\xi_{9,ijt}^o]$ exists and it is bounded. Since $h_t(\theta^o)$, e_t, and x_t are strict stationary ergodic processes, Theorem 3.5.8 of Stout (1974) with the structure (A.17) implies that $\xi_{9,ijt}^o$ is also strict stationary ergodic. By the ULLN for stationary ergodic process (Theorem A.2.2 of White (1994)) with $E[\sup_{\theta\in\Theta}|\xi_{9,ijt}^o|] < \infty$,

$$\left| \frac{1}{T}\sum_{t=1}^{T} \xi_{9,ijt}^{o} - E[\xi_{9,ijt}^{o}] \right| \xrightarrow{a.s.} 0,$$

for i ($i = 1,..., p$) and j ($j = 1,..., k$). By the structure, $E[\xi_{9,ijt}^0 \mid \mathfrak{I}_{t-1}] = 0$, and hence the law of iterated expectation indicates $E[\xi_{10,ijt}^0] = 0$. Therefore, we obtain $\frac{1}{T}\sum_{t=1}^{T} \Xi_{9t}(\theta^o)| \xrightarrow{a.s.} 0$. Since the almost sure convergence implies the convergence in probability, which is equivalent to the definition of $o_p(1)$, the result establishes Lemma 15. □

Lemma 16. *Define*

$$\Xi_{10t}^{o} = [\lambda_t(\beta^o) - x_t B_T^o]' \frac{\psi_t(\beta^o)}{[h_t(\theta^o)]^2} \frac{\partial h_t(\theta^o)}{\partial \delta'}. \tag{A.18}$$

Under Assumptions 1-5 and H_0, $\frac{1}{T}\sum_{t=1}^{T} \Xi_{10t}(\theta^o) = o_p(1)$.

Proof. Noting that $B_T^o = 0$ by Lemma 11 and:

$$\frac{\partial h_t(\theta^o)}{\partial \beta'} = \left[1\ u_{t-1}^2\ \cdots\ u_{t-q}^2\right],$$

we can write the (i, j)th element of Ξ_{10t}^o under H_0 as:

$$\xi^o_{10,ijt} = \begin{cases} \frac{e_t e_{t-i}}{[h_t(\theta^o)]^2} & \text{for } j=1, \\ \frac{e_t e_{t-i} e^2_{t-i}}{[h_t(\theta^o)]^2} & \text{for } j=2,\ldots,q+1, \end{cases}$$

for i (i = 1,..., p) and j (j = 1,..., k). By Assumption 3,

$$\sup_{\theta\in\Theta} \left|\xi^o_{10,ijt}\right| \le K\left|e_t e_{t-i}\right|,$$

for j = 1, and:

$$\sup_{\theta\in\Theta} \left|\xi^o_{10,ijt}\right| \le K\left|e_t e_{t-i} e^2_{t-j}\right|,$$

for j = 2,..., q + 1. Since $E|e_t e_{t-i}| < \infty$ and $E|e_t\, e_{t-i}\, e^2_{t-j}| < \infty$ by Assumption 5, $E[\xi^o_{10,ijt}]$ exists and it is bounded. Since $h_t(\theta^o)$ and e_t are strict stationary ergodic processes, Theorem 3.5.8 of Stout (1974) with the structure (A.18) implies that $\xi^o_{10,ijt}$ is also strict stationary and ergodic. By the ULLN for stationary ergodic process (Theorem A.2.2 of White (1994)) with $E[\sup_{\theta\in\Theta}|\xi^o_{10,ijt}|] < \infty$,

$$\left|\frac{1}{T}\sum_{t=1}^{T} \xi^o_{10,ijt} - E[\xi^o_{10,ijt}]\right| \xrightarrow{a.s.} 0,$$

for i (i = 1,..., p) and j (j = 1,..., k). By the structure, $E[\xi^0_{10,ijt} \mid \Im_{t-1}] = 0$, and hence the law of iterated expectation indicates $E[\xi^0_{10,ijt}] = 0$. Therefore, we obtain $\frac{1}{T}\sum_{t=1}^{T} \Xi_{10t}(\theta^o)| \xrightarrow{a.s.} 0$. Since the almost sure convergence implies the convergence in probability, which is equivalent to the definition of $o_p(1)$, the result establishes Lemma 16. □

Lemma 17. *Define*

$$\ddot{\Omega}^o_T = \frac{1}{T}\sum_{t=1}^{T} \Omega^o_t, \tag{A.19}$$

where

$$\Omega^o_t = E\left[\left[\frac{\psi_t(\beta^o)}{h_t(\theta^o)}\right]^2 \lambda_t(\beta^o)'\lambda_t(\beta^o)\middle| \Im_{t-1}\right].$$

Under Assumptions 1-5 and H_0,

(*i*) $\ddot{\Omega}^o_T$ *is positive definite for large T;*

(*ii*) $\left|T^{-1}\sum_{t=1}^{T} \omega^o_{ijt} - E[\omega^o_{ijt}]\right| \xrightarrow{p} 0$ for all i and j (i = 1,..., p),

where ω^o_{ijt} is the (i,j)th element of Ω^o_T.

Proof. Noting that

$$\Omega^o_t = \frac{\sigma^2_u}{[h_t(\theta^o)]^2}\lambda_t(\beta^o)'\lambda_t(\beta^o),$$

we can write the (i,j)th element of Ω^o_t under H_0 as:

$$\omega^o_{ijt} = \frac{\sigma^2_e}{[h_t(\theta^o)]^2} e_{t-i}e_{t-j}$$

for i and j (i = 1,..., p). By Assumption 3

$$\left|\omega_{ijt}^{o}\right| \leq K\left|e_{t-i}e_{t-j}\right|,$$

for all i and j $(i, j = 1,..., p)$. Since $E\left[e_{t-i}\, e_{t-j}\right] < \infty$ by Assumption 5, $E\left[\omega_{ijt}^{o}\right]$ exists and it is bounded. By the structure, $\ddot{\Omega}_{T}^{o}$ is the sample mean of the outer product of random vector $[\sigma_e / h_t(\theta^o)]\lambda_t(\beta^o)$, thus its determinant is non-negative. Since $\lambda_t(\beta^o)$ is linearly independent by Assumption 1, the rank of $\ddot{\Omega}_{T}^{o}$ is p, which guarantees that the inverse of the matrix exists almost surely when $T > p$. Combined with (A.6), we obtain Lemma 17(i).

Since $h_t(\theta^o)$ and e_t are strict stationary ergodic processes, Theorem 3.5.8 of Stout (1974) with the structure (A.19) implies that ω_{ijt}^{o} is strict stationary and ergodic. The uniform law of large numbers (ULLN) for stationary ergodic process (Theorem A.2.2 of White (1994)) with the result $E[|\omega_{ijt}^{o}|] < \infty$ indicates that:

$$\left|T^{-1}\sum_{t=1}^{T}\omega_{ijt}^{o} - E[\omega_{ijt}^{o}]\right| \xrightarrow{a.s.} 0,$$

for all i and j (i, $j = 1,..., p$). By the almost sure convergence, we obtain the weak convergence in Lemma 17(ii). □

Lemma 18. *Define*

$$\ddot{\zeta}_{t}^{o} = \frac{\psi_t(\beta^o)}{h_t(\theta^o)}\lambda_t(\beta^o)'. \tag{A.20}$$

Under Assumptions 1-5,

$$\ddot{\Omega}^{o-1/2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\ddot{\zeta}_{t}^{o} \xrightarrow{d} N(0, I_k),$$

where $\ddot{\Omega}^{o} = E[\Omega_{t}^{o}]$, *where* Ω_{t}^{o} *is stated in Lemma 17.*

Proof. By the definition, $E[\ddot{\zeta}_{t}^{0}|\Im_{t-1}] = 0$ and $V[\ddot{\zeta}_{t}^{0}\,|\,\Im_{t-1}] = \Omega_{t}^{o}$. Since e_t and $h_t(\theta^o)$ are strictly stationary and ergodic, Theorem 3.5.8 of Stout (1974) with the structure (A.21) implies that $\ddot{\zeta}_{t}^{0}$ is strictly stationary ergodic matringale difference under H_0. Since $\ddot{\Omega}^{o}$ is finite and positive definite by Lemma 17, we can define the symmetric positive definite matrix, $\ddot{\Omega}^{o\,-1/2}$, such that $(\ddot{\Omega}^{o\,-1/2})^2 = \ddot{\Omega}^{o-1}$. Lemma 17 implies that $\ddot{\Omega}^{o-1/2}$ is uniformly bounded. By Lemma 3.2 of White (1980a),

$$\left|\left\{\ddot{\Omega}^{o-1/2}\left[\frac{1}{T}\sum_{t=1}^{T}\Omega_{t}^{o}\right]\ddot{\Omega}^{o-1/2}\right\}_{i,j} - \{I_p\}_{i,j}\right| \xrightarrow{a.s.} 0, \tag{A.21}$$

for all i and j $(i, j = 1,..., p)$. For the ith element of $\ddot{\zeta}_{t}^{0}$, $\ddot{\zeta}_{it}^{0}(i = 1,..., p)$, Chebyshev's inequality indicates:

$$P\left(\left|\frac{\ddot{\zeta}_{it}^{o}}{\sqrt{T}}\right| > \epsilon\right) \leq \frac{V(\ddot{\zeta}_{it}^{o})}{T\epsilon^2} = \frac{\ddot{\omega}_{ij}^{o}}{T\epsilon^2} \to 0,$$

as $T \to \infty$, where $\ddot{\omega}_{ij}^{o}$ is the (i, j)th element of $\ddot{\Omega}^{o}$. Hence,

$$\max_{1\leq t\leq T}\left|\frac{\ddot{\zeta}_{it}^{o}}{\sqrt{T}}\right| \xrightarrow{p} 0. \tag{A.22}$$

As equations (A.21) and (A.22) satisfy the regularity conditions for the CLT for the strict stationary ergodic martingale difference (Theorem 24.3 of Davidson (1994)), we obtain

$$\ddot{\Omega}^{o-1/2}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\ddot{\zeta}_t^o \xrightarrow{d} N(0, I_p), \tag{A.23}$$

which establishes Lemma 18. □

Lemma 19. *Define*

$$\Xi_{11t}(\theta) = \frac{[\psi_t(\beta)]^2}{[h_t(\theta)]^2}\lambda_t(\beta)'\lambda_t(\beta). \tag{A.24}$$

Under Assumptions 1-5 and H_0*,*

(i) $\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T}\xi_{11,ijt}(\theta) - E[\xi_{11,ijt}(\theta)]\right| \xrightarrow{p} 0$ *for all i and j* $(i, j = 1,..., p)$, *where* $\xi_{11,ijt}(\theta)$ *is the* (i, j)*th element of* $\Xi_{11t}(\theta)$*;*

(ii) $\left\{T^{-1}\sum_{t=1}^{T}E[\Xi_{11t}(\theta)] : \theta \in \Theta, T = 1, 2,...\right\}$ *is* $O(1)$ *and continuous on* Θ *uniformly in* T.

Proof. We can write the (i, j)th element of $\Xi_{11t}(\theta)$ under H_0 as:

$$\xi_{11,ijt}(\theta) = -[h_t(\theta)]^{-2}[e_t - x_t(\beta-\beta^o)]^2 e_{t-i}e_{t-j},$$

for i and j $(i, j = 1,..., p)$. Noting that $h_t(\theta) \geq \delta_0 > 0$, Assumptions 3 and 4, we obtain:

$$\begin{aligned}
|\xi_{11,ijt}(\theta)| &\leq [h_t(\delta,\beta)]^{-3}\left|[e_t - x_t(\beta-\beta^o)]^2 e_{t-i}e_{t-j}\right| \\
&\leq K\sum_{l=1}^{q}\Bigg|e_t^2 e_{t-i}e_{t-j} - 2e_{t-i}e_{t-j}\sum_{r=1}^{k}x_{rt}(\beta_r-\beta_r^o) \\
&\qquad + e_{t-i}e_{t-j}\sum_{r=1}^{k}\sum_{l=1}^{k}x_{rt}x_{lr}(\beta_r-\beta_r^o)(\beta_l-\beta_l^o)\Bigg| \\
&\leq K\sum_{l=1}^{q}\Bigg[\left|e_t^2 e_{t-i}e_{t-j}\right| + 2\left|e_t e_{t-i}e_{t-j}\right|\sum_{r=1}^{k}|x_{rt}|(\beta_r-\beta_r^o) \\
&\qquad + |e_{t-i}e_{t-j}|\sum_{r=1}^{k}\sum_{l=1}^{k}|x_{rt}x_{lt}|(\beta_r-\beta_r^o)(\beta_l-\beta_l^o)\Bigg],
\end{aligned}$$

for all i and j $(i, j = 1,..., k)$. Since $|\beta_l - \beta_l^o|$ is bounded by the discussion of the proof of Lemma 7,

$$\sup_{\theta\in\Theta}|\xi_{11,ijt}(\theta)| \leq K_1\sum_{l=1}^{q}\left[\left|e_t^2 e_{t-i}e_{t-j}\right| + K_2\left|e_t e_{t-i}e_{t-j}\right|\sum_{r=1}^{k}|x_{rt}| + K_3|e_{t-i}e_{t-j}|\sum_{r=1}^{k}\sum_{l=1}^{k}|x_{rt}x_{lt}|\right],$$

for all i and j $(i, j = 1,..., p)$. By Assumptions 2 and 5, we obtain $E[\sup_{\theta\in\Theta}|\xi_{11,ijt}(\theta)|] < \infty$. Since $h_t(\theta)$, $\psi_t(\beta)$, and x_t are strict stationary ergodic processes, Theorem 3.5.8 of Stout (1974) with the structure (A.15) implies that $\xi_{11,ijt}(\theta)$ is strict stationary and ergodic. The ULLN for stationary ergodic processes (Theorem A.2.2 of White (1994)) with the result $E[\sup_{\theta\in\Theta}|\xi_{11,ijt}(\theta)|] < \infty$ indicates that:

$$\sup_{\theta\in\Theta}\left|T^{-1}\sum_{t=1}^{T}\xi_{11,ijt}(\theta)-E[\xi_{11,ijt}(\theta)]\right|\xrightarrow{a.s.}0,$$

for all i and j ($i, j = 1,\ldots, p$). By the almost sure convergence, we obtain the weak convergence in Lemma 19(i). By the proof of Lemma 19(i), $E[\Xi_{11t}(\theta)]$ exists, and it does not depend on t and continuous on Θ by the structure. Thus Lemma 19(ii) holds. □

Proof of Proposition 2 Noting that $\hat{B}_T - B_T^o = o_p(1)$ by Lemma 8, we rewrite (15) as:

$$\ddot{\zeta}_T=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\hat{\beta})}{h_t(\hat{\theta})}\left[\lambda_t(\hat{\beta})-x_tB_T^o\right]'-\left(\hat{B}_T-B_T^o\right)'\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\hat{\beta})}{h_t(\hat{\theta})}x_t'.$$

We first consider the second term excluding $\hat{B}_T - B_T^o$. Noting that $\sqrt{T}(\hat{\theta}-\theta^o) = O_p(1)$ by Lemma 3 and Proposition 1, a standard mean value expansion about θ^o and Lemma 1 produce:

$$\begin{aligned}\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\hat{\beta})}{h_t(\hat{\theta})}x_t' =&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\beta^o)}{h_t(\theta^o)}x_t'+\frac{1}{T}\sum_{t=1}^{T}\left\{\frac{1}{h_t(\theta^o)}x_t'\frac{\partial\psi_t(\beta^o)}{\partial\beta}\right\}\sqrt{T}(\hat{\beta}-\beta^o)\\ &-\frac{1}{T}\sum_{t=1}^{T}\frac{\psi_t(\beta^o)}{[h_t(\theta^o)]^2}x_t'\frac{\partial h_t(\theta^o)}{\partial\theta'}\sqrt{T}(\hat{\theta}-\theta^o)+o_p(1)\end{aligned}\qquad\text{(A.25)}$$

For the right-hand-side of (A.25), the first term is $O_p(1)$ by Lemma 9. Since $\frac{\partial\psi_t(\beta)}{\partial\beta} = -x_t'$, the second term is $[-T^{-1}\sum_{t=1}^{T}\Xi_{1t}]T^{-1/2}(\hat{\beta}-\beta^o)$, which is $O_p(1)$ by Lemmas 3 and 6. Since

$$\frac{\partial h_t(\theta)}{\partial\theta'}=\left[\frac{\partial h_t(\theta)}{\partial\delta'}\;\;\frac{\partial h_t(\theta)}{\partial\beta'}\right]=\left[\kappa_t(\beta)\;\;(-2)\sum_{i=1}^{q}\delta_i(y_{t-1}-x_{t-i}\beta)x_{t-i}\right],$$

the third term of the right-hand-side of (A.25) is

$$-\left[\frac{1}{T}\sum_{t=1}^{T}[\Xi_{4t}(\theta)\;\;\Xi_{5t}(\theta)]\right]\sqrt{T}(\hat{\theta}-\theta^o),$$

which is $O_p(1)$ by Lemmas 3, 10 and 11 and Proposition 1. Therefore

$$\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\hat{\beta})}{h_t(\hat{\theta})}x_t'=O_p(1).$$

Accompanied by $\hat{B}_T - B_T^o = o_p(1)$, this results show that:

$$\ddot{\zeta}_T=\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\hat{\beta})}{h_t(\hat{\theta})}\left[\lambda_t(\hat{\beta})-x_tB_T^o\right]'+o_p(1).$$

Noting that $\sqrt{T}(\hat{\theta}-\theta^o) = O_p(1)$ by Lemma 3 and Proposition 1, a mean value expansion about θ^o and Lemma 1 produce:

$$\begin{aligned}\ddot{\zeta}_T=&\frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\beta^o)}{h_t(\theta^o)}[\lambda_t(\beta^o)-x_tB_T^o]'\\ &+\frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{h_t(\theta^o)}[\lambda_t(\beta^o)-x_tB_T^o]'\frac{\partial\psi_t(\beta^o)}{\partial\beta}\right]\sqrt{T}(\hat{\beta}-\beta^o)\end{aligned}\qquad\text{(A.26)}$$

$$- \frac{1}{T}\sum_{t=1}^{T}\left[\frac{\psi_t(\beta^o)}{h_t(\theta^o)}\left[\frac{\partial\lambda_t(\theta^o)}{\partial\theta}\right]' - [\lambda_t(\beta^o) - x_t B_T^o]' \frac{\psi_t(\beta^o)}{[h_t(\theta^o)]^2}\frac{\partial h_t(\theta^o)}{\partial\theta'}\right]$$
$$\times \sqrt{T}(\hat{\theta} - \theta^o) + o_p(1).$$

For the second term of the right hand side of (A.26),

$$\sqrt{T}(\hat{\beta} - \beta^o) = O_p(1), \quad \frac{1}{T}\sum_{t=1}^{T}\left[\frac{1}{h_t(\theta^o)}[\lambda_t(\beta^o) - x_t B_T^o]' \frac{\partial\psi_t(\beta^o)}{\partial\beta}\right] = o_p(1),$$

by Lemmas 3 and 13, respectively. Hence the second term of the right hand side of (A.26) is $o_p(1)$. For the third term of the right hand side of (A.26), Lemmas 14-16 indicate that

$$\frac{1}{T}\sum_{t=1}^{T}\left[\frac{\psi_t(\beta^o)}{h_t(\theta^o)}\left[\frac{\partial\lambda_t(\theta^o)}{\partial\theta}\right]' - [\lambda_t(\beta^o) - x_t B_T^o]' \frac{\psi_t(\beta^o)}{[h_t(\theta^o)]^2}\frac{\partial h_t(\theta^o)}{\partial\theta'}\right] = o_p(1).$$

As $\sqrt{T}\,(\hat{\theta} - \theta^o) = O_p(1)$ by Lemma 3 and Proposition 1, the third term of the right hand side of (A.26) is $o_p(1)$. With $B_T^o = 0$ by Lemma 12,

$$\ddot{\zeta}_T = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\frac{\psi_t(\beta^o)}{h_t(\theta^o)}\lambda_t(\beta^o)' + o_p(1) = \frac{1}{\sqrt{T}}\sum_{t=1}^{T}\zeta_t^o + o_p(1),$$

where ζ_t^o is stated in Lemma 18. By Lemma 17, the covariance matrix of $\ddot{\zeta}_T$ is positive definite for large T. Moreover, $\ddot{\Omega}^{o-1/2}\ddot{\zeta}_T \xrightarrow{d} N(0, I_p)$ under H_0 by Lemma 18. Thus, $\ddot{\zeta}_T'\ddot{\Omega}^{o-1}\ddot{\zeta}_T \xrightarrow{d} \chi^2(p)$ under H_0. Applying Lemma 1 with $\sqrt{T}(\hat{\theta} - \theta^o) = O_p(1)$, which is obtained by Lemma 3 and Proposition 1, Lemma 19 ensures that $\ddot{\Omega}_T$ is a consistent estimator of $\ddot{\Omega}^o$. Therefore, $\ddot{\zeta}_T'\ddot{\Omega}_T^{-1}\ddot{\zeta}_T \xrightarrow{d} \chi^2(p)$ under H_0, which establishes Proposition 2. □