Weak Convergence for Variational Inequalities with Inertial-Type Method

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Abstract

Weak convergence of inertial iterative method for solving variational inequalities is the focus of this paper. The cost function is assumed to be non-Lipschitz and monotone. We propose a projection-type method with inertial terms and give weak convergence analysis under appropriate conditions. Some test results are performed and compared with relevant methods in the literature to show the efficiency and advantages given by our proposed methods.

1 Introduction

Suppose C is a nonempty, closed and convex subset of a real Hilbert space H and $F : C \to H$ a continuous mapping. The variational inequality problem (for short, $\mathrm{VI}(F, C)$) is defined as: find $x \in C$ such that

$$\langle F(x), y - x\rangle \geq 0, \quad \forall y \in C. \tag{1}$$

We shall denote by SOL the solution set of $\mathrm{VI}(F, C)$ in (1). Various applications of variational inequality can be found in [7, 8, 23–25, 33–35, 43].

Projection-type method for solving $\mathrm{VI}(F, C)$ (1) have been considered severally in the literature (see, for example, [16–18,22,29,40–42,45,48,56]). Several other related methods to extragradient method and (2) for solving $\mathrm{VI}(F, C)$ (1) in real Hilbert spaces when F is monotone and L-Lipschitz-continuous mapping have been studied in the literature (see, for example, [16–18, 22, 29, 36, 40–42, 45, 56]). Some of these methods involve computing projection onto the feasible set C twice per iteration and this can affect the efficiency of the methods.

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In [19], Censor et al. introduced the subgradient extragradient method: $x_1 \in H$,

$$\begin{cases} y_n = P_C(x_n - \lambda F(x_n)), \\ T_n := \{w \in H : \langle x_n - \lambda F(x_n) - y_n, w - y_n \rangle \le 0\}, \\ x_{n+1} = P_{T_n}(x_n - \lambda F(y_n)) \end{cases} \tag{2}$$

and gave weak convergence result when F is monotone and L-Lipschitz-continuous mapping where $\lambda \in (0, \frac{1}{L})$.

In order to accelerate the convergence of subgradient extragradient method (2) and using the idea of in [2–6, 10, 12, 13, 20, 37, 38, 46, 47], Thong and Hieu [55] introduced the following inertial subgradient extragradient method: $x_0, x_1 \in H$,

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F(w_n)), \\ T_n := \{w \in H : \langle w_n - \lambda F(w_n) - y_n, w - y_n \rangle \le 0\}, \\ x_{n+1} = P_{T_n}(w_n - \lambda F(y_n)) \end{cases} \tag{3}$$

and proved that $\{x_n\}$ generated by (3) converges weakly to a solution of VI(F, C) (1) when F is monotone and L-Lipschitz-continuous mapping F where $0 < \lambda L \le \frac{\frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2 - \delta}{\frac{1}{2} - \alpha + \frac{1}{2}\alpha^2}$ for some $0 < \delta < \frac{1}{2} - 2\alpha - \frac{1}{2}\alpha^2$ and $\{\alpha_n\}$ is a non-decreasing sequence with $0 \le \alpha_n \le \alpha < \sqrt{5} - 2$.

The step-sizes in above methods (2) and (3) are bounded by the inverse of the Lipschitz constant and this is quite inefficient, since in most applications a global Lipschitz constant (if it indeed exists at all) of F cannot be accurately estimated, and is usually overestimated. This leads to too small step-sizes, which, of course, is not practical. Therefore, algorithms (2) and (3) are not applicable in most cases of interest. This can be overcome by using an Armijo type line search procedure (see [33, 43, 53]).

We provide a simple example of a variational inequality problem where the method (2) proposed in [19] and method (3) proposed in [55] cannot be applied.

Example 1.1. Suppose $F : [0, \infty) \to \mathbb{R}$ is defined by $F(x) := e^x, \ x \in [0, \infty)$. It is easy to see that F is not Lipschitz continuous on $[0, \infty)$. By the mean value theorem, one has for an arbitrary $r > 0$,

$$|F(x) - F(y)| \le e^r |x - y|$$

with $|x|, |y| \le r$. Hence, F is uniformly continuous on bounded subsets of $C := [0, \infty)$. Consequently, one can easily see that F is monotone on $[0, \infty)$ since

$$\langle F(x) - F(y), x - y \rangle = (F(x) - F(y))(x - y) \ge 0, \quad \forall x, y \in [0, \infty).$$

Finally, SOL of VI(F, C) is nonempty since $0 \in$ SOL.

Motivated by Example 1.1, it would be of interest to propose an iterative method for solving VI(F, C) (1) for which the underline cost function F is uniformly continuous on bounded subsets of C but not Lipschitz continuous on C.

Our interest in this paper is to obtain weak convergence results using inertial projection-type algorithm for VI(F, C) (1) when the underline operator F is monotone and uniformly continuous. We do not assume the cost function to be Lipschitz continuous as assumed in [18, 19, 22, 40, 41, 55]. Our proposed method is much more practical and outperforms the methods (2) and (3) numerically.

We organize the paper as follows: Basic definitions and results are given in Section 2 and the proposed method is introduced in Section 3. We give weak convergence analysis of the proposed method in Section 4 and give some numerical comparisons of our method with methods (2) and (3) in Section 5. Finally, we some concluding remarks in Section 6.

2 Preliminaries

Suppose we take H as a real Hilbert space and $X \subseteq H$ be a nonempty subset.

Definition 2.1. *A mapping $F : X \to H$ is called*

(a) monotone *on X if $\langle F(x) - F(y), x - y\rangle \geq 0$ for all $x, y \in X$;*

(b) Lipschitz continuous *on X if there exists a constant $L > 0$ such that*

$$\|F(x) - F(y)\| \leq L\|x - y\|, \ \forall x, y \in X.$$

(c) sequentially weakly continuous *if for each sequence $\{x_n\}$ we have: $\{x_n\}$ converges weakly to x implies $\{F(x_n)\}$ converges weakly to $F(x)$.*

Given any point $u \in H$, there exists a unique point $P_C u \in C$ (see, e.g., [9]) such that

$$\|u - P_C u\| \leq \|u - y\|, \ \forall y \in C.$$

This P_C is called the *metric projection* of H onto C. It is known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in H. \tag{4}$$

In particular, we get from (4) that

$$\langle x - y, x - P_C y\rangle \geq \|x - P_C y\|^2, \quad \forall x \in C, y \in H. \tag{5}$$

Another property of $P_C x$ is :

$$P_C x \in C \quad \text{and} \quad \langle x - P_C x, P_C x - y\rangle \geq 0, \ \forall y \in C. \tag{6}$$

More details on P_C can be found, for example, in Section 3 of [26].

The following results are needed in the next section.

Lemma 2.2. *The following statements hold in H:*

(a) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$ *for all* $x, y \in H$*;*

(b) $2\langle x-y, x-z\rangle = \|x-y\|^2 + \|x-z\|^2 - \|y-z\|^2$ *for all* $x, y, z \in H$*;*

(c) $\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x-y\|^2$ *for all* $x, y \in H$ *and* $\alpha \in \mathbb{R}$.

Lemma 2.3. *(see [1, Lem. 3]) Let* $\{\psi_n\}$, $\{\delta_n\}$ *and* $\{\alpha_n\}$ *be the sequences in* $[0, +\infty)$ *such that* $\psi_{n+1} \leq \psi_n + \alpha_n(\psi_n - \psi_{n-1}) + \delta_n$ *for all* $n \geq 1$, $\sum_{n=1}^{\infty} \delta_n < +\infty$ *and there exists a real number* α *with* $0 \leq \alpha_n \leq \alpha < 1$ *for all* $n \geq 1$*. Then the following hold:*
(i) $\sum_{n\geq 1}[\psi_n - \psi_{n-1}]_+ < +\infty$*, where* $[t]_+ = \max\{t, 0\}$*;*
(ii) there exists $\psi^* \in [0, +\infty)$ *such that* $\lim_{n\to+\infty} \psi_n = \psi^*$.

Lemma 2.4. *(see [9, Lem. 2.39]) Let* C *be a nonempty set of* H *and* $\{x_n\}$ *be a sequence in* H *such that the following two conditions hold:*
(i) for any $x \in C$, $\lim_{n\to\infty} \|x_n - x\|$ *exists;*
(ii) every sequential weak cluster point of $\{x_n\}$ *is in* C.
Then $\{x_n\}$ *converges weakly to a point in* C.

Lemma 2.5. *([28]) Let* C *be a nonempty closed and convex subset of* H*. Let* h *be a real-valued function on* H *and define* $K := \{x : h(x) \leq 0\}$*. If* K *is nonempty and* h *is Lipschitz continuous on* C *with modulus* $\theta > 0$*, then*

$$\operatorname{dist}(x, K) \geq \theta^{-1} \max\{h(x), 0\},\ \forall x \in C,$$

where $\operatorname{dist}(x, K)$ *denotes the distance function from* x *to* K.

Lemma 2.6. *Let* C *be a nonempty closed and convex subset of* H, $y := P_C(x)$ *and* $x^* \in C$*. Then*

$$\|y - x^*\|^2 \leq \|x - x^*\|^2 - \|x - y\|^2. \tag{7}$$

Lemma 2.7. *([31, Prop. 2.11], [30, Prop. 4]) Let* H_1 *and* H_2 *be two real Hilbert spaces. Suppose* $F : H_1 \to H_2$ *is uniformly continuous on bounded subsets of* H_1 *and* M *is a bounded subset of* H_1*. Then* $F(M)$ *is bounded.*

Lemma 2.8. *([54, Lem. 7.1.7]) Let* C *be a nonempty, closed, and convex subset of* H*. Let* $F : C \to H$ *be a continuous, monotone mapping and* $z \in C$*. Then*

$$z \in \mathrm{SOL} \iff \langle F(x), x - z\rangle \geq 0 \quad \textit{for all } x \in C.$$

3 Proposed Method

We give some assumptions on the feasible set C, the cost fucntion F and the iterative parameter $\{\alpha_n\}$ below.

Assumption 3.1. Suppose that the following hold:

(a) The feasible set C is a nonempty closed affine subset of the real Hilbert space H.

(b) $F : C \to H$ is monotone and uniformly continuous on bounded subsets of H.

(c) The solution set SOL of VI(F, C) is nonempty.

Assumption 3.2. Suppose the real sequence $\{\alpha_n\}$ satisfy the following condition:

- $\{\alpha_n\} \subset (0, 1)$ with $0 \le \alpha_n \le \alpha_{n+1} \le \alpha < \frac{1}{3}$ for all n.

Suppose we define

$$r(x) := x - P_C(x - F(x))$$

as the residual equation. Then if $y = x - F(x)$ in (5), we obtain

$$\langle F(x), r(x)\rangle \ge \|r(x)\|^2, \ \forall x \in C. \tag{8}$$

We next give our proposed inertial projection-type method.

Algorithm 1 Inertial Projection Method

1: Choose sequence $\{\alpha_n\}$ and $\sigma \in (0, 1)$ such that the conditions from Assumption 3.2 hold, and take $\gamma \in (0, 1)$. Let $x_0 = x_1 \in H$ be a given starting point. Set $n := 1$.

2: Set

$$w_n := x_n + \alpha_n(x_n - x_{n-1}).$$

Compute $z_n := P_C(w_n - F(w_n))$. If $r(w_n) = w_n - z_n = 0$: STOP.

3: Compute $y_n = w_n - \gamma^{m_n} r(w_n)$, where m_n is the smallest nonnegative integer satisfying

$$\langle F(y_n), r(w_n)\rangle \ge \frac{\sigma}{2}\|r(w_n)\|^2. \tag{9}$$

Set $\eta_n := \gamma^{m_n}$.

4: Compute

$$x_{n+1} = P_{C_n}(w_n), \tag{10}$$

where $C_n = \{x : h_n(x) \le 0\}$ and

$$h_n(x) := \langle F(y_n), x - y_n\rangle. \tag{11}$$

5: Set $n \leftarrow n + 1$ and **goto 2**.

If $r(w_n) = 0$, then w_n is a solution of VI(F, C) (1). In the analysis we assume that $r(w_n) \ne 0$ for infinitely many iterations, so that Algorithm 1 generates an infinite sequence satisfying $r(w_n) \ne 0$ for all $n \in \mathbb{N}$.

Remark 3.3. (a) Our proposed Algorithm 1 requires, at each iteration, only one projection onto the feasible set C and another projection onto the half-space C_n (which has a closed form solution, [15]) and this is numerically less expensive than the twice computation of projection onto C per iteration in extragradient method [36].

(b) As we have mentioned before, Algorithm 1 is much more applicable than (2) and (3) because the Lipschtz constant of the cost function F is not needed during implementations. ♢

Lemma 3.4. *Let the function h_n be defined by* (11)*. Then*

$$h_n(w_n) \geq \frac{\sigma\eta_n}{2}\|w_n - z_n\|^2.$$

In particular, if $w_n \neq z_n$, then $h_n(w_n) > 0$. If $x^* \in SOL$, then $h_n(x^*) \leq 0$.

Proof. Since $y_n = w_n - \eta_n(w_n - z_n)$, using (9) we have

$$\begin{aligned} h_n(w_n) &= \langle F(y_n), w_n - y_n\rangle \\ &= \eta_n\langle F(y_n), w_n - z_n\rangle \geq \eta_n\frac{\sigma}{2}\|w_n - z_n\|^2 \geq 0. \end{aligned}$$

If $w_n \neq z_n$, then $h_n(w_n) > 0$. Furthermore, suppose $x^* \in$ SOL. Then by Lemma 2.8 we have $\langle F(x), x - x^*\rangle \geq 0 \quad$ for all $x \in C$. In particular, $\langle F(y_n), y_n - x^*\rangle \geq 0$ and hence $h_n(x^*) \leq 0$. □

4 Convergence Analysis

Let us give weak convergence analysis of our proposed Algorithm 1 in this section.

Lemma 4.1. *Let $\{x_n\}$ be generated by Algorithm 1. Then under Assumptions 3.1 and 3.2, we have that*
(i) *$\{x_n\}$ is bounded, and*
(ii) $\lim_{n\to\infty}\|x_{n+1} - w_n\| = 0$.

Proof. Let $x^* \in$ SOL. By Lemma 2.6 we get (since $x^* \in C_n$) that

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_{C_n}(w_n) - x^*\|^2 \leq \|w_n - x^*\|^2 - \|x_{n+1} - w_n\|^2 \\ &= \|w_n - x^*\|^2 - \text{dist}^2(w_n, C_n). \end{aligned} \tag{12}$$

Now, using Lemma 2.2 (c), we have

$$\begin{aligned} \|w_n - x^*\|^2 &= \|(1+\alpha_n)(x_n - x^*) - \alpha_n(x_{n-1} - x^*)\|^2 \\ &= (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\| \\ &\quad +\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{13}$$

Also,

$$\begin{aligned} \|x_{n+1} - w_n\|^2 &= \|x_{n+1} - (x_n + \alpha_n(x_n - x_{n-1}))\|^2 \\ &= \|x_{n+1} - x_n\|^2 + \alpha_k^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\ &\geq \|x_{n+1} - x_n\|^2 + \alpha_k^2\|x_n - x_{n-1}\|^2 - 2\alpha_n\|x_{n+1} - x_n\|\|x_n - x_{n-1}\| \\ &\geq \|x_{n+1} - x_n\|^2 + \alpha_k^2\|x_n - x_{n-1}\|^2 - \alpha_n\|x_{n+1} - x_n\|^2 \\ &\quad -\alpha_n\|x_n - x_{n-1}\|^2 \\ &= (1-\alpha_n)\|x_{n+1} - x_n\|^2 + (\alpha_n^2 - \alpha_n)\|x_n - x_{n-1}\|^2. \end{aligned} \tag{14}$$

Combining (12), (13) and (14), we get

$$\|x_{n+1} - x^*\|^2 \;\leq\; (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2$$

$$
\begin{aligned}
&\quad +\alpha_n(1+\alpha_n)\|x_n - x_{n-1}\|^2 - (1-\alpha_n)\|x_{n+1} - x_n\|^2 \\
&\quad -(\alpha_n^2 - \alpha_n)\|x_n - x_{n-1}\|^2 \\
&= (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 \\
&\quad -(1-\alpha_n)\|x_{n+1} - x_n\|^2 + (\alpha_n(1+\alpha_n) - (\alpha_n^2 - \alpha_n))\|x_n - x_{n-1}\|^2 \\
&= (1+\alpha_n)\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 \\
&\quad -(1-\alpha_n)\|x_{n+1} - x_n\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2. \qquad (15)
\end{aligned}
$$

Using the fact that $\alpha_n \leq \alpha_{n+1}$, we obtain from (15) that

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq\; & (1+\alpha_{n+1})\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 \\
& -(1-\alpha_n)\|x_{n+1} - x_n\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2. \qquad (16)
\end{aligned}
$$

By (16), we get

$$
\begin{aligned}
&\|x_{n+1} - x^*\|^2 - \alpha_{n+1}\|x_n - x^*\|^2 + 2\alpha_{n+1}\|x_{n+1} - x_n\|^2 \leq \|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 \\
&+2\alpha_n\|x_n - x_{n-1}\|^2 + 2\alpha_{n+1}\|x_{n+1} - x_n\|^2 - (1-\alpha_n)\|x_{n+1} - x_n\|^2 \\
= \;&\|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2 + (2\alpha_{n+1} - 1 + \alpha_n)\|x_{n+1} - x_n\|^2.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 \leq\; & \|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
& +(2\alpha_{n+1} - 1 + \alpha_n)\|x_{n+1} - x_n\|^2. \qquad (17)
\end{aligned}
$$

Let us define

$$
\Gamma_n := \|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2.
$$

Then we have from (17) that

$$
\Gamma_{n+1} - \Gamma_n \leq (2\alpha_{n+1} - 1 + \alpha_n)\|x_{n+1} - x_n\|^2. \qquad (18)
$$

Since $0 \leq \alpha_n \leq \alpha_{n+1} \leq \alpha < \frac{1}{3}$, we get $-2\alpha_{n+1} \geq -2\alpha$ and $-\alpha_n \geq -\alpha$. This implies that $-(2\alpha_{n+1} - 1 + \alpha_n) = -2\alpha_{n+1} + 1 - \alpha_n \geq -2\alpha + 1 - \alpha \geq 1 - 3\alpha > 0$ since $\alpha < \frac{1}{3}$. Now, let us define $\sigma := 1 - 3\alpha$. Then

$$
2\alpha_{n+1} - 1 + \alpha_n \leq -\sigma. \qquad (19)
$$

Putting (19) into (18), we have

$$
\Gamma_{n+1} - \Gamma_n \leq -\sigma\|x_{n+1} - x_n\|^2. \qquad (20)
$$

From (20), we see that $\{\Gamma_n\}$ is monotone nonincreasing. Furthermore,

$$
\begin{aligned}
\Gamma_n &= \|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2 + 2\alpha_n\|x_n - x_{n-1}\|^2 \\
&\geq \|x_n - x^*\|^2 - \alpha_n\|x_{n-1} - x^*\|^2. \qquad (21)
\end{aligned}
$$

So,

$$
\|x_n - x^*\|^2 \leq \alpha_n\|x_{n-1} - x^*\|^2 + \Gamma_n
$$

$$
\begin{aligned}
&\le \alpha\|x_{n-1} - x^*\|^2 + \Gamma_1 \\
&\vdots \\
&\le \alpha^k\|x_0 - x^*\|^2 + (1 + \alpha + \alpha^2 + \ldots + \alpha^{k-1})\Gamma_1 \\
&= \alpha^k\|x_0 - x^*\|^2 + \frac{\Gamma_1}{1-\alpha}. \qquad (22)
\end{aligned}
$$

From (22), we can infer that $\{x_n\}$ is bounded. Using the definition of Γ_n, we have

$$
\begin{aligned}
\Gamma_{n+1} &= \|x_{n+1} - x^*\|^2 - \alpha_{n+1}\|x_n - x^*\|^2 + 2\alpha_{n+1}\|x_{n+1} - x_n\|^2 \\
&\ge -\alpha_{n+1}\|x_n - x^*\|^2. \qquad (23)
\end{aligned}
$$

Using (22) in (23), we get

$$
\begin{aligned}
-\Gamma_{n+1} &\le -\alpha_{n+1}\|x_n - x^*\|^2 \le \alpha\|x_n - x^*\|^2 \\
&\le \alpha^{k+1}\|x_0 - x^*\|^2 + \frac{\alpha\Gamma_1}{1-\alpha}. \qquad (24)
\end{aligned}
$$

From (20), we get

$$\sigma\|x_{n+1} - x_n\|^2 \le \Gamma_n - \Gamma_{n+1}$$

and so

$$
\begin{aligned}
\sigma\sum_{j=1}^{n}\|x_{j+1} - x_j\|^2 &\le \Gamma_1 - \Gamma_{n+1} \\
&\le \Gamma_1 + \alpha^{n+1}\|x_0 - x^*\|^2 + \frac{\alpha\Gamma_1}{1-\alpha} \\
&\le \alpha^{n+1}\|x_0 - x^*\|^2 + \frac{\Gamma_1}{1-\alpha} \\
&\le \|x_0 - x^*\|^2 + \frac{\Gamma_1}{1-\alpha}.
\end{aligned}
$$

Therefore, since $x_0 = x_1$, we get

$$
\begin{aligned}
\sum_{k=1}^{\infty}\|x_{n+1} - x_n\|^2 &\le \frac{1}{\sigma}\Big(\|x_0 - x^*\|^2 + \frac{\Gamma_1}{1-\alpha}\Big) \\
&= \frac{1}{\sigma}\|x_0 - x^*\|^2 + \frac{1-\alpha_1}{1-\alpha}\|x_0 - x^*\|^2 \\
&= \Big(\frac{1}{1-3\alpha} + \frac{1-\alpha_1}{1-\alpha}\Big)\|x_0 - x^*\|^2 < \infty.
\end{aligned}
$$

Observe that

$$
\begin{aligned}
\|x_{n+1} - w_n\| &= \|x_{n+1} - x_n - \alpha_n(x_n - x_{n-1})\| \\
&\le \|x_{n+1} - x_n\| + \alpha_n\|x_n - x_{n-1}\| \\
&\le \|x_{n+1} - x_n\| + \alpha\|x_n - x_{n-1}\|. \qquad (25)
\end{aligned}
$$

Using (25), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - w_n\| = 0. \qquad (26)$$

□

Lemma 4.2. *Let $\{x_n\}$ generated by Algorithm 1 above and Assumptions 3.1 and 3.2 hold. Then*

(a) $\lim_{n\to\infty} \eta_n \|w_n - z_n\|^2 = 0$;

(b) $\lim_{n\to\infty} \|w_n - z_n\| = 0$.

Proof. Let $x^* \in$ SOL. Since F is uniformly continuous on bounded subsets of X, then $\{F(x_n)\}, \{z_n\}, \{w_n\}$ and $\{F(y_n)\}$ are bounded. In particular, there exists $M > 0$ such that $\|F(y_n)\| \leq M$ for all $n \in \mathbb{N}$. Combining Lemma 2.5 and Lemma 3.4, we get

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|P_{C_n}(w_n) - x^*\|^2 \leq \|w_n - x^*\|^2 - \|x_{n+1} - w_n\|^2 \\
&= \|w_n - x^*\|^2 - \text{dist}^2(w_n, C_n) \\
&\leq \|w_n - x^*\|^2 - \Big(\frac{1}{M} h_n(w_n)\Big)^2 \\
&\leq \|w_n - x^*\|^2 - \Big(\frac{1}{2M}\sigma\eta_n\|r(w_n)\|^2\Big)^2 \\
&= \|w_n - x^*\|^2 - \Big(\frac{1}{2M}\sigma\eta_n\|w_n - z_n\|^2\Big)^2. \qquad (27)
\end{aligned}$$

Since $\{x_n\}$ is bounded, we obtain from (27) that

$$\begin{aligned}
\Big(\frac{1}{2M}\sigma\eta_n\|w_n - z_n\|^2\Big)^2 &\leq \|w_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= \Big(\|w_n - x^*\| - \|x_{n+1} - x^*\|\Big)\Big(\|w_n - x^*\| + \|x_{n+1} - x^*\|\Big) \\
&\leq \|w_n - x^*\| - \|x_{n+1} - x^*\| M_1 \\
&\leq \|w_n - x_{n+1}\| M_1, \qquad (28)
\end{aligned}$$

where $M_1 := \sup_{n\geq 1}\{\|w_n - x^*\| + \|x_{n+1} - x^*\|\}$. This establishes (a).

To establish (b), We distinguish two cases depending on the behaviour of (the bounded) sequence of step-sizes $\{\eta_n\}$.
Case 1: Suppose that $\liminf_{n\to\infty} \eta_n > 0$. Then

$$0 \leq \|r(w_n)\|^2 = \frac{\eta_n\|r(w_n)\|^2}{\eta_n}$$

and this implies that

$$\begin{aligned}
\limsup_{n\to\infty} \|r(w_n)\|^2 &\leq \limsup_{n\to\infty}\Big(\eta_n\|r(w_n)\|^2\Big)\Big(\limsup_{n\to\infty}\frac{1}{\eta_n}\Big) \\
&= \Big(\limsup_{n\to\infty}\eta_n\|r(w_n)\|^2\Big)\frac{1}{\liminf_{n\to\infty}\eta_n} \\
&= 0.
\end{aligned}$$

Hence, $\limsup_{n\to\infty}\|r(w_n)\| = 0$. Therefore,

$$\lim_{n\to\infty}\|w_n - z_n\| = \lim_{n\to\infty}\|r(w_n)\| = 0.$$

Case 2: Suppose that $\liminf_{n\to\infty} \eta_n = 0$. Subsequencing if necessary, we may assume without loss of generality that $\lim_{n\to\infty} \eta_n = 0$ and $\lim_{n\to\infty} \|w_n - z_n\| = a \geq 0$. Define $\bar{y}_n := \frac{1}{\gamma}\eta_n z_n + \left(1 - \frac{1}{\gamma}\eta_n\right) w_n$ or, equivalently, $\bar{y}_n - w_n = \frac{1}{\gamma}\eta_n(z_n - w_n)$. Since $\{z_n - w_n\}$ is bounded and since $\lim_{n\to\infty} \eta_n = 0$ holds, it follows that

$$\lim_{n\to\infty} \|\bar{y}_n - w_n\| = 0. \tag{29}$$

From the step-size rule and the definition of $\bar{y}_k$, we have

$$\langle F(\bar{y}_n), w_n - z_n\rangle < \frac{\sigma}{2}\|w_n - z_n\|^2, \ \forall n \in \mathbb{N},$$

or equivalently

$$2\langle F(w_n), w_n - z_n\rangle + 2\langle F(\bar{y}_n) - F(w_n), w_n - z_n\rangle < \sigma\|w_n - z_n\|^2, \ \forall n \in \mathbb{N}.$$

Setting $t_n := w_n - F(w_n)$, we obtain form the last inequality that

$$2\langle w_n - t_n, w_n - z_n\rangle + 2\langle F(\bar{y}_n) - F(w_n), w_n - z_n\rangle < \sigma\|w_n - z_n\|^2, \ \forall n \in \mathbb{N}.$$

Using Lemma 2.2 (b) we get

$$2\langle w_n - t_n, w_n - z_n\rangle = \|w_n - z_n\|^2 + \|w_n - t_n\|^2 - \|z_n - t_n\|^2.$$

Therefore,

$$\|w_n - t_n\|^2 - \|z_n - t_n\|^2 < (\sigma - 1)\|w_n - z_n\|^2 - 2\langle F(\bar{y}_n) - F(w_n), w_n - z_n\rangle \ \forall n \in \mathbb{N}.$$

Since F is uniformly continuous on bounded subsets of H and (29), if $a > 0$ then the right hand side of the last inequality converges to $(\sigma - 1)a < 0$ as $n \to \infty$. From the last inequality we have

$$\limsup_{n\to\infty} \left(\|w_n - t_n\|^2 - \|z_n - t_n\|^2\right) \leq (\sigma - 1)a < 0.$$

For $\epsilon = -(\sigma - 1)a/2 > 0$, there exists $N \in \mathbb{N}$ such that

$$\|w_n - t_n\|^2 - \|z_n - t_n\|^2 \leq (\sigma - 1)a + \epsilon = (\sigma - 1)a/2 < 0 \quad \forall n \in \mathbb{N}, n \geq N,$$

leading to

$$\|w_n - t_n\| < \|z_n - t_n\| \quad \forall n \in \mathbb{N}, n \geq N,$$

which is a contradiction to the definition of $z_n = P_C(w_n - F(w_n))$. Hence $a = 0$, which completes the proof. □

Lemma 4.3. *Let Assumptions 3.1 and 3.2 hold. Furthermore let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging weakly to a limit point p. Then $p \in SOL$.*

Proof. By the definition of z_{n_k} together with (6), we have

$$\langle w_{n_k} - F(w_{n_k}) - z_{n_k}, x - z_{n_k}\rangle \leq 0, \ \forall x \in C,$$

which implies that

$$\langle w_{n_k} - z_{n_k}, x - z_{n_k}\rangle \leq \langle F(w_{n_k}), x - z_{n_k}\rangle, \ \forall x \in C.$$

Hence,

$$\langle w_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle F(w_{n_k}), z_{n_k} - w_{n_k}\rangle \leq \langle F(w_{n_k}), x - w_{n_k}\rangle, \ \forall x \in C. \quad (30)$$

Fix $x \in C$ and let $k \to \infty$ in (30). Since $\lim_{k\to\infty} \|w_{n_k} - z_{n_k}\| = 0$, we have

$$0 \leq \liminf_{k\to\infty} \langle F(w_{n_k}), x - w_{n_k}\rangle \quad (31)$$

for all $x \in C$. It follows from (30) and the monotonicity of F that

$$\begin{aligned} \langle w_{n_k} - z_{n_k}, x - z_{n_k}\rangle + \langle F(w_{n_k}), z_{n_k} - w_{n_k}\rangle &\leq \langle F(w_{n_k}), x - w_{n_k}\rangle \\ &\leq \langle F(x), x - w_{n_k}\rangle \quad \forall x \in C. \end{aligned}$$

Letting $k \to +\infty$ in the last inequality, remembering that $\lim_{k\to\infty} \|w_{n_k} - z_{n_k}\| = 0$ for all k, we have

$$\langle F(x), x - p\rangle \geq 0 \quad \forall x \in C.$$

In view of Lemma 2.8, this implies $p \in$ SOL. □

Theorem 4.4. *Let Assumptions 3.1 and 3.2 hold. Then the sequence $\{x_n\}$ generated by Algorithm 1 weakly converges to a point in SOL.*

Proof. We have shown that
(i) $\lim_{n\to\infty} \|x_n - x^*\|$ exists;
(ii) $\omega_w(x_n) \subset$ SOL, where $\omega_w(x_n) := \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.
Then, by Lemma 2.4, we have that $\{x_n\}$ converges weakly to a point in SOL. □

Remark 4.5. (a) One can still obtain weak convergence for Algorithm 1 when C is a nonempty, closed and convex subset of H.

(b) In finite-dimensional spaces, Theorem 4.4 holds when F is monotone and continuous.

(c) Lemmas 3.5, 4.1, 4.2 and Theorem 4.4 can be obtained when F pseudomonotone and weakly sequentially continuous (i.e., for all $x, y \in H$, $\langle F(x), y - x\rangle \geq 0 \Longrightarrow \langle F(y), y - x\rangle \geq 0$;). The reader can see, for example, [49]. ◊

Remark 4.6. Our proposed method in this paper gives weak convergence results in infinite dimensional Hilbert space. There exists strong convergence methods in the literature for solving variational inequality problem in infinite dimensional Hilbert space (see, for example, [16, 18, 32, 39, 42, 44, 45, 52]). These methods use ideas of viscosity terms, Halpern iterations and hybrid methods. It has been shown numerically in [32] that viscosity and Halpern-type strongly convergent methods outperform those of hybrid methods. Nonetheless, proposed viscosity and Halpern-type strongly convergent methods involve the iterative parameter that is both diminishing and non-summable. These conditions on the iterative parameters make the viscosity and Halpern-type strongly convergent methods to be slower than our proposed method in this paper in terms number of iterations and CPU time.

5 Numerical Experiments

In this section, we discuss the numerical behaviour of Algorithm 1 using different test examples taken from the literature which are describe below and compare our method with (2), (3) and Algorithm 3.2 of Shehu and Iyiola [51].

Example 5.1. Equilibrium-optimization Model
In this example, we consider an equilibrium-optimization model (see, for example, [50]) which can be regarded as an extension of a Nash-Cournot oligopolistic equilibrium model in electricity markets.

In this equilibrium model, we assume that there are m companies, each company i may possess I_i generating units. Suppose we denote by x, the vector whose entry x_j stands for the power generating by unit j. Suppose the price $p_i(s)$ is a decreasing affine function of s where $s := \sum_{j=1}^{N} x_j$ where N is the number of all generating units. Thus, $p_i(s) := \alpha - \beta_i s$. Then the profit made by company i is given by $f_i(x) := p_i(s)\sum_{j\in I_i} x_j - \sum_{j\in I_i} c_j(x_j)$, where $c_j(x_j)$ is the cost for generating x_j by generating unit j . Let us assume that K_i is the strategy set of company i, which implies that $\sum_{j\in I_i} x_j \in K_i$ for each i. Then the strategy set of the model is $C := K_1 \times K_2 \times \ldots \times K_m$.

A commonly used approach when each company wants to maximize its profit by choosing the corresponding production level under the presumption that the production of the other companies are parametric input is the Nash equilibrium concept.

We recall that a point $x^* \in C = K_1 \times K_2 \times \ldots \times K_m$ is an equilibrium point if

$$f_i(x^*) \geq f_i(x^*[x_i]) \forall x_i \in K_i, \quad i = 1, 2, \ldots, m,$$

where the vector $x^*[x_i]$ stands for the vector obtained from x^* by replacing x_i^* with x_i. Define

$$f(x, y) := \psi(x, y) - \psi(x, x)$$

with

$$\psi(x, y) := -\sum_{i=1}^{n} f_i(x^*[y_i]).$$

Then the problem of finding a Nash equilibrium point of our model can be formulated as

$$X^* \in C : f(x^*, x) \geq 0 \ \ \forall x \in C. \tag{32}$$

Suppose for every j, the cost c_j for production and the environmental fee g are increasingly convex functions. The convexity assumption here means that both the cost and fee for producing a unit production increases as the quantity of the production gets larger. Under this convexity assumption, it is not hard to see that (32) is equivalent to (see, [58])

$$x \in C : \langle Bx - a + \nabla\varphi(x), y - x\rangle \geq 0 \ \ \forall y \in C, \tag{33}$$

where

$$a := (\alpha, \alpha, \dots, \alpha)^T$$

$$B_1 = \begin{pmatrix} \beta_1 & 0 & 0 & \dots & 0 \\ 0 & \beta_2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \beta_m \end{pmatrix} B = \begin{pmatrix} 0 & \beta_1 & \beta_1 & \dots & \beta_1 \\ \beta_2 & 0 & \beta_2 & \dots & \beta_2 \\ \dots & \dots & \dots & \dots & \dots \\ \beta_m & \beta_m & \beta_m & \dots & \beta_m \end{pmatrix}$$

$$\varphi(x) := x^T B_1 x + \sum_{j=1}^{N} c_j(x_j).$$

Note that when c_j is differentiable convex for every j.

We tested the proposed algorithm with the cost function given by

$$c_j(x_j) = \frac{1}{2} x_j^T D x_j + d^T x_j.$$

The parameters β_j for all $j = 1, \dots, m$, matrix D and vector d were generated randomly in the interval $(0, 1]$, $[1, 40]$ and $[1, 40]$ respectively.

We perform numerical implementations using different choices of 10, and 20, different initial choices x_1 generated randomly in the interval $[1, 40]$ and $m = 10$ with the stopping criterion as $\|x_{n+1} - x_n\| \leq 10^{-2}$. Let us assume that each company have the same lower production bound 1 and upper production bound 40, that is,

$$K_i := \{x_i : 1 \leq x_i \leq 40\}, \quad i = 1, \dots, 10.$$

We compare our proposed Algorithm 1 with algorithm 3.2 proposed by Shehu and Iyiola in [51].

Table 1: Example 5.1 Comparison: Proposed Alg. 1 and Shehu & Iyiola Alg. 3.2 (SI Alg.) for $\sigma = 0.5$

	N=10				N=20			
	No. of Iter.		CPU time (10^{-2})		No. of Iter.		CPU time (10^{-2})	
γ	Alg. 1	SI Alg.	Alg. 1	SI Alg.	Alg. 1	SI Alg.	Alg. 1	SI Alg.
0.01	223	435	2.6822	8.8916	228	520	7.9536	22.352
0.1	38	518	1.4433	13.108	36	473	1.0797	14.871
0.5	10	434	0.8232	7.9301	9	285	0.4196	9.6452
0.8	9	514	12.3590	13.1390	8	320	0.4596	8.3524

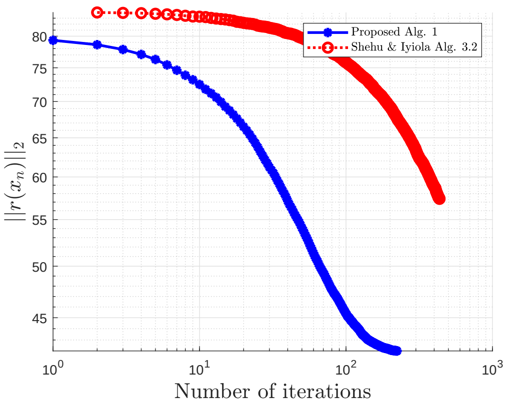

Figure 1: Example 5.1: $\gamma = 0.01$, $N = 10$

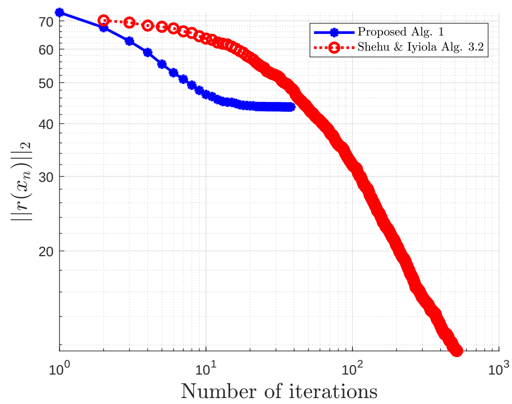

Figure 2: Example 5.1: $\gamma = 0.1$, $N = 10$

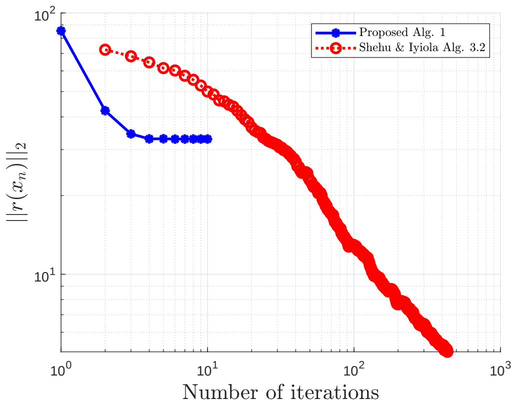

Figure 3: Example 5.1: $\gamma = 0.5$, $N = 10$

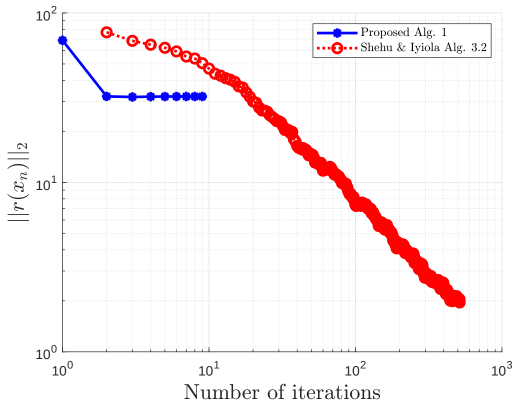

Figure 4: Example 5.1: $\gamma = 0.7$, $N = 10$

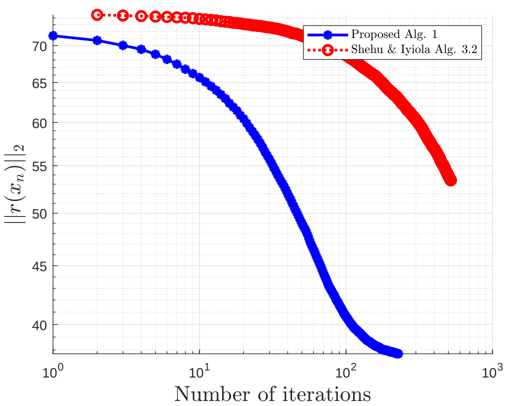

Figure 5: Example 5.1: $\gamma = 0.01$, $N = 20$

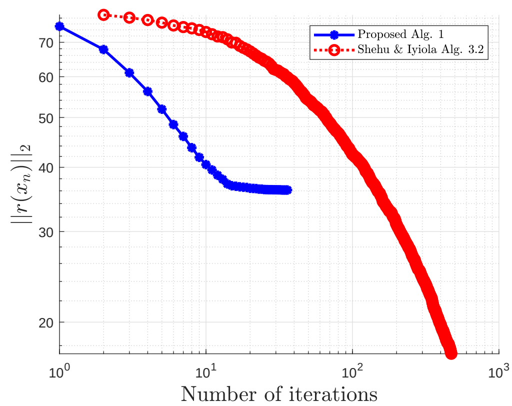

Figure 6: Example 5.1: $\gamma = 0.1$, $N = 20$

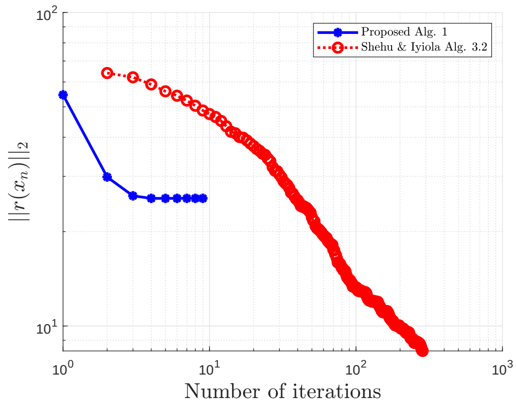

Figure 7: Example 5.1: $\gamma = 0.5$, $N = 20$

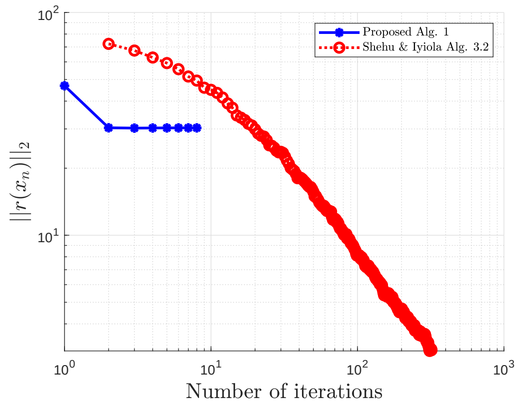

Figure 8: Example 5.1: $\gamma = 0.7$, $N = 20$

Example 5.2. This example is taken from [27] and has been considered by many authors for numerical experiments (see, for example, [29, 42, 53]). The operator A is defined by $A(x) := Mx + q$, where $M = BB^T + S + D$, with $B, S, D \in \mathbb{R}^{m\times m}$ randomly generated matrices such that S is skew-symmetric (hence the operator does not arise from an optimization problem), D is a positive definite diagonal matrix (hence the variational inequality has a unique solution) and $q = 0$. The feasible set C is described by linear inequality constraints $Bx \leq b$ for some random matrix $B \in \mathbb{R}^{k\times m}$ and a random vector $b \in \mathbb{R}^k$ with nonnegative entries. Hence the zero vector is feasible and therefore the unique solution of the corresponding variational inequality. These projections are computed using the MATLAB solver `fmincon`. Hence, for this class of problems, the evaluation of A is relatively inexpensive, whereas projections are costly. We present the corresponding numerical results (number of iterations and CPU times in seconds) using six different dimensions m and two different numbers of inequality constraints k.

We choose the stopping criterion as $\|x^k\| \leq \epsilon = 0.001$. The size $k = 30, 50$ and $m = 10, 20, 30, 40, 50, 60$. The matrices B, S, D and the vector b are generated randomly. We choose $\gamma = 0.8$, $\sigma = 0.5$, $\alpha_n = 0.2$ in Algorithm (1). In (2), we choose $\sigma = 0.8$, $\rho = 0.1$, $\mu = 0.2$. In (3), we choose $L = \|M\|$. Here, we compare our proposed Algorithm 1 with the subgradient extragradient method (SEM) (2), and the inertial subgradient extragradient method (Thong & Hieu) (3).

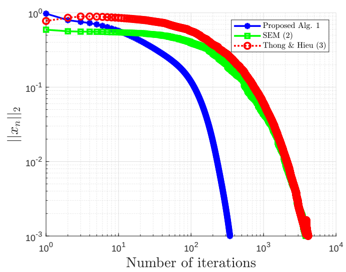

Figure 9: Example 5.2: $k = 30$, $m = 10$

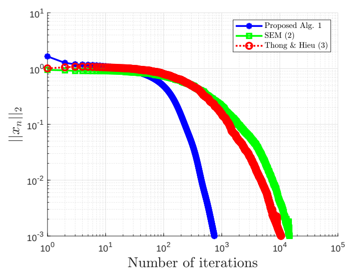

Figure 10: Example 5.2: $k = 30$, $m = 20$

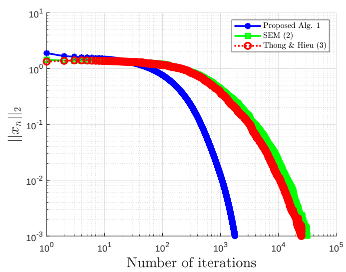

Figure 11: Example 5.2: $k = 30$, $m = 30$

Table 2: Example 5.2 Comparison: Proposed Alg. 1 vs SEM (2) vs Thong & Hieu (3) (T & H (3))

		No. of Iterations			CPU time			Norm sol. (10^{-3})		
	m	**Alg. 1**	**SEM (2)**	**T & H (3)**	**Alg. 1**	**SEM (2)**	**T & H (3)**	**Alg. 1**	**SEM (2)**	**T & H (3)**
$k = 30$	**10**	344	3867	4123	2.8707	31.5956	30.3300	0.99605	0.99927	0.99938
	20	747	14683	10493	5.6957	117.7458	84.7406	0.99608	0.99996	0.99984
	30	1777	31668	24968	13.891	269.2987	211.1937	0.99955	0.99999	0.99980
	40	2612	40224	36119	21.5972	358.3453	320.2933	0.99790	1.00000	0.99994
	50	3710	70321	51143	32.074	655.8354	469.0297	0.99981	0.99997	0.99995
	60	5619	56670	50619	50.4537	554.3951	491.5552	0.99929	0.99992	0.99998
$k = 50$	**10**	200	6213	5518	1.90869	60.1212	47.6937	0.98471	0.99969	0.99945
	20	835	14354	10372	6.4942	126.429	96.9197	0.99909	0.99980	0.99991
	30	1978	25519	19357	16.4674	240.61	208.910	0.99794	0.99991	0.99990
	40	2832	47661	26790	30.3799	539.729	314.588	0.99734	0.99991	0.99938
	50	3933	43773	53055	45.8745	562.959	800.371	0.99985	0.99925	0.99999
	60	6025	97772	65820	100.304	1515.76	589.180	0.99955	0.99995	0.99994

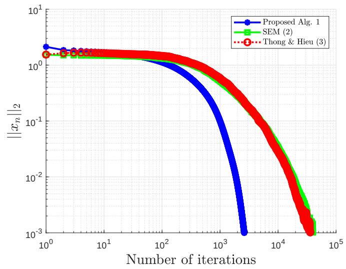

Figure 12: Example 5.2: $k = 30$, $m = 40$

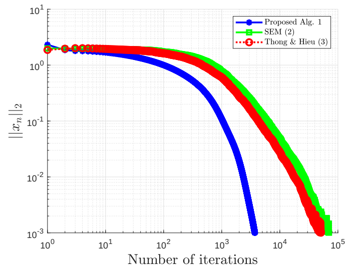

Figure 13: Example 5.2: $k = 30$, $m = 50$

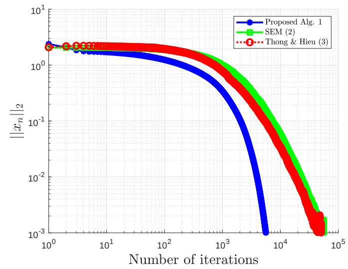

Figure 14: Example 5.2: $k = 30$, $m = 60$

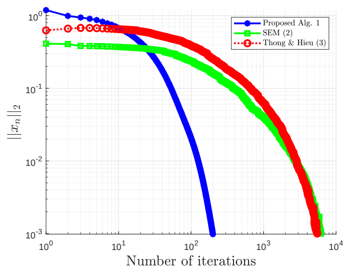

Figure 15: Example 5.2: $k = 50$, $m = 10$

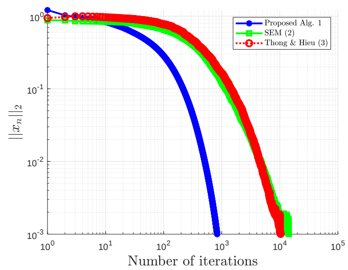

Figure 16: Example 5.2: $k = 50$, $m = 20$

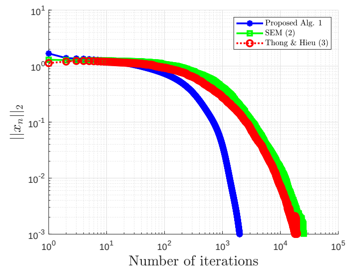

Figure 17: Example 5.2: $k = 50$, $m = 30$

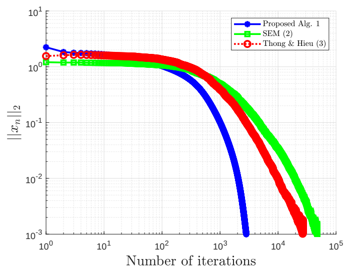

Figure 18: Example 5.2: $k = 50$, $m = 40$

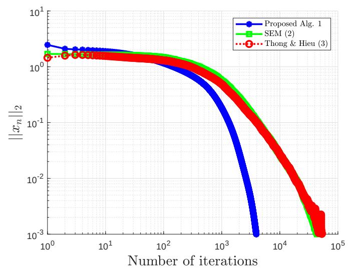

Figure 19: Example 5.2: $k = 50$, $m = 50$

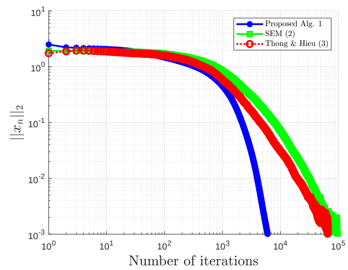

Figure 20: Example 5.2: $k = 50$, $m = 60$

We give an example in infinite dimensional Hilbert spaces. We give comparison of our proposed Algorithm 1 with Algorithm (2), Algorithm (3) and the non-inertial case of Algorithm 1 (when $\alpha_n = 0$).

Example 5.3. Let $H := L^2([0,1])$ with norm and inner product given as $\|x\| := \left(\int_0^1 x(t)^2 dt\right)^{\frac{1}{2}}$ and $\langle x, y\rangle := \int_0^1 x(t)y(t)dt, \quad x, y \in H$ respectively. We define the feasible set C as: $C := \{x \in L^2([0,1]) : \int_0^1 tx(t)dt = 2\}$. Let us define the Volterra integral operator $F : L^2([0,1]) \to L^2([0,1])$ by

$$Fx(t) := \int_0^t x(s)ds, \quad x \in L^2([0,1]), t \in [0,1].$$

Then, F is monotone, bounded and linear with $L = \frac{2}{\pi}$ (see Exercises 20.12 of [9]). Observe that SOL $\neq \emptyset$ since $0 \in$ SOL and that (see [15])

$$P_C(x)(t) := x(t) - \frac{\int_0^1 tx(t)dt - 2}{\int_0^1 t^2 dt} t, \quad t \in [0,1].$$

We set the stopping criterion to be $e_n := \|x_{n+1} - x_n\| < \epsilon$ where $\epsilon = 10^{-3}$. In our proposed algorithm (1), we choose $\gamma = 0.2$, $\sigma = 0.05$, $\alpha_n = 0.3$ and for subgradient extragradient method (SEM) (i.e., Algorithm (2)), we choose $\lambda = \frac{1}{2L}$ while for the Thong and Hieu Algorithm (3), we choose $\alpha_n = 0.3$ and $\lambda = \frac{0.01}{2L}$. We compared all four algorithms with different initial points.

Table 3: Example 5.3 Comparison: Proposed Alg. 1, Non-inertial Alg. ($\alpha_n = 0$), SEM Alg., and Thong & Hieu (T & H) Alg.

$x_0 = x_1$		Alg. 1	Alg. 1 ($\alpha_n = 0$)	SEM Alg.	T & H Alg.
$\frac{37}{5} te^t$	No. of Iter.	63	87	919	16902
	CPU time	1.0443×10^{-2}	1.5735×10^{-2}	0.13886	2.5550
$\frac{97}{12}(t^2 + 7t)$	No. of Iter.	69	97	912	16605
	CPU time	9.8188×10^{-3}	1.0992×10^{-2}	0.13403	2.5040

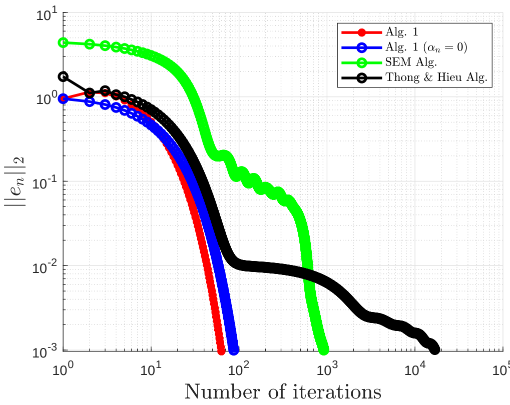

Figure 21: Ex. 5.3: $x_0 = x_1 = \frac{37}{5} te^t$

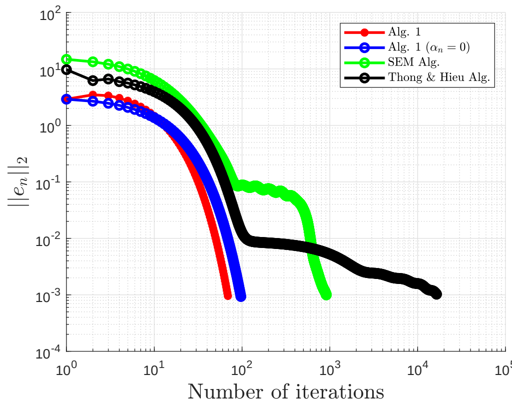

Figure 22: Ex. 5.3: $x_0 = x_1 = \frac{97}{12}(t^2 + 7t)$

Clearly, from all the three Examples presented above, our proposed algorithm 1 outperforms and highly improves the non-inertial algorithm ($\alpha_n = 0$), Shehu and Iyiola Algorithm (3.2) in [51], subgradient extragradient method (SEM) (2), and the inertial subgradient extragradient method (Thong & Hieu) (3) with respect to number of iterations required and CPU time and achieved norm of the solution. See Tables 1 - 3 and Figures 1 - 22.

6 Final Remarks

We propose an inertial projection method for solving variational inequality problem and give weak convergence result. The cost function is assumed to be monotone and non-Lipschitz continuous. Our numerical implementations show that our method is more efficient and outperforms some other related methods in the literature. Our result is more applicable than the results on variational inequality where the Lipschitz constant of the cost function is needed. Our future project is focused on how to extend the range of inertial factor α_n beyond 1/3 and extend our results to infinite dimensional Banach spaces.

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