



Document downloaded from the institutional repository of the University of Alcalá: <http://ebuah.uah.es/dspace/>

This is a postprint version of the following published document:

Blasco, Ángel & Pérez-Díaz, Sonia. 2019, "An in depth analysis, via resultants, of the singularities of a parametric curve", Computer Aided Geometric Design, vol. 68 (Enero 2019), pp. 22-47

Available at <https://doi.org/10.1016/j.cagd.2018.12.003>

© 2019 Elsevier

*(Article begins on next page)*



This work is licensed under a

Creative Commons Attribution-NonCommercial-NoDerivatives  
4.0 International License.

# An in Depth Analysis, via Resultants, of the Singularities of a Parametric Curve

Angel Blasco and Sonia Pérez-Díaz  
Departamento de Física y Matemáticas  
Universidad de Alcalá  
28871-Alcalá de Henares, Madrid, Spain  
angel.blasco@uah.es, sonia.perez@uah.es

---

## Abstract

Let  $\mathcal{C}$  be an algebraic space curve defined by a rational parametrization  $\mathcal{P}(t) \in \mathbb{K}(t)^\ell$ ,  $\ell \geq 2$ . In this paper, we consider the  $T$ -function,  $T(s)$ , which is a polynomial constructed from  $\mathcal{P}(t)$  by means of a univariate resultant, and we show that  $T(s)$  contains essential information concerning the singularities of  $\mathcal{C}$ . More precisely, we prove that  $T(s) = \prod_{i=1}^n \mathcal{H}_{P_i}(s)$ , where  $P_i$ ,  $i = 1, \dots, n$ , are the (ordinary and non-ordinary) singularities of  $\mathcal{C}$  and  $\mathcal{H}_{P_i}$ ,  $i = 1, \dots, n$ , are polynomials, each of them associated to a singularity, whose factors are the fibre functions of those singularities as well as those other belonging to their corresponding neighborhoods. That is,  $\mathcal{H}_Q(s) = H_Q(s)^{m-1} \prod_{j=1}^k H_{Q_j}(s)^{m_j-1}$ , where  $Q$  is an  $m$ -fold point,  $Q_j$ ,  $j = 1, \dots, k$ , are the neighboring singularities of  $Q$ , and  $m_j$ ,  $j = 1, \dots, k$ , are their corresponding multiplicities ( $H_P$  denotes the fibre function of  $P$ ). Thus, by just analyzing the factorization of  $T$ , we can obtain all the singularities (ordinary and non-ordinary) as well as interesting data relative to each of them, like its multiplicity, character, fibre or number of associated tangents. Furthermore, in the case of non-ordinary singularities, we can easily get the corresponding number of local branches and delta invariant.

*Keywords:* Rational parametrization; Singularities of an algebraic curve; Multiplicity of a point; Ordinary singularity; Non-ordinary singularity;  $T$ -function; Fibre function

---

## 1. Introduction

The study of algebraic varieties from the parametric point of view has been a subject of attention for many researchers in the last years ([7], [9], [15], [16], [17], [23], [27]). The increasing interest in this area is partially motivated by the convenience of manipulating algebraic varieties through their corresponding parametric equations. The computations required for many practical applications in computer aided geometric design (CAGD), can often be carried out in an easier way by working with the parametric expression. Important examples of these assertions can be found for instance, in visualization problems (see [16], [17]) or modeling problems (see [15] or [19]).

One of the main topics in this area is the study of singularities. Here, rational parametrizations provide interesting approaches from the computational point of view. For instance, for the case of parametric plane curves, some interesting results are provided in [9], where the singular points are computed using the implicitization matrix derived from the  $\mu$ -basis of the curve. In addition, a conjecture is presented which concerns the computation of the parameter values corresponding to all the singularities, from the Smith normal forms of certain Bézout resultant matrices derived from  $\mu$ -bases. A similar result is proved in [18]. In [30], a natural one to one correspondence is derived between the singular points of a rational planar curve and the axial moving lines that follow that curves. This correspondence is applied to compute and analyze the singular points of low degree rational planar curves, by using  $\mu$ -bases. In [7], it is introduced a new implicit representation of the curve which consists in the locus where the rank of a single matrix drops. From this representation, one may compute the singularities of the given curve. In [8], it is given a complete factorization of the invariant factors of resultant matrices, built from birational parameterizations of rational plane curves, in terms of the singular points of the curve and their multiplicity graph. This also allows to prove the validity of some conjectures introduced in [9]. A new technique for detecting singularities is introduced in [29]. The idea is to compute a  $\mu$ -basis for the parametrization and to generate, from this  $\mu$ -basis, three planar algebraic curves of different bidegrees, whose intersection points correspond to the parameters of the singularities. In order to find these intersection points, a new sparse resultant matrix for these three bivariate polynomials is constructed. Afterwards, authors compute the parameter values corresponding to the singularities by applying

Gaussian elimination to the resultant matrix. All these works are based on the use of  $\mu$ -basis. Besides, there is a second line of work, based on the use of univariate resultants. For instance, in [22], a method for detecting and analyzing the singularities of a rational curve (including the non-ordinary ones) by computing a univariate resultant, is provided. This approach is based on the ideas introduced in Subsection 4.3 in [27], and it generalizes some previous results presented in [1], [13], [20] and [31]. Finally, one also should mention the recent work [2], where a different approach is considered. There, in order to study the singularities of a plane rational curve  $\mathcal{C}$  of degree  $n$ , authors use the fact that the parameterization of  $\mathcal{C}$  defines a projection  $\pi : \mathbb{P}^n \rightarrow \mathbb{P}^2$ , from the rational normal curve  $\mathcal{C}_n \subset \mathbb{P}^n$ , and  $\pi(\mathcal{C}_n) = \mathcal{C} \subset \mathbb{P}^2$  ( $\pi$  is generically one-to-one). If  $P$  is a singular point of multiplicity  $m$  of  $\mathcal{C}$ , then there is an  $(m - 1)$ -dimensional  $m$ -secant space  $H$  to  $\mathcal{C}_n$  such that  $\pi(H) = P$ . The center of projection of  $\pi$  is a  $(n - 3)$ -linear space  $\Pi$ , and  $\Pi \cap H$  has to be  $(m - 2)$ -dimensional, in order to have that  $\pi(H)$  is a point. It is proved that  $H \cap \mathcal{C}_n$  and  $H \cap \Pi$  contain all the information about the singularity  $P$ . In order to extract this information, authors consider the spaces  $\mathbb{P}^k \cong \mathbb{P}^k(\mathbb{K}[s, t]_k)$  that parameterize the  $k$ -secant variety of  $((k - 1)$ -dimensional)  $k$ -secant spaces to  $\mathcal{C}$  and their intersection with the center of projection  $\Pi$  (which is determined by the parameterization of  $\mathcal{C}$ ). Such study yields to considering 0-dimensional schemes,  $X_k \subset \mathbb{P}^k$ , which parameterize the  $k$ -secant  $(k - 1)$ -spaces that get contracted to a point by  $\pi$ , so that they encode all the information on the singularities of  $\mathcal{C}$ .

The study of singularities in parametric space curves has been addressed, for instance, in [7], [25], [29] and [30]. In this case, also the most important methods used have to do with the computation of  $\mu$ -bases and univariate resultants. For the case of surfaces, some works have also been published (see e.g. [24]).

In this paper, by using some previous results in [3] and [22], we develop a resultant based methodology which allows us to obtain all the (ordinary and non-ordinary) singularities of a rational curve (plane or space curve in any dimension) from its parametric representation. Our approach allows us to easily get relevant information about each of those singularities, as its multiplicity, its character, its fibre or the tangent lines to the curve at this point. Furthermore, in the case of non-ordinary singularities our method also provides the number of local branches and the delta invariant. The most important point of the paper is that all this information is easily deduced

from the structure of a polynomial which is carefully characterized. This polynomial, which is called the T–function, is obtained by just computing an univariate resultant constructed from the input parametrization of the given curve. The above mentioned parameters are obtained in a relatively simple way, by just analyzing the factors of this polynomial.

More precisely, let  $\mathcal{P}(t) \in \mathbb{P}^2(\mathbb{K}(t))$  be a rational projective proper parametrization of an algebraic plane curve  $\mathcal{C}$  over an algebraically closed field of characteristic zero,  $\mathbb{K}$ . We denote by  $\mathcal{F}_{\mathcal{P}}(P)$  the fibre of a point  $P \in \mathcal{C}$  via  $\mathcal{P}(t)$ ; that is  $\mathcal{F}_{\mathcal{P}}(P) = \{t \in \mathbb{K} \mid \mathcal{P}(t) = P\}$ . Intuitively speaking,  $\mathcal{P}(t)$  proper means that the parametrization traces the curve once, except for at most a finite number of points. In [22], it is shown that those points are the singularities of  $\mathcal{C}$  and it is introduced the notion of *fibre function*. The fibre function of a point  $P \in \mathcal{C}$  is a polynomial,  $H_P(t)$ , which provides a lot of information about the fibre of  $P$ . In particular, we have that  $t_0 \in \mathcal{F}_{\mathcal{P}}(P)$  if and only if  $H_P(t_0) = 0$ . In [22], it is proved that if  $H_P(t) = \prod_{i=1}^n (t-s_i)^{k_i}$  then,  $\mathcal{C}$  has  $n$  tangents at  $P$  of multiplicities  $k_1, \dots, k_n$ , respectively. In addition, each of these tangents can be computed by using  $\mathcal{P}(t)$  and the root of the corresponding fibre function. Furthermore, it is shown that  $\text{mult}_P(\mathcal{C}) = \deg(H_P)$ . A deeper analysis of the relation between the fibre of a point and its multiplicity can be found in [3]. There, it is introduced the *T–function*,  $T(s)$ , a polynomial which is defined by means of a univariate resultant constructed from  $\mathcal{P}(t)$  and contains essential information about the singularities of  $\mathcal{C}$ . The main theorem of [3] states that, if  $\mathcal{C}$  only has ordinary singularities, then  $T$  can be factorized as  $T(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}$  where  $P_i, i = 1, \dots, n$ , are the singularities and  $m_i, i = 1, \dots, n$ , are their respective multiplicities. Thus, a complete classification of the singularities of  $\mathcal{C}$ , via the factorization of a resultant, is obtained.

In this work, we extend the methodology introduced in [3] and we generalize the main result of that work by proving that the T–function also contains the fibre functions of the non–ordinary singularities. More precisely, we show that  $T(s) = \prod_{i=1}^n \mathcal{H}_{P_i}(s)$ , where  $P_i, i = 1, \dots, n$ , are the (ordinary and non–ordinary) singularities of  $\mathcal{C}$  and  $\mathcal{H}_{P_i}, i = 1, \dots, n$ , are polynomials, each of them associated to a singularity, whose factors are the fibre functions of those singularities as well as those other belonging to their corresponding neighborhoods. That is, given an  $m$ -fold point  $Q$ , we have that  $\mathcal{H}_Q(s) = H_Q(s)^{m-1} \prod_{j=1}^k H_{Q_j}(s)^{m_j-1}$ , where  $Q_j, j = 1, \dots, k$ , are the neighboring singularities of  $Q$  and  $m_j, j = 1, \dots, k$ , are their respective mul-

tiplicities. We will see that, by appropriately reading these factors, one can obtain all the ordinary and non-ordinary singularities of the curve, as well as interesting data relative to each of them, like its multiplicity, character, fibre or number of associated tangents. Furthermore, in the case of non-ordinary singularities we can easily get the corresponding number of local branches and delta invariant.

The results above described can easily be adapted for the study of singularities of rational space curves in any dimension. For this purpose, we introduce an extension of the T-function which allows us to perfectly generalize the results of plane curves to the space case. Any factor of the new T-function ( $T_E(s)$ ) corresponds to a singularity of the space curve. Thus, we provide, as in the case of plane curves, a complete classification of the singularities (ordinary and non-ordinary) of a given rational space curve, via the factorization of a univariate resultant. In this way, we are generalizing some previous results (cited above) that partially approach the computation and analysis of singularities for rational space curves. In particular, [1], where the factorization of the T-function is carried out for a given polynomial parametrization, and [8], where authors provide a generalization of Abhyankar's formula for the case of rational parametrizations (not necessarily polynomial). This second work is based on the concept of singular factors introduced in [9], and it involves the construction of  $\mu$ -basis. However, our approach is totally different since we generalize Abhyankar's formula by using the methods and techniques presented in [22]. Amongst other things, this allows us to group the factors of the T-function to easily obtain the fibre functions of the different singularities and, hence, the set of relevant parameters above mentioned: multiplicity, number of branches, delta invariant, etc.

In summary, our approach exhibits certain specific features and advantages with respect to other existing methods, that we list below: 1) it allows us to study the singularities of the curve by just computing a single univariate resultant, 2) it gives us both, the ordinary and the non-ordinary singularities, 3) for each singularity, one can easily get its multiplicity, character, fibre and number of tangents, as well as the number of local branches and the delta invariant in the case of non-ordinary singularities, and 4) it may be easily adapted for the study of space curves. In addition, we recall that, in a direct method, in order to compute the singularities, one would introduce algebraic numbers during the computations. However, in this paper,

in order to deal with this problem, we consider *families of conjugated parametric points*. This notion allows us to determine the singularities of a curve without directly introducing algebraic numbers in the computations.

The structure of the paper is as follows: Section 2 is devoted to introduce some basic concepts and previous results, mainly concerned with the analysis of the fibre and its application to the study of ordinary singularities. In particular, the T–function is defined and some of its most important properties are summarized. In order to completely describe a non–ordinary singularity we need to “blow up” the curve. For this purpose, in Section 3, we first summarize the blowing up process and we explain how to perform it from the parametric expression of a rational curve. Afterwards, the main result of the paper, Theorem 4, and some technical lemmas and corollaries, are stated. Theorem 4 claims that the factors of the T–function are the fibre functions of the ordinary and non–ordinary singularities of the curve. These fibre functions provide interesting information concerning each singularity. However, for unfamiliar users, the T–function may be difficult to read, since factors corresponding to fibre values of different singularities, use to appear scrambled and raised to different powers. Thus, in order to obtain the different fibre functions, we need to correctly group those factors. For this purpose, an efficient algorithm, which can be used as a guideline through the whole process, has been included in Section 4. In Section 5, we show how to generalize our method for the study of parametric space curves in any dimension. Finally, the proofs of some technical results are presented in Section 6.

## 2. Basis concepts and previous results

Let  $\mathcal{C}$  be a rational (projective) plane curve defined by the projective parametrization

$$\mathcal{P}(t) = (p_1(t), p_2(t), p(t)) \in \mathbb{P}^2(\mathbb{K}(t)),$$

where  $\gcd(p_1, p_2, p) = 1$ , and  $\mathbb{K}$  is an algebraically closed field of characteristic zero . We assume that  $\mathcal{C}$  is not a line (note that a line does not have multiple points). Let  $d_1 = \deg(p_1)$ ,  $d_2 = \deg(p_2)$ ,  $d_3 = \deg(p)$ , and

$d = \max\{d_1, d_2, d_3\}$ . Thus, we may write  $p_1$ ,  $p_2$  and  $p$  as

$$\begin{cases} p_1(t) = a_0 + a_1t + a_2t^2 + \cdots + a_d t^d \\ p_2(t) = b_0 + b_1t + b_2t^2 + \cdots + b_d t^d \\ p(t) = c_0 + c_1t + c_2t^2 + \cdots + c_d t^d. \end{cases}$$

Associated with  $\mathcal{P}(t)$ , we consider the induced rational map  $\psi_{\mathcal{P}} : \mathbb{K} \longrightarrow \mathcal{C} \subset \mathbb{P}^2(\mathbb{K}); t \longmapsto \mathcal{P}(t)$ . We denote by  $\deg(\psi_{\mathcal{P}})$  the degree of the rational map  $\psi_{\mathcal{P}}$  (for further details see e.g. [28] pp.143, or [14] pp.80). As an important result, we recall that the birationality of  $\psi_{\mathcal{P}}$ , i.e. the properness of  $\mathcal{P}(t)$ , is characterized by  $\deg(\psi_{\mathcal{P}}) = 1$  (see [14] and [28]). Also, we recall that the degree of a rational map can be seen as the cardinality of the fibre of a generic element (see Theorem 7, pp. 76 in [28]). We will use this characterization in our reasoning. For this purpose, we denote by  $\mathcal{F}_{\mathcal{P}}(P)$  the fibre of a point  $P \in \mathcal{C}$  via the parametrization  $\mathcal{P}(t)$ ; that is

$$\mathcal{F}_{\mathcal{P}}(P) = \mathcal{P}^{-1}(P) = \{t \in \mathbb{K} \mid \mathcal{P}(t) = P\}.$$

In general, it holds that  $P \in \mathcal{C}$  if and only if  $\mathcal{F}_{\mathcal{P}}(P) \neq \emptyset$ , although an exception can be found for *the limit point* of the parametrization.

**Definition 1.** *We define the limit point of the parametrization  $\mathcal{P}(t)$  as*

$$P_L = \lim_{t \rightarrow \infty} \mathcal{P}(t)/t^d = (a_d : b_d : c_d).$$

Note that  $P_L \in \mathcal{C}$  since  $\mathcal{P}(t)/t^d = \mathcal{P}(t) \in \mathcal{C}$ , for  $t \in \mathbb{K}$ , and  $\mathcal{C}$  is a closed set. Furthermore, we observe that, given a parametrization  $\mathcal{P}(t)$ , there always exists an associated limit point, and it is unique.

The limit point is reachable via the parametrization  $\mathcal{P}(t)$ , if there exists  $t_0 \in \mathbb{K}$  such that  $\mathcal{P}(t_0) = P_L$ . However, the value  $t_0 \in \mathbb{K}$  could not exist, and then  $\mathcal{F}_{\mathcal{P}}(P_L) = \emptyset$ . If  $P_L$  is not an affine point or it is a reachable affine point, we have that  $\mathcal{P}(t)$  is a *normal parametrization*. Otherwise, we say that  $\mathcal{P}(t)$  is not normal and  $P_L$  is the *critical point* (see Subsection 6.3 in [27]). Further properties of the limit point are stated and proved in [4].

In Subsection 2.2. in [27], it is shown that the degree of a dominant rational map between two varieties of the same dimension is the cardinality of the fiber of a generic element. Therefore, in the case of the mapping  $\psi_{\mathcal{P}}$ , this implies that almost all points of  $\mathcal{C}$  (except at most a finite number



of points) are generated via  $\mathcal{P}(t)$  by the same number of parameter values, and this number is the degree of  $\psi_{\mathcal{P}}$ . Thus, intuitively speaking, the degree measures the number of times the parametrization traces the curve when the parameter takes values in  $\mathbb{K}$ . Taking into account this intuitive notion, the degree of the mapping  $\psi_{\mathcal{P}}$  is also called *the tracing index* of  $\mathcal{P}(t)$ . In order to compute the tracing index, the following polynomials are considered,

$$\begin{cases} G_1(s, t) := p_1(s)p(t) - p(s)p_1(t) \\ G_2(s, t) := p_2(s)p(t) - p(s)p_2(t) \\ G_3(s, t) := p_1(s)p_2(t) - p_2(s)p_1(t) \end{cases} \quad (1)$$

and  $G(s, t) = \gcd(G_1(s, t), G_2(s, t), G_3(s, t))$ . These functions satisfy the following properties (see Remark 1 in [3]):

- $G_i(s, t) = -G_i(t, s)$  for  $i = 1, 2, 3$ .
- $\deg_s(G_i) = \deg_t(G_i)$  for  $i = 1, 2, 3$ , and  $\deg_s(G) = \deg_t(G)$ .
- $\deg_t(G_1) = \max\{d_1, d_3\}$ ,  $\deg_t(G_2) = \max\{d_2, d_3\}$  and  $\deg_t(G_3) = \max\{d_1, d_2\}$ .
- $G(s, t) = \gcd(G_i(s, t), G_j(s, t))$  where  $i, j = 1, 2, 3$  and  $i \neq j$ .

The following theorem has been proved in [27] (see Subsection 4.3). It allows us to compute the tracing index of  $\mathcal{P}(t)$  using the polynomial  $G(s, t)$ .

**Theorem 1.** *It holds that  $\deg(\psi_{\mathcal{P}}) = \deg_t(G)$ .*

Throughout this paper, we assume that  $\mathcal{P}(t)$  is proper, that is  $\deg(\psi_{\mathcal{P}}) = 1$ . Otherwise, we can reparametrize the curve using, for instance, the results in [21]. Under these conditions, it holds that the degree of  $\mathcal{C}$  is  $d$  (see Theorem 6 in [22]). In addition,  $G(t, s) = t - s$ , and the cardinality of the fibre for a generic point of  $\mathcal{C}$  is 1, although it can be different for a particular point.

Given a point  $P \in \mathcal{C}$ , the fibre of  $P$  consists of the values  $t \in \mathbb{K}$  such that  $\mathcal{P}(t) = P$ . In particular, if  $P = \mathcal{P}(s_0)$  for some  $s_0 \in \mathbb{K}$ , those values are the common roots of the *fibre equations*, given by  $G_i(s_0, t) = 0$ ,  $i = 1, 2, 3$ . This fact motivates the following definition (see Corollary to Theorem 4.28 in Section 4.3 in [27]):

**Definition 2.** Given the rational parametrization  $\mathcal{P}(t) \in \mathbb{P}^2(\mathbb{K}(t))$  and the point  $P = \mathcal{P}(s_0)$ , we define the fibre function of  $P$  as

$$H_P(t) := \gcd(G_1(s_0, t), G_2(s_0, t), G_3(s_0, t)).$$

Thus,  $t_0 \in \mathcal{F}_{\mathcal{P}}(P)$  if and only if  $H_P(t_0) = 0$ .

In [3], it is shown that one may compute  $H_P(t)$  as follows:

$$H_P(t) = \begin{cases} \gcd(G_1(s_0, t), G_2(s_0, t)) & \text{if } P \text{ is an affine point} \\ \gcd(p(t), G_3(s_0, t)) & \text{if } P \text{ is an infinity point.} \end{cases} \quad (2)$$

### 2.1. Detection and analysis of ordinary singularities

The analysis of the fibre allows us to study the singularities of a rational curve. We recall that  $P$  is a *point of multiplicity*  $\ell$  on  $\mathcal{C}$  if and only if all the derivatives of  $F$  (where  $F$  denotes the implicit polynomial defining  $\mathcal{C}$ ) up to and including those of  $(\ell - 1)$ -th order, vanish at  $P$  but at least one  $\ell$ -th derivative does not vanish at  $P$ . We denote it by  $\text{mult}_P(\mathcal{C}) = \ell$ . The point  $P$  is called a *simple point* on  $\mathcal{C}$  if  $\text{mult}_P(\mathcal{C}) = 1$ . If  $\text{mult}_P(\mathcal{C}) = \ell > 1$ , then we say that  $P$  is a multiple or singular point (or singularity) of multiplicity  $\ell$  on  $\mathcal{C}$  or an  $\ell$ -fold point. Clearly  $P \notin \mathcal{C}$  if and only if  $\text{mult}_P(\mathcal{C}) = 0$ .

Observe that the multiplicity of  $P$  is given by the order of the Taylor expansion of  $F$  at  $P$ . The tangents of  $\mathcal{C}$  at  $P$  are the irreducible factors of the first non-vanishing form in that Taylor expansion, and the multiplicity of each tangent is the multiplicity of the corresponding factor. If all the  $\ell$  tangents at the  $\ell$ -fold point  $P$  are different, then this singularity is called *ordinary*, and *non-ordinary* otherwise. Thus, we say that the character of  $P$  is either ordinary or non-ordinary.

In [22], it is shown how to compute the singular points of a given rational plane curve from its parametric expression. Furthermore, it is provided a method for computing the multiplicity of each singular point as well as the tangents of the curve at that point. In particular, the following theorem and corollary are proved.

**Theorem 2.** Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P \neq P_L$  be a point of  $\mathcal{C}$  and let  $H_P(t) = \prod_{i=1}^n (t - s_i)^{k_i}$  be its fibre function. Then,  $\mathcal{C}$  has  $n$  tangents at  $P$  of multiplicities  $k_1, \dots, k_n$ , respectively.

**Corollary 1.** *Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P \neq P_L$  be a point of  $\mathcal{C}$  and let  $H_P(t)$  be its fibre function. Then,  $\text{mult}_P(\mathcal{C}) = \deg(H_P(t))$ .*

From these results and using the input parametrization, in [3], we construct a polynomial, the  $T$ -function, which provides essential information about the singularities of the curve. In order to define it, we need to introduce the following notation:

$$\delta_i := \deg_t(G_i), \quad \lambda_{ij} := \min\{\delta_i, \delta_j\}, \quad G_i^*(s, t) := \frac{G_i(s, t)}{t - s} \in \mathbb{K}[s, t]$$

and

$$R_{ij}(s) := \text{Res}_t(G_i^*, G_j^*) \in \mathbb{K}[s] \quad \text{for } i, j = 1, 2, 3, \quad i < j.$$

**Definition 3.** *The  $T$ -function of the parametrization  $\mathcal{P}(t)$  is*

$$T(s) = R_{12}(s)/p(s)^{\lambda_{12}-1}.$$

In [3], it is shown that  $T(s)$  is a polynomial and that it may also be expressed as:

$$T(s) = \frac{R_{13}(s)}{p_1(s)^{\lambda_{13}-1}} = \frac{R_{23}(s)}{p_2(s)^{\lambda_{23}-1}}. \quad (3)$$

In addition, the following lemma shows that the fibre function of any ordinary singularity is a factor of  $T(s)$ . This lemma is proved in [3].

**Lemma 1.** *Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P \neq P_L$  be an ordinary singular point of multiplicity  $m$ . It holds that*

$$T(s) = H_P(s)^{m-1}T^*(s),$$

where  $T^*(s) \in \mathbb{K}[s]$  and  $\gcd(H_P(s), T^*(s)) = 1$ .

Lemma 1 is the key for the proof of Theorem 3, which states that the factorization of the  $T$ -function provides the fibre functions of all the ordinary singularities in the curve. This theorem is also proved in [3].

**Theorem 3.** *Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P_1, \dots, P_n$  be the singular points of  $\mathcal{C}$ , with multiplicities  $m_1, \dots, m_n$  respectively. Let us assume that all of them are ordinary singularities and that  $P_i \neq P_L$  for  $i = 1, \dots, n$ . Then, it holds that*

$$T(s) = \prod_{i=1}^n H_{P_i}(s)^{m_i-1}.$$

The T–function totally describes the ordinary singularities of the curve, since it gives us the corresponding fiber functions. We recall that, from the fibre function of a point  $P$ , we can obtain its multiplicity, its fibre, and the tangent lines of the curve at  $P$  (see Theorem 2 and Corollary 1). In [4], it is proved that the theorem also holds if  $P_L$  is a singularity. An alternative approach for computing this factorization, based on the construction of  $\mu$ –basis, can be found in [8] (see also [1], [9], [7]). From Theorem 3, we can derive the following corollary.

**Corollary 2.** *Let  $\mathcal{C}$  be a rational plane curve of degree  $d$  such that all its singularities are ordinary. Let  $\mathcal{P}(t)$  be a proper parametrization of  $\mathcal{C}$  such that  $P_L$  is regular. It holds that  $\deg(T) = (d - 1)(d - 2)$ .*

This result may be used, for instance, for checking if  $P_L$  is a regular point. More precisely,  $\deg(T) < (d - 1)(d - 2)$  implies that  $P_L$  is not regular and the assumptions of Theorem 3 does not hold. Then, in order to use Theorem 1, an appropriate reparametrization should be applied (see Section 3 in [4]).

### 3. Detection and analysis of non–ordinary singularities

The methods described in Section 2 are valid for the detection and analysis of ordinary singularities. However, when non–ordinary singularities are involved, some new difficulties may appear. For instance, a non–ordinary singularity may have other singularities in its “neighborhood”, which can not be detected in a direct way. The analysis of these neighboring singularities is fundamental in order to compute the delta invariant, which measures, for instance, the contribution of the singularity to the genus formula. In order to describe the neighborhood of a non–ordinary singularity, some quadratic transformations called “blow-ups” may be applied to the curve (see [6], [12] or [32]).

In Subsection 3.1, we summarize the process of blowing up a given curve  $\mathcal{C}$  at a non-ordinary singularity  $P$ , and we introduce some notions, the *delta invariant* and *the number of local branches*, which are associated to  $P$ . Afterwards, in Subsection 3.2, we explain how to blow up a rational curve from a given parametrization. Finally, in Subsection 3.3, we generalize Theorem 3 (see Subsection 2.1) to non-ordinary singularities (see Theorem 4) which provides a method for studying also the non-ordinary singularities.

The main result of this paper, Theorem 4, claims that the factorization of the  $T$ -function provides the fibre functions of all the singularities in the curve, including the ordinary and the non-ordinary ones. Furthermore, for each non-ordinary singularity,  $T(s)$  also gives the fibre functions of all the singularities in its neighborhood. By analyzing these fibre functions, we can obtain essential information about each singularity, as its multiplicity, its fibre and the tangent lines of the curve at this point. In addition, for each non-ordinary singularity  $P$ , we can easily obtain the number of local branches centered at  $P$  as well as its corresponding delta invariant.

### 3.1. Description of the blowing-up process

In order to completely describe the non-ordinary singularities of a given curve, it must be “blown up”. The blowing up process consists in recursively applying certain quadratic transformations (blow-ups) for birationally transforming the curve into a new one with only ordinary singularities. More precisely, let  $P$  be a non-ordinary singularity of multiplicity  $m$  of a curve  $\mathcal{C}$ . The blowing up process can be summarized as follows (see [6], [12] or [32]):

**Step 1.** Apply a linear change of coordinates,  $\mathcal{L}$ , such that  $P$  is moved to  $(0 : 0 : 1)$ , none of its tangents is an irregular line (i.e. a line  $x_1 = 0, x_2 = 0$  or  $x_3 = 0$ ), and no other point on an irregular line is a singular point on  $\mathcal{C}$ .

**Step 2.** Apply the quadratic transformation  $\mathcal{T} = (x_2x_3, x_1x_3, x_1x_2)$  to  $\mathcal{C}$ , getting the transformed curve  $\mathcal{C}_1$ . It holds that:

- Outside of the irregular lines,  $\mathcal{T}$  preserves the multiplicity of points and their tangents (and thus, its character).
- New ordinary singularities might be created at the points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  and  $(0 : 0 : 1)$  (called the fundamental points).

- The new curve  $\mathcal{C}_1$  might have singularities, also non-ordinary ones, on the irregular line  $x_3 = 0$ . These singularities come from  $P$ . That is,  $P$  is replaced on  $\mathcal{C}_1$  by points of the form  $(1 : \gamma : 0)$ , with  $\gamma \neq 0$ . We denote by  $\xi_1 := \{P_1^1, \dots, P_{\alpha_1}^1\}$ , the set of points of multiplicities  $\{m_1^1, \dots, m_{\alpha_1}^1\}$ ,  $m_j^1 \geq 2$ , where  $P_i^1 = (1 : \gamma_i : 0)$ ,  $\gamma_i \neq 0$ ,  $i = 1, \dots, \alpha_1$ . We say that  $\xi_1$  is the *first neighborhood* of  $P$ .

**Step 3.** As we stated before, some of the singularities in the first neighborhood of  $P$  may be non-ordinary. In this case, we apply again Steps 1 and 2 to each non-ordinary singularity in  $\xi_1$ . Then, we get the second neighborhood of  $P$  as the union of the first neighborhoods of these non-ordinary singular points. We denote the second neighborhood of  $P$  as  $\xi_2 := \{P_1^2, \dots, P_{\alpha_2}^2\}$ . In general, we will call any point in one of the neighborhoods of  $P$ , a *neighboring point* of  $P$ .

It is proved that there are at most a finite number of singular points in the neighborhoods of any point of an irreducible curve (see [32], pp. 82). Hence the analysis of a singularity in terms of neighboring singularities is a finite process, which leads to a complete classification of all singular points. Observe that the process finishes when  $\xi_m = \emptyset$ , for some  $m \in \mathbb{N}$  and thus, this method always achieves an irreducible curve having only ordinary singularities in a finite number of steps (see [12]).

We also note that the above process can be generalized for the case of space curves of any dimension. For instance, for a curve in the 3-dimensional space, one would consider the transformation  $\mathcal{T}_s = (x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3)$ .

Let  $P \in \mathcal{C}$  be a singularity of multiplicity  $m_P$ . One can associate to  $P$  its *delta invariant*,  $\delta_P$ , and *the number of local branches*,  $r_P$ . The delta invariant is a very important number since for instance, the genus of an irreducible plane curve is the number  $(d-1)(d-2)/2 - \sum_{P \in \mathcal{S}} \delta_P$ , where  $d$  is the degree of the given curve, and  $\mathcal{S}$  is the set of singular points. Intuitively speaking, the delta invariant  $\delta_P$  measures the number of “equivalent” double points concentrated at  $P$ ; i.e. it takes into account the multiplicity of  $P$  and all its neighboring points (see Subsection 7.4.1 in [19], Subsection 2.5.4 in [5], Subsection 8.1 in [10] or Subsection 9.2.5 in [11]). Thus, the delta invariant of  $P$  is given by

$$\delta_P = m_P(m_P - 1)/2 + \sum_{j=1}^n \sum_{i=1}^{\alpha_j} m_i^j(m_i^j - 1)/2, \quad (4)$$

where  $n$  is the *number of neighborhoods*,  $\xi_j := \{P_1^j, \dots, P_{\alpha_j}^j\}$ ,  $j = 1, \dots, n$  (in order to compute  $\delta_P$ , one also may apply the formula using intersection index of Puiseux expansion; see e.g. Subsection 2.5.4 in [5]).

In general  $r_P \leq m_P$  and  $m_P(m_P - 1)/2 \leq \delta_P$ , and both of these inequalities are equalities when  $P$  is an ordinary singularity of multiplicity  $m_P$ . If  $P$  is a non-ordinary singularity the following situations may appear (see [12] or [32]):

- S<sub>1</sub>.  $r_P < m_P$ : in this case there exist  $m_P$  values of the parameter  $t$ , namely  $s_1, \dots, s_{m_P}$ , such that  $\mathcal{P}(s_j) = P$  but only  $r_P$  of them are different.
- S<sub>2</sub>.  $m_P(m_P - 1)/2 < \delta_P$ : in this case we have that  $\delta_{P^1} = \delta_P - m_P(m_P - 1)/2$ , where  $\delta_{P^1} := \sum_{j=1}^{\alpha_1} \delta_{P_j^1}$  and, in general,

$$\delta_{P^i} = \delta_{P^{i-1}} - \sum_{j=1}^{\alpha_{i-1}} m_j^{i-1} (m_j^{i-1} - 1)/2,$$

where  $\delta_{P^i} := \sum_{j=1}^{\alpha_i} \delta_{P_j^i}$  for  $i = 1, \dots, n$  (we denote  $P^0 := P$ ). In the above expression, the  $i$ -th neighborhood of  $P$  is given by the set of points  $\xi_i := \{P_1^i, \dots, P_{\alpha_i}^i\}$  of multiplicities  $\{m_1^i, \dots, m_{\alpha_i}^i\}$ , and delta invariants  $\{\delta_1^i, \dots, \delta_{\alpha_i}^i\}$ , respectively (for  $i = 1, \dots, n$ ). In addition,  $\delta_{P^{n+1}} = 0$  and  $\xi_{n+1} = \emptyset$ .

From the situations above considered, one may classify any non-ordinary singularity  $P$  as follows:

- We say that  $P$  is a *type I non-ordinary singularity* if only S<sub>1</sub> holds.
- We say that  $P$  is a *type II non-ordinary singularity* if only S<sub>2</sub> holds.
- We say that  $P$  is a *type III non-ordinary singularity* if both, S<sub>1</sub> and S<sub>2</sub>, hold.

Note that if none of these situations hold, we have that  $P$  is ordinary.

### 3.2. Blowing up of a rational curve from a given parametrization

In the following, we show how to blow up a rational plane curve  $\mathcal{C}$  from a given parametrization  $\mathcal{P}(t)$ . For this purpose, we apply the three steps introduced in Subsection 3.1 and we see how the parametrization, after blowing up, can be computed from the parametrization before blowing up.

### Step 1

Let  $P$  be an  $m$ -fold point of  $\mathcal{C}$ . We apply a linear change of coordinates,  $\mathcal{L}$ , such that  $P$  is moved to  $(0 : 0 : 1)$ . Thus, let  $P = (0 : 0 : 1) \in \mathcal{C}$  be a point of multiplicity  $m$ . We assume w.l.o.g that  $P$  is not the limit point (see Definition 1). Otherwise, we consider a reparametrization of  $\mathcal{P}$  (see Section 3 in [4]). From (2), we may write  $p_j(t) = H_P(t)\bar{p}_j(t)$ ,  $j = 1, 2$  with  $\gcd(\bar{p}_1, \bar{p}_2) = 1$ ,  $\gcd(H_P, p) = \gcd(p_1, p_2, p) = 1$  and  $\deg(H_P) = m$  (see Corollary 1). Then,

$$\mathcal{P}(t) = (p_1(t), p_2(t), p(t)) = (H_P(t)\bar{p}_1(t), H_P(t)\bar{p}_2(t), p(t)).$$

In this first step of the blowing up process, we have to impose the following two assumptions:

- a) None of the tangents of  $P$  is an irregular line. For this purpose, we have to ensure that  $\mathcal{P}'(s_0) \neq (1, 0, \alpha)$  and  $\mathcal{P}'(s_0) \neq (0, 1, \alpha)$  for any  $s_0$  in the fibre of  $P$  (i.e., such that  $H_P(s_0) = 0$ ) and any  $\alpha \in \mathbb{K}$  (if  $\mathcal{P}'(s_0) = (0, 0, 0)$ , we reason with the second derivative; in general, we reason with the first derivative different from zero). Thus, we need that  $\gcd(p'_i, H_P) = 1$  for  $i = 1, 2$ .

We observe that  $p_i(t) = H_P(t)\bar{p}_i(t)$ , which implies that

$$p'_i(s_0) = H'_P(s_0)\bar{p}_i(s_0) + H_P(s_0)\bar{p}'_i(s_0) = H'_P(s_0)\bar{p}_i(s_0).$$

That is,  $p'_i(s_0) = 0$  implies that  $\bar{p}_i(s_0) = 0$  or  $H'_P(s_0) = 0$ . Note that  $H'_P(s_0) = 0$  is equivalent to  $(t - s_0)^2$  divides  $H_P(t)$  and in this case, we would have a tangent of multiplicity at least 2. In this case, we would need that  $p''_i(s_0) \neq 0$ , which implies that  $H''_P(s_0)\bar{p}_i(s_0) \neq 0$ . Note that  $H''_P(s_0) = 0$  implies that  $(t - s_0)^3$  divides  $H_P(t)$  and we would have a tangent of multiplicity at least 3, and so on. Thus,  $p'_i(s_0) = 0$  is equivalent to  $\bar{p}_i(s_0) = 0$  and the condition to impose is

$$\gcd(H_P(t), \bar{p}_i(t)) = 1, \quad \text{for } i = 1, 2. \quad (5)$$

- b) No other point on an irregular line is a singular point. That is, any point  $\mathcal{P}(a) \neq P$  such that  $p_1(a) = 0$ ,  $p_2(a) = 0$  or  $p(a) = 0$  must be regular. From Definition 2 and Corollary 1, we need that

$$H_{\mathcal{P}(a)}(t) = \gcd(G_1(a, t), G_2(a, t), G_3(a, t)) = t - a$$



where

$$G_j(a, t) = p_j(a)p(t) - p(a)p_j(t), \quad j = 1, 2, \quad G_3(a, t) = p_1(a)p_2(t) - p_2(a)p_1(t).$$

If  $p_2(a) = 0$  we have that  $G_2(a, t) = -p(a)p_2(t)$  and  $G_3(a, t) = p_1(a)p_2(t)$ , so  $H_{\mathcal{P}(a)}(t) = \gcd(p_1(a)p(t) - p(a)p_1(t), p_2(t))$ . Now, taking into account that  $\gcd(H_P, H_{\mathcal{P}(a)}) = 1$ , we deduce that

$$H_{\mathcal{P}(a)}(t) = \gcd(p_1(a)p(t) - p(a)p_1(t), \bar{p}_2(t)).$$

Similarly, if  $p_1(a) = 0$  we have that

$$H_{\mathcal{P}(a)}(t) = \gcd(p_2(a)p(t) - p(a)p_2(t), \bar{p}_1(t)).$$

In addition, if  $p(a) = 0$  we have that

$$H_{\mathcal{P}(a)}(t) = \gcd(p_1(a)p_2(t) - p_2(a)p_1(t), p(t)).$$

Hence, in order to get this assumption, we impose that

$$\gcd(G_i(a, t), \bar{p}_j(t)) = t - a, \quad \gcd(G_3(a, t), p(t)) = t - a \quad (6)$$

where  $\bar{p}_i(a) = p(a) = 0, i, j \in \{1, 2\}, i \neq j$ .

We have to check whether  $\mathcal{P}(t)$  satisfies both conditions. In the negative case, we should apply a linear change of coordinates to  $\mathcal{P}(t)$ .

## Step 2

The second step in the blowing up process consists in applying the quadratic transformation  $\mathcal{T} = (x_2x_3, x_1x_3, x_1x_2)$  to  $\mathcal{C}$ , getting the transformed curve  $\mathcal{C}_1$ . Thus, by applying  $\mathcal{T}$  to  $\mathcal{P}(t)$ , we get the projective parametrization of  $\mathcal{C}_1$  defined by

$$\begin{aligned} \mathcal{M}_1(t) &:= \mathcal{T}(\mathcal{P}(t)) = \\ &= (p_2(t)p(t), p_1(t)p(t), p_1(t)p_2(t)) = (\bar{p}_2(t)p(t), \bar{p}_1(t)p(t), H_P(t)\bar{p}_1(t)\bar{p}_2(t)). \end{aligned}$$

Now, we check whether there exist singular points of the form  $(1 : \gamma : 0) \in \mathcal{C}_1$ , with  $\gamma \neq 0$  (note that if these points exist, they should be reached by the values of  $t$  being roots of the polynomial  $H_P(t)$ ). The singular points obtained in this way constitute the first neighborhood of  $P$ , which we denote by  $\xi_1$ .

### Step 3

If all the singularities in  $\xi_1$  are ordinary the blowing up process is over. However, some of them might be non-ordinary and, in this case, we should apply the blowing up to each of them. Let  $P^1 := (1 : \gamma : 0) \in \mathcal{C}_1$ , with  $\gamma \neq 0$ , be one of these non-ordinary singularities. We first apply a change of coordinates such that  $P^1$  is moved to  $(0 : 0 : 1)$  and the conditions appearing in equalities (5) and (6) hold. Then, we apply again the transformation  $\mathcal{T}$ , and we obtain a parametrization  $\mathcal{M}_2$  that defines a new curve  $\mathcal{C}_2$ . The singular points  $(1 : \delta : 0) \in \mathcal{C}_2$ , with  $\delta \neq 0$ , constitute the first neighborhood of  $P^1$  and, consequently, they take part of the second neighborhood of  $P$ , which we denote by  $\xi_2$ . This process must be recursively applied until we get a curve without non-ordinary singularities.

We note that the above process can be generalized for the case of space curves of any dimension.

In the following example, we illustrate the method above described and we blow up a given irreducible plane curve defined by a parametrization  $\mathcal{P}(t)$ . We compute its singularities and, for each singularity  $P$ , we compute the delta invariant ( $\delta_P$ ) and the number of local branches ( $r_P$ ).

**Example 1.** *Let  $\mathcal{C}$  be the plane curve over  $\mathbb{C}$  defined by the parametrization  $\mathcal{P}(t) = (t^2, t^5, 1)$ . One may check (see, for instance, [22] or Chapter 2 of [27]) that  $\mathcal{C}$  has two singularities:  $P_1 = (0 : 0 : 1)$  of multiplicity  $m_{P_1} = 2$ , and  $P_2 = (0 : 1 : 0)$  of multiplicity  $m_{P_2} = 3$ . Let us analyze both points:*

- *We start with  $P_1 = (0 : 0 : 1)$  and we observe that  $H_{P_1}(t) = t^2$  (see Definition 2). Thus, the number of parameters  $t$  corresponding to  $P_1$  is 2 ( $t = 0, 0$ ), and  $r_{P_1} = 1 < m_{P_1} = 2$ , so  $S_1$  holds (see Subsection 3.1).*

*Now, we analyze the neighboring points of  $P_1$ . For this purpose, we first note that  $\gcd(H_{P_1}(t), \bar{p}_2(t)) = t^2$ , so the conditions for the blowing up do not hold (see (5)). Hence, we apply the change of coordinates  $\mathcal{L} = (1/2x_1 - 1/2x_2, 1/2x_2 + 1/2x_1, x_3)$  and we obtain a new curve that can be parametrized by*

$$\mathcal{L}(\mathcal{P}(t)) = (1/2t^2 - 1/2t^5, 1/2t^5 + 1/2t^2, 1).$$

*One may check that the conditions for the blowing-up are now satisfied:  $\mathcal{L}(P_1) = (0 : 0 : 1)$ , none of its tangents is an irregular line, and no*

other point on an irregular line is singular. Then, we consider the parametrization

$$\mathcal{M}_1(t) = \mathcal{T}(\mathcal{L}(\mathcal{P}(t))) = (1 + t^3, 2/3 - t^3, -1/6t^5 + 1/3t^2 - 1/2t^8),$$

where  $\mathcal{T} = (x_2x_3, x_1x_3, x_1x_2)$ . We apply the results in [22] (see Corollary 1), and we get that  $P_1^1 = (1 : 1 : 0)$  is an ordinary singularity of multiplicity 2, the points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  are singularities of multiplicity 3, and  $(0 : 0 : 1)$  is a singularity of multiplicity 5. Thus, the first neighborhood of  $P_1$  is given by  $P_1^1$  and the process finishes with this point (see step 2 of the blowing-up process).

Note that  $\delta_{P_1} = 2 > m_{P_1}(m_{P_1} - 1)/2 = 1$ , so  $S_2$  holds and  $P_1$  is a type III non-ordinary singularity (see Subsection 3.1).

- Now, we reason similarly for  $P_2 = (0 : 1 : 0)$ . However, note that this is the limit point of the parametrization (see Definition 1). To deal with this situation we can use the reparametrization  $\mathcal{U}(t) := \mathcal{P}(1/t)$ , which allows us to reach the limit point as  $P_L = \mathcal{U}(0)$  (see Section 3 in [4]). Now we have that  $P_2 = \mathcal{U}(0)$  and  $H_{P_2}(t) = t^3$ . Thus, the number of parameters  $t$  corresponding to  $P_2$  is 3 ( $t = 0, 0, 0$ ), and  $r_{P_2} = 1 < m_{P_2} = 3$ , so  $S_1$  holds (see Subsection 3.1).

Let us analyze the neighboring points of  $P_2$ . Recall that  $P_2 = (0 : 1 : 0)$ , so we first have to apply a change of coordinates that moves it to  $(0 : 0 : 1)$ . We take  $\mathcal{L} = (1/2x_1 + 1/2x_3, -1/2x_3 + 1/2x_1, x_2)$  (see step 1 of the blowing-up process) and we check that  $\mathcal{L}(\mathcal{U}(t))$  satisfies the conditions for the blowing up (see (5) and (6)). Now, we apply the blowing up by considering the following parametrization:

$$\mathcal{M}_1(t) = \mathcal{T}(\mathcal{L}(\mathcal{U}(t))) = (6 + 6t^2, 4 - 6t^2, 2t^3 - t^5 - 3t^7)$$

(see step 2 of the blowing-up process). We apply the results in [22] (see Corollary 1), and we get that  $P_2^1 = (6 : 4 : 0)$  is an ordinary singularity of multiplicity 2, and the points  $(1 : 0 : 0)$ ,  $(0 : 1 : 0)$  are singularities of multiplicity 2. The first neighborhood of  $P_2$  is given by  $P_2^1$  and thus, the process finishes with the point  $P_2^1$ .

Finally, we note that  $\delta_{P_2} = 4 > m_{P_2}(m_{P_2} - 1)/2 = 3$ , so  $P_2$  is a type III non-ordinary singularity (see Subsection 3.1).

### 3.3. The $T$ -function and the non-ordinary singularities

In Section 2.1 we have summarized a method, developed in [3], for obtaining the ordinary singularities of a rational curve from its parametric expression. This approach is based on the construction of the  $T$ -function, a polynomial whose factors give us the fibre functions of the different ordinary singularities (see Theorem 3). From the fibre function of a point, one can obtain relevant information like its multiplicity, its fibre or the tangent lines of the curve at that point.

The main result presented in this paper, Theorem 4, shows how the  $T$ -function can also be used to obtain and characterize non-ordinary singularities. Furthermore, by analyzing this function we can get a partial knowledge of the different neighborhoods, which allows us, for instance, to directly compute the delta invariant of each non-ordinary singularity.

The following result provides a first approach to the proof of Theorem 4. Observe that it is similar to Lemma 1, but in this case the singular point  $P$  is non-ordinary. To enhance the reading flow, the proof of the lemma is presented in Section 6.

**Lemma 2.** *Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P \neq P_L$  be a non-ordinary singular point of multiplicity  $m$ , and let  $\xi_1 = \{P^1\}$ , where  $P^1$  is an ordinary singular point of multiplicity  $m_1$ . Then it holds that*

$$T(s) = H_P(s)^{m-1} H_{P^1}(s)^{m_1-1} K(s),$$

where  $K(s) \in \mathbb{K}[s]$ ,  $H_{P^1}(s)$  divides  $H_P(s)$ , and  $\gcd(H_P(s), K(s)) = 1$ .

The next lemma generalizes Lemma 2 and allows us to deal with more difficult situations. More precisely, we consider a point  $P$  that has several singular points in its different neighborhoods. The proof of this lemma is also presented in Section 6.

**Lemma 3.** *Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P \neq P_L$  be a singular point of multiplicity  $m$ , let  $\xi_i := \{P_1^i, \dots, P_{\alpha_i}^i\}$  be its  $i$ -th neighborhood and let  $m_j^i$  be the multiplicity of  $P_j^i$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, \alpha_i$  ( $\delta_{P^{n+1}} = 0$  and  $\xi_{n+1} = \emptyset$ ). It*

holds that

$$T(s) = H_P(s)^{m-1} \prod_{i=1}^n \prod_{j=1}^{\alpha_i} H_{P_j^i}(s)^{m_j^i-1} T^*(s), \quad (7)$$

where  $T^*(s) \in \mathbb{K}[s]$ ,  $H_{P_j^i}(s)$  divides  $H_P(s)$ , and  $\gcd(H_P(s), T^*(s)) = 1$ ,  $j = 1, \dots, \alpha_i$ ,  $i = 1, \dots, n$ .

**Remark 1.** a) Equality (7) can be written as  $T(s) = \mathcal{H}_P(s)T^*(s)$  where

$$\mathcal{H}_P(s) = H_P(s)^{m-1} \prod_{i=1}^n \prod_{j=1}^{\alpha_i} H_{P_j^i}(s)^{m_j^i-1}. \quad (8)$$

In the following,  $\mathcal{H}_P(s)$  will be called the contribution of  $P$  to the  $T$ -function. It contains a lot of information about  $P$  since its factors are the fibre functions of  $P$  and all its neighboring singularities. From Lemma 3, we have that  $\gcd(\mathcal{H}_P(s), T^*(s)) = 1$ .

b) The delta invariant of  $P$  can be easily obtained from  $\mathcal{H}_P$  since, from (4), we have that

$$2\delta_P = m(m-1) + \sum_{i=1}^n \sum_{j=1}^{\alpha_i} m_j^i(m_j^i-1) = \deg(\mathcal{H}_P).$$

c) We observe that, in (8), the product  $\prod_{i=1}^n \prod_{j=1}^{\alpha_i} H_{P_j^i}(s)^{m_j^i-1}$  contains the fibre functions of the singularities in the neighborhood of  $P$ . If  $\xi_1 = \dots = \xi_n = \emptyset$ , we have that  $H_{P_j^i}(s) = 1$  for every  $j = 1, \dots, \alpha_i$ ,  $i = 1, \dots, n$  and then,  $\mathcal{H}_P(s) = H_P(s)^{m-1}$ . In particular, if  $P$  is ordinary, Lemma 3 states that  $T(s) = H_P(s)^{m-1}T^*(s)$ , which agrees with Lemma 2 in [3] (see also Lemma 1).

Now, we are ready to state the main result of the paper, Theorem 4, which can be directly proved from Lemma 3. It claims that the factorization of the  $T$ -function provides the contributions of all the ordinary and non-ordinary singularities of the curve. We will see that, by analyzing these contributions one may easily obtain the multiplicity, character, number of branches and delta invariant of every singularity (see Example 2).

**Theorem 4. (Main theorem)** *Let  $\mathcal{C}$  be a rational plane curve and let  $\mathcal{P}(t)$  be a proper parametrization of  $\mathcal{C}$  such that  $P_L$  is regular. Let  $P_1, \dots, P_\ell$  be the singular points of  $\mathcal{C}$  and let  $\mathcal{H}_{P_1}, \dots, \mathcal{H}_{P_\ell}$  be their corresponding contributions to the  $T$ -function. Then, it holds that*

$$T(s) = \prod_{k=1}^{\ell} \mathcal{H}_{P_k}(s).$$

**Proof:** Taking into account Lemma 3, Remark 1, statement (a), for each singular point  $P_k$ , we have that

$$T(s) = \mathcal{H}_{P_k}(s)T_k^*(s),$$

where  $T_k^*(s) \in \mathbb{K}[s]$  and  $\gcd(\mathcal{H}_k(s), T_k^*(s)) = 1$ . In addition, note that  $\gcd(\mathcal{H}_{P_i}(s), \mathcal{H}_{P_j}(s)) = 1$  for  $i \neq j$  (otherwise, there would exist  $s_0 \in \mathbb{K}$  such that  $H_{P_i}(s_0) = H_{P_j}(s_0) = 0$ , that is,  $\mathcal{P}(s_0) = P_i = P_j$ ). Then, we get that

$$T(s) = \prod_{k=1}^{\ell} \mathcal{H}_{P_k}(s)V(s),$$

where  $V(s) \in \mathbb{K}[s]$  and  $\gcd(\mathcal{H}_{P_k}, V) = 1$  for  $k = 1, \dots, \ell$ .

Finally, we prove that  $V(s) \in \mathbb{K}$ . For this purpose, we recall that

$$T(s) = \frac{R_{13}(s)}{p_1(s)^{\lambda_{13}-1}},$$

(see (3)).

Note that if  $V(s_0) = 0$ , then  $T(s_0) = 0$  and, thus,  $R_{12}(s_0) = R_{13}(s_0) = R_{23}(s_0) = 0$ . From  $R_{13}(s_0) = \text{Res}_t(G_1^*(s, t), G_3^*(s, t))(s_0) = 0$ , using the properties of the resultants (see e.g. [27]), we deduce that one of the following two statements hold:

1. There exists  $s_1 \in \mathbb{K}$  such that  $G_1^*(s_0, s_1) = G_3^*(s_0, s_1) = 0$ . Thus,  $\deg(H_P) \geq 2$  where for  $P = \mathcal{P}(s_0) = \mathcal{P}(s_1)$ . This implies that  $P$  is a singular point of multiplicity at least 2 (see Corollary 1), so there is some  $k = 1, \dots, \ell$  such that  $P = P_k$  and, hence,  $\gcd(\mathcal{H}_{P_k}, V) \neq 1$ , which is impossible.

2. It holds that  $\gcd(\text{lc}_t(G_1^*), \text{lc}_t(G_3^*))(s_0) = 0$ . However, this is also a contradiction since we would have that

$$\text{lc}_t(G_1^*)(s_0) = \text{lc}_t(G_1)(s_0) = p_1(s_0)c_d - p(s_0)a_d = 0$$

and

$$\text{lc}_t(G_3^*)(s_0) = \text{lc}_t(G_3)(s_0) = p_1(s_0)b_d - p_2(s_0)a_d = 0.$$

From these two equalities we deduce that  $\mathcal{P}(s_0) = (a_d : b_d : c_d) = P_L$  and, thus, the limit point is reached by the parametrization. This implies that  $P_L$  is a singularity (see Proposition 3.4 in [4]), which contradicts the assumptions of the theorem.

Thus,  $V(s) \in \mathbb{K}$  and we conclude that, up to constants in  $\mathbb{K} \setminus \{0\}$ ,

$$T(s) = \prod_{k=1}^{\ell} \mathcal{H}_{P_k}(s).$$

□

**Corollary 3.** *Let  $\mathcal{C}$  be a rational plane curve of degree  $d$ . Let  $\mathcal{P}(t)$  be a proper parametrization of  $\mathcal{C}$  such that  $P_L$  is regular. It holds that*

$$\deg(T) = (d-1)(d-2).$$

**Proof:** Let  $P_1, \dots, P_\ell$  be the singular points of  $\mathcal{C}$ , with delta invariants  $\delta_1, \dots, \delta_\ell$ , respectively. Then, from Theorem 4 and Remark 1 (statement (b)), we get that

$$\deg(T(s)) = \sum_{k=1}^{\ell} \deg(\mathcal{H}_k(s)) = 2 \sum_{k=1}^{\ell} \delta_{P_k}.$$

Since the genus of an irreducible plane curve of degree  $d$  is the number  $(d-1)(d-2)/2 - \sum_{k=1}^{\ell} \delta_{P_k}$  (see [27]), and  $\mathcal{C}$  is a rational curve (i.e. its genus is 0), we get that  $\deg(T(s)) = 2 \sum_{k=1}^{\ell} \delta_{P_k} = (d-1)(d-2)$ . □

**Remark 2.** *Corollary 3 may be used for checking if  $P_L$  is a regular point. More precisely, if  $\deg(T) < (d-1)(d-2)$  then  $P_L$  is not regular and the assumptions of Theorem 4 do not hold. Therefore, in order to use this theorem, an appropriate reparametrization should be applied (see Section 3 in [4]).*

**Example 2.** Let  $\mathcal{C}$  be the plane curve over  $\mathbb{C}$  defined by the parametrization

$$\mathcal{P}(t) = (2t^4 - 8t^3 + 2t^2 + 20t - 16, t^4 - 2t^3 - 11t^2 + 32t - 20, t^2 + 1).$$

We will obtain its singularities using the  $T$ -function. First, we compute the polynomials  $G_1(s, t)$ ,  $G_2(s, t)$  and  $G_3(s, t)$  introduced in (1):

$$G_1(s, t) = 18t^2 - 8t^3 - 8t^3s^2 + 2t^4 + 2t^4s^2 + 20t + 20ts^2 - 18s^2 + 8s^3 + 8s^3t^2 - 2s^4 - 2s^4t^2 - 20s - 20st^2.$$

$$G_2(s, t) = 9t^2 - 2t^3 - 2t^3s^2 + t^4 + t^4s^2 + 32t + 32ts^2 - 9s^2 + 2s^3 + 2s^3t^2 - s^4 - s^4t^2 - 32s - 32st^2.$$

$$G_3(s, t) = 112t - 112s - 216t^2 - 92s^3t^2 + 24s^4t^2 + 284st^2 - 216t^3s + 44t^4s - 4t^3s^4 + 4t^4s^3 + 216ts^3 - 44ts^4 + 92t^3s^2 - 24t^4s^2 - 284ts^2 + 128t^3 - 24t^4 + 216s^2 - 128s^3 + 24s^4.$$

We have that  $G(s, t) = \gcd(G_1(s, t), G_2(s, t), G_3(s, t)) = s - t$ , so we get that  $\mathcal{P}$  is a proper parametrization. Let  $G_i^*(s, t) = G_i(s, t)/(s - t)$  for  $i = 1, 2, 3$ . Then,

$$R_{12}(s) = \text{Res}_t(G_1^*(s, t), G_2^*(s, t)) = 1280(5t^2 - 24t + 37)(t-1)^2(t-2)^2(1+t^2)^3.$$

In addition, note that  $d_1 = 4$ ,  $d_2 = 4$  and  $d_3 = 2$ . Hence,  $\delta_1 = \delta_2 = 4$  and  $\lambda_{12} = 4$ . Now, by applying Definition 3, we obtain the  $T$ -function:

$$T(s) = R_{12}(s)/p(s)^{\lambda_{12}-1} = 1280(5t^2 - 24t + 37)(t-1)^2(t-2)^2.$$

The factors of  $T$  are the fibre functions of the singularities. Let us analyze each of them:

- The factor  $(5t^2 - 24t + 37)$  has the conjugate complex roots  $s_1 = 12/5 + \sqrt{41}/5I$  and  $s_2 = 12/5 - \sqrt{41}/5I$ . Both of them point to the singularity

$$P_1 = \mathcal{P}(s_1) = \mathcal{P}(s_2) = (-1450, -1900, 4959),$$

which corresponds to the affine point  $(-171/50, -261/100)$ . The fibre function of  $P_1$  is  $H_{P_1}(t) = 5t^2 - 24t + 37$ . Hence, we deduce that  $P_1$  is a double point, since  $m_1 = \deg(H_{P_1}) = 2$ , and that  $\mathcal{C}$  has two branches at this point ( $H_{P_1}$  has two simple roots). In addition, the contribution of  $P_1$  to the  $T$ -function is  $\mathcal{H}_{P_1}(s) = H_{P_1}(s)$ . From Remark 1, statement (b), we deduce that the delta invariant is  $\delta_{P_1} = \deg(\mathcal{H}_{P_1})/2 = 1$ . Thus,  $P_1$  is an ordinary double point (see Subsection 3.1).



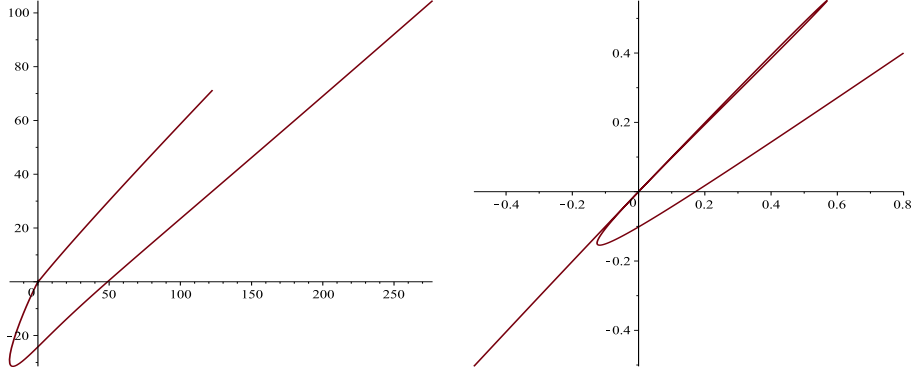


Figure 1: General view of curve  $\mathcal{C}$  (left) and detailed view of the non-ordinary double point  $P_2 = (0 : 0 : 1)$  (right)

- The factor  $(t - 1)$  gives us a third root of the  $T$ -function,  $s_3 = 1$ , that points to the singularity  $P_2 = \mathcal{P}(1) = (0 : 0 : 1)$ . Its fibre function is  $H_{P_2}(s) = (t - 1)(t - 2)$ . Then, reasoning as above, we deduce that  $P_2$  is a double point and that  $\mathcal{C}$  has two different branches centered at this point. However, the contribution of  $P_2$  to the  $T$ -function is  $\mathcal{H}_{P_2}(s) = (t - 1)^2(t - 2)^2$  and, consequently, the delta invariant is  $\delta_{P_2} = \deg(\mathcal{H}_{P_2})/2 = 2$ . Thus,  $P_2$  is a type II non-ordinary double point (see Subsection 3.1).

We recall that type II singularities do always have other singularities in its neighborhood, whose fibre functions are given by

$$\frac{\mathcal{H}_P(s)}{H_P(s)^{m-1}} = \prod_{i=1}^n \prod_{j=1}^{\alpha_i} H_{P_j^i}(s)^{m_j^i-1} = (t - 1)(t - 2)$$

(see Remark 1, statement (c)). Thus, in this case, we can deduce that  $\xi_1 = \{P_2^1\}$ , where  $H_{P_2^1}(s) = (t - 1)(t - 2)$  (it is a double point), and  $\xi_2 = \dots = \xi_n = \emptyset$ .

In Figure 1, we plot the curve  $\mathcal{C}$  and we observe that it has two branches centered at  $P_2$  although both of them have the same tangent. We can not appreciate  $P_1$  since it is an isolated point.

**Remark 3.** *In general, different conjugate roots of the T–function appear all together under a unique irreducible polynomial. These roots are associated to families of conjugated parametric points (see Definition 4 in [3]). In [22] (Theorem 16), it is shown that all the points in such a family have the same multiplicity and character.*

*Let us assume that  $T(s)$  includes a factor  $m(s)^{k-1}$ , where  $m(s)$  is an irreducible polynomial of degree  $l$ . Then,  $m(s)$  contains the fibre functions of  $l/k$  singular points of multiplicity  $k$  (see Theorem 5 in [3]). Note that we have just faced this situation in Example 2, where the fibre function of  $P_1$  was given by the irreducible polynomial  $m(t) = 5t^2 - 24t + 37$ .*

#### 4. Algorithm and example

In Section 3, we show that the T–function may be used to obtain essential information concerning the singularities of the curve. In particular, the factorization of  $T(s)$  provides the contributions of the different singularities to the T–function. We recall that, by analyzing these contributions, we can compute the corresponding multiplicities, number of branches and delta invariants.

However, the contributions of the different singularities use to appear scrambled and it may be difficult, for unfamiliar users, to get conclusions by simply watching the T–function. In the following, we present an algorithm that allows us to extract and organize all the information provided by  $T(s)$ . For this purpose, we recall that the contribution of an  $m$ -fold point  $P$  to the T–function is given by

$$\mathcal{H}_P(s) = H_P(s)^{m-1} \prod_{i=1}^n \prod_{j=1}^{\alpha_i} H_{P_j^i}(s)^{m_j^i-1},$$

where  $\xi_i := \{P_1^i, \dots, P_{\alpha_i}^i\}$  is the  $i$ -th neighborhood and  $\{m_j^i\}$  is the multiplicity of  $P_j^i$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, \alpha_i$  ( $\delta_{P^{n+1}} = 0$  and  $\xi_{n+1} = \emptyset$ ).

We can easily obtain  $\mathcal{H}_P(s)$  from  $T(s)$  if we know the fibre function  $H_P(s)$ . Note that  $H_{P_j^i}(s)|H_P(s)$  and  $\gcd(\mathcal{H}_P, \mathcal{H}_Q) = 1$  if  $P \neq Q$  (see the proof of Lemma 3). Thus,  $\mathcal{H}_P$  is composed by all the factors of  $H_P$ , each of them

raised to its power in the T–function. That is, let

$$H_P(s) = \prod_{i=1}^r (s - s_i)^{f_i},$$

where  $s_i \neq s_j$  for  $i \neq j$ ,  $i, j = 1, \dots, r$ . Note that  $\sum_{i=1}^r f_i = m$  and  $r$  is the number of branches of the curve centered at  $P$ . Then  $\mathcal{H}_P(s)$  has the form

$$\mathcal{H}_P(s) = \prod_{i=1}^r (s - s_i)^{g_i},$$

where  $g_i \geq f_i$  for  $i = 1, \dots, r$ . Note that  $\sum_{i=1}^r g_i = 2\delta_P$ . Now, we may classify the singularity  $P$  as follows (see Subsection 3.1):

- If  $r = m$  and  $2\delta_P = m(m - 1)$  then,  $P$  is an ordinary singularity.
- If  $r < m$  and  $2\delta_P = m(m - 1)$  then,  $P$  is a type I non–ordinary singularity.
- If  $r = m$  and  $2\delta_P > m(m - 1)$  then,  $P$  is a type II non–ordinary singularity.
- If  $r < m$  and  $2\delta_P > m(m - 1)$  then,  $P$  is a type III non–ordinary singularity.

The above statements are used for developing the following algorithm. We assume that the conditions of Theorem 4 are verified that is, we have a proper parametrization with a regular limit point. Note that, if  $\mathcal{P}$  is not proper, one can always get a proper reparametrization (see [21]). In addition, in [3] and [4] one can find some linear changes of variables that may be applied for the limit point to be regular (see also Example 1).

**Algorithm Classification of Singularities.**

Given a parametrization,  $\mathcal{P}(t)$ , of a rational plane curve  $\mathcal{C}$ , the algorithm returns the singularities as well as their multiplicity and character. For the non-ordinary singularities, the algorithm classifies the singularities as type I, II or III (see Subsection 3.1) and returns their number of branches and delta invariant.

1. Compute  $T(s)$  and get its factorization.
2. Let  $T_{aux}(s) = T(s)$  and  $k = 1$  and repeat steps 2.1 to 2.5 until the algorithm finishes.
  - 2.1. Take one factor of  $T_{aux}(s)$ , namely  $(s - s_0)$ . If  $T_{aux}(s) = 1$  the algorithm finishes.
  - 2.2. Compute  $P_k := \mathcal{P}(s_0)$  and  $H_{P_k}(t)$  (see Definition 2). The multiplicity of  $P_k$  is  $m_k = \deg(H_{P_k})$ . The number of different factors in  $H_{P_k}$  is the number of branches,  $r_k$ .
  - 2.3. Compute  $\mathcal{H}_{P_k}(s)$  by taking all the factors of  $H_{P_k}$  raised to its power in  $T_{aux}$ . The delta invariant of  $P_k$  is  $\delta_{P_k} = \deg(\mathcal{H}_{P_k})/2$ .
  - 2.4. Choose one of the following options:
    - a) If  $r_k = m_k$  and  $2\delta_{P_k} = m_k(m_k - 1)$ , RETURN  $P_k$  is an ordinary singularity of multiplicity  $m_k$ .
    - b) If  $r_k < m_k$  and  $2\delta_{P_k} = m_k(m_k - 1)$ , RETURN  $P_k$  is a type I non-ordinary singularity with  $r_k$  branches.
    - c) If  $r_k = m_k$  and  $2\delta_{P_k} > m_k(m_k - 1)$ , RETURN  $P_k$  is a type II non-ordinary singularity with delta invariant  $\delta_{P_k}$ .
    - d) If  $r_k < m_k$  and  $2\delta_{P_k} > m_k(m_k - 1)$ , RETURN  $P_k$  is a type III non-ordinary singularity with  $r_k$  branches and delta invariant  $\delta_{P_k}$ .
  - 2.5. Let  $T_{aux}(s) := T_{aux}(s)/\mathcal{H}_{P_k}(s)$  and  $k := k + 1$ .

In the following, we illustrate the performance of algorithm Classification of Singularities with an example.

**Example 3.** Let  $\mathcal{C}$  be the plane curve over  $\mathbb{C}$  defined by the projective parametrization  $\mathcal{P}(t) =$

$$(35/2t^4 + 2t^3 - 5t^2 + 1/2 - 25t^6 - 10t^5 + 12t^8 + 8t^7, t^4 + 2t^3 - 5t^6 - 10t^5 + 4t^8 + 8t^7, t^8).$$

First, we note that from [27] (see Chapter 4), the degree of the curve is  $d = \max\{\deg(p_1), \deg(p_2), \deg(p)\} = 8$ . Furthermore, by using Theorem 1, we get that  $\mathcal{P}$  is a proper parametrization. Hence, we can study the singularities of  $\mathcal{C}$  by applying algorithm Classification of Singularities:

**Step 1:** Compute  $T(s)$  and get its factorization. We proceed as in Example 2 and we obtain:

$$T(s) = (-1 - t + t^2)(t + 1)^6(2t - 1)^6(2t + 1)^7(t - 1)^7t^{14}.$$

At this point, we should check that the limit point is regular. For this purpose, we use Remark 2. Since  $\deg(T) = 42 = (d - 1)(d - 2)$ , we get that  $P_L$  is regular.

**Step 2:** Let  $T_{aux}(s) = T(s)$  and  $k = 1$ .

**Step 2.1:** We take one factor of  $T_{aux}$ , namely  $(-1 - t + t^2)$ . Hence we obtain the conjugate roots:  $(1 \pm \sqrt{5})/2$ . Let  $s_0 = (1 + \sqrt{5})/2$ .

**Step 2.2:** We compute the first singularity  $P_1 = \mathcal{P}(s_0) = (15/2 : 5 : 1)$ . We have that its fibre function is  $H_{P_1}(t) = -1 - t + t^2$ . The degree of  $H_{P_1}$  is  $m_1 = 2$  and then,  $P_1$  is a double point. In addition, since  $H_{P_1}$  has two different simple roots, we deduce that the curve has  $r_1 = 2$  branches centered at  $P_1$ .

**Step 2.3:** We determine the contribution of  $P_1$  to the  $T$ -function:  $\mathcal{H}_{P_1}(t) = -1 - t + t^2$ . The delta invariant of  $P_1$  is  $\delta_{P_1} = \deg(\mathcal{H}_{P_1})/2 = 1$ .

**Step 2.4:** We have that  $r_1 = m_1 = 2$  and  $\delta_{P_1} = m_1(m_1 - 1)/2 = 1$ . Then, the algorithm returns that  $\boxed{P_1 = (15/2 : 5 : 1)}$  is an ordinary double point.

**Step 2.5:** Let  $T_{aux}(s) = T_{aux}(s)/\mathcal{H}_{P_1}(s) = (t + 1)^6(2t - 1)^6(2t + 1)^7(t - 1)^7t^{14}$  and  $k = 2$ . We repeat steps 2.1 to 2.5.

**Step 2.1:** We take one factor of  $T_{aux}$ , namely  $(t + 1)$ . Hence, we obtain the root  $s_0 = -1$ .

**Step 2.2:** We compute the second singularity  $P_2 = \mathcal{P}(s_0) = (0 : 0 : 1)$ . The fibre function of  $P_2$  is  $H_{P_2}(t) = (t - 1)(2t + 1)(2t - 1)(t + 1)$ . The multiplicity of this point is  $m_2 = \deg(H_{P_2}) = 4$ . Since  $H_{P_2}$  has

four different simple roots, we get that the curve has  $r_2 = 4$  branches centered at  $P_2$ .

**Step 2.3:** We determine the contribution  $\mathcal{H}_{P_2}(t) = (t+1)^6(2t-1)^6(2t+1)^7(t-1)^7$ , and we have that  $\delta_{P_2} = \deg(\mathcal{H}_{P_2})/2 = 13$ .

**Step 2.4:** Since  $r_2 = m_2 = 4$  and  $\delta_{P_2} = 13 > m_2(m_2 - 1)/2 = 6$ , the algorithm returns that  $\boxed{P_2 = (0 : 0 : 1)}$  is a type II non-ordinary 4-fold point with four associated branches, and  $\delta_{P_2} = 13$ .

**Step 2.5:** Let  $T_{aux}(s) = T_{aux}(s)/\mathcal{H}_{P_2}(s) = t^{14}$  and  $k = 3$ . We repeat steps 2.1 to 2.5.

**Step 2.1:** We take one factor of  $T_{aux}$ , namely  $t$ , which corresponds to the root  $s_0 = 0$ .

**Step 2.2:** We compute the third singularity  $P_3 = \mathcal{P}(s_0) = (1 : 0 : 0)$ . The fibre function of  $P_3$  is  $H_{P_3}(t) = t^3$ . The degree of  $H_{P_3}$  is  $m_3 = 3$ , so  $P_3$  is a triple point. In addition, since  $H_{P_3}$  has one only root, we deduce that  $\mathcal{C}$  has one only branch centered at  $P_3$ . That is,  $r_3 = 1$ .

**Step 2.3:** We determine the contribution of  $P_3$  to the  $T$ -function:  $\mathcal{H}_{P_3}(t) = t^{14}$ . The delta invariant of  $P_3$  is  $\delta_{P_3} = \deg(\mathcal{H}_{P_3})/2 = 7$ .

**Step 2.4:** We have that  $r_3 = 1 < m_3 = 3$  and  $\delta_{P_3} = 7 > m_3(m_3 - 1)/2 = 3$ . Then, the algorithm returns that  $\boxed{P_3 = (1 : 0 : 0)}$  is a type III non-ordinary triple point with one associated branch, and  $\delta_{P_3} = 7$ .

**Step 2.5:** Let  $T_{aux}(s) = T_{aux}(s)/\mathcal{H}_{P_3}(s) = 1$  and  $k = 3$ . We repeat steps 2.1 to 2.5.

**Step 2.1:** We have that  $T_{aux}(s) = 1$ . Then, the algorithm finishes.

Thus, the algorithm returns that  $\mathcal{C}$  has three singularities:

- one ordinary double point at  $P_1 = (15/2 : 5 : 1)$
- one type II non-ordinary 4-fold point at  $P_2 = (0 : 0 : 1)$  with  $\delta_{P_2} = 13$
- one type III non-ordinary triple point at  $P_3 = (1 : 0 : 0)$  with  $r_3 = 1$  and  $\delta_{P_3} = 7$

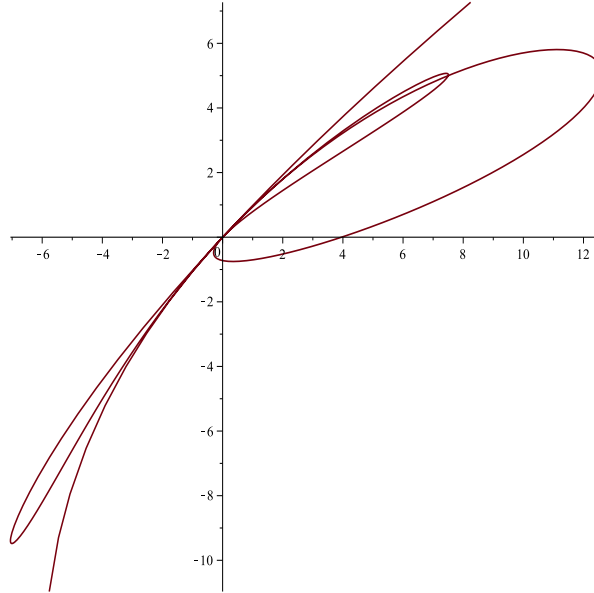


Figure 2: The curve  $\mathcal{C}$  has two affine singularities: the double point  $(15/2, 5)$  and the 4-fold point  $(0, 0)$ .

*In Figure 2, we plot the curve  $\mathcal{C}$ . We can appreciate the two affine singularities,  $P_1$  and  $P_2$ .*

## 5. The general case for rational space curves

Reasoning similarly as in [3] (see Section 4), we may adapt Theorem 4 for studying the singularities of rational space curves in any dimension. More precisely, in this section, we generalize Theorem 6, in [3], for the case that the given curve,  $\mathcal{C}$ , is a rational space curve with also non-ordinary singularities. In this case, a deeply analysis is necessary to prove that the invariants associated to the non-ordinary singularity (as the delta invariant) are preserved (see Proposition 2).

As in [3], we use an equivalent polynomial to the T-function, the  $T_E(s)$  polynomial, which totally describes the singularities of  $\mathcal{C}$ . In particular, the factorization of  $T_E(s)$  provides the fibre functions of both, the ordinary and the non-ordinary singularities. We recall that from the fibre function of a

point  $P$ , one may determine the multiplicity of  $P$  as well as its fibre  $\mathcal{F}_{\mathcal{P}}(P)$  and the tangent lines of  $\mathcal{C}$  at  $P$  (see Theorem 2 in Section 2.1). In addition, we can compute the contribution of each singularity to  $T_E(s)$  (see Remark 1) and, hence, we also obtain its delta invariant. The method presented generalizes the results obtained in previous works (see Section 1), since a complete classification of the singularities of a given space curve, via the factorization of a univariate resultant, is obtained.

For this purpose, we consider a proper parametrization

$$\mathcal{P}(t) = (p_1(t), \dots, p_n(t), p(t)) \in \mathbb{P}^n(\mathbb{K}(t)), \quad \gcd(p_1, \dots, p_n, p) = 1,$$

of a given rational space curve,  $\mathcal{C}$ . In addition, we define the associated rational parametrization over  $\mathbb{K}(Z)$ , where  $Z = (Z_1, \dots, Z_{n-2})$  and  $Z_1, \dots, Z_{n-2}$  are new variables, given by

$$\begin{aligned} \widehat{\mathcal{P}}(t) &= (\widehat{p}_1(t), \widehat{p}_2(t), \widehat{p}(t)) = \\ &= (p_1(t), p_2(t) + Z_1 p_3(t) + \dots + Z_{n-2} p_n(t), p(t)) \in \mathbb{P}^2((\mathbb{K}(Z))(t)). \end{aligned}$$

Note that  $\widehat{\mathcal{P}}(t) = R(\mathcal{P}(t))$ , where

$$R(\bar{x}) = (x_1, x_2 + Z_1 x_3 + \dots + Z_{n-2} x_n, x_{n+1}) \in \mathbb{P}^2((\mathbb{K}(Z))(\bar{x})), \quad \bar{x} = (x_1, \dots, x_{n+1}).$$

This notation is used for the sake of simplicity, but we note that  $\widehat{\mathcal{P}}(t)$  depends on  $Z$ . Observe that  $\widehat{\mathcal{P}}(t)$  is a proper parametrization of a rational plane curve  $\widehat{\mathcal{C}}$  defined over the algebraic closure of  $\mathbb{K}(Z)$ .

There exists a correspondence between the points of  $\mathcal{C}$  and the points of  $\widehat{\mathcal{C}}$ . More precisely, for each point  $P = (a_1 : a_2 : a_3 : \dots : a_n : a_{n+1}) \in \mathcal{C}$  we have another point  $\widehat{P} = (a_1 : a_2 + Z_1 a_3 + \dots + Z_{n-2} a_n : a_{n+1}) \in \widehat{\mathcal{C}}$ . Moreover, this correspondence is bijective for the points satisfying that  $a_1 \neq 0$  or  $a_{n+1} \neq 0$ . For these points, it holds that  $\mathcal{F}_{\mathcal{P}}(P) = \mathcal{F}_{\widehat{\mathcal{P}}}(\widehat{P})$ , which implies that  $H_P(s) = H_{\widehat{P}}(s)$ . Note that the polynomial  $H_P$  represents the fibre function of a point  $P$  in the space curve  $\mathcal{C}$  computed from  $\mathcal{P}(t)$ ; i.e. the roots of  $H_P$  are the fibre of  $P \in \mathcal{C}$  (this notion was introduced in Definition 2 for a given plane curve but it can be easily generalized for space curves).

The study of the singularities of  $\mathcal{C}$  through those of  $\widehat{\mathcal{C}}$ , arises an additional difficulty when  $\mathcal{C}$  contains two or more points of the form  $(0 : a_2 : a_3 : \dots : a_n : 0)$ . We call them *bad points* since the correspondence above introduced



is not bijective when they appear. However, we may assume w.l.o.g. that  $\mathcal{C}$  does not have two or more bad points. Otherwise, a change of coordinates may be applied in order to remove them from the curve (see [3] for further details).

We observe that additional points, which can not be written in the form  $(a_1 : a_2 + Z_1 a_3 + \cdots + Z_{n-2} a_n : a_{n+1})$ ,  $a_i \in \mathbb{K}$ ,  $i = 1, \dots, n+1$ , may appear in the curve  $\widehat{\mathcal{C}}$ . Such points are obtained from  $\widehat{\mathcal{P}}(t)$  for  $t \in \mathbb{K}(Z) \setminus \mathbb{K}$  and they do not have a correspondence with any point of  $\mathcal{C}$ . However, we will see later that these points are not a problem, since we can easily put them aside and focus on those which have a correspondence in  $\mathcal{C}$ .

Under these conditions, we note that the above correspondence can also be established between the places of  $\mathcal{C}$  and  $\widehat{\mathcal{C}}$  centered at  $P$  and  $\widehat{P}$ , respectively. That is, for each place  $\varphi(t) = (\varphi_1(t), \varphi_2(t), \varphi_3(t), \dots, \varphi_n(t), \varphi_{n+1}(t))$  of  $\mathcal{C}$  centered at  $P$  we have the place  $\widehat{\varphi}(t) = (\varphi_1(t), \varphi_2(t) + Z_1 \varphi_3(t) + \cdots + Z_{n-2} \varphi_n(t), \varphi_{n+1}(t))$  of  $\widehat{\mathcal{C}}$  centered at  $\widehat{P}$ . Hence, the number of tangents of  $\mathcal{C}$  at  $P$  is the same that the number of tangents of  $\widehat{\mathcal{C}}$  at  $\widehat{P}$  and, as a consequence, the number of branches of each curve at  $P$  and  $\widehat{P}$  and the character of each point is the same. Furthermore,  $\text{mult}_P(\mathcal{C}) = \text{mult}_{\widehat{P}}(\widehat{\mathcal{C}})$  (we recall that the curve tangents of  $\mathcal{C}$  at  $P$ , respectively of  $\widehat{\mathcal{C}}$  at  $\widehat{P}$ , consist of the tangents to the places of the curve that are centered at  $P$ , resp. at  $\widehat{P}$ ; see e.g [15]). This assertion can also be easily proved using results in Subsection 3.2. More precisely, we have the following proposition.

**Proposition 1.** *It holds that  $(a : b : c : d) \in \mathcal{C}$  is a singular point of multiplicity  $m$  if and only if  $(a : b + Zc : d) \in \widehat{\mathcal{C}}$  is a singular point of multiplicity  $m$ .*

**Proof:** For the sake of simplicity, we assume that  $\mathcal{C}$  is a space curve defined by the proper parametrization

$$\mathcal{P}(t) = (p_1(t), p_2(t), p_3(t), p(t)) \in \mathbb{P}^3(\mathbb{K}(t)), \quad \gcd(p_1, p_2, p_3, p) = 1.$$

Then,  $\widehat{\mathcal{C}}$  is defined by the proper parametrization

$$\widehat{\mathcal{P}}(t) = (\widehat{p}_1(t), \widehat{p}_2(t), \widehat{p}(t)) = (p_1(t), p_2(t) + Zp_3(t), p(t)) \in \mathbb{P}^2((\mathbb{K}(Z))(t)).$$

In addition, we assume that  $P = (0 : 0 : 0 : 1) \in \mathcal{C}$  is a point of multiplicity  $m$ . Thus, using Subsection 3.2 (see Step 1), we may write

$$p_j(t) = H_P(t) \bar{p}_j(t), \quad j = 1, 2, 3$$

with  $\gcd(\bar{p}_1, \bar{p}_2, \bar{p}_3) = 1$ ,  $\gcd(H_P, p) = \gcd(p_1, p_2, p_3, p) = 1$  and  $\deg(H_P) = m$  (see Corollary 1). Furthermore,  $\gcd(H_P(t), \bar{p}_i(t)) = 1$  for  $i = 1, 2, 3$  (see equality (5)). Under these conditions, we may write

$$\widehat{\mathcal{P}}(t) = (H_P(t)\bar{p}_1(t), H_P(t)(\bar{p}_2(t) + Z\bar{p}_3(t)), p(t)) \in \mathbb{P}^2((\mathbb{K}(Z))(t)),$$

where  $\gcd(\bar{p}_1, \bar{p}_2(t) + Z\bar{p}_3(t)) = 1$  (since  $\gcd(\bar{p}_1, \bar{p}_2, \bar{p}_3) = 1$ ). Thus, from the results in Subsection 3.2 (see Step 1), we deduce that  $\widehat{P} = (0 : 0 : 1) \in \widehat{\mathcal{C}}$  is a point of multiplicity  $m$ .

Reciprocally, we reason similarly and we conclude that  $(a : b : c : d) \in \mathcal{C}$  is a singular point of multiplicity  $m$  if and only if  $(a : b + Zc : d) \in \widehat{\mathcal{C}}$  is a singular point of multiplicity  $m$ .  $\square$

We also have to check that the correspondence above defined preserves the neighboring points. For this purpose, considering the notation introduced above and using Subsections 3.1 and 3.2, we prove the following proposition. We recall that  $\xi_j$ ,  $j = 1, \dots, n$ , represents the neighborhoods of a certain singularity  $P \in \mathcal{C}$ . We denote by  $\widehat{\xi}_j$ ,  $j = 1, \dots, n$ , the neighborhoods of the corresponding singularity  $\widehat{P} \in \widehat{\mathcal{C}}$ .

**Proposition 2.** *There exists a bijective correspondence between the points in  $\xi_j$  and the points in  $\widehat{\xi}_j$  (for any  $j$ ). In addition, the multiplicity of each point is preserved.*

**Proof:** For the sake of simplicity, we assume that we are in the conditions of Proposition 1. That is,  $\mathcal{C}$  is a space curve defined by the proper parametrization

$$\mathcal{P}(t) = (p_1(t), p_2(t), p_3(t), p(t)) \in \mathbb{P}^3(\mathbb{K}(t)), \quad \gcd(p_1, p_2, p_3, p) = 1,$$

and then,  $\widehat{\mathcal{C}}$  is defined by the proper parametrization

$$\widehat{\mathcal{P}}(t) = (\widehat{p}_1(t), \widehat{p}_2(t), \widehat{p}(t)) = (p_1(t), p_2(t) + Zp_3(t), p(t)) \in \mathbb{P}^2((\mathbb{K}(Z))(t)).$$

We recall that

$$\widehat{\mathcal{P}}(t) = R(\mathcal{P}(t)), \quad \text{where } R(\bar{x}) = (x_1, x_2 + Zx_3, x_4) \in \mathbb{P}^2((\mathbb{K}(Z))(\bar{x})).$$

Let  $P = (0 : 0 : 0 : 1) \in \mathcal{C}$  be a point of multiplicity  $m$ , and we write

$$p_j(t) = H_P(t)\bar{p}_j(t), \quad j = 1, 2, 3$$

with  $\gcd(\bar{p}_1, \bar{p}_2, \bar{p}_3) = 1$ ,  $\gcd(H_P, p) = \gcd(p_1, p_2, p_3, p) = 1$ ,  $\deg(H_P) = m$  and  $\gcd(H_P(t), \bar{p}_i(t)) = 1$ ,  $i = 1, 2, 3$  (see Step 1 in Subsection 3.2). Hence, from Proposition 1, we get that  $\widehat{P} = (0 : 0 : 1) \in \widehat{\mathcal{C}}$  is a point of multiplicity  $m$ .

In order to study the neighboring points, we apply Steps 2 and 3 in Subsection 3.2. We distinguish two different cases:

1. First, we study the correspondence between the points in the first neighborhood of  $P$  and  $\widehat{P}$  (see Step 2 in Subsection 3.2). For this purpose, we apply  $\mathcal{T}$  to  $\widehat{\mathcal{P}}(t)$ , and we get the projective parametrization of the transformed curve  $\widehat{\mathcal{C}}_1$  defined by

$$\begin{aligned} \widehat{\mathcal{M}}_1(t) &:= \mathcal{T}(\widehat{\mathcal{P}}(t)) = (\widehat{q}_1(t), \widehat{q}_2(t), \widehat{q}(t)) = \\ &= ((\bar{p}_2(t) + Z\bar{p}_3)p(t), \bar{p}_1(t)p(t), H_P(t)\bar{p}_1(t)(\bar{p}_2(t) + Z\bar{p}_3)). \end{aligned}$$

We assume that there exists a singular point of multiplicity  $m_1$  that belongs to the first neighborhood of  $\widehat{P}$ . This point should be of the form  $(a : b : 0) \in \widehat{\mathcal{C}}_1$ , with  $ab \neq 0$ , and it is reached by the values of  $t$  being roots of the polynomial  $H_P(t)$ . That is,

$$(\bar{p}_2(t) + Z\bar{p}_3)p(t) - a = H_P^*(t)\widehat{q}_1, \quad \bar{p}_1(t)p(t) - b = H_P^*(t)\widehat{q}_2 \quad (9)$$

where  $H_P^*(t)$  divides  $H_P(t)$ ,  $\gcd(\widehat{q}_1, \widehat{q}_2) = 1$ , and  $\deg(H_P^*) = m_1$ . Then, we deduce that  $a = a_1 + Za_2$ , and

$$\bar{p}_2(t)p(t) - a_1 = H_P^*(t)\widehat{q}_{1,1}, \quad \bar{p}_3p(t) - a_2 = H_P^*(t)\widehat{q}_{1,2} \quad (10)$$

where  $\widehat{q}_1 = \widehat{q}_{1,1} + Z\widehat{q}_{1,2}$ , and  $\gcd(\widehat{q}_{1,1}, \widehat{q}_{1,2}, \widehat{q}_2) = 1$ . We note that  $a_1a_2 \neq 0$  since  $\gcd(H_P^*(t), \bar{p}_2) = \gcd(H_P^*(t), \bar{p}_3) = 1$  (we recall that  $\gcd(H_P(t), \bar{p}_i(t)) = 1$  for  $i = 1, 2, 3$ , and  $H_P^*(t)$  divides  $H_P(t)$ ).

Now, we apply the transformation  $\mathcal{T}_s$  to  $\mathcal{C}$ , getting the transformed curve  $\mathcal{C}_1$ . Thus, we get the projective parametrization of  $\mathcal{C}_1$  defined by

$$\begin{aligned} \mathcal{M}_1(t) &:= \mathcal{T}_s(\mathcal{P}(t)) = \\ &= (\bar{p}_2(t)\bar{p}_3(t)p(t), \bar{p}_1(t)\bar{p}_3(t)p(t), \bar{p}_1(t)\bar{p}_2(t)p(t), H_P(t)\bar{p}_1(t)\bar{p}_2(t)\bar{p}_3(t)). \end{aligned}$$

Note that since  $\widehat{\mathcal{P}}(t) = R(\mathcal{P}(t))$ ,  $\mathcal{M}_1(t) = \mathcal{T}_s(\mathcal{P}(t))$  and  $\widehat{\mathcal{M}}_1(t) := \mathcal{T}(\widehat{\mathcal{P}}(t))$ , we get that

$$\widehat{\mathcal{M}}_1(t) = R_1(\mathcal{M}_1(t)), \text{ where } R_1 = \mathcal{T} \circ R \circ \mathcal{T}_s \in \mathbb{P}^2((\mathbb{K}(Z))(t))$$

(recall that  $\mathcal{T}_s^{-1} = \mathcal{T}_s$ ).

Observe that the parametrization  $\mathcal{M}_1$  can be written as

$$\mathcal{M}_1(t) = (q_1(t), q_2(t), q_3(t), q(t)) =$$

$$(\bar{p}_2(t)\bar{p}_3(t)p(t)^2, \bar{p}_1(t)\bar{p}_3(t)p(t)^2, \bar{p}_1(t)\bar{p}_2(t)p(t)^2, H_P(t)\bar{p}_1(t)\bar{p}_2(t)\bar{p}_3(t)p(t)).$$

Under these conditions, and using equalities (9) and (10), we get that

$$\begin{aligned}\bar{p}_2(t)\bar{p}_3(t)p(t)^2 - a_1a_2 &= H_P^*(t)\bar{q}_1 \\ \bar{p}_1(t)\bar{p}_3(t)p(t)^2 - a_2b &= H_P^*(t)\bar{q}_2 \\ \bar{p}_1(t)\bar{p}_2(t)p(t)^2 - a_1b &= H_P^*(t)\bar{q}_3.\end{aligned}$$

In addition, it holds that  $\gcd(\bar{q}_1, \bar{q}_2, \bar{q}_3) = 1$ . Indeed: from equalities (9) and (10), we have that

$$\begin{aligned}\bar{q}_1 &= a_1\bar{\tilde{q}}_{1,2} + a_2\bar{\tilde{q}}_{1,1} + H_P^*(t)\bar{\tilde{q}}_{1,1}\bar{\tilde{q}}_{1,2} \\ \bar{q}_2 &= a_2\bar{\tilde{q}}_2 + b\bar{\tilde{q}}_{1,2} + H_P^*(t)\bar{\tilde{q}}_{1,2}\bar{\tilde{q}}_2 \\ \bar{q}_3 &= a_1\bar{\tilde{q}}_2 + b\bar{\tilde{q}}_{1,1} + H_P^*(t)\bar{\tilde{q}}_{1,1}\bar{\tilde{q}}_2.\end{aligned}$$

If there exists  $r \in \mathbb{K}$  such that  $\bar{q}_j(r) = 0$ ,  $j = 1, 2, 3$ , since  $a_1a_2b \neq 0$ , the three above equations imply that  $\bar{\tilde{q}}_{1,1}(r) = \bar{\tilde{q}}_{1,2}(r) = \bar{\tilde{q}}_2(r) = 0$ , which is impossible since  $\gcd(\bar{\tilde{q}}_{1,1}, \bar{\tilde{q}}_{1,2}, \bar{\tilde{q}}_2) = 1$ .

Thus, we obtain that  $(a_1a_2 : a_2b : a_1b : 0) \in \mathcal{C}_1$ , with  $a_1a_2b \neq 0$ , is a singular point of multiplicity  $m_1$  (note that  $\deg(H_P^*) = m_1$ ). This point belongs to the first neighborhood of  $P$ .

Reciprocally, let  $(a : b : c : 0) \in \mathcal{C}_1$ , with  $abc \neq 0$ , be a singular point of multiplicity  $m_1$ . Then, reasoning similarly as above, we have that

$$\begin{aligned}\bar{p}_2(t)\bar{p}_3(t)p(t)^2 - a &= H_P^*(t)\bar{q}_1 \\ \bar{p}_1(t)\bar{p}_3(t)p(t)^2 - b &= H_P^*(t)\bar{q}_2 \\ \bar{p}_1(t)\bar{p}_2(t)p(t)^2 - c &= H_P^*(t)\bar{q}_3,\end{aligned}$$

where  $\gcd(\bar{q}_1, \bar{q}_2, \bar{q}_3) = 1$ ,  $H_P^*(t)$  divides  $H_P(t)$  and  $\deg(H_P^*) = m_1$ . Thus, we may write  $a = a_2a_3$ ,  $b = a_1a_3$ ,  $c = a_1a_2$ , where

$$\bar{p}_i(t)p(t) - a_i = H_P^*(t)\bar{\tilde{q}}_i, \quad i = 1, 2, 3,$$

and  $\gcd(\bar{\tilde{q}}_1, \bar{\tilde{q}}_2, \bar{\tilde{q}}_3) = 1$  (note that  $a_1 = \sqrt{abc}/a$ ,  $a_2 = \sqrt{abc}/b$ , and  $a_3 = \sqrt{abc}/c$ ; also the solution  $a_1 = -\sqrt{abc}/a$ ,  $a_2 = -\sqrt{abc}/b$ , and

$a_3 = -\sqrt{abc}/c$  is possible but in any case, the point  $(a_1 : a_2 : a_3 : 0)$  is the same). Hence

$$(\bar{p}_2(t) + Z\bar{p}_3)p(t) - (a_2 + Za_3) = H_P^*(t)(\bar{q}_2 + Z\bar{q}_3), \quad \bar{p}_1(t)p(t) - a_1 = H_P^*(t)\bar{q}_1$$

where  $\gcd(\bar{q}_1, \bar{q}_2 + Z\bar{q}_3) = 1$  (note that  $\gcd(\bar{q}_1, \bar{q}_2, \bar{q}_3) = 1$ ), and  $\deg(H_P^*) = m_1$ . Then, we deduce that  $(a_2 + Za_3 : a_1 : 0) \in \widehat{\mathcal{C}}_1$  is a singular point of multiplicity  $m_1$ .

Therefore  $(a : b : c : 0) = (a_2a_3 : a_1a_3 : a_1a_2 : 0) \in \mathcal{C}_1$ , with  $a_1a_2a_3 \neq 0$ , is a singular point of multiplicity  $m_1$  if and only if  $(a_2 + Za_3 : a_1 : 0) \in \widehat{\mathcal{C}}_1$  is a singular point of multiplicity  $m_1$ .

2. For the second neighborhood, we apply Step 3 in Subsection 3.2. For this purpose, we first observe that the correspondence proved in case 1., is also satisfied for the curves defined by the parametrizations  $L(\widehat{\mathcal{M}}_1(t))$  and  $L_s(\mathcal{M}_1(t))$ , where  $L$  and  $L_s$  are linear change of coordinates. In particular, we consider  $\widehat{\mathcal{Q}}_1(t) := L(\widehat{\mathcal{M}}_1(t))$  and  $\mathcal{Q}_1(t) := L_s(\mathcal{M}_1(t))$ , where  $L$  and  $L_s$  are the linear change of coordinates one should apply to  $\widehat{\mathcal{M}}_1$  and  $\mathcal{M}_1$ , respectively, to move the singular point obtained in the first neighborhood to the origin point (see Step 3 in Subsection 3.2). Hence, using case 1., we get that there exists a correspondence between the singularities (and their multiplicities) defined by the curves parametrized by  $\widehat{\mathcal{Q}}_1$  and  $\mathcal{Q}_1$ .

Now, we may reason as in case 1., and we prove that since  $\widehat{\mathcal{Q}}_1(t) = \bar{R}_1(\mathcal{Q}_1(t))$  (where  $\bar{R}_1 = L(R_1(L_s^{-1}))$ ),  $\mathcal{M}_2(t) = \mathcal{T}_s(\mathcal{Q}_1(t))$  and  $\widehat{\mathcal{M}}_2(t) := \mathcal{T}(\widehat{\mathcal{Q}}_1(t))$ , it holds that

$$\widehat{\mathcal{M}}_2(t) = R_2(\mathcal{M}_2(t)), \quad \text{where } R_2 = \mathcal{T} \circ \bar{R}_1 \circ \mathcal{T}_s \in \mathbb{P}^2((\mathbb{K}(Z))(t)).$$

Thus, we are in the same conditions as in case 1., and hence we may reason in a similar way to get that there exists a bijective correspondence between the points in  $\xi_2$  and  $\widehat{\xi}_2$ . In addition, the multiplicity of each point is preserved.

In general, and reasoning similarly for each neighborhood, we get that there exists a bijective correspondence between the points in  $\xi_j$  and the points in  $\widehat{\xi}_j$  (for any  $j$ ). In addition, the multiplicity of each point is preserved.  $\square$

Taking into account the definition of the delta invariant (see Subsection 3.1) and Proposition 2, we easily get the following corollary. We recall that  $\delta_P$  represents the delta invariant of a certain singularity  $P \in \mathcal{C}$ , and we denote by  $\widehat{\delta}_{\widehat{P}}$  the delta invariant of the corresponding singularity  $\widehat{P} \in \widehat{\mathcal{C}}$ .

**Corollary 4.** *It holds that  $\delta_P = \widehat{\delta}_{\widehat{P}}$ .*

Once the main properties of the considered correspondence are proved, we show how to generalize the concept of the T–function for the space case and how to compute it. For this purpose, let  $\widehat{G}_1, \widehat{G}_2$  and  $\widehat{G}_3$  be the equivalent polynomials to  $G_1, G_2$  and  $G_3$  (defined in (1)), constructed from the parametrization  $\widehat{\mathcal{P}}(t)$ . In addition, let  $\widehat{\delta}_i := \deg_t(\widehat{G}_i)$  and  $\widehat{\lambda}_{ij} := \min\{\widehat{\delta}_i, \widehat{\delta}_j\}$ ,  $i, j = 1, 2, 3$ ,  $i < j$ ,

$$\widehat{G}_i^*(s, t) := \frac{\widehat{G}_i(s, t)}{t - s} \in (\mathbb{K}[Z])[s, t], \quad i = 1, 2, 3,$$

and

$$\widehat{R}_{ij}(s) := \text{Res}_t(\widehat{G}_i^*, \widehat{G}_j^*) \in (\mathbb{K}[Z])[s], \quad i, j = 1, 2, 3, \quad i < j.$$

Then, the T–function of the parametrization  $\widehat{\mathcal{P}}(t)$  is given by

$$\widehat{T}(s) = \widehat{R}_{12}(s) / \widehat{p}(s)^{\widehat{\lambda}_{12}-1}$$

(see Definition 3).

In the following, we show how this function can be used to define an equivalent polynomial to the T–function introduced for plane curves (see Definition 3). The factorization of this polynomial, which will be denoted as  $T_E(s)$ , provides essential information about the singularities of the space curve. In particular, for each  $m$ -fold point  $P$ ,  $T_E(s)$  has a factor of the form:

$$\mathcal{H}_P(s) = H_P(s)^{m-1} \prod_{i=1}^n \prod_{j=1}^{\alpha_i} H_{P_j^i}(s)^{m_j^i-1},$$

where  $\xi_i := \{P_1^i, \dots, P_{\alpha_i}^i\}$  is the  $i$ -th neighborhood of  $P$  and  $m_j^i$  is the multiplicity of  $P_j^i$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, \alpha_i$  ( $\delta_{P^{n+1}} = 0$  and  $\xi_{n+1} = \emptyset$ ). We say that  $\mathcal{H}_P(s)$  is *the contribution of the singularity  $P$  to the T–function*. This notion was introduced in Remark 1 for a given plane curve but it can easily be generalized for space curves.

Theorem 5, below, is obtained as a consequence of Theorem 4 (see Section 3). We recall that we have assumed that the input curve does not have two or more bad points.

**Theorem 5.** *Let  $\mathcal{C}$  be a rational algebraic space curve and let  $\mathcal{P}(t)$  be a proper parametrization of  $\mathcal{C}$  such that  $P_L$  is regular. Let  $P_1, \dots, P_\ell$  be the singular points of  $\mathcal{C}$  and let  $\mathcal{H}_{P_1}, \dots, \mathcal{H}_{P_\ell}$  be their corresponding contributions to the  $T$ -function (see Remark 1, statement (a)). Then, it holds that*

$$T_E(s) = \prod_{k=1}^{\ell} \mathcal{H}_{P_k}(s),$$

where  $T_E(s) = \text{Content}_Z \left( \widehat{T}(s) \right) \in \mathbb{K}[s]$ .

**Proof:** From the statements before Theorem 5, we observe that there exists a bijective correspondence between the points  $\widehat{P} = (a_1 : a_2 + Z_1 a_3 + \dots + Z_{n-2} a_n : a_{n+1})$ ,  $a_i \in \mathbb{K}$ ,  $i = 1, \dots, n+1$ , of  $\widehat{\mathcal{C}}$  and the points  $P = (a_1 : a_2 : a_3 : \dots : a_n : a_{n+1})$  of  $\mathcal{C}$ . Consequently, we have that  $\text{mult}_{\widehat{P}}(\widehat{\mathcal{C}}) = \text{mult}_P(\mathcal{C})$ , which implies that  $\widehat{P}$  is a singularity of  $\widehat{\mathcal{C}}$  of multiplicity  $m$  if and only if  $P$  is a singularity of  $\mathcal{C}$  of multiplicity  $m$ . Hence, using Theorem 4, we deduce that

$$\widehat{T}(s) = \prod_{k=1}^{\ell} \mathcal{H}_{P_k}(s) L(s, Z).$$

We observe that the factor  $L(s, Z) \in \mathbb{K}[s, Z] \setminus \mathbb{K}[s]$  is a product of the fibre functions corresponding to the singularities of  $\widehat{\mathcal{C}}$  that can not be written as  $(a_1 : a_2 + Z_1 a_3 + \dots + Z_{n-2} a_n : a_{n+1})$ ,  $a_i \in \mathbb{K}$ ,  $i = 1, \dots, n+1$  (these singularities do not have an equivalent singularity in  $\mathcal{C}$ , and its fibre function necessarily is a polynomial in  $\mathbb{K}[s, Z] \setminus \mathbb{K}[s]$ ). Then, we conclude that

$$T_E(s) = \text{Content}_Z \left( \widehat{T}(s) \right) = \prod_{k=1}^{\ell} \mathcal{H}_{P_k}(s).$$

□

**Remark 4.** *From Propositions 1 and 2, and Corollary 4, one gets that the polynomial  $T_E(s)$  perfectly describes the singularities of  $\mathcal{C}$ .*

**Example 4.** Let  $\mathcal{C}$  be the rational space curve defined by the projective parametrization  $\mathcal{P}(t) = (p_1(t), p_2(t), p_3(t), p(t)) \in \mathbb{P}^3(\mathbb{C}(t))$ , where

$$\begin{aligned} p_1(t) &= -t^6 - 12t^5 + t^4 + 38t^3 + 30t^2 + 4t, \\ p_2(t) &= -t^6 - 5t^5 + 2t^4 + 17t^3 + 13t^2 + 2t, \\ p_3(t) &= -t^7 - 4t^6 - (13/8)t^5 + (93/8)t^4 + (175/8)t^3 + (131/8)t^2 + (19/4)t, \\ p(t) &= 1. \end{aligned}$$

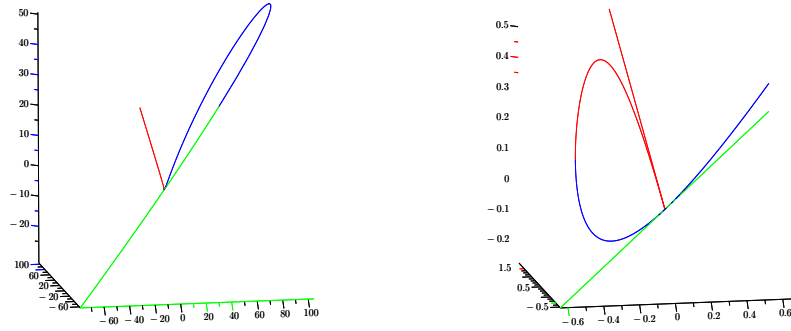


Figure 3: General view of curve  $\mathcal{C}$  (left) and detailed view of the non-ordinary singular point  $P = (0,0,0)$  (right)

We consider the plane curve  $\widehat{\mathcal{C}}$  defined by the parametrization

$$\widehat{\mathcal{P}}(t) = (p_1(t) : p_2(t) + Zp_3(t) : p(t)) \in \mathbb{P}^2((\mathbb{C}(Z))(t)).$$

From  $\widehat{\mathcal{P}}(t)$ , we compute  $\widehat{G}_i(s, t)$ ,  $i = 1, 2, 3$  and the corresponding  $T$ -function. We get that

$$T_E(s) = 1/32768s^4(s - 2)^4(s + 1)^6.$$

Now, we apply Algorithm Classification of Singularities. Thus, we take one factor of  $T_E(s)$ , namely  $s$ , which corresponds to the root  $s_0 = 0$ , and we get the singularity  $P = \mathcal{P}(0) = (0 : 0 : 0 : 1)$ . Its fibre function is

$$H_P(t) := \gcd(\widehat{G}_1(0, t), \widehat{G}_2(0, t), \widehat{G}_3(0, t)) = (t - 2)(t + 1)^2t.$$



Hence, we deduce that  $P$  is the only singularity of  $\mathcal{C}$  and it has multiplicity  $m = \deg(H_P) = 4$ . In addition, the curve has  $r = 3$  branches centered at  $P$  ( $H_P$  has 3 different roots), and the contribution of this point to the  $T$ -function is

$$\mathcal{H}_P(s) = s^4(s-2)^4(s+1)^6,$$

which implies that  $\delta_P = \deg(\mathcal{H}_P)/2 = 7$ . Finally, reasoning as in Example 2, we have that

$$\frac{\mathcal{H}_P(s)}{H_P(s)^{m-1}} = s(s-2),$$

is the part of  $\mathcal{H}_P(s)$  which corresponds to the neighboring singularities of  $P$ . Hence, we deduce that  $P$  has only one double point at its first neighborhood.

In Figure 3, we plot the curve  $\mathcal{C}$ . If we focus on the area around the 4-fold point,  $P$ , we observe that it is crossed by 3 branches which yield 2 different tangents.

## 6. Proofs of Lemmas 2 and 3 in Section 3

In order to prove Lemmas 2 and 3, we need first to recall the following technical result, which summarizes some useful properties of the resultant of two polynomials. This lemma has been proved in [3] (see Section 5).

**Lemma 4.** *Let  $A(s, t), B(s, t), C(s, t) \in \mathbb{K}[s, t]$ , and  $K(s) \in \mathbb{K}[s]$ . The following properties hold:*

1.  $\text{Res}_t(A, K) = K^{\deg_t(A)}$ .
2.  $\text{Res}_t(A, B \cdot C) = \text{Res}_t(A, B) \cdot \text{Res}_t(A, C)$ .
3. If  $B$  divides  $A$ , it holds that  $\text{Res}_t(A/B, C) = \text{Res}_t(A, C)/\text{Res}_t(B, C)$ .
4.  $\text{Res}_t(A, B+CA) = \text{lc}(A)^k \text{Res}_t(A, B)$ , where  $k = \deg_t(B+CA) - \deg_t B$ .

Now we present the proof of Lemma 2.

**Statement of Lemma 2:** *Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P \neq P_L$  be a non-ordinary singular point of multiplicity  $m$ , and let  $\xi_1 = \{P^1\}$ , where  $P^1$  is an ordinary singular point of multiplicity  $m_1$ . Then it holds that*

$$T(s) = H_P(s)^{m-1} H_{P^1}(s)^{m_1-1} K(s),$$

where  $K(s) \in \mathbb{K}[s]$ ,  $H_{P_1}(s)$  divides  $H_P(s)$ , and  $\gcd(H_P(s), K(s)) = 1$ .

**Proof:** In order to prove the lemma, one may consider three cases: (a) let  $P = (0 : 0 : 1)$ , (b) let  $P$  be an affine point of the form  $(a : b : 1)$  and (c) let  $P$  be an infinity point of the form  $(a : b : 0)$ . However, the proof for cases (b) and (c) can be derived from that for case (a) by applying a change of coordinates (see the proof of Lemma 2 in [3]). Therefore, in the following we assume that the given singularity is the point  $P = (0 : 0 : 1)$ . Thus, we may write

$$p_1(t) = H_P(t)\bar{p}_1(t), \quad p_2(t) = H_P(t)\bar{p}_2(t), \quad (11)$$

where  $\bar{p}_1(t)$  and  $\bar{p}_2(t)$  are polynomials satisfying that  $\gcd(\bar{p}_1, \bar{p}_2) = 1$ . Furthermore, it holds that  $\gcd(H_P(t), p(t)) = 1$ , since  $\gcd(p_1, p_2, p) = 1$ . In addition, we assume that none of the tangents of  $P$  are an irregular line and no other point on an irregular line is a singular point (see the first step of the blowing up process in Subsection 3.1). In Subsection 3.2, we showed that both conditions hold if equalities (5) and (6) hold. In the negative case, we apply a linear change of coordinates to  $\mathcal{P}(t)$ . Finally, we assume that  $(0 : 1 : 0), (1 : 0 : 0) \notin \mathcal{C}$ ; i.e.  $\gcd(\bar{p}_i, p) = 1$  for  $i = 1, 2$  (otherwise, we apply a linear change of coordinates).

We recall that the T–function can be computed as (see (3))

$$T(s) = R_{13}(s)/p_1(s)^{\lambda_{13}-1}, \quad (12)$$

where  $\delta_i := \deg_t(G_i)$ ,  $\lambda_{ij} := \min\{\delta_i, \delta_j\}$ ,  $G_i^*(s, t) := \frac{G_i(s, t)}{t - s} \in \mathbb{K}[s, t]$  and

$$R_{ij}(s) := \text{Res}_t(G_i^*, G_j^*) \in \mathbb{K}[s] \quad \text{for } i, j = 1, 2, 3, \quad i < j.$$

In addition, we have that  $G_3(s, t) = p_1(s)p_2(t) - p_2(s)p_1(t)$  (see (1)). By substituting (11) in this expression, we get that

$$G_3(s, t) = H_P(s)H_P(t)(\bar{p}_1(s)\bar{p}_2(t) - \bar{p}_2(s)\bar{p}_1(t))$$

and hence,

$$G_3^*(s, t) = H_P(s)H_P(t)\bar{G}_3^*(s, t),$$

where

$$\bar{G}_3^*(s, t) := \frac{\bar{p}_1(s)\bar{p}_2(t) - \bar{p}_2(s)\bar{p}_1(t)}{s - t}.$$

Then,

$$R_{13}(s) = \text{Res}_t(G_1^*(s, t), H_P(s)H_P(t)\bar{G}_3^*(s, t))$$

and by applying the properties stated in Lemma 4 (for further details see the proof of Lemma 2 in [3]), we get that

$$R_{13}(s) = H_P(s)^{\delta_1-1} \bar{p}_1(s)^m H_P(s)^{m-1} \operatorname{Res}_t(G_1^*(s, t), \bar{G}_3^*(s, t)).$$

Therefore, (12) can be expressed as

$$T(s) = \frac{H_P(s)^{\delta_1-1} \bar{p}_1(s)^m H_P(s)^{m-1} \operatorname{Res}_t(G_1^*(s, t), \bar{G}_3^*(s, t))}{p_1(s)^{\lambda_{13}-1}}.$$

Observe that  $\lambda_{13} = \delta_1$  since  $\delta_1 \leq \delta_3$ . Otherwise, if  $\delta_1 > \delta_3$ , we would have that  $\max\{d_1, d_3\} > \max\{d_1, d_2\}$  and then,  $d_3 > d_1, d_2$ . However, this would imply that  $P = P_L$  (see Definition 1), which contradicts the assumptions. Thus, taking into account that  $p_1(s) = H_P(s) \bar{p}_1(s)$ , we conclude that

$$T(s) = H_P(s)^{m-1} T^*(s), T^*(s) = \frac{\operatorname{Res}_t(G_1^*(s, t), \bar{G}_3^*(s, t))}{\bar{p}_1(s)^{\delta_1-1-m}}. \quad (13)$$

In [3] (see Lemma 2), it is proved that  $T^*(s) \in \mathbb{K}[s]$ .

In the second step of the blowing up process, we apply the transformation  $\mathcal{T}$  to the original curve  $\mathcal{C}$  and we get a new curve  $\mathcal{C}_1$  that is defined by the projective parametrization

$$\begin{aligned} \mathcal{M}_1(t) &= \mathcal{T}(\mathcal{P}(t)) = \\ &= (p_2(t)p(t), p_1(t)p(t), p_1(t)p_2(t)) = (\bar{p}_2(t)p(t), \bar{p}_1(t)p(t), H_P(t)\bar{p}_1(t)\bar{p}_2(t)). \end{aligned}$$

By hypotheses, we know that there exists one ordinary singular point  $P^1 := (1 : \gamma : 0) \in \mathcal{C}_1$ , with  $\gamma \neq 0$ . Note that this point is reached by values of  $t$  being roots of the polynomial  $H_P(t)$ . Hence, there exists a polynomial  $H_{P^1}(t)$ , which divides  $H_P(t)$ , such that for every  $s_0 \in \mathbb{K}$  with  $H_{P^1}(s_0) = 0$ , it holds that  $\mathcal{M}_1(s_0) = P^1$ . That is,

$$\mathcal{M}_1(s_0) = \left( 1 : \frac{\bar{p}_1(s_0)}{\bar{p}_2(s_0)} : \frac{H_P(s_0)\bar{p}_1(s_0)}{p(s_0)} \right) = (1 : \gamma : 0). \quad (14)$$

We observe that  $H_P(t)$  and  $H_{P^1}(t)$  are fibre functions defined from different parametrizations ( $\mathcal{P}$  and  $\mathcal{M}_1$  respectively). However, for the sake of simplicity, we have not remarked this fact in the notation.

Now, let us apply a second change of coordinates in order to move the point  $(1 : \gamma : 0)$  to the point  $(0 : 0 : 1)$ ; for instance, let us consider the change  $\mathcal{L} = (x_2 - \gamma x_1, x_3, x_1)$ . The transformed curve can be parametrized by  $\mathcal{Q}_1(t) := \mathcal{L}(\mathcal{M}_1(t))$ . We have that

$$\mathcal{Q}_1(t) = (q_1(t), q_2(t), q_3(t)) = (\bar{p}_1(t)p(t) - \gamma\bar{p}_2(t)p(t), H_P(t)\bar{p}_1(t)\bar{p}_2(t), \bar{p}_2(t)p(t)).$$

Note that the fibre of the point  $Q^1 := (0 : 0 : 1)$  in the parametrization  $\mathcal{Q}_1$  is the fibre of  $P^1 = (1 : \gamma : 0)$  in the parametrization  $\mathcal{M}_1$ . Thus,  $H_{Q^1}(t) = H_{P^1}(t)$  and, reasoning as in (11), we get

$$q_i(t) = H_{Q^1}(t)\bar{q}_i(t) = H_{P^1}(t)\bar{q}_i(t), \quad i = 1, 2, \quad (15)$$

where  $\bar{q}_1, \bar{q}_2 \in \mathbb{K}[t]$  and  $\gcd(\bar{q}_1, \bar{q}_2) = 1$ . Furthermore, since  $\gcd(\bar{p}_i, p) = 1$  for  $i = 1, 2$ , we have that  $\gcd(q_1, q_2, q) = 1$ , and then  $\gcd(H_{Q^1}, q) = 1$ .

From the above parametrization, we construct the polynomials

$$G_{1,1}^*(s, t) = \frac{q_1(s)q(t) - q(s)q_1(t)}{t - s} = p(t)p(s)\bar{G}_3^*(s, t), \quad (16)$$

where  $\bar{G}_3^*(s, t) = (\bar{p}_1(s)\bar{p}_2(t) - \bar{p}_2(s)\bar{p}_1(t))/(t - s)$ ,

$$G_{2,1}^*(s, t) = \frac{q_2(s)q(t) - q(s)q_2(t)}{t - s} = \bar{p}_2(t)\bar{p}_2(s)G_1^*(s, t), \quad (17)$$

and

$$G_{3,1}^*(s, t) = \frac{q_1(s)q_2(t) - q_2(s)q_1(t)}{t - s} = H_{P^1}(s)H_{P^1}(t)\bar{G}_{3,1}^*(s, t),$$

where  $\bar{G}_{3,1}^*(s, t) = (\bar{q}_1(s)\bar{q}_2(t) - \bar{q}_2(s)\bar{q}_1(t))/(t - s)$ . Note that  $\deg(\bar{p}_i) = d_i - m$ ,  $i = 1, 2$  and hence,  $\deg_t(G_{1,1}^*) := \delta_{1,1} = d_3 + \delta_3 - 1 - m$  and  $\deg_t(G_{2,1}^*) := \delta_{2,1} = d_2 + \delta_1 - 1 - m$ . In addition, we denote  $\deg_t(G_{3,1}^*) := \delta_{3,1}$ .

The polynomials  $G_{i,1}^*$  are equivalent to the polynomials  $G_i^*$ , but constructed from the parametrization  $\mathcal{Q}_1$ . Taking this into account, we may apply Definition 3 for computing  $T_1$  (that is, a polynomial equivalent to  $T$ , but constructed from  $\mathcal{Q}_1$ ). Thus, we obtain

$$T_1(s) = \frac{\text{Res}_t(G_{1,1}^*(s, t), G_{2,1}^*(s, t))}{q(s)^{\lambda_{12} - 1}}, \quad (18)$$

where  $\lambda_{12}^1 := \min\{\delta_{1,1}, \delta_{2,1}\}$ . Now, we reason as above (see (13)), and we get that

$$T_1(s) = H_{Q^1}(s)^{m_1-1}U_1(s), U_1(s) := \frac{\text{Res}_t \left( G_{1,1}^*(s, t), \overline{G}_{3,1}^*(s, t) \right)}{\overline{q}_1(s)^{\delta_{1,1}-m_1-1}} \in \mathbb{K}[s].$$

From the proof of Lemma 2 in [3], we have that  $\gcd(H_{Q^1}, U_1) = 1$ , since  $Q^1$  is an ordinary singularity and it is not the limit point of  $\mathcal{Q}_1$  (by assumption,  $P$  is not the limit point of  $\mathcal{P}(t)$ , and we have just applied a change of coordinates). On the other hand, as we have remarked above,  $H_{Q^1}(s) = H_{P^1}(s)$ , so we have that

$$T_1(s) = H_{P^1}(s)^{m_1-1}U_1(s), U_1(s) := \frac{\text{Res}_t \left( G_{1,1}^*(s, t), \overline{G}_{3,1}^*(s, t) \right)}{\overline{q}_1(s)^{\delta_{1,1}-m_1-1}} \in \mathbb{K}[s] \quad (19)$$

with  $\gcd(H_{P^1}, U_1) = 1$ .

In order to complete the proof, we need to find a connection between equations (13) and (19), which allows us to relate the T-functions,  $T(s)$  and  $T_1(s)$ . For this purpose, we substitute (16) and (17) in (18) and, by applying statements 1 and 2 of Lemma 4, we get that

$$\begin{aligned} & \text{Res}_t \left( G_{1,1}^*(s, t), G_{2,1}^*(s, t) \right) = \\ & = p(s)^{\delta_1-1+d_2-m} \overline{p}_2(s)^{\delta_3-1+d_3-m} \text{Res}_t \left( p(t) \overline{G}_3^*(s, t), \overline{p}_2(t) G_1^*(s, t) \right). \end{aligned}$$

Now, using again Lemma 4, and we have that

$$\begin{aligned} & \text{Res}_t \left( p(t) \overline{G}_3^*(s, t), \overline{p}_2(t) G_1^*(s, t) \right) = \\ & = \text{Res}_t \left( p(t), G_1^*(s, t) \right) \text{Res}_t \left( \overline{G}_3^*(s, t), \overline{p}_2(t) \right) \text{Res}_t \left( \overline{G}_3^*(s, t), G_1^*(s, t) \right). \end{aligned}$$

Let us analyze the first two factors. On the one hand, we have that

$$\begin{aligned} \text{Res}_t \left( p(t), G_1^*(s, t) \right) &= \text{Res}_t \left( p(t), \frac{p_1(s)p(t) - p(s)p_1(t)}{t-s} \right) = \\ &= \frac{\text{Res}_t(p(t), -p(s)p_1(t))}{\text{Res}_t(p(t), t-s)} = \frac{\text{Res}_t(p(t), p(s))}{p(s)} = p(s)^{d_3-1}. \end{aligned}$$

On the other hand, reasoning similarly as above, we get that

$$\operatorname{Res}_t \left( \overline{G}_3^*(s, t), \overline{p}_2(t) \right) = \overline{p}_2(s)^{d_2-m-1}$$

Therefore,

$$\begin{aligned} & \operatorname{Res}_t \left( G_{1,1}^*(s, t), G_{2,1}^*(s, t) \right) = \\ & = p(s)^{\delta_1-2+d_2-m+d_3} \overline{p}_2(s)^{\delta_3-2+d_3-2m+d_2} \operatorname{Res}_t \left( \overline{G}_3^*(s, t), G_1^*(s, t) \right). \end{aligned}$$

Now, using that  $q(s) = p(s)\overline{p}_2(s)$ , and substituting it in (18), we get

$$T_1(s) = \frac{\operatorname{Res}_t \left( \overline{G}_3^*(s, t), G_1^*(s, t) \right)}{p(s)^a \overline{p}_2(s)^b},$$

where  $a := \lambda_{12}^1 - \delta_1 - d_2 - d_3 + m + 1$  and  $b := \lambda_{12}^1 - \delta_3 - d_2 - d_3 + 2m + 1$ . This equality is, somehow, the key of the proof, since it relates some elements obtained from the parametrization  $\mathcal{P}$  and some others obtained from  $\mathcal{Q}_1$ , which provides us the desired connection between equations (13) and (19).

We substitute  $\operatorname{Res}_t \left( \overline{G}_3^*(s, t), G_1^*(s, t) \right) = T_1(s)p(s)^a \overline{p}_2(s)^b$  on (13), and we get that

$$T(s) = H_P(s)^{m-1} \frac{T_1(s)p(s)^a \overline{p}_2(s)^b}{\overline{p}_1(s)^{\delta_1-1-m}}.$$

Thus, from (19), we obtain

$$T(s) = H_P(s)^{m-1} H_{P_1}(s)^{m_1-1} K(s), \text{ where } K(s) := p(s)^a \overline{p}_2(s)^b \overline{p}_1(s)^{m+1-\delta_1} U(s).$$

Observe that since  $T(s) \in \mathbb{K}[s]$ ,  $U(s) \in \mathbb{K}[s]$ ,  $H_{P_1}$  divides  $H_P$ , and  $\gcd(H_P, p) = \gcd(H_P, \overline{p}_2) = \gcd(H_P, \overline{p}_1) = 1$  (see (5)), we deduce that  $K(s) \in \mathbb{K}[s]$ .

Now, we have to prove that  $\gcd(H_P(s), K(s)) = 1$ . Since  $\gcd(H_P, p) = \gcd(H_P, \overline{p}_i) = 1, i = 1, 2$  (see (5)), we have that  $\gcd(H_P, K) = \gcd(H_P, U)$ . On the other hand, since  $H_{P_1}(t)$  divides  $H_P(t)$ , there exist  $L(t) \in \mathbb{K}[t]$  such that  $H_P(t) = L(t)H_{P_1}(t)$  and, from (19), we know that  $\gcd(H_{P_1}, U) = 1$ , so  $\gcd(H_P, U) = \gcd(L, U)$ . Now, we remind that

$$U(s) := \overline{q}_1(s)^{m_1+1-\delta_{1,1}} \operatorname{Res}_t \left( G_{1,1}^*(s, t), \overline{G}_{3,1}^*(s, t) \right) \in \mathbb{K}[s],$$

and thus,

$$\gcd(L, U) = \gcd(L, \overline{q}_1)^{m_1+1-\delta_{1,1}} \gcd \left( L, \operatorname{Res}_t \left( G_{1,1}^*(s, t), \overline{G}_{3,1}^*(s, t) \right) \right).$$

Note that  $\gcd(L, \bar{q}_1) = 1$  (otherwise, since  $\bar{q}_2(t) = L(t)\bar{p}_1(t)\bar{p}_2(t)$ , we would have that  $\gcd(\bar{q}_1, \bar{q}_2) \neq 1$ , which is impossible). Hence,

$$\gcd(H_P, K) = \gcd\left(L, \operatorname{Res}_t(G_{1,1}^*(s, t), \bar{G}_{3,1}^*(s, t))\right).$$

Now, let us assume that there exists  $k_0 \in \mathbb{K}$  such that  $L(k_0) = 0$  and  $\operatorname{Res}_t\left(G_{1,1}^*(s, t), \bar{G}_{3,1}^*(s, t)\right)(k_0) = 0$ . Then, one of the following statements hold:

1. There exists  $k_1 \in \mathbb{K}$  such that  $G_{1,1}^*(k_0, k_1) = \bar{G}_{3,1}^*(k_0, k_1) = 0$  which implies that  $G_{1,1}(k_0, k_1) = G_{3,1}(k_0, k_1) = 0$ . Thus, the fibre function  $H_{\mathcal{Q}_1(k_0)}(t) = \gcd(G_{1,1}(k_0, t), G_{3,1}(k_0, t))$  has degree at least 2. Therefore,  $\mathcal{Q}_1(k_0)$  is a singular point (note that  $\mathcal{Q}_1$  is proper). On the other hand,  $L(k_0) = 0$  implies that  $H_P(k_0) = 0$  so, if we substitute  $k_0$  in (14), we get that

$$\mathcal{M}_1(k_0) = \left(1 : \frac{\bar{p}_1(k_0)}{\bar{p}_2(k_0)} : \frac{H_P(k_0)\bar{p}_1(k_0)}{p(k_0)}\right) = (1 : \beta : 0), \beta \neq 0.$$

Note that  $\bar{p}_1(k_0)\bar{p}_2(k_0)p(k_0) \neq 0$ , since we had that  $\gcd(H_P(t), p(t)) = 1$  and that  $\gcd(H_P(t), \bar{p}_i(t)) = 1$  for  $i = 1, 2$  (see (5)). Now, we distinguish two different cases and we show that both of them contradict the assumptions of the proof:

- (a) Let  $\beta = \gamma$ . This implies that  $\mathcal{M}_1(k_0) = P^1$ , so  $H_{P^1}(k_0) = 0$ . However, this is not possible since  $\operatorname{Res}_t\left(G_{1,1}^*(s, t), \bar{G}_{3,1}^*(s, t)\right)(k_0) = 0$  implies that  $U(k_0) = 0$  (recall that  $\gcd(L, \bar{q}_1) = 1$ ) and we know that  $\gcd(H_{P^1}, U) = 1$ .
  - (b) Let  $\beta \neq \gamma$ . This implies that  $\mathcal{C}_1$  has a singularity of the form  $\mathcal{Q}_1(k_0) = (1 : \beta : 0) \neq P^1$ , with  $\beta \neq 0$ , which contradicts the assumption that  $\xi_1 = \{P^1\}$ .
2. It holds that  $\gcd(\operatorname{lc}_t(G_{1,1}^*), \operatorname{lc}_t(\bar{G}_{3,1}^*))(k_0) = 0$ . This implies that  $\mathcal{Q}_1 = \mathcal{Q}_1(k_0)$  is the limit point of the parametrization (see the proof of Lemma 2 in [3]). However, as we stated before, this is not possible under the assumption that  $P \neq P_L$ .

The cases 1 and 2 above considered, provide a contradiction with the assumptions of the lemma. Therefore, we deduce that the polynomials  $L$  and  $\operatorname{Res}_t(G_{1,1}^*(s, t), \bar{G}_{3,1}^*(s, t))$  do not have common roots and  $\gcd(H_P, K) = 1$ .  $\square$

Finally, we present the proof of Lemma 3.

**Statement of Lemma 3:** *Let  $\mathcal{C}$  be a rational algebraic curve defined by a proper parametrization  $\mathcal{P}(t)$ , with limit point  $P_L$ . Let  $P \neq P_L$  be a singular point of multiplicity  $m$ , let  $\xi_i := \{P_1^i, \dots, P_{\alpha_i}^i\}$  be its  $i$ -th neighborhood and let  $m_j^i$  be the multiplicity of  $P_j^i$ , for  $i = 1, \dots, n$  and  $j = 1, \dots, \alpha_i$  ( $\delta_{P^{n+1}} = 0$  and  $\xi_{n+1} = \emptyset$ ). It holds that*

$$T(s) = H_P(s)^{m-1} \prod_{i=1}^n \prod_{j=1}^{\alpha_i} H_{P_j^i}(s)^{m_j^i-1} T^*(s),$$

where  $T^*(s) \in \mathbb{K}[s]$ ,  $H_{P_j^i}(s)$  divides  $H_P(s)$ , and  $\gcd(H_P(s), T^*(s)) = 1$ ,  $j = 1, \dots, \alpha_i$ ,  $i = 1, \dots, n$ .

**Proof:** First, we assume that  $P$  has two ordinary singularities  $P^1$  and  $P^2$ , of multiplicities  $m_1$  and  $m_2$ , in its first neighborhood. That is,  $\xi_1 = \{P^1, P^2\}$ . Then, from Lemma 2, we have that

$$T(s) = H_P(s)^{m-1} H_{P^1}(s)^{m_1-1} K_1(s) = H_P(s)^{m-1} H_{P^2}(s)^{m_2-1} K_2(s),$$

where  $K_1(s), K_2(s) \in \mathbb{K}[s]$  and  $\gcd(H_P(s), K_i(s)) = 1$ ,  $i = 1, 2$ . In addition, note that  $\gcd(H_{P^1}(s), H_{P^2}(s)) = 1$  (otherwise, if  $s_0$  was a common root of  $H_{P^1}(s)$  and  $H_{P^2}(s)$ , we had that  $P^1 = \mathcal{Q}_1(s_0) = P^2$ ). Hence, we deduce that  $H_{P^2}(s)^{m_2-1}$  divides  $K_1(s)$  and thus,

$$T(s) = H_P(s)^{m-1} H_{P^1}(s)^{m_1-1} H_{P^2}(s)^{m_2-1} K(s) \quad (20)$$

where  $K(s) \in \mathbb{K}[s]$  and  $\gcd(H_P(s), K(s)) = 1$ .

Now, we assume that  $P^1 \in \xi_1$  is a non-ordinary singularity which in turn has an ordinary singularity  $P^2$  at its first neighborhood. That is,  $\xi_1 = \{P^1\}$  and  $\xi_2 = \{P^2\}$ . Then, we observe that equality (19) appearing in the proof of Lemma 2, may be written, in this case, as

$$T_1(s) = H_{P^1}(s)^{m_1-1} H_{P^2}(s)^{m_2-1} U(s), \quad U(s) \in \mathbb{K}[s].$$

Reasoning similarly as in that proof, we also get (20).

The lemma follows by induction on the two cases above considered.  $\square$



## 7. Conclusions

Let  $\mathcal{C}$  be an algebraic space curve defined by a rational parametrization  $\mathcal{P}(t) \in \mathbb{K}(t)^\ell$ ,  $\ell \geq 2$ . In this paper, we construct the  $T$ -function,  $T(s)$ , which is a polynomial obtained from  $\mathcal{P}(t)$  by means of a univariate resultant, and we carefully study the structure of  $T(s)$  showing that it contains essential information concerning the (ordinary and non-ordinary) singularities of  $\mathcal{C}$ . More precisely, we prove that  $T(s) = \prod_{i=1}^n \mathcal{H}_{P_i}(s)$ , where  $P_i$ ,  $i = 1, \dots, n$ , are the singularities of  $\mathcal{C}$  and  $\mathcal{H}_{P_i}$ ,  $i = 1, \dots, n$ , are polynomials, each of them associated to a singularity, whose factors are the fibre functions of those singularities as well as those other belonging to their corresponding neighborhoods. That is,  $\mathcal{H}_Q(s) = H_Q(s)^{m-1} \prod_{j=1}^k H_{Q_j}(s)^{m_j-1}$ , where  $Q$  is an  $m$ -fold point,  $Q_j$ ,  $j = 1, \dots, k$ , are the neighboring singularities of  $Q$ , and  $m_j$ ,  $j = 1, \dots, k$ , are their corresponding multiplicities ( $H_P$  is the fibre function of  $P$ ). Therefore, by just analyzing the factorization of  $T$ , we can obtain all the singularities (ordinary and non-ordinary) as well as interesting data relative to each of them, like its multiplicity, character, fibre or number of associated tangents. Furthermore, in the case of non-ordinary singularities, we can easily get the corresponding number of local branches and delta invariant.

The problem of computing the singularities from the parametrization defining a curve has been much treated in the literature (see Section 1). In the present paper, we describe the structure of singular points by using the parameterization, but from a different point of view with respect to the previous works. This point of view is just based on the analysis of the structure of a polynomial, the  $T$ -function, constructed by a univariate resultant. The results presented in this paper and some appearing in previous literature are directly related and all allow compute satisfactorily the singularities. However, the approaches are totally different and, perhaps, complementary (see Section 1). For instance, in [7], it is introduced the notion of *inversion formula* of a point  $P$  on a rational curve  $\mathcal{C}$ . It is a polynomial whose roots provide the fibre of the point  $P$  in a given homogeneous parametrization of  $\mathcal{C}$ . In this paper, we work with non-homogeneous parametrizations and we define the equivalent concept of *fibre function* which, in fact, was introduced in [22] (see Theorem 17 and Corollary 1).

In the context of planar curves, in [1], Abhyankar shows how to compute the product of the inversion formulae of the different singularities by means of

a resultant. However, his approach only works if one considers a polynomial parametrization. In [8], it is obtained a similar formula for the case of a generic rational parametrization, not necessarily polynomial. There, using  $\mu$ -bases, authors deal with the singular factors introduced in [9], and it is proved that the  $k$ -th singular factor provides the product of the inversion formulae of the  $k$ -fold singularities (see Theorem 11). This allows to obtain the product of the inversion formulae as the product of the singular factors.

Our paper is based on the approach of [22]. We consider the case of a generic rational parametrization, and we aim to generalize Abhyankar's formula by just computing a univariate resultant and dealing with two important and new situations, the presence of the non-ordinary singularities as well as the case of space curves in any dimension. For this purpose, we observe that in [22], it is proved that the inversion formulae of the different singularities divide the resultant (see Theorem 10). Now, we improve that result and we show how to obtain, in an exact way, the product of the inversion formulae (which we call the T-function) by removing the residual factors of the resultant (in this sense, we note that the extraneous factor removing from the resultant computed for defining the T-function,  $T(s)$ , is only a power of the denominator of the parametrization). Thus, we get these inversion formulae, but using the resultant and generalizing Abhyankar's result.

In addition, we show how to group the different factors of the resultant to obtain the fibre functions of the different singularities (ordinary and non-ordinary), their multiplicity, number of branches and delta invariants (see the algorithm presented). Furthermore, we show how to deal with singularities that are reached by algebraic values of the parameter (see Remark 3).

Regarding the study of space curves, in [7],  $\mu$ -basis are used to obtain an extension of the singular factors and it is proved that the inversion formula of any  $k$ -fold singularity divides the  $k$ -th singular factor (see Corollary 17 of Section 5.2). However, there can be other elements dividing the singular factor that do not correspond to any inversion formula.

In this sense, we introduce an extension of the T-function which allows us to perfectly generalize the results of plane curves to the space case. The new T-function ( $T_E$ ) provides, in an exact way, the product of the inversion formulae, so that any factor of the T-function corresponds to a singularity of the curve.

Therefore, we can conclude that in this paper we provide some new achievements on this topic. More precisely: 1) we provide a complete classification of the singularities of the curve by just analyzing the structure of a univariate resultant, 2) we deal with both, the case of ordinary and the case of non-ordinary singularities, 3) for each singularity, we get its multiplicity, character, fibre and number of tangents, as well as the number of local branches and the delta invariant for the case of non-ordinary singularities and 4), we show how the results can be obtained for the case of space curves in any dimension (for this purpose, we construct a plane curve which is in correspondence with the input space curve). In addition, we recall that, in a direct method, in order to compute the singularities, one would introduce algebraic numbers during the computations. However, in this paper, in order to deal with this problem, we consider families of conjugated parametric points. This new notion allows us to determine the singularities of a curve without directly introducing algebraic numbers in the computations.

## Acknowledgements

This work has been partially supported by the Spanish Ministerio de Ciencia, Innovación y Universidades and by the European Regional Development Fund (ERDF), under the project MTM2017-88796-P. The second author belongs to the Research Group ASYNACS (Ref. CCEE2011/R34).

## References

- [1] Abhyankar, S.S. (1990). *Algebraic geometry for scientists and engineers*. Math. Surveys Monogr., vol. 35, American Mathematical Society, Providence, RI.
- [2] Bernardi, A., Gimigliano, A., Idà, M. (2018). *Singularities of plane rational curves via projections*. Journal of Symbolic Computation. Vol. 86, pp. 189–214.
- [3] Blasco, A., Pérez-Díaz, S. (2017). *Resultants and singularities of parametric curves*. See arXiv:1706.08430v3 [math.AG].
- [4] Blasco, A., Pérez-Díaz, S. (2018). *The limit point and the T-function*. To appear in Journal of Symbolic Computation. DOI: 10.1016/j.jsc.2018.06.009.

- [5] Bobenko, A.I., Klein, C. (2011). *Computational approach to riemann surfaces*. Lecture Notes in Mathematics. Volume 2013.
- [6] Brieskorn, E., Knoerrer, H. (1986). *Plane algebraic curves*. Birkhäuser Verlag Basel.
- [7] Busé, L., Thang, L.B. (2010) *Matrix-based implicit representations of rational algebraic curves and applications*. Computer Aided Geometric Design 27 vol. 9, 681–699.
- [8] Busé, L., D’Andrea, C. (2012) *Singular factors of rational plane curves*. Journal of Algebra. Vol. 357, pp. 322–346.
- [9] Chen, F., Wang, W., Liu, Y. (2008) *Computing singular points of plane rational curves*. J. Symbolic Comput. Vol. 43 (2), pp. 92–117.
- [10] Emiris, I.Z., Sottile, F., Theobald, T. (2010). *Nonlinear computational geometry*. The IMA Volumes in Mathematics and its Applications. Volume 151.
- [11] Farouki, R.T. (2008). *Pythagorean–hodograph curves*. Series: Geometry and Computing. Springer.
- [12] Fulton, W. (1989). *Algebraic crves – An introduction to algebraic geometry*. Addison-Wesley, Redwood City, CA.
- [13] Gutierrez, J., Rubio, R., Yu, J.-T. (2002). *Resultant for rational functions*. Proceedings of the American Mathematical Society. Vol. 130 (8). pp. 2237–2246.
- [14] Harris, J. (1995). *Algebraic geometry. A first course*. Springer-Verlag.
- [15] Hoffmann, C. (1989). *Geometric and solid modeling: An introduction*. Morgan Kaufmann Publishers, Inc. California.
- [16] Hoffmann, C.M., Sendra, J.R., Winkler, F. (1997). *Parametric algebraic curves and applications*. J. Symbolic Computation. Vol. 23.
- [17] Hoschek, J., Lasser, D. (1993). *Fundamentals of computer aided geometric design*. A.K. Peters Wellesley MA., Ltd.

- [18] Jia, X., Goldman, R. (2012). *Using Smith forms and  $\mu$ -bases to compute all the singularities of rational planar curves*. Computer Aided Geometric Design. Vol. 29. pp. 296–314.
- [19] Jüttler, B., Piene, R. (2007). *Geometric modeling and algebraic geometry*. Springer Science and Business Media 2007.
- [20] Park, H. (2002). *Effective computation of singularities of parametric affine curves*. Journal of Pure and Applied Algebra. Vol. 173, pp. 49–58.
- [21] Pérez-Díaz, S. (2006). *On the problem of proper reparametrization for rational curves and surfaces*. Computer Aided Geometric Design. Vol. 23(4). pp. 307–323.
- [22] Pérez-Díaz, S. (2007). *Computation of the singularities of parametric plane curves*. Journal of Symbolic Computation. Vol. 42/8. pp. 835–857.
- [23] Pérez-Díaz, S., Sendra, J.R. (2008). *A univariate resultant-based implicitization algorithm for surfaces*. Journal of Symbolic Computation. Vol. 43/2. pp. 118–139.
- [24] Pérez-Díaz, S., Sendra, J.R., Villarino, C. (2015). *Computing the singularities of rational surfaces*. Mathematics of Computation. Vol. 84, N. 294, pp. 1991—2021.
- [25] Rubio, R., Serradilla, J.M., Vélez, M.P. (2009). *Detecting real singularities of a space curve from a real rational parametrization*. Journal of Symbolic Computation. Vol. 44, Issue 5, pp. 490–498.
- [26] Sendra, J.R., Winkler, F. (2001). *Tracing index of rational curve parametrizations*. Computer Aided Geometric Design. Vol. 18 (8). pp. 771–795.
- [27] Sendra, J.R., Winkler, F., Pérez-Díaz, S. (2007). *Rational algebraic curves: A computer algebra approach*. Series: Algorithms and Computation in Mathematics. Vol. 22. Springer Verlag.
- [28] Shafarevich I.R., (1994). *Basic algebraic geometry Schemes; 1 Varieties in projective space*. Berlin New York: Springer-Verlag. Vol. 1.

- [29] Shi, X., Jia, X., Goldman, R. (2013). *Using a bihomogeneous resultant to find the singularities of rational space curves*. Journal of Symbolic Computation. Volume 53. pp. 1–25.
- [30] Song, N., Chen, F., Goldman, R. (2007). *Axial moving lines and singularities of rational planar curves*. Comput. Aided Geom. Design. Vol. 24(4). pp. 200–209.
- [31] Van den Essen, A., Yu, J.-T. (1997). *The D-resultant, singularities and the degree of unfaithfulness*. Proceedings of the American Mathematical Society. Vol. 125 (3), pp. 689–695.
- [32] Walker, R.J. (1950). *Algebraic curves*. Princeton University Press.