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Parameterization of rational translational surfaces

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Abstract

A rational translational surface is a typical modelling surface used in computer-aided design and the architecture industry. In this study, we determine whether a given algebraic surface implicitly defined as $\mathcal{V}$ is a rational translational surface or not. This problem is reduced to finding the rational parameterizations of two space curves. More important, our discussions are constructive, and thus if $\mathcal{V}$ is translational, we provide a parametric representation of $\mathcal{V}$ of the form $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$.

Key words: rational parameterization, reparameterization, translational surface

1. Introduction

In computer-aided geometric design and geometric modelling, common surfaces are often used to construct complex models. These common surfaces are generally known as basic modelling surfaces and they have some advantages, such as being simple and widely used. By "simple",we refer to surfaces with low degree such as quadratic surfaces [4, 7, 28] and cubic surfaces [2, 3]. By "widely used", we refer to surfaces that are common in industrial design such as ruled surfaces [5, 22, 24], swept surfaces [20, 27], and translational surfaces [12, 13, 18, 19]. The first task is to obtain a deep understanding of these basic modelling surfaces. Indeed, the best way to represent these surfaces is the first problem that needs to be addressed.

It is well known that two representation forms are generally used: parametric and implicit forms. The parametric representation is the most popular geometric representation in computer graphics and computer-aided design (e.g., see [8] and [9]). This representation is easy to render and helpful for some geometric operations such as the computation of curvature or bounds and the control of position or tangency. However, some problems are difficult to deal with if the surfaces are given parametrically, such as positional relationship determination and collision detection. These reasons explain

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why many researchers have recently shown an increased interest in geometric modelling with implicit equations [26, 30].

In this study, we deal with translational surfaces that are commonly used in industrial design, computer-aided design, and the architecture industry (e.g., see [12, 13, 18, 19]). A translational surface is a simple structure generated by two auxiliary space curves. In particular, let $\mathcal{P}_i(t_i)$, $i = 1, 2$ be parameterizations of two space curves $\mathcal{C}_{\mathcal{P}_i}$, $i = 1, 2$, respectively, which intersect at a common point $\bar{a} = \mathcal{P}_1(\alpha) = \mathcal{P}_2(\alpha)$. A translational surface is defined by $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2) - \bar{a}$ (see Chapter 15.3 in [8]). Without loss of generality (w.l.o.g.), throughout this study, we set $\bar{a} = (0, 0, 0)$.

Many researchers have addressed various problems related translational surfaces, which are motivated by the practical requirements of architectural design. These geometric problems have led to interesting research and results in the areas of geometry processing, computer-aided geometric design, and discrete differential geometry (e.g., [8, 12, 13, 14, 15, 18, 19, 25]).

In this study, because of the increasing interest in modelling from implicit equations, we consider a problem concerning translational surfaces but starting from the implicit point of view. Thus, given an implicitly defined algebraic surface $\mathcal{V}$, we analyze whether $\mathcal{V}$ is translational and if this is true, we compute a parameterization of $\mathcal{V}$. The parameterization of a given surface is not unique due to the parameter transformations and different approaches can be applied for its computation. However, to ensure better control and design, we are interested in finding auxiliary space curves $\mathcal{C}_{\mathcal{P}_i}$, $i = 1, 2$, and computing their parametric representations $\mathcal{P}_i(t_i)$, $i = 1, 2$, thereby obtaining a parameterization $\mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ for $\mathcal{V}$. First, we find a rational space curve, which plays the role $\mathcal{C}_{\mathcal{P}_1}$, and we compute a parameterization, $\mathcal{P}_1(t_1)$, of $\mathcal{C}_{\mathcal{P}_1}$. Next, if $\mathcal{V}$ is translational, we find $\mathcal{C}_{\mathcal{P}_2}$ and compute a parameterization, $\mathcal{P}_2(t_2)$, of $\mathcal{C}_{\mathcal{P}_1}$. Finally, we obtain a parameterization $\mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ for the given algebraic translational surface.

The remainder of this paper is organized as follows. In Section 2, we introduce some preliminaries and some previous theoretical results, which we use to obtain the main results presented in Section 3. In Section 3, we first construct a candidate for the first auxiliary curve (see Theorems 1 and 2). Next, we use Theorem 3 and we compute the second auxiliary curve. These two auxiliary curves provide a parameterization for $\mathcal{V}$ in the form of $\mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$, if $\mathcal{V}$ is translational (see Theorem 5). Some results related to improved computational aspects are also presented (see Theorem 4 and Corollary 1). Finally, based on the results obtained in Section 3, we derive an algorithm, which we illustrate with examples (see Section 4).

2. Preliminaries and previous theoretical results

Let $\mathcal{V}$ be a surface over an algebraically closed field of characteristic zero $\mathbb{K}$, and let $f(\,\overline{x}\,) \in \mathbb{K}[\,\overline{x}\,]$, $\overline{x} := (x_1, x_2, x_3)$, be an irreducible polynomial that implicitly defines

$\mathcal{V}$. For practical applications, we may consider that $\mathbb{K}$ is the field of complex numbers.

In the following, we analyze whether $\mathcal{V}$ is a *rational translational surface*, i.e., whether $\mathcal{V}$ admits a parameterization of the standard form

$$\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2), \tag{1}$$

where

$$\mathcal{P}_1(t_1)=(p_{11}(t_1),p_{12}(t_1),p_{13}(t_1))\in\mathbb{K}(t_1)^3\setminus\mathbb{K}^3,$$
$$\mathcal{P}_2(t_2)=(p_{21}(t_2),p_{22}(t_2),p_{23}(t_2))\in\mathbb{K}(t_2)^3\setminus\mathbb{K}^3$$

and if this is true, then we compute $\mathcal{P},\mathcal{P}_1$ and $\mathcal{P}_2$. We denote $\mathcal{C}_{\mathcal{P}_i}$, $i=1,2$, as the space curves over $\mathbb{K}$ defined by the rational parameterizations $\mathcal{P}_i$, $i=1,2$, respectively. We refer to these curves as *auxiliary curves of* $\mathcal{V}$. Note that if $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are different parameterizations of $\mathcal{C}_{\mathcal{P}_1}$ and $\mathcal{C}_{\mathcal{P}_2}$, respectively, then $\mathcal{Q}(t_1,t_2):=\mathcal{Q}_1(t_1)+\mathcal{Q}_2(t_2)$ is also a parameterization of $\mathcal{V}$ ($R_i(t_i)\in\mathbb{K}(t_i)\setminus\mathbb{K}$ exist such that $\mathcal{Q}_i(R_i(t_i))=\mathcal{P}_i(t_i)$, $i=1,2$, and thus $\mathcal{Q}(R_1(t_1),R_2(t_2))=\mathcal{P}(t_1,t_2)$; e.g., see [17]).

In addition, we also observe that auxiliary curves are not unique. For instance, if $\mathcal{V}$ is a translational surface and $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$ parameterizes $\mathcal{V}$, then we may write $\mathcal{P}(t_1,t_2)=(\mathcal{P}_1(t_1)+\bar{c})+(\mathcal{P}_2(t_2)-\bar{c})$, $\bar{c}\in\mathbb{K}^3$, and in this case, two auxiliary curves $\mathcal{C}_{\overline{\mathcal{P}}_1}$ and $\mathcal{C}_{\overline{\mathcal{P}}_2}$, are parameterized by $\overline{\mathcal{P}}_1(t_1):=\mathcal{P}_1(t_1)+\bar{c}$, and $\overline{\mathcal{P}}_2(t_2):=\mathcal{P}_2(t_2)-\bar{c}$, respectively.

A translational surface degenerates to a cylindrical surface if one of the auxiliary curves can be defined by a rational parametrization of degree one. Throughout this study, we assume that $\mathcal{V}$ is not a cylindrical surface. We can check that $\mathcal{V}$ is a cylindrical surface if and only if a constant vector $(a_1,a_2,a_3)\in\mathbb{K}^3\setminus\{(0,0,0)\}$ exists such that $\nabla f(\,\overline{x}\,)\cdot(a_1,a_2,a_3)=a_1f_{x_1}(\,\overline{x}\,)+a_2f_{x_2}(\,\overline{x}\,)+a_3f_{x_3}(\,\overline{x}\,)=0$, where $f_{x_i}(\,\overline{x}\,)$ denotes the partial derivative of the polynomial f w.r.t. the variable x_i. In this case, we may compute a parameterization of $\mathcal{V}$ by applying, for instance, the results in [23].

In the following, we consider a translational surface $\mathcal{V}$ parameterized by $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$, and we present some properties concerning two auxiliary space curves $\mathcal{C}_{\mathcal{P}_i}$, $i=1,2$, and their parameterizations $\mathcal{P}_i(t_i)\in\mathbb{K}(t_i)^3\setminus\mathbb{K}^3$. These theoretical results play an important role in Section 3.

Proposition 1. *Let $\mathcal{V}$ be a translational surface with parameterization $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$. The following properties hold.*

1. *If $\mathcal{P}$ is proper, then $\mathcal{P}_1$ and $\mathcal{P}_2$ are both proper parameterizations.*
2. *There exist $\overline{\mathcal{P}}_1$ and $\overline{\mathcal{P}}_2$ proper such that $\overline{\mathcal{P}}_1(t_1)+\overline{\mathcal{P}}_2(t_2)$ parameterizes $\mathcal{V}$.*
3. *The auxiliary curves are not lines.*

Proof.

1. Let us prove that $\mathcal{P}_1$ is a proper parameterization (similarly, we show that $\mathcal{P}_2$ is proper). For this purpose, we assume that $\mathcal{P}_1$ is not proper. Then, $\phi_i(s_1) \in \overline{\mathbb{K}(s_1)}$, $i = 1, 2$, $\phi_1 \neq \phi_2$, exist such that $\mathcal{P}_1(\phi_1(s_1)) = \mathcal{P}_1(\phi_2(s_1)) = \mathcal{P}_1(s_1)$ ($\overline{\mathbb{K}(s_1)}$ denotes the algebraic closure of $\mathbb{K}(s_1)$, and s_1 is a new variable). Thus, $\mathcal{P}(\phi_1(s_1), s_2) = \mathcal{P}(\phi_2(s_1), s_2) = \mathcal{P}(s_1, s_2)$ (s_2 is a new variable). This implies that $\mathcal{P}$ is not proper, which is a contradiction. Therefore, we conclude that $\mathcal{P}_1$ is proper.

2. Considering, for instance, the results in [17], proper parametrizations $\overline{\mathcal{P}}_i$ and rational functions $R_i(t_i) \in \mathbb{K}(t_i) \setminus \mathbb{K}$ must exist such that $\overline{\mathcal{P}}_i(R(t_i)) = \mathcal{P}_i(t_i)$, $i = 1, 2$. Thus, $\overline{\mathcal{P}}(t_1, t_2) = \overline{\mathcal{P}}_1(t_1) + \overline{\mathcal{P}}_2(t_2)$ satisfies that $\overline{\mathcal{P}}(R_1(t_1), R_2(t_2)) = \mathcal{P}(t_1, t_2)$. Hence $\overline{\mathcal{P}}$ is a parameterization of $\mathcal{V}$ of the form given in Eq. (1), and $\overline{\mathcal{P}}_i$, $i = 1, 2$ are both proper parameterizations.

3. Let us assume that an auxiliary curve, $\mathcal{C}_{\mathcal{P}_2}$, is a line (similarly, if $\mathcal{C}_{\mathcal{P}_1}$ is a line). Then, a proper parameterization of $\mathcal{C}_{\mathcal{P}_2}$ is given by $\mathcal{P}_2(t_2) = (a_1t_2 + b_1, a_2t_2 + b_2, a_3t_2 + b_3) \in \mathbb{K}(t_2)^3 \setminus \mathbb{K}^3$. Since $f(\mathcal{P}(t_1, t_2)) = 0$, we find that $\nabla f(\mathcal{P}(t_1, t_2)) \cdot \mathcal{P}_2'(t_2) = \nabla f(\mathcal{P}(t_1, t_2)) \cdot (a_1, a_2, a_3) = 0$. Thus, $\mathcal{P}$ parameterizes the surface $\mathcal{V}$ defined by $f(\overline{x})$ and the surface defined by $h(\overline{x}) := \nabla f(\overline{x}) \cdot (a_1, a_2, a_3)$. Since f is irreducible and $\deg(h) < \deg(f)$, we find that $h(\overline{x}) = 0$, which is impossible since $\mathcal{V}$ is not a cylindrical surface. □

Remark 1. *In Section 3, we compute proper parametrizations $\mathcal{P}_i$, $i = 1, 2$ (see statement 2 in Proposition 1). However, note that the properness of $\mathcal{P}_i$, $i = 1, 2$ does not imply the properness of $\mathcal{P}$.*

If $\mathcal{V}$ is a translational surface, its auxiliary curves can be assumed to satisfy an important property that is proved in the following lemma. This property facilitates the computation of the auxiliary curves in Section 3.

Lemma 1. *Let $\mathcal{V}$ be a translational surface with parameterization $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$. A nonempty Zariski open subset $\Omega_{\mathcal{P}_2} \subset \mathbb{K}$ exists such that for every $t_2^0 \in \Omega_{\mathcal{P}_2}$, there are two space curves $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i = 1, 2$, defined by proper parameterizations $\overline{\mathcal{P}}_i$, $i = 1, 2$, respectively, which satisfy that:*

(1.) $\overline{\mathcal{P}}_2(t_2^0) = (0, 0, 0)$, (2.) $\overline{\mathcal{P}}'_2(t_2^0) \neq (0, 0, 0)$, (3.) $\overline{\mathcal{P}}_1(t_1) + \overline{\mathcal{P}}_2(t_2)$ *parameterizes* $\mathcal{V}$.

Hence, $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i = 1, 2$ are auxiliary curves of $\mathcal{V}$.

Proof. Let $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$ be a parameterization of $\mathcal{V}$ such that $\mathcal{P}_i,\ i=1,2$, are proper (see statement 2 in Proposition 1). In addition, let

$$\Omega_{\mathcal{P}_2}:=\{t\in\mathbb{K}\,|\,\mathcal{P}_2(t_2)\text{ and }\mathcal{P}_2'(t_2)\text{ be defined at }t_2=t,\ \text{ and }\ \mathcal{P}_2'(t)\neq(0,0,0)\}.$$

Note that $\Omega_{\mathcal{P}_2}\subset\mathbb{K}$ is a Zariski open subset and it is nonempty. Indeed, if $\Omega_{\mathcal{P}_2}=\emptyset$, then $\mathcal{P}_2(t_2)\in\mathbb{K}^3$, which is impossible since $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$ parameterizes the surface $\mathcal{V}$. Under these conditions, for every $t_2^0\in\Omega_{\mathcal{P}_2}$, the space curves defined by the parameterizations $\overline{\mathcal{P}}_1(t_1)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2^0)$, and $\overline{\mathcal{P}}_2(t_2)=\mathcal{P}_2(t_2)-\mathcal{P}_2(t_2^0)$, respectively, satisfy conditions (1.), (2.), and (3.). Observe that $\overline{\mathcal{P}}_i$ are proper since $\mathcal{P}_i$ are proper (for $i=1,2$). In addition, $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i=1,2$, are auxiliary curves of $\mathcal{V}$. □

3. Parameterizing the translational surface

In this section, we provide a necessary and sufficient condition for an algebraic surface $\mathcal{V}$ being translational. The proofs are constructive and a method for computing $\mathcal{P}$ is then developed (see Section 4, where an algorithm and illustrative examples are presented).

We construct a candidate for the first auxiliary curve (see Theorems 1 and 2) of a translational surface. Then, using Theorem 3, we compute the second auxiliary curve (see also Theorem 4). These two auxiliary curves provide a parameterization for $\mathcal{V}$ in the form given in Eq. (1). If these curves do not exist, we conclude that $\mathcal{V}$ is not translational (see Theorem 5 in Section 4).

In the following, we assume we are in the conditions stated in Section 2.

Theorem 1. *Let $\mathcal{V}$ be a translational surface defined parametrically by $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$, and implicitly by $f(\overline{x})\in\mathbb{K}[\overline{x}]$. Let $\Omega_{\mathcal{P}_2}$ be the nonempty Zariski open subset introduced in Lemma 1. Then, for every $t_2^0\in\Omega_{\mathcal{P}_2}$, the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$, where*

$$g_{\overline{a}}(\overline{x}):=\nabla f(\overline{x})\cdot\overline{a}=a_1f_{x_1}(\overline{x})+a_2f_{x_2}(\overline{x})+a_3f_{x_3}(\overline{x}),\quad \overline{a}:=\mathcal{P}_2'(t_2^0)=(a_1,a_2,a_3),$$

contains a space curve parameterized by $\overline{\mathcal{P}}_1(t_1):=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2^0)\in\mathbb{K}(t_1)^3$. In addition, $\overline{\mathcal{P}}_1+\overline{\mathcal{P}}_2$ is a parameterization of $\mathcal{V}$, where $\overline{\mathcal{P}}_2(t_2):=\mathcal{P}_2(t_2)-\mathcal{P}_2(t_2^0)\in\mathbb{K}(t_2)^3$, and thus the space curves $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i=1,2$, parameterized by $\overline{\mathcal{P}}_i$, $i=1,2$, respectively, are auxiliary curves of $\mathcal{V}$.

Proof. Since $\mathcal{V}$ admits a parameterization of the form given in Eq. (1), we find that $f(\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2))=0$, and in particular,

$$f(\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2^0))=0$$

for every $t_2^0 \in \Omega_{\mathcal{P}_2}$ (see Lemma 1). In addition, from $f(\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2))=0$, we also find that $\nabla f(\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)) \cdot \mathcal{P}_2'(t_2)=0$. Thus,

$$g_{\overline{a}}(\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2^0)) = \nabla f(\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2^0)) \cdot \overline{a} = 0,$$

where $\overline{a} := \mathcal{P}_2'(t_2^0) = (a_1,a_2,a_3) \in \mathbb{K}^3 \setminus \{(0,0,0)\}$ (see Lemma 1).

Finally, we prove that the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$ contains a space curve that is properly parameterized by $\overline{\mathcal{P}}_1$. Indeed, since $\mathcal{V}$ is not a cylindrical surface, we have that $g_{\overline{a}}(\overline{x}) \notin \mathbb{K}$ for every $\overline{a} \in \mathbb{K}^3 \setminus \{(0,0,0)\}$ (note that $g_{\overline{a}}(\overline{\mathcal{P}}_1)=0$, and thus if $g_{\overline{a}}(\overline{x})=c\in\mathbb{K}$, then $c=0$, which would imply that $g_{\overline{a}}(\overline{x})=0$, so $\mathcal{V}$ is a cylindrical surface, which is impossible). In addition, since $0<\deg(g_{\overline{a}})<\deg(f)$, f is irreducible and $f(\overline{\mathcal{P}}_1)=g_{\overline{a}}(\overline{\mathcal{P}}_1)=0$, we conclude that the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$ contains a space curve parameterized by $\overline{\mathcal{P}}_1$. In addition, $\overline{\mathcal{P}}_1$ is proper since $\mathcal{P}_1$ is proper, and $\overline{\mathcal{P}}_1+\overline{\mathcal{P}}_2$ is a parameterization of $\mathcal{V}$, where $\overline{\mathcal{P}}_2(t_2) := \mathcal{P}_2(t_2)-\mathcal{P}_2(t_2^0) \in \mathbb{K}(t_2)^3$ (note that $\overline{\mathcal{P}}_1+\overline{\mathcal{P}}_2=\mathcal{P}$). Hence, by definition, the space curves $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i=1,2$, parameterized by $\overline{\mathcal{P}}_i$, $i=1,2$, respectively, are auxiliary curves of $\mathcal{V}$. □

Remark 2. *1. Theorem 1 provides a necessary condition for the existence of a first auxiliary curve (this results also shows how it is computed). However, there may be suitable vectors $\overline{a}$ not defined from $\mathcal{P}_2'(t_2^0)$ that provide more (and different) auxiliary curves (see Example 5).*

2. We recall that any space curve $\mathcal{D}_s$ can be birationally projected onto a plane curve $\mathcal{D}_p$, and the problem of deciding the rationality of $\mathcal{D}_s$ as well as computing a parameterization of $\mathcal{D}_s$ can be reduced to the problem of deciding whether $\mathcal{D}_p$ is rational and computing a parameterization of $\mathcal{D}_p$ (e.g., see [1], [10], [11]). Some algorithmic methods for dealing with these problems in the case of plane curves were developed in [21] (see Chapter 4).

Example 1. *Let $\mathcal{V}$ be the surface implicitly defined by the polynomial $f(\overline{x}) = x_3+5x_1^2-6x_1x_2+2x_2^2 \in \mathbb{C}[\overline{x}]$. We may check that $\mathcal{V}$ satisfies the assumptions introduced in Section 2. We consider the polynomials*

$$f(\overline{x}) \quad \text{and} \quad g_{\overline{a}}(\overline{x}) := \nabla f(\overline{x}) \cdot \overline{a} = a_1 f_{x_1}(\overline{x}) + a_2 f_{x_2}(\overline{x}) + a_3 f_{x_3}(\overline{x}),$$

where $\overline{a} := (a_1,a_2,a_3)$ is an undetermined vector in $\mathbb{C}^3 \setminus \{(0,0,0)\}$ (note that $\mathcal{P}_2$ is unknown and in fact, we do not know if $\mathcal{V}$ is translational. Thus, we consider that $\overline{a}$ is undetermined; see Theorem 1). By applying statement 2 in Remark 2, we find that $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$ define an irreducible parametric space curve for "almost all" values of the vector $\overline{a}$. In fact, a parameterization of this space curve is given by

$$\left(t_1, \frac{-10a_1t_1+6a_2t_1-a_3}{2(-3a_1+2a_2)}, \frac{-12t_1^2a_1a_2+4t_1^2a_2^2+2t_1a_3a_1+10t_1^2a_1^2+a_3^2}{-2(9a_1^2-12a_1a_2+4a_2^2)}\right) \in \mathbb{C}(t_1)^3.$$

We note that if $\mathcal{V}$ is translational and $\mathcal{P}_2$ is a parameterization as in Eq. (1), then we may consider that $\overline{a} = \mathcal{P}_2'(t_2^0)$ (for some $t_2^0 \in \Omega_{\mathcal{P}_2}$), and thus the parameterization given above would define an auxiliary curve (see Theorem 1).

In Example 1, we obtain a parameterization that depends on the unknown vector $\overline{a}$. For a specific value of this vector, we obtain a particular parameterization that "could" provide a first candidate for the auxiliary curves of $\mathcal{V}$ (if $\overline{a} = \mathcal{P}_2'(t_2^0)$, for some $t_2^0 \in \Omega_{\mathcal{P}_2}$, and $\mathcal{P}_2$ is given as in Eq. (1); however, note that $\mathcal{P}_2$ is unknown since we only have the implicit representation $f(\overline{x})$). In the following, we show the computation of the first candidate auxiliary curve of $\mathcal{V}$ can be improved (computationally speaking) by choosing any particular value for the vector $\overline{a}$, which is not necessarily given by $\mathcal{P}_2'$ (see Theorem 1). In particular, Theorem 2 allows us to consider $\overline{a} \in \mathcal{W}$, where $\mathcal{W}$ is a special surface. In addition, some of these points in $\mathcal{W}$ can be computed in advance, without explicitly computing $\mathcal{W}$.

First, we need to prove the following technical lemma where some particular points on $\mathcal{W}$ are determined in advance, i.e., without explicitly computing $\mathcal{W}$.

Lemma 2. *Let $\mathcal{W}$ be a surface defined by a parameterization $\mathcal{Q}(t_1, t_2) = (t_1q_1(t_2), t_1q_2(t_2), t_1q_3(t_2)) \in \mathbb{K}(t_1, t_2)^3$, where $q_i(t_2)$, $i = 1, 2, 3$, are any rational functions in $\mathbb{K}(t_2)$. The following statements hold:*

1. *$(1, \lambda, 0) \in \mathcal{W}$ or $(0, 1, 0) \in \mathcal{W}$, for some $\lambda \in \mathbb{K}$.*
2. *$(\lambda, 0, 1) \in \mathcal{W}$ or $(1, 0, 0) \in \mathcal{W}$, for some $\lambda \in \mathbb{K}$.*
3. *$(0, 1, \lambda) \in \mathcal{W}$ or $(0, 0, 1) \in \mathcal{W}$, for some $\lambda \in \mathbb{K}$.*

Proof. We prove statement 1. For this purpose, we first note that $\mathcal{Q}(t_1, t_2)$ defines a rational conical surface and the implicit polynomial defining $\mathcal{W}$, $f_{\mathcal{W}}(\overline{x})$ is homogeneous w.r.t. the variables x_1, x_2, x_3. Then, we set $x_3 = 0$ and we find that the polynomial $\overline{f}(x_1, x_2) := f_{\mathcal{W}}(x_1, x_2, 0) = 0$ is homogeneous in x_1 and x_2. Hence, if $(\alpha, \beta) \in \mathbb{K}^2 \setminus \{(0, 0)\}$ is such that $\overline{f}(\alpha, \beta) = 0$, we find that $f_{\mathcal{W}}(1, \beta/\alpha, 0) = 0$ (if $\alpha \neq 0$) or $f_{\mathcal{W}}(0, 1, 0) = \overline{f}(0, 1) = 0$ (if $\alpha = 0$). Thus, $(1, \lambda, 0) \in \mathcal{W}$ (where $\lambda = \beta/\alpha$) or $(0, 1, 0) \in \mathcal{W}$.

The reasoning is similar for statements 2 and 3 (for these cases, we set $x_2 = 0$ and $x_1 = 0$, respectively). □

Remark 3. *Note that since $\mathcal{W}$ is defined by a homogeneous polynomial, it holds that $(a, b, c) \in \mathcal{W}$ if and only if $(\gamma a, \gamma b, \gamma c) \in \mathcal{W}$ for every $\gamma \in \mathbb{K} \setminus \{0\}$.*

In the following, given a translational surface $\mathcal{V}$ with parameterization $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$, we set $\mathcal{P}_2'(t_2) := (q_1(t_2), q_2(t_2), q_3(t_2))$, and we consider the surface $\mathcal{W}_{\mathcal{P}_2}$ defined by the parameterization $\mathcal{Q}(t_1, t_2) = (t_1q_1(t_2), t_1q_2(t_2), t_1q_3(t_2))$. Note that $\mathcal{Q}$ parameterizes a surface since its Jacobian has rank 2; otherwise, $q_iq_j' = q_i'q_i$, which implies that $(q_i/q_j)' = 0$ and thus $q_i = c_j q_j$ for $i, j \in \{1, 2, 3\}$, $i \neq j$ and $c_j \in \mathbb{K}$. Thus, $\mathcal{P}_2(t_2) = (p(t_2), \alpha_1 p(t_2) + \beta_1, \alpha_2 p(t_2) + \beta_2)$, $\alpha_i, \beta_i \in \mathbb{K}$, which is impossible since $\mathcal{C}_{\mathcal{P}_2}$ is not a line (see statement 3 in Proposition 1).

In Theorem 2, we show that the computation of auxiliary curves can be improved by taking a particular value for the vector $\overline{a}$ introduced in Theorem 1 (compare Examples 1 and 2). We remark that the vector $\overline{a}$ is unknown since taking into account Theorem 1, if $\mathcal{V}$ is translational, $\overline{a} = \mathcal{P}_2'(t_2^0)$ (for some $t_2^0 \in \Omega_{\mathcal{P}_2}$), and $\mathcal{P}_2$ is given as in Eq. (1). However, $\mathcal{P}_2$ is the parameterization that we seek.

In particular, Theorem 2 states that for each vector $\overline{a} \in \mathcal{W}_{\mathcal{P}_2}$, we may construct an auxiliary curve of $\mathcal{V}$ (i.e., $\overline{a}$ is not necessarily given from $\mathcal{P}_2'$). In fact, some of these vectors $\overline{a} \in \mathcal{W}_{\mathcal{P}_2}$ can be obtained in advance, without explicitly computing $\mathcal{W}_{\mathcal{P}_2}$; in particular, one of these vectors can be $(1, \lambda, 0)$ (for some $\lambda \in \mathbb{K}$) or $(0, 1, 0)$ (we also may consider $(\lambda, 0, 1)$ or $(1, 0, 0)$, and $(0, 1, \lambda)$ or $(0, 0, 1)$; see Remark 4). Similarly, as in Theorem 1, Theorem 2 provides a necessary condition for the existence of a first auxiliary curve (this result also shows how it is computed). However, there may exist suitable vectors $\overline{a}$ not lying on the surface $\mathcal{W}_{\mathcal{P}_2}$ that provide more (and different) auxiliary curves (see Example 6).

If $\mathcal{V}$ is translational, an auxiliary curve computed by using Theorem 2 will be combined with a space curve obtained from Theorem 3 (which will be another auxiliary curve). From these two curves, we obtain a parameterization for $\mathcal{V}$ in the form given in Eq. (1). If any of these curves do not exist, then we may conclude that $\mathcal{V}$ is not translational (see Theorem 5 in Section 4).

Theorem 2. *Let $\mathcal{V}$ be a translational surface defined parametrically by $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$, and implicitly by $f(\overline{x}) \in \mathbb{K}[\overline{x}]$. Let $\mathcal{W}_{\mathcal{P}_2}$ be the surface introduced above, and let $\Omega_{\mathcal{P}_2}$ be the nonempty Zariski open subset obtained in Lemma 1. Then, for every nonzero vector $\overline{a} := (a_1, a_2, a_3) \in \mathcal{W}_{\mathcal{P}_2}$, the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$, where*

$$g_{\overline{a}}(\overline{x}) := \nabla f(\overline{x}) \cdot \overline{a} = a_1 f_{x_1}(\overline{x}) + a_2 f_{x_2}(\overline{x}) + a_3 f_{x_3}(\overline{x}),$$

contains a space curve parameterized by $\overline{\mathcal{P}}_1(t_1) := \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2^0) \in \mathbb{K}(t_1)^3$, for some $t_2^0 \in \mathbb{K}$. In addition, $\overline{\mathcal{P}}_1(t_1) + \overline{\mathcal{P}}_2(t_2)$ is a parameterization of $\mathcal{V}$, where $\overline{\mathcal{P}}_2(t_2) := \mathcal{P}_2(t_2) - \mathcal{P}_2(t_2^0) \in \mathbb{K}(t_2)^3$, and thus the space curves $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i = 1, 2$, parameterized by $\overline{\mathcal{P}}_i$, $i = 1, 2$, respectively, are auxiliary curves of $\mathcal{V}$.

Proof. In order to prove this theorem, we distinguish two different cases as follows.

1. We assume that $(t_1^0, t_2^0) \in \mathbb{K}^2$ exists such that $\mathcal{Q}(t_1^0, t_2^0) = \overline{a}$. Note that $t_1^0 \neq 0$ since $\overline{a} \neq (0, 0, 0)$. We assume w.l.o.g. that $t_1^0 = 1$ (otherwise, we consider the reparameterization $\mathcal{Q}(t_1 t_1^0, t_2)$). Hence, $\mathcal{P}_2'(t_2^0) = \overline{a}$, and thus $t_2^0 \in \Omega_{\mathcal{P}_2}$. Now,

we apply Theorem 1 and we deduce that the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$ contains a space curve parameterized by $\overline{\mathcal{P}}_1(t_1) := \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2^0) \in \mathbb{K}(t_1)^3$. In addition, $\overline{\mathcal{P}}_1 + \overline{\mathcal{P}}_2$ is a parameterization of $\mathcal{V}$, where $\overline{\mathcal{P}}_2(t_2) := \mathcal{P}_2(t_2) - \mathcal{P}_2(t_2^0) \in \mathbb{K}(t_2)^3$. Hence, the space curves $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i = 1, 2$, parameterized by $\overline{\mathcal{P}}_i$, $i = 1, 2$, respectively, are auxiliary curves of $\mathcal{V}$.

2. We assume that there does not exist $(t_1^0, t_2^0) \in \mathbb{K}^2$ such that $\mathcal{Q}(t_1^0, t_2^0) = \overline{a}$, where $\overline{a} = (a_1, a_2, a_3)$ (we assume w.l.o.g that $a_3 \neq 0$ since $\overline{a} \neq \overline{0}$). Note that this statement is equivalent to assuming that there does not exist $t_2^0 \in \mathbb{K}$ satisfying that $\mathcal{N}(t_2^0) = (a_1/a_3, a_2/a_3)$, where $\mathcal{N}(t_2) := (q_1(t_2)/q_3(t_2), q_2(t_2)/q_3(t_2))$. Then, we consider a reparameterization of the plane curve defined by $\mathcal{N}(t_2)$ such that $\mathcal{N}(R(s)) = \mathcal{N}^*(s)$, $R(s) \in \mathbb{K}(s) \setminus \mathbb{K}$, and $\mathcal{N}^*(s^0) = (a_1/a_3, a_2/a_3)$, $s_0 \in \mathbb{K}$ (see Section 6.3 in [21]). Thus, $\mathcal{Q}^*(t, s) = \mathcal{Q}(t, R(s)) = t\mathcal{P}_2'(R(s))$ is a reparameterization of $\mathcal{Q}$ satisfying that $\mathcal{Q}^*(t^0, s^0) = \overline{a}$ for some $t_0 \in \mathbb{K}$. Clearly, $\mathcal{Q}^*(t, s)$ is again a parameterization of the surface $\mathcal{W}_{\mathcal{P}_2}$. Note that since $R \notin \mathbb{K}$, then $\frac{\partial R}{\partial s} \neq 0$. Under these conditions, since $f(\mathcal{P}_1(t_1) + \mathcal{P}_2(R(s))) = 0$, we find that

$$\nabla f(\mathcal{P}_1(t_1) + \mathcal{P}_2(R(s))) \cdot \mathcal{P}_2'(R(s)) \frac{\partial R}{\partial s}(s) = 0.$$

Then, $\nabla f(\mathcal{P}_1(t_1) + \mathcal{P}_2(R(s))) \cdot \mathcal{P}_2'(R(s)) = 0$, and thus

$$\nabla f(\mathcal{P}_1(t_1) + \mathcal{P}_2(R(s))) \cdot \mathcal{P}_2'(R(s))t = 0.$$

Hence, $\nabla f(\mathcal{P}_1(t_1) + \mathcal{P}_2(R(s))) \cdot \mathcal{Q}^*(t, s) = 0$, and in particular,

$$\nabla f(\mathcal{P}_1(t_1) + \mathcal{P}_2(R(s^0))) \cdot \mathcal{Q}^*(t^0, s^0) = \nabla f(\mathcal{P}_1(t_1) + \mathcal{P}_2(R(s^0))) \cdot \overline{a} = 0.$$

Therefore, the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$ contains a space curve parameterized by $\overline{\mathcal{P}}_1(t_1) := \mathcal{P}_1(t_1) + \mathcal{P}_2(R(s^0)) \in \mathbb{K}(t_1)^3$. In addition, $\overline{\mathcal{P}}_1 + \overline{\mathcal{P}}_2$ is a parameterization of $\mathcal{V}$, where $\overline{\mathcal{P}}_2(t_2) := \mathcal{P}_2(t_2) - \mathcal{P}_2(R(s^0)) \in \mathbb{K}(t_2)^3$ (note that $\mathcal{P}(t_1, t_2) = \overline{\mathcal{P}}_1(t_1) + \overline{\mathcal{P}}_2(t_2)$), and thus the space curves $\mathcal{C}_{\overline{\mathcal{P}}_i}$, $i = 1, 2$, parameterized by $\overline{\mathcal{P}}_i$, $i = 1, 2$, respectively, are auxiliary curves of $\mathcal{V}$. □

Remark 4. *From Lemma 2, we deduce that in Theorem 2, we may consider the vector $\overline{a} = (1, 0, 0)$ and check whether the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x}) := \nabla f(\overline{x}) \cdot \overline{a}$ contains a rational space curve. If $V(f, g_{\overline{a}})$ does not contain a rational space curve, we reason in a similar manner for $\overline{a} = (0, 1, 0)$ and/or $\overline{a} = (0, 0, 1)$. If $V(f, g_{\overline{a}})$ does not contain a rational space curve for these three vectors, we consider $\overline{a} = (1, \lambda, 0)$, and we compute $\lambda \in \mathbb{K}$ such that the variety defined by the polynomials $f(\overline{x})$ and $g_{\overline{a}}(\overline{x})$ contains a rational space curve (see statement 2 in Remark 2). From Theorem 2, we deduce that this $\lambda \in \mathbb{K}$ exists if $\mathcal{V}$ is a translational surface.*

In the following example, we consider the surface $\mathcal{V}$ introduced in Example 1, and we show how the computation of the first candidate of the auxiliary curve can be improved by applying Theorem 2. In particular, we consider the vector $(1,0,0)$ and we reason as in Remark 4. A parameterization of $\mathcal{V}$ in the form $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$ is computed in Example 3.

Example 2. *Let $\mathcal{V}$ be the surface implicitly defined by the polynomial $f(\overline{x})=x_3+5x_1^2-6x_1x_2+2x_2^2\in\mathbb{C}[\overline{x}]$ introduced in Example 1. We consider the polynomials*

$$f(\overline{x}) \quad \textit{and} \quad g_{\overline{a}}(\overline{x}):=f_{x_1}(\overline{x}).$$

We may check that they define an irreducible rational space curve (in fact, for every value of $\lambda\in\mathbb{C}$, the polynomials $f(\overline{x})$, $g_{\overline{a}}(\overline{x})=\nabla f(\overline{x})\cdot\overline{a}$, with $\overline{a}=(1,\lambda,0)$, define a rational space curve). A proper parameterization of this curve is computed by applying Remark 2 (statement 2). We find that

$$\left(t_1,\frac{5}{3}t_1,-\frac{5}{9}t_1^2\right)\in\mathbb{C}(t_1)^3.$$

In Theorem 3, we give a characterization of the second auxiliary curve of a translational surface $\mathcal{V}$ defined parametrically by $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$, and implicitly by $f(\overline{x})\in\mathbb{K}[\overline{x}]$. In order to prove this result, we first consider

$$h(\overline{x},t_1):=f(\mathcal{P}_1(t_1)+(x_1,x_2,x_3))=\widetilde{h}(\overline{x})\Psi(\overline{x},t_1)p(t_1),$$

where $\Psi(\overline{x},t_1):=\psi_0(\overline{x})+\psi_1(\overline{x})t_1\cdots+\psi_n(\overline{x})t_1^n\in\mathbb{K}[\overline{x},t_1]$, $\gcd(\psi_0,\ldots,\psi_n)=1$, and $p(t_1)\in\mathbb{K}(t_1)$, $\widetilde{h}(\overline{x})\in\mathbb{K}[\overline{x}]$. We denote $V(\psi_0,\ldots,\psi_n)$ as the variety defined by the polynomials $\psi_0,\ldots,\psi_n$.

The following lemma provides some theoretical properties of $\Psi(\overline{x},t_1),\widetilde{h}(\overline{x})$ and $p(t_1)$, which are used in Theorems 3, 4, and 5, as well as in Corollary 1.

Lemma 3. *Let $\mathcal{V}$ be a translational surface defined parametrically by $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$, and implicitly by $f(\overline{x})\in\mathbb{K}[\overline{x}]$. Let*

$$h(\overline{x},t_1):=f(\mathcal{P}_1(t_1)+(x_1,x_2,x_3))=\widetilde{h}(\overline{x})\Psi(\overline{x},t_1)p(t_1),$$

where $\Psi(\overline{x},t_1)\in\mathbb{K}[\overline{x},t_1]$, $p(t_1)\in\mathbb{K}(t_1)$, $\widetilde{h}(\overline{x})\in\mathbb{K}[\overline{x}]$. The following statements hold:

1. *$f(\mathcal{P}_1(t_1)+\overline{x})\neq 0$ (and then $\widetilde{h}(\overline{x})\Psi(\overline{x},t_1)p(t_1)\neq 0$). In addition, $p(t_1^0)\neq 0$ for every $t_1^0\in\mathbb{K}$ for which $\mathcal{P}_1(t_1)$ is defined.*

2. *$\Psi(\overline{x},t_1)\in\mathbb{K}[\overline{x},t_1]\setminus\mathbb{K}[\overline{x}]$.*

3. *$\Psi(\overline{x},t_1)\in\mathbb{K}[\overline{x},t_1]\setminus\mathbb{K}[t_1]$.*

Proof.

1. Let us assume that $f(\mathcal{P}_1(t_1) + \overline{x}\,) = 0$. Then, $\nabla f(\mathcal{P}_1(t_1^0) + \overline{x}\,) \cdot (1, 1, 1) = 0$ for every $t_1^0 \in \mathbb{K}$ where $\mathcal{P}_1(t_1)$ is defined. By applying the change of variable $\overline{x} \to \overline{x} - \mathcal{P}_1(t_1^0)$, we get that $\nabla f(\,\overline{x}\,) \cdot (1, 1, 1) = 0$, which contradicts our assumption that $\mathcal{V}$ is not a cylindrical surface.
   In addition, we also have that $p(t_1^0) \neq 0$ for every $t_1^0 \in \mathbb{K}$; otherwise, $f(\mathcal{P}_1(t_1^0) + \overline{x}\,) = 0$, which implies that $\nabla f(\mathcal{P}_1(t_1^0) + \overline{x}\,) \cdot (1, 1, 1) = 0$ and this leads to a contradiction according to the reasoning given earlier.

2. Let us assume that $\Psi(\,\overline{x}\,, t_1) \in \mathbb{K}[\overline{x}]$, i.e., $f(\mathcal{P}_1(t_1) + \overline{x}\,) = \widetilde{h}(\,\overline{x}\,)\Psi(\,\overline{x}\,)p(t_1)$. Let $t_1^0 \in \mathbb{K}$ such that $p(t_1)$, $\mathcal{P}_1(t_1)$ and $\mathcal{P}_1'(t_1)$ are defined at $t_1 = t_1^0$, $\mathcal{P}_1'(t_1^0) \neq 0$, and $\deg_{\overline{x}}(f(\mathcal{P}_1(t_1) + \overline{x}\,)) = \deg_{\overline{x}}(f(\mathcal{P}_1(t_1^0) + \overline{x}\,))$. Clearly, this t_1^0 exists since the above conditions determine a nonempty open subset of $\mathbb{K}$. Then, we have that

   $$p(t_1^0)f(\mathcal{P}_1(t_1) + \overline{x}\,) = p(t_1)f(\mathcal{P}_1(t_1^0) + \overline{x}\,)$$

   (from statement 1 above, we find that $p(t_1^0) \neq 0$ for every $t_1^0 \in \mathbb{K}$). Deriving w.r.t. t_1, we obtain

   $$p(t_1^0)\nabla f(\mathcal{P}_1(t_1) + \overline{x}\,) \cdot \mathcal{P}_1'(t_1) = f(\mathcal{P}_1(t_1^0) + \overline{x}\,)p'(t_1).$$

   Since $\deg_{\overline{x}}(\nabla f(\mathcal{P}_1(t_1) + \overline{x}\,)) < \deg_{\overline{x}}(f(\mathcal{P}_1(t_1) + \overline{x}\,)) = \deg_{\overline{x}}(f(\mathcal{P}_1(t_1^0) + \overline{x}\,))$, we obtain $p'(t_1) = 0$, and therefore $p(t_1) = p(t_1^0) \in \mathbb{K}$. Thus,

   $$\nabla f(\mathcal{P}_1(t_1^0) + \overline{x}\,) \cdot \mathcal{P}_1'(t_1) = \nabla f(\mathcal{P}_1(t_1) + \overline{x}\,) \cdot \mathcal{P}_1'(t_1) = 0,$$

   and by applying the change of variable $\overline{x} \to \overline{x} - \mathcal{P}_1(t_1^0)$, we get that $\nabla f(\,\overline{x}\,) \cdot \mathcal{P}_1'(t_1^0) = 0$, which contradicts our assumption that $\mathcal{V}$ is not a cylindrical surface. Therefore, we conclude that $\Psi(\,\overline{x}\,, t_1) \in \mathbb{K}[\,\overline{x}\,, t_1] \setminus \mathbb{K}[\,\overline{x}\,]$.

3. We reason similarly as in statement 2 to show that $\Psi(\,\overline{x}\,, t_1) \in \mathbb{K}[\,\overline{x}\,, t_1] \setminus \mathbb{K}[t_1]$. □

Theorem 3. *Let $f(\,\overline{x}\,) \in \mathbb{K}[\,\overline{x}\,]$ define a translational surface $\mathcal{V}$ and let $\mathcal{P}_1(t_1)$ be a parameterization of its first auxiliary curve. Then, the second auxiliary curve satisfies that $\mathcal{C}_{\mathcal{P}_2} \subset V(\psi_0, \dots, \psi_n)$.*

In addition, for every rational space curve $\mathcal{C}_{\overline{\mathcal{P}}_2} \subset V(\psi_0, \dots, \psi_n)$, it holds that $\mathcal{P}_1(t_1) + \overline{\mathcal{P}}_2(t_2)$ parameterizes $\mathcal{V}$, where $\overline{\mathcal{P}}_2(t_2) \in \mathbb{K}(t_2)^3$ is a parameterization of $\mathcal{C}_{\overline{\mathcal{P}}_2}$. Hence, the space curve $\mathcal{C}_{\overline{\mathcal{P}}_2}$ parameterized by $\overline{\mathcal{P}}_2(t_2)$ is an auxiliary curve of $\mathcal{V}$.

Proof. If $\mathcal{V}$ admits a parameterization of the form given in Eq. (1), then $h(\mathcal{P}_2(t_2), t_1) = 0$. Thus, $\Psi(\mathcal{P}_2(t_2), t_1) = 0$. Indeed, let us assume that $\Psi(\mathcal{P}_2(t_2), t_1) \neq 0$, and then $\widetilde{h}(\mathcal{P}_2(t_2)) = 0$. Let us prove that this is impossible. First, we consider $t_1^0 \in \mathbb{K}$ such that

$p(t_1)$ and $\mathcal{P}_1(t_1)$ are defined at $t_1 = t_1^0$, and $\deg_{\overline{x}}(f(\mathcal{P}_1(t_1)+\overline{x})) = \deg_{\overline{x}}(f(\mathcal{P}_1(t_1^0)+\overline{x}))$. Now, considering that

$$f(\mathcal{P}_1(t_1) + \overline{x}) = \widetilde{h}(\overline{x})\Psi(\overline{x}, t_1)p(t_1), \qquad \text{(I)}$$

we apply the change of variable $\overline{x} \to \overline{x} - \mathcal{P}_1(t_1^0)$, and we obtain

$$f(\mathcal{P}_1(t_1) + \overline{x} - \mathcal{P}_1(t_1^0)) = \widetilde{h}(\overline{x} - \mathcal{P}_1(t_1^0))\Psi(\overline{x} - \mathcal{P}_1(t_1^0), t_1)p(t_1).$$

Then, for $t_1 = t_1^0$, we find that

$$f(\overline{x}) = \widetilde{h}(\overline{x} - \mathcal{P}_1(t_1^0))\Psi(\overline{x} - \mathcal{P}_1(t_1^0), t_1^0)p(t_1^0).$$

Since $f(\overline{x})$ is irreducible and $\widetilde{h}(\overline{x})$ is not a constant (note that $\widetilde{h}(\mathcal{P}_2(t_2)) = 0$, and thus if $\widetilde{h}(\overline{x}) = c \in \mathbb{K}$, then $c = 0$; hence, $\widetilde{h}(\overline{x}) = 0$, which is impossible by statement 1 in Lemma 3), we get that $\Psi(\overline{x} - \mathcal{P}_1(t_1^0), t_1^0)p(t_1^0) = \alpha \in \mathbb{K} \setminus \{0\}$, and thus $f(\mathcal{P}_1(t_1) + \overline{x} - \mathcal{P}_1(t_1^0)) = \alpha\widetilde{h}(\overline{x} - \mathcal{P}_1(t_1^0))$, which is equivalent to

$$f(\overline{x} + \mathcal{P}(t_1^0)) = \alpha\widetilde{h}(\overline{x})$$

(we apply the change of variable $\overline{x} \to \overline{x} + \mathcal{P}_1(t_1^0)$). By substituting in (I), we obtain

$$f(\mathcal{P}_1(t_1) + \overline{x}) = 1/\alpha\, f(\mathcal{P}_1(t_1^0) + \overline{x})\Psi(\overline{x}, t_1)p(t_1), \qquad \text{(II)}.$$

Since $\deg_{\overline{x}}(f(\mathcal{P}_1(t_1)+\overline{x})) = \deg_{\overline{x}}(f(\mathcal{P}_1(t_1^0)+\overline{x}))$, we get that $\Psi(\overline{x}, t_1)p(t_1) = c(t_1) \in \mathbb{K}(t_1)$ and $c(t_1^0) = \alpha$ (note that $\Psi(\overline{x} - \mathcal{P}_1(t_1^0), t_1^0)p(t_1^0) = \alpha$). Deriving w.r.t. t_1 in (II), we obtain

$$\nabla f(\mathcal{P}_1(t_1) + \overline{x}) \cdot \mathcal{P}_1'(t_1) = 1/\alpha\, f(\mathcal{P}_1(t_1^0) + \overline{x})c'(t_1).$$

However, $\deg_{\overline{x}}(\nabla f(\mathcal{P}_1(t_1) + \overline{x})) < \deg_{\overline{x}}(f(\mathcal{P}_1(t_1) + \overline{x})) = \deg_{\overline{x}}(f(\mathcal{P}_1(t_1^0) + \overline{x}))$, so $c'(t_1) = 0$, which implies that $c(t_1) = c(t_1^0) = \alpha \in \mathbb{K} \setminus \{0\}$,

$$f(\mathcal{P}_1(t_1) + \overline{x}) = 1/\alpha\, f(\mathcal{P}_1(t_1^0) + \overline{x})\Psi(\overline{x}, t_1)p(t_1) = f(\mathcal{P}_1(t_1^0) + \overline{x}).$$

Thus, from the above equalities, we get that

$$\nabla f(\mathcal{P}_1(t_1^0) + \overline{x}) \cdot \mathcal{P}_1'(t_1) = \nabla f(\mathcal{P}_1(t_1) + \overline{x}) \cdot \mathcal{P}_1'(t_1) = 0,$$

and by applying the change of variable $\overline{x} \to \overline{x} - \mathcal{P}_1(t_1^0)$, we have that $\nabla f(\overline{x}) \cdot \mathcal{P}_1'(t_1) = 0$, which can only occur if $\mathcal{V}$ is a cylindrical surface. This contradicts our assumption, and thus we conclude that $\Psi(\mathcal{P}_2(t_2), t_1) = 0$.

Therefore, since $\Psi(\mathcal{P}_2(t_2), t_1) = 0$ and $\mathcal{P}_2(t_2) \in \mathbb{K}(t_2)^3 \setminus \mathbb{K}^3$ does not depend on t_1, we find that $\psi_i(\mathcal{P}_2) = 0$, $0 \le i \le n$, which implies that $\mathcal{C}_{\mathcal{P}_2} \subset V(\psi_0, \ldots, \psi_n)$.

Finally, we note that for every rational space curve $\mathcal{C}_{\overline{\mathcal{P}}_2} \subset V(\psi_0, \ldots, \psi_n)$ and $\overline{\mathcal{P}}_2(t_2) \in \mathbb{K}(t_2)^3$ a parameterization of $\mathcal{C}_{\overline{\mathcal{P}}_2}$, it holds that $\mathcal{P}_1(t_1) + \overline{\mathcal{P}}_2(t_2)$ parameterizes $\mathcal{V}$ since $f(\mathcal{P}_1(t_1) + \overline{\mathcal{P}}_2(t_2)) = \widetilde{h}(\overline{\mathcal{P}}_2(t_2))\Psi(\overline{\mathcal{P}}_2(t_2), t_1)p(t_1) = 0$. Hence, the space curve $\mathcal{C}_{\overline{\mathcal{P}}_2}$ parameterized by $\overline{\mathcal{P}}_2(t_2)$ is an auxiliary curve of $\mathcal{V}$.

□

Remark 5. *If $\mathcal{V}$ is a translational surface, any rational space curve $\mathcal{C}_{\overline{\mathcal{P}}_2} \subset V(\psi_0, \ldots, \psi_n)$ can be considered for the second auxiliary curve. Thus, in the following, we refer to this curve as $\mathcal{C}_{\mathcal{P}_2}$, and we let $\mathcal{P}_2(t_2) \in \mathbb{K}(t_2)^3$ be a parameterization of $\mathcal{C}_{\mathcal{P}_2}$.*

In the following theorem, we show that the computation of $\mathcal{C}_{\mathcal{P}_2}$ can be improved in the sense that we do not need to explicitly compute the variety $V(\psi_0, \ldots, \psi_n)$, but instead we compute a simpler one that generates the same space curves as $V(\psi_0, \ldots, \psi_n)$. In particular, we prove that for *"almost all"* pairs of values $s_1, s_2 \in \mathbb{K}$, it holds that any rational space curve in $V(\psi_0, \ldots, \psi_n)$ can be defined by the polynomials $\Psi(\overline{x}, s_i) \in \mathbb{K}[\overline{x}]$, $i = 1, 2$.

Theorem 4. *Let $\mathcal{V}$ be a translational surface defined parametrically by $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$, and implicitly by $f(\overline{x}) \in \mathbb{K}[\overline{x}]$. Let $V(\psi_0, \ldots, \psi_n)$ be the variety defined by the polynomials $\psi_0, \ldots, \psi_n$, where*

$$\Psi(\overline{x}, t_1) := \psi_0(\overline{x}) + \psi_1(\overline{x})t_1 \cdots + \psi_n(\overline{x})t_1^n \in \mathbb{K}[\overline{x}, t_1], \ \gcd(\psi_0, \ldots, \psi_n) = 1,$$

and $h(\overline{x}, t_1) := f(\mathcal{P}_1(t_1) + (x_1, x_2, x_3)) = \widetilde{h}(\overline{x})\Psi(\overline{x}, t_1)p(t_1)$. Let $\mathcal{D}$ be a rational space curve such that $\mathcal{D} \subset V(\psi_0, \ldots, \psi_n)$. A nonempty Zariski open subset $\Sigma \subset \mathbb{K}^2$ exists such that for every $(s_1, s_2) \in \Sigma$, it holds that $\mathcal{D} \subset V(g_1, g_2)$, where $g_i(\overline{x}) := \Psi(\overline{x}, s_i) \in \mathbb{K}[\overline{x}]$, $i = 1, 2$.

Proof. Let $\mathcal{M}(t_2) = (m_1(t_2), m_2(t_2), m_3(t_2))$ be a proper parameterization of $\mathcal{D}$, and let $R(x_1, x_2, y_1, y_2) := \mathrm{Res}_{x_3}(\Psi(\overline{x}, y_1), \Psi(\overline{x}, y_2))$, where y_1, y_2 are new variables. By considering the construction of $\Psi(\overline{x}, t_1)$ and Lemma 3, we find that $R \neq 0$. We also note that since $\Psi(\mathcal{M}(t_2), t_1) = 0$ for every t_1, then by the properties of the resultants (e.g., see Chapter 3 in [6]), we obtain that $R(\widehat{\mathcal{M}}(t_2), y_1, y_2) = 0$ for every y_1, y_2, where $\widehat{\mathcal{M}}(t_2) := (m_1(t_2), m_2(t_2))$. Furthermore, $G(m_3(t_2), t_2, y_1, y_2) = 0$, where $G(x_3, t_2, y_1, y_2) = \gcd(\Psi(\widehat{\mathcal{M}}(t_2), x_3, y_1), \Psi(\widehat{\mathcal{M}}(t_2), x_3, y_2))$.
Under these conditions, we consider

$$\Sigma := \{(s_1, s_2) \in \mathbb{K}^2 \,|\, R(x_1, x_2, s_1, s_2)\ell(x_1, x_2, s_1)L(t_2, s_1, s_2) \neq 0\} \subset \mathbb{K}^2,$$

where

$$L(t_2, y_1, y_2) = \mathrm{Res}_{x_3}(\Psi_1^*(x_3, t_2, y_1), \Psi_2^*(x_3, t_2, y_2)),$$
$$\Psi(\widehat{\mathcal{M}}(t_2), x_3, y_j) = \Psi_j^*(x_3, t_2, y_j)G(x_3, t_2, y_1, y_2), \ j = 1, 2,$$

and $\ell \in \mathbb{K}[x_1, x_2, y_1]$ denotes the leading coefficient of $\Psi(\overline{x}, y_1)$ w.r.t. the variable x_3. Clearly, Σ is a nonempty Zariski open subset of $\mathbb{K}^2$. Now, we apply the properties of the specialization of the resultants (see Lemma 4.3.1, p. 96 in [29]) and gcds (see Lemma 7 in [16]), and we find that for $(s_1, s_2) \in \Sigma$

$$R(x_1, x_2, s_1, s_2) = \ell(x_1, x_2, s_1)^n \mathrm{Res}_{x_3}(\Psi(\overline{x}, s_1), \Psi(\overline{x}, s_2)), \ n \in \mathbb{N}, \quad \text{and}$$

$$G(x_3, t_2, s_1, s_2) = \gcd(\Psi(\widehat{\mathcal{M}}(t_2), x_3, s_1), \Psi(\widehat{\mathcal{M}}(t_2), x_3, s_2)).$$

Therefore, for $(s_1, s_2) \in \Sigma$, and since $R(\widehat{\mathcal{M}}(t_2), s_1, s_2) = G(m_3(t_2), t_2, s_1, s_2) = 0$ and $\Psi(\overline{x}, s_1), \Psi(\overline{x}, s_2)$ are linearly independent (note that $R(x_1, x_2, s_1, s_2) \neq 0$), we deduce that $\mathcal{D}$ is contained in the variety generated by $\Psi(\overline{x}, s_i) \in \mathbb{K}[\overline{x}]$, $i = 1, 2$, for $(s_1, s_2) \in \Sigma$. □

Remark 6. *Given $\mathcal{D} \subset V(\psi_0, \ldots, \psi_n)$, Theorem 4 proves that $\mathcal{D} \subset V(g_1, g_2)$, $g_i(\overline{x}) = \Psi(\overline{x}, s_i)$, and a constructive method for computing this curve is provided. However, we should note that $\Psi(\overline{x}, s_1), \Psi(\overline{x}, s_2)$ could have more than one component, and thus after a rational space curve is computed in $V(g_1, g_2)$, we should check that this curve is included in $V(\psi_0, \ldots, \psi_n)$. This can be achieved easily by checking that $\Psi(\mathcal{M}(t_2), t_1) = 0$ ($\mathcal{M}$ is a rational parameterization of the space curve computed) or by checking that $f(\mathcal{P}_1 + \mathcal{M}) = 0$ (also see Remark 7, and the algorithm in Section 4).*

The following corollary allows us to explicitly compute the polynomials $\Psi(\overline{x}, s_i)$, $i = 1, 2$ obtained in Theorem 4. For this purpose, we consider Σ, the nonempty Zariski open subset of $\mathbb{K}^2$ introduced in Theorem 4.

Corollary 1. *Let $\mathcal{V}$ be a translational surface and let $\mathcal{P}_1(t_1) = (p_{11}(t_1), p_{12}(t_1), p_{13}(t_1)) \in \mathbb{K}(t_1)^3$, $p_{1j} = \frac{p_{1j1}}{p_{1j2}}$ for $j = 1, 2, 3$, be the parameterization of the auxiliary curve $\mathcal{C}_{\mathcal{P}_1}$. Let*

$$\Gamma := \Sigma \cap \{(s_1, s_2) \in \mathbb{K}^2 \,|\, p_1(s_1)p_1(s_2) \neq 0\} \subset \mathbb{K}^2, \quad \text{where} \quad p_1 := \operatorname{lcm}(p_{112}, p_{122}, p_{132}).$$

For every $(s_1, s_2) \in \Gamma$, it holds that up to the constants in $\mathbb{K} \setminus \{0\}$,

$$\Psi(\overline{x}, s_i) = f(\mathcal{P}_1(s_i) + \overline{x})/G(\overline{x}), \;\; i = 1, 2,$$

where $G(\overline{x}) := \gcd(f(\mathcal{P}_1(s_1) + \overline{x}), f(\mathcal{P}_1(s_2) + \overline{x}))$.

Proof. First, we observe that Γ is clearly a nonempty Zariski open subset of $\mathbb{K}^2$. Now, given that $f(\mathcal{P}_1(t_1) + \overline{x}) = \widetilde{h}(\overline{x})\Psi(\overline{x}, t_1)p(t_1)$, the construction of Γ, and that $p(\alpha) \neq 0$ for every $\alpha \in \mathbb{K}$ (see statement 1 in by Lemma 3), we get that up to the constants in $\mathbb{K} \setminus \{0\}$,

$$\Psi(\overline{x}, s_i) = f(\mathcal{P}_1(s_i) + \overline{x})/G(\overline{x}), \;\; i = 1, 2,$$

where $(s_1, s_2) \in \Gamma$. □

Remark 7. *Theorem 4 and Corollary 1 work by taking (s_1, s_2) in a nonempty open subset of $\mathbb{K}^2$. However, instead, we may consider any random point $(s_1, s_2) \in \mathbb{K}^2$ and compute the irreducible rational space curve contained in the variety generated by the polynomials $g_i(\overline{x}) := \Psi(\overline{x}, s_i)$, $i = 1, 2$. The result is correct with a probability of almost one (we may test that the result is correct by checking that $f(\mathcal{P}_1 + \mathcal{P}_2) = 0$, where $\mathcal{P}_2$ is the parameterization obtained using Theorem 4 or Corollary 1; also see Remark 6). This approach is heuristic. For the deterministic approach, the open subsets introduced in Theorem 4 and Corollary 1, and the variety $V(\psi_0, \ldots, \psi_n)$ given in Theorem 3, must be computed.*

In the following example, we consider the surface $\mathcal{V}$ introduced in Examples 1 and 2, and we show how to compute the auxiliary curve $\mathcal{C}_{\mathcal{P}_2}$ by applying Corollary 1. A parameterization of the form $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$ is obtained for the input surface $\mathcal{V}$. Thus, we conclude that $\mathcal{V}$ is a translational surface.

Example 3. *Let $\mathcal{V}$ be the surface introduced in Example 1 implicitly defined by the polynomial $f(\overline{x})=x_3+5x_1^2-6x_1x_2+2x_2^2\in\mathbb{C}[\overline{x}]$. In Example 2, we obtain a space curve, $\mathcal{C}_{\mathcal{P}_1}$, and its parameterization, which is given by*

$$\mathcal{P}_1(t_1)=\left(t_1,\frac{5}{3}t_1,-\frac{5}{9}t_1^2\right)\in\mathbb{C}(t_1)^3.$$

Now, we apply Remark 7 (also see Theorems 3 and 4, Remark 6, and Corollary 1) and we consider a rational space curve $\mathcal{C}_{\mathcal{P}_2}$ that is contained in the variety generated by the polynomials $g_1(\overline{x})=f(\mathcal{P}_1(1)+\overline{x})$, and $g_2(\overline{x})=f(\mathcal{P}_1(-3)+\overline{x})$ (note that $G(\overline{x})=\gcd(f(\mathcal{P}_1(1)+\overline{x}),f(\mathcal{P}_1(-3)+\overline{x}))=1$). In this case, a rational proper parameterization of $\mathcal{C}_{\mathcal{P}_2}$ is given by

$$\mathcal{P}_2(t_2)=\left(t_2,0,-5t_2^2\right)\in\mathbb{C}(t_2)^3.$$

Finally, we obtain the parameterization

$$\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)=\left(t_1+t_2,\frac{5}{3}t_1,-\frac{5}{9}t_1^2-5t_2^2\right).$$

We may check that $f(\mathcal{P}(t_1,t_2))=0$, and thus $\mathcal{P}$ is a rational parameterization of the translational surface $\mathcal{V}$.

4. Algorithm and examples

In this section, we present an algorithm for deciding whether a given implicitly defined surface $\mathcal{V}$ is a translational surface and if this is true, then we compute a parameterization of $\mathcal{V}$ in the standard form $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$. We illustrate this algorithm with several examples.

This algorithm is derived from the results presented in Section 3. The idea involves computing a candidate for the first auxiliary curve that will be contained in the variety $V(f,g_{\overline{a}})$, where $g_{\overline{a}}(\overline{x}):=\nabla f(\overline{x})\cdot\overline{a}$, and the vector $\overline{a}$ is considered as in Remark 4 (also see Theorem 2). We denote this curve as $\mathcal{C}_{\mathcal{P}_1}$, and let $\mathcal{P}_1(t_1)\in\mathbb{K}(t_1)^3$ be its parameterization (we note that this curve exists if $\mathcal{V}$ is a translational surface). Next, we compute a candidate for the second auxiliary curve $\mathcal{C}_{\mathcal{P}_2}$ and its parameterization, $\mathcal{P}_2(t_2)\in\mathbb{K}(t_2)^3$ (see Theorem 3). At this point, the algorithm returns a parameterization of the translational surface $\mathcal{V}$ given by $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$. In Theorem 5, we prove that the algorithm is correct and that $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$ is a parameterization of $\mathcal{V}$.

Algorithm: Parameterization of a Translational Surface

- **Input:** A surface $\mathcal{V}$ defined by an irreducible polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$.
- **Output:** The message "*$\mathcal{V}$ is not a translational surface*" or *a parameterization $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ of the translational surface $\mathcal{V}$.*

1. Let $g_{\overline{a}}(\overline{x}) := \nabla f(\overline{x}) \cdot \overline{a}$. Set $\overline{a} = (1, 0, 0)$. If $V(f, g_{\overline{a}})$ does not contain an irreducible rational space curve, then set $\overline{a} = (0, 1, 0)$ or $\overline{a} = (0, 0, 1)$.
If $V(f, g_{\overline{a}})$ does not contain an irreducible rational space curve for these three vectors, then set $\overline{a} = (1, \lambda, 0)$ and compute $\lambda \in \mathbb{K}$ such that $V(f, g_{\overline{a}})$ contains an irreducible rational space curve (see statement 2 in Remark 2).
If such a $\lambda \in \mathbb{K}$ does not exist, then RETURN "*$\mathcal{V}$ is not a translational surface*" else let $\mathcal{C}_{\mathcal{P}_1}$ be this space curve and $\mathcal{P}_1(t_1) \in \mathbb{K}(t_1)^3$ a proper parameterization of $\mathcal{C}_{\mathcal{P}_1}$ (see Theorems 1 and 2, and Remark 4).
2. Consider a rational space curve $\mathcal{C}_{\mathcal{P}_2} \subset V(\psi_0, \ldots, \psi_n)$ and let $\mathcal{P}_2(t_2) \in \mathbb{K}(t_2)^3$ be a parameterization of $\mathcal{C}_{\mathcal{P}_2}$ (see Theorems 3 and 4, Corollary 1, and Remarks 6 and 7).
If such a curve does not exist, then RETURN "*$\mathcal{V}$ is not a translational surface*" else RETURN $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ "*is a parameterization of the translational surface $\mathcal{V}$*" (see Theorem 5).

In the following, we prove that the proposed algorithm is correct. In particular, after computing (in Step 1 of the algorithm) a candidate for the first auxiliary curve ($\mathcal{C}_{\mathcal{P}_1}$) and its parameterization ($\mathcal{P}_1$), we show that $\mathcal{V}$ is a translational surface if and only if we can find a candidate for the second auxiliary curve (see Step 2 of the algorithm). Note that if a candidate is not found in Step 1, $\mathcal{V}$ is not a translational surface (see Theorem 1 or Theorem 2).

Theorem 5. *Let $\mathcal{V}$ be a surface defined by an irreducible polynomial $f(\overline{x}) \in \mathbb{K}[\overline{x}]$. The following statements hold.*

1. *Let $\mathcal{C}_{\mathcal{P}_1}$ be the rational space curve computed in Step 1 of the algorithm* Parameterization of a Translational Surface, *and let $\mathcal{P}_1(t_1)$ be a parameterization of $\mathcal{C}_{\mathcal{P}_1}$. If a rational space curve $\mathcal{C}_2$ parameterized by $\mathcal{P}_2(t_2)$ exists such that $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ parameterizes $\mathcal{V}$, then $\mathcal{V}$ is a translational surface.*
2. *If $\mathcal{V}$ is a translational surface, then Steps 1 and 2 of the algorithm* Parameterization of a Translational Surface *provide two auxiliary space curves and its parameterizations, $\mathcal{P}_1, \mathcal{P}_2$, such that $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ parameterizes $\mathcal{V}$.*

Proof.

1. According to the definition, $\mathcal{V}$ is a translational surface if a parameterization $\mathcal{P}_2(t_2) \in \mathbb{K}(t_2)^3$ exists such that $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ parameterizes $\mathcal{V}$.
2. Conversely, for a given translational surface $\mathcal{V}$ defined implicitly by $f(\overline{x})$, we can find the first auxiliary curve by Theorem 1 (or Theorem 2) and the second auxiliary curve by Theorem 3. Let $\mathcal{P}_i(t_i) \in \mathbb{K}(t_i)^3$ be parameterizations of the two auxiliary rational space curves $\mathcal{C}_{\mathcal{P}_i}$, $i = 1, 2$, output by Steps 1 and 2 of the algorithm Parameterization of a Translational Surface. Then, since $\mathcal{C}_{\mathcal{P}_2} \subset V(\psi_0, \ldots, \psi_n)$ (see Theorem 3), we find that $\psi_i(\mathcal{P}_2) = 0$, $i \in \{0, \ldots, n\}$. Therefore, $h(\mathcal{P}_2(t_2), t_1) = 0$, and hence $f(\mathcal{P}) = 0$, where $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$. Furthermore, according to Lemma 3 (see statements 2 and 3), $V(\psi_0, \ldots, \psi_n)$ defines an algebraic set that is independent of t_1, which means that $\mathcal{P}(t_1, t_2)$ defines a surface. □

In the following, we illustrate the algorithm with some examples.

Example 4. *We consider the surface $\mathcal{V}$ over $\mathbb{C}$ implicitly defined by the polynomial*

$f(\overline{x}) = -9 - x_3^3x_1^2 - 3x_1x_3x_2 - 2x_1x_3^2x_2 + 4x_1^2x_3x_2 + 2x_1x_3^3x_2 + x_1^2x_2 - x_1^3x_2 - 2x_1 - 11x_3 + 12x_2 + x_3^4x_2 + 5x_1x_2 + 2x_1x_3^5 + 2x_3^5x_2 - 4x_1^2x_3^4 + 5x_1x_3^2 + 9x_3^3x_1 - 4x_1^2x_3 + 2x_3^3x_2 - 2x_1x_2^2 + 6x_3x_1 + 4x_3x_2 - 3x_1^2x_3^2 - 2x_3^2x_2^2 + 10x_3^2x_2 - 15x_3^2 - 2x_3^5 - x_3^7 - 6x_2^2 + x_1^3 - 8x_3^4 - x_3^3 + x_2^3 + x_3^3x_1^3 - x_3^3x_2^2 - x_1^2.$

We apply Step 1 of the algorithm. For this purpose, we consider the vector $\overline{a} = (1, 0, 0)$ and get that $V(f, g_{\overline{a}})$, where

$g_{\overline{a}}(\overline{x}) := f_{x_1}(\overline{x}) = -11 + 3x_3^3x_1^2 - 4x_1x_3x_2 + 6x_1^2x_3x_2 + 6x_1^2x_2 + 4x_1^3x_2 - 30x_1 + 6x_3 + 4x_2 + 20x_1x_2 - 6x_1x_3^2 + 27x_1^2x_3 - 4x_1x_2^2 + 10x_3x_1 - 3x_3x_2 - 3x_1^2x_3^2 + 4x_3^2x_2 - 4x_3^2 - 32x_1^3 - 3x_1^2 - 7x_1^6 - 16x_3^2x_1^3 - 10x_1^4 - 3x_1^2x_2^2 + 10x_1^4x_2 + 10x_3x_1^4,$

contains a rational space curve $\mathcal{C}_{\mathcal{P}_1}$. We compute a proper rational parametrization of $\mathcal{C}_{\mathcal{P}_1}$. We get

$$\mathcal{P}_1(t_1) = \left(t_1, 1 + t_1^2, \frac{1}{t_1}\right) \in \mathbb{C}(t_1)^3$$

Now, we apply Step 2 of the algorithm and look for a rational space curve $\mathcal{C}_{\mathcal{P}_2} \subset V(\psi_0, \ldots, \psi_n)$. Instead applying Theorem 3, we may use Theorem 4 and Corollary 1 since the computation of auxiliary curves is improved. However, one has to take into account that the result in this case is probabilistic and thus, one has to check that $f(\mathcal{P}_1 + \mathcal{P}_2) = 0$ (see Remarks 6 and 7).

Hence, we obtain an irreducible rational space curve contained in the variety generated by the polynomials $g_1(\overline{x}) = f(\mathcal{P}_1(1) + \overline{x})$ and $g_2(\overline{x}) = f(\mathcal{P}_1(-3) + \overline{x})$ (note that $G(\overline{x}) = \gcd(f(\mathcal{P}_1(1) + \overline{x}), f(\mathcal{P}_1(-3) + \overline{x})) = 1$). Let $\mathcal{C}_{\mathcal{P}_2}$ denote this space curve, and let

$$\mathcal{P}_2(t_2) = \left(t_2, t_2^3, t_2^2\right) \in \mathbb{C}(t_2)^3$$

be a rational parametrization of $\mathcal{C}_{\mathcal{P}_2}$*. Finally, one checks that* $f(\mathcal{P}_1+\mathcal{P}_2)=0$ *(see Remarks 6 and 7), and then we conclude that* $\mathcal{V}$ *is a translational surface and*

$$\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)=\left(t_1+t_2, 1+t_1^2+t_2^3, \frac{1}{t_1}+t_2^2\right)\in\mathbb{C}(t_1,t_2)^3$$

is a rational parametrization of $\mathcal{V}$*.*

Example 5 illustrates statement 1 in Remark 2. More precisely, Theorem 1 provides a necessary (but not sufficient) condition for the existence of the first auxiliary space curve $\mathcal{C}_{\mathcal{P}_1}$. This theorem is proved by taking $\overline{a}:=\mathcal{P}_2'(t_2^0)$, for $t_2^0\in\Omega_{\mathcal{P}_2}$. However, there may exist suitable vectors $\overline{a}$ not only given by $\mathcal{P}_2'(t_2^0)$ that provide more (and different) auxiliary curves that solve the problem we are dealing with.

In Example 5, we also illustrate the situation presented in the proof of Theorem 2. That is, the parametrization of the surface $\mathcal{W}_{\mathcal{P}_2}$ given by $\mathcal{Q}(t_1,t_2)=t_1\mathcal{P}_2(t_2)$ could satisfy that there does not exist $(t_1^0,t_2^0)\in\mathbb{K}^2$ such that $\mathcal{Q}(t_1^0,t_2^0)=\overline{a}$, where $\overline{a}$ is the vector considered.

Example 5. *Let* $\mathcal{V}$ *be the surface over* $\mathbb{C}$ *implicitly defined by the polynomial*

$$f(\,\overline{x}\,)=x_1^4-2\,x_3+7\,x_3x_1+2\,x_2^2-5\,x_2x_3+x_3^2+2\,x_1^3-10\,x_1^2x_2-2\,x_3x_1^2+7\,x_1x_2^2-x_2^3.$$

We apply Step 1 of the algorithm. For this purpose, we consider the vector $\overline{a}=(1,0,0)$*, and we get that* $V(f,g_{\overline{a}})$*, where* $g_{\overline{a}}(\,\overline{x}\,):=f_{x_1}(\,\overline{x}\,)$*, contains a rational space curve* $\mathcal{C}_{\mathcal{P}_1}$*. We compute a proper rational parametrization of* $\mathcal{C}_{\mathcal{P}_1}$*. We obtain*

$$\mathcal{P}_1(t_1)=(t_1,t_1,t_1^2)\in\mathbb{C}(t_1)^3. \quad (2)$$

Now, we apply Step 2 of the algorithm, and we look for a rational space curve $\mathcal{C}_{\mathcal{P}_2}\subset V(\psi_0,\ldots,\psi_n)$*. We reason as in Example 4, and we determine* $\mathcal{C}_{\mathcal{P}_2}$ *and a proper parametrization of it. We get*

$$\mathcal{P}_2(t_2)=(t_2,t_2^2,t_2^3)\in\mathbb{C}(t_2)^3. \quad (3)$$

Since $f(\mathcal{P}_1+\mathcal{P}_2)=0$ *(see Remarks 6 and 7), we conclude that* $\mathcal{P}(t_1,t_2)=\mathcal{P}_1(t_1)+\mathcal{P}_2(t_2)$ *is a parametrization of* $\mathcal{V}$*. Observe that* $\mathcal{P}_2(0)=(0,0,0)$ *and* $\mathcal{P}_2'(0)=(1,0,0)$ *(see Lemma 1).*
In addition, we also note that the parametrization of the surface $\mathcal{W}_{\mathcal{P}_2}$ *given by* $\mathcal{Q}(t_1,t_2)=t_1\mathcal{P}_2(t_2)=(t_1,2t_1t_2,3t_1t_2^2)$ *satisfies that* $\mathcal{Q}(1,0)=(1,0,0)\in\mathcal{W}_{\mathcal{P}_2}$ *(see statement* 1 *in the proof of Theorem 2).*

If we apply Step 1 of the algorithm with the vector $\overline{a}=(0,0,1)$ *(see Remark 4) and we reason as above, we get that the irreducible rational space curve* $\mathcal{C}_{\mathcal{P}_1}$ *is contained in* *the variety generated by the polynomials* $f(\overline{x})$ *and* $g_{\overline{a}}(\overline{x}) := f_{x_3}(\overline{x})$*. Applying Steps 1 and 2 as in the previous examples, we get the proper parametrizations*

$$\mathcal{P}_1(t_1) = \left(t_1, t_1 - \frac{1}{4}, \frac{3}{8} - t_1 + t_1^2\right) \in \mathbb{C}(t_1)^3,$$

$$\mathcal{P}_2(t_2) = \left(t_2, t_2 + t_2^2, \frac{3}{4}\,t_2 + \frac{3}{2}\,t_2^2 + t_2^3\right) \in \mathbb{C}(t_2)^3,$$

and that $\mathcal{V}$ *is a translational surface parametrized by* $\mathcal{P}(t_1,t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$*. In this case, we note that there does not exist* $t_2^0 \in \mathbb{K}$ *such that* $\mathcal{P}_2'(t_2^0) = (0,0,1)$ *(see statement 1 in Remark 2). However, we may consider different auxiliary curves by considering the parametrizations*

$$\overline{\mathcal{P}}_1(t_1) = \mathcal{P}_1(t_1) + \mathcal{P}_2(-1/2) = \left(t_1 - \frac{1}{2}, (t_1 - \frac{1}{2}), (t_1 - \frac{1}{2})^2\right) \in \mathbb{C}(t_1)^3, \qquad (4)$$

and

$$\overline{\mathcal{P}}_2(t_2) = \mathcal{P}_2(t_2) - \mathcal{P}_2(-1/2) = \left(t_2 + \frac{1}{2}, (t_2 + \frac{1}{2})^2, (t_2 + \frac{1}{2})^3\right) \in \mathbb{C}(t_2)^3 \qquad (5)$$

(see Section 2). Now $\overline{\mathcal{P}}_1(t_1) + \overline{\mathcal{P}}_2(t_2)$ *is a parametrization of* $\mathcal{V}$ *(note that* $\overline{\mathcal{P}}_1(t_1) + \overline{\mathcal{P}}_2(t_2) = \mathcal{P}(t_1,t_2)$*), and* $\overline{\mathcal{P}}_2(-\frac{1}{2}) = (0,0,0)$ *and* $\overline{\mathcal{P}}_2'(-\frac{1}{2}) = (1,0,0)$*, which support Lemma 1 and Theorem 1. In fact, one notes that the parametrizations given in Eq. (2) and Eq. (4) define the same space curve* $\mathcal{C}_{\mathcal{P}_1}$*. Similarly, parametrizations in Eq. (3) and Eq. (5) are proper parametrizations of the same space curve* $\mathcal{C}_{\mathcal{P}_2}$ *(see Section 2).*

Finally, we note that using these new parametrizations, the rational parametrization of the surface $\mathcal{W}_{\overline{\mathcal{P}}_2}$ *given by* $\mathcal{Q}(t_1,t_2) = t_1\overline{\mathcal{P}}_2(t_2) = (t_1, t_1(1+2\,t_2), t_1(\frac{3}{4} + 3\,t_2 + 3\,t_2^2))$ *satisfies that there does not exist* $(t_1^0,t_2^0) \in \mathbb{K}^2$ *such that* $\mathcal{Q}(t_1^0,t_2^0) = (0,0,1)$ *(however the surface* $\mathcal{W}_{\overline{\mathcal{P}}_2}$ *is defined by the polynomial* $4x_1x_3 - 3x_2$ *and* $(0,0,1) \in \mathcal{W}_{\overline{\mathcal{P}}_2}$*; see statement* 2 *in the proof of Theorem 2).*

In the following example, we show how Theorem 2 provides a necessary (but not sufficient) condition for the existence of the first auxiliary space curve $\mathcal{C}_{\mathcal{P}_1}$. That is, Example 6 shows that there may exist suitable vectors not lying on the surface $\mathcal{W}_{\mathcal{P}_2}$ that allow to construct auxiliary curves.

Example 6. *Let* $\mathcal{V}$ *be the surface introduced in Example 5 implicitly defined by the polynomial*

$$f(\overline{x}) = x_1^4 - 2\,x_3 + 7\,x_3x_1 + 2\,x_2^2 - 5\,x_2x_3 + x_3^2 + 2\,x_1^3 - 10\,x_1^2x_2 - 2\,x_3x_1^2 + 7\,x_1x_2^2 - x_2^3.$$

We consider an irreducible rational space curve contained in $V(f, g_{\overline{a}})$*, where* $g_{\overline{a}}(\overline{x}) = \nabla f(\overline{x}) \cdot \overline{a}$*, and* $\overline{a} = (1,1,1)$*. A parametrization of this space curve,* $\mathcal{C}_{\mathcal{P}_1}$*, is given by*

$$\mathcal{P}_1(t_1) = \left(t_1, t_1 - \frac{1}{4}, \frac{3}{8} - t_1 + t_1^2\right) \in \mathbb{C}(t_1)^3.$$

Reasoning as in the previous examples, we get a proper parametrization of $\mathcal{C}_{\mathcal{P}_2}$ given by

$$\mathcal{P}_2(t_2) = \left(t_2, t_2 + t_2^2, \frac{3}{4}\,t_2 + \frac{3}{2}\,t_2^2 + t_2^3\right) \in \mathbb{C}(t_2)^3.$$

One can check that $\mathcal{P}(t_1, t_2) = \mathcal{P}_1(t_1) + \mathcal{P}_2(t_2)$ is a parametrization of the surface $\mathcal{V}$ (see Remarks 6 and 7). In this case, the parametrization of the surface $\mathcal{W}_{\mathcal{P}_2}$ is $\mathcal{Q}(t_1, t_2) = t_1\mathcal{P}_2(t_2) = (t_1, t_1(1 + 2\,t_2), t_1(\frac{3}{4} + 3\,t_2 + 3\,t_2^2))$, and $\mathcal{W}_{\mathcal{P}_2}$ is implicitly defined by the polynomial $4x_1x_3 - 3x_2$. However, $(1, 1, 1) \notin \mathcal{W}_{\mathcal{P}_2}$.

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