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Marshall Olkin exponential Gompertz distribution: Properties and applications

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ABSTRACT

Generalizing distribution is an important area in probability theory. Many distributions are not suitable for modeling data, that are either symmetric or heavily skewed. In this paper, a new compound distribution termed as Marshall Olkin Exponential Gompertz (MOEGo) is introduced. Several essential statistical properties of MOEGo distribution were studied and investigated. The estimation of distribution parameters was performed using the maximum likelihood estimation method. Two real data (symmetric and right-skewed) were adopted to illustrate the flexibility of MOEGo distribution. This flexibility enables the use of MOEGo distribution in various application areas.

Keywords: Marshall Olkin distribution, Gompertz distribution, Exponential distribution, Statistical properties, Maximum likelihood estimation method

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1. Introduction

In probability theory, the generalization of distribution is an important area. Most distributions are not appropriate for data modeling, whether symmetrical or strongly skewed. In the last few years or so, several authors have been investigating the addition of new parameter(s) to the expansion of traditional well-known distributions to achieve more flexibility. In many areas of application "such as lifetime research, including, medical sciences, biological, engineering, physics, economics, environmental, finance, and insurance" there is a strong need for extended types of traditional distributions, i.e. new distributions that are more flexibility for modeling real data in these fields since the data can be highly skewed. In this area, among others, the authors of [1] have suggested an interesting technique for adding a new parameter to a baseline distribution that yields a family of probability distributions. Many distributions have been proposed based on this family, such as Marshall–Olkin exponential Weibull distribution introduced by [2] that provided a better fit than several other related models like gamma, Weibull, exponential–Weibull and Kumaraswamy modified Weibull distributions. Marshall-Olkin Kumaraswamy distribution was introduced and studied in [3] with some mathematical properties. Marshall-Olkin length-biased exponential distribution was proposed and discussed in [4] with some of its statistical properties in addition to estimating their parameters by maximum likelihood estimation method. Based on the Marshall-Olkin family, this paper introduced a new proposed family that holds for any baseline distribution besides a new distribution named Marshall Olkin Exponential Gompertz.

The rest of this paper is structured as follows: In Section 2, Marshall-Olkin family of probability distributions is discussed. In Section 3, Exponential-G family of probability distributions is discussed. In Section 4, a new family based on composing Marshall-Olkin family with an Exponential-G family named Marshall-Olkin Exponential-G family is introduced. Furthermore, a new continuous probability distribution named Marshall-Olkin Exponential-Gompertz along with its statistical properties and maximum likelihood estimation method of their parameters are introduced. In Section 5, two real data set are adopted to illustrate the behavior of new distribution. Finally, conclusions are addressed in Section 6.

2. Marshall-Olkin (MO) family of probability distributions

For a given baseline distribution with cdf and pdf, the researchers of [1] suggested a new flexible family of probability distributions based on defined a new survival function through presenting an additional parameter ($\alpha > 0$) known as the tilt parameter. The survival function of the Marshall-Olkin (MO) family is defined by [5] as

$$\bar{F}(x)_{MO} = \frac{\alpha\, \bar{H}(x)}{1 - \bar{\alpha}\, \bar{H}(x)}\ ;\ \ \bar{\alpha} = 1 - \alpha\,, \bar{H}(x) = 1 - H(x) \tag{1}$$

Consequently, the cdf and pdf of MO family are defined by [5] as

$$F(x)_{MO} = 1 - \bar{F}(x) = \frac{H(x)}{1 - \bar{\alpha}\, \bar{H}(x)} \tag{2}$$

$$f(x)_{MO} = \frac{d}{dx}\,[F(x)] = \frac{\alpha\, h(x)}{\left(1 - \bar{\alpha}\, \bar{H}(x)\right)^2} \tag{3}$$

Since $1 - \bar{\alpha}\, \bar{H}(x) = \alpha + \bar{\alpha}\, H(x)$, the cdf and pdf of MO family in (2) and (3) can be re-written as

$$F(x)_{MO} = \frac{H(x)}{\alpha + \bar{\alpha}\, H(x)} \tag{4}$$

$$f(x)_{MO} = \frac{\alpha\, h(x)}{\left(\alpha + \bar{\alpha}\, H(x)\right)^2} \tag{5}$$

For more details about Marshall-Olkin family see [6], [7], and [8].

3. Exponential-G (E-G) family of probability distributions

For a given baseline distribution with cdf, G(x), and pdf, g(x), [6] defined a family of probability distributions based on exponential distribution named Exponential-G (E-G) family. The E-G family providing a rather large and flexible structure for statistical analysis. It also offers a relatively flexible framework to suit a wide range of data sets in the real world.

The cdf and pdf of E-G family are defined for $x > 0$ and $\lambda \geq 0$ by [6] as

$$H(x)_{E-G} = \frac{1}{1 - e^{-\lambda}}\left(1 - e^{-\lambda G(x)}\right) \tag{6}$$

$$h(x)_{E-G} = \frac{\lambda}{1 - e^{-\lambda}}\ g(x)\ e^{-\lambda G(x)} \tag{7}$$

4. Marshall-Olkin Exponential-G (MOE-G) family

In this section, based on composing the MO family with an E-G family, a new family of probability distributions that hold for any baseline distribution is introduced.
Inserting (6) and (7) in (4) and (5), the cdf and associated pdf for a new family of probability distributions named Marshall-Olkin Exponential-G (symbolized by $\mathrm{MOE-G}$) family are given by

$$F(x)_{\text{MOE-G}} = \frac{1 - e^{-\lambda G(x)}}{(1 - e^{-\lambda})\left(\alpha + \bar{\alpha}\ \frac{1 - e^{-\lambda G(x)}}{1 - e^{-\lambda}}\right)} \tag{8}$$

$$f(x)_{\text{MOE-G}} = \frac{\alpha\lambda g(x)\ e^{-\lambda G(x)}}{(1 - e^{-\lambda})\left(\alpha + \bar{\alpha}\ \frac{1 - e^{-\lambda G(x)}}{1 - e^{-\lambda}}\right)^2} \tag{9}$$

and then (8) and (9) can be re-written as

$$F(x)_{\text{MOE-G}} = \frac{1 - e^{-\lambda G(x)}}{\alpha(1 - e^{-\lambda}) + \bar{\alpha}\ (1 - e^{-\lambda G(x)})} \tag{10}$$

$$f(x)_{\text{MOE-G}} = \frac{\alpha\lambda\left(1 - e^{-\lambda}\right) g(x)\ e^{-\lambda G(x)}}{\left(\alpha(1 - e^{-\lambda}) + \bar{\alpha}\ (1 - e^{-\lambda G(x)})\right)^2} \tag{11}$$

Also, the general expanded formula to $f(x)_{\text{MOE-G}}$ in (11) can be obtained as follows

$$f^E(x)_{\text{MOE-G}} = \alpha\lambda\left(1 - e^{-\lambda}\right)g(x)e^{-\lambda G(x)}\left(\alpha\left(1 - e^{-\lambda}\right) + \bar{\alpha}\left(1 - e^{-\lambda G(x)}\right)\right)^{-2}$$

Now

$$\left(\alpha\left(1 - e^{-\lambda}\right) + \bar{\alpha}\left(1 - e^{-\lambda G(x)}\right)\right)^{-2} = \left(1 - \frac{-\bar{\alpha}\left(1 - e^{-\lambda G(x)}\right)}{\alpha(1 - e^{-\lambda})}\right)^{-2}$$

Using $(1 - z)^{-b} = \Sigma_{i=0}^{\infty} \frac{\Gamma(b+i)}{i!\,\Gamma(b)}\ z^i;\ |z| < 1, b > 0$, we get,

$$\left(\alpha\left(1 - e^{-\lambda}\right) + \bar{\alpha}\left(1 - e^{-\lambda G(x)}\right)\right)^{-2} = \Sigma_{i=0}^{\infty}\ \frac{\Gamma(2+i)}{i!\ \Gamma(2)}\left(\frac{-\bar{\alpha}\left(1 - e^{-\lambda G(x)}\right)}{\alpha(1 - e^{-\lambda})}\right)^i$$

$$= \Sigma_{i=0}^{\infty}\ \frac{(-1)^i\Gamma(2+i)}{i!}\left(\frac{\bar{\alpha}}{\alpha(1 - e^{-\lambda})}\right)^i\left(1 - e^{-\lambda G(x)}\right)^i$$

Using $(1 - z)^b = \Sigma_{j=0}^{\infty}\ (-1)^j\ C_j^b\ z^j\ ;\ |z| < 1\ , b > 0$ we get

$$\left(1 - e^{-\lambda G(x)}\right)^i = \Sigma_{j=0}^{\infty}\ (-1)^j\ C_j^i\ \left(e^{-\lambda G(x)}\right)^j = \Sigma_{j=0}^{\infty}\ (-1)^j\ C_j^i\ e^{-\lambda j G(x)}$$

Then

$$\left(\alpha\left(1 - e^{-\lambda}\right) + \bar{\alpha}\left(1 - e^{-\lambda G(x)}\right)\right)^{-2} = \Sigma_{i=0}^{\infty}\ \frac{(-1)^i\Gamma(2+i)}{i!}\left(\frac{\bar{\alpha}}{\alpha(1 - e^{-\lambda})}\right)^i \Sigma_{j=0}^{\infty}\ (-1)^j\ C_j^i\ e^{-\lambda j G(x)}$$

$$= \Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}\ \frac{(-1)^{i+j}\Gamma(2+i)}{i!} C_j^i \left(\frac{\bar{\alpha}}{\alpha(1 - e^{-\lambda})}\right)^i e^{-\lambda j G(x)}$$

Then

$$f^E(x)_{\text{MOE-G}} = \alpha\lambda\left(1 - e^{-\lambda}\right)g(x)e^{-\lambda G(x)}\Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}\ \frac{(-1)^{i+j}\Gamma(2+i)}{i!} C_j^i \left(\frac{\bar{\alpha}}{\alpha(1 - e^{-\lambda})}\right)^i e^{-\lambda j G(x)}$$

$$= \lambda g(x) \Sigma_{i=0}^{\infty} \Sigma_{j=0}^{\infty} \frac{(-1)^{i+j}\Gamma(2+i)}{i!} C_j^i \frac{(\bar{\alpha})^i}{\left(\alpha(1-e^{-\lambda})\right)^{i-1}} e^{-\lambda G(x)(1+j)}$$

$$= \lambda g(x) \Sigma_{i=0}^{\infty} \Sigma_{j=0}^{\infty} (-1)^{i+j}(i+1) C_j^i \frac{(\bar{\alpha})^i}{\left(\alpha(1-e^{-\lambda})\right)^{i-1}} e^{-\lambda G(x)(1+j)}$$

Using $e^{-z} = \Sigma_{k=0}^{\infty} \frac{(-1)^k}{k!} z^k$ we get

$$f^E(x)_{\text{MOE-G}} = \lambda g(x) \Sigma_{i=0}^{\infty} \Sigma_{j=0}^{\infty} (-1)^{i+j}(i+1) C_j^i \frac{(\bar{\alpha})^i}{\left(\alpha(1-e^{-\lambda})\right)^{i-1}} \Sigma_{k=0}^{\infty} \frac{(-1)^k}{k!} (\lambda G(x)(1+j))^k$$

$$= \Sigma_{i=0}^{\infty} \Sigma_{j=0}^{\infty} \Sigma_{k=0}^{\infty} \frac{(-1)^{i+j+k}}{k!} (i+1)\,(j+1)^k\, C_j^i \frac{(\bar{\alpha})^i}{\left(\alpha(1-e^{-\lambda})\right)^{i-1}} \lambda^{k+1}\, g(x)\, (G(x))^k$$

Therefore, the expansion formula of pdf of MOE-G family is given by:

$$f^E(x)_{\text{MOE-G}} = S\, g(x)\, (G(x))^k \tag{12}$$

where

$$S = \Sigma_{i=0}^{\infty} \Sigma_{j=0}^{\infty} \Sigma_{k=0}^{\infty} \frac{(-1)^{i+j+k}}{k!} (i+1)\,(j+1)^k\, C_j^i \frac{(\bar{\alpha})^i\, \lambda^{k+1}}{\left(\alpha(1-e^{-\lambda})\right)^{i-1}} \tag{13}$$

4.1 Marshall-Olkin Exponential-Gompertz (MOEGo) distribution

The Gompertz (Go) distribution is a flexible distribution that can be skewed to the both sides (right and left). This distribution is an exponential distribution generalization and is widely used in many applied problems, particularly in modeling survival (reliability) and human mortality. For more details about the Go distribution and its applications, see [9] and [10]. The cdf and pdf of Go distribution are defined for $x \geq 0$ with $\theta > 0$ and $\beta > 0$ by [11] as

$$G(x)_{\text{Go}} = 1 - e^{-\frac{\theta}{\beta}(e^{\beta x}-1)} \tag{14}$$

$$g(x)_{\text{Go}} = \theta e^{\beta x} e^{-\frac{\theta}{\beta}(e^{\beta x}-1)} \tag{15}$$

Inserting (14) and (15) in (10), (11) and (12), the cdf, pdf and expansion pdf for a new continuous probability distribution named Marshall-Olkin Exponential-Gompertz (symbolized by MOEGo) distribution are given respectively by:

$$F(x)_{\text{MOEGo}} = \frac{1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}}{\alpha(1-e^{-\lambda}) + \bar{\alpha}\left(1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}\right)} \tag{16}$$

$$f(x)_{\text{MOEGo}} = \frac{\alpha\lambda\left(1-e^{-\lambda}\right) \theta e^{\beta x} e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\, e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}}{\left(\alpha(1-e^{-\lambda}) + \bar{\alpha}\left(1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}\right)\right)^2} \tag{17}$$

$$f^E(x)_{\text{MOEGo}} = S\,\theta e^{\beta x} e^{-\frac{\theta}{\beta}(e^{\beta x}-1)} \left(1 - e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)^k \tag{18}$$

where S as in (13).

Now, it is easily to obtain the survival function $\bar{F}(x)$ and hazard rate function $\tau(x) = \frac{f(x)}{\bar{F}(x)}$ as:

$$\bar{F}(x)_{\text{MOEGo}} = 1 - \frac{1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}}{\alpha(1-e^{-\lambda}) + \bar{\alpha}\left(1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}\right)} \tag{19}$$

$$\tau(x)_{\text{MOEGo}} = \frac{\dfrac{\alpha\lambda(1-e^{-\lambda})\,\theta e^{\beta x} e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\, e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}}{\alpha(1-e^{-\lambda}) + \bar{\alpha}\left(1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}\right)}}{\left[\alpha(1-e^{-\lambda}) + \bar{\alpha}\left(1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}\right)\right] - \left(1 - e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)}\right)} \tag{20}$$

Plots of the cdf and the pdf are presented in Figures 1 and 2 for certain values of the four parameters where Figure 2 illustrates the flexibility of new MOEGo distribution.

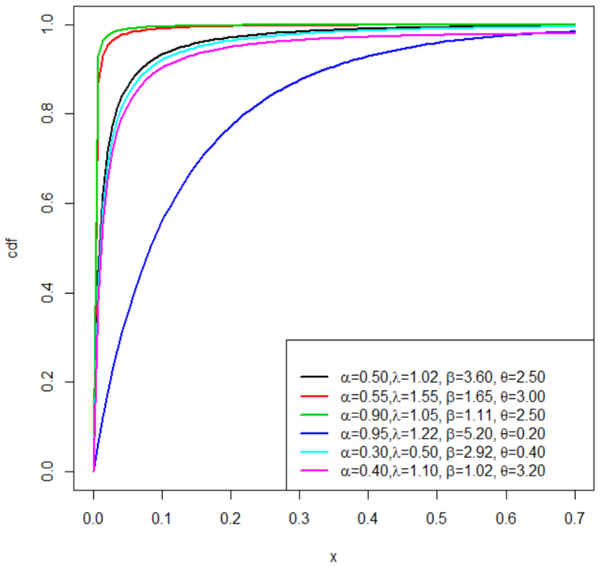

Figure 1. Plot of cdf of MOEGo for certain values of the parameters

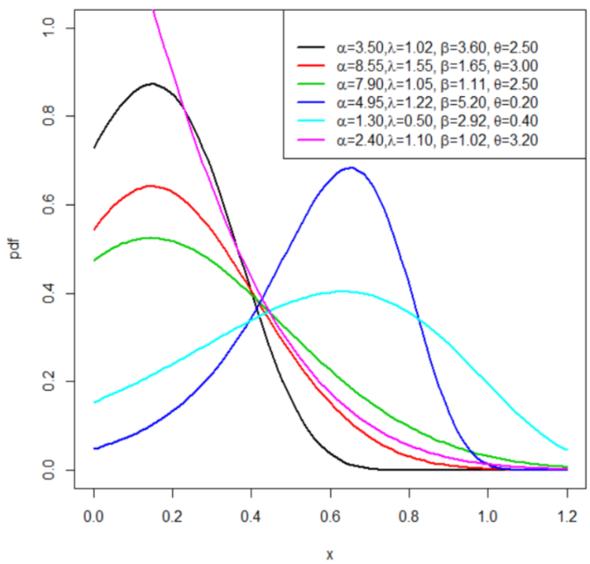

Figure 2. Plot of pdf of MOEGo for certain values of the parameters

4.2 Statistical properties of MOEGo distribution

Here, we present the most essential statistical properties of MOEGo distribution.

r-th moment: The r-th moment of MOEGo distribution can be obtained from $\int_0^\infty x^r f(x)_{\text{MOEGo}}\, dx$. Based on (18), the r-th moment can be obtained as follows

$$E(X^r)_{\text{MOEGo}} = \int_0^\infty x^r S\,\theta\, e^{\beta x} e^{-\frac{\theta}{\beta}(e^{\beta x}-1)} \left(1 - e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)^k \; dx$$

$$= \frac{S}{(k+1)} \int_0^\infty x^r \theta(k+1)\, e^{\beta x} e^{-\frac{\theta}{\beta}(e^{\beta x}-1)} \left(1 - e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)^k \; dx$$

The result of the above integration represents the r-th moment of the generalized Gompertz distribution (see [12]) with parameters θ, β and $k+1$ as in:

$$\int_0^\infty x^r \theta(k+1)\, e^{\beta x} e^{-\frac{\theta}{\beta}(e^{\beta x}-1)} \left(1 - e^{-\frac{\theta}{\beta}(e^{\beta x}-1)}\right)^k \; dx = (k+1)\theta\, \Gamma(r+1)$$

$$\Sigma_{l=0}^\infty \Sigma_{t=0}^\infty\, C_l^k \frac{(-1)^{l+t}}{\Gamma(t+1)} e^{\frac{\theta}{\beta}(l+1)} \left(\frac{\theta}{\beta}(l+1)\right)^t \left(-\frac{1}{\beta(t+1)}\right)^{r+1}$$

The r-th moment of the MOEGo distribution is given by

$$E(X^r)_{\text{MOEGo}} = S\,\theta\,\Gamma(r+1) \Sigma_{l=0}^\infty \Sigma_{t=0}^\infty\, C_l^k \frac{(-1)^{l+t}}{\Gamma(t+1)} e^{\frac{\theta}{\beta}(l+1)} \left(\frac{\theta}{\beta}(l+1)\right)^t \left(-\frac{1}{\beta(t+1)}\right)^{r+1} \tag{21}$$

where S as in (13).

<u>The characteristic function:</u> The characteristic function of MOEGo distribution can be obtained from $E\left(e^{itX}\right) = \Sigma_{r=0}^\infty \frac{(it)^r}{r!} E(X^r)_{\text{MOEGo}}$ as :

$$\phi_X(t)_{\text{MOEGo}} = S\,\theta\, \Sigma_{r=0}^\infty \Sigma_{l=0}^\infty \Sigma_{t=0}^\infty\, C_l^k \frac{(-1)^{l+t}\,\Gamma(r+1)(it)^r}{\Gamma(t+1)\, r!} e^{\frac{\theta}{\beta}(l+1)} \left(\frac{\theta}{\beta}(l+1)\right)^t \left(-\frac{1}{\beta(t+1)}\right)^{r+1} \tag{22}$$

where S as in (13).

<u>The quantile function and simulated data:</u> The quantile function of MOEGo $\left(x_{q-\text{MOEGo}}\right)$ random variable can be obtained through inverting the cdf in (16) as:

$$x_{q-\text{MOEGo}} = \frac{1}{\beta} \ln\left[1 - \frac{\beta}{\theta} \ln\left\{1 + \frac{1}{\lambda} \ln\left(1 - \frac{\alpha\left(1-e^{-\lambda}\right)}{q^{-1} - \bar{\alpha}}\right)\right\}\right] \tag{23}$$

By setting $q = 0.5$, we obtain the median of MOEGo random variable.

The MOEGo random variable can be simulated as:

$$x_{\text{MOEGo}} = \frac{1}{\beta} \ln\left[1 - \frac{\beta}{\theta} \ln\left\{1 + \frac{1}{\lambda} \ln\left(1 - \frac{\alpha\left(1-e^{-\lambda}\right)}{U^{-1} - \bar{\alpha}}\right)\right\}\right] \tag{24}$$

where U has the standard uniform distribution.

<u>Shannon entropy:</u> In the information theory, Shannon entropy (SH) plays an essential role. It is defined as an uncertainty measure. The SH of the MOEGo distribution can be obtained from $-\int_0^\infty \ln(f(x)_{\text{MOEGo}})\, f(x)_{\text{MOEGo}}\, dx$. Taking the natural logarithm of the pdf in (17) we get:

$$\ln(f(x)_{\text{MOEGo}}) = \ln\left(\alpha\lambda\left(1-e^{-\lambda}\right)\theta\right) + \beta x - \frac{\theta}{\beta}\left(e^{\beta x}-1\right) - \lambda\left(1-e^{-\frac{\theta}{\beta}\left(e^{\beta x}-1\right)}\right)$$
$$-2\ln\left(\alpha\left(1-e^{-\lambda}\right) + \bar{\alpha}\left(1-e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}\left(e^{\beta x}-1\right)}\right)}\right)\right)$$

The SH of the MOEGO distribution can be obtained as:

$$SH_{\text{MOEGo}} = -\left\{\begin{matrix}\ln\left(\alpha\lambda\left(1-e^{-\lambda}\right)\theta\right) + \frac{\theta}{\beta} - \lambda + \beta E(X) - \frac{\theta}{\beta}E\left(e^{\beta X}\right) \\ +\lambda E\left(e^{-\frac{\theta}{\beta}\left(e^{\beta X}-1\right)}\right) - 2\,I^{*}\end{matrix}\right\} \tag{25}$$

where $I^{*} = E\left[\ln\left(1 + \frac{\bar{\alpha}\left(1-e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}\left(e^{\beta X}-1\right)}\right)}\right)}{\alpha\left(1-e^{-\lambda}\right)}\right)\right]$.

Now, based on using the following formulas
$e^{z} = \Sigma_{i=0}^{\infty}\frac{z^{i}}{i!}$, $e^{-z} = \Sigma_{i=0}^{\infty}\frac{(-1)^{i}}{i!}z^{i}$, $(a+b)^{n} = \Sigma_{k=0}^{\infty} C_{k}^{n}\, a^{k}\, b^{n-k}$ for $n \geq 0$ we get,

$$E\left(e^{\beta X}\right) = \sum_{i=0}^{\infty}\frac{\beta^{i}}{i!}E\left(X^{i}\right) \tag{26}$$

$$E\left(e^{-\frac{\theta}{\beta}\left(e^{\beta X}-1\right)}\right) = \Sigma_{j=0}^{\infty}\Sigma_{k=0}^{\infty}\Sigma_{r=0}^{\infty}\frac{(-1)^{2j-k}}{j!\ r!}\,C_{k}^{j}\,(\beta k)^{r}\left(\frac{\theta}{\beta}\right)^{j} E(X^{r}) \tag{27}$$

Furthermore, based on using $\ln(1-z) = -\Sigma_{i=1}^{\infty}\frac{z^{i}}{i}$ for $|z| < 1$,
$(1-z)^{b} = \Sigma_{i=0}^{\infty}(-1)^{i}\,C_{i}^{b}\,z^{i}$ for $|z| < 1$, $b > 0$ (see [13]) we get

$$I^{*} = \Sigma_{i=1}^{\infty}\Sigma_{m=0}^{\infty}\Sigma_{t=0}^{\infty}\Sigma_{l=0}^{\infty}\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{\infty}\Sigma_{r=0}^{\infty}\frac{(-1)^{i+m+t+l+2j-k+1}}{i\ t!\ j!\ r!}\,C_{m}^{i}C_{l}^{t}C_{k}^{j}$$
$$\left(\frac{\bar{\alpha}}{\alpha(1-e^{-\lambda})}\right)^{i}(\lambda m)^{t}(\beta k)^{r}\left(\frac{\theta l}{\beta}\right)^{j}E(X^{r}) \tag{28}$$

Therefore the SH of the MOEGO distribution in (25) will be

$$SH_{\text{MOEGo}} = \ln\left(\frac{1}{\alpha\lambda(1-e^{-\lambda})\,\theta}\right) - \frac{\theta}{\beta} + \lambda - \beta E(X) + \frac{\theta}{\beta}E\left(e^{\beta X}\right)$$
$$-\lambda E\left(e^{-\frac{\theta}{\beta}\left(e^{\beta X}-1\right)}\right) + 2\,I^{*} \tag{29}$$

where $E(X)$ as in (21) with $r = 1$, $E\left(e^{\beta X}\right)$, $E\left(e^{-\frac{\theta}{\beta}\left(e^{\beta X}-1\right)}\right)$ and I^{*} as in (26), (27) and (28) respectively.

<u>The relative entropy:</u> The relative entropy (RE) is a measure of the difference between two probability distributions. More closely, the relative entropy is the amount of information lost when used to approximate the first distribution by the second distribution. The RE of the MOEGo distribution can be obtained from $\int_{0}^{\infty}\ln\left(\frac{f(x)_{\text{MOEGo}}}{f_{1}(x)_{\text{MOEGo}}}\right)f(x)_{\text{MOEGo}}\,dx$. Taking the natural logarithm of $f(x)_{\text{MOEGo}}$ in (17) relative to $f_{1}(x)_{\text{MOEGo}}$ with parameters $(\alpha_{1}, \lambda_{1}, \beta_{1}, \theta_{1})$ we get

$$\ln\left(\frac{f(x)_{\text{MOEGo}}}{f_1(x)_{\text{MOEGo}}}\right) = \ln\left(\alpha\lambda\left(1-e^{-\lambda}\right)\theta\right) + \beta x - \frac{\theta}{\beta}\left(e^{\beta x}-1\right)$$
$$-\lambda\left(1-e^{-\frac{\theta}{\beta}\left(e^{\beta x}-1\right)}\right) - 2\ln\left(\alpha\left(1-e^{-\lambda}\right) + \bar{\alpha}\left(1-e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}\left(e^{\beta x}-1\right)}\right)}\right)\right)$$
$$-\ln\left(\alpha_1\lambda_1\left(1-e^{-\lambda_1}\right)\theta_1\right) - \beta_1 x + \frac{\theta_1}{\beta_1}\left(e^{\beta_1 x}-1\right) + \lambda_1\left(1-e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 x}-1\right)}\right)$$
$$+2\ln\left(\alpha_1\left(1-e^{-\lambda_1}\right) + \bar{\alpha}_1\left(1-e^{-\lambda_1\left(1-e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 x}-1\right)}\right)}\right)\right)$$

Now, the RE of the MOEGO distribution is given by

$$RE_{\text{MOEGo}} = \ln\left(\frac{\alpha\lambda\left(1-e^{-\lambda}\right)\theta}{\alpha_1\lambda_1\left(1-e^{-\lambda_1}\right)\theta_1}\right) + \frac{\theta}{\beta} - \frac{\theta_1}{\beta_1} - \lambda + \lambda_1 + (\beta-\beta_1)E(X) - \frac{\theta}{\beta}E\left(e^{\beta X}\right)$$
$$+\frac{\theta_1}{\beta_1}E\left(e^{\beta_1 X}\right) + \lambda E\left(e^{-\frac{\theta}{\beta}\left(e^{\beta X}-1\right)}\right) - \lambda_1 E\left(e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 X}-1\right)}\right) - 2I^* + 2I_1^* \quad (30)$$

where $E(X)$ as in (21) with $r=1$, and with specified parameters, $E\left(e^{\beta X}\right)$ and $E\left(e^{\beta_1 X}\right)$ as in (26), $E\left(e^{-\frac{\theta}{\beta}\left(e^{\beta X}-1\right)}\right)$ and $E\left(e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 X}-1\right)}\right)$ as in (27), I^* and I_1^* as in (28).

<u>The stress strength:</u> Given Y and X as a stress strength independent random variables follows MOEGO distribution with different parameters $(\alpha_1, \lambda_1, \beta_1, \theta_1)$, then the stress strength of MOEGO distribution can be obtained by

$$SS_{\text{MOEGO}} = P(Y<X) = \int_0^\infty f_X(x)_{\text{MOEGO}}\, F_Y(x)\, dx$$

where,

$$F_Y(x) = \frac{1-e^{-\lambda_1\left(1-e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 x}-1\right)}\right)}}{\alpha_1\left(1-e^{-\lambda_1}\right) + \bar{\alpha}_1\left(1-e^{-\lambda_1\left(1-e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 x}-1\right)}\right)}\right)}$$
$$=\left[\bar{\alpha}_1 + \frac{\alpha_1\left(1-e^{-\lambda_1}\right)}{1-e^{-\lambda_1\left(1-e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 x}-1\right)}\right)}}\right]^{-1}$$
$$=\frac{1}{\bar{\alpha}_1}\left[1 + \frac{\alpha_1\left(1-e^{-\lambda_1}\right)}{\bar{\alpha}_1\left(1-e^{-\lambda_1\left(1-e^{-\frac{\theta_1}{\beta_1}\left(e^{\beta_1 x}-1\right)}\right)}\right)}\right]^{-1}$$

Using $(1-z)^{-b} = \Sigma_{i=0}^{\infty}\frac{\Gamma(b+i)}{i!\,\Gamma(b)}z^i$ for $|z|<1, b>0$, $e^{-z} = \Sigma_{i=0}^{\infty}\frac{(-1)^i}{i!}z^i$, and $(1-z)^b = \Sigma_{i=0}^{\infty}(-1)^i C_i^b z^i$ for $|z|<1$, $b>0$, we get,

$$F_Y(x) = \Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{\infty}\,\Sigma_{l=0}^{\infty}\Sigma_{j_1=0}^{\infty}\Sigma_{k_1=0}^{\infty}\Sigma_{r=0}^{\infty}\,\frac{(-1)^{i+k+l+2j_1-k}}{k!\;j_1!\;r!}\,C_l^k C_{k_1}^{j_1}$$

$$\frac{\Gamma(i+j)}{j!\;\Gamma(i)}\,\frac{\left(\alpha\left(1-e^{-\lambda}\right)\right)^i}{(\bar{\alpha})^{i+1}}(\lambda j)^k(\beta k_1)^r\left(\frac{\theta l}{\beta}\right)^{j_1}\,x^r$$

Therefor based on the above formula of $F_Y(x)$, the stress strength of the MOEGo distribution can be obtained as

$$SS_{\mathrm{MOEGO}} = \Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}\Sigma_{k=0}^{\infty}\,\Sigma_{l=0}^{\infty}\Sigma_{j_1=0}^{\infty}\Sigma_{k_1=0}^{\infty}\,\Sigma_{r=0}^{\infty}\,\frac{(-1)^{i+k+l+2j_1-k}}{k!\;j_1!\;r!}\,C_l^k C_{k_1}^{j_1}$$

$$\frac{\Gamma(i+j)}{j!\;\Gamma(i)}\,\frac{\left(\alpha\left(1-e^{-\lambda}\right)\right)^i}{(\bar{\alpha})^{i+1}}(\lambda j)^k(\beta k_1)^r\left(\frac{\theta l}{\beta}\right)^{j_1}\,E(X^r) \quad (31)$$

where $E(X^r)$ as in (21).

4.3 Estimation of MOEGo parameters

Here the maximum likelihood estimation method is considered to estimate the MOEGo parameters from complete sample.
Consider a complete random sample $x_1, x_2, \dots, x_n$ of the MOEGo distribution with parameter vector $\varphi = (\alpha, \lambda, \beta, \theta)^T$. According to (17), the natural logarithm likelihood function for φ is

$$\ell(\varphi|\underline{\mathrm{x}}) = n\ln(\alpha) + n\ln(\lambda) + n\ln\left(\left(1-e^{-\lambda}\right)\right) + n\ln(\theta) + \;\beta\Sigma_{i=0}^{\infty}x_i$$
$$-\frac{\theta}{\beta}\Sigma_{i=0}^{\infty}\left(e^{\beta x_i}-1\right) - \lambda\Sigma_{i=0}^{\infty}\left(1-e^{-\frac{\theta}{\beta}\left(e^{\beta x_i-1}\right)}\right)$$
$$-\,2\,\Sigma_{i=0}^{\infty}\ln\left(\alpha\left(1-e^{-\lambda}\right)+\bar{\alpha}\left(1-e^{-\lambda\left(1-e^{-\frac{\theta}{\beta}\left(e^{\beta x_{i}-1}\right)}\right)}\right)\right) \quad (32)$$

The maximum likelihood estimates (MLEs) of four parameters can be obtained by solving the nonlinear natural logarithm likelihood system equations $\frac{\partial\ell(\varphi|\underline{\mathrm{x}})}{\partial\varphi} = \left(\frac{\partial\ell(\varphi|\underline{\mathrm{x}})}{\partial\alpha}, \frac{\partial\ell(\varphi|\underline{\mathrm{x}})}{\partial\lambda}, \frac{\partial\ell(\varphi|\underline{\mathrm{x}})}{\partial\beta}, \frac{\partial\ell(\varphi|\underline{\mathrm{x}})}{\partial\theta}\right)^T = 0$ through numerical iterative techniques.

5. Applications

A real life application is presented in this section to demonstrate the potentials of the Marshall Olkin Exponential Gompertz distribution by applying it to two data sets**.**

The usefulness of the Marshall Olkin Exponential Gompertz distribution (MOEGo) is comparison with the Beta Gompertz (BGo), Kumaraswamy Gompertz (KuGo), Exponentiated Generalized Gompertz (EGGo), and Weibull Gompertz (WeGo) distributions. The R software was used to compute their negative log-likelihood (NLL), Akaike Information Criteria (AIC), Consistent Akaike Information Criteria (CAIC), Bayesian Information Criteria (BIC), Hanan and Quinn Information Criteria (HQIC), and parameter estimation values according to MLE method.

5.1 Data1

The dataset1 used reports consists of 63 observations of the strengths of 1.5 cm glass fibers. The dataset has previously been analyzed by [14], [15], [16], and [17] to fit models. The observations are:

0.55, 0.74, 0.77, 0.81, 0.84, 1.24, 0.93, 1.04, 1.11, 1.13, 1.30, 1.25, 1.27, 1.28,1.29, 1.48, 1.36, 1.39, 1.42, 1.48, 1.51, 1.49, 1.49, 1.50, 1.50,1.55, 1.52, 1.53, 1.54, 1.55, 1.61, 1.58,1.59, 1.60, 1.61, 1.63,1.61, 1.61, 1.62, 1.62, 1.67, 1.64, 1.66, 1.66, 1.66, 1.70, 1.68,1.68, 1.69, 1.70, 1.78, 1.73, 1.76, 1.76, 1.77, 1.89, 1.81, 1.82,1.84, 1.84, 2.00, 2.01, 2.24 .
The result is presented in Table 1.

Table 1. Results of Data1

Distribution	NLL	AIC	CAIC	BIC	HQIC	MLE
MOEGo	**11.7667**	**31.5334**	**32.2231**	**40.1059**	**34.9050**	$\hat{\alpha}$=11.104 $\hat{\lambda}$=11.569 $\hat{\beta}$=2.6271 $\hat{\theta}$=0.0107
BGo	14.1443	36.2886	36.9782	44.8611	39.6602	$\hat{\alpha}$=1.6338 $\hat{\lambda}$=1.0620 $\hat{\beta}$=2.8492 $\hat{\theta}$=0.0357
KuGo	14.0322	36.0645	36.7542	44.6370	39.4361	$\hat{\alpha}$=1.9457 $\hat{\lambda}$=4.1909 $\hat{\beta}$=2.2596 $\hat{\theta}$=0.0340
EGGo	14.1452	36.2904	36.9800	44.8629	39.6620	$\hat{\alpha}$=0.2203 $\hat{\lambda}$=1.6060 $\hat{\beta}$=2.8833 $\hat{\theta}$=0.1616
WeGo	14.4020	36.8041	37.4938	45.3767	40.1757	$\hat{\alpha}$=2.8720 $\hat{\lambda}$=3.5457 $\hat{\beta}$=1.0222 $\hat{\theta}$=0.8305

The newly developed MOEGo displays a very good potential in Table 1 as it has the lowest values for the NLL, AIC, CAIC, BIC, and HQIC criterion. To further validate the results obtained, the histogram plot of the dataset1 with the distributions compared is presented in Figure 3 and the corresponding empirical cdf plot is presented in Figure 4.

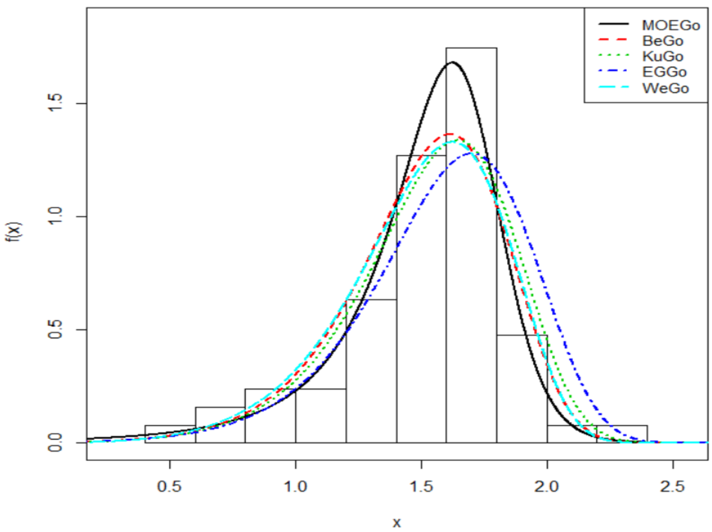

Figure 3. Histogram plot of the dataset1 with the compared distributions

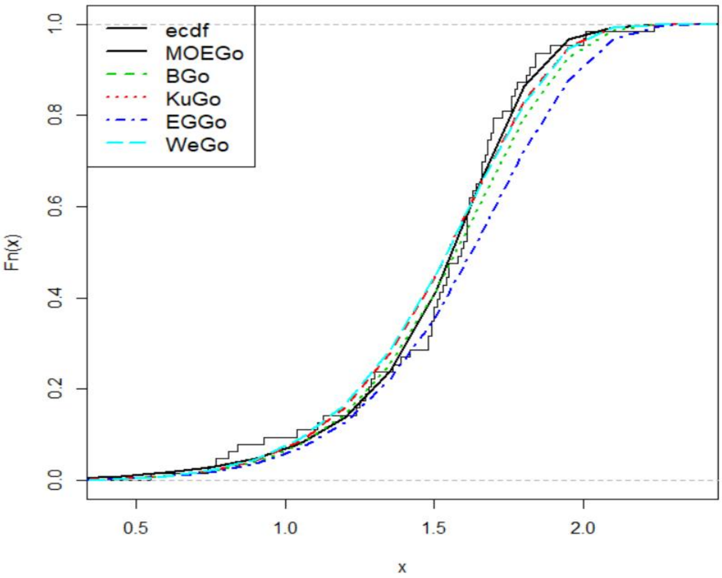

Figure 4. Empirical cdf of the dataset1 with the compared distributions

Figures 3 and 4 show that the MOEGo distribution fits the dataset1 better than the other distributions. The probability-probability (p-p) plots for each of the distributions with respect to the dataset used are presented in Figure 5.

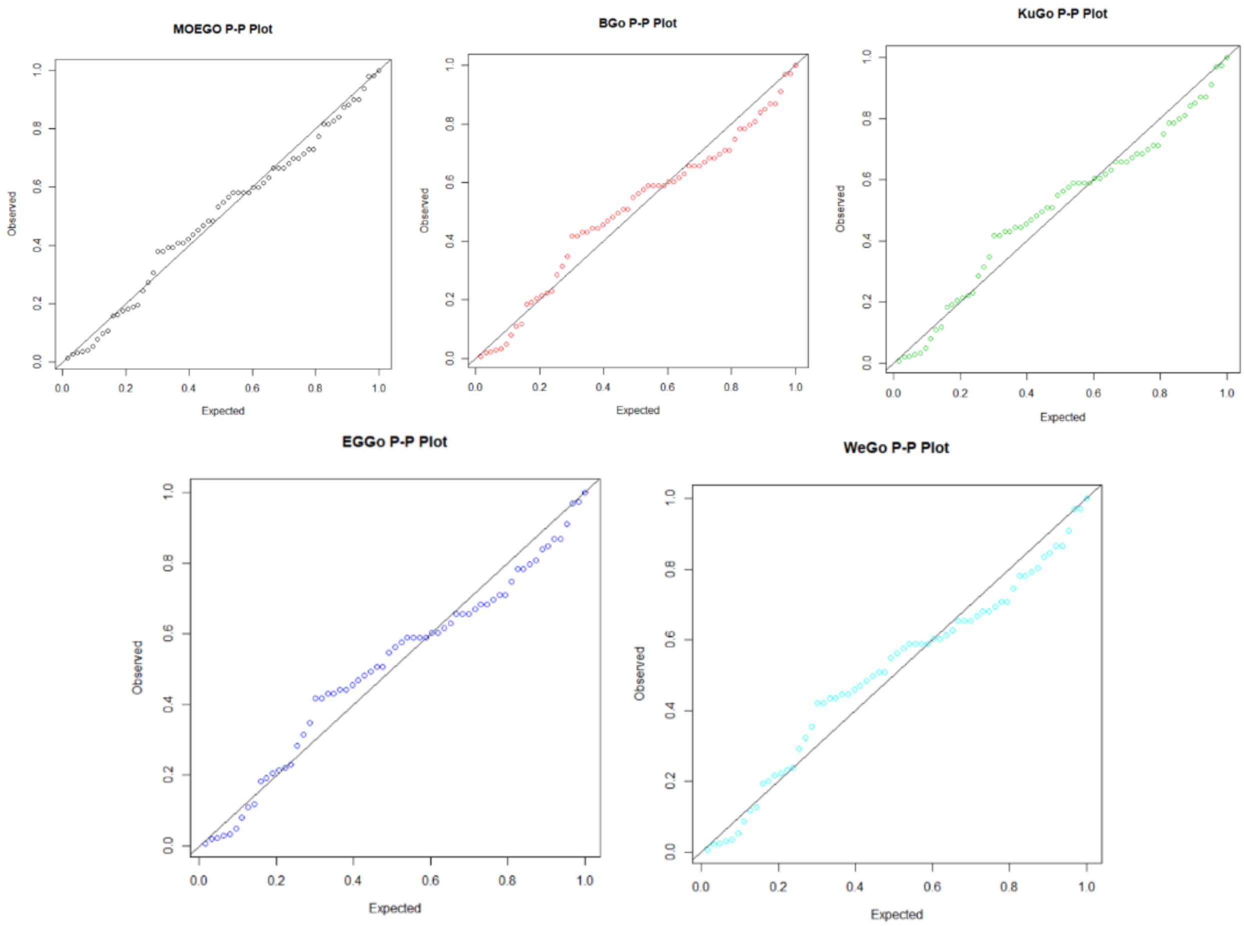

Figure 5. p-p plot for MOEGo, BGo, KuGo, EGGo, WeGo distributions of dataset1

The plots in Figure 5 support the results in Table 1, showing that the MOEGo distribution fits the dataset better than the other compared distributions.

5.2 Data2

The dataset2 used consists the life of fatigue fracture of Kevlar 373/epoxy that are subject to constant pressure at the 90% stress level until all had failed. The dataset2 has previously been extracted from [18]. The observations are:

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773, 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.7746, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.2100, 2.2460, 2.2878, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.4045, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295, 6.5541, 9.0960

The result is presented in Table 2.

Table 2. Results of data2

Distribution	NLL	AIC	CAIC	BIC	HQIC	MLE
MOEGo	**120.2490**	**248.4980**	**249.0613**	**257.8209**	**252.2238**	$\hat{\alpha}$=13.6052 $\hat{\lambda}$=11.3846 $\hat{\beta}$=0.10699 $\hat{\theta}$=0.18196
BGo	122.2130	252.4261	252.9895	261.7490	256.1520	$\hat{\alpha}$=1.70643 $\hat{\lambda}$=2.36972 $\hat{\beta}$=0.01653 $\hat{\theta}$=0.33543
KuGo	121.7980	251.5978	252.1612	260.9207	255.3237	$\hat{\alpha}$=1.61724 $\hat{\lambda}$=9.80554 $\hat{\beta}$=0.08516 $\hat{\theta}$=0.14358
EGGo	122.2430	252.4871	253.0505	261.8100	256.2130	$\hat{\alpha}$=0.50931 $\hat{\lambda}$=.707339 $\hat{\beta}$=0.00077 $\hat{\theta}$=1.37725
WeGo	121.6990	251.3986	251.9620	260.7215	255.1245	$\hat{\alpha}$=1.58689 $\hat{\lambda}$=1.28623 $\hat{\beta}$=0.13679 $\hat{\theta}$=0.72142

The newly developed MOEGo displays a very good potential in Table 2 as it has the lowest values for the NLL, AIC, CAIC, BIC, and HQIC criterion.
To further validate the results obtained, the histogram plot of the dataset1 with the distributions compared is presented in Figure 6 and the corresponding empirical cdf plot is presented in Figure 7.

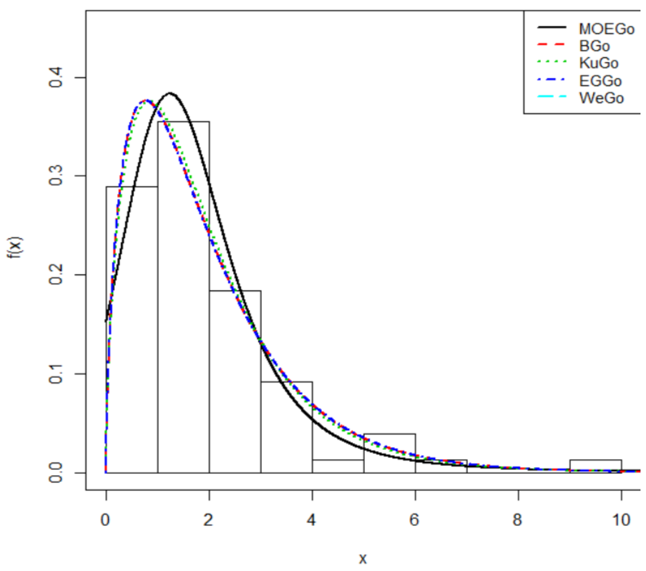

Figure 6. Histogram plot of the dataset2 with the compared distributions

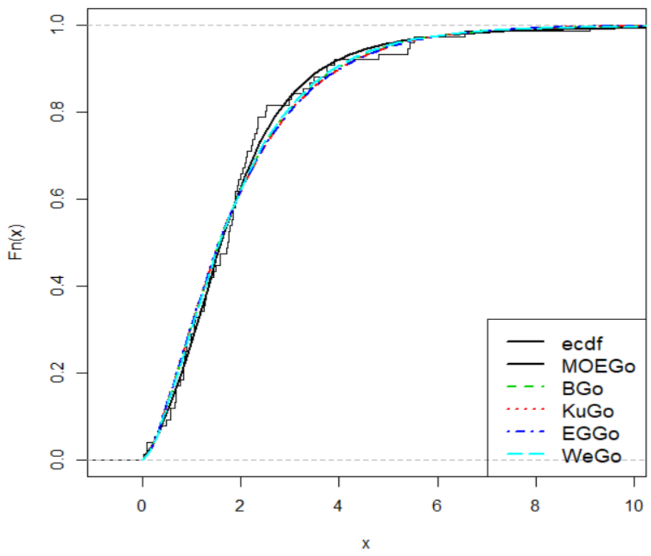

Figure 7. Empirical cdf of the dataset2 with the compared distributions

Figures 6 and 7 show that the MOEGo distribution fits the dataset1 better than the other distributions. The probability-probability (p-p) plots for each of the distributions with respect to the dataset used are presented in Figure 8.

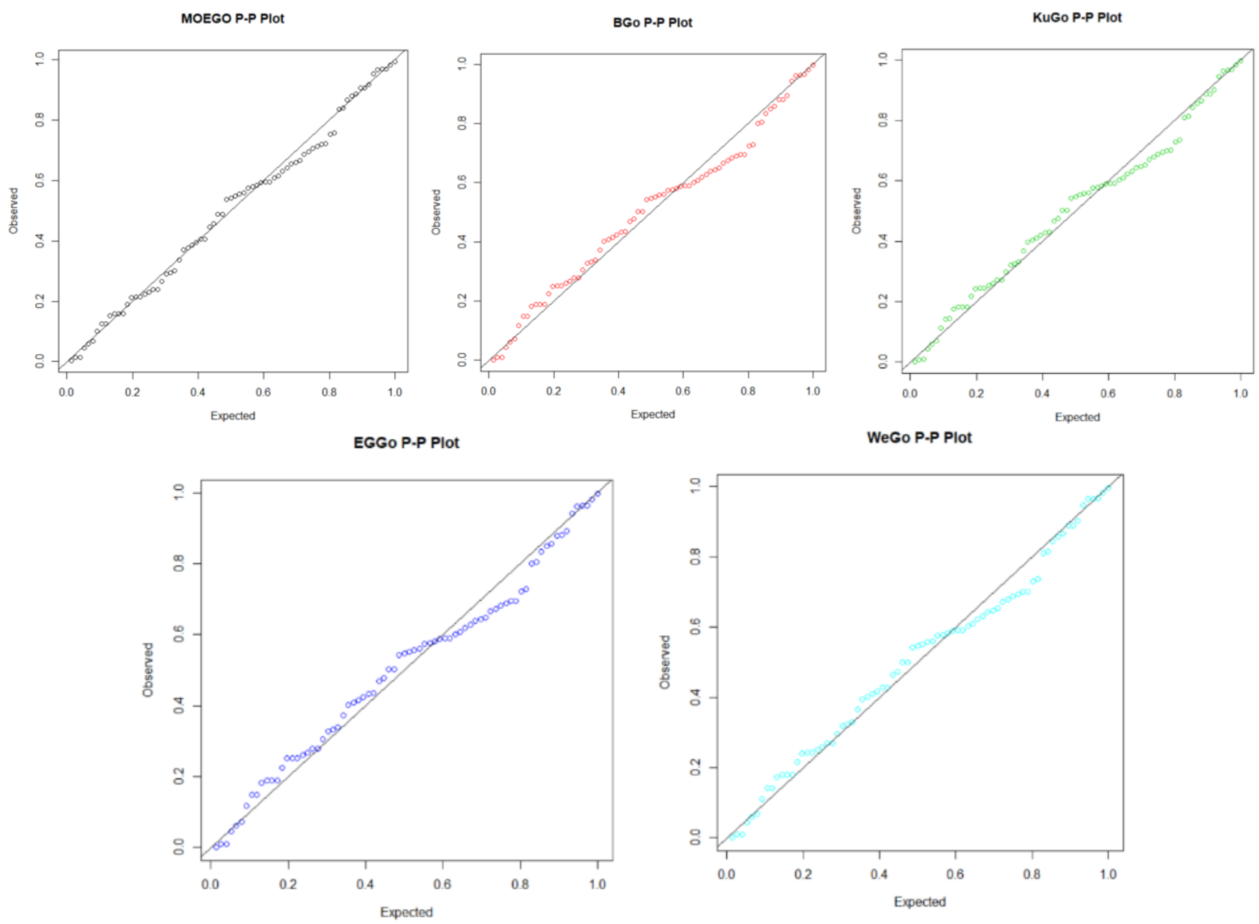

Figure 8. p-p plot for MOEGo, BGo, KuGo, EGGo, WeGo distributions of dataset2

The plots in Figure 8 support the results in Table 2, showing that the MOEGo distribution fits the dataset better than the other compared distributions.

6. Conclusions

This paper introduced a newly proposed family that holds for any baseline distribution besides introduced a new distribution with four-parameter named Marshall Olkin Exponential Gompertz (MOEGo). The vital statistical properties of MOEGo distribution such that r-moment, characteristic function, quantile function, simulated data, Shannon entropy, relative entropy, and stress-strength model are introduced. The MOEGo parameters are estimated through maximum likelihood estimation method. Applications with two real data sets (symmetric and right-skewed) are conducted. The newly MOEGo distribution has the lowest value of NLL, AIC, CAIC, BIC, and HQIC, so it is shown to provide the best fit and most flexible than several other compared models like Beta Gompertz (BGo), Kumaraswamy Gompertz (KuGo), Exponentiated Generalized Gompertz (EGGo), and Weibull Gompertz (WeGo) distributions. This flexibility enables the use of the MOEGo distribution in different application areas.

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