Periodicals of Engineering and Natural Sciences ISSN 2303-4521
Vol. 8, No. 1, March 2020, pp.242-247

Bayesian estimation for two parameter exponential distribution using linear transformation of reliability function

Ban Ghanim AL-Ani[1], Raya Salim AL-Rassam[2], Safwan Nathem Rashed[3]
[1,2,3]Department of Statistics and Informatics, University of Mosul

ABSTRACT

The estimation of two-parameter exponential distribution using Bayes approach need a prior distribution for the two parameters. It is difficult to know this joint prior distribution, so it requires sometimes the approximation or to some assumptions which depends on previous experience. An estimation method was proposed by using linear transformation of reliability function of two-parameter exponential distribution in order to get simple linear regression model. Its parameters can be estimated by using Bayes approach, and then get the estimated parameters of the distribution from the relationship between the distribution parameters and regression model parameters. Simulation experiments at different sample sizes were applied in order to make a comparison between Bayes estimators yield from approximation method and estimators from proposed method. The findings show that the proposed method estimators were more efficient that from approximation method estimators by using mean squares error (MSE) as a criterion for comparison. Also, the results of estimation methods were applied on actual data taken from Babil Tires Factory, where the data represents the working time (hours) between successive failures.

Keywords: Exponential distribution, Bayes estimator, Linear regression, Linear transformation

Corresponding Author:
Ban Ghanim AL-Ani
Department of Statistics and Informatics
University of Mosul
Mosul, Iraq
E-mail: drbanalani2019@gmail.com

1. Introduction

The two-parameter exponential distribution plays an important role in survival analysis and has extensive applications in the field of reliability, engineering, queueing theory, and medical sciences. The researchers studied different methods for estimating two-parameter exponential distribution. The most important one of these methods is Bayesian method. It depends on its applications in the assumption in which the parameters are estimated as random variables with a probability density function. The researchers faced the problem of determining the prior probability distribution of the parameters in the two cases with information and non-information. They often use approximate methods to identify them as in the investigations reported in [1-5].

In this paper, we will consider a proposed method for estimating the parameters of the two-parameter distribution by modeling a simple linear regression model based on the cumulative distribution function of the distribution. This proposed method is to find the estimating of the distribution parameters based on the Bayes method and then comparing the proposed method with the approximation method [1] using the mean squares error.

The probability density function for the two-parameter exponential distribution (θ,η) is:

$$f(x;\theta,\eta) = \theta^{-1}e^{-\frac{1}{\theta}(x-\eta)} \qquad 0 < \eta < x \quad , \theta > 0$$

$x_1,x_2,\text{---},x_n$ are random variables from the two-parameter exponential distribution, the likelihood function is given by:

$$L(\eta,\theta) = \theta^{-n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}(x_i-\eta)}$$
$$= \theta^{-n}e^{-\frac{1}{\theta}\{S+n(x_{(1)}-\eta)\}}$$

where $x_{(1)}$ is the first order statistic in the sample x=($x_1,x_2,---,x_n$) such that $x_{(1)} < x_{(2)} <$ -----< $x_{(n)}$ and $S = \sum_{i=1}^{n}\left(x_i - x_{(1)}\right)$

If we use Jefferry's procedure [6], the joint Jefferry's prior will be:

$$P(\eta,\theta) \propto \frac{1}{\theta} \qquad \eta,\theta > 0$$

According to above, the posterior density for θ and η is :

$$P(\eta,\theta|x)\ \alpha\ p(\eta,\theta)\ L(\eta,\theta)$$

$$P(\eta,\theta|x) \propto \theta^{-(n+1)} e^{-\frac{1}{\theta}\{S+n(x_{(1)}-\eta)\}}$$

$$= \frac{k}{\theta^{n+1}} e^{-\frac{1}{\theta}\{S+n(x_{(1)}-\eta)\}}$$

where $k = \frac{nS^{n-1}}{\Gamma(n-1)}$

$$P(\eta,\theta|x) = \frac{nS^{n-1}}{\Gamma(n-1)\theta^{n+1}} e^{-\frac{1}{\theta}\{S+n(x_{(1)}-\eta)\}}$$

The marginal posterior density of η is given by

$$P(\eta|x) = \int_0^\infty P(\eta,\theta|x)d\theta$$

$$= n(n-1)\frac{S^{n-1}}{\left\{S+n\left(x_{(1)}-\eta\right)\right\}^{n-1}}$$

$$\hat{\eta} = E(\eta|x) = x_{(1)} - \frac{S}{n(n-2)} \qquad (1)$$

The marginal posterior density of θ is given by

$$P(\theta|x) = \int_0^{x_{(1)}} P(\eta,\theta|x)d\eta$$

$$= \frac{S^{n-1}}{\Gamma(n-1)}.\frac{e^{-\frac{S}{\theta}}}{\theta^n}$$

$$\hat{\theta} = E(\theta|x) = \frac{S}{(n-2)} \qquad (2)$$

where $S = \sum_{i=1}^{n}(x_i - x_{(1)})$

2. Proposed method

We will use the method of transforming the reliability function of the two-parameter exponential distribution (η,θ) into a linear function [7] in order to estimate the distribution parameters by taking the natural logarithm to facilitate the solution as follows:

$$R(x_i) = e^{-\frac{(x_i-\eta)}{\theta}}$$

Taking the logarithm for both sides we get:

$$-\ln R(x_i) = \theta^{1} x_i - \eta\,\theta^{1}$$

This equation is similar to the following simple linear regression model:

$$y_i = \beta_0 + \beta_1 x_i + e_i \qquad i=1, 2,\ldots,n$$

where:

$y_i = -\ln R(t_i)$, $\beta_0 = -\eta\,\theta^{1}$ and $\beta_1 = \theta^{1}$

So, the estimators of the two-parameter exponential distribution are as follows:

$$\hat{\theta} = \hat{\beta}_{1L}^{-1} \qquad (3)$$

$$\hat{\eta} = -\hat{\beta}_{0L}\,\hat{\theta} \qquad (4)$$

where $\hat{\beta}_{0L}$ and $\hat{\beta}_{1L}$ represent the Bayes estimators of the simple linear regression model.

3. Bayesian estimation for regression model parameters

If you have the simple regression model:

$$\underline{Y} = X\underline{\beta} + \underline{U} \quad , \quad \underline{U} \sim N_2(\underline{0}, \sigma^2 I_2)$$

Where $\underline{Y}$ and $\underline{U}$ are vectors of sizes $(n \times 1)$, X is a matrix of size $(n \times 2)$ and $\underline{\beta}$ is a vector of regression parameters of size (2×1).
The likelihood function is:

$$L\left(\underline{\beta}, \sigma^2 | X\right) = \frac{e^{-\frac{1}{2\sigma^2}(\underline{Y}-X\underline{\beta})'(\underline{Y}-X\underline{\beta})}}{(2\pi)^{\frac{n}{2}}(\sigma^2)^{\frac{n}{2}}} \tag{5}$$

As $\underline{\beta}$ and σ^2 are unknown, the conditional conjugate prior distribution for $\underline{\beta}$ is given by [8]:

$$\underline{\beta}|\sigma^2 \sim N_2(\underline{\beta}_0, \sigma^2 V_0)$$

where V_0 is a symmetric positive definite matrix of size (2×2).

$$P\left(\underline{\beta}|\sigma^2\right) = \frac{1}{(2\pi)\sigma^2 |V_0|^{\frac{1}{2}}} e^{-\frac{1}{2\sigma^2}(\underline{\beta}-\underline{\beta}_0)' V_0^{-1}(\underline{\beta}-\underline{\beta}_0)} \tag{6}$$

The prior distribution for σ^2 is $IG\left(\frac{a_0}{2}, \frac{b_0}{2}\right)$ where:

$$P(\sigma^2) = \frac{\left(\frac{b_0}{2}\right)^{-\frac{a_0}{2}}}{\Gamma\left(\frac{a_0}{2}\right)} \sigma^{2^{-\left(\frac{a_0}{2}+1\right)}} e^{-\frac{b_0}{2}\sigma^2} \tag{7}$$

Then, the joint prior distribution for $\underline{\beta}$ and σ^2 is defined as:

$$P\left(\underline{\beta}, \sigma^2\right) = P\left(\underline{\beta}|\sigma^2\right) P(\sigma^2)$$

From (6) and (7), we get:

$$P\left(\underline{\beta}, \sigma^2\right) \propto \frac{1}{\sigma^2} e^{-\frac{1}{2\sigma^2}(\underline{\beta}-\underline{\beta}_0)' V_0^{-1}(\underline{\beta}-\underline{\beta}_0)} \sigma^{2^{-\left(\frac{a_0}{2}+1\right)}} e^{-\frac{b_0}{2}\sigma^2} \tag{8}$$

After obtaining the likelihood function and prior distribution, we obtain the posterior distribution [9]. The joint posterior distribution for $(\underline{\beta}, \sigma^2)$ can be found by equations (8) and (5) as follows:

$$P\left(\underline{\beta}, \sigma^2 \middle| data\right) \propto (\sigma^2)^{-\left(\frac{a_0+n}{2}+1\right)} e^{-\frac{(b_0+(n-2)S_e^2)}{(2\sigma^2)}} \frac{1}{\sigma^2} e^{-\frac{1}{2\sigma^2}\left[\left(\underline{\beta}-\underline{\beta}_0\right)' V_0^{-1}\left(\underline{\beta}-\underline{\beta}_0\right)+\left(\underline{\beta}-\underline{\hat{\beta}}_{LS}\right)' I_2\left(\underline{\beta}-\underline{\hat{\beta}}_{LS}\right)\right]} \tag{9}$$

Where $\underline{\hat{\beta}}_{LS}$ is the least squares estimator for simple linear regression parameters.
To simplify Eq. (9), we use the quadratic form as reported by [10]. Then, we get the joint posterior distribution as follows:

$$P\left(\underline{\beta}, \sigma^2 | data\right) \propto (\sigma^2)^{-\left(\frac{a_0+n}{2}+1\right)} e^{-\frac{S_T^2}{2\sigma^2}} \frac{1}{\sigma^2} e^{-\frac{1}{2\sigma^2}\left(\underline{\hat{\beta}}_{LS}-\underline{C}\right)' D\left(\underline{\hat{\beta}}_{LS}-\underline{C}\right)} \tag{10}$$

It represents the kernel of $IG\left(\frac{a_{0+n}}{2}, \frac{S_T^2}{2}\right). N_2(\underline{C}, \sigma^2 D^{-1})$
Where,

$$\underline{C} = \left(I_2 + V_0^{-1}\right)^{-1} . \left(V_0^{-1}\underline{\beta}_0 + \underline{\hat{\beta}}\right) \tag{11}$$

$$D = V_0^{-1} + I_2 \tag{12}$$

This can be written as follows:

$$D^{-1} = \begin{bmatrix} d_{11}^* & d_{12}^* \\ d_{21}^* & d_{22}^* \end{bmatrix}$$

$$S_T^2 = b_0 + (n-1)S_e^2 + d$$

$$S_e^2 = \frac{(\underline{Y} - X\underline{\hat{\beta}}_{LS})'(\underline{Y} - X\underline{\hat{\beta}}_{LS})}{n-1}$$

$d = \left(\underline{\beta}_0 - \underline{\hat{\beta}}_{LS}\right)' (V_0 + I_2)^{-1} \left(\underline{\beta}_0 - \underline{\hat{\beta}}_{LS}\right)$, this is constant.

$$\underline{\beta} = \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix}$$

This means that the marginal conditional posterior distribution for β_0 and β_1 are described respectively as:
$(\beta_0|\sigma^2, data) \sim N(C_1, \sigma^2 d_{11}^*)$
$(\beta_1|\sigma^2, data) \sim N(C_2, \sigma^2 d_{22}^*)$
Bayesian estimators for β_0 and β_1 are defined as the followed values respectively:

$$\hat{\beta}_{0L} = C_1$$

$$\hat{\beta}_{1L} = C_2$$

where C_1 and C_2 were defined in Eq.(11).
By substitute values of $\hat{\beta}_{0L}$, $\hat{\beta}_{1L}$ in the Eq. (3) and Eq.(4) then the estimated values of the exponential distribution parameters in the proposed method will be as follows:

$$\hat{\theta}_L = C_2^{-1} \quad (13)$$

$$\hat{\eta}_L = -C_1 C_2^{-1} \quad (14)$$

where C_1 and C_2 were defined in Eq.(11)

4. Practical part

4.1 Simulation

In order to apply what mentioned in the theoretical part, the simulation approach was used. The Monte Carlo method was used for the cumulative distribution function and to generate data followed two-parameter exponential distribution. Samples were generated with sizes (n=10, 25, 50, 100), as for the two-parameter exponential distribution, the values of scale and location parameters are (θ=2, η=2), where these values were most appropriate. The initial values of matrix V were V_{120} = V_{210} =1, V_{220} =2, V_{110} =3. A program was designed using the MATLAB language and MSE used as a criterion for comparing between the approximate and proposed formulas by using the simple linear regression model. MSE was calculated at each value of scale parameter θ and location parameter η as follows:

$$Mse(\hat{\theta}) = \frac{\sum_{i=1}^{R}(\hat{\theta}_i-\theta)^2}{R}, \qquad Mse(\hat{\eta}) = \frac{\sum_{i=1}^{R}(\hat{\eta}_i-\eta)^2}{R}$$

Here, the number of replicated cases is *R=1000*.

4.2 Empirical results

The results have presented in the following tables including a comparison between the estimations of scale and location. The θ and η parameters were estimated from approximate formula as in Eq.(1) and Eq.(2), and based on proposed formula using simple linear regression model as in Eq. (13) and (14) with different initial values for θ_0, η_0, β_{00}, β_{01}. These values are close to estimated values by using different sample sizes. The results of MSE criteria values for that estimators were presented.

Table 1. Initial values for β_{00}= *-1* & β_{01} *=0.5* using different sample sizes

n	$\hat{\theta}_{bayes}$	$\hat{\eta}_{bayes}$	$\hat{\theta}_{PL\,bayes}$	$\hat{\eta}_{PL\,bayes}$	$MSE_{\hat{\theta}_{bayes}}$	$MSE_{\hat{\eta}_{bayes}}$	$MSE_{\hat{\theta}_{PL\,bayes}}$	$MSE_{\hat{\eta}_{PL\,bayes}}$
10	2.2217	1.9747	2	2	0.5784	0.0458	0.0197e-28	0.4900e-28
25	2.0862	1.9941	2	2	0.1861	0.0065	0.0197e-28	0.4900e-28
50	2.0438	1.9986	2	2	0.0905	0.0015	0.0197e-28	0.4900e-28
100	72.0201	1.99986	2	2	0.0405	0.00043	0.0197e-28	0.4900e-28

Table 2. Initial values for β_{00}= *-1.008* & β_{01} *=0.5* using different sample sizes

n	$\hat{\theta}_{bayes}$	$\hat{\eta}_{bayes}$	$\hat{\theta}_{PL\,bayes}$	$\hat{\eta}_{PL\,bayes}$	$MSE_{\hat{\theta}_{bayes}}$	$MSE_{\hat{\eta}_{bayes}}$	$MSE_{\hat{\theta}_{PL\,bayes}}$	$MSE_{\hat{\eta}_{PL\,bayes}}$
10	2.2453	1.9662	1.9971	2.0015	0.5771	0.0419	0.8438e-5	0.2110e-5
25	2.0746	1.9969	1.9971	2.0015	0.1840	0.0061	0.8438e-5	0.2110e-5
50	2.0445	1.9988	1.9971	2.0015	0.0897	0.0017	0.8438e-5	0.2110e-5
100	2.0137	1.9999	1.9971	2.0015	0.0424	0.0004	0.8438e-5	0.2110e-5

Table 3. Initial values for β_{00}= -2 & β_{01} =0.3, θ_0=η_0=2 using different sample sizes

n	$\hat{\theta}_{bayes}$	$\hat{\eta}_{bayes}$	$\hat{\theta}_{PL\ bayes}$	$\hat{\eta}_{PL\ bayes}$	$MSE_{\hat{\theta}_{bayes}}$	$MSE_{\hat{\eta}_{bayes}}$	$MSE_{\hat{\theta}_{PL\ bayes}}$	$MSE_{\hat{\eta}_{PL\ bayes}}$
10	2.2577	1.9759	1.9298	2.0947	0.6199	0.0497	0.004925	0.008975
25	2.0556	1.9999	1.9298	2.0947	0.1742	0.0062	0.004925	0.008975
50	2.0500	1.9993	1.9298	2.0947	0.0854	0.0016	0.004925	0.008975
100	2.0244	1.9995	1.9298	2.0947	0.0376	0.0004	0.004925	0.008975

From Tables 1-4, we note that the MSE values of the estimators in the proposed method are less than its value using the approximate Bayes method and for all the initial values of exponential distribution parameters and simple linear regression. We also note that the MSE values of the exponential distribution parameters using the proposed method have not changed by the size of samples. This means that the sample size does not affect the MSE value of the proposed method parameters. In the approximate Bayes method, we note that the MSE value of the parameters decreases as the sample size increases.

4.3 Actual data

The following data were selected from Babel Tires Factory, where the working time (hours) between failures were deduced by the time recorded in the internal statements of the factory for six months. A test was carried out by EasyFit program where the results of the test showed that the data have two-parameter exponential distribution with scale parameter (θ=97.85) and location parameter (η=0.25) as shown in Figure. 1.

140.5 312 22.5 48.75 72.5 49.75 218.25 22.25 9.25 68.25 75.5 22.5 23.25 22 63 23 58.75
237.5 193.5 30.25 17.75 141 146.5 127.5 257.75 352 42.5 138.5 51 35.75 173 41.75

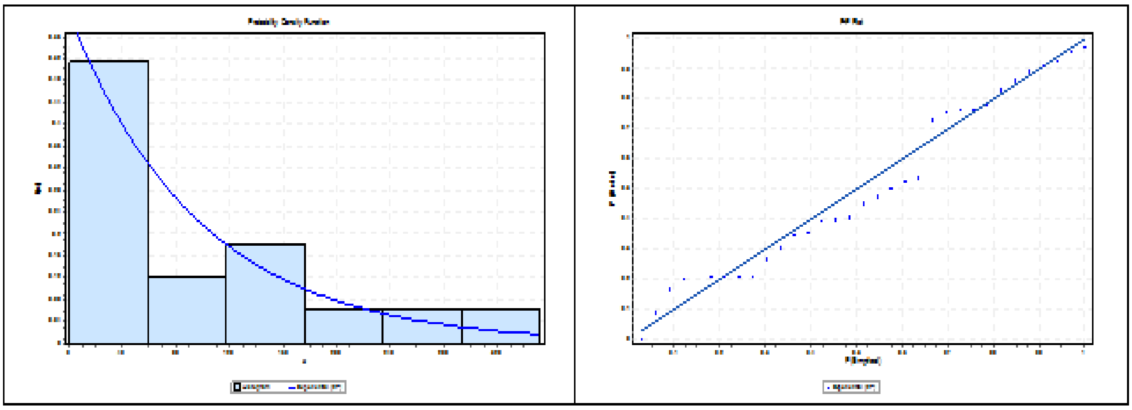

Figure 1. Fitting the Two parameter exponential distribution of the research data

The following table depicts the estimated value by proposed and approximate Bayes methods when the values of positive definite matrix are fixed at v_{110}=8;v_{220}=8;v_{120}=v_{210}=0.1;and selecting initial values of parameters which are close to estimated values $\beta_{00} = -0.025, \beta_{01} = 0.01, \theta_0 = 97, \eta_0 = 0.25$ using different sample sizes.

Table 4. Estimated values and its MSE of parameters using proposed and approximate Bayes Methods

n					$MSE(\hat{\theta})$	[illegible]		
35	98.0750	6.1852	97.0629	0.4920	1.1556	35.2261	0.003959	0.05855

5. Conclusions

1. The proposed method of estimation provides possible estimating for the distribution parameters without the need for the prior distribution of parameters from transforming the reliability function into a linear function that facilitate finding the posterior Bayes estimators of the parameters.
2. Form the empirical results, it is clear that the proposed estimation method is better than the approximate Bayes method and the MSE value of the estimator in both cases.

Acknowledgments

The research is supported by College of Computer Sciences and Mathematics, University of Mosul, Republic of Iraq.

References

[1] S.P. Ahmad and B.A. Bhat, “Posterior Estimates of Two-Parameter Exponential Distribution using S-PLUS Software,” *Journal of Reliability and Statistical Studies*, vol. 3, no. 2, pp. 27-34, 2010.

[2] R. Singh, S.K. Singh, U. Singh and G.P. Singh, "Bayes Estimator of Generalized-Exponential parameters under Linex Loss Function Using Lindley’s Approximation" , *Data Science Journal*, Vol. 7, 5 May 2008

[3] S.K. Upadhyay and U. Singh, “Bayes Estimators for Two-Parameter Exponential Distribution,” *Commun. Statist. Theor. Meth*., vol. 24, no. 1, pp. 227-240, 2007.

[4] K. Lam, B.K.Sinha and Z. Wu, “Estimation of Parameters in a Two-Parameter Exponential Distribution using Ranked Set Sample”, *Ann. Inst. Statist. Math*., vol. 46, no. 4, pp. 723-736, 1994.

[5] H.S.AL-Kutubi and N.A.Ibrahim, " Bayes Estimator for Exponential Distribution with Extension of Jeffery Prior Information" , *Malaysian Journal of Mathematical Sciences,* vol.3,no.2, pp.297-313, 2009.

[6] R. Arabi Belaghi, H. Bevrani and M. Mohammadi," The Analysis of the Two Parameter Exponential Distribution Based on Progressive Type II Censored Data", *Journal of the Turkish Statistical Association***,** vol. 8, no. 1-2, pp. 15–23, 2015.

[7] C. E. Ebeling, "An Introduction to Reliability and Maintainability Engineering" McGraw-Hill Book Company, 2003

[8] S. James Press , "Applied Multivariate Analysis: Using Bayesian and Frequentist Methods of Inference, Second Edition" Dover Publications, Florida, USA, 2005.

[9] D R S Saputro, F Amalia, P Widyaningsih and R C Affan, " Parameter estimation of multivariate multiple regression model using bayesian with non-informative Jeffreys’ priordistribution" IOP Conf. Series: *Journal of Physics*: Conf. Series 1022, 2018.

[10] G.E.P. Box and G. Tiao, “Bayesian Inference in Statistical Analysis,” Reading Addison-Wesley, New York., 1973.