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Abstract

We define a binary metric as a symmetric, distributive lattice ordered magma-valued function of two variables, satisfying a "triangle inequality". Using the notion of a Kuratowski topology, in which topologies are specified by closed sets rather than open sets, we prove that every topology is induced by a binary metric. We conclude with a discussion on the relation between binary metrics and some separation axioms.

Keywords: binary metric, generalized metric, partial metric
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1. Introduction

Topology generalizes the theory of metric spaces. In [**?**], quasi-metrics are used as the generalized metric. In [**?**], generalized metrics were explored whose codomain is an ordered commutative monoid, in [**?**] the codomain is an abelian ℓ-group, and in [**?**] the codomain is a value lattice which was chosen "to allow as many of the usual constructions as possible". By constructing continuity spaces [**?**], Kopperman showed that any topology may be induced by a generalized metric having values in an additive semigroup $(\Gamma, +)$. The paper [**?**] refined that work by showing that one can replace Γ by $\{0,1\}^{\mathcal{I}}$ for some indexing set $\mathcal{I}$. For any topological space $(X, \mathcal{T})$, the $\{0,1\}$–valued quasi-metric $\xi : X \times X \to \{0,1\}^{\mathcal{T}}$ was defined as follows: for each $U \in \mathcal{T}$ and for every $x, y \in X$,

$$\pi_U(\xi(x,y)) = \begin{cases} 1, & \text{if } x \in U \text{ and } y \notin U, \\ 0, & \text{otherwise.} \end{cases}$$

We note here that the terminology "quasi-metric" used in [**?**] matches what [**?** , Definition 6.1.1] calls a "hemi-metric" or what [**?**] calls a "quasi-pseudo-metric". Both of those references use "quasi-metric" to refer to a T_0 hemi-metric, i.e. one for which $d(x,y) = d(y,x) = 0$ implies $x = y$. Thus, the main result in [**?**] that all topologies may be represented as a "$\{0,1\}$-valued quasi-metric" is a generalization of Wilson's Theorem [**?** , Theorem 6.3.13] that all second-countable topological spaces are hemi-metrizable.

In this setting, one might think that a natural generalization of the metric open balls is given as follows: for every $x \in X$ and $\epsilon \in \{0,1\}^{\mathcal{I}}$,

$$B_\epsilon(x) = \{y \in X \big| \xi(x,y) < \epsilon\}.$$

Unfortunately, if we use this definition of open balls, any intersection of an arbitrary collection of open sets containing x can be written as an open ball around x. This would imply that every non $\mathcal{T}_0$ topology is discrete, which is obviously false. In the article [**?**], the authors did not use the above definition, but rather chose to use a definition in which $\epsilon = 0$ or $\epsilon = 1$, circumventing this issue.

In this paper, we define a generalized metric that is valued in a power, $\{0,1\}^{\mathcal{T}}$, of $\{0,1\}$. In doing so we switch our gaze from the conventional approach to topology, which focuses on the open sets of a space, to its equivalent, closure system based counterpart, which we call the Kuratowski topology. This view of topology originated with Kazimierz Kuratowski [**?**] using closure operators, and in some recent work, one finds that it is sometimes preferable, for example in the study of representation spaces [**?**]. Below is an equivalent definition.

Definition 1.1. *Consider a set X and let $\mathcal{C}$ be a family of subsets of X. We say that $\mathcal{C}$ is a Kuratowski topology on X if and only if*

1. $\{\emptyset, X\} \subseteq \mathcal{C}$,
2. *any intersection of elements of $\mathcal{C}$ is an element of $\mathcal{C}$, and*
3. *any finite union of elements of $\mathcal{C}$ is an element of $\mathcal{C}$,*

in which case, we call $(X, \mathcal{C})$ a Kuratowski space.
An element of the Kuratowski topology is called a closed set while an element whose complement is in the Kuratowski topology is called an open set.

Other authors [**?** , p.48] have used Kuratowski's definition of a topological space in terms of closed set systems as a starting point for investigation of lattices, called Kuratowski lattices, in pointless topology.

Definition 1.2. *Let $(X, \mathcal{C})$ be a Kuratowski space. We define*

$$\mathcal{T}_{\mathcal{C}} := \left\{X \setminus A \big| A \in \mathcal{C}\right\}.$$

It is easy to see that $(X, \mathcal{T}_{\mathcal{C}})$ is a conventional topological space. Every conventional topology has a basis of open sets; similarly the Kuratowski topology has a basis of closed sets, and we will refer to it as a closed basis of the Kuratowski topology.

Definition 1.3. *Consider a Kuratowski space $(X, \mathcal{C})$. A subset $\mathfrak{B}$ of $\mathcal{C}$ is called a closed basis of $\mathcal{C}$ if and only if for every $x \in X$,*

1. *there exists $B \in \mathfrak{B}$ not containing x,*
2. *for any $A \in \mathcal{C}$ not containing x, there exists $B \in \mathfrak{B}$ not containing x such that $A \subseteq B$.*

The elements of $\mathfrak{B}$ are called basic closed sets. We say that $\mathfrak{B}$ generates $\mathcal{C}$ since for any closed set A in $\mathcal{C}$, there is a collection basic closed sets $\{{}^jB | j \in \mathcal{J}\} \subseteq \mathfrak{B}$ such that

$$A = \bigcap_{j \in \mathcal{J}} {}^jB.$$

On the other hand, given any set X, a closed basis is a collection of subsets of X whose arbitrary intersection generates a Kuratowski topology on X.

Definition 1.4. *Consider a set X. A collection $\mathfrak{B}$ of subsets of X is called a closed basis on X if and only if for every $x \in X$,*

1. *there exists $B \in \mathfrak{B}$ not containing x,*
2. *for any $B_1, B_2 \in \mathfrak{B}$ not containing x, there exists $B \in \mathfrak{B}$ not containing x such that $B_1 \cup B_2 \subseteq B$.*

In practice, any collection $\mathcal{H}$ of subsets of X satisfying the first axiom is called a closed subbasis on X, and can be extended to a closed basis on X in the following manner

$$\mathfrak{B}_{\mathcal{H}} = \left\{ \bigcup_{i=1}^{n} {}^iH \middle| n \in \mathbb{N}, {}^iH \in \mathcal{H} \right\} \bigcup \{\emptyset, X\}.$$

In **??**, we introduce a lattice ordered magma, the algebra needed to work with our generalized metric having values in $\{0,1\}^{\mathcal{I}}$, thus providing the prerequisites for the study of binary metrics. In Section 3, we continue to lay the ground work for the study of binary metrics and their associated spaces. Specifically, we there define binary metrics and closed balls, and we present some of their fundamental properties. In **??**, we prove that any Kuratowski topology is induced by a binary metric, illustrating the robustness of the theory of binary metrics. In **??**, we define some separation axioms for Kuratowski spaces, characterize them in terms of binary metrics, and correlate them to their analogs in conventional topology.

2. Binary Lattice

Notation 2.1. *Let $\mathcal{I}$ be an indexing set. We denote*

$$\Gamma = \{-1, 0, 1\},\ \Gamma^{\mathcal{I}} = \{-1, 0, 1\}^{\mathcal{I}},\ \textit{and}\ \Gamma^{\mathcal{I}+} = \{0, 1\}^{\mathcal{I}}.$$

As we will see in **??**, in $\Gamma^{\mathcal{I}}$, $\Gamma^{\mathcal{I}+}$ is a natural analogue to a nonnegative cone. For completeness, we define

$$\Gamma^{\mathcal{I}-} = \{-1, 0\}^{\mathcal{I}}.$$

For any element $a \in \Gamma^{\mathcal{I}}$, $b \in \Gamma$,and $i \in \mathcal{I}$, we denote

1. $a_i = \pi_i(a)$ i.e. the projection of a along the i^{th} coordinate,
2. $-a = \prod_{i\in\mathcal{I}}(-a_i)$,
3. $\bar{b} = \prod_{i\in\mathcal{I}}(b)$, i.e.

$$-\bar{1} = \prod_{i\in\mathcal{I}}(-1),\ \bar{0} = \prod_{i\in\mathcal{I}}(0), \text{and } \bar{1} = \prod_{i\in\mathcal{I}}(1).$$

Definition 2.2. *For any a and b in $\Gamma^{\mathcal{I}}$, we define $\leq^{\mathcal{I}}$ as the product order on $\Gamma^{\mathcal{I}}$:*

$$a \leq^{\mathcal{I}} b \text{ if and only if for any index } i \in \mathcal{I}, a_i \leq b_i.$$

Definition 2.3. *We define a binary operation $\oplus$ on Γ by*

$\oplus$	-1	0	1
-1	-1	-1	0
0	-1	0	1
1	0	1	1

We denote $(a \oplus^{\mathcal{I}} b)_i = a_i \oplus b_i$.

The operation $\oplus$ will be our substitute for the addition of traditional metric spaces.

Definition 2.4. *We define a"subtraction" operation $\ominus^{\mathcal{I}}$ on $\Gamma^{\mathcal{I}}$ by*

$$a \ominus^{\mathcal{I}} b = a \oplus^{\mathcal{I}} (-b).$$

Working in $(\Gamma^{\mathcal{I}+}, \oplus^{\mathcal{I}}, \leq^{\mathcal{I}})$ is quite straightforward. We list some properties below:

Proposition 2.5. *Consider the lattice ordered magma $(\Gamma^{\mathcal{I}+}, \oplus^{\mathcal{I}}, \leq^{\mathcal{I}})$. If a, b, c, and d are elements of $\Gamma^{\mathcal{I}+}$, then*

1. *$a \oplus^{\mathcal{I}} a = a$,*
2. *for every $n \in \mathbb{N}$, $(n+1)a = a$; hence, by convention $0a = \bar{0}$,*
3. *$a \oplus^{\mathcal{I}} b = b \oplus^{\mathcal{I}} a$,*
4. *$a \oplus^{\mathcal{I}} \bar{0} = a$,*
5. *if $a \leq^{\mathcal{I}} b$ and $c \leq^{\mathcal{I}} d$ then $a \oplus^{\mathcal{I}} c \leq^{\mathcal{I}} b \oplus^{\mathcal{I}} d$,*

6. *if* $a \leq^{\mathcal{I}} b$ *then* $a \ominus^{\mathcal{I}} c \leq^{\mathcal{I}} b \ominus^{\mathcal{I}} c$*,*

7. $a \oplus^{\mathcal{I}} \bar{1} = \bar{1}$*,*

8. $(a \oplus^{\mathcal{I}} b) \oplus^{\mathcal{I}} c = a \oplus^{\mathcal{I}} (b \oplus^{\mathcal{I}} c)$*,*

9. $a \leq^{\mathcal{I}} b$ *if and only if* $a \oplus^{\mathcal{I}} b = b$*,*

10. $(a \ominus^{\mathcal{I}} b) \oplus^{\mathcal{I}} (c \ominus^{\mathcal{I}} d) = (a \ominus^{\mathcal{I}} d) \oplus^{\mathcal{I}} (c \ominus^{\mathcal{I}} b)$*,*

11. $(a \oplus^{\mathcal{I}} b) \ominus^{\mathcal{I}} c \leq^{\mathcal{I}} a \oplus^{\mathcal{I}} (b \ominus^{\mathcal{I}} c) = (a \ominus^{\mathcal{I}} c) \oplus^{\mathcal{I}} b$*,*

12. $\bar{1} = \max(\Gamma^{\mathcal{I}+})$ *and* $\bar{0} = \min(\Gamma^{\mathcal{I}+})$*; hence,* $\Gamma^{\mathcal{I}+}$ *is bounded,*

13. $a \oplus^{\mathcal{I}} b = \prod_{i \in \mathcal{I}} (a_i \oplus b_i) = \max\{a, b\}$ *making* $\Gamma^{\mathcal{I}+}$ *a lattice.*

Unfortunately, for $(\Gamma^{\mathcal{I}}, \oplus^{\mathcal{I}}, \leq^{\mathcal{I}})$, **??** (7)–(11) do not hold, and (12) and (13) are replaced by the following property.

Property 2.6. *If* a *and* b *are elements in* $(\Gamma^{\mathcal{I}}, \oplus^{\mathcal{I}}, \leq^{\mathcal{I}})$*, then*

1. $-(a \oplus^{\mathcal{I}} b) = (-a) \oplus^{\mathcal{I}} (-b)$*,*

2. $\bar{1} = \max(\Gamma^{\mathcal{I}})$ *and* $-\bar{1} = \min(\Gamma^{\mathcal{I}})$*; hence,* $\Gamma^{\mathcal{I}}$ *is bounded,*

3. $a \oplus^{\mathcal{I}} b = \prod_{i \in \mathcal{I}} (a_i \oplus b_i) = \prod_{i \in \mathcal{I}} sgn(a_i + b_i) \left\lceil \frac{|a_i + b_i|}{2} \right\rceil$.

As the following lemma shows, $(\Gamma^{\mathcal{I}}, \oplus^{\mathcal{I}})$ is not associative.

Lemma 2.7. *Let* a *be a maximum of an ordered magma* $(\Omega, *, \leq)$ *with identity* $e \neq a$*. If* $a^{-1} \in \Omega$ *then* $*$ *is not associative.*

Proof. The element a of Ω is the maximum, hence $e \leq a$. In an ordered magma, the operation is compatible with the order. Therefore,

$$e * a \leq a * a \Leftrightarrow a \leq a * a \Leftrightarrow a = a * a.$$

Now

$$(a^{-1} * a) * a = (e) * a = a,$$

whereas

$$a^{-1} * (a * a) = a^{-1} * (a) = e,$$

completing the proof. □

3. Binary Metric

The following definition is from [**?**].

Definition 3.1. *Consider a set X and an indexing set $\mathcal{I}$. For every $i \in \mathcal{I}$, let $g^i : X \times X \to \{0,1\}$. Then, g^i is said to be a $\{0,1\}-$valued generalized quasi-metric if it satisfies the following axioms: for all $x, y, z \in X$, for every $i \in \mathcal{I}$,*
(g−lbnd): $0 = g^i(x,x) \leq g^i(x,y)$,
(g−inq): $g^i(x,y) \leq g^i(x,z) + g^i(z,y)$.

As we will show in **??**, any Kuratowski topology can be induced by what we will call a binary metric.

Definition 3.2. *Consider a set X and an indexing set $\mathcal{I}$. Let $\xi : X \times X \to \Gamma^{\mathcal{I}+}$. Then, ξ is said to be a binary metric if it satisfies the following axioms: for all $x, y, z \in X$*
(ξ–lbnd): $\xi(x,x) \leq^{\mathcal{I}} \xi(x,y)$, *(also known as small self-distance),*
(ξ–sym): $\xi(x,y) = \xi(y,x)$, *and*
(ξ–inq): $\xi(x,y) \leq^{\mathcal{I}} \xi(x,z) \oplus^{\mathcal{I}} [\xi(z,y) \ominus^{\mathcal{I}} \xi(z,z)]$.

We note here that our binary metric could be described as a "generalized partial pseudo-metric" because it is not real-valued and obeys a weakening of the axioms of partial metrics appearing in [**?** , Definition 3.1]. We will retain the terminology "binary metric" because it is intuitive.

Remark 3.3. *By **??** (10),*

$$\xi(x,z) \oplus^{\mathcal{I}} [\xi(z,y) \ominus^{\mathcal{I}} \xi(z,z)] = [\xi(x,z) \ominus^{\mathcal{I}} \xi(z,z)] \oplus^{\mathcal{I}} \xi(z,y).$$

*Additionally, $\xi(x,x) = \bar{0}$ may seem as an important restriction. But, as shown in **??** below, it is not needed.*

Remark 3.4. *The reader may notice the lack of a separation axiom i.e. an axiom used to deduce $x = y$ by looking at $\xi(x,y)$. This is in fact intentional. One of the main purposes of this paper is to prove that any Kuratowski topology can be induced by a binary metric. If a topology is not $\mathcal{T}_0$, then there should be at least two distinct points x and y that are not topologically distinguishable from each other. Hence, the corresponding induced binary metric should not be able to distinguish them either.*

Definition 3.5. *Consider a set X and an indexing set $\mathcal{I}$. Let $\xi \colon X \times X \to \Gamma^{\mathcal{I}+}$ be a binary metric.*

1. *We say that ξ is a strong binary metric if and only if (ξ-lbnd) is replaced by a strict inequality.*

2. *We say that ξ is a separating binary metric if and only if it is a binary metric along with the extra axiom*
(ξ–sep): $\xi(x,x) = \xi(x,y) = \xi(y,y)$ *if and only if* $x = y$.

Clearly, any strong binary metric is a separating binary metric, and any separating binary metric is a binary metric. With extra conditions, we can achieve separation in binary metric spaces using the following more restrictive binary metrics, as we will show in **??**. Also, similar to earlier, a separating binary metric may be called a "generalized partial metric".

Given a set X and a Kuratowski topology $\mathcal{C}$ on X, we can define a binary metric on X by taking a closed basis $\mathfrak{B}$ of $\mathcal{C}$ as the indexing set.

Proposition 3.6. *Let $(X, \mathcal{C})$ be a Kuratowski space with a closed basis $\mathfrak{B}$. Define a function $\xi : X \times X \to \Gamma^{\mathcal{C}+}$ by: for every $x, y \in X$, for every closed set $A \in \mathfrak{B}$,*

$$\xi(x,y)_A = \begin{cases} 0 & \text{if } x \in A \text{ and } y \in A, \\ 1 & \text{otherwise.} \end{cases}$$

Then, ξ is a binary metric.

Proof. (<u>ξ−lbnd</u>): Let $A \in \mathfrak{B}$, if $\xi(x,x)_A = 1$ then $x \notin A$ i.e $\xi(x,y)_A = 1$ and, hence,

$$\xi(x,x) \leq^{\mathfrak{B}} \xi(x,y).$$

(<u>ξ−sym</u>): Is straightforward.
(<u>ξ−inq</u>): Let $x, y, z \in X$ and $A \in \mathfrak{B}$, if $\xi(x,y)_A = 1$ then either $x \in A$ and $y \notin A$, $x \notin A$ and $y \in A$, or $x \notin A$ and $y \notin A$.
Case 1: $x \in A$ and $y \notin A$
If $z \in A$

$$\xi(x,z)_A \oplus [\xi(z,y)_A \ominus \xi(z,z)_A] = 0 \oplus 1 = 1.$$

If $z \notin A$

$$\xi(x,z)_A \oplus [\xi(z,y)_A \ominus \xi(z,z)_A] = 1 \oplus 0 = 1.$$

Case 2: $x \notin A$ and $y \in A$
If $z \in A$

$$\xi(x,z)_A \oplus [\xi(z,y)_A \ominus \xi(z,z)_A] = 1 \oplus 0 = 1.$$

If $z \notin A$

$$\xi(x,z)_A \oplus [\xi(z,y)_A \ominus \xi(z,z)_A] = 1 \oplus 0 = 1.$$

Case 3: $x \notin A$ and $y \notin A$
If $z \in A$

$$\xi(x,z)_A \oplus [\xi(z,y)_A \ominus \xi(z,z)_A] = 1 \oplus 1 = 1.$$

If $z \notin A$

$$\xi(x,z)_A \oplus [\xi(z,y)_A \ominus \xi(z,z)_A] = 1 \oplus 0 = 1.$$

And, hence

$$\xi(x,y) \leq^{\mathfrak{B}} \xi(x,z) \oplus^{\mathfrak{B}} [\xi(z,y) \ominus \xi(z,z)],$$

completing the proof. □

Definition 3.7. *Let $(X, \mathcal{C})$ be a Kuratowski space with a closed basis $\mathfrak{B}$. The canonical binary metric determined by $\mathfrak{B}$ is the binary metric illustrated above in* **??**.

In [**?**], an equivalent version of the binary metric is given to refine the work done in [**?**]. The balls they proposed were open balls but defined on the projections of $\Gamma^{\mathcal{I}+}$ rather than $\Gamma^{\mathcal{I}+}$ itself. To be able to define a ball with radius in $\Gamma^{\mathcal{I}+}$, we needed to consider closed balls and, therefore, Kuratowski topologies. We start by defining $\xi-$closed balls.

Definition 3.8. *Let ξ be a binary metric on a set X with $\mathcal{I}$ as the indexing set. For every $x \in X$ and $\epsilon \in \Gamma^{\mathcal{I}+}$, the $\xi-$closed ball around x of radius ϵ is defined by*

$$B_\epsilon(x) = \{y \in X | \xi(x,y) \ominus^{\mathcal{I}} \xi(x,x) \leq^{\mathcal{I}} \epsilon\}.$$

We notice that due to a lack of separation axiom, when $\epsilon = \bar{0}$, the ball may contain elements other than x. Additionally, for any $\epsilon \in \Gamma^{\mathcal{I}+}$, $x \in B_\epsilon(x)$. By **??**, *the set of $\xi-$closed balls can be used as a subbasis i.e.*

$$\mathfrak{B} = \left\{ \bigcup_{i=1}^{n} B_{{}^i\epsilon}({}^i x) \middle| n \in \mathbb{N}, {}^i x \in X \text{ and } {}^i\epsilon \in \Gamma^{\mathcal{I}+} \right\} \bigcup \{\emptyset, X\}$$

forms a closed basis on X.

Lemma 3.9. *Let ξ be a binary metric on a set X with $\mathcal{I}$ as the indexing set. For every $x, y \in X$, for every $\epsilon \in \Gamma^{\mathcal{I}+}$,*

$$y \in B_\epsilon(x) \text{ if and only if } B_\epsilon(y) \subseteq B_\epsilon(x).$$

Proof. If $y \in B_\epsilon(x)$, i.e. $\xi(x,y) \ominus^{\mathcal{I}} \xi(x,x) \leq^{\mathcal{I}} \epsilon$, then for every $z \in B_\epsilon(y)$, i.e. $\xi(y,z) \ominus^{\mathcal{I}} \xi(y,y) \leq^{\mathcal{I}} \epsilon$ we have

$$\xi(x,z) \ominus^{\mathcal{I}} \xi(x,x)$$

by ($\xi-$inq)

$$\leq^{\mathcal{I}} \big(\xi(x,y) \oplus^{\mathcal{I}} [\xi(y,z) \ominus^{\mathcal{I}} \xi(y,y)]\big) \ominus^{\mathcal{I}} \xi(x,x)$$

by **??** (10)

$$\leq^{\mathcal{I}} \big([\xi(x,y) \ominus^{\mathcal{I}} \xi(x,x]) \oplus^{\mathcal{I}} [\xi(y,z) \ominus^{\mathcal{I}} \xi(y,y)]\big)$$

by **??** (8)

$$\leq^{\mathcal{I}} \epsilon \oplus^{\mathcal{I}} \epsilon = \epsilon$$

and, hence, $z \in B_\epsilon(x)$. The converse is trivial, completing the proof. □

4. Kuratowski Topology

Now that we have shown how to obtain a Kuratowski topology from a binary metric, we show that any Kuratowski topology $\mathcal{C}$ can be induced by the canonical binary metric determined by a closed basis $\mathfrak{B}$ of $\mathcal{C}$. Our approach is similar to [**?**] albeit, using closed sets and a different generalized metric. Also, the following result generalizes the results in [**?**] to arbitrary topological spaces.

Theorem 4.1. *Let $(X, \mathcal{C})$ be a Kuratowski space having $\mathfrak{B}$ as a closed basis. The canonical binary metric determined by $\mathfrak{B}$ induces $\mathcal{C}$.*

Proof. Let ξ be the canonical binary metric determined by $\mathfrak{B}$ as defined in **??**. We start by showing that every the $\xi-$closed ball $B_\epsilon(x)$ is in $\mathcal{C}$. For every $x \in X$ and $\epsilon \in \Gamma^{\mathfrak{B}+}$, let

$$\mathcal{H}_\epsilon(x) = \{A \in \mathfrak{B} \big| x \in A \text{ and } \epsilon_A = 0\}$$

and

$$\mathcal{K} = \bigcap_{A \in \mathcal{H}_\epsilon(x)} A.$$

Note that $\mathcal{K}$ is an element of $\mathcal{C}$. We claim that in fact, $\mathcal{K} = B_\epsilon(x)$. To prove this let $y \in B_\epsilon(x)$, i.e. $\xi(x, y) \leq^{\mathfrak{B}} \epsilon$. For every $A \in \mathcal{H}_\epsilon(x)$, we know that $x \in A$ and $0 \leq \xi(x, y)_A \leq^{\mathfrak{B}} \epsilon_A = 0$ therefore, $y \in A$ and hence,

$$y \in \bigcap_{A \in \mathcal{H}_\epsilon(x)} A$$

giving us

$$B_\epsilon(x) \subseteq \mathcal{K}.$$

Conversely, let $y \in \mathcal{K}$, and $D \in \mathfrak{B}$,

Case 1: If $\epsilon_D = 1$ then it is trivial that

$$\xi(x, y)_D \ominus \xi(x, x)_D \leq \epsilon_D.$$

Case 2: If $\epsilon_D = 0$ and $x \notin D$ then

$$\xi(x, y)_D \ominus \xi(x, x)_D = 1 \ominus 1 = 0 \leq \epsilon_D.$$

Case 3: If $\epsilon_D = 0$ and $x \in D$, then $D \in \mathcal{H}_\epsilon(x)$ and, hence, $y \in \mathcal{K} \subseteq D$ giving us that

$$\xi(x, y)_D = 0 \leq \epsilon_D.$$

Therefore,

$$\mathcal{K} \subseteq B_\epsilon(x)$$

and hence,

$$B_\epsilon(x) = \mathcal{K}.$$

To show that $\xi-$closed balls determine $\mathcal{C}$, it is not sufficient to merely show that every $\xi-$closed ball is closed. We will show that every closed set is a $\xi-$ closed ball.

Let D be a nontrivial closed set in $\mathcal{C}$. Because $\mathfrak{B}$ is a basis of $\mathcal{C}$, it follows that

$$D = \bigcap_{j \in \mathcal{J}} {}^j B \text{ such that, for all } j \in \mathcal{J} \subseteq \mathcal{I}, {}^j B \in \mathfrak{B}.$$

Let $x \in D$, we now define $\delta \in \Gamma^{\mathfrak{B}+}$ as: for every $A \in \mathfrak{B}$,

$$\delta_A = \begin{cases} 0 & A = {}^j B \text{ for some } j \in \mathcal{J}, \\ 1 & \text{otherwise.} \end{cases}$$

Then,

$$D = B_\delta(x).$$

Step 1: We need only consider the case where $\delta_A = 0$ i.e. $A = {}^j B$. For every $y \in D$, $x, y \in {}^j B$, so that $\xi(x,y)_{{}^j B} = 0 \leq \delta_{{}^j B} = 0$, we obtain

$$D \subseteq B_\delta(x).$$

Step 2: For every $y \in B_\delta(x)$, $\xi(x,y)_{{}^j B} \leq \delta_{{}^j B} = 0$ and hence, $y \in {}^j B$. Therefore,

$$B_\delta(x) = \bigcap_{j \in \mathcal{J}} {}^j B = D,$$

completing the proof. □

Notation 4.2. *Let ξ be a binary metric on a set X. We denote by $\mathfrak{B}_\xi$ the closed basis on X induced by ξ as in* **??**. *We denote by (X,ξ) the Kuratowski space on X generated by $\mathfrak{B}_\xi$.*

5. Separation Propositions

We now explore separation axioms from a Kuratowski perspective, translate them to binary metric language, and compare them to the standard separation axioms. To do that, we establish the following lemma.

Lemma 5.1. *Let (X,ξ) be a binary metric space. For any two distinct points x and y in X, $\xi(x,y) \neq \xi(x,x)$ if and only of there is a closed set A such that $x \in A$ and $y \notin A$.*

Proof. ($\Rightarrow$): We take A to be the $\xi-$closed ball

$$B_{0^{\mathcal{I}}}(x) = \left\{ z \in X \middle| \xi(x,z) \ominus^{\mathcal{I}} \xi(x,x) \leq^{\mathcal{I}} 0^{\mathcal{I}} \right\}.$$

Since $\xi(x,y) \neq \xi(x,x)$ and, by ($\xi - lbnd$), $\xi(x,x) \leq \xi(x,y)$, then there is an $i \in \mathcal{I}$ such that ${}^i\xi(x,x) = 0$ and ${}^i\xi(x,y) = 1$. Therefore, $\xi(x,y) \ominus^{\mathcal{I}} \xi(x,x) > 0^{\mathcal{I}}$ and hence, $y \notin B_{0^{\mathcal{I}}}(x)$.
($\Leftarrow$): If there exists a closed set A such that $x \notin A$ and $y \in A$, then by **??** and **??**, there exists a finite number of $\xi-$closed balls such that

$$x \in A \subseteq \bigcup_{i=1}^{n} B_{{}^i\epsilon}({}^i x) \text{ and } y \notin \bigcup_{i=1}^{n} B_{{}^i\epsilon}({}^i x).$$

Hence, there exists an $i \in \{1, ..., n\}$ such that

$$x \in B_{{}^i\epsilon}({}^i x) \text{ and } y \notin B_{{}^i\epsilon}({}^i x)$$

by **??**

$$x \in B_{^i\epsilon}(x) \subseteq B_{^i\epsilon}(^ix) \text{ and hence, } y \notin B_{^i\epsilon}(x)$$

therefore, $\xi(x, y) \ominus^{\mathcal{I}} \xi(x, x) > 0^{\mathcal{I}}$. □

The following definition is a Kuratowski space analogue of a topological space being $\mathcal{T}_0$.

Definition 5.2. *Let $(X, \mathcal{C})$ be a Kuratowski space. We say that $(X, \mathcal{C})$ is $\mathcal{K}_0$ if and only if for any two distinct elements x and y in X, there is a closed set A in $\mathcal{C}$ such that*

$$[x \in A \textit{ and } y \notin A] \textit{ or } [x \notin A \textit{ and } y \in A].$$

The following theorem characterizes the property of being $\mathcal{K}_0$ in terms of the binary metric.

Theorem 5.3. *A binary metric space (X, ξ) is $\mathcal{K}_0$ if and only if ξ is a separating binary metric i.e. for any two distinct elements x and y in X*

$$\xi(x, y) \neq \xi(x, x) \textit{ or } \xi(x, y) \neq \xi(y, y).$$

The next theorem demonstrates that Definition **??** gives a natural analogue of the $\mathcal{T}_0$ property and its proof follows directly from Theorem **??**.

Theorem 5.4. *A Kuratowski space $(X, \mathcal{C})$ is $\mathcal{K}_0$ if and only if $(X, \mathcal{T}_{\mathcal{C}})$ is $\mathcal{T}_0$.*

Similarly, we define a Kuratowski space analogue of being $\mathcal{T}_1$.

Definition 5.5. *Let $(X, \mathcal{C})$ be a Kuratowski space. We say that $(X, \mathcal{C})$ is $\mathcal{K}_1$ if and only if for any two distinct elements x and y in X, there is a closed set A in $\mathcal{C}$ such that*

$$[x \in A \textit{ and } y \notin A] \textit{ and } [x \notin A \textit{ and } y \in A].$$

We now characterize being $\mathcal{K}_1$ in terms of the binary metric.

Theorem 5.6. *A binary metric space (X, ξ) is $\mathcal{K}_1$ if and only if ξ is a strong binary metric, i.e. for any two distinct elements x and y in X,*

$$\xi(x, y) \neq \xi(x, x) \textit{ and } \xi(x, y) \neq \xi(y, y).$$

The next theorem demonstrates that Definition **??** gives a natural analogue to the $\mathcal{T}_1$ property and its proof follows directly from Theorem **??**.

Theorem 5.7. *A Kuratowski space $(X, \mathcal{C})$ is $\mathcal{K}_1$ if and only if $(X, \mathcal{T}_{\mathcal{C}})$ is $\mathcal{T}_1$.*

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6. Conclusion

We presented the definition of a binary metric as an alternative to that of a $\{0,1\}$-valued generalized quasi-metric introduced by Ercan and Vural in [**?**]. We have shown that any topology may be realized in a natural way using a binary metric. In addition, we elucidated the relationship between Kuratowski spaces, binary metric spaces, and topological spaces as presented traditionally. This includes characterizations of important separation properties in terms of binary metrics. Further work can investigate other separation axioms, characterizations of compactness, traditional metrizability, and topological completeness in terms of binary metrics.

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