Thermal effects and damping mechanisms in the forced radial oscillations of gas bubbles in liquids*

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A linearized theory of the forced radial oscillations of a gas bubble in a liquid is presented. Particular attention is devoted to the thermal effects. It is shown that both the effective polytropic exponent and the thermal damping constant are strongly dependent on the driving frequency. This dependence is illustrated with the aid of graphs and numerical tables which are applicable to any noncondensing gas–liquid combination. The particular case of an air bubble in water is also considered in detail.

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INTRODUCTION

In a recent work a new experimental technique was suggested which appears capable of permitting accurate measurements of the damping as well as other characteristics of the forced radial oscillations of a gas bubble in a liquid.[1] In the course of that study, it was realized that a clear and straightforward theory of this process sufficiently general to describe any noncondensing gas–liquid combination, and at the same time easily applicable to specific cases, was not available in the literature. For example, the frequently quoted paper by Devin,[2] based on the earlier work of Pfriem,[3] contains several approximations which, while affecting the applicability of the results obtained, can be very easily removed. In addition, the approach of these authors is based on an *ad hoc* Lagrangian formulation which tends to obscure the physical aspects of the process. For these reasons it appeared appropriate to reconsider the problem anew, and to give a straightforward treatment of the small-amplitude forced oscillations of gas bubbles based on the conservation equations of continuum mechanics. In analogy with previous studies,[2–5] our analysis is based on the linearized formulation of the problem. However, we dispense with the approximation of uniform pressure inside the bubble, which was used in Refs. 2 and 3. It is found that pressure nonuniformities may have interesting effects at very high frequency.

As is well known, the aspect of this problem that tends to complicate the analysis is the thermal interaction of the bubble with the surrounding liquid. This effect determines the magnitude of the natural frequency of the bubble or, alternatively, of the effective polytropic exponent which is often used in a pressure–volume relationship to describe the oscillations. The thermal damping is also very important, because it normally is the dominant mechanism of energy dissipation. A secondary objective of this work is to present results for both these quantities in an easily accessible form. To this end we make use of the fact that, to a very good approximation, it is possible to combine the driving frequency, the bubble radius, and the relevant physical properties of gas and liquid into two dimensionless quantities, which are then used for the construction of charts and tables. In addition, the particular case of an air bubble in water is considered in detail, and the functional dependence of the polytropic exponent and the damping on the driving frequency and the radius is shown.

We need not dwell here on the reasons of practical interest in bubble oscillations, which are amply documented in the references given below. It is sufficient to mention the fields of ultrasonic cavitation, sound propagation in the ocean, and the flow of bubbly liquids.

I. THE EQUATION OF MOTION FOR THE OSCILLATIONS

We consider a spherical gas bubble in an unbounded liquid, subject to a steady sound field of wavelength that is large compared with the bubble radius. Under these conditions the bubble is set in forced radial oscillations, whose governing equation can be readily obtained from the condition of continuity of the normal stress across the bubble interface

$$P_{in} - P_{ext} = 2\sigma/R + 4\mu\dot{R}/R \;, \qquad (1)$$

where P_{in}, P_{ext} are the pressures acting on the inner and outer sides of the interface, σ and μ are the surface tension and the viscosity of the liquid, and $\dot{R} = dR/dt$. Viscous effects of the gas contained in the bubble are neglected. In the linearized theory, the external pressure is given by the superposition of the contributions of the driving sound field and the acoustic radiation from the bubble

$$P_{ext} = P_0(1 + \epsilon e^{i\omega t}) + P_{rad} \,. \qquad (2)$$

Here P_0 is the average ambient pressure and ϵ is the dimensionless amplitude of the sound field whose frequency is ω. It is assumed that $\epsilon \ll 1$; the limitations posed by this assumption are briefly considered at the end of this section. The radiated pressure P_{rad} is related to the velocity potential in the liquid, χ, by

$$P_{rad} = -\rho_l \, \partial\chi(R, t)/\partial t \;, \qquad (3)$$

where ρ_l is the liquid density. With the aid of the result[6,7]

$$\chi(r, t) = -\dot{R}R_0^2 r^{-1}(1 + i\omega R_0/c)^{-1} \exp[-i\omega(r - R_0)/c] \,, \qquad (4)$$

where r is the radial coordinate measured from the center of the bubble, c the speed of sound in the liquid, and R_0 the equilibrium radius of the bubble, Eq. (2) becomes

$$P_{ext} = P_0(1+\epsilon e^{i\omega t}) + \rho_l R_0 \ddot{R}(1+i\omega R_0/c)^{-1} . \quad (5)$$

The value of the pressure P_{in} acting on the gas side of the bubble wall can only be obtained by solving the complete (linearized) fluid mechanical problem in the gas. For the time being, we assume that its deviation from the equilibrium value $P_{in,eq}$ is small so that, if we let

$$P_{in} = P_{in,eq} + P_0 p(R_0, t) , \quad (6)$$

the dimensionless quantity p is of order ϵ, $|p| \ll 1$. From Eq. (1) the well-known relation between the equilibrium values of pressure and radius follows immediately,

$$P_{in,eq} - P_0 = 2\sigma R_0^{-1} .$$

To treat the case of small-amplitude oscillations we linearize the right-hand side of Eq. (1) by letting

$$R = R_0(1+x) ,$$

with $|x| \ll 1$, and we neglect all powers of x and of its derivatives higher than the first. By Eqs. (5) and (6), Eq. (1) then becomes

$$\frac{\rho_l R_0^2 \ddot{x}}{1+i\omega R_0/c} + 4\mu\dot{x} - 2\sigma R_0^{-1} x = P_0[p(R_0,t) - \epsilon e^{i\omega t}] . \quad (7)$$

The quantity x may be taken to have the same time dependence as the sound field, so that $\dot{x} = i\omega x$, $\ddot{x} = -\omega^2 x$. With these substitutions, from Eq. (7) we obtain

$$x = -\alpha \frac{\epsilon e^{i\omega t} - p(R_0,t)}{-\omega^2(1+i\omega R_0/c)^{-1} - \alpha w + 4 i\nu\omega R_0^{-2}} , \quad (8)$$

where $\nu = \mu/\rho_l$ is the kinematic viscosity of the liquid and the notation

$$\alpha = P_0/\rho_l R_0^2, \quad w = 2\sigma/R_0 P_0$$

has been introduced.

The computation of the perturbation pressure $p(R_0, t)$ is lengthy but straightforward and is given in the last section. Let us only remark here that, in its derivation, the linearized equations of mass and momentum in the gas, and of energy both in the gas and in the liquid, are solved exactly with the appropriate boundary conditions. The momentum equation in the liquid is satisfied by Eq. (3) and the continuity equation by Eq. (4). The final result for x is

$$x = -[-\omega^2(1+i\omega R_0/c)^{-1} - \alpha w + 4 i\nu\omega R_0^{-2} + \omega^2(\rho_g/\rho_l)\varphi]^{-1} \epsilon\alpha e^{i\omega t} . \quad (9)$$

In this equation ρ_g is the equilibrium density of the gas and the complex quantity φ is a function of ω, R_0, and the physical properties of the liquid and the gas, and includes all thermal effects. Its discussion will be the object of Sec. II.

Rather than taking into account in detail the processes occurring in the bubble interior, it is often convenient to make use of a pressure–volume relationship of the polytropic type and to describe the thermal losses by means of an *ad hoc* dissipative term which may be thought of as arising from an effectively increased liquid viscosity. The dependence of these quantities on the parameters of the problem can be readily determined if one writes

$$P_{in} = P_{in,eq}(R_0/R)^{3\kappa} - 4\mu_{th}\dot{x} ,$$

where κ is the polytropic exponent and μ_{th} the additional effective "thermal" viscosity. Then, from Eq. (6),

$$P_0 p(R_0,t) = P_{in,eq}[(R_0/R)^{3\kappa} - 1] - 4\mu_{th}\dot{x} \simeq -3\kappa P_{in,eq} x - 4\mu_{th}\dot{x} .$$

Substitution of this expression into Eq. (7) and rearrangement of terms gives[8]

$$\ddot{x} + \left(4\frac{\mu+\mu_{th}}{\rho_l R_0^2} + \frac{\omega R_0/c}{1+(\omega R_0/c)^2}\omega\right)\dot{x} + \left(3\kappa\frac{P_{in,eq}}{\rho_l R_0^2} - \frac{2\sigma}{\rho_l R_0^3} + \frac{(\omega R_0/c)^2}{1+(\omega R_0/c)^2}\omega^2\right)x = -\epsilon\alpha e^{i\omega t} . \quad (10)$$

Comparing this equation with that of a damped harmonic oscillator,

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = -\epsilon\alpha e^{i\omega t} , \quad (11)$$

in which β is the damping constant and ω_0 the natural frequency, it is possible to identify the viscous, thermal, and acoustic contributions to β ($\beta = \beta_{vis} + \beta_{th} + \beta_{ac}$) as

$$\beta_{vis} = 2\mu/\rho_l R_0^2 ,$$

$$\beta_{th} = 2\mu_{th}/\rho_l R_0^2 ,$$

$$\beta_{ac} = \tfrac{1}{2}\omega(\omega R_0/c)[1+(\omega R_0/c)^2]^{-1} ,$$

and the natural frequency as

$$\omega_0^2 = 3\kappa(P_{in,eq}/\rho_l R_0^2) - 2\sigma/\rho_l R_0^3 + (\omega R_0/c)^2[1+(\omega R_0/c)^2]^{-1}\omega^2 . \quad (12)$$

If in this last equation the acoustic contribution is disregarded, a well-known expression for the natural frequency of the bubble is obtained, which, in terms of the ambient pressure P_0, may be written[5,9]

$$\omega_0^2 = 3\kappa P_0 \frac{1+2\sigma/P_0 R_0}{\rho_l R_0^2} - \frac{2\sigma}{\rho_l R_0^3} . \quad (13)$$

The acoustic contribution represents an addition to the stiffness of the system and owes its presence to the momentum imparted to the liquid.[10] Since c is of the order of 10^5 cm/sec, this effect tends to be important only for relatively large bubbles at high frequencies.

The unknown quantities μ_{th} and κ can now be determined by setting $\dot{x} = i\omega x$, $\ddot{x} = -\omega^2 x$ in Eq. (10), solving for x, and comparing with Eq. (9). The result, which will be discussed in the following section, is

$$\mu_{th} = \tfrac{1}{4}\omega\rho_g R_0^2 \,\mathrm{Im}\,\varphi , \quad (14)$$

$$\kappa = \tfrac{1}{3}(\omega^2\rho_g R_0^2/P_{in,eq})\,\mathrm{Re}\,\varphi . \quad (15)$$

The average energy E absorbed by the bubble during one cycle is given by

$$E = \frac{-\omega}{2\pi}\int_0^{2\pi/\omega} P_0(1+\epsilon e^{i\omega t}) 4\pi R^2 \dot{R}\, dt ,$$

and can be readily computed from Eq. (11) with the result

$$E = \omega^2(\epsilon P_0)^2\beta(4\pi R_0/\rho_l)[(\omega_0^2 - \omega^2)^2 + 4\beta^2\omega^2]^{-1} ,$$

which is formally the same as one would find for a harmonic oscillator of equivalent mass $4\pi R_0^3\rho_l$ subject to an external force $4\pi R_0^2\epsilon P_0 e^{i\omega t}$. One may define a cross section σ_a dividing E by the incident energy flux I, which for a plane wave is given by $I=(\epsilon P_0)^2/c\rho_l$ (Ref. 6. Sec. 64); the result is

$$\sigma_a = 4\pi\beta\omega^2 R_0 c\,[(\omega_0^2-\omega^2)^2+4\beta^2\omega^2]^{-1} .$$

Alternatively, the damping may be expressed in nondimensional form in the manner of Refs. 2 and 3, defining a loss angle δ by

$$\delta = 2\beta\omega/\omega_0^2 .$$

The most stringent assumption introduced in the present study is that of the smallness of the amplitude of the bubble-wall motion. It is interesting to discuss what limitation this poses on the dimensionless pressure amplitude ϵ. First of all, it should be noticed that, as was shown elsewhere,[11] for small bubbles the "effective" pressure amplitude ξ is strongly influenced by surface tension and is given by

$$\xi = (1+w)^{-1}\epsilon ,$$

and hence ξ, rather than ϵ, should be small. Setting an actual upper bound on ξ is rather difficult. In general, the amplitude A of the oscillations will be of the order of

$$A \sim \{[(\omega/\omega_0)^2-1]^2+4(\beta/\omega_0)^2\}^{-1/2}\xi ,$$

so that the error in our results should be within about 10%, provided that this quantity is smaller than ~ 0.3. The applicability of this estimate, however, is limited by the fact that a linearized theory cannot account for the harmonic and subharmonic components in the response. In the vicinity of the frequency regions corresponding to these resonances [i.e., for $\omega\sim(m/n)\omega_0$, with m and n integers] the amplitude of the motion may be significantly greater than A as defined above, in spite of the fact that, formally, it is of order ξ^{m+n}. From the results of Ref. 11 we may infer that for $\beta \gtrsim 0.1$ and $\xi \lesssim 0.2$ our expressions are accurate within 10%–15%, even near the first harmonic ($\omega\sim\frac{1}{2}\omega_0$). The error decreases with increasing β and/or decreasing ξ.

II. THERMAL EFFECTS

As can be seen from Eqs. (14) and (15) the complex quantity φ describes all the thermal effects associated to small-amplitude bubble oscillations. The quantities on which it is found to depend most strongly can be grouped into two dimensionless parameters defined as

$$G_1 = MD_g\omega/\gamma R_g T_\infty ,$$

$$G_2 = \omega R_0^2/D_g ,$$

where M is the molecular weight of the gas contained in the bubble, $D_g = k_g/\rho_g c_{v,g}$ is its thermal diffusivity, $\gamma = c_{p,g}/c_{v,g}$ is the ratio of its specific heats, R_g is the universal gas constant, and T_∞ is the liquid temperature.

The physical meaning of G_1 can be made clear by noting that the speed of sound in the gas is given by $c_g^2 = \gamma(R_g/M)T_\infty$, and that the penetration depth of the thermal effects below the bubble surface in one cycle is of the order of $(D_g/\omega)^{1/2}$. G_1 is therefore essentially the square of the ratio between the thickness of the layer in which conduction causes significant temperature changes and the wavelength of the sound in the gas:

$$G_1 = \left[\frac{(D_g/\omega)^{1/2}}{c_g/\omega}\right]^2 .$$

Alternatively, since $D_g\sim\bar\lambda c_g$, where $\bar\lambda$ is the mean free path of the molecules in the gas,[12] we see that

$$G_1 \sim \frac{\bar\lambda}{c_g/\omega} ,$$

i.e., that G_1 is of the order of magnitude of the ratio between the mean free path and the wavelength, so that it can be expected to be a very small quantity for any liquid-gas combination under a very wide set of conditions. The parameter G_2 is essentially the square of the ratio between the bubble radius and the thermal penetration depth.

Additional quantities that enter in the expression for φ are the parameters

$$G_3 = \omega R_0^2/D_l ,$$

where $D_l = k_l/\rho_l c_{v,l}$ is the thermal diffusivity of the liquid, and

$$k = k_l/k_g$$

is the ratio between the liquid and the gas thermal conductivities. It is shown in Sec. IV below that the explicit expression for φ is

$$\varphi = \frac{kf(\Gamma_2-\Gamma_1)+\lambda_2\Gamma_2-\lambda_1\Gamma_1}{kf(\lambda_2\Gamma_1-\lambda_1\Gamma_2)-\lambda_1\lambda_2(\Gamma_2-\Gamma_1)} , \qquad (16)$$

where[13]

$$\begin{aligned} &\Gamma_{1,2} = i + G_1 \pm [(i-G_1)^2+4iG_1/\gamma]^{1/2} ,\\ &\lambda_i = \beta_i \coth\beta_i - 1, \quad i=1,2 ,\\ &\beta_{1,2} = (\tfrac{1}{2}\gamma G_2\{i-G_1\pm[(i-G_1)^2+4iG_1/\gamma]^{1/2}\})^{1/2} ,\\ &f = 1+(1+i)(\tfrac{1}{2}G_3)^{1/2} . \end{aligned} \qquad (17)$$

In view of the smallness of G_1 the above expressions can be approximated as follows:

$$\begin{aligned} &\Gamma_1 = 2(i+G_1/\gamma)+O(G_1^2) ,\\ &\Gamma_2 = 2[(\gamma-1)/\gamma]G_1+O(G_1^2) ,\\ &\beta_1 = (1+i)(\tfrac{1}{2}\gamma G_2)^{1/2}\{1+\tfrac{1}{2}i[(\gamma-1)/\gamma]G_1+O(G_1^2)\} ,\\ &\beta_2 = (G_1G_2)^{1/2}\{i+\tfrac{1}{2}[(\gamma-1)/\gamma]G_1+O(G_1^2)\} . \end{aligned} \qquad (18)$$

The absolute value of the quantity kf is in most cases a large number, both because the thermal conductivity of the liquid is much larger than that of the gas, and the thermal diffusion length in the liquid $(D_l/\omega)^{1/2}$ is small compared to the radius in the range of frequencies and radii of interest. Hence one can approximate Eq. (16) by

$$\varphi = \Phi[1+(kf)^{-1}E_1+O(kf)^{-2}] ,$$

where

$$\Phi = \frac{\Gamma_1-\Gamma_2}{\lambda_1\Gamma_2-\lambda_2\Gamma_1} , \qquad (19)$$

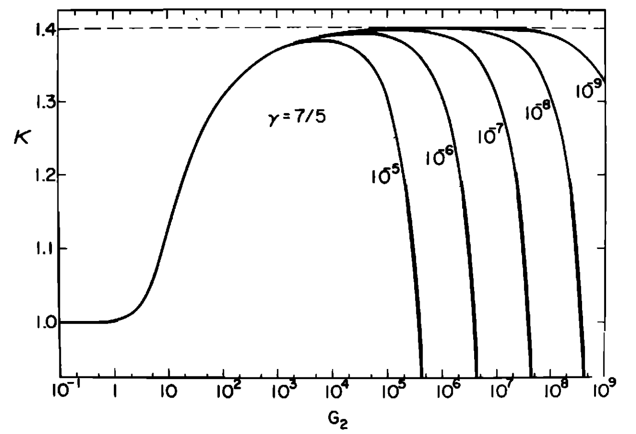

FIG. 1. The effective polytropic exponent κ, Eq. (21), as a function of the parameter G_2 for a diatomic gas ($\gamma=\frac{7}{5}$). The numbers labeling the curves are the corresponding values of G_1.

$$E_1=\frac{\Gamma_1\Gamma_2}{\Gamma_1-\Gamma_2}\frac{(\lambda_1-\lambda_2)^2}{\lambda_1\Gamma_2-\lambda_2\Gamma_1}\ . \tag{20}$$

In view of the complex analytical expression of the function φ, it is perhaps of some interest to present in graphical form the information contained in Eq. (16). For simplicity, we consider only the zero-order approximation Φ and the first-order correction E_1, since both these quantities depend only on the parameters G_1 and G_2. The specific case of an ~~air bubble in~~ water will be considered in Sec. III.

From Eq. (15) it is seen that the polytropic exponent is given by

$$\kappa=\tfrac{1}{3}\gamma G_1G_2\ \mathrm{Re}\varphi\ ,$$

or, approximately,

$$\kappa\simeq\tfrac{1}{3}\gamma G_1G_2\ \mathrm{Re}\Phi\ . \tag{21}$$

This relation is illustrated in Figs. 1 and 2 for fixed values of G_1 for the case of a diatomic gas ($\gamma=\frac{7}{5}$) and of a monatomic gas ($\gamma=\frac{5}{3}$), respectively. When the bubble radius is smaller than the thermal diffusion length it is seen that the thermal behavior is isothermal for small values of G_2. This result can be easily established in general for the complete function φ by taking the limit $G_1\to 0$, $G_2\to 0$ in Eq. (16). For larger values of G_2 the polytropic exponent increases toward the adiabatic limit $\kappa=\gamma$,[14] which is attained provided that G_1 is not too large; over a large range of values of G_2 this behavior is practically independent of G_1. For still greater values of G_2, κ decreases very rapidly and (although not shown in the figures) actually starts oscillating between positive and negative values.

These characteristics can be readily explained by a consideration of the pressure distribution within the bubble. From the results of Sec. IV below, it is seen that one has the following perturbation pressure distribution within the bubble:

$$p(r,t)\simeq\epsilon e^{i\omega t}(R_0/r)A_2\{\sinh(\beta_2 r/R_0)+[(\gamma-1)/\gamma]G_1\exp[-(\tfrac{1}{2}\gamma G_2)^{1/2}]A_0\sinh(\beta_1 r/R_0)\}\ , \tag{22}$$

where A_0 and A_2 are two complex quantities of order one. Clearly, the second term is normally much smaller than the first one, so that for $|\beta_2|$ small [i.e., approximately $(G_1G_2)^{1/2}$ small, see Eq. (18)], the pressure distribution is very nearly independent of r. Notice that, according to the interpretation of G_1 and G_2 given above, one has

$$(G_1G_2)^{1/2}\sim\frac{R_0}{c_g/\omega}\ , \tag{23}$$

so that the pressure is practically uniform in the bubble as long as the wavelength in the gas is much larger than the bubble radius. In this situation the value of the polytropic exponent depends only on the thermal behavior of the bubble as a whole. We find an isothermal pressure–volume relationship if the thermal diffusion length is of the order of the bubble radius or larger (i.e., $G_2<1$), and an adiabatic one if the thermal diffusion length is smaller than the radius; this behavior will also be very nearly independent of G_1. However, when G_1, and hence ω, are so large that pressure nonuniformities in the bubble are important, the effective pressure–volume relationship experienced by the bubble wall in its motion is the one pertaining to the *local* pressure distribution in the vicinity of $r=R_0$. It is clear from Eq. (22) that the pressure distribution corresponds to a standing wave, so that $p(R_0,t)$ may be positive as well as negative during a volume contraction, depending on the spacial phase of this wave, i.e., essentially, on $|\beta_2|$. In the former case one has a "normal" pressure–volume relationship, in the sense that the pressure increases with decreasing volume and κ is positive. In the latter case, however, the pressure decreases with decreasing volume, and $\kappa<0$. Since $|\beta_2|$ increases without bound as the frequency increases, it is seen that there is no asymptotic limit to κ as $\omega\to\infty$. It is clear, however, that if κ is negative, the oscillatory regime under investigation is unstable, and hence the model adopted in the present study is invalid. It appears likely that the steady state corresponding to this situation would be one of large-amplitude oscillations, which, however, may be impossible to observe because of the instability of the spherical shape. From Eq. (23) we may estimate that these abnormal situations begin to be possible when $\omega R_0\sim c_g$, and hence may

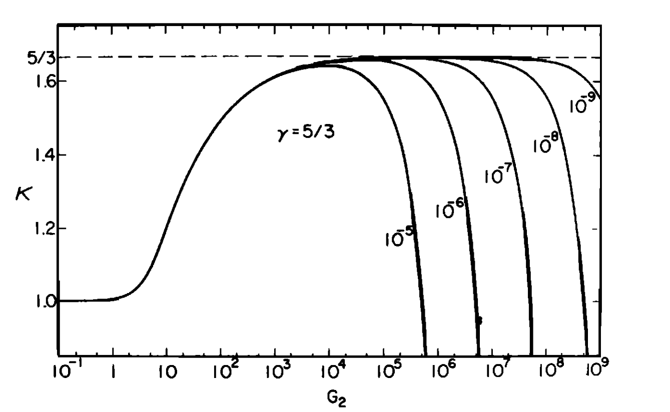

FIG. 2. The effective polytropic exponent κ, Eq. (21), as a function of the parameter G_2 for a monatomic gas ($\gamma=\frac{5}{3}$). The numbers labeling the curves are the corresponding values of G_1.

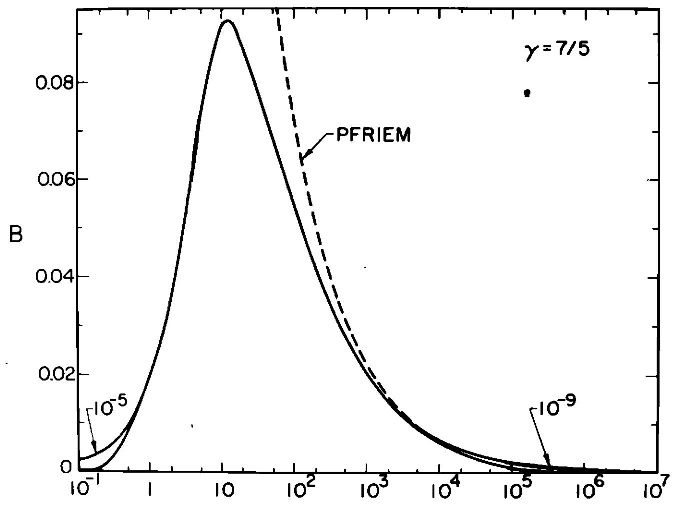

FIG. 3. The nondimensional thermal damping constant B, Eq. (25), as a function of the parameter G_2 for a diatomic gas ($\gamma=\frac{7}{5}$). The two curves correspond to the values $G_1=10^{-9}$ and $G_1=10^{-5}$. The dashed line is Pfriem's result, Eq. (26).

be observed in a frequency domain in which acoustic effects in the liquid are still small.

Clearly, when the internal pressure nonuniformity becomes appreciable, the polytropic exponent κ loses its thermal meaning, but retains a mechanical one in the sense that it determines the pressure acting on the bubble wall during the oscillations, i.e., the effective compressibility of the bubble. Therefore there is no contradiction between the present findings and the fact that, as discussed by Plesset,[15] the high-frequency oscillations are isothermal. Indeed, this statement refers to the thermal behavior of the thermodynamic system constituted by the bubble as a whole, rather than to the pressure–volume relationship. We may illustrate this

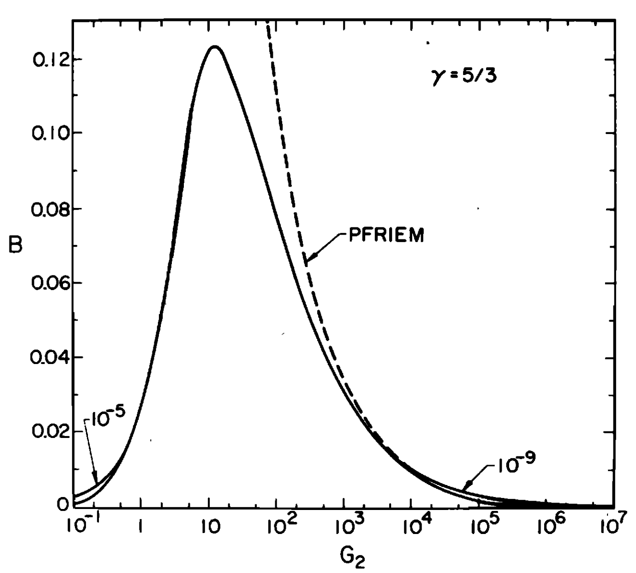

FIG. 4. The nondimensional thermal damping constant B, Eq. (25), as a function of the parameter G_2 for a monatomic gas ($\gamma=\frac{5}{3}$). The two curves correspond to the values $G_1=10^{-9}$ and $G_1=10^{-5}$. The dashed line is Pfriem's result, Eq. (26).

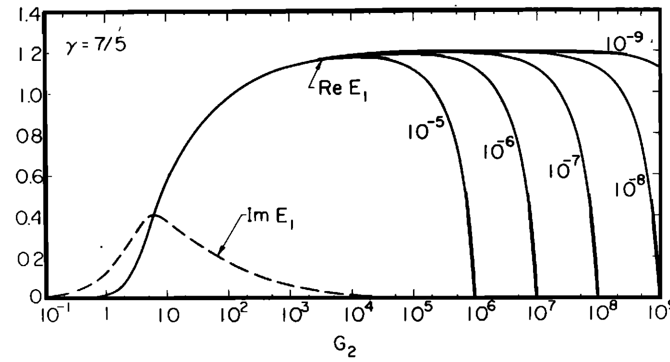

FIG. 5. The real part (full lines) and the imaginary part (broken line) of the first order correction E_1, Eq. (20), for a diatomic gas as a function of the parameter G_2. The numbers along the curves of $\mathrm{Re}E_1$ are the corresponding values of G_1. The values of $\mathrm{Im}\,E_1$ corresponding to different values of G_1 are indistinguishable within the thickness of the line.

overall thermodynamic behavior by noting that the instantaneous increase in the internal energy of the bubble is given by

$$\Delta U_b(t)=4\pi c_{v,g}\int_0^R [T(r,t)-T_\infty]r^2\,dr\ . \tag{24}$$

The temperature field in the bubble is also associated with a standing wave. As the frequency increases, the number of wavelengths contained in the bubble increases without bound, so that the regions where $T(r,t)>T_\infty$ tend to compensate those where $T(r,t)<T_\infty$, and the integral in (24) becomes very small.[16]

The imaginary part of Φ, which determines the thermal dissipation, is shown in Figs. 3 and 4. Actually, to avoid the necessity of an excessive number of orders of magnitude on the ordinate scale, the quantity plotted is

$$B=\tfrac{1}{3}G_1G_2\,\mathrm{Im}\Phi\ , \tag{25}$$

which is essentially the dimensionless thermal damping constant[17] $\delta_{th}=2\beta_{th}\omega/\omega_0^2$ defined by Pfriem[3] and Devin.[2] In both these figures the only slight dependence on the parameter G_1 should be noticed. A more detailed illustration of the characteristics of the thermal damping will be made in the following section.

Notice that for very large G_2 and $G_1\sim 0$, one readily obtains from Eq. (19)

$$B\sim 3(\gamma-1)(2\gamma G_2)^{-1/2}\ , \tag{26}$$

a result first given by Pfriem.[3] This curve is also plotted in Figs. 3 and 4, and it is seen to be in agreement with the exact one for G_2 very large and $G_1=10^{-9}$. However, since this result holds only for a bubble with a uniform internal pressure, one expects it to be in error for greater values of G_1, as indeed is seen in the figures.

Finally, in Figs. 5 and 6 we plot the first-order correction E_1, Eq. (20). It is seen that in the range of parameters of concern in practice, this quantity is at most of order 1, so that the error in using the approximate function Φ instead of the complete result φ is of order $|kf|^{-1}$, and hence generally small. For instance, for

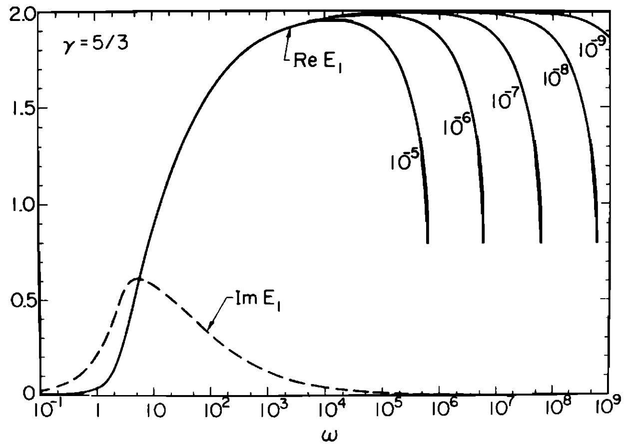

FIG. 6. The real part (full lines) and the imaginary part (broken line) of the first order correction E_1, Eq. (20), for a diatomic gas as a function of the parameter G_2. The numbers along the curves of $\mathrm{Re}\,E_1$ are the corresponding values of G_1. The values of $\mathrm{Im}\,E_1$ corresponding to different values of G_1 are indistinguishable within the thickness of the line.

an air bubble in water at 20 °C one has $kf \simeq 23.2[(1+i)8.6G_3^{1/2}+1]$.

III. EXAMPLE: AIR BUBBLES IN WATER

The characteristics of the various effects discussed in the previous sections can be better appreciated in a specific example. In view of its practical interest we consider here the case of an air bubble in water.[18] Figures 7–11 show the dimensional damping constant β as a function of the sound frequency for bubble radii of 1, 10^{-1}, 10^{-2}, 10^{-3}, and 10^{-4} cm, respectively. The open circle on the curves marks the damping at the resonant frequency, which coincides with the values given by Chapman and Plesset.[5] It is seen that, except for the smallest value of the radius, β_{th} is the dominant contribution at the lower frequencies, but that it steadily decreases and becomes less important as ω increases. These features may seem inconsistent with the fact

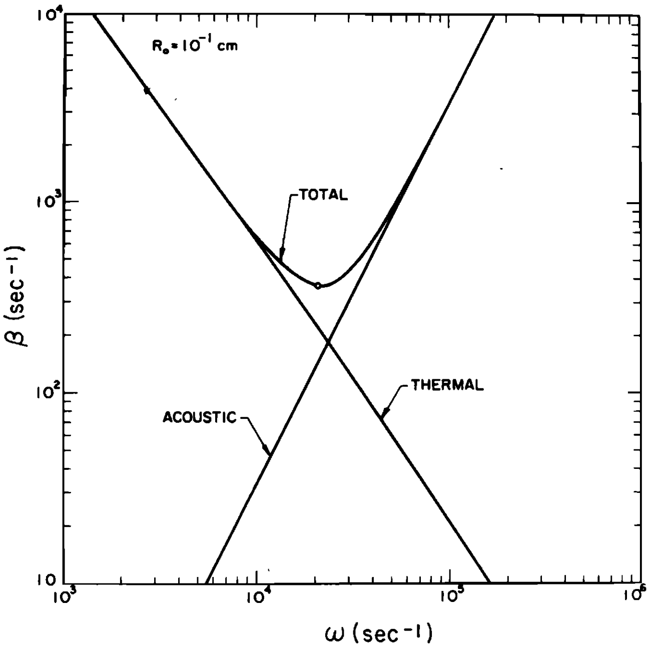

FIG. 8. The total dimensional damping constant β for $R_0=10^{-1}$ cm as a function of frequency ω. The thermal and acoustic contributions are shown. Not shown is the viscous one which has the constant value $\beta_{vis}=2$. The open circle marks the value of the damping constant at resonance.

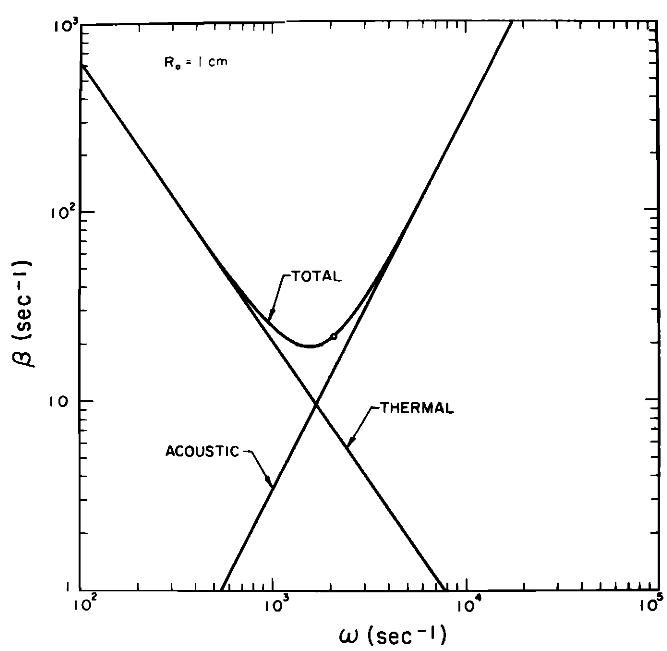

FIG. 7. The total dimensional damping constant β for $R_0=1$ cm as a function of frequency ω. The thermal and acoustic contributions are shown. Not shown is the viscous one which has the constant value $\beta_{vis}=0.02$. The open circle marks the value of the damping constant at resonance.

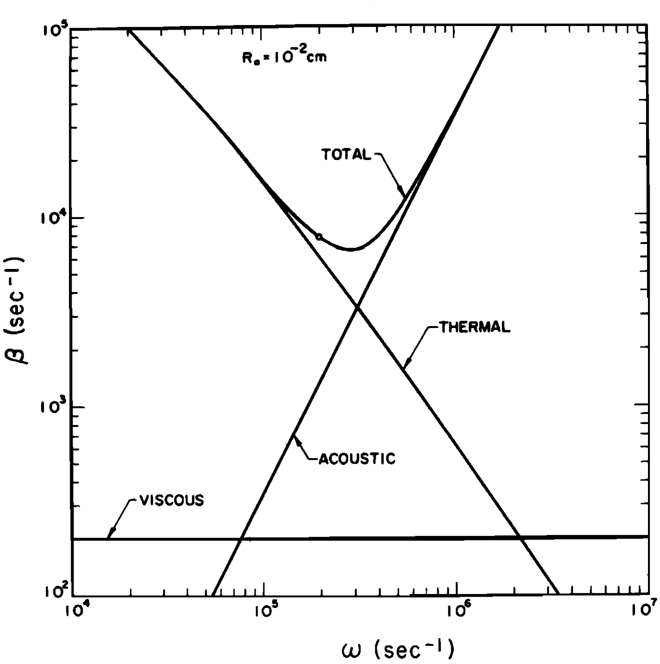

FIG. 9. The total dimensional damping constant β for $R_0=10^{-2}$ cm as a function of frequency ω. The thermal and acoustic contributions are shown. The open circle marks the value of the damping constant at resonance.

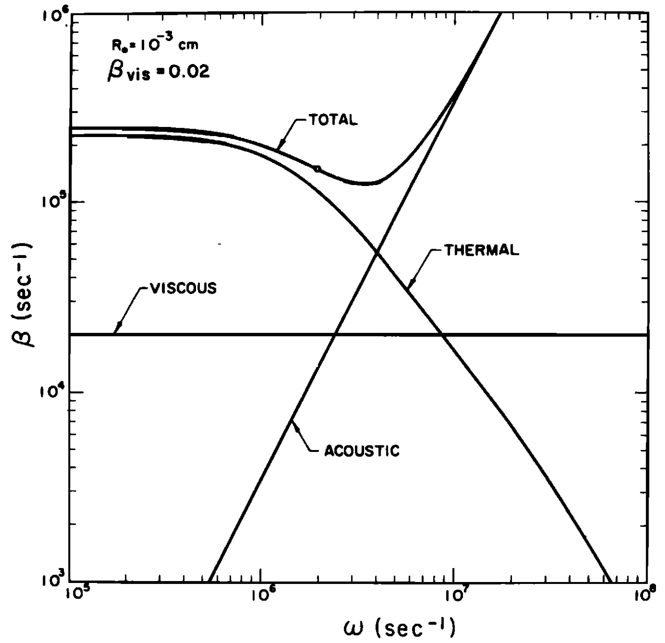

FIG. 10. The total dimensional damping constant β for $R_0 = 10^{-3}$ cm as a function of frequency ω. The thermal, acoustic, and viscous contributions are shown. The open circle marks the value of the damping constant at resonance.

stated above that both the low- and the high-frequency oscillations are essentially isothermal. Actually, one must realize that at low frequency the behavior is isothermal because the thermal energy associated to compressions and expansions of the bubble is conducted away by the liquid at nearly the same rate at which it is produced. On the other hand, at high frequency the bubble oscillates isothermally because, as was discussed in connection with Eq. (24), the *net* internal energy variations are small.

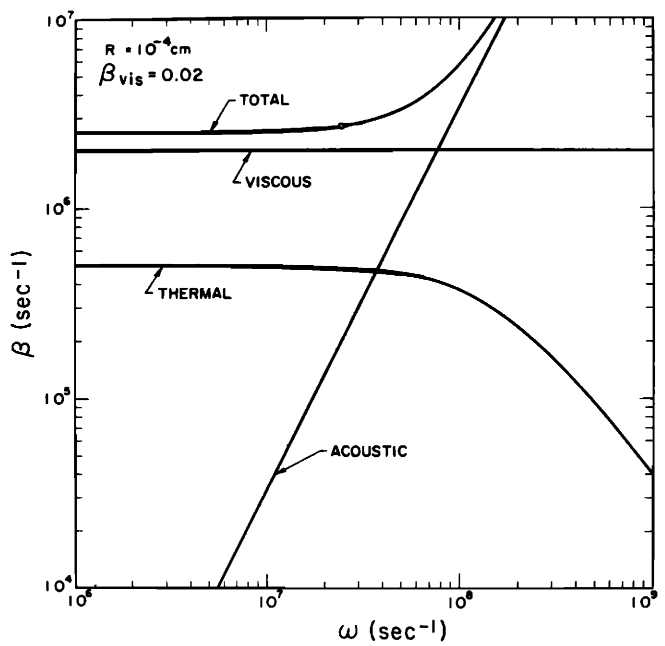

FIG. 11. The total dimensional damping constant β for $R_0 = 10^{-4}$ cm as a function of frequency ω. The thermal, acoustic, and viscous contributions are shown. The open circle marks the value of the damping constant at resonance.

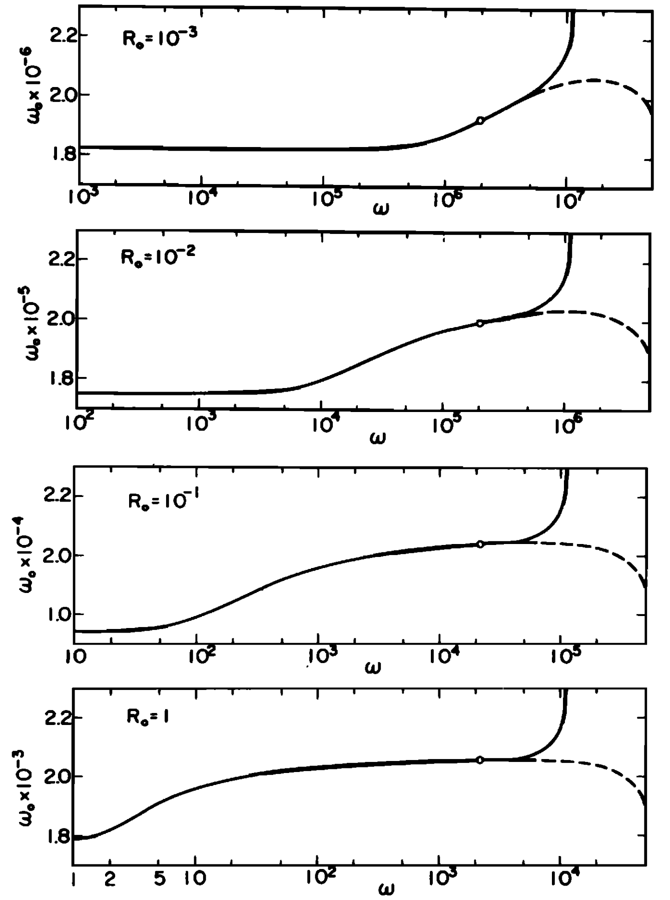

FIG. 12. The effective resonant frequency of the bubble ω_0 as a function of the impressed frequency ω, Eq. (12). The broken line represents the purely thermal contribution, Eq. (13).

The acoustic contribution β_{ac} dominates at high frequency. It appears from the figures that the two contributions from β_{th} and β_{ac} balance near the resonant frequency of the bubble.[19] This fact is a coincidence peculiar to the system investigated, as is clear from the fact that, for fixed ω and R_0, the thermal dissipation depends primarily on the parameters of the gas (i.e., G_1 and G_2, see Sec. II), whereas the acoustic one is determined by the sound velocity in the liquid. The viscous damping β_{vis} is independent of the frequency and, as is well known, its importance increases rapidly with decreasing radius.

Thermal and acoustic processes affect also the natural frequency of the bubble, which thus becomes a function of ω. Figure 12, which is a plot of Eq. (12), illustrates this relationship. The isothermal behavior at low frequencies lasts longer for the smaller bubbles reflecting the proportionality of G_2 to ωR_0^2. At a frequency of about ten times the resonant frequency, acoustic effects become important and the natural frequency increases very sharply. At the same time, the effective polytropic exponent starts decreasing, with a corresponding decrease in the purely thermal contribution to the natural frequency defined by Eq. (13) and indicated in the figure by a broken line.

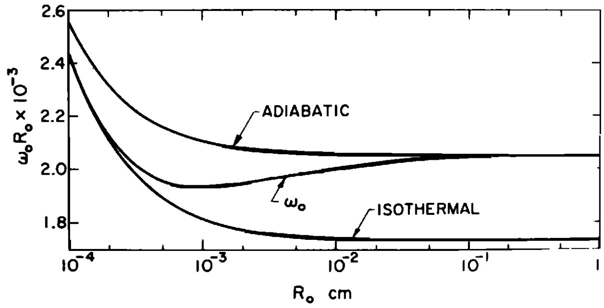

FIG. 13. The quantity $\omega_0 R_0$ as a function of the equilibrium radius R_0 for an air bubble in water. The curves corresponding to purely isothermal ($\kappa = 1$) and purely adiabatic ($\kappa = \frac{7}{5}$) oscillations are also shown.

Finally, in Fig. 13 the value at resonance of the quantity $\omega_0 R_0$ (i.e., the natural frequency normalized by the equilibrium radius) is plotted against the equilibrium radius. For comparison, the natural frequency corresponding to purely isothermal ($\kappa = 1$) and purely adiabatic ($\kappa = \frac{7}{5}$) oscillations is also shown.

IV. THE BUBBLE INTERIOR[20]

Let us denote by ρ, P, and T_g the local values of density, pressure, and temperature in the gas, and by T_l the temperature in the liquid. We assume that the deviation of these quantities from their equilibrium value is small, and we let

$$\rho = \rho_g(1+\eta), \qquad P = P_0(1+w+p),$$
$$T_g = T_\infty(1+\theta_g), \quad T_l = T_\infty(1+\theta_l).$$

All second- and higher-order terms in η, p, θ_g, and θ_l will be neglected. The equation of state of the gas $P = (R_g/M)\rho T$ becomes then

$$p = (1+w)(\theta_g + \eta), \tag{27}$$

and the equations of conservation of mass and momentum in the gas can be written

$$\frac{\partial \eta}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 u)}{\partial r} = 0, \tag{28}$$

$$\frac{\partial u}{\partial t} + \frac{P_0}{\rho_g}\frac{\partial p}{\partial r} = 0, \tag{29}$$

where u is the gas velocity. The equations of conservation of energy in the gas and in the liquid are

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta_g}{\partial r}\right) + \frac{P_0(1+w)}{k_g T_\infty}\frac{\partial \eta}{\partial t} = \frac{1}{D_g}\frac{\partial \theta_g}{\partial t}, \tag{30}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \theta_l}{\partial r}\right) = \frac{1}{D_l}\frac{\partial \theta_l}{\partial t}. \tag{31}$$

The boundary conditions associated to this system of equations are those of continuity of temperature and heat flux at $r = R_0$,

$$\theta_l(R_0, t) = \theta_g(R_0, t), \tag{32a}$$

$$-k_l \frac{\partial \theta_l}{\partial r}(R_0, t) = -k_g \frac{\partial \theta_g}{\partial r}(R_0, t), \tag{32b}$$

and the kinematical condition

$$u(R_0, t) = dR/dt. \tag{32c}$$

For this purely radial motion the tangential stresses vanish identically, so that the requirement of their continuity is trivially satisfied. The condition on the normal stresses, Eq. (1), has already been imposed.

It is now assumed that the time dependence of the various quantities is that of the sound field, so that the operator $\partial/\partial t$ can be replaced by $i\omega$. In this way combining Eqs. (28) and (29) we have

$$\eta = -\frac{P_0}{\rho_g \omega^2 r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p}{\partial r}\right). \tag{33}$$

Furthermore, expressing θ_g in terms of the equation of state [Eq. (27)], from Eqs. (30) and (33) we obtain

$$\frac{\partial^4 (rp)}{\partial r^4} + \left[\frac{\rho_g \omega^2}{P_0(1+w)} - i\gamma \frac{\omega}{D_g}\right]\frac{\partial^2 (rp)}{\partial r^2} - i\frac{\rho_g \omega^3}{D_g P_0 (1+w)}(rp) = 0,$$

where the relation $\gamma = 1 + P_0(1+w)D_g/k_g T_\infty$ has been used. The solution of this equation which is finite at the origin can be expressed as

$$p = (R_0/r)[A_1 \sinh(\beta_1 r/R_0) + A_2 \sinh(\beta_2 r/R_0)]\epsilon\, e^{i\omega t}, \tag{34}$$

where β_1 and β_2 are given by Eq. (17) and A_1, A_2 are complex constants to be determined from the boundary conditions. From Eq. (34) expressions for η, θ_g, and u are readily derived and will not be written down. The relevant solution of Eq. (31) bounded at infinity is

$$\theta_l = A_3(R_0/r)\exp[-(1+i)(\tfrac{1}{2}G_2)^{1/2} r/R_0]\epsilon\, e^{i\omega t}, \tag{35}$$

with A_3 another complex constant. The three boundary conditions Eq. (32) are now used to determine A_1, A_2, and A_3. The result is

$$A_1 = -\Gamma_2(\lambda_2 + kf)/(\Delta \sinh\beta_1), \tag{36a}$$

$$A_2 = \Gamma_1(\lambda_1 + kf)/(\Delta \sinh\beta_2), \tag{36b}$$

$$A_3 = \Gamma_1\Gamma_2(\lambda_1 - \lambda_2)\exp[(1+i)(\tfrac{1}{2}G_2)^{1/2}]/[G_1(1+w)\Delta],$$

where the various symbols have the same meaning as in Sec. II, and

$$\Delta = \Gamma_1\lambda_1 - \Gamma_2\lambda_2 + kf(\Gamma_1 - \Gamma_2)$$
$$+ \omega^{-2}(\rho_l/\rho_g)[(\Gamma_1 - \Gamma_2)\lambda_1\lambda_2 + kf(\lambda_2\Gamma_1 - \lambda_1\Gamma_2)]$$
$$\times[\omega^2(1 + i\omega R_0/c)^{-1} + \alpha w - 4\nu i\omega R_0^{-2}].$$

With the aid of Eqs. (34) and (36), the step leading from Eq. (8) to (9) is readily proven.

In conclusion, we would like to make some comments on an apparent inconsistency in our treatment of the liquid. The solution of the momentum and continuity equations in terms of a potential [Eqs. (3) and (4)] requires the approximation of a compressible fluid undergoing isentropic motion, whereas the form of the energy equation (31) is appropriate for an incompressible medium of varying temperature. This procedure is justified, however, if one can show that thermal effects in the dynamic equations are small, and similarly that compressibility effects in the energy equation are small. Concerning the first aspect of the matter, notice that temperature changes in the liquid in practice extend only to a very thin shell, the thickness of which is approximately given by the diffusion length $(D_l/\omega)^{1/2}$. This nonisothermal layer would become important dynamically only at frequencies so high that it is comparable to the wavelength of sound in the liquid. For water these frequencies are of the order of 10^{14} $\sec^{-1}$, which shows the negligible importance of this effect. It may also be observed that the speed of sound in liquids is not very sensitive to temperature. For instance, data for water show that in the neighborhood of 25 °C, $dc/dT_l \simeq 240$ cm/sec, which introduces an error in the value of c of less than 3%, even for temperature changes as large as ± 20 °C.[21]

The second question of compressibility effects in the energy equation of the liquid can be clarified by observing that if liquid diffusivity is defined as $D_l = k_l/\rho_l c_{v,l}$, compressibility introduces an additional term in the right-hand side of Eq. (31) which may be written as $(\alpha/D_l)\nabla \cdot u_l$. Here $\alpha = (\rho_l/c_{v,l})(\partial S/\partial \rho)_T$ or, equivalently, $\alpha = -(\rho_l/T)(\partial T/\partial \rho_l)_S$, and S denotes the entropy of the liquid. Inclusion of this correction adds to Eq. (35) the new term

$$\alpha\epsilon(\omega R_0/c)^2(1+i\omega R_0/c)(1-i\omega D_l/c)^{-1}(R_0/r)\exp i\omega(t-r/c)\,.$$

While small terms of order $(\omega R_0/c)^2$ have been retained in the calculations presented here, the (dimensionless) constant α is usually so small that this term can be disregarded. For instance, again in the case of water at room temperature, α is of the order of 5×10^{-2}.

ACKNOWLEDGMENT

The author wishes to express his gratitude to Professor Milton S. Plesset for many helpful discussions.

TABLE A1. Numerical values of κ, Eq. (21), and B, Eq. (25), for $G_1 = 10^{-7}$.

$G_1=10^{-7}$	$\gamma=\frac{7}{5}$		$\gamma=\frac{5}{3}$	
G_2	κ	$\gamma B\times10$	κ	$\gamma B\times10$
0.1	1.000	0.0265	1.000	0.0444
0.2	1.000	0.0533	1.000	0.0888
0.5	1.001	0.133	1.001	0.221
1	1.003	0.263	1.005	0.438
2	1.011	0.509	1.019	0.841
5	1.053	1.029	1.091	1.659
10	1.124	1.283	1.201	2.023
20	1.194	1.212	1.310	1.947
50	1.264	0.967	1.425	1.615
10^2	1.302	0.767	1.491	1.310
2×10^2	1.330	0.588	1.540	1.021
5×10^2	1.355	0.399	1.586	0.704
10^3	1.368	0.292	1.609	0.520
2×10^3	1.378	0.212	1.626	0.379
5×10^3	1.386	0.137	1.641	0.246
10^4	1.390	0.0978	1.648	0.177
2×10^4	1.393	0.0697	1.654	0.126
5×10^4	1.395	0.0443	1.658	0.0804
10^5	1.396	0.0314	1.660	0.0571
2×10^5	1.396	0.0223	1.660	0.0404
5×10^5	1.394	0.0141	1.659	0.0255
10^6	1.390	9.88×10^{-3}	1.654	0.0180
2×10^6	1.381	6.90×10^{-3}	1.643	0.0125
5×10^6	1.352	4.19×10^{-3}	1.610	7.61×10^{-3}
10^7	1.304	2.75×10^{-3}	1.552	5.00×10^{-3}
2×10^7	1.201	1.65×10^{-3}	1.430	3.01×10^{-3}
5×10^7	0.847	5.20×10^{-4}	1.008	0.946×10^{-3}

TABLE A2. Numerical values of κ Eq. (21) and B Eq. (25), for $G_1 = 10^{-8}$.

$G_1=10^{-8}$	$\gamma=\frac{7}{5}$		$\gamma=\frac{5}{3}$	
G_2	κ	$\gamma B\times10$	κ	$\gamma B\times10$
0.1	1.000	0.0220	1.000	0.0409
0.2	1.000	0.0521	1.000	0.0880
0.5	1.001	0.132	1.001	0.221
1	1.003	0.263	1.005	0.438
2	1.011	0.508	1.019	0.841
5	1.053	1.029	1.091	1.658
See Table A1 for $5<G_2<5\times10^4$				
5×10^4	1.395	0.0444	1.658	0.0805
10^5	1.397	0.0315	1.661	0.0571
2×10^5	1.398	0.0223	1.662	0.0405
5×10^5	1.398	0.0141	1.664	0.0257
10^6	1.398	0.0100	1.664	0.0182
2×10^6	1.397	7.07×10^{-3}	1.663	0.0128
5×10^6	1.395	4.45×10^{-3}	1.660	8.10×10^{-3}
10^7	1.390	3.13×10^{-3}	1.655	5.69×10^{-3}
2×10^7	1.381	2.18×10^{-3}	1.644	3.97×10^{-3}
5×10^7	1.353	1.33×10^{-3}	1.610	2.41×10^{-3}
10^8	1.304	8.71×10^{-4}	1.552	1.58×10^{-3}
2×10^8	1.202	5.23×10^{-4}	1.430	9.51×10^{-4}
5×10^8	0.847	1.64×10^{-4}	1.008	2.99×10^{-4}

APPENDIX

We present here in tabular form part of the numerical values used in the construction of Figs. 1–4. The data for $G_1 = 10^{-7}$ and $0.1 \le G_2 \le 5\times10^7$ are given in Table A1. Table A2 refers to the case $G_1 = 10^{-8}$. The entries corresponding to $5 < G_2 < 5\times10^4$ have been omitted since they coincide to the last digit given in the table with those of Table A1. For the sake of brevity a similar procedure has been adopted for $G_1 = 10^{-9}$ (Table A3),

TABLE A3. Numerical values of κ, Eq. (21), and B, Eq. (25), for $G_1 = 10^{-9}$.

$G_1=10^{-9}$	$\gamma=\frac{7}{5}$		$\gamma=\frac{5}{3}$	
G_2	κ	$\gamma B\times10$	κ	$\gamma B\times10$
0.2	1.000	7.03×10^{-3}	1.000	0.0500
0.5	1.001	0.120	1.001	0.210
1	1.003	0.259	1.005	0.434
2	1.011	0.507	1.019	0.840
5	1.053	1.029	1.091	1.658
See Table A1 for $5<G_2<5\times10^4$				
5×10^4	1.396	0.0444	1.658	0.0805
10^5	1.397	0.0315	1.661	0.0571
2×10^5	1.398	0.0223	1.663	0.0405
5×10^5	1.399	0.0141	1.664	0.0257
10^6	1.399	0.0100	1.665	0.0182
2×10^6	1.399	7.08×10^{-3}	1.665	0.0129
5×10^6	1.399	4.48×10^{-3}	1.665	8.15×10^{-3}
10^7	1.399	3.17×10^{-3}	1.665	5.76×10^{-3}
2×10^7	1.398	2.24×10^{-3}	1.664	4.07×10^{-3}
5×10^7	1.395	1.41×10^{-3}	1.661	2.56×10^{-3}
10^8	1.391	9.90×10^{-4}	1.655	1.80×10^{-3}
2×10^8	1.381	6.91×10^{-4}	1.644	1.26×10^{-3}
5×10^8	1.353	4.19×10^{-4}	1.610	7.62×10^{-4}
10^9	1.304	2.75×10^{-4}	1.552	5.01×10^{-4}

TABLE A4. Numerical values of κ, Eq. (21), and B, Eq. (25), for $G_1 = 10^{-6}$.

$G_1 = 10^{-6}$	$\gamma = \frac{7}{5}$		$\gamma = \frac{5}{3}$	
G_2	κ	$\gamma B \times 10$	κ	$\gamma B \times 10$
0.1	1.000	0.0267	1.000	0.0444
See Table A1 for $0.1 < G_2 < 2\times 10^3$.				
2×10^3	1.377	0.212	1.626	0.379
5×10^3	1.385	0.137	1.640	0.246
10^4	1.389	0.0977	1.647	0.176
2×10^4	1.391	0.0695	1.652	0.126
5×10^4	1.391	0.0441	1.653	0.0799
10^5	1.388	0.0311	1.650	0.0564
2×10^5	1.379	0.0217	1.640	0.0395
5×10^5	1.351	0.0132	1.608	0.0240
10^6	1.303	8.69×10^{-3}	1.551	0.0158
2×10^6	1.201	5.23×10^{-3}	1.430	9.50×10^{-3}
5×10^6	0.847	1.65×10^{-3}	1.008	3.00×10^{-3}

TABLE A5. Numerical values of κ, Eq. (21), and B, Eq. (25), for $G_1 = 10^{-5}$.

$G_1 = 10^{-5}$	$\gamma = \frac{7}{5}$		$\gamma = \frac{5}{3}$	
G_2	κ	$\gamma B \times 10$	κ	$\gamma B \times 10$
0.1	1.000	0.0267	1.000	0.0444
See Table A1 for $0.1 < G_2 < 5\times 10^2$.				
5×10^2	1.355	0.398	1.585	0.704
10^3	1.367	0.292	1.608	0.520
2×10^3	1.376	0.211	1.624	0.378
5×10^3	1.381	0.136	1.635	0.245
10^4	1.381	0.0966	1.638	0.174
2×10^4	1.374	0.0679	1.632	0.123
5×10^4	1.348	0.0415	1.603	0.0753
10^5	1.301	0.0274	1.547	0.0497
2×10^5	1.200	0.0165	1.427	0.0300
5×10^5	0.847	5.25×10^{-3}	1.008	9.53×10^{-3}

TABLE A6. Bounds for the validity of the approximation (A2) for $\gamma = \frac{7}{5}$.

G_1	A_l	A_u	B_l	B_u	C_l	C_u	D_l	D_u
10^{-9}	5	10^9	20	2×10^8	200	5×10^8	5×10^3	10^8
10^{-8}	5	10^8	20	2×10^7	200	5×10^7	5×10^3	10^7
10^{-7}	5	10^7	20	2×10^6	200	10^7	5×10^3	10^6
10^{-6}	5	10^6	20	2×10^5	200	10^8	5×10^3	10^5
10^{-5}	5	10^5	20	2×10^4	400	5×10^4	···	···

$G_1 = 10^{-6}$ (Table A4), and $G_1 = 10^{-5}$ (Table A5); the reader is referred to Table A1 for all the entries omitted in these tables.

Finally, we observe that the expression

$$\tilde{\Phi} = 3(G_1 G_2)^{-1}\left\{1 + i\frac{\gamma - 1}{\gamma}\left[2G_1 + 3G_2^{-1}(\beta_{1a} - 1)\right]\right\}, \tag{A1}$$

with β_{1a} given by

$$\beta_{1a} = (1 + i)(\tfrac{1}{2}\gamma G_2)^{1/2}\left(1 + \tfrac{1}{2} i \frac{\gamma - 1}{\gamma} G_1\right) \tag{A2}$$

approximates Φ, Eq. (19), in certain ranges of values of the parameters G_1 and G_2 as follows:

TABLE A7. Bounds for the validity of the approximation (A2) for $\gamma = \frac{5}{3}$.

G_1	A_l	A_u	B_l	B_u	C_l	C_u	D_l	D_u
10^{-9}	10	10^9	50	2×10^8	500	5×10^8	10^4	10^8
10^{-8}	10	10^8	40	4×10^7	500	7×10^7	10^4	1.5×10^7
10^{-7}	10	1.5×10^7	40	3×10^6	500	7×10^6	10^4	1.5×10^6
10^{-6}	10	1.2×10^6	30	3×10^5	500	7×10^5	10^4	10^5
10^{-5}	10	1.2×10^5	30	3×10^4	500	6×10^4	···	···

$$A_l < G_2 < A_u, \quad \left|\mathrm{Re}(\Phi - \tilde{\Phi})/\mathrm{Re}\Phi\right| < 0.10\,,$$
$$B_l < G_2 < B_u, \quad \left|\mathrm{Re}(\Phi - \tilde{\Phi})/\mathrm{Re}\Phi\right| < 0.02\,,$$
$$C_l < G_2 < C_u, \quad \left|\mathrm{Im}(\Phi - \tilde{\Phi})/\mathrm{Im}\Phi\right| < 0.10\,,$$
$$D_l < G_2 < D_u, \quad \left|\mathrm{Im}(\Phi - \tilde{\Phi})/\mathrm{Im}\Phi\right| < 0.02\,.$$

The constants A_l, A_u, B_l, B_u, C_l, C_u, D_l, and D_u are functions of both G_1 and γ. Approximate values for these quantities are given in Table A6 ($\gamma = \frac{7}{5}$) and in Table A7 ($\gamma = \frac{5}{3}$). Equation (A1) is obtained by observing that, for $|z|$ large, one has $\coth z \sim 1$, so that, for G_2 large enough, $\lambda_1 \sim \beta_1 - 1$. Also, for small G_1, by a straightforward Taylor expansion, one has

$$\lambda_2\Gamma_1 - \lambda_1\Gamma_2 = -\tfrac{2}{3} i G_1 G_2\left[1 - iG_1 - 3i\frac{\gamma - 1}{\gamma}\frac{\beta_{1a} - 1}{G_2} + O(G_1^2)\right].$$

Inserting this expression in the denominator of Eq. (19) and expanding again for small G_1 one readily obtains Eq. (A1).

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[5]R. B. Chapman and M. S. Plesset, J. Basic Eng. **93**, 373–376 (1971).

[6]L. Landau and E. Lifshitz, *Fluid Mechanics* (Addison–Wesley, Reading, MA, 1959), Sec. 73.

[7]An assumption implicit in Eq. (4) is that the viscosity of the liquid may be neglected in the computation of the radiated wave. A rough estimate of the orders of magnitude of $|\partial q/\partial t|$ and $\nu|\nabla^2 q|$ (where q is the particle velocity in the liquid) shows that this approximation is permissible, provided that $\nu \ll \omega R_0^2$.

[8]In view of the rather large numerical value of 2π here and in the following we retain the square of the quantity $\omega R_0/c$ in spite of the fact that our assumption on the relative magnitude of R_0 and the wavelength in the liquid implies that $\omega R_0/2\pi c \ll 1$.

[9]M. Minnaert, Philos. Mag. **16**, 235–248 (1933).

[10]Lord Rayleigh, *The Theory of Sound* (Dover, New York, 1945), 2nd ed., Sec. 325; I. Malecki, *Physical Foundations of Technical Acoustics* (Pergamon, Oxford, 1963), p. 174.

[11]A. Prosperetti, J. Acoust. Soc. Am. **56**, 878–885 (1974).

[12]See e.g., S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge University, Cambridge, England, 1952), p. 103.

[13]The square root in the expression for $\Gamma_{1,2}$ and the inner one

in Eq. (17) are unambiguous since the expression for φ is symmetric upon interchange of the subscripts 1 and 2; the outer square root in Eq. (17) is defined in such a way that $\mathrm{Re}\,\beta_l \geq 0$.

[14]For a recent experimental verification see F. B. Jensen, J. Fluid Eng. **96**, 389–393 (1974).

[15]M. S. Plesset, in *Cavitation in Real Liquids*, edited by R. Davies (Elsevier, Amsterdam, Holland, 1964), pp. 1–18.

[16]For frequencies so high that the wavelength in the liquid becomes of the order of the bubble size or smaller, the analytical details of the above argument must be modified in an obvious way, but its physical content remains applicable.

[17]It is readily seen from Eq. (14) that, if surface tension and acoustic effects are neglected, one has $B \simeq (\kappa/\gamma)\delta_{th}$.

[18]The values of the physical properties used for this example are $\gamma = 1.4$, $M = 28.88$ g, $D_g = 0.208$ cm^2/sec, $D_l = 1.42 \times 10^{-3}$ cm^2/sec, $k_g = 2.54 \times 10^3$ erg/cm sec K, $k_l = 5.9 \times 10^4$ erg/cm sec K, $\sigma = 72.8$ erg/cm^2, $c = 1.486 \times 10^5$ cm/sec, $\rho_l = 1$ g/cm^3, and $\nu = 0.01$ cm^2/sec. The results presented in this section have been obtained from the full expression for φ, Eq. (16).

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[20]The analysis in this section follows that of Refs. 4 and 5.

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