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On the Classical Theory of the Electron.

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Summary. — A classical theory of the electron, proposed by one of us several years ago and based on finite-difference equations, is discussed by considering the three possible following cases: radiating electron, absorbing electron and nonradiating, nonabsorbing electron. In particular the so-called trasmission laws necessary to determine, in conjunction with the dynamical equations, the motion of a charged particle corresponding to given initial values of position and velocity are critically reconsidered. The general characteristics of the one-dimensional motion in the non-relativistic approximation are discussed in detail. It is found that in the case of the radiating electron the particle position tends asimptotically to the point of stable equilibrium. The present theory is, therefore, free from the unphysical phenomenon of runaway solutions. These general results are illustrated by studying the motion of a particle under the action of a restoring elastic force and under the action of purely time-dependent forces.

1. – Introduction.

A series of recent papers (1-6) is witness to the continued interest that the classical theory of charged particles arises even after the brilliant success of quantum electrodynamics. Many reasons for this state of affairs can be found

(1) C. TEITELBORN: *Phys. Rev. D*, **1**, 1572 (1970).
(2) T. C. MO and C. H. PAPAS: *Phys. Rev. D*, **4**, 3566 (1971).
(3) M. SORG: *Zeits. Naturf.*, **29** a, 1671 (1974); **31** a, 644, 1133 (1976); **32** a, 101, 659 (1977).
(4) I. PETZOLD and M. SORG: *Zeits. f. Phys.*, **283** A, 207 (1977).
(5) H. LEVINE, E. J. MONIZ and D. H. SHAPP: *Amer. Journ. Phys.*, **45**, 75 (1977).
(6) G. H. GOEDECKE: *Nuovo Cimento*, **28** B, 225 (1975).

in the books by ROHRLICH (7) and ARZELIÈS (8). To these reasons we add that it may be easier to get a valuable insight into the consequences of some deep modifications of fundamental physical assumptions in a classical rather than in a quantum context. The present paper deals exactly with one such modification, in so far as the dynamical equation for charged particles that we examine is a finite-difference rather than a differential equation. This equation, which was put forward by one of us some time ago, is studied here in the nonrelativistic limit. We present several closed-form solutions for simple one-dimensional motions and we consider some properties of the solutions in more general cases. It is found that the solutions of the equation in question exhibit a physically reasonable behaviour and are free from the difficulties besetting other classical theories of charged particles. In addition, the importance of the so-called « transmission law », *i.e.* of the relation between position and velocity in the elementary interval of time typical of the theory considered, is illustrated, and a new form of it is presented. With this modification of the transmission law it is found that our theory goes over into the Lorentz-Dirac one (also when the force acting on the particle depends explicitly on the position variables) as the elementary interval of time, the chronon, is made vanishingly small.

2. – The Dirac classical equation.

In his theory LORENTZ (9) represented the electron as a small sphere contractile in the direction of its velocity: by calculating the reaction force using the Wiechert and Liénard potentials, he derived an equation of motion that in the nonrelativistic approximation reads

$$m_0 \frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \boldsymbol{F} + \boldsymbol{\Gamma} .$$

The coefficient m_0, which has the physical meaning of a rest mass, is given by

$$m_0 = \frac{2}{3} \frac{e^2}{Rc} ,$$

(7) F. ROHRLICH: *Classical Charged Particles* (Reading, Mass., 1965).
(8) H. ARZELIÈS: *Rayonnement et dynamique du corpuscle chargé fortement accéleré* (Paris, 1966).
(9) H. A. LORENTZ: *The Theory of Electron* (Leipzig, 1916).

where R is the radius of the electron at rest, $\boldsymbol{F}$ is the external force and

$$\boldsymbol{\Gamma} = \frac{2}{3}\frac{e^2}{c^3}\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2} + (\ )R + (\ )R^2 + \dots$$

is the reaction force.

Notice that the coefficients of the powers of R contain higher-order derivatives of $\boldsymbol{v}$ and furthermore their expression would change if one assumed a different model for the electron. In any case, in the limit $R = 0$, corresponding to a point electron, only the first term on the right-hand side contributes. However, in this limit, m_0 is infinite. This result is obviously absurd. Furthermore, other serious difficulties are connected with the Lorentz theory (7,8).

To avoid these difficulties DIRAC (10), starting from Maxwell's equations, evaluated the flux of the energy-momentum vector through a small tube of radius $\varepsilon \ll R$ surrounding the world-line of the electron. In this way he obtained the following *exact* equation:

$$m_0 \frac{\mathrm{d}u_\alpha}{\mathrm{d}s} = F_\alpha + \Gamma_\alpha \qquad (u_\alpha u_\alpha = -c^2)\,,$$

where the rest mass

$$m_0 = \frac{1}{2}\frac{e^2}{c^2}\frac{1}{\varepsilon} - k(\varepsilon)$$

is the difference of two quantities, each of which becomes infinite for $\varepsilon \to 0$ in such a way that m_0 remains finite. The terms on the right-hand side are given by

$$F_\alpha = \frac{e}{c}F_{\alpha\beta}u_\beta\,, \qquad \Gamma_\alpha = \frac{2}{3}\frac{e^2}{c}\left(\frac{\mathrm{d}^2u_\alpha}{\mathrm{d}s^2} + \frac{u_\alpha u_\beta}{c^2}\frac{\mathrm{d}^2u_\beta}{\mathrm{d}s^2}\right)$$

and are the four-vectors representing the external force and the radiation reaction, respectively. The Dirac «exact» equation for the classical radiating electron is therefore the following:

$$m_0 c\frac{\mathrm{d}u_\alpha}{\mathrm{d}s} - \frac{2}{3}\frac{e^2}{c}\left(\frac{\mathrm{d}^2u_\alpha}{\mathrm{d}s^2} + \frac{u_\alpha u_\beta}{c^2}\frac{\mathrm{d}^2u_\beta}{\mathrm{d}s^2}\right) = \frac{e}{c}\boldsymbol{F}_{\alpha\beta}u_\beta\,, \tag{1}$$

which in the nonrelativistic approximation becomes

$$m_0\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} - \frac{2}{3}\frac{e^2}{c^3}\frac{\mathrm{d}^2\boldsymbol{v}}{\mathrm{d}t^2} = \boldsymbol{F} \tag{2}$$

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(10) P. A. M. DIRAC: *Proc. Roy. Soc.*, **161** A, 148 (1938).

with

$$F = e\left(E + \frac{v}{c}\times B\right).$$

But, also adopting equations of the type (1) or (2), we still meet many difficulties connected expecially with the fact that their solutions contain more than two arbitrary constants, since the equations involve second-order derivatives of the velocity. For a detailed discussion of such difficulties we refer the reader to some review papers [11-13] and to the already-mentioned books [8,9].

3. – Relativistic finite-difference equations.

A way to overcome the difficulties presented by the Dirac classical equations has been indicated by one of us [13] about twenty years ago by proposing the substitution of eq. (1) by the following relativistic finite-difference *equation for the radiating electron*:

$$\frac{m_0}{\tau_0}\left\{u_\alpha(\tau) - u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}\left[u_\beta(\tau) - u_\beta(\tau-\tau_0)\right]\right\} = \frac{e}{c}F_{\alpha\beta}(\tau)\,u_\beta(\tau)\,, \tag{3}$$

that can also be written in the equivalent form

$$-\frac{m_0}{\tau_0}\left[u_\alpha(\tau-\tau_0) + \frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}\,u_\beta(\tau-\tau_0)\right] = \frac{e}{c}F_{\alpha\beta}(\tau)u_\beta(\tau)\,. \tag{4}$$

Here $\tau = s/c$ is the proper time of the particle and τ_0 is an invariant interval of time given by

$$\tau_0 = 2\theta_0\,, \tag{5}$$

where

$$\theta_0 = \frac{2}{3}\,\frac{e^2}{m_0 c^3} \tag{6}$$

is the so-called *chronon*.

Let us observe that in the nonrelativistic approximation eq. (3) reduces to

$$\frac{m_0}{\tau_0}\left[v(t) - v(t-\tau_0)\right] = F(r(t);\, v(t);\, t)\,. \tag{7}$$

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[11] T. ERBER: *Forts. der Phys.*, **9**, 343 (1961).
[12] G. N. PLASS: *Rev. Mod. Phys.*, **93**, 37 (1961).
[13] P. CALDIROLA: *Suppl. Nuovo Cimento*, **3**, 297 (1956).

This equation coincides with that obtained by many authors [6,14-16] considering an extended electron, the charge of which is distributed on a spherical shell.

Another equation already considered by one of us [13] is the following (*equation for the absorbing electron*):

$$\frac{m_0}{\tau_0}\left\{u_\alpha(\tau+\tau_0)-u_\alpha(\tau)+\frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}[u_\beta(\tau+\tau_0)-u_\beta(\tau)]\right\}=\frac{e}{c}F_{\alpha\beta}(\tau)u_\beta(\tau) \tag{8}$$

equivalent to

$$\frac{m_0}{\tau_0}\left[u_\alpha(\tau+\tau_0)+\frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}u_\beta(\tau+\tau_0)\right]=\frac{e}{c}F_{\alpha\beta}(\tau)u_\beta(\tau)\,. \tag{9}$$

As will be shown below, this equation may be interpreted as describing the motion of an ordinary particle of charge e absorbing energy from the external space.

The corresponding nonrelativistic approximation is

$$\frac{m_0}{\tau_0}[\boldsymbol{v}(t+\tau_0)-\boldsymbol{v}(t)]=\boldsymbol{F}(\boldsymbol{r}(t);\,\boldsymbol{v}(t);\,t)\,. \tag{10}$$

Finally let us recall that for a *nonradiating and nonabsorbing electron* it is possible [13] to assume a symmetrical finite-difference relativistic equation (*)

$$\frac{m_0}{2\theta_0}\left\{u_\alpha(\tau+\theta_0)-u_\alpha(\tau-\theta_0)+\frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}[u_\beta(\tau+\theta_0)-u_\beta(\tau-\theta_0)]\right\}=\frac{e}{c}F_{\alpha\beta}(\tau)u_\beta(\tau)\,, \tag{11}$$

which in the nonrelativistic approximation becomes

$$\frac{m_0}{2\theta_0}[\boldsymbol{v}(t+\theta_0)-\boldsymbol{v}(t-\theta_0)]=\boldsymbol{F}(\boldsymbol{r}(t);\,\boldsymbol{v}(t);\,t)\,. \tag{12}$$

Let us mention that this last equation suggested to one of us the starting point for the introduction of the chronon in quantum mechanics writing a suitable finite-difference Schrödinger equation [17].

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[14] L. PAGE: *Phys. Rev.*, **9**, 376 (1918).
[15] D. BOHM and M. WEINSTEIN: *Phys. Rev.*, **74**, 1789 (1948).
[16] C. J. ELIEZER: *Proc. Camb. Phys. Soc.*, **46**, 198 (1950).
(*) This equation is slightly different from the equation proposed before [13].
[17] P. CALDIROLA: *Lett. Nuovo Cimento*, **16**, 151 (1976); **17**, 461 (1976).

4. – Initial conditions and the transmission law.

In the general case, $F_{\alpha\beta}$ or $\boldsymbol{F}$ depend explicitly on the positional variables x_α or $\boldsymbol{r}$. The dynamical law (3) (and the analogous (8) and (11)), therefore, is not sufficient by itself to calculate $u_\alpha(n\tau_0)$, $x_\alpha(n\tau_0)$ (or, in the nonrelativistic approximation, $\boldsymbol{v}(n\tau_0)$ and $\boldsymbol{r}(n\tau_0)$). A further finite-difference equation is necessary to close the system. To this equation we shall refer as to the *transmission law*. The importance of this point was already stressed in our original paper ([13]).

Changing slightly our original assumption, we shall assume the following form for the transmission law associated to eq. (3) for the relativistic *radiating electron*:

$$x_\alpha(n\tau_0) - x_\alpha[(n-1)\tau_0] = \tfrac{1}{2}\tau_0\{u_\alpha(n\tau_0) + u_\alpha[(n-1)\tau_0]\}\,. \tag{13$'$}$$

The reason for the choice of this particular transmission law will be discussed below.

The nonrelativistic approximation of (13′) is given by

$$\boldsymbol{r}(n\tau_0) - \boldsymbol{r}[(n-1)\tau_0] = \tfrac{1}{2}\tau_0\{\boldsymbol{v}(n\tau_0) + \boldsymbol{v}[(n-1)\tau_0]\}\,. \tag{13$''$}$$

In a similar way we shall associate to eq. (8) for the *absorbing electron* the relativistic transmission law

$$x_\alpha[(n+1)\tau_0] - x_\alpha(n\tau_0) = \tfrac{1}{2}\tau_0\{u_\alpha[(n+1)\tau_0] + u_\alpha(n\tau_0)\}\,, \tag{14$'$}$$

which in the nonrelativistic approximation reduces to

$$\boldsymbol{r}[(n+1)\tau_0] - \boldsymbol{r}(n\tau_0) = \tfrac{1}{2}\tau_0\{\boldsymbol{v}[(n+1)\tau_0] + \boldsymbol{v}(n\tau_0)\}\,. \tag{14$''$}$$

Finally we recall also that the transmission law associated to the relativistic symmetrical equation (11) for the *nonradiating and nonabsorbing electron* is assumed to have the following form:

$$x_\alpha[(n+1)\theta_0] - x_\alpha[(n-1)\theta_0] = 2\theta_0 u_\alpha(n\theta_0)\,, \tag{15$'$}$$

which in the nonrelativistic approximation becomes

$$\boldsymbol{r}[(n+1)\theta_0] - \boldsymbol{r}[(n-1)\theta_0] = 2\theta_0\boldsymbol{v}(n\theta_0)\,. \tag{15$''$}$$

By means of the equation of motion and of the transmission law, it is now possible to evaluate $x_\alpha(\tau)$ and $u_\alpha(\tau)$ at any time $\tau = n\tau_0$.

The resulting motion, defined in a succession of discrete instants of time separated from each other by the interval τ_0, will be called *macroscopical motion* of the electron. We remark that this motion remains unaltered if, instead of a constant value of $u_\alpha(\tau)$ in the interval $(n-1)\tau_0 \leqslant \tau \leqslant n\tau_0$, we assume any other velocity satisfying the transmission law; in other words, we may take for $u_\alpha(\tau)$ any function variable in the interval considered in such a way that its average value is equal to $\frac{1}{2}[u_\alpha((n-1)\tau_0) + u_\alpha(n\tau_0)]$ for the radiating electron. Similar assumptions may be taken for the absorbing electron and for the neither radiating nor absorbing electron. This means in particular that the macroscopical motion remains unaltered by the superposition on the uniform motion, involved in the transmission law, of any other *internal* or *microscopical* motion, provided the average velocity of the latter is zero in an elementary interval of duration τ_0 and the conditions $u_\alpha^{(m)}(n\tau_0) = 0$ and $x_\alpha^{(m)}(n\tau_0) = 0$ are satisfied. These conditions ensure that, at any instant $\tau = n\tau_0$, the effective position $x_\alpha(\tau) = x_\alpha^{(M)}(\tau) + x_\alpha^{(m)}(\tau)$ and the effective velocity $u_\alpha(\tau) = u_\alpha^{(M)}(\tau) + u_\alpha^{(m)}(\tau)$ of the electron are equal to the corresponding quantities for the macroscopical motion only.

In any case, even if we are interested in the macroscopical motion only, the condition

$$u_\alpha(\tau)u_\alpha(\tau) = -c^2 \tag{16}$$

is still to be considered verified for any $\tau = n\tau_0$.

Of particular interest seem to be the *periodical internal motions* ([13]) and the *random internal motions* ([18]). We shall recall also that all the internal motions *are nonradiating* and *nonabsorbing motions*.

The solutions of eq. (3) and eq. (7) for the radiating electron (which is the case of direct physical interest) are easily obtained ([13]) for some problems concerning motions under the action of constant (*) or of time-dependent forces (*i.e.* when $F_{\alpha\beta}$ is a function of τ only or $\boldsymbol{F}$ of t only). In these cases, given the initial value of the velocity, it is possible to evaluate without any ambiguity, by means of the equation of motion only, the values of the same quantity at any other instant. The successive and independent application of the transmission law leads to the determination of the position. The results are completely satisfactory: in particular the strange behaviour of an electron according to the Dirac classical equation is no longer present. The situation is more complicated when the external forces depend explicity on the positional variables of the electron. In this case the possibility of evaluating the velocity and the

([18]) P. Caldirola: *Lett. Nuovo Cimento*, **15**, 489 (1976).

(*) See in particular the detailed discussion of our relativistic finite-difference equation for the radiating electron in the case of hyperbolic motion carried out by Lanz ([19]).

([19]) L. Lanz: *Nuovo Cimento*, **23**, 195 (1962).

position is clearly related to the simultaneous solution of the equation of motion and of the equation expressing the transmission law.

5. – General properties of the rectilinear motion in the nonrelativistic approximation.

In this section we wish to elucidate some general features of the dynamical equations (7) and (10) and the associated transmission laws (13″) and (14″) for the simple cases of nonrelativistic, rectilinear motion, respectively for the radiating and for the absorbing electron, under the action of forces independent of time.

The dynamical equations under investigation are, therefore,

$$\frac{m_0}{\tau_0}[v(t)-v(t-\tau_0)] = F\big(x(t);\, v(t)\big) \qquad (t = n\tau_0) \tag{17}$$

for the radiating electron, and

$$\frac{m_0}{\tau_0}[v(t+\tau_0)-v(t)] = F\big(x(t),\, v(t)\big) \qquad (t = n\tau_0) \tag{18}$$

for the absorbing electron.

We can carry both cases at the same time by writing

$$\frac{m_0}{\tau_0}[v(t+\tau_0)-v(t)] = \lambda F\big(x(t+\tau_0),\, v(t+\tau_0)\big) + \mu F\big(x(t);\, v(t)\big)\,, \tag{19}$$

where

$$\begin{cases} \lambda = 1\,, \quad \mu = 0 & \text{for the radiating electron,} \\ \lambda = 0\,, \quad \mu = 1 & \text{for the absorbing electron.} \end{cases} \tag{20}$$

To eq. (19) we add the transmission law, (13″) and (14″), in the generalized form

$$x(t+\tau_0)-x(t) = \tau_0[\alpha v(t+\tau_0)+\beta v(t)]\,. \tag{21}$$

Incidentally we remark that the original form of the transmission law given in our old paper ([13]) corresponds to $\alpha = 1$, $\beta = 0$, whereas eqs. (13″) and (14″) are obtained from (21) by putting

$$\alpha = \tfrac{1}{2}\,, \qquad \beta = \tfrac{1}{2}\,. \tag{22}$$

Notice that, for eq. (19) and eq. (21) reduce to the ordinary form of Newtonian mechanics in the limit $\tau_0 \to 0$, it is necessary that

$$\lambda + \mu = 1 , \quad \alpha + \beta = 1 . \tag{22'}$$

We shall consider the behaviour of the solutions of the system of equations (17) and (21) in a neighborhood of a point of stable equilibrium $(\bar{x}, 0)$, *i.e.* a point such that

$$F(\bar{x}, 0) = 0 . \tag{23}$$

Stability of equilibrium requires also the existence of a region $\mathscr{R}_0$ of positive measure containing $(\bar{x}, 0)$ as an interior point and such that

$$\frac{\partial F(x, v)}{\partial x} < 0 , \quad \frac{\partial F(x, v)}{\partial v} < 0 \quad \text{for } (x, v) \in \mathscr{R}_0; \tag{24}$$

we shall further assume these derivatives to be continuous in $\mathscr{R}_0$.

For our purpose it is expedient to consider eqs. (19) and (21) as defining a transformation (*mapping*) of the phase space (x, v) into itself:

$$(x', v') = T(x, v) , \tag{25}$$

according to the following law:

$$v' = v + \frac{\tau_0}{m_0}[\lambda F(x', v') + \mu(F(x, v)] , \qquad x' = x + \tau_0(\alpha v' + \beta v) . \tag{26}$$

Consider now all the points of a region $\mathscr{R} \subseteq \mathscr{R}_0$, and apply the transformation (25) to obtain the set of points $\mathscr{R}' = T\mathscr{R}$. It is well known [20] that, if the Jacobian

$$J = \frac{\partial(x', v')}{\partial(x, v)}$$

of the transformation T is less than 1 at all points of $\mathscr{R}$, the mapping is a contracting mapping having as fixed point the equilibrium point $(\bar{x}, 0)$. Under these circumstances all the solutions of (26) having initial conditions within $\mathscr{R}$ will tend asymptotically to the equilibrium point, which is the behaviour that one would expect from a particle losing energy by radiation. Conversely, if $J > 1$ in $\mathscr{R}$, then the solution will move away from the equilibrium point, a behaviour appropriate for a particle absorbing energy.

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[20] See, for example, I. G. MALKIN: *Theory of stability of motion* (United States AEC-tr-3352).

From (26) it is a simple matter to establish that

$$J = \frac{1-(\tau_0/m_0)\mu(\beta\tau_0(\partial F/\partial x)-\partial F/\partial v))}{1-(\tau_0/m_0)\lambda(\alpha\tau_0(\partial F/\partial x')+\partial F/\partial v')}\,. \tag{27}$$

Let us take the region $\mathscr{R}$ small enough that $\mathscr{R}' \subseteq \mathscr{R}_0$. Then all the partial derivatives appearing in (27) have the sign specified by (24). It is evident therefore that, in order to have $J < 1$ for any force satisfying (23) and (24), it is necessary to take $\mu = 0$ in agreement with (20).

The situation for absorbing particles is slightly more complicated. Consider a velocity-independent force first. Then, as before, in order that $J > 1$ irrespectively of the particular structure of the force, one must take $\lambda = 0$, as had been anticipated. If the force now depends on the velocity, then eq. (26) implies that the rate of energy absorption is not sufficient to overcome the dissipative influence of the force, unless $\beta\tau_0|\partial F/\partial x| > |\partial F/\partial v|$. All these characteristics are quite reasonable and confirm the physical interpretation of eq. (7) and eq. (10) given above.

The extension of these considerations to the three-dimensional case is straightforward and need not be pursued here.

A similar discussion may be performed for the symmetrical equations (12) and (15″). Here it is necessary to introduce the auxiliary variable $u(t) = v(t-\theta_0)$ and $y(t) = x(t-\theta_0)$. The transformation of the space into itself defined by (12) and (15″) is then

$$y' = x\,, \qquad u' = v\,,$$

$$v' = u + \frac{\tau_0}{m_0}F(x, v)\,,$$

$$x' = y + \tau_0 v\,.$$

A straightforward computation leads then to

$$\frac{\partial(y', u', x', v')}{\partial(y, u, x, v)} = 1\,,$$

in agreement with the physical interpretation of these equations.

6. – Example: linear restoring force.

The preceding considerations can be illustrated with the aid of some specific examples. In the first place we will consider the case of the linear restoring force $F = -m_0\omega_0^2 x$. Equation (17) becomes

$$v(t) - v(t-\tau_0) = -\tau_0^2\omega_0^2 x(t)\,, \tag{28}$$

which is to be solved simultaneously with eq. (13″) under the initial conditions $x(0) = x_0$, $v(0) = v_0$. The solution is readily found by standard techniques to be

$$x(n\tau_0) = \varrho^{-n}(x_0 \cos n\vartheta + A \sin n\vartheta)\,, \tag{28a}$$

$$v(n\tau_0) = \varrho^{-n}(n_0 \cos n\vartheta + B \sin n\vartheta)\,, \tag{28b}$$

where the amplitude factor ϱ and the angle ϑ are defined by

$$\varrho = \left(1 + \frac{\omega_0^2}{2}\tau_0^2\right)^{-\frac{1}{2}}, \tag{29}$$

$$\vartheta = \mathrm{tg}^{-1}\left[\omega_0\tau_0\left(\frac{1+\frac{1}{4}\omega_0^2\tau_0^2}{1-\frac{1}{4}\omega_0^2\tau_0^2}\right)^{\frac{1}{2}}\right] \tag{30}$$

and the two constants A, B are given by

$$A = \frac{\tau_0 v_0 + [1-(1+\frac{1}{2}\omega_0^2\tau_0^2)^{\frac{1}{2}}\cos\vartheta]x_0}{(1+\frac{1}{2}\omega_0^2\tau_0^2)^{\frac{1}{2}}\sin\vartheta}\,,$$

$$B = \frac{[1+\omega_0^2\tau_0^2-(1+\frac{1}{2}\omega_0^2\tau_0^2)^{\frac{1}{2}}\cos\vartheta]v_0-\omega_0^2\tau_0 x_0}{(1+\omega_0^2\tau_0^2)^{\frac{1}{2}}\sin\vartheta}\,.$$

By putting $n = t/\tau_0$, eq. (28a) can be put in the standard form of a damped oscillatory motion

$$x(t) = \exp\left[-\frac{\gamma}{2}t\right](x_0\cos\omega t + A\sin\omega t)\,,$$

where the damping coefficient γ and the natural frequency ω are given by (*)

$$\gamma = \frac{1}{\tau_0}\ln\left(1+\frac{1}{2}\omega_0^2\tau_0^2\right),$$

$$\omega = \frac{1}{\tau_0}\mathrm{tg}^{-1}\left[\omega_0\tau_0\left(\frac{1+\frac{1}{4}\omega_0^2\tau_0^2}{1-\frac{1}{4}\omega_0^2\tau_0^2}\right)^{\frac{1}{2}}\right].$$

In the limit $\tau_0 \to 0$ these quantities reduce to the well-known form given by the classical Lorentz theory, since

$$\gamma = \frac{1}{2}\tau_0\omega_0^2 + O(\tau_0^3) = \frac{2}{3}\frac{e^2}{m_0c^3}\omega_0^2 + O(\tau_0^3)\,, \tag{31a}$$

$$\omega = \omega_0 + O(\tau_0^2)\,. \tag{32a}$$

(*) It is obvious from these equations that the motion is truly periodic only for $\omega_0 > 2\tau_0^{-1}$. A change to aperiodic behaviour occurs for $\omega_0 = 2\tau_0^{-1}$.

A similar limit operation performed on the constants A and B yields the classical values $\left(v_0+\frac{\gamma}{2}x_0\right)\Big/\omega_0^2$ and $-\omega_0 x_0$, respectively.

It may be of some interest to consider the solution of the more general set of equations (19) and (21) of an absorbing or a radiating electron for the simple elastic force under consideration. This solution is still of the form (28), but the quantities ϱ and ϑ are now given by

$$\varrho=\left(\frac{1+\beta\mu\omega_0^2\tau_0^2}{1+\alpha\lambda\omega_0^2\tau_0^2}\right)^{\frac{1}{2}}, \tag{32}$$

$$\vartheta=\operatorname{tg}^{-1}\frac{\omega_0\tau_0[(\alpha+\beta)(\lambda+\mu)-\frac{1}{4}(\alpha\mu-\beta\lambda)^2\omega_0\tau_0^2]^{\frac{1}{2}}}{1-\frac{1}{2}(\alpha\mu+\beta\lambda)\omega_0^2\tau_0^2}. \tag{33}$$

It is clear from (31) that in the case of eq. (18) (*i.e.* $\lambda=0$, $\mu=1$) the amplitude of the oscillatory motion increases with time in agreement with the interpretation of eq. (18) as describing an absorbing, rather than a radiating, electron.

It is interesting to consider the limit $\tau_0\to 0$ of eqs. (32) and (33) to obtain the natural frequency and the damping coefficient. The result is

$$\gamma=(\alpha\lambda-\beta\mu)\tau_0\omega_0^2+O(\tau_0^3),$$

$$\omega=[(\alpha+\beta)(\lambda+\mu)]^{\frac{1}{2}}\omega_0+O(\omega_0^2).$$

Therefore, in order to recover the Lorentz result (31) for the radiating electron ($\lambda=1$, $\mu=0$), it is necessary to set $\alpha=\beta=\frac{1}{2}$.

For the absorbing electron($\lambda=0$, $\mu=1$) we require $\gamma=-\frac{1}{2}\omega_0^2\tau_0$, which again implies $\alpha=\beta=\frac{1}{2}$. Thus we verify in this particular example what was said above, namely that the form of the transmission law (14″) is the only one capable of yielding the results of the Lorentz theory of the radiating electron in the limit $\tau_0\to 0$.

For completeness let us note here that in this case the solution of eq. (12) and eq. (15″), corresponding to an electron which neither radiates nor absorbs, is given by

$$x(n\theta_0)=x_0\cos(n\eta)+C\sin(n\eta)$$

with a similar expression for $v(n\tau_0/2)$. The angle η is given by

$$\eta=\operatorname{arctg}\frac{\theta_0\omega_0}{\sqrt{1-\theta_0^2\omega_0^2}},$$

and the constant C is determinaed by imposing the initial condition on the velocity. This result clearly corresponds to an oscillatory motion of fixed amplitude.

Perhaps a further comment should be added on eq. (32). It is clear from this expression that, if $\alpha\lambda = \beta\mu$, then $\varrho = 1$ and the oscillatory motion maintains a fixed amplitude. This result corresponds to a particle of fixed energy and, therefore, it may appear that other equations might describe a nonradiating, nonabsorbing electron in addition to eq. (11) or eq. (12). In reality this is not true, as a consideration of eq. (27) readily shows. What we find here is purely a consequence of the particular form of the force under investigation, for which

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial x'} = -m_0\omega_0^2 .$$

7. – Example: time-dependent forces.

For the case of forces dependent on time, but not on positional variables, one can obtain the general solution of eqs. (19), (21) subject to the initial conditions $x(0) = x_0$, $v(0) = v_0$; the result is

$$v(n\tau_0) = v_0 - \frac{\tau_0}{m_0}[\lambda F(0) + \mu F(n\tau_0)] + \frac{\tau_0}{m_0}(\lambda + \mu)\sum_{k=0}^{n} F(k\tau_0) , \tag{34}$$

$$x(n\tau_0) = x_0 - \tau_0[\alpha v_0 + \beta v(n\tau_0)] + (\alpha + \beta)\tau_0 \sum_{k=0}^{n} v(k\tau_0) . \tag{35}$$

The extension of these results to three-dimensional motion is straightforward. For purposes of illustration we consider here two specific examples.

In the case of a constant force, $F(n\tau_0) = K$, eqs. (34), (35) give

$$v(n\tau_0) = v_0 + (\lambda + \mu)\frac{K}{m_0} n\tau_0 , \tag{36}$$

$$x(n\tau_0) = x_0 + (\alpha + \beta)\, v_0 n\tau_0 + \frac{1}{2}(\lambda + \mu)\frac{K}{m_0}\tau_0^2 n[(\alpha + \beta)n + (\alpha - \beta)] . \tag{37}$$

For the case of constant acceleration the nonrelativistic Lorentz equation does not differ from Newton equation and we, therefore, must require that eqs. (36) and (37) reduce to the law of uniformly accelerated motion for $\tau_0 \to 0$ with $n\tau_0 = t$ fixed. Clearly this can only happen if $\lambda + \mu = 1$, $\alpha + \beta = 1$ and $\alpha - \beta = 0$. Thus we find again $\alpha = \beta = \frac{1}{2}$, as was assumed in (14). Notice that for this case we cannot discriminate between radiating ($\lambda = 1$, $\mu = 0$) or absorbing ($\lambda = 0$, $\mu = 1$) electrons, because these effects are connected with the sign of the term containing the second derivative of the acceleration, which vanishes identically. Hence, the only requirement to be satisfied is $\lambda+\mu=1$ with no separate conditions on λ and μ.

A more interesting example is that of an exponential time dependence of the force

$$F(n\tau_0)=A\exp[\sigma n\tau_0]\,.$$

In this case eq. (34) reduces to

$$v(n\tau_0)=v_0+\frac{A\tau_0}{m_0}\,\frac{\lambda+\mu\exp[-\sigma\tau_0]}{1-\exp[-\sigma\tau_0]}\left(\exp[\sigma n\tau_0]-1\right).$$

The expansion for small τ_0 with $n\tau_0=t$ fixed gives

$$v(t)=v_0+\frac{A}{m_0\sigma}\,\frac{\lambda+\mu-\mu\sigma\tau_0}{1-\frac{1}{2}\tau_0\sigma}\left(\exp[\sigma t]-1\right),$$

to be compared with the solution of the Lorentz equation

$$v(t)=v_0+\frac{A}{m_0\sigma}\,\frac{\exp[\sigma\tau]-1}{1-\frac{1}{2}\sigma\tau_0}\,. \tag{38}$$

For the two results to coincide one must require $\lambda+\mu=1$, $\mu=0$ as expected. With this choice for λ and μ eq. (35) gives for the position

$$x(n\tau_0)=x_0+(\alpha+\beta)v_0n\tau_0+\\+\frac{A\tau_0^2}{m_0(1-\exp[-\sigma\tau_0])}\left\{\frac{\alpha\exp[\sigma\tau_0]+\beta}{\exp[\sigma\tau_0]-1}[\exp[\sigma n\tau_0]-1]-(\alpha+\beta)n\right\}.$$

For this expression to be identical with the integral of eq. (38) in the small τ_0 limit, it is readily shown that one must require $\alpha+\beta=1$, $\alpha=\frac{1}{2}$ in full accord with our preceding rusults.

8. – Conclusions.

The results presented so far justify the form of the transmission laws (13″), (14″), (15″) which we have associated to the different equations of the electron deriving from our theory in the nonrelativistic approximation. It seems quite reasonable to assume that their immediate generalizations (13′), (14′), (15′) give the correct form of the transmission laws in the relativistic case.

Therefore, as already anticipated in sect. 4, the complete fundamental equations of our classical theory of the electron may be written:

a) for radiating electron, the equation of motion

$$\frac{m_0}{\tau_0}\left\{u_\alpha(n\tau_0)-u_\alpha((n-1)\tau_0)+\frac{u_\alpha(n\tau_0)u_\beta(n\tau_0)}{c^2}\left[u_\beta(n\tau_0)-u_\beta((n-1)\tau_0)\right]\right\}=$$
$$=\frac{e}{c}F_{\alpha\beta}(n\tau_0)u_\beta(n\tau_0)$$

and the transmission law

$$x_\alpha(n\tau_0)-x_\alpha((n-1)\tau_0)=\frac{\tau_0}{2}\left[u_\alpha(n\tau_0)+u_\alpha((n-1)\tau_0)\right];$$

b) for absorbing electron, the equation of motion

$$\frac{m_0}{\tau_0}\left\{u_\alpha((n+1)\tau_0)-u_\alpha(n\tau_0)+\frac{u_\alpha(n\tau_0)u_\beta(n\tau_0)}{c^2}\left[u_\beta((n+1)\tau_0)-u_\beta(n\tau_0)\right]\right\}=$$
$$=\frac{e}{c}F_{\alpha\beta}(n\tau_0)u_\beta(n\tau_0)$$

and the transmission law

$$x_\alpha((n+1)\tau_0)-x_\alpha(n\tau_0)=\frac{\tau_0}{2}\left[u_\alpha((n+1)\tau_0)+u_\alpha(n\tau_0)\right];$$

c) for nonradiating, nonabsorbing electron, the equation of motion ($\theta_0=\tau_0/2$)

$$\frac{m_0}{2\theta_0}\left\{u_\alpha((n+1)\theta_0)-u_\alpha((n-1)\theta_0)+\right.$$
$$\left.+\frac{u_\alpha(n\theta_0)u_\beta(n\theta_0)}{c^2}\left[u_\beta((n+1)\theta_0)-u_\beta((n-1)\theta_0)\right]\right\}=\frac{e}{c}F_{\alpha\beta}(n\theta_0)u_\beta(n\theta_0)$$

and the transmission law

$$x_\alpha((n+1)\theta_0)-x_\alpha((n-1)\theta_0)=2\theta_0u_\alpha(n\theta_0)\,.$$

The univocally determined solutions in the points $\tau=n\tau_0$ of the different equations considered give the so-called macroscopical motion of the electron. To this motion an arbitrary internal motion, which gives the behaviour of $u_\alpha(\tau)$ and $x_\alpha(\tau)$ in the interior of any elementary interval of time $(n-1)\cdot\tau_0\leqslant\tau\leqslant n\tau_0$, may be superposed. This possibility seems particularly attractive because it may allow the description of families of particles the members of which differ in intrinsic characteristics such as mass, magnetic moment, etc.

● RIASSUNTO

Si discute una teoria classica dell'elettrone proposta qualche anno fa da uno degli autori e basata su equazioni alle differenze finite, considerando i tre casi possibili: elettrone irraggiante, elettrone assorbente, elettrone nè irraggiante nè assorbente. In particolare si riconsiderano, attraverso un'analisi critica, le cosidette « leggi di trasmissione » necessarie per determinare, unitamente alle equazioni dinamiche, il moto di una particella carica in corrispondenza di determinati valori iniziali della posizione e della velocità. Si discutono poi particolareggiatamente, nella approssimazione non relativistica, le caratteristiche generali del moto unidimensionale. Si trova, in particolare, che nel caso dell'elettrone irraggiante la posizione della particella tende asintoticamente a un punto di equilibrio stabile. La presente teoria è pertanto esente dal fenomeno non fisico delle cosiddette « runaway solutions ». Questi risultati generali si illustrano studiando il moto di una particella sotto l'azione di una forza elastica di richiamo e di una forza che dipende dal solo tempo.

Классическая теория электрона.

Резюме (*). — Обсуждается классическая теория электрона, предложенная одним из авторов несколько лет назад и основанная на уравнениях в конечных разностях. Рассматрваются три возможных случая: излучающий электрон, поглощаюий электрон и не поглощающий — не излучающий электрон. В частности, заново рассматриваются так называемые законы прохождения, необходимые вместе с динамическими уравнениями для определения движения заряженной частицы, которое соответствует заданным начальным значениям координат и скорости. Подробно обсуждаются общие характеристики одномерного движения в нерелятивистском приближении. Получено, что в случае излучающего электрона положение частицы стремится асимптотически к точке устойчивого равновесия. Таким образом, предложенная теория свободна от нефизического явления быстро растущих решений. Обшие результаты иллюстрируются на примере движения частицы под действием возвращающей упругой силы и под действием сил, зависящих от времени.

(*) *Переведено редакцией.*