Bubble oscillations in the vicinity of a nearly plane free surface

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The linear oscillation frequency of a bubble in the vicinity of a distorted plane free surface is calculated by a perturbation method. The approximate expression found is compared with numerical results valid for surface deformations of arbitrary magnitude. It is found that the approximate analytical result is quite good, provided that the deformation is small compared with the depth of immersion of the bubble. It is also shown that, unless the deformation of the free surface extends to distances at least of the order of an acoustic wavelength, the "image" bubble has the same source strength of the real bubble so that a dipolar acoustic emission can be expected in spite of the deformation of the surface.

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INTRODUCTION

Recent work shows that a substantial contribution to oceanic ambient noise is caused by the free oscillations of air bubbles entrained by breaking waves, by impacting rain drops or sprays, or possibly by capillary waves of limiting form.[1–7] In all these cases, at the instant when the surface closes, the resulting bubble is not, in general, in equilibrium. The excess initial energy is dissipated in the course of shape and volume oscillations that give rise to the acoustic emission.

Although a great deal is known about the oscillations of bubbles in unbounded liquids (see, e.g., Refs. 8–13), it is not obvious that this information is applicable to the circumstances described above because, during the acoustic emission, the bubble is near the ocean surface, and this surface is not plane.

A typical example is shown in Fig. 1 (with the permission of H. C. Pumphrey and L. A. Crum), which is a frame from a high-speed movie film of a water drop falling on an undisturbed water surface. The left side of the frame shows a bubble that has just detached from the bottom of the crater created by the impact. The right side of the frame shows the oscilloscope trace produced by a hydrophone driven by the sound pulse emitted by the bubble. The frequency of the damped sinusoid measured from this photograph is approximately 7 kHz, and the bubble diameter, measured from a later frame in which the bubble is in equilibrium, is 0.95 mm. It is most remarkable that, in spite of the proximity of the highly distorted free surface, this frequency is in almost perfect agreement with the natural frequency of a bubble of the same radius in an unbounded fluid, which equals 6.78 kHz.

In the present paper, we try to find a theoretical justification for this fact on the basis of a perturbative solution of the problem. The approximation results are checked against numerical ones, and their domain of validity is established. In the last section, we offer some comments on the expected acoustic radiation from the bubble under these conditions.

The basis of our approximation consists in the fact that, in many situations of interest, the distortion of the free surface has a scale larger than the bubble radius and the distance of the bubble from the free surface is not small. The first circumstance arises because, in general, the events that render the free surface multiply connected are so energetic as to produce disturbances on a much larger scale than the bubble. Second, in the neighborhood of the point from which the bubble detaches, the curvature is very large. As soon as the bubble is formed, therefore, under the action of surface tension, this local, high-curvature deformation disappears very

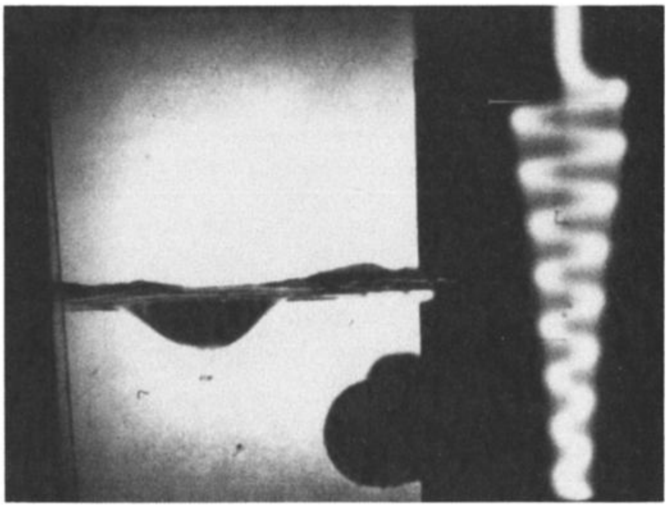

FIG. 1. A frame from a high-speed cinematographic sequence showing the entrapment of an air bubble when a drop hits a liquid surface (courtesy of H. C. Pumphrey and L. A. Crum). The left half of the frame shows the physical event. One can see the free surface indented by the crater caused by the impact and, directly under it, the bubble that has just detached. The shadow in the lower right is a hydrophone that drives an oscilloscope the trace of which is shown, vertically, in the right half of the frame. The right half of the picture shows therefore, in a sense, the "soundtrack" of the left half. The damped sinusoid of the oscilloscope trace is the acoustic signal produced by the volume oscillations of the bubble. A study of the angular distribution of the signal reveals the expected dipole pattern. See Refs. 3–5 for further details.

quickly and the bubble finds itself practically instantaneously at a distance from the free surface of at least a few radii. Both features can clearly be seen in Fig. 1. After the initial stage in which the bubble is formed, the free surface moves on time scales that are slow compared with the oscillation period of the bubble. This fact is ultimately also a consequence of the difference in the characteristic length scales, since these quantities determine the mass, and therefore the inertia, of the mass of fluid involved in the motion. A contributing factor is the fact that the restoring forces for the bubble motion, i.e., internal pressure perturbations and surface tension, are stronger than the gravity and capillary forces that tend to restore the plane equilibrium shape of the water surface.

The approximate framework suggested by the previous considerations is clearly that of "freezing" the shape of the free surface in the neighborhood of the bubble and of treating its deviations from planarity perturbatively. We shall show that, for a number of cases in which the approximate analytical results obtained in this way can be checked numerically, a very good agreement is found.

The conclusion of our study is that, in general, the effect of the free-surface distortion is quite small, so that the oscillation frequency of the bubble in the vicinity of a deformed free surface is very close to its value in the presence of a plane liquid surface. This statement can be considerably strengthened because, as mentioned above and as can be seen in Fig. 1, the depth of immersion of the bubble is frequently at least a few radii. As a consequence, the bubble is at such a depth below the undisturbed free surface that results strictly appropriate for an unbounded liquid are quite accurate, as in the cases investigated in Refs. 3–5.

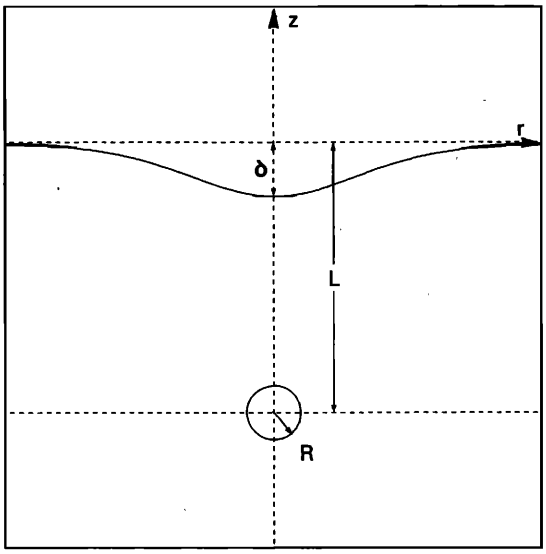

FIG. 2. Geometry of the problem studied in the present paper. The situation is assumed to be axially symmetric, and the axis of symmetry is taken as the z axis of a cylindrical coordinate system (r,z). The solid line represents the free surface and is given by Eq. (30). The bubble radius is R, the depth of its center below the undisturbed plane free surface is L, and, on the axis, the displacement of the free surface below its plane, equilibrium configuration has the value δ.

I. EXPRESSION FOR THE NATURAL FREQUENCY

The geometry of the problem is shown in Fig. 2. Although the general case can be treated in a similar manner, for simplicity, we limit ourselves to the axisymmetric configuration. Thus the free surface is taken to be given by

$$S_f(r,z) \equiv z + \delta f(r) = 0, \tag{1}$$

where δ is the depth of the surface depression, or crater, so that $f(0) = 1$. Let L be the depth of the bubble center under the underformed plane surface. Our perturbation expansion is based on the assumption that

$$\epsilon = \delta / L \ll 1,$$

is small.

The solution of the problem is greatly facilitated by the use of an approximate expression for the natural frequency of the bubble that is now derived in a manner similar to that of Strasberg.[14] The relation that will be obtained is actually the simplest of an infinite number of similar ones that can be found from the general result presented in Ref. 15.

As discussed, e.g., in Refs. 8, 11, and 13, under a wide set of conditions, the pressure p_i inside the bubble can be considered uniform. For linear oscillations, which is the only case we address, any variable can be considered a function of any other one, and, in particular, we may write

$$\frac{dp_i}{dt} = \frac{dp_i}{dV}\frac{dV}{dt}, \tag{2}$$

where V is the bubble volume. The bubble compressibility dp_i/dV is, in general, a complex quantity accounting for both the stiffness of the bubble and the thermal losses affecting the oscillations. This quantity is discussed in considerable detail in Refs. 9–11.

With the neglect of viscosity, the balance of normal stresses across the bubble interface stipulates that

$$p_i = p_B + \sigma C, \tag{3}$$

where p_B is the pressure in the liquid at the interface, σ is the surface-tension coefficient, and C is the local curvature of the interface.[16] For linear oscillations, we have

$$p_B = -\rho \frac{\partial \phi}{\partial t} + p_\infty, \tag{4}$$

where ϕ is the velocity potential, ρ is the liquid density, and p_∞ is the ambient pressure. Note also that

$$\frac{dV}{dt} = -\int_S \mathbf{u}\cdot\mathbf{n}\, dS = \int_S \frac{\partial \phi}{\partial n} dS, \tag{5}$$

where $\mathbf{u}$ is the liquid velocity, $\mathbf{n}$ is the unit normal directed into the bubble, and S is the bubble surface. Let averages over the bubble surface be denoted by angle brackets:

$$\langle(\cdots)\rangle = \frac{1}{S}\int_S (\cdots) dS, \tag{6}$$

where $(\cdots)$ indicates any quantity. In the linear approximation the operations of averaging and time differentiation commute so that, combining Eqs. (2)–(5), we find

$$\rho \frac{d^2}{dt^2}\langle\phi\rangle = \frac{dp_i}{dV} S \left\langle \frac{\partial \phi}{\partial n} \right\rangle + \sigma \frac{d}{dt}\langle C \rangle. \tag{7}$$

This relation becomes particularly useful with the further approximation that the bubble oscillates with only minor deviations from sphericity. This approximation is adequate to capture the effect of the volume changes occurring during the bubble's motion, which give rise to the largest contribution to the radiated sound. According to recent speculations,[17] shape deformations might also be important, but only insofar as they can excite the volume mode. Our results will therefore not be imcompatible with such other mechanisms of noise radiation. Since, for a spherical bubble,

$$\frac{d}{dt}\langle C\rangle = -\frac{2}{R^2}\frac{dR}{dt} = \frac{2}{R^2}\left\langle\frac{\partial\phi}{\partial n}\right\rangle,$$

upon the substitution $d/dt \to i\omega$, we find

$$\omega^2 = -\frac{1}{\rho}\left(S\frac{dp_i}{dV} + \frac{2\sigma}{R^2}\right)\frac{1}{\langle\phi\rangle}\left\langle\frac{\partial\phi}{\partial n}\right\rangle. \tag{8}$$

It is readily checked that this result reduces to the usual one (see, e.g., Ref. 11) in the case of a spherical bubble in an infinite liquid, namely,

$$\omega_0^2 = -\frac{1}{R\rho}\left(S\frac{dp_i}{dV} + \frac{2\sigma}{R^2}\right). \tag{9}$$

We may, therefore, write

$$\frac{\omega^2}{\omega_0^2} = \frac{R}{\langle\phi\rangle}\left\langle\frac{\partial\phi}{\partial n}\right\rangle. \tag{10}$$

II. THE VELOCITY POTENTIAL

The bubble can only remain close to spherical if the velocity potential ϕ is very nearly uniform over its surface. A consistent approximation is therefore to calculate the surface averages appearing in Eq. (10) by solving Laplace's equation $\nabla^2\phi = 0$ subject to the boundary condition $\phi = 1$ on the bubble's surface and

$$\phi = 0 \quad \text{on } S_f = 0. \tag{11}$$

The first condition arises from the linearity of the problem and the division by $\langle\phi\rangle$ in (10), the second one from the assumption that S_f behaves as a pressure-release boundary. This formulation for the potential problem is identical to that for the calculation of the capacitance of a conductor in the presence of a grounded conducting surface, an analogy already noticed and exploited by Strasberg.[14]

We look for an approximate solution to this problem in the form

$$\phi = \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \cdots. \tag{12}$$

It readily follows from (11) that, on $z = 0$,

$$\phi_0 = 0, \quad \phi_1 = Lf(r)\frac{\partial\phi_0}{\partial z},$$

$$\phi_2 = Lf(r)\frac{\partial\phi_1}{\partial z} - \frac{1}{2}L^2 f^2(r)\frac{\partial^2\phi_0}{\partial z^2},\ldots, \tag{13}$$

while $\phi_0 = 1$, $\phi_1 = \phi_2 = \cdots = 0$ on the bubble surface.

At this point it is convenient to introduce bispherical coordinates (ξ,η) related to the cylindrical coordinates (r,z) by

$$r = R\sinh\alpha[\sin\eta/(\cosh\xi - \cos\eta)], \tag{14}$$

$$z = -R\sinh\alpha[\sinh\xi/(\cosh\xi - \cos\eta)], \tag{15}$$

where

$$\alpha = \log[L/R + \sqrt{(L/R)^2 - 1}]. \tag{16}$$

Since in the present problem the physical domain corresponds to $z \leqslant 0$, in Eq. (15) we have introduced a minus sign so that $\xi \geqslant 0$ in the region occupied by the liquid. The surface $\xi = \alpha > 0$ is the bubble, while the undisturbed free surface $z = 0$ corresponds to $\xi = 0$. For $\xi = 0$, $\cosh\eta \to 1$ corresponds to $r \to \infty$.

The boundary conditions on the bubble surface are satisfied by writing

$$\phi_0(\eta,\xi) = (\cosh\xi - \cos\eta)^{1/2}\sum_{n=0}^{\infty}\sqrt{2}e^{-(n+1/2)\alpha} \times[\sinh(n+\tfrac{1}{2})\xi/\sinh(n+\tfrac{1}{2})\alpha]P_n(\cos\eta), \tag{17}$$

$$\phi_1(\eta,\xi) = (\cosh\xi - \cos\eta)^{1/2}\sum_{n=0}^{\infty}A_n[\sinh(n+\tfrac{1}{2}) \times(\alpha-\xi)/\sinh(n+\tfrac{1}{2})\alpha]P_n(\cos\eta), \tag{18}$$

$$\phi_2(\eta,\xi) = (\cosh\xi - \cos\eta)^{1/2}\sum_{n=0}^{\infty}B_n[\sinh(n+\tfrac{1}{2}) \times(\alpha-\xi)/\sinh(n+\tfrac{1}{2})\alpha]P_n(\cos\eta), \tag{19}$$

where the P_n's are the Legendre polynomials, and A_n and B_n must be determined from the boundary conditions on the free surface. A direct substitution into Eq. (13) leads, after some manipulations, to

$$\sum_{n=0}^{\infty}A_nP_n(\cos\eta) = -\sqrt{2}\frac{L}{R\sinh\alpha}f(\cos\eta)(1-\cos\eta) \times\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})\exp[-(n+\frac{1}{2})\alpha]}{\sinh(n+\frac{1}{2})\alpha} \times P_n(\cos\eta), \tag{20}$$

$$\sum_{n=0}^{\infty}B_nP_n(\cos\eta) = \frac{L}{R\sinh\alpha}f(\cos\eta)(1-\cos\eta) \times\sum_{n=0}^{\infty}\frac{(2n+1)A_n}{\tanh(n+\frac{1}{2})\alpha}P_n(\cos\eta), \tag{21}$$

where we write $f(\cos\eta)$ for the value of $f(r)$ obtained after substitution of (14) with $\xi = 0$. If the shape function f is given, the coefficients A_n and B_n can be calculated, at least in principle, by taking the scalar product of these relations with P_k. In this way we find

$$A_k = -\sqrt{2}\left(k+\frac{1}{2}\right)\frac{L}{R\sinh\alpha} \times\sum_{n=0}^{\infty}\frac{(n+\frac{1}{2})\exp[-(n+\frac{1}{2})\alpha]}{\sinh(n+\frac{1}{2})\alpha} \times\int_{-1}^{1}f(\mu)(1-\mu)P_n(\mu)P_k(\mu)d\mu, \tag{22}$$

where $\mu = \cos\eta$, and similarly for B_k.

III. RESULTS

With ϕ determined, $\partial\phi/\partial n$ is given by the sum of the three quantities

$$\left.\frac{\partial\phi_0}{\partial n}\right|_{\xi=\alpha} = \frac{(\cosh\alpha - \cos\eta)^{1/2}}{\sqrt{2}R} \times \sum_{n=0}^{\infty} e^{-(n+1/2)\alpha} P_n(\cos\eta) + \frac{(\cosh\alpha - \cos\eta)^{3/2}}{R\sinh\alpha} \times \sum_{n=0}^{\infty} \frac{\sqrt{2}(n+\frac{1}{2})e^{-(n+1/2)\alpha}}{\tanh(n+\frac{1}{2})\alpha} P_n(\cos\eta),$$

$$\left.\frac{\partial\phi_1}{\partial n}\right|_{\xi=\alpha} = -\epsilon\frac{(\cosh\alpha - \cos\eta)^{3/2}}{R\sinh\alpha} \times \sum_{n=0}^{\infty} \frac{\sqrt{2}(n+\frac{1}{2})A_n}{\sinh(n+\frac{1}{2})\alpha} P_n(\cos\eta),$$

$$\left.\frac{\partial\phi_1}{\partial n}\right|_{\xi=\alpha} = -\epsilon^2\frac{(\cosh\alpha - \cos\eta)^{3/2}}{R\sinh\alpha} \times \sum_{n=0}^{\infty} \frac{\sqrt{2}(n+\frac{1}{2})B_n}{\sinh(n+\frac{1}{2})\alpha} P_n(\cos\eta).$$

Upon averaging over the bubble and substituting the results into the expression (10) for the natural frequency, we find

$$\omega^2/\omega_0^2 = S_0 + \epsilon S_1 + \epsilon^2 S_2 + O(\epsilon^3), \qquad (23)$$

where

$$S_0 = 2\sum_{n=0}^{\infty} C_n, \qquad (24)$$

$$S_1 = -\sqrt{2}\sum_{n=0}^{\infty} A_n C_n, \qquad (25)$$

$$S_2 = -\sqrt{2}\sum_{n=0}^{\infty} B_n C_n, \qquad (26)$$

with

$$C_n = \frac{\sqrt{(L/R)^2-1}}{[L/R+\sqrt{(L/R)^2-1}]^{(2n+1)}-1} = \frac{\sinh\alpha}{e^{(2n+1)\alpha}-1}. \qquad (27)$$

A graph of S_0 and its square root as functions of L/R is shown in Fig. 3. This function diverges logarithmically as the bubble approaches the plane free surface, $R \to L$.[18] However, the deviation from 1 is significant only for depths of immersion of the bubble smaller than about three bubble radii.

To illustrate the results for a deformed surface with a specific example, we take the following one-parameter family of functions to describe the deformation:

$$f(r) = [(L^2-R^2)/(L^2-R^2+r^2)]^s, \qquad (28)$$

where the exponent s is the free parameter. Upon substitution of (14) with $\xi = 0$, we find

$$f(\cos\eta) = [(1-\cos\eta)/2]^s, \qquad (29)$$

a form previously used by Longuet-Higgins[19] for a similar purpose. The shape of the free surface corresponding to (1) is given by

$$S_f(r,z) \equiv z + \delta\,[(L^2-R^2)/(L^2-R^2+r^2)]^s = 0, \qquad (30)$$

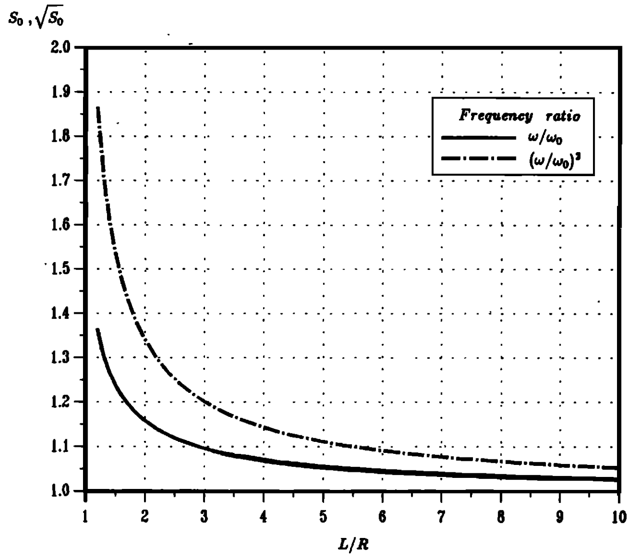

FIG. 3. The dashed line represents the ratio of the oscillation frequency of a bubble near a plane surface to its value in an unbounded fluid. The abscissa is the distance from the plane divided by the bubble radius. The solid line is the square of the preceding quantity and is therefore a graph of the function S_0 defined in Eq. (24).

and is shown in Fig. 4 for $L/R = 10$ and $s = 5$ [Fig. 4(a)] and $s = 50$ [Fig. 4(b)]. From these figures, or directly from (28) or (29), it is possible to see that the parameter s determines the sharpness of the deformation. Indeed, for large s, it can be shown that

$$f(r) \approx \exp\{-s[r^2/(L^2-R^2)]\}. \qquad (31)$$

The function f possesses the following expansion in terms of Legendre polynomials:

$$\left(\frac{1-\mu}{2}\right)^s = \sum_{m=0}^{\infty} \frac{(-1)^m(2m+1)(s!)^2}{(s-m)!(s+m+1)!} P_m(\mu). \qquad (32)$$

Using this result and, for the product of Legendre polynomials, the known representation

$$P_m(\mu)P_n(\mu) = \sum_{k=0}^{m} \frac{\alpha_{m-k}\alpha_k\alpha_{n-k}}{\alpha_{n+m-k}}\left(\frac{2n+2m-4k+1}{2n+2m-2k+1}\right) \times P_{n+m-2k}(\mu), \qquad (33)$$

where $\alpha_k = (2k-1)!!/k!$, we can calculate S_1 and S_2.

The values of the first- and second-order corrections S_1, Eq. (25), and S_2, Eq. (26), for the surface deformation (28) are plotted as functions of the ratio L/R for various values of s in Fig. 5(a) and (b), respectively. As expected, the correction due to the deformation vanishes quickly as the ratio L/R increases.

The accuracy and range of applicability of the perturbative solution can be checked by solving the potential problem numerically. A boundary integral formulation is suitable for this purpose. Since this part of our work is standard,[20,21] it is not necessary to give details on the technique or its numerical implementation.

In Fig. 6(a) and (b) we compare the perturbation solution with the numerical one as a function of the magnitude of the surface deformation as measured by the parameter $\epsilon = \delta/L$ for $s = 5$ and 50. Here, we have taken $L/R = 10$,

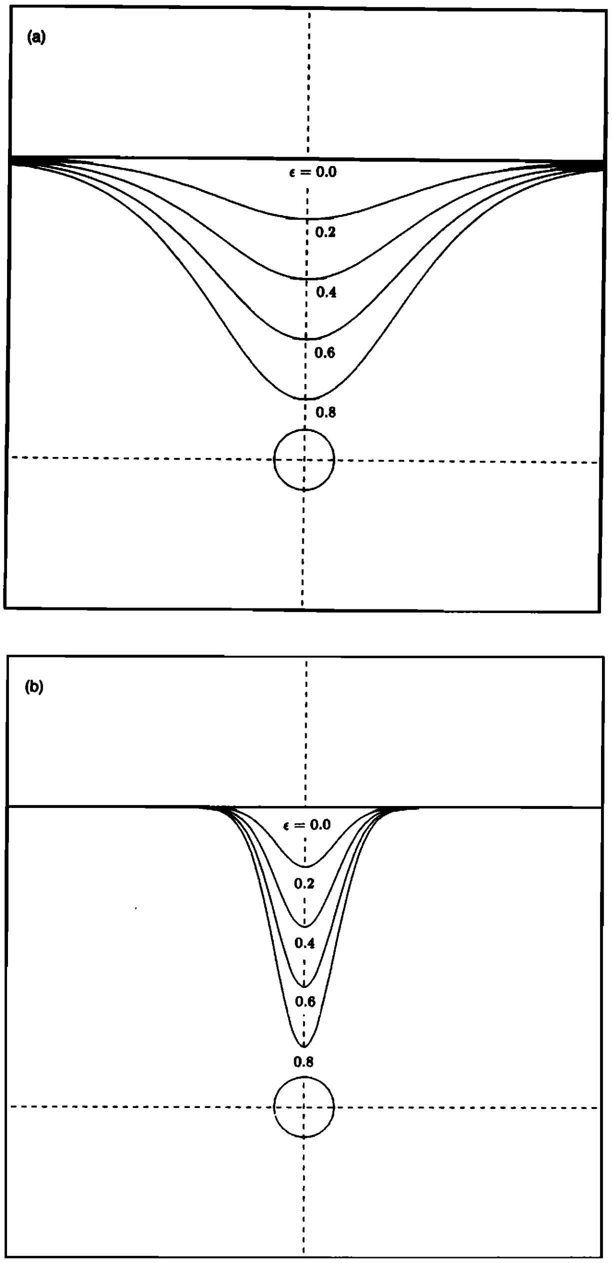

FIG. 4. Shapes of the distorted free surface according to Eq. (30) with increasing $\epsilon = \delta/L$ for (a) $s = 5$ and (b) $s = 50$.

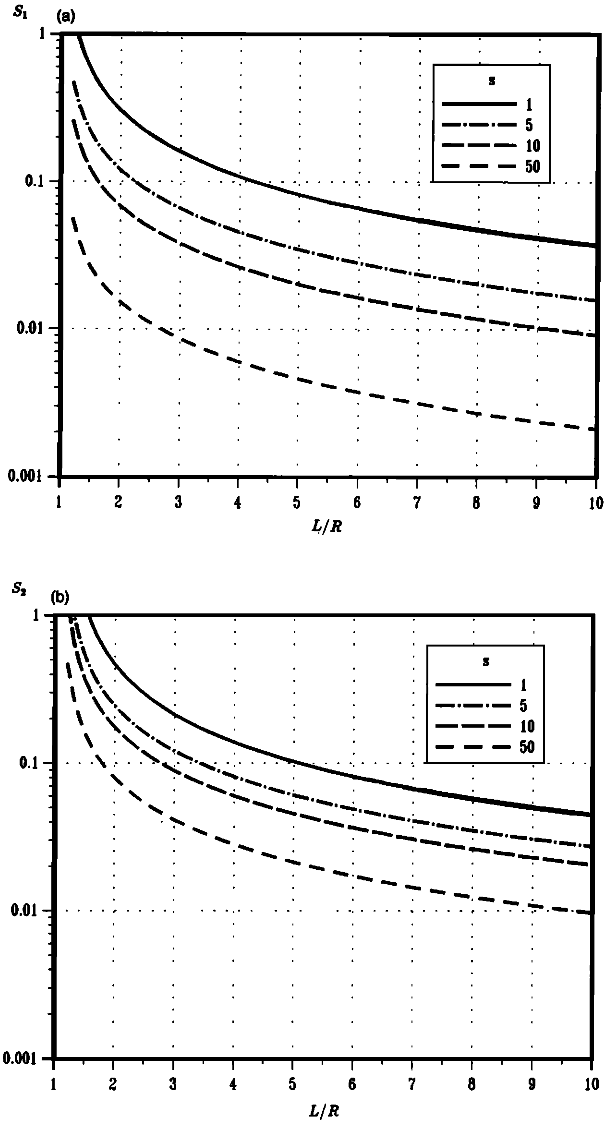

FIG. 5. Graphs of the first- and second-order corrections (a) S_1, Eq. (24), and (b) S_2, Eq. (25), to $(\omega/\omega_0)^2$ for the surface deformation (28) as functions of the ratio L/R for various values of s.

which is typical of the order of magnitude of the depth of the bubble in the experiments of Pumphrey. These figures show the squares of the ratio ω/ω_0 vs ϵ, which may be viewed as a measure of the distortion of the free surface as seen by the bubble. The perturbation solution agrees very well with the numerical one up to $\epsilon \sim 0.5$. Interestingly enough, the error is about the same for $s = 5$ and 50, even though the shapes are very different as can be seen from Fig. 4(a) and (b).

A more complete picture of the effect of the surface deformation is given in Fig. 7, which shows the squares of the ratio ω/ω_0 calculated numerically by the boundary integral method again for $L/R = 10$, but on a different scale for more values of s. It may be noted that $\epsilon = 0.7$ and 0.8 correspond to thicknesses of $2R$ and R, respectively, of the layer of liquid separating the bubble from the free surface. The small effect of the surface deformation on the frequency of oscillation is quite remarkable.

IV. ACOUSTIC RADIATION

The presence of a plane free surface has the effect of turning the monopole radiation that a bubble would produce in an unbounded liquid into a dipole acoustic emission so that the total radiated acoustic energy is smaller than what it would otherwise be by a factor of the order of $(kL)^2$. For a bubble oscillating at its natural frequency in water, the wavenumber k is roughly given by $k \simeq 0.013/R$ and therefore $(kL)^2 \sim 10^{-4}(L/R)^2$. With a typical order of magnitude $L/R \sim 10$, the acoustic radiation is reduced approximately a 100-fold with respect to the unbounded liquid case. It is therefore of interest to inquire whether the deformation of the plane free surface can restore a monopole component. An argument in favor of this possibility might be that a

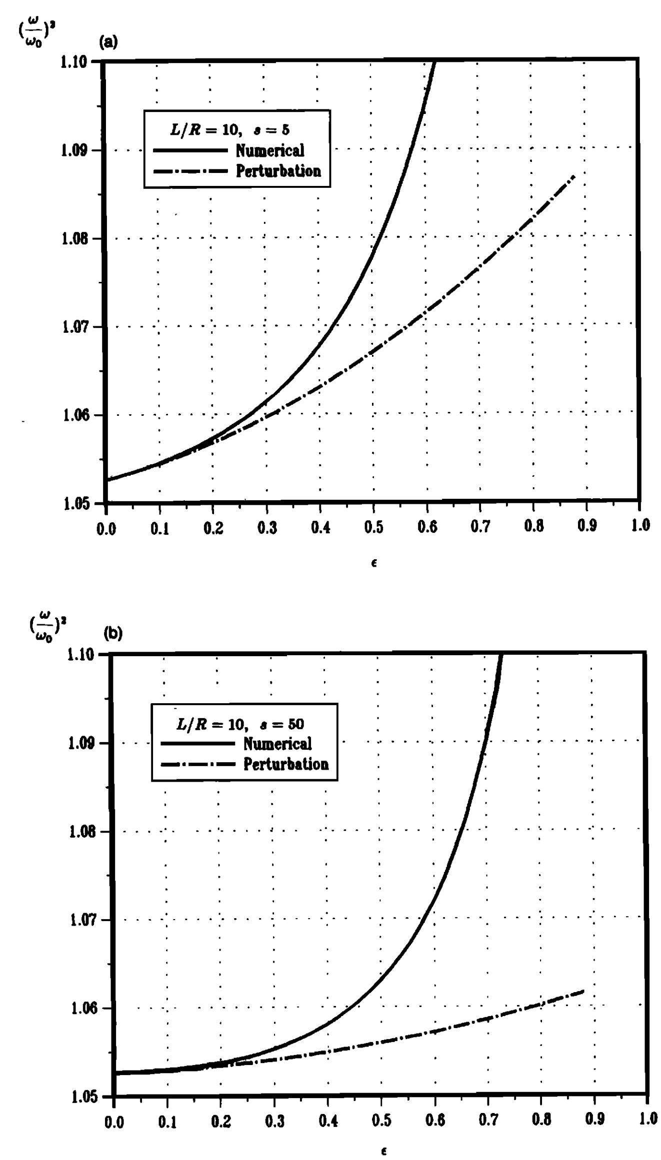

FIG. 6. Comparison of the perturbation and numerical solutions as functions of the dimensionless deformation amplitude $\epsilon = \delta/R$. Here, $L/R = 10$ and (a) $s = 5$ (b) $s = 50$.

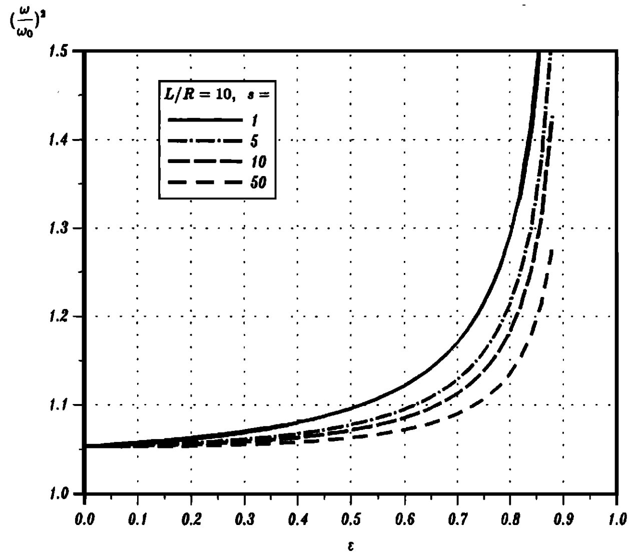

FIG. 7. Graph of the square of the ratio ω/ω_0 calculated numerically by the boundary integral method as a function of the dimensionless deformation amplitude $\epsilon = \delta/R$ for various values of s. Here, $L/R = 10$.

source radiating near a spherical pressure-release boundary does retain a fraction of its monopole nature because the image source has a strength smaller than the real one. Our conclusion is, however, negative for the case of present concern, as we now show.

If the potential has a component that behaves like $Q/4\pi\sqrt{r^2+z^2}$ at large distances from the bubble, the monopole source strength Q can be calculated from

$$Q = 2\int_{S_\infty} \frac{\partial \phi}{\partial n}\, dS, \tag{34}$$

where S_∞ represents a surface tending to infinity in the limit, and the factor 2 has been introduced to account for the fact that the fluid region in the present problem occupies only half of the total solid angle. It is possible to calculate exactly this source strength by noting that the total mass flow rate must vanish:

$$\int_S \frac{\partial \phi}{\partial n}\, dS + \int_{S_f} \frac{\partial \phi}{\partial n}\, dS + \int_{S_\infty} \frac{\partial \phi}{\partial n}\, dS = 0,$$

so that

$$Q = -2\int_S \frac{\partial \phi}{\partial n}\, dS - 2\int_{S_f} \frac{\partial \phi}{\partial n}\, dS. \tag{35}$$

The first integral in this equation has already been used for calculation of the natural frequency, and it is given, up to the first order in ϵ, by

$$\int_S \frac{\partial \phi}{\partial n}\, dS = 4\sqrt{2}\pi R \sinh\alpha \sum_{n=0}^{\infty} \frac{\sqrt{2} - \epsilon A_n}{e^{(2n+1)\alpha} - 1}. \tag{36}$$

The integral over the free surface S_f is complicated by the distortion of the surface, but can be easily calculated approximately to the first order in ϵ. To this end we begin by expressing it in bispherical coordinates, writing

$$\int_{S_f} \frac{\partial \phi}{\partial n}\, dS = 2\pi R \sinh\alpha \int_{-1}^{1} \left(\frac{\partial \phi}{\partial \xi}\frac{1}{\cosh\xi - \mu}\right)_{z=-\delta f(r)} d\mu. \tag{37}$$

Next, we expand the integral in Taylor series to find

$$[1/(\cosh\xi - \mu)]_{z=-\delta f(r)} = [1/(1-\mu)] + O(\epsilon^2),$$

and

$$\left(\frac{\partial \phi}{\partial \xi}\right)_{z=\delta f(x)} = \frac{\partial \phi_0}{\partial \xi}\bigg|_{(\xi=0)} + \frac{\partial \phi_1}{\partial \xi}\bigg|_{(\xi=0)} + O(\epsilon^2).$$

It may be noted that here the effect of the perturbation only enters through ϕ_1 since $\partial^2\phi_0/\partial\xi^2$ vanishes on the plane $\xi = 0$. After substitution of these approximations in Eq. (37), we obtain, to first order,

$$\int_{S_f} \frac{\partial \phi}{\partial n} dS = -2\sqrt{2}\pi R \sinh \alpha \times \left(\sum_{n=0}^{\infty} \frac{2\sqrt{2}}{e^{(2n+1)\alpha} - 1} - \epsilon \sum_{n=0}^{\infty} \frac{A_n}{\tanh(n+\frac{1}{2})\alpha} \right). \quad (38)$$

Finally, the addition of (36) and (38), after some simplifications, gives the monopole component of the potential as

$$Q = 2\pi R \epsilon \sqrt{2} \sinh \alpha \sum_{k=0}^{\infty} A_k. \quad (39)$$

Since the A_k's depend on the form of the shape function, this relation might suggest, in general, the presence of a monopole component. However, it is easy to prove that the summation is actually zero whatever the form of the surface deformation. Indeed, upon subsitution into (39) of the expression (22) for A_k, interchange of the order of summation over k and over n, and of the order of the integration and summation over k, we find

$$Q = -4\pi\epsilon L \sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})\exp[-(n+\frac{1}{2})\alpha]}{\sinh[(n+\frac{1}{2})\alpha]} \times \int_{-1}^{1} f(\mu)(1-\mu)\left[\sum_{k=0}^{\infty}\left(k+\frac{1}{2}\right)P_k(\mu)\right]P_n(\mu)d\mu.$$

With the help of the generating function for the Legendre polynomials, it can be shown that the summation over k equals $\delta(1-\mu)$, a result that can also be verified directly using the definition of a distribution as a functional over the space of test functions. Since $P_n(1) = 1$, the integral over μ becomes

$$\int_{-1}^{1} f(\mu)(1-\mu)\delta(1-\mu)d\mu,$$

and for it not to vanish, $f(\mu)$ should be singular near $\eta = 0$, i.e., at infinity, which is evidently impossible both physically and in the present perturbation framework. We therefore conclude that the deformation of the free surface cannot restore a monopole component, at least to the order considered.

The result that we have obtained becomes more interesting and physically transparent if it is viewed in a broader context. It is well known that, in a situation with acoustically compact sources, the assumption of incompressible flow can be used to obtain a valid inner approximation to the complete problem.[22–25] From the point of view of this approximation, *infinity* is the region where the incompressible inner field matches the acoustic outer solution. The previous result can therefore be interpreted as implying that, unless the surface distortion extends to distances of the order of an acoustic wavelength or more from the bubble, the acoustic emission will retain a dipolar nature. In the laboratory setting in which the photo shown in Fig. 1 was taken, the plane surface was only disturbed near the bubble, and therefore no monopolar component would be expected, in agreement with observation.[5] In the case of actual rain falling on a liquid surface, however, surface perturbations extending over many acoustic wavelengths will be present in the neighborhood of the bubble, and therefore a limited reinforcement of the dipolar radiation cannot be ruled out.

The surface deformation does not alter the dipole nature of the radiated sound, but it does of course affect the magnitude of the dipole itself. To determine this change quantitatively, we use the fact that, since no monopole component is present, the farfield behavior of the potential must be of the form

$$\phi_0 + \epsilon\phi_1 \approx -[(\mu_0 - \epsilon\mu_1)/\rho^2]P_1(\cos\theta),$$

where ρ and θ indicate polar coordinates centered at the origin, and P_1 is the second Legendre polynomial. The correction μ_1, as defined here, will be found to be positive, corresponding to a decrease of the dipole moment for a surface deformed toward the bubble, for which $\epsilon > 0$. This result is due to the fact that, in this case, the bubble is closer to the surface than in the plane case. Along the negative z axis, where $\rho = -z$ and $\theta = \pi$, we then have

$$\phi_0 + \epsilon\phi_1 \approx (\mu_0 - \epsilon\mu_1)/z^2. \quad (40)$$

On the other hand, for $|z| \gg R$, we have, from (15),

$$1/z^2 \approx \xi^2/(2R\sinh\alpha)^2 \ll 1, \quad (41)$$

since $\eta = 0$ along the z axis. We can then find expressions for μ_0 and μ_1 by expanding ϕ_0 and ϕ_1 in Taylor series in ξ and keeping only terms of order ξ^2. In this way, and using the fact that

$$\sum_{n=0}^{\infty} A_n = 0, \quad (42)$$

we find

$$\mu_0 = (2R\sinh\alpha)^2 \sum_{n=0}^{\infty} e^{-(n+1/2)\alpha} \frac{(n+\frac{1}{2})}{\sinh(n+\frac{1}{2})\alpha}, \quad (43)$$

$$\mu_1 = 2\sqrt{2}(R\sinh\alpha)^2 \sum_{n=0}^{\infty} A_n\left(n+\frac{1}{2}\right)\frac{\cosh(n+\frac{1}{2})\alpha}{\sinh(n+\frac{1}{2})\alpha}. \quad (44)$$

In Fig. 8 we plot the plane-surface dipole moment μ_0 normalized by $2L/R$, its asymptotic value for $R/2L \to 0$, and

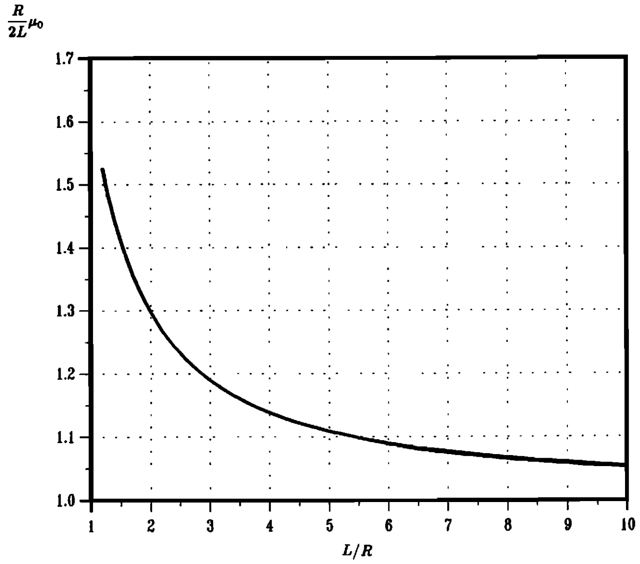

FIG. 8. Graph of the plane-surface dipole moment μ_0 scaled with $R/2L$ as a function of L/R.

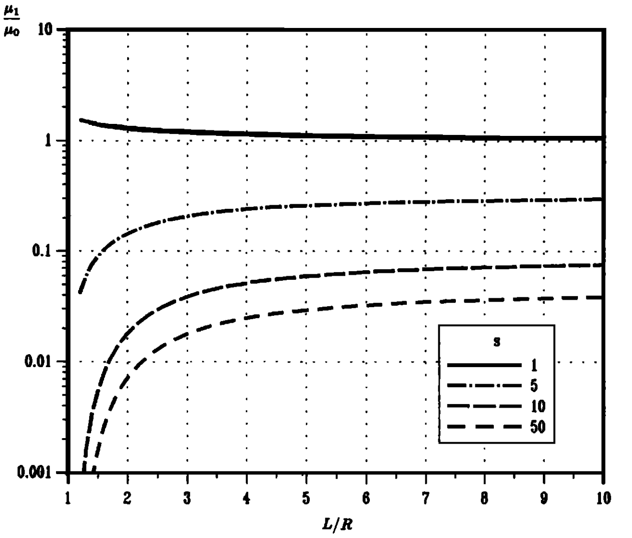

FIG. 9. Graph of the dipole moment ratio μ_1/μ_0 as a function of L/R for various values of s.

in Fig. 9 the ratio μ_1/μ_0 as function of L/R for various values of s. It can be readily shown from (28) that the horizontal scale of the deformation is a decreasing function of s. The smaller the value of s, therefore, the bigger the effect that one anticipates. This expectation is borne out by the numerical results of Fig. 9.

V. CONCLUSIONS

We have studied the oscillations of a gas bubble in the neighborhood of a nearly plane pressure-release boundary. It has been found that the effect of the distortion of the boundary is generally small and can be predicted by a perturbation approach. This conclusion is in agreement with the limited available experimental evidence. The approximate results obtained by the perturbation scheme have been checked against a fully numerical solution, and an excellent agreement has been found over a large range of values of the perturbation parameter.

A consideration of the bubble's acoustic emission shows that the surface distortion cannot enhance the radiation of sound over that of a dipole, unless it extends beyond distances of the order of an acoustic wavelength from the bubble. This conclusion might suggest that rain falling on a body of water would be a somewhat stronger acoustic radiator than would be expected on the basis of laboratory experiments with single drops, where the surface of the receiving liquid is plane, except for the localized perturbation induced by the impacting drop. In the situation described, however, the drops would hit the liquid surface at an angle, which is expected to adversely affect their ability to entrain bubbles.[26] The net result of these two effects on the underwater noise of rain cannot therefore be established *a priori*.

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