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The Motion of a Charged Particle in a Uniform Magnetic Field.

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Summary. — We study the motion of a classical (nonquantal) charged particle in a uniform magnetic field by means of i) the Abraham-Lorentz equation, ii) the Dirac relativistic equation and iii) the Caldirola nonrelativistic, finite-difference equation. In cases i) and iii) closed-form solutions are obtained. For case ii) we apply for the first time the two-variable asymptotic method which enables us to obtain a uniformly valid approximate solution free of the secular terms present in the results of previous authors.

1. – Introduction.

In the present study we consider the motion of a classical (*i.e.* nonquantal) charged particle in a uniform magnetic field in terms of the Abraham-Lorentz nonrelativistic equation, of the Dirac relativistic equation and of the Caldirola nonrelativistic, finite-difference equation. This paper has been motivated primarily by the renewed interest in the classical motion of charged particles, both for technological applications and for fundamental reasons ([1-5]). In particular, the chronon theory of Caldirola ([6]) exhibits an intriguing variety

([1]) W. Maas and J. Petzold: *J. Phys. A*, **11**, 1211 (1978).
([2]) M. Sorg: *Z. Naturforsch.*, **29**a, 1671 (1974); **31**a, 644, 1133 (1976); **32**a, 101, 659 (1977).
([3]) G. H. Goedecke: *Nuovo Cimento B*, **28**, 225 (1975).
([4]) H. Levine, E. J. Moniz and D. H. Shapp: *Am. J. Phys.*, **45**, 75 (1977).
([5]) P. Caldirola, G. Casati and A. Prosperetti: *Nuovo Cimento A*, **43**, 127 (1978).
([6]) P. Caldirola: *Nuovo Cimento Suppl.*, **3**, 297 (1956); *Nuovo Cimento A*, **45**, 548 (1978); **49**, 497 (1979); *Lett. Nuovo Cimento*, **15**, 489 (1976); **16**, 151 (1976); **17**, 461 (1976); **18**, 465 (1977); **20**, 519, 632 (1977); **21**, 250 (1978); **23**, 83 (1978); *Riv. Nuovo Cimento*, **2**, N. 13 (1979).

of aspects which deserve to be investigated in view of their possible bearing on the current microscopic description of the physical universe. For the Dirac model we apply here for the first time the asymptotic method of two-variable expansion [7] which is found to be ideally suited for the present purpose and to hold considerable promise for other applications of the Dirac equation.

The unifying theme of this study is the effect of radiation reaction on the motion of the charged particle [8]. In the Abraham-Lorentz model this reaction is computed from classical electrodynamics as the effect of the electromagnetic field produced by the charge distribution of the particle on itself [9,10]. DIRAC provided a relativistic generalization of this result on the basis of the conservation laws for the energy and momentum of the electromagnetic field [11-13].

In the equation of Caldirola, radiation reaction appears naturally as a consequence of the fundamental postulate of the theory, namely the discontinuous interaction between the particle and the external forces acting on it at successive instants separated by a fundamental time interval, the *chronon* [5,6].

2. – Preliminary considerations.

We perform first of all an approximate analysis of the problem with the purpose of determining the important physical quantities. The basis for this treatment is the assumption that radiation reaction is small so that, to first order, it can be considered as a perturbation of the motion of a nonradiating particle. Then, to zero order, the motion of the particle is determined solely by the Lorentz force through Newton's equation

$$m\frac{V^2}{r}=\frac{e}{c}BV\,, \tag{1}$$

where $V=|\boldsymbol{V}|$ is the magnitude of the velocity component perpendicular to the magnetic field, r is the radius of the orbit, B is the magnitude of the magnetic field, c is the speed of light and m, e are the mass and electric charge of

(7) See, *e.g.*, J. D. COLE: *Perturbation Methods in Applied Mathematics* (Waltham, Mass., 1968).
(8) T. ERBER: *Fortschr. Phys.*, **9**, 343 (1961).
(9) J. D. JACKSON: *Classical Electrodynamics* (New York, N. Y., 1962, 1975).
(10) W. K. H. PANOFSKY and M. PHILLIPS: *Classical Electricity and Magnetism* (Reading, Mass., 1955).
(11) P. A. M. DIRAC: *Proc. R. Soc. London Ser. A*, **161**, 148 (1938).
(12) A. O. BARUT: *Electrodynamics and Classical Theory of Fields and Particles* (New York, N. Y., 1964).
(13) F. ROHRLICH: *Classical Charged Particles* (Reading, Mass., 1965).

the particle. Equation (1) shows that the trajectory of the particle is a circle of radius

$$r = \frac{1}{\omega_L} V, \tag{2}$$

where

$$\omega_L = \frac{eB}{mc} \tag{3}$$

is the Larmor frequency, *i.e.* the angular frequency of the unperturbed motion.

The perturbation induced by radiation on the motion described by (1) and (2) can now be computed approximately equating the time derivative of the kinetic energy, $\frac{1}{2}mV^2$, to minus the radiated power P given, according to the well-known result of Larmor [9,10], by

$$P = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{d\boldsymbol{V}}{dt}\right)^2.$$

In this way we may write, using (1) to compute $(d\boldsymbol{V}/dt)^2$,

$$\frac{d}{dt}\left(\frac{1}{2}mV^2\right) = -\frac{2}{3}\frac{e^2}{c^3}\left(\frac{eB}{mc}V\right)^2,$$

from which

$$\frac{dV}{dt} = -\frac{2}{3}\frac{e^2\omega_L^2}{mc^3}V. \tag{4}$$

The solution of this equation is

$$V = V_0 \exp[-\theta\omega_L^2 t], \tag{5}$$

where $V_0 = V(0)$ and θ, the so-called *chronon* [6], is given by

$$\theta = \frac{2}{3}\frac{e^2}{mc^3}. \tag{6}$$

The radius r can be obtained from (5) and (2) as

$$r = r_0 \exp[-\theta\omega_L^2 t], \tag{7}$$

where $r_0 = V_0/\omega_L$. It is seen that in this approximate treatment the trajectory of the particle is a spiral of slowly decreasing radius. If the velocity component parallel to the magnetic field is zero, the particle will eventually come to rest at the centre of the orbit. Clearly, for the results (5) and (6) to be an acceptable approximation, the characteristic time for the decrease of the velocity must be much greater than that for the completion of one revolution, *i.e.*

$$\theta\omega_{\mathrm{L}} \ll 1\,. \tag{8}$$

3. – The Abraham-Lorentz equation.

A more precise treatment of the problem at hand can be made on the basis of the Abraham-Lorentz equation of motion for a charged particle [9,10] which incorporates in a systematic way the energy loss by radiation. For the present case this equation is

$$m\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t} - \frac{2}{3}\frac{e^2}{c^3}\frac{\mathrm{d}^2\boldsymbol{V}}{\mathrm{d}t^2} = \frac{e}{c}\boldsymbol{V}\times\boldsymbol{B}\,. \tag{9}$$

Taking $\boldsymbol{B}$ directed along the positive z-axis and projecting on the x and y axes, we obtain a system of two equations which can be compactly written as

$$\left(\frac{\mathrm{d}}{\mathrm{d}t} - \theta\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)\boldsymbol{V} = \omega_{\mathrm{L}} A\boldsymbol{V}\,, \tag{10}$$

where

$$A = \begin{vmatrix} 0 & 1 \\ -1 & 0 \end{vmatrix}\,, \qquad \boldsymbol{V} = \begin{vmatrix} u \\ v \end{vmatrix} \tag{11}$$

and u, v are the x and y components of $\boldsymbol{V}$.

Equation (10) can readily be solved by expanding the vector $\boldsymbol{V}$ on the basis formed the eigenvectors of A. In this way we find solutions with a time dependence of the form

$$p_i \exp[\mu_i t] + q_i \exp[\nu_i t]\,, \qquad i = 1, 2\,,$$

where

$$\mu_i = -b \pm i\omega\,, \qquad \nu_i = b' \pm i\omega\,,$$

$$b = \tau^{-1}(2^{-\frac{1}{2}}\{[1 + 4\omega_{\mathrm{L}}^2\tau^2]^{\frac{1}{2}} + 1\}^{\frac{1}{2}} - 1)\,, \tag{12}$$

$$\omega = 2^{-\frac{1}{2}}\tau^{-1}\{[1 + 4\omega_{\mathrm{L}}^2\tau^2]^{\frac{1}{2}} - 1\}^{\frac{1}{2}}\,, \tag{13}$$

$$b' = \tau^{-1}(2^{-\frac{1}{2}}\{[1 + 4\omega_{\mathrm{L}}^2\tau^2]^{\frac{1}{2}} + 1\}^{\frac{1}{2}} + 1)$$

and the notation

$$\tau = 2\theta = \frac{4}{3}\frac{e^2}{mc^3} \tag{14}$$

has been used. It is clear that $\operatorname{Re} \nu_i = b' > 0$, so that, for the solution to be bounded for $t \to \infty$, we must set $q_i = 0$, $i = 1.2$. The appearance of these non-physical, unbounded solutions is a well-known problem with the Abraham-Lorentz and Dirac theories which is a consequence of the presence of the second time derivative of the velocity in (9) (9-13). This difficulty has led to various reformulations of the theory (*e.g.* in terms of an integro-differential equation, see ref. (13)) and has also been one of the elements that has induced other authors to abandon altogether the differential-equation formulation in favour of finite-difference ones (6).

The remaining two constants p_i are to be determined from the initial conditions $u(0) = u_0$, $v(0) = v_0$. Finally we find

$$\left\{\begin{aligned} u &= \exp[-bt](u_0 \cos\omega t + v_0 \sin\omega t)\,, \\ v &= \exp[-bt](-u_0 \sin\omega t + v_0 \cos\omega t)\,. \end{aligned}\right. \tag{15}$$

The magnitude V of the velocity is given by

$$V = (u^2 + v^2)^{\frac{1}{2}} = V_0 \exp[-bt] \tag{16}$$

and is, therefore, seen to decrease exponentially with time as had been obtained in the approximate treatment of the previous section. In order to compare the rate of decrease, notice that, for small values of $\omega_L\theta$,

$$b = \omega_L^2\theta[1 - 5(\omega_L\theta)^2 + O(\omega_L\theta)^4]\,. \tag{17}$$

Hence, in this limit, we recover the previous result, as was to be expected. Similarly, the frequency ω with which the orbit is described is for small values of $\omega_L\theta$

$$\omega = \omega_L[1 - 2(\omega_L\theta)^2 + O(\omega_L\theta)^4]\,, \tag{18}$$

in agreement with (3).

Another integration of (15) gives the co-ordinates $x(t)$, $y(t)$ of the particle. We find

$$x(t) = x_0 + \frac{\omega v_0 + b u_0}{b^2 + \omega^2} + \frac{\exp[-bt]}{b^2 + \omega^2}[(\omega u_0 - b v_0)\sin\omega t - (b u_0 + \omega v_0)\cos\omega t]\,, \tag{19a}$$

$$y(t) = y_0 - \frac{\omega u_0 - b v_0}{b^2 + \omega^2} + \frac{\exp[-bt]}{b^2 + \omega^2}[(\omega v_0 + b u_0)\sin\omega t + (\omega u_0 - b v_0)\cos\omega t]\,, \tag{19b}$$

where $x_0 = x(0)$, $y_0 = y(0)$. The co-ordinates x_∞, y_∞ at which the particle will eventually come to rest clearly are

$$x_\infty = x_0 + \frac{\omega v_0 + b u_0}{b^2 + \omega^2}, \qquad y_\infty = y_0 - \frac{\omega u_0 - b v_0}{b^2 + \omega^2}. \tag{20}$$

The distance $r(t)$ from this point can readily be found to be given by

$$r(t) = r_0 \exp[-bt], \tag{21}$$

where r_0 is the initial distance from (x_∞, y_∞) given by

$$r_0 = \frac{V_0}{(b^2 + \omega^2)^{\frac{1}{2}}}. \tag{22}$$

The trajectory is thus still a spiral and the result (22) coincides with (7) for small values of $\omega_L \theta$.

4. – The Dirac relativistic equation.

The relativistic generalization of (9), as obtained by DIRAC (11) and WHEELER and FEYNMAN (14), is

$$\dot{u}_\mu = \frac{e}{mc} F_{\mu\nu} u^\nu + \theta \left(\ddot{u}_\mu - \frac{1}{c^2} u_\mu \dot{u}^\nu \dot{u}_\nu \right), \tag{23}$$

where the dot denotes differentiation with respect to the proper time, the summation convention on repeated indices is used, and the metric is such that $u^\nu u_\nu = -c^2$. Again we limit ourselves to the plane case, for which the spatial part of eq. (23) can be compactly written as

$$\dot{\boldsymbol{V}} = \omega_L A \boldsymbol{V} + \theta \left[\ddot{\boldsymbol{V}} - \frac{1}{c^2} \frac{c^2 \dot{\boldsymbol{V}} \cdot \dot{\boldsymbol{V}} + (\boldsymbol{V} \times \dot{\boldsymbol{V}})^2}{c^2 + \boldsymbol{V} \cdot \boldsymbol{V}} \boldsymbol{V} \right], \tag{24}$$

where ω_L, $\boldsymbol{V}$, A have been defined in (3) and (11).

This equation does not appear to be amenable to an exact solution of physical significance. In the past PLASS (15,16) has attempted an approximate solution by a regular perturbation method, which, however, is unsuitable for the present problem because it gives rise to secular terms of the type

(14) J. A. WHEELER and R. P. FEYNMAN: *Rev. Mod. Phys.*, **17**, 157 (1945).
(15) G. N. PLASS: *Rev. Mod. Phys.*, **33**, 37 (1961).
(16) H. ARZELIÈS: *Rayonnement et dynamique du corpuscule chargé fortement accéleré* (Paris, 1966).

$t\cos\omega t$ or $t\sin\omega t$ which limit its validity to times much smaller than the decay time of the motion, which is of the order of $(\theta\omega_L^2)^{-1}$. Here we shall treat eq. (24) by the method of multiple time scales, especially designed to avoid secular terms in the perturbation expansion [7]. To our knowledge this is the first time that this method has been applied to the Dirac equation.

According to the method, instead of the proper time T we introduce a «fast» and a «slow» (dimensionless) time $\tilde{t}$, $\bar{t}$ defined by

$$\tilde{t} = \omega_L T(1 + \varepsilon^2 l_2 + \dots)\,, \tag{25a}$$

$$\bar{t} = \varepsilon\omega_L T(1 + \varepsilon m_1 + \dots)\,, \tag{25b}$$

where the parameter ε is given by

$$\varepsilon = \omega_L\theta\,. \tag{26}$$

In the following we shall take ε be to small, so that our perturbation expansion will be suitable in the case of weak energy loss by radiation. The constants $l_2, \dots, m_1, \dots,$ are to be determined by imposing the absence of secular terms in the solution. The unknown $\boldsymbol{V}(T)$ is now to be considered as a function of the two variables (25), $\boldsymbol{V}(\tilde{t}, \bar{t})$, so that, for instance,

$$\dot{\boldsymbol{V}} = \omega_L\left[(1 + \varepsilon^2 l_2 + \dots)\frac{\partial}{\partial\tilde{t}} + \varepsilon(1 + \varepsilon m_1 + \dots)\frac{\partial}{\partial\bar{t}}\right]\boldsymbol{V}(\tilde{t}, \bar{t}) \tag{27}$$

and similarly for $\ddot{\boldsymbol{V}}$. We also expand $\boldsymbol{V}$ in powers of ε as

$$\boldsymbol{V} = \boldsymbol{V}_0 + \varepsilon\boldsymbol{V}_1 + \varepsilon^2\boldsymbol{V}_2 + \dots\,. \tag{28}$$

Substituting (25), (28) into the Dirac equation and separating the terms according to their order in ε, we obtain a set of equations which can be solved recursively according to the procedure which now we describe. The initial conditions of $\boldsymbol{V}$ will be imposed on $\boldsymbol{V}_0$, so that $\boldsymbol{V}_1, \boldsymbol{V}_2, \dots$ will be taken to satisfy homogeneous initial conditions.

The physical basis of the method to be used is that, in the case of weak radiation, the motion can be considered as given by a «modulated» circular motion, *i.e.* a circular motion of «slowly» varying radius. The orbit of this motion is described in a time of order $2\pi/\omega_L$ (*i.e.* an interval $\Delta\tilde{t}$ of order 1), whereas the time scale on which the radius varies appreciably is much longer, of order $2\pi/\varepsilon\omega_L$ (*i.e.* an interval $\Delta\bar{t}$ of order 1).

Order ε^0. To this order we simply find

$$\frac{\partial\boldsymbol{V}_0}{\partial\tilde{t}} = A\boldsymbol{V}_0\,, \tag{29}$$

which describes an undamped circular motion. We can express the solution as

$$V_0 = \alpha(\bar{t})\,\boldsymbol{R}(\tilde{t}) + \beta(\bar{t})\,\boldsymbol{S}(\tilde{t})\,, \tag{30}$$

where

$$\boldsymbol{R}(\tilde{t}) = \begin{vmatrix} \cos\tilde{t} \\ -\sin\tilde{t} \end{vmatrix}, \qquad \boldsymbol{S}(\tilde{t}) = \begin{vmatrix} \sin\tilde{t} \\ \cos\tilde{t} \end{vmatrix} \tag{31}$$

and α, β are suitable integration constants. Notice that, since (29) gives information only on the dependence of $\boldsymbol{V}_0$ on the « fast » variable $\tilde{t}$, α and β are allowed to depend on the « slow » variable $\bar{t}$, which has been explicitly indicated in (30). The functional form of this dependence will be determined at the following step.

Order ε. If we take into account (29), the equation for this order can be written as

$$(c^2 + \boldsymbol{V}_0\cdot\boldsymbol{V}_0)\left(\frac{\partial \boldsymbol{V}_1}{\partial \tilde{t}} - A\boldsymbol{V}_1 - \frac{\partial^2 \boldsymbol{V}_0}{\partial \tilde{t}^2} + \frac{\partial \boldsymbol{V}_0}{\partial \bar{t}}\right) = -\frac{1}{c^2}\left[\left(\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}}\right)^2 c^2 + \left(\boldsymbol{V}_0\times\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}}\right)^2\right]\boldsymbol{V}_0\,. \tag{32}$$

Upon substitution of (30) we find

$$\frac{\partial \boldsymbol{V}_1}{\partial \tilde{t}} - A\boldsymbol{V}_1 = -\left[\frac{\mathrm{d}\alpha}{\mathrm{d}\bar{t}} + \left(1 + \frac{\alpha^2+\beta^2}{c^2}\right)\alpha\right]\boldsymbol{R} - \left[\frac{\mathrm{d}\beta}{\mathrm{d}\bar{t}} + \left(1 + \frac{\alpha^2+\beta^2}{c^2}\right)\beta\right]\boldsymbol{S}\,. \tag{33}$$

The condition for the solution to be free of secular terms is that the right-hand side of this equation be orthogonal to the solutions of the homogeneous equation, which are proportional to $\boldsymbol{R}$ and $\boldsymbol{S}$ (7). (The truth of the above statement can be proven in an elementary way if (33) is written out in component form and is solved by elimination. Then one would find equations of the form $\partial^2 u_1/\partial\tilde{t}^2 + u_1 = p\sin\tilde{t} + q\cos\tilde{t}$, the solution of which would indeed contain terms of the type $\tilde{t}\sin\tilde{t}$ and $\tilde{t}\cos\tilde{t}$ unless $p = q = 0$.)

Therefore, we must require the coefficients of $\boldsymbol{R}$ and $\boldsymbol{S}$ to vanish, which leads to the following system of equations determining α and β:

$$\frac{\mathrm{d}\alpha}{\mathrm{d}\bar{t}} + \left(1 + \frac{\alpha^2+\beta^2}{c^2}\right)\alpha = 0\,, \tag{34a}$$

$$\frac{\mathrm{d}\beta}{\mathrm{d}\bar{t}} + \left(1 + \frac{\alpha^2+\beta^2}{c^2}\right)\beta = 0\,. \tag{34b}$$

This system is readily seen to be Hamiltonian, so that its solution is straightforward. We find

$$\alpha(\bar{t}) = \frac{c \sin \varphi}{(Q \exp [2\bar{t}] - 1)^{\frac{1}{2}}}, \tag{35a}$$

$$\beta(\bar{t}) = \frac{c \cos \varphi}{(Q \exp [2\bar{t}] - 1)^{\frac{1}{2}}}, \tag{35b}$$

where

$$Q = \frac{c^2}{V^2(0)} + 1, \tag{36}$$

and the angle φ is to be determined from the unitial conditions, $\operatorname{tg} \varphi = v_0/u_0$.

Having determined α and β, we can now solve for $\boldsymbol{V}_1$, to find

$$\boldsymbol{V}_1 = \alpha_1(\bar{t})\,\boldsymbol{R}(\tilde{t}) + \beta_1(\bar{t})\,\boldsymbol{S}(\tilde{t}). \tag{37}$$

The integration constants α_1, β_1 will be determined from the second-order equation. Notice that, as was already stated, we take $\boldsymbol{V}_1(0) = 0$, so that we must impose

$$\alpha_1(\bar{t} = 0) = \beta_1(\bar{t} = 0) = 0.$$

Order ε^2. Again taking into account (29), we find the following equation at the order ε^2:

$$\begin{aligned}(c^2 + \boldsymbol{V}_0\cdot\boldsymbol{V}_0)\Big(\frac{\partial \boldsymbol{V}_2}{\partial \tilde{t}} - A\boldsymbol{V}_2 + l_2\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}} + m_1\frac{\partial \boldsymbol{V}_0}{\partial \bar{t}} + \frac{\partial \boldsymbol{V}_1}{\partial \bar{t}} -\\ -2\frac{\partial^2 \boldsymbol{V}_0}{\partial \tilde{t}\,\partial \bar{t}} - \frac{\partial^2 \boldsymbol{V}_1}{\partial \tilde{t}^2}\Big) + 2\boldsymbol{V}_0\cdot\boldsymbol{V}_1\Big(\frac{\partial \boldsymbol{V}_1}{\partial \tilde{t}} - A\boldsymbol{V}_1 - \frac{\partial^2 \boldsymbol{V}_0}{\partial \tilde{t}^2} + \frac{\partial \boldsymbol{V}_0}{\partial \bar{t}}\Big) =\\ = -\frac{1}{c^2}\Bigg(\Big[c^2\Big(\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}}\Big)^2 + \Big(\boldsymbol{V}_0\times\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}}\Big)^2\Big]\boldsymbol{V}_1 + 2\Big\{c^2\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}}\cdot\Big(\frac{\partial \boldsymbol{V}_0}{\partial \bar{t}} + \frac{\partial \boldsymbol{V}_1}{\partial \tilde{t}}\Big) +\\ + \Big(\boldsymbol{V}_0\times\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}}\Big)\cdot\Big[\boldsymbol{V}_1\times\frac{\partial \boldsymbol{V}_0}{\partial \tilde{t}} + \boldsymbol{V}_0\times\Big(\frac{\partial \boldsymbol{V}_0}{\partial \bar{t}} + \frac{\partial \boldsymbol{V}_1}{\partial \tilde{t}}\Big)\boldsymbol{V}_0\Big]\Big\}\Bigg).\end{aligned} \tag{38}$$

Substituting (30), (37) we find for $\boldsymbol{V}_2$ an equation similar to (33), for which again we must impose the vanishing of the coefficients of $\boldsymbol{R}$ and $\boldsymbol{S}$ in the right-hand side. In this way we obtain the following equations for $\alpha_1(\bar{t})$ and $\beta_1(\bar{t})$:

$$\begin{aligned}\frac{d\alpha_1}{d\bar{t}} + \Big(1 + \frac{\alpha^2+\beta^2}{c^2} + \frac{4\alpha^2}{c^2}\Big)\alpha_1 + \frac{4\alpha\beta}{c^2}\beta_1 =\\ = m_1\Big(1 + \frac{\alpha^2+\beta^2}{c^2}\Big)\alpha - \Big[l_2 + 2\Big(1 + \frac{\alpha^2+\beta^2}{c^2}\Big)\Big]\beta,\end{aligned} \tag{39a}$$

$$(39b)\qquad \frac{d\beta_1}{d\bar{t}} + \left(1 + \frac{\alpha^2+\beta^2}{c^2} + 4\frac{\beta^2}{c^2}\right)\beta_1 + \frac{4\alpha\beta}{c^2}\alpha_1 = \\ = \left[l_2 + 2\left(1 + \frac{\alpha^2+\beta^2}{c^2}\right)\right]\alpha + m_1\left(1 + \frac{\alpha^2+\beta^2}{c^2}\right)\beta\,,$$

in which α and β are given by (35). To solve this system it is useful to set

$$\alpha_1 = \frac{a_1(\bar{t})}{(Q\exp[2\bar{t}]-1)^{\frac{1}{2}}}\,,\qquad \beta_1 = \frac{b_1(\bar{t})}{(Q\exp[2\bar{t}]-1)^{\frac{1}{2}}}\,.$$

With this substitution eqs. (39) can be compactly written as

$$\frac{d}{d\bar{t}}\begin{vmatrix} a_1 \\ b_1 \end{vmatrix} + \frac{4}{Q(\exp[2\bar{t}]-1)}M\begin{vmatrix} a_1 \\ b_1 \end{vmatrix} = \frac{c}{Q(\exp[2\bar{t}]-1)}\begin{vmatrix} p \\ q \end{vmatrix}\,,$$

where

$$M = \begin{vmatrix} \sin^2\varphi & \sin\varphi\cos\varphi \\ \sin\varphi\cos\varphi & \cos^2\varphi \end{vmatrix}\,,$$

$$p = Q\exp[2\bar{t}][m_1\sin\varphi - (2+l_2)\cos\varphi] + l_2\cos\varphi\,,$$

$$q = Q\exp[2\bar{t}][m_1\cos\varphi + (2+l_2)\sin\varphi] - l_2\sin\varphi\,.$$

Diagonalizing the matrix M we are led to the following pair of (now separated) equations

$$\frac{dc_1}{d\bar{t}} = -cl_2 - \frac{2cQ\exp[2\bar{t}]}{Q\exp[2\bar{t}]-1}\,,$$

$$\frac{dd_1}{d\bar{t}} + \frac{4}{Q\exp[2\bar{t}]-1}d_1 = \frac{m_1 Qc\exp[2\bar{t}]}{Q\exp[2\bar{t}]-1}\,,$$

where

$$c_1 = \cos\varphi\, a_1 - \sin\varphi\, b_1\,,$$

$$d_1 = \sin\varphi\, a_1 + \cos\varphi\, b_1\,.$$

These equations are very readily solved to find

$$(40a)\qquad c_1(\bar{t}) = c\left[\log\frac{Q-1}{Q-\exp[2\bar{t}]} - (l_2+2)\bar{t}\right],$$

$$(40b)\qquad d_1(\bar{t}) = \frac{1}{2}m_1 Qc\frac{\exp[4\bar{t}]-\exp[2\bar{t}]}{Q\exp[2\bar{t}]-1}\,,$$

where homogeneous initial conditions have already been imposed. It is clear that c_1 and d_1 diverge as $\bar{t} \to \infty$ unless

$$l_2 = -2\,, \qquad m_1 = 0\,. \tag{41}$$

Thus we see that, if these constants are chosen in this way, expansion (28) will be uniformly valid in time to the present order in ε.

Collecting the previous results we have the following expression for $\boldsymbol{V}$ correct to first order in ε

$$\begin{aligned}\boldsymbol{V} = {} & \frac{c}{(Q\exp[2\varepsilon\omega_{\mathrm{L}}T]-1)^{\frac{1}{2}}}\left(\sin\varphi + \varepsilon\cos\varphi\log\frac{Q-1}{Q-\exp[-2\varepsilon\omega_{\mathrm{L}}T]}\right)\boldsymbol{R} + \\ & + \frac{c}{(Q\exp[2\varepsilon\omega_{\mathrm{L}}T]-1)^{\frac{1}{2}}}\left(\cos\varphi - \varepsilon\sin\varphi\log\frac{Q-1}{Q-\exp[-2\varepsilon\omega_{\mathrm{L}}T]}\right)\boldsymbol{S}\,,\end{aligned} \tag{42}$$

where

$$\boldsymbol{R} = \begin{vmatrix} \cos(1-2\varepsilon^2)\omega_{\mathrm{L}}T \\ -\sin(1-2\varepsilon^2)\omega_{\mathrm{L}}T \end{vmatrix}, \qquad \boldsymbol{S} = \begin{vmatrix} \sin(1-2\varepsilon^2)\omega_{\mathrm{L}}T \\ \cos(1-2\varepsilon^2)\omega_{\mathrm{L}}T \end{vmatrix}. \tag{43}$$

From these expressions we can easily trace the origin of the difficulty encountered by PLASS. Indeed, for *fixed* T we have

$$\sin(1-2\varepsilon^2)\omega_{\mathrm{L}}T \simeq \sin\omega_{\mathrm{L}}T - 2\varepsilon^2\omega_{\mathrm{L}}T\cos\omega_{\mathrm{L}}T + \dots\,.$$

Clearly a perturbation method which generates the result in the form of the series on the right-hand side of this equation is bound to give results non-uniformly valid if only a finite number of terms is retained. This problem would of course disappear if *all* the terms in the series were retained. The multiple time scale method is superior in so far as at each time step it adds new terms to the perturbation expansion but, at the same time, it adjusts the previously found terms by bringing up to date their time dependence through $\tilde{t}$ and $\bar{t}$ so that no secular terms arise.

5. – Comments on the relativistic solution.

From eqs. (43) we see that, to order ε^2, the angular frequency with which the charged particle goes around its orbit is

$$\omega = \omega_{\mathrm{L}}[1 - 2\varepsilon^2 + O(\varepsilon^3)]\,. \tag{44}$$

Comparing with (18) we see that, to order ε^2, this is precisely the frequency predicted by the nonrelativistic Abraham-Lorentz theory. The relativistic effect enters only in the rate at which the particle velocity decreases. If we let

$V = |\boldsymbol{V}|$, we find to first order in ε

$$V(T) = c(Q \exp[2\varepsilon\omega_{\mathrm{L}} T] - 1)^{-\frac{1}{2}} . \tag{45}$$

For $c \to \infty$, this result becomes

$$V(T) \simeq V(0) \exp\left[-\varepsilon\omega_{\mathrm{L}} T[1 + O(\varepsilon^2)]\right] ,$$

which, by (17), coincides with the nonrelativistic one up to order ε^2 included. (In fact, this correspondence can be carried further to cover the individual components of $\boldsymbol{V}$ since the logarithm in eq. (42) vanishes for $c \to \infty$.)

A particularly interesting feature of the technique adopted to obtain our approximate relativistic solution is the fact the unphysical «runaway» solutions that are contained in the general solution of the Dirac equation are eliminated automatically by the mathematical procedure which takes care of the secular terms. Thus the outcome of the calculation is an expression which approximates only the physically significant solution.

Finally it may be of some interest to consider the energy balance for the relativistic case. From the fourth component of (23) we readily obtain in the standard way (see, *e.g.*, ref. [15])

$$\frac{\mathrm{d}}{\mathrm{d}T}\left[(c^2 + V^2)^{\frac{1}{2}} - \theta \boldsymbol{V}\cdot\dot{\boldsymbol{V}}\left(1 + \frac{V^2}{c^2}\right)^{-\frac{1}{2}}\right] = -\theta c \cdot \frac{c^2 \dot{\boldsymbol{V}}\cdot\dot{\boldsymbol{V}} + (\boldsymbol{V}\times\dot{\boldsymbol{V}})^2}{(c^2 + V^2)^{\frac{1}{2}}} . \tag{46}$$

The instantaneous radiated power is found to be given by

$$P_{\mathrm{rad}} = m\theta\omega_{\mathrm{L}}^2 V^2(1 + V^2/c^2)^{\frac{1}{2}}, \tag{47}$$

which agrees with the previously quoted nonrelativistic result for $c \to \infty$. The energy lost in one revolution (*i.e.* between two time instants separated by $2\pi/\omega$ with ω given by (44)) is

$$\Delta E = m\left\{[c^2 + V^2(T + 2\pi/\omega)]^{\frac{1}{2}} - [c^2 + V^2(T)]^{\frac{1}{2}}\right\} . \tag{48}$$

(Notice that the second term in brackets in (46), the so-called Schott energy [8], is of second order in ε and, therefore, negligible to the present approximation.) If the two velocities appearing in (48) are not very different, we have [15]

$$\Delta E \simeq 2\pi\varepsilon m V^2(1 + V^2/c^2)^{\frac{1}{2}} . \tag{49}$$

6. – The Caldirola finite-difference equation.

A totally different approach to the dynamics of charged particles has been proposed by CALDIROLA [6] on the basis of a finite-difference equation which, in the nonrelativistic approximation, is

$$\frac{m}{\tau}[\boldsymbol{V}(t)-\boldsymbol{V}(t-\tau)]=\boldsymbol{F}[\boldsymbol{X}(t),\boldsymbol{V}(t),t]\,, \tag{50}$$

where $\boldsymbol{F}$ is the force, $\boldsymbol{X}$ is the position, and τ is given by (14). This equation must be complemented by the so-called transmission law, which establishes a connection between position and velocity,

$$\frac{1}{\tau}[\boldsymbol{X}(t)-\boldsymbol{X}(t-\tau)]=\frac{1}{2}[\boldsymbol{V}(t)+\boldsymbol{V}(t-\tau)]\,. \tag{51}$$

The reader is referred to the references cited for a discussion of the advantages and of the interesting possibilities that this finite-difference formulation has over the Abraham-Lorentz equation. Suffice it to say here that eqs. (50), (51) do not exhibit any unbounded solutions and that, in the vicinity of an equilibrium point, all solutions tend to this equilibrium point [5].

Let $t_n=n\tau$ be the time measured from the initial instant and set $\boldsymbol{V}_n=\boldsymbol{V}(t_n)$. Then, using the Lorentz force for $\boldsymbol{F}$ in (50) and projecting on the x, y axes normal to the field $\boldsymbol{B}$, we obtain a system of two difference equations, which can be written compactly as

$$(I-\omega_{\mathrm{L}}\tau A)\,\boldsymbol{V}_n=\boldsymbol{V}_{n-1}\,, \tag{52}$$

where I is the identity two-matrix and A and $\boldsymbol{V}$ are defined by (11). Let $\boldsymbol{w}_i$, $i=1,2$, be the eigenvectors of the operator appearing in (52) and σ_i the corresponding eigenvalues:

$$(I-\omega_{\mathrm{L}}\tau A)\,\boldsymbol{w}_i=\sigma_i\boldsymbol{w}_i\,,\qquad i=1,2\,.$$

A straightforward calculation gives

$$\sigma_{1,2}=1\pm i\omega_{\mathrm{L}}\tau\,, \tag{53}$$

$$\boldsymbol{w}_1=2^{-\frac{1}{2}}\begin{vmatrix}1\\ i\end{vmatrix}\,,\qquad \boldsymbol{w}_2=2^{-\frac{1}{2}}\begin{vmatrix}1\\ -i\end{vmatrix}\,. \tag{54}$$

Now set

$$\boldsymbol{V}_n=c_n^{(1)}\boldsymbol{w}_1+c_n^{(2)}\boldsymbol{w}_2\,, \tag{55}$$

substitute into (52) and project on the two eigenvectors to find the following equations for $c_n^{(i)}$:

$$\sigma_i c_n^{(i)}=c_{n-1}^{(i)}\,,\qquad i=1,2,$$

from which

$$c_n^{(i)} = c_0^{(i)} \sigma_i^{-n} ,$$

where $c_0^{(i)}$ are the initial data determined as before. In this way the following solution is found:

$$u_n = \frac{1}{2}\frac{u_0 + iv_0}{(1 + i\omega_L \tau)^n} + \frac{1}{2}\frac{u_0 - iv_0}{(1 - i\omega_L \tau)^n},$$

$$v_n = \frac{1}{2}\frac{v_0 + iu_0}{(1 - i\omega_L \tau)^n} + \frac{1}{2}\frac{v_0 - iu_0}{(1 + i\omega_L \tau)^n}.$$

This solution can be written alternatively as

$$\left\{\begin{aligned} u_n &= (1 + \omega_L^2 \tau^2)^{-n/2}(u_0 \cos n\delta + v_0 \sin n\delta) , \\ v_n &= (1 + \omega_L^2 \tau^2)^{-n/2}(v_0 \cos n\delta - u_0 \sin n\delta) , \end{aligned}\right. \tag{56}$$

where the angle δ is defined by

$$\operatorname{tg} \delta = \omega_L \tau \qquad \left(-\frac{\pi}{2} < \delta < \frac{\pi}{2}\right). \tag{57}$$

The comparison with the previous result (15) is facilitated by setting $n = t/\tau$,

$$b = \frac{1}{2}\frac{1}{\tau} \log (1 + \omega_L^2 \tau^2) , \tag{58}$$

$$\omega = \frac{1}{\tau} \operatorname{arctg} \omega_L \tau , \tag{59}$$

since then eqs. (56) become identical in form with eqs. (15). Clearly a result identical to (16) can also be obtained.

Expanding eqs. (58), (59) for b and ω for small $\varepsilon = \omega_L \theta$, we find

$$b = \theta\omega_L^2(1 - 2\varepsilon^2 + \dots) , \qquad \omega = \omega_L(1 - \tfrac{4}{3}\varepsilon^2 + \dots) , \tag{60}$$

which agrees, to first order in ε, with the results obtained from the Abraham-Lorentz equation and from the approximate treatment of sect. **2**. It is interesting to notice that a difference between (58), (59) and (12), (13) appears already at second order in ε, so that, if the Caldirola equation is indeed the correct equation of motion for classical charged particles, the domain of validity of the Abraham-Lorentz theory would not be greater than that of the simple *a priori* estimate of sect. **2**.

It is readily shown that the transmission law (5), viewed as an equation in $\boldsymbol{X}$ for given $\boldsymbol{V}$, has the solution

$$\boldsymbol{X}_n = \boldsymbol{X}_0 - \tfrac{1}{2}\tau(\boldsymbol{V}_n + \boldsymbol{V}_0) + \tau\sum_{k=0}^{n}\boldsymbol{V}_k\,, \tag{61}$$

where $\boldsymbol{X}_0$ is the initial condition. Upon substitution of (56) into (61) it is found that the summation indicated in this equation involves the following sums:

$$R_n = \sum_{k=0}^{n}\frac{\cos k\delta}{(1+\omega_{\mathrm{L}}^2\tau^2)^{k/2}}\,,\qquad I_n = \sum_{k=0}^{n}\frac{\sin k\delta}{(1+\omega_{\mathrm{L}}^2\tau^2)^{k/2}}\,.$$

A closed-form expression for these quantities can be obtained by observing that the combination $R_n + iI_n$ can readily be evaluated since it is just a geometric progression of argument $\exp[i\delta](1+\omega_{\mathrm{L}}^2\tau^2)^{-\frac{1}{2}}$. In this way, separating the real and imaginary part of the result, we have

$$\begin{cases} \displaystyle\sum_{k=0}^{n}\frac{\cos k\delta}{(1+\omega_{\mathrm{L}}^2\tau^2)^{k/2}} = \frac{\sin n\delta}{\sin\delta}(1+\omega_{\mathrm{L}}^2\tau^2)^{-(n+1)/2} + 1\,, \\[2ex] \displaystyle\sum_{k=0}^{n}\frac{\sin k\delta}{(1+\omega_{\mathrm{L}}^2\tau^2)^{k/2}} = \operatorname{ctg}\delta - \frac{\cos n\delta}{\sin\delta}(1+\omega_{\mathrm{L}}^2\tau^2)^{-(n+1)/2}\,. \end{cases} \tag{62}$$

These expressions lead to the following results for $\boldsymbol{X}_n = (x_n, y_n)$:

$$x_n = x_0 + \frac{1}{2}\tau\left(u_0 + \frac{2v_0}{\omega_{\mathrm{L}}\tau}\right) + \frac{(2u_0 - \omega_{\mathrm{L}}\tau v_0)\sin n\delta - (2v_0 + \omega_{\mathrm{L}}\tau u_0)\cos n\delta}{2\omega_{\mathrm{L}}(1+\omega_{\mathrm{L}}^2\tau^2)^{n/2}}\,, \tag{63a}$$

$$y_n = y_0 + \frac{1}{2}\tau\left(v_0 - \frac{2u_0}{\omega_{\mathrm{L}}\tau}\right) + \frac{(2v_0 + \omega_{\mathrm{L}}\tau u_0)\sin n\delta + (2u_0 - \omega_{\mathrm{L}}\tau v_0)\cos n\delta}{2\omega_{\mathrm{L}}(1+\omega_{\mathrm{L}}^2\tau^2)^{n/2}}\,. \tag{63b}$$

The position (x_∞, y_∞) at which the particle will eventually come to rest is

$$x_\infty = x_0 + \frac{1}{2}\tau\left(u_0 + \frac{2v_0}{\omega_{\mathrm{L}}\tau}\right),$$

$$y_\infty = y_0 + \frac{1}{2}\tau\left(v_0 - \frac{2u_0}{\omega_{\mathrm{L}}\tau}\right)$$

and the distance r_n from this point follows the law

$$r_n = r(0)(1+\omega_{\mathrm{L}}^2 r^2)^{-n/2}\,,$$

where $r(0)$, the distance of (x_0, y_0) from (x_∞, y_∞), is given by

$$r(0) = (1+\tfrac{1}{4}\omega_{\mathrm{L}}^2\tau^2)^{\frac{1}{2}}V_0/\omega_{\mathrm{L}}\,.$$

It is obvious that all these results (and in particular the last one) bear a strong resemblance with those of the previous section if (58), (59) are used.

In order to complete the comparison of the Caldirola model with the classical Dirac one, it would be necessary to obtain an approximate solution of the relativistic Caldirola finite-difference equation. For this purpose a finite-difference analogue of the two-timing method should be developed. Efforts in this direction are currently under way.

* * *

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● RIASSUNTO

Si studia il moto non quantistico di una particella carica in un campo magnetico uniforme mediante l'equazione non relativistica di Abraham e Lorentz, quella relativistica di Dirac e quella alle differenze finite non relativistica di Caldirola. Nel primo e nel terzo caso si ottengono soluzioni in forma chiusa. Alla equazione di Dirac si applica invece per la prima volta il metodo perturbativo della doppia variabile temporale, giungendo ad una soluzione priva dei termini secolari presenti nelle soluzioni approssimate ottenute da altri autori.

Движение заряженной частицы в однородном магнитном поле.

Резюме (*). - - Мы исследуем движение классической (не квантовой) заряженной частицы в однородном магнитном поле, используя 1) уравнение Абрагама-Лоренца, 2) релятивистское уравнение Дирака и 3) нерелятивистское конечно-разностное уравнение Калдиролы. В первом и третьем случаях получаются решения в замкнутой форме. Для второго случая мы впервые применяем асимптотический метод по двум переменным, который позволяет нам получить приближенное решение, свободное от секулярных членов, присутствующих в результатах других авторов.

(*) *Переведено редакцией.*