On γ-hyperelliptic Weierstrass semigroups of genus $6\gamma+1$ and 6γ

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Let (C, P) be a pointed non-singular curve such that the Weierstrass semigroup $H(P)$ of P is a γ-hyperelliptic numerical semigroup. Torres showed that there exists a double covering $\pi : C \longrightarrow C'$ such that the point P is a ramification point of π if the genus g of C is larger than or equal to $6\gamma + 4$. Kato and the authors also showed that the same result holds in the case $g = 6\gamma + 3$ or $6\gamma + 2$. In this paper we prove that there exists a double covering $\pi : C \longrightarrow C'$ satisfying the above condition even if $g = 6\gamma + 1$, 6γ and $H(P)$ does not contain 4.

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1 Introduction

Let C be a complete nonsingular irreducible curve over an algebraically closed field k of characteristic 0, which is called a *curve* in this paper. For a point P of C, we set

(1) $$H(P) = \{\alpha \in \mathbb{N}_0 \mid \text{ there exists } f \in k(C) \text{ with } (f)_\infty = \alpha P\},$$

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which is called the *Weierstrass semigroup of* P where $\mathbb{N}_0$ denotes the additive monoid of non-negative integers and $k(C)$ is the field of rational functions on C. A submonoid H of $\mathbb{N}_0$ is called a *numerical semigroup* if its complement $\mathbb{N}_0\backslash H$ is a finite set. The cardinality of $\mathbb{N}_0\backslash H$ is called the *genus* of H, which is denoted by $g(H)$. It is known that the Weierstrass semigroup of a point on a curve of genus g is a numerical semigroup of genus g. For a numerical semigroup H we denote by $d_2(H)$ the set consisting of the elements h with $2h \in H$. Then $d_2(H)$ is also a numerical semigroup. Let $\pi : C \longrightarrow C'$ be a double covering of a curve with a ramification point P. Then it is proved that $d_2(H(P)) = H(\pi(P))$. Let γ be a non-negative integer. A numerical semigroup H is said to be *γ-hyperelliptic* if $m_1, \ldots, m_\gamma$ are even, $m_\gamma = 4\gamma$ and $4\gamma + 2 \in H$ where $H = \{0 = m_0 < m_1 < m_2 < \cdots\}$. By Lemma 2.6 in [10] we have $g(d_2(H)) = \gamma$ if H is γ-hyperelliptic. We consider the following problem:

We express the condition that (C, P) *is a pointed curve of genus* g *such that* $H(P)$ *is a* γ*-hyperelliptic numerical semigroup* in $D(C, P; g, \gamma)$. When we assume $D(C, P; g, \gamma)$, is C a double cover of some curve such that P is its ramification point?

Torres [10] solved the problem on the condition that $g \geqq 6\gamma + 4$, namely, he showed the following:

Theorem (Torres) *Let* $g \geqq 6\gamma + 4$*. Assume* $D(C, P; g, \gamma)$*. Then* C *is a double cover of some curve such that* P *is its ramification point.*

Torres's result is very important in the history of the study on Weierstrass semigroups. Buchweitz gave the first non-Weierstrass numerical semigroup H , which means that the numerical semigroup H is not attained by the Weierstrass semigroup $H(P)$ for any pointed curve (C, P). His method depends on the cohomology dimensions of the multi-folds of the canonical sheaf on a curve. As an application of the above theorem Torres [10] gave non-Weierstrass numerical semigroups which cannot be gained by the Buchweitz's method. But to construct non-Weierstrass numerical semigroups by Torres's method a non-Weierstrass numerical semigroup which is given by Buchweitz's method is needed. In [8] the authors recently found non-Wierstrass numerical semigroups which are obtained by neither using the way of Buchweitz nor using the way of Torres. The above theorem proved by Torres is used to prove the main theorem in [8]. But a non-Weierstrass numerical semigroup gained by Buchweitz's method is not needed to get new non-Weierstrass numerical semigroups. Our result in this paper is the following:

Main Theorem *Let* $g = 6\gamma + 1$ *or* 6γ*. For* $g = 6\gamma + 1$ *(resp.* 6γ*) we suppose* $\gamma \geqq 6$ *(resp.* $\gamma \geqq 10$*). Assume* $D(C, P; g, \gamma)$*. If* $H(P)$ *does not contain* 4*, then* C *is a double cover of some curve such that* P *is its ramification point.*

2 Case $g = 6\gamma + 1$

Let γ be an integer with $\gamma \geqq 3$. Assume $D(C, P; 6\gamma+1, \gamma)$. Then $6\gamma+2$ is a non-gap at P by Lemma 2.6 in [10]. Let $\phi = \varphi_{|(6\gamma+2)P|} : C \longrightarrow \mathbb{P}^{2\gamma+1}$ be the morphism corresponding to the complete linear system $|(6\gamma+2)P|$. If ϕ is not birational, the proof of Main Theorem is done by Proof of Theorem A (iii)$\Longrightarrow$(i) in [10]. Assume that $\phi : C \longrightarrow C_0(\subset \mathbb{P}^{2\gamma+1})$ is a birational morphism where C_0 is the image of C by ϕ. Then by Theorem 3.7 in [5] we obtain $6\gamma + 1 = g(C) \leqq p_a(C_0) \leqq 6\gamma + 3$.

If $p_a(C_0) = g(C)$, then the morphism ϕ is étale at P. Hence, the morphism ϕ is locally isomorphic (for example, see [9]). Thus, $6\gamma + 1$ is a non-gap. By Lemma 2.2 in [10] this contradicts the assumption that $H(P)$ is γ-hyperelliptic. Therefore, we get $6\gamma+1 = g(C) < p_a(C_0) \leqq 6\gamma+3$, which implies that $p_a(C_0) > 6\gamma + 1$ and $p_a(C_0) - g(C) = 1$ or 2.

We calculate the number π_1 (see Theorem (3.15) in [5]) associated with C_0. We have $\left[\dfrac{6\gamma + 2 - 1}{2\gamma + 1}\right] = 2$ and

$$\epsilon_1 = 6\gamma + 2 - 2(2\gamma + 1) - 1 = 2\gamma - 1 \neq (2\gamma + 1) - 1. \tag{2}$$

Hence, we get

$$\pi_1 = \pi_1(6\gamma + 2, 2\gamma + 1) = 2\gamma + 1 + 2(2\gamma - 1 + 1) + 0 = 6\gamma + 1. \tag{3}$$

Since $p_a(C_0) > 6\gamma+1 = \pi_1(6\gamma+2, 2\gamma+1)$, by the second Castelnuovo's inequality (see Theorem (3.15) i) in [5]) there exists a surface S of degree 2γ in $\mathbb{P}^{2\gamma+1}$ such that

$$\phi = \varphi_{|(6\gamma+2)P|} : C \longrightarrow C_0 \subset S \subset \mathbb{P}^{2\gamma+1}. \tag{4}$$

Let $\pi : \tilde{S} \longrightarrow S$ be the minimal resolution of S. Since $\tilde{S}$ is a rational ruled surface Σ_e, we have $\mathrm{Pic}(\tilde{S}) = \mathbb{Z}H \oplus \mathbb{Z}F$ with $(H^2) = 2\gamma$, $(H, F) = 1$ and $K_{\tilde{S}} = -2H + (2\gamma - 2)F$, where F is a fiber and H is a hyperplane section of Σ_e with $\pi(\Sigma_e) = S \subset \mathbb{P}^{2\gamma+1}$ (for example, see p.121 in [1]). Let T_e be a minimal section of Σ_e. Then we have $H \sim T_e + mF$ where $2m = e + 2\gamma$. From now on, $C_0 \subset \tilde{S}$ means the proper transformation of $C_0 \subset S$. Let $C_0 \sim aH + bF$. Then we obtain

$$6\gamma + 2 = (H, C_0) = (H, aH + bF) = 2a\gamma + b, \tag{5}$$

which implies that $b = 6\gamma + 2 - 2\gamma a$. Moreover, we have

$$2p_a(C_0) - 2 = (K_{\tilde{S}} + C_0, C_0) = ((a - 2)H + (b + 2\gamma - 2)F, aH + bF) \tag{6}$$
$$= 2a(a - 2)\gamma + (6\gamma + 2 - 2\gamma a)(a - 2) + a(8\gamma - 2\gamma a) \tag{7}$$
$$= -2\gamma a^2 + (14\gamma + 2)a - 2(6\gamma + 2), \tag{8}$$

which implies that

$$p_a(C_0) = -\gamma a^2 + (7\gamma + 1)a - 6\gamma - 1. \tag{9}$$

If $a \leqq 2$, then $p_a(C_0) \leqq 4\gamma + 1$. If $a = 3$, then $p_a(C_0) = 6\gamma + 2$, which implies that $C_0 \sim 3H + 2F$. If $a = 4$, then $p_a(C_0) = 6\gamma + 3$, which implies that $C_0 \sim 4H - (2\gamma - 2)F$. If $a \geqq 5$, then $p_a(C_0) \leqq 4\gamma + 4$. Since $6\gamma + 2 \leqq p_a(C_0)$, a must be 3 or 4.

Lemma *A γ-hyperelliptic numerical semigroup of genus larger than or equal to $3\gamma + 2$ with $\gamma \geqq 3$ cannot be attained by the Weierstrass semigroup of any point on a trigonal curve.*

Proof. Let C be a trigonal curve and $\pi : C \longrightarrow \mathbb{P}^1$ a unique covering of degree 3. Let P be a point of C such that $H(P)$ is γ-hyperelliptic.

Assume that P is a total ramification point of π. Then the minimum positive integer in $H(P)$ is 3, which implies that $\gamma = 0$. This is a contradiction.

Assume that P is a ramification point of π with ramification number 2. Then by Coppens [2], [3] and Kato-Horiuchi [6] there exists an integer n with $(g-1)/3 \leqq n \leqq g/2$ such that $H(P)$ is either a $(2n+1)$-semigroup or a $(2n+2)$-semigroup where an m-semigroup means that the minimum positive integer in the numerical semigroup is m. In view of $\gamma > 0$ we see that $H(P)$ is a $(2n+2)$-semigroup. More explicitly, the semigroup $H(P)$ is equal to

$$\{0 < 2n+2 < 2n+4 < \cdots < 2n + 2(g-2n-1) < 2g - 2n \longrightarrow\} \tag{10}$$

where for an integer m the symbol $m \longrightarrow$ means the consequent integers larger than or equal to m. Hence, we have $d_2(H(P)) = \{0, n+1 \longrightarrow\}$, which implies that $g(d_2(H(P)) = n$. By Lemma 2.6 in Torres [10] we have $g(d_2(H)) = \gamma$, because $H(P)$ is γ-hyperelliptic. In view of $(g-1)/3 \leqq n = \gamma$ we get $g \leqq 3\gamma + 1$, which contradicts $g \geqq 3\gamma + 2$.

Assume that P is a non-ramification point of π. If P is not a Weierstrass point, then we have $H(P) = \{0, g+1 \longrightarrow\}$. If g is even, then $\gamma = 0$, which is a contradiction. If g is odd, then we have $\gamma = 1$. This is a contradiction. We may assume that P is a Weierstrass point. By Kim [7] we obtain

$$H(P) = \{0, b, \dots, b + (s-g), s+2 \longrightarrow\}, \tag{11}$$

where we set $s = \max\{m \mid mP \text{ is special}\}$. Let $s = g$. Then we have

$$H(P) = \{0, b, g+2 \longrightarrow\}. \tag{12}$$

If g is odd (resp. even), then we have $\gamma \leqq 1$ (resp. $\gamma \leqq 2$), which contradicts $\gamma \geqq 3$. Hence, we may assume that $s - g > 0$. Then we obtain

$$H(P) = \{0, b, b+1, \dots, b + (s-g), s+2 \longrightarrow\}. \tag{13}$$

Since $H(P)$ is γ-hyperelliptic, we get $\gamma \leqq 1$, which is a contradiction. $\square$

Hence, we may assume that $a = 4$, which implies that $C_0 \sim 4H - (2\gamma - 2)F$ and $p_a(C_0) = 6\gamma + 3$, which implies that $p_a(C_0) - g(C) = 2$.

We will show that the minimal resolution $\pi : \tilde{S} \longrightarrow S$ is isomorphic. It suffices to show that the hyperplane section H is very ample. Since a general member H of $|T_e + mF|$ is irreducible, we have $m \geqq e$. Assume that H is not very ample. Then we get $m = e$, which implies that $e = 2\gamma$, because $2m = e + 2\gamma$. Moreover, we have

$$C_0 \sim 4H - (2\gamma - 2)F \sim 4(T_e + mF) - (2\gamma - 2)F \tag{14}$$
$$= 4T_e + (4m - 2\gamma + 2)F = 4T_e + (6\gamma + 2)F. \tag{15}$$

Since C_0 is irreducible, we obtain

$$0 \leqq 6\gamma + 2 - 4e = 6\gamma + 2 - 8\gamma = 2 - 2\gamma. \tag{16}$$

This is a contradiction, because $\gamma \geqq 3$. Hence S is the rational ruled surface.

Case 1: Assume that $\gamma \geqq 4$ and C_0 has distinct two singularities. Let $\phi(P) = P_0 \in C_0$.

If P_0 is not a singular point, then $\phi : C \longrightarrow C_0$ is étale at P, because ϕ is birational. Thus, $6\gamma + 1$ is a non-gap. This is a contradiction.

We may assume that P_0 is a double point of C_0. Let $\rho : V \longrightarrow S = \Sigma_e$ be the blow-up at P_0 and $\tilde{C}_0$ its proper transform of C_0. We have $\rho^* C_0 = \tilde{C}_0 + 2E$ where E is an exceptional divisor. Moreover, we obtain $0 = (\rho^* C_0, E) = (\tilde{C}_0, E) + 2(E, E) = (\tilde{C}_0, E) - 2$, which implies that $(\tilde{C}_0, E) = 2$. Let F_1 be a fiber of Σ_e such that $P_0 \in F_1$. Then $\rho^* F_1 - E$ and E intersect transversally. Hence $\tilde{C}_0$ and $\rho^* F_1 - E$ intersect at $\tilde{P}_0$ with multiplicity 1 where $\tilde{P}_0$ is the point of $\tilde{C}_0$ over P_0. Let $\mu : V \longrightarrow \Sigma_\theta$ be the contraction of $\rho^* F_1 - E$ where Σ_θ is a rational ruled surface with $\theta = e - 1$ and $e + 1$ if $P_0 \notin T_e$ and $P_0 \in T_e$ respectively. Here T_e denotes a minimal section of Σ_e. Then $\mu(\tilde{C}_0)$ is smooth at $P_1 = \mu(\tilde{P}_0)$. Consider the linear system $\delta_{P_0} = \{D \in |H| \mid D \ni P_0\}$ on Σ_e. Then we have $\dim \delta_{P_0} = \dim |H| - 1 = 2\gamma + 1 - 1 = 2\gamma$. We consider the linear system $\rho^* \delta_{P_0} - E$ on V, which is $\mu^* \eta$ for some linear system η on Σ_θ with $\dim \eta = 2\gamma$, because

$$(\rho^* \delta_{P_0} - E, \rho^* F_1 - E) = (\rho^*(T_e + mF) - E, \rho^* F_1 - E) = \tag{17}$$
$$(T_e + mF, F_1) + (E, E) = (T_e, F_1) + (mF, F_1) - 1 = 1 + 0 - 1 = 0. \tag{18}$$

Then we get

$$(\rho^* \delta_{P_0} - E)|_{\tilde{C}_0} = (6\gamma + 2)\tilde{P}_0 - 2\tilde{P}_0 = 6\gamma \tilde{P}_0, \tag{19}$$

because $E|_{\tilde{C}_0} = 2\tilde{P}_0$. Consider the morphism $\varphi_{|6\gamma P|} : C \longrightarrow \mathbb{P}^{2\gamma}$. Then the image $\varphi_{|6\gamma P|}(C)$ is contained in S' which is the image of the morphism $f_1 : \Sigma_\theta \longrightarrow \mathbb{P}^{2\gamma}$ corresponding to η. Let H' is a hyperplane section which determines the morphism $f_1 : \Sigma_\theta \longrightarrow \mathbb{P}^{2\gamma}$. Then we obtain $H' \sim T_\theta + m'F'$ where T_θ and F' are a minimal section and a fibre of Σ_θ respectively. We note that

$$C_0 \sim 4H - (2\gamma - 2)F \sim \tag{20}$$

$$4(T_e + mF) - (2\gamma - 2)F = 4T_e + (4m - (2\gamma - 2))F. \tag{21}$$

The irreducibility of C_0 implies that

$$4m - (2\gamma - 2) - 4e = 4(m - e - 1) - (2\gamma - 6) \geqq 0. \tag{22}$$

Since $\gamma \geqq 4$, we get $m' - \theta > 0$, because $m' - \theta = m - e$ or $m - e - 1$. Hence, H' is very ample. Thus, f_1 is an embedding, which implies that Σ_θ and the image S' are isomorphic. Thus, we regard as $\mu(\tilde{C}_0) \subset S'$. The image of P by $\varphi_{|6\gamma P|}$ is smooth, because $\mu(\tilde{C}_0)$ is smooth at $P_1 = \mu(\tilde{P}_0)$. Hence the morphism $\varphi_{|6\gamma P|} : C \longrightarrow \mathbb{P}^{2\gamma}$ is étale at P. Therefore, $6\gamma - 1$ is a non-gap. This is a contradiction.

Case 2 : Assume that $\gamma \geqq 6$ and C_0 has only one singularity. We may assume that P_0 is a double point such that there is an infinitely near singularity to P_0. Let H be the hyperplane section of Σ_e whose pullback to C is $(6\gamma + 2)P$.

Case 2$-$i): We consider the case where H is irreducible. Let F_1 be a fiber on Σ_e with $F_1 \ni P_0$. We have $(H, F_1) = 1$. Let $\rho : S_1 \longrightarrow \Sigma_e$ be the blow-up at P_0 and E its exceptional divisor. We get

$$(\rho^* F_1 - E, \rho^* H - E) = (H, F_1) + (E^2) = 1 - 1 = 0 \tag{23}$$

which implies that $(\rho^* H - E) \cap (\rho^* F_1 - E) = \emptyset$, because H is irreducible. Since there is an infinitely near singularity to P_0, we get $\rho^* C_0 = C_1 + 2E$ and $C_1 \cap E = \{P_1\}$ where C_1 is the proper transform of C_0. Since $p_a(C_0) - g(C) = 2$, there is no infinitely near point to P_1, hence the blow-up of C_1 at P_1 is non-singular. Let $\phi_0 : C_1 \longrightarrow C_0$ be the blow-up at P_0. There exists a morphism $\phi_1 : C \longrightarrow C_1$ with $\phi = \varphi_{|(6\gamma+2)P|} = \phi_0 \circ \phi_1$. Then we have $\phi(P) = P_0$ and $\phi_1(P) = P_1$. Since the pullback of H to C is $(6\gamma+2)P$, we have $\rho^* H \cap C_1 = \{P_1\}$ and $P_1 \in \rho^* H - E$. Hence, $(\rho^* H - E) \cap (\rho^* F_1 - E) = \emptyset$ implies that $P_1 \notin \rho^* F_1 - E$. Consider the blow-up $\rho_1 : S_2 \longrightarrow S_1$ at P_1. Let E_1 be its exceptional divisor. Then the total transformation and the proper transformation of $\rho^* F_1 - E$ coincide. We have $\rho_1^* C_1 = C_2 + 2E_1$ where C_2 is the proper transform of C_1 of ρ_1. Let $\phi_2 : C \longrightarrow C_2$ be the morphism with $\phi_1 = \rho_1|_{C_2} \circ \phi_2$. We denote by $P_2 \in C_2$ the smooth point over P_1. Let $\mu_1 : S_2 \longrightarrow \Sigma$ be the contraction of $\rho_1^*(\rho^* F_1 - E)$ and $\mu : \Sigma \longrightarrow \Sigma_\theta$ the contraction of $\rho_1^* E - E_1$. Moreover, we obtain

$$\phi_2^*((\rho_1^* \rho^* H - \rho_1^* E - E_1)|_{C_2}) \sim (6\gamma + 2)P - 2P - 2P = (6\gamma - 2)P. \tag{24}$$

It follows from $(\rho^* H - E) \cap (\rho^* F_1 - E) = \emptyset$ that

$$(\rho_1^* \rho^* H - \rho_1^* E - E_1) \cap \rho_1^*(\rho^* F_1 - E) = \emptyset. \tag{25}$$

Moreover, we get

$$(26)\quad (\rho_1^*\rho^*H - \rho_1^*E - E_1, \rho_1^*E - E_1) = (\rho^*H - E, E) + (E_1, E_1) = 0.$$

Hence the linear system $\rho_1^*\rho^*H - \rho_1^*E - E_1$ defines a morphism $f_2 : \Sigma_\theta \longrightarrow \mathbb{P}^{2\gamma-1}$. Let H' be a hyperplane section defining the morphism f_2 and $H' \sim T_\theta + m'F'$ where T_θ and F' are a minimal section and a fiber on Σ_θ respectively. We have $m' - \theta = m - e$ or $m - e - 1$ or $m - e - 2$. Since C_0 with $C_0 \sim 4H - (2\gamma - 2)F_1 \sim 4T_e + (4m - 2\gamma + 2)F_1$ is irreducible, we get

$$(27)\quad 0 \leqq 4m - 2\gamma + 2 - 4e = 4(m - e - 2) - (2\gamma - 10)$$
$$(28)\quad \leqq 4(m' - \theta) - (2\gamma - 10),$$

which implies that $4(m' - \theta) \geqq 2(\gamma - 5)$. Since $\gamma \geqq 6$, we have $m' - \theta > 0$. Hence f_2 is an embedding. Thus, the image of $\varphi_{|(6\gamma-2)P|} : C \longrightarrow \mathbb{P}^{2\gamma-1}$ is contained in a rational ruled surface Σ_θ. The image P_2 of P by $\varphi_{|(6\gamma-2)P|}$ is a smooth point. Hence, $6\gamma - 3$ is a non-gap at P. This is a contradiction.

Case 2−ii): We assume that H is reducible. We set $H = A + B$. Then we have $1 = (H, F) = (A, F) + (B, F)$ where F is a fiber on Σ_e. Since for any $D \geqq 0$ we have $(D, F) \geqq 0$, we may assume that $(A, F) = 1$ and $(B, F) = 0$. Hence, we may set $B = F_1 + F_2 + \cdots + F_\alpha$ and $A \sim T_e + (m - \alpha)F$. Since $(A + B)|_{C_0} = (6\gamma + 2)P_0$ and $(C_0, F_i) = (4H - (2\gamma - 2)F, F) = 4$, we have $F_i \cap C_0 = \{P_0\}$ and $F_i|_{C_0} = 4P_0$. Thus, we have $F_1 = \ldots = F_\alpha = F$ and $F|_{C_0} = 4P$ for any $i = 1, 2, \ldots, \alpha$. Hence, we obtain $h^0(4P) = 2$. Since $H(P)$ is γ-hyperelliptic, we must have

$$(29)\quad m_1 = 4, m_2 = 8, \ldots, m_\gamma = 4\gamma,$$

where $H(P) = \{0 < m_1 < m_2 < \cdots < m_\gamma < \cdots\}$. Hence, $H(P)$ is a 4-semigroup. □

Indeed, there is a pointed curve (C, P) with a γ-hyperelliptic 4-semigroup $H(P)$ such that $\varphi_{|(6\gamma+2)P|}$ is a birational morphism from C to its image.

Remark. We apply Theorem 22 in [4] to the case where $n = 4$ and $s = 4\gamma + 2$. Then by the theorem the linear system $|(4\gamma + 2)P|$ is simple. Hence, $|(6\gamma + 2)P|$ is simple.

3 Case $g = 6\gamma$

Let γ be an integer with $\gamma \geqq 4$. Assume $D(C, P; 6\gamma, \gamma)$. Consider the morphism $\phi = \varphi_{|(6\gamma-2)P|} : C \longrightarrow \mathbb{P}^{2\gamma-1}$. Let $\tilde{C}$ be the normalization of the image $C_0 = \phi(C)$. If the morphism $\tilde{\phi} : C \longrightarrow \tilde{C}$ is of degree t, then we get

$$(30)\quad 2\gamma - 1 \leqq \frac{6\gamma - 2}{t},$$

which implies that $t \leqq 3$. Since $6\gamma - 2$ cannot be divided by 3, we get $t = 1$ or 2. We may assume that $t = 1$, that is to say, $\tilde{\phi}$ is birational. Then by Castelnuovo's bound in [5] we obtain $6\gamma = g \leqq p_a(C_0) \leqq 6\gamma + 3$. If $p_a(C_0) = g(C)$, $6\gamma - 3$ is a non-gap, which is a contradiction. Therefore, we get $6\gamma = g(C) < p_a(C_0) \leqq 6\gamma + 3$, which implies that $p_a(C_0) > 6\gamma$ and $p_a(C_0) - g(C) = 1$ or 2 or 3.

The number $\pi_1 = \pi_1(6\gamma - 2, 2\gamma - 1)$ is $3(2\gamma - 1) + 3(0 + 1) + 0 = 6\gamma$. By the second Castelnuovo's inequality in [5] there exists a surface S of degree $2\gamma - 2$ in $\mathbb{P}^{2\gamma-1}$ such that

$$\phi = \varphi_{|(6\gamma-2)P|} : C \longrightarrow C_0 \subset S \subset \mathbb{P}^{2\gamma-1}. \tag{31}$$

Let $\pi : \tilde{S} = \Sigma_e \longrightarrow S$ be the minimal resolution of S. We have

$$\mathrm{Pic}(\tilde{S}) = \mathbb{Z}H \oplus \mathbb{Z}F \text{ with } (H^2) = 2\gamma - 2,\ (H, F) = 1 \tag{32}$$

and $K_{\tilde{S}} \sim -2H + (2\gamma - 4)F$, where F, H and T_e are as in Case $g = 6\gamma + 1$. Then we have $H \sim T_e + mF$ where $2m = e + 2\gamma - 2$. From now on, $C_0 \subset \tilde{S}$ means the proper transformation of $C_0 \subset S$. Let $C_0 \sim aH + bF$. Then we obtain

$$6\gamma - 2 = (H, C_0) = (H, aH + bF) = a(2\gamma - 2) + b, \tag{33}$$

which implies that $b = 6\gamma + 2a - 2 - 2\gamma a$. Moreover, we have

$$\begin{aligned} 2p_a(C_0) - 2 = (K_{\tilde{S}} + C_0, C_0) &= ((a-2)H + (b + 2\gamma - 4)F, aH + bF) && (34)\\ &= (a-2)a(2\gamma - 2) + (a-2)b + (b + 2\gamma - 4)a && (35)\\ &= -2(\gamma - 1)a^2 + 2(7\gamma - 4)a - 2(6\gamma - 2), && (36)\end{aligned}$$

which implies that

$$p_a(C_0) = -(\gamma - 1)a^2 + (7\gamma - 4)a - 6\gamma + 3. \tag{37}$$

If $a \leqq 2$, then $p_a(C_0) \leqq 4\gamma - 1$. If $a = 3$, then $p_a(C_0) = 6\gamma$. If $a = 4$, then $p_a(C_0) = 6\gamma + 3$. If $a \geqq 5$, then $p_a(C_0) \leqq 4\gamma + 8$. Since $6\gamma + 1 \leqq p_a(C_0)$, a must be 4. Thus, we obtain $C_0 \sim 4H - (2\gamma - 6)F$ and $p_a(C_0) = 6\gamma + 3$. We note that $p_a(C_0) - g(C) = 3$.

We can show that the minimal resolution $\pi : \tilde{S} \longrightarrow S$ is isomorphic. Indeed, the same proof as in Case $g = 6\gamma + 1$ works well if we replace $e = 2\gamma - 2$ and $2m = e + 2\gamma$ by $e = 2\gamma - 4$ and $2m = e + 2\gamma - 2$ respectively. We get a contradiction, because $\gamma \geqq 4$.

Case 1: Assume that $\gamma \geqq 6$ and C_0 has distinct three singularities. We may assume that P_0 is a double point. We replace $\varphi_{|(6\gamma+2)P|} : C \longrightarrow \mathbb{P}^{2\gamma+1}$ and $C_0 \sim 4H - (2\gamma - 2)F$ by $\varphi_{|(6\gamma-2)P|} : C \longrightarrow \mathbb{P}^{2\gamma-1}$ and $C_0 \sim 4H - (2\gamma - 6)F$ in Case 1 of Case $g = 6\gamma + 1$ respectively. Then it follows that $6\gamma - 5$ is a non-gap. This is a contradiction.

Case 2: Assume that $\gamma \geqq 8$ and C_0 has distinct two singularities. We may assume that P_0 is a double point such that there is an infinitely near singularity to P_0. Let H be the hyperplane section of Σ_e whose pullback to C is $(6\gamma-2)P$.

We consider the case where H is irreducible. We replace $\varphi_{|(6\gamma+2)P|}: C \longrightarrow \mathbb{P}^{2\gamma+1}$ and $C_0 \sim 4H-(2\gamma-2)F$ by $\varphi_{|(6\gamma-2)P|}: C \longrightarrow \mathbb{P}^{2\gamma-1}$ and $C_0 \sim 4H-(2\gamma-6)F$ in Case 2-i) of Case $g=6\gamma+1$ respectively. Then it follows that $6\gamma-7$ is a non-gap. This is a contradiction.

If H is reducible, by the same proof as that of the case where $g=6\gamma+1$, the semigroup $H(P)$ contains 4.

Case 3: Assume that C_0 has only one singularity. Let H be the hyperplane section of Σ_e as in Case 2. We may assume that H is irreducible.

Case 3-i): Assume that $\gamma \geqq 10$ and P_0 is a double point. Let F, $\rho: S_1 \longrightarrow \Sigma_e$, E, P_1, $\rho_1: S_2 \longrightarrow S_1$, E_1, C_2 and $\phi_2: C \longrightarrow C_2$ be as in Case 2 of Case $g=6\gamma+1$. We denote by $P_2 \in C_2$ the infinitely near singularity to P_1. Let $\rho_2: S_3 \longrightarrow S_2$ be the blow-up at P_2 and E_2 its exceptional divisor. Let C_3 be the proper transformation of C_2. Let $\phi_3: C \longrightarrow C_3$ be the morphism with $\phi_2 = \rho_2|_{C_2} \circ \phi_3$. We denote by $P_3 \in C_3$ the nonsingular point over P_2. Let $\mu_1: S_2 \longrightarrow \Sigma_1$ be the contraction of $\rho_2^*\rho_1^*(\rho^*F-E)$, $\mu_2: \Sigma_1 \longrightarrow \Sigma_2$ the contraction of $\rho_2^*(\rho_1^*E-E_1)$ and $\mu_3: \Sigma_2 \longrightarrow \Sigma_\theta$ the contraction of $\rho_2^*E_1-E_2$. We note that Σ_θ is a rational ruled surface with invariant θ. Moreover, we obtain

$$\phi_3^*((\rho_2^*\rho_1^*\rho^*H-\rho_2^*\rho_1^*E-\rho_2^*E_1-E_2)|_{C_3}) \tag{38}$$
$$\sim (6\gamma-2)P-2P-2P-2P=(6\gamma-8)P. \tag{39}$$

It follows from $(\rho^*H-E)\cap(\rho^*F-E)=\emptyset$ that

$$(\rho_2^*\rho_1^*\rho^*H-\rho_2^*\rho_1^*E-\rho_2^*E_1-E_2, \rho_2^*\rho_1^*(\rho^*F-E))=0. \tag{40}$$

Moreover, we get

$$(\rho_2^*\rho_1^*\rho^*H-\rho_2^*\rho_1^*E-\rho_2^*E_1-E_2, \rho_2^*(\rho_1^*E-E_1) \tag{41}$$
$$=(\rho_1^*\rho^*H-\rho_1^*E-E_1, \rho_1^*E-E_1)=0 \tag{42}$$

and

$$(\rho_2^*\rho_1^*\rho^*H-\rho_2^*\rho_1^*E-\rho_2^*E_1-E_2, \rho_2^*E_1-E_2)) \tag{43}$$
$$=(\rho_1^*\rho^*H-\rho_1^*E-E_1, E_1)+(E_2,E_2)=0. \tag{44}$$

Hence the linear system $\rho_2^*\rho_1^*\rho^*H-\rho_2^*\rho_1^*E-\rho_2^*E_1-E_2$ defines a morphism $f_3: \Sigma_\theta \longrightarrow \mathbb{P}^{2\gamma-4}$. Let H'' be a hyperplane section defining the morphism f_3 and $H'' \sim T_\theta + m''F''$ where T_θ and F'' are a minimal section and a fiber on Σ_θ respectively. We have $m''-\theta = m-e$ or $m-e-1$ or $m-e-2$ or $m-e-3$. Since $C_0 \sim 4T_e+(4m-2\gamma+6)F$ is irreducible, we get

$$0 \leqq 4m-(2\gamma-6)-4e \tag{45}$$
$$=4(m-e-3)-(2\gamma-18) \leqq 4(m''-\theta)-(2\gamma-18), \tag{46}$$

which implies that $4(m'' - \theta) \geqq 2(\gamma - 9)$. Since $\gamma \geqq 10$, we have $m'' - \theta > 0$. Hence f_3 is an embedding. Thus, the image C_3 of $\varphi_{|(6\gamma-8)P|} : C \longrightarrow \mathbb{P}^{2\gamma-4}$ is contained in a rational ruled surface Σ_θ. The image P_3 of P by $\varphi_{|(6\gamma-8)P|}$ is a smooth point. Hence, $6\gamma - 9$ is a non-gap at P. This is a contradiction.

Case 3-ii): Assume that $\gamma \geqq 5$ and C_0 has a triple ordinary point. We may assume that P_0 is a triple point of C_0. We use the same notation in Case 1 of Case $g = 6\gamma + 1$. Then we have $(\tilde{C}_0, E) = 3$. Since $\gamma \geqq 5$, H' is base point free. Moreover, $(\rho^*\delta_{P_0} - E)|_{\tilde{C}_0} = (6\gamma - 5)\tilde{P}_0$. Therefore, $6\gamma - 5$ is a non-gap. This is a contradiction.

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