University of Arizona

Honors Thesis

Mathematics

Extension of the Group Invariance Theorem for Perceptrons

Author
Andrew Gordon

Thesis Advisor
Professor Kevin Lin

December 18, 2019

Abstract

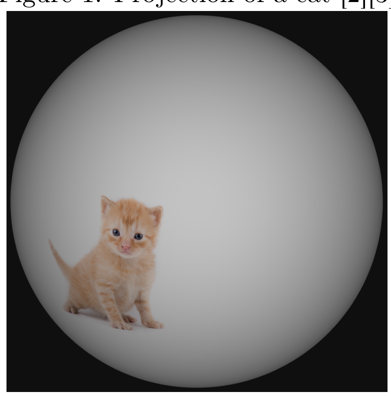
Most of statistical machine learning relies on deep neural nets, whose underlying theory and mathematical properties are currently not well explored. This thesis focuses on the theory of Perceptrons, a form of shallow neural network mostly explored in Minsky & Papert's book "Perceptrons: an Introduction to Computational Geometry". Their analysis relied on the use of groups that describe the global symmetries of what is to be computed. However, all real images have an input space with finite length and height, and it is often the case that machine learning systems have disconnected input spaces. This means that real-world systems must actually rely on a concept of *local* symmetry, if any at all. This paper explores the implications of having a bounded and possibly disconnected input space, and generalizes the Minsky & Papert result, proving that the implications of the group invariance theorem can be retained by using groupoids instead.

Introduction

All real images have an input space with finite length and height, and it is often the case that machine learning systems have disconnected input spaces. This means that real-world image systems must actually rely on a concept of *local* symmetry, if they are to rely on any concept of symmetry at all. Groups model a concept of *global* symmetry: they allow multiplications without finite bound, they assume that whatever surface they are acting on is infinite (or loops around like the surface of a torus), they assume that any action that was previously applied to an object is still legal to apply after any number of other legal changes. These assumptions are simply too ideal to deal with many real world problems. Groupoids, on the other hand, capture much of the analytical power of groups while avoiding the above pitfalls. For these reasons, they have applications to many of the fundamental problems in computer vision and machine learning.

It is useful to first build intuition about what is a group, what is a groupoid, and why they apply to the problem before proceeding to the next question. Consider the following image of a cat projected onto a sphere [14]:

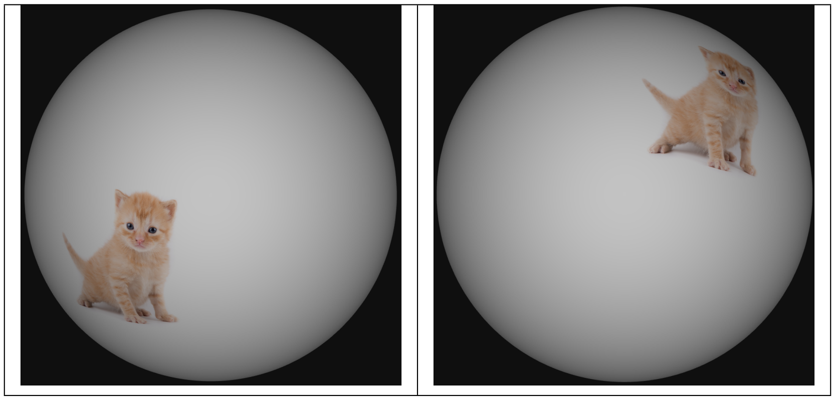
Figure 1: Projection of a cat [2][3]



Clearly, spheres can be rotated without changing the space that they occupy. Furthermore, any image on the surface of a sphere isn't deformed after a rotation, its position is merely moved:

Table 1: Rotations of the Sphere [2][3]

Before Rotation | After Rotation



This makes rotations of a sphere a natural candidate for a group acting on the sphere. Consider the set:

$G \subset M_{3x3}(\mathbb{R})$ s.t. $\forall g \in G$, $\det g = \pm 1$

(the set of all 3x3 matrices over $\mathbb{R}$ with determinant 1 or -1). Then:

Associativity is inherited from matrix multiplication.

Each has $g \in G$ nonzero determinant, and thus is invertible.

$\det\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = 1$, thus there is an identity element in G.

$1 = \det\left(\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right) = \det(g \cdot g^{-1}) = \det(g)\det(g^{-1}) \implies \det(g) = \det(g^{-1}) = \pm 1 \implies g^{-1} \in G$. Thus not only does each g have an inverse, each of those inverses are also actually in G

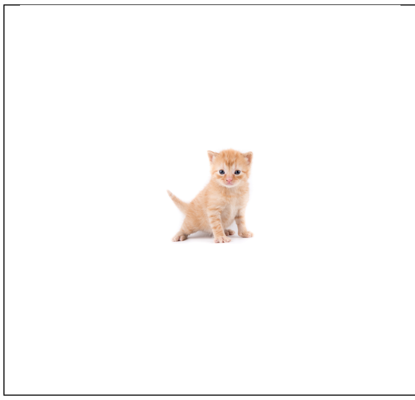
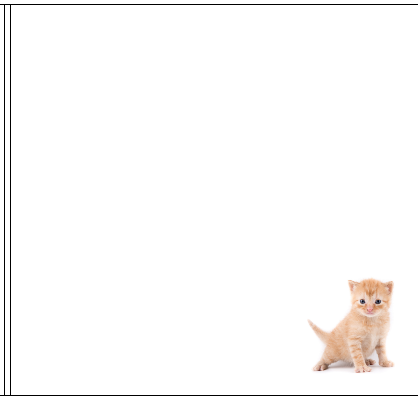
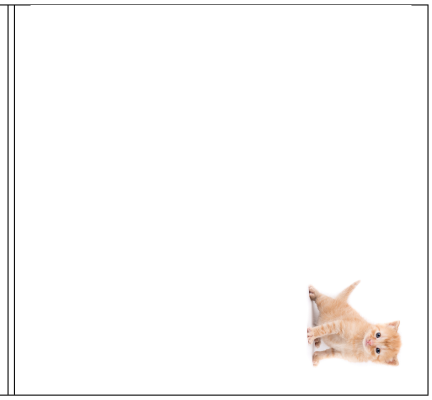
Thus G is a group, with the group operation being matrix multiplication. In fact, this group is called the 3D rotation group, or $SO(3)$ [13]. This group acts on not just the sphere, but also naturally acts on the image of the cat which is projected onto the sphere. In fact, if an orthogonal basis $\{x, y, z\}$ of $\mathbb{R}^3$ is chosen such that z points to the center of the cat, then:

rotation around z (multiplication by matrices of the form $\begin{bmatrix}* & * & 0\\ * & * & 0\\ 0 & 0 & 1\end{bmatrix}$) is equivalent to a rotation of the cat image,

rotation around y (multiplication by matrices of the form $\begin{bmatrix}* & 0 & *\\ 0 & 1 & 0\\ * & 0 & *\end{bmatrix}$) is equivalent to a translation of the cat image along the x-axis,

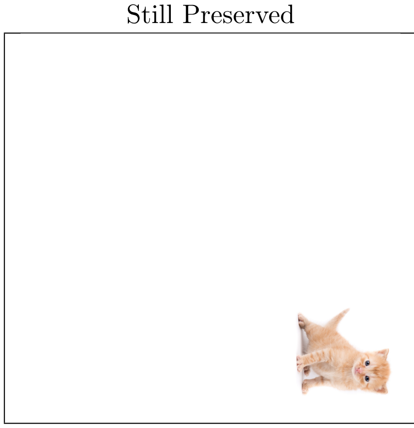
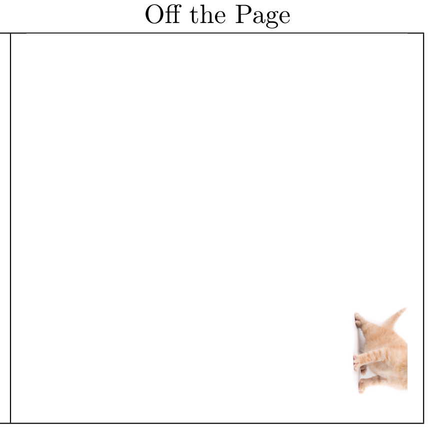
rotation around x (multiplication by matrices of the form $\begin{bmatrix}1 & 0 & 0\\ 0 & * & *\\ 0 & * & *\end{bmatrix}$) is equivalent to a translation of the cat image along the y-axis:

Table 2: Action on the finite plane [3]

Initial State	Translation	Rotation
		

The image of the cat is preserved in all these cases, which means that it would seem that a group action is the natural way to explain the movement of the cat image on the finite plane. Except there is a problem:

Table 3: Groupishness Fails [3]

Still Preserved	Off the Page
	

It is impossible to move the cat "off the page" when rotating the sphere, but easy to do so when translating the cat in the finite plane. Thus there is a "nearly group action" inherited from most of the composition of multiplication in $SO(3)$ with the inverse projection map - associativity and inverses are fine, except some translations that go too far would move the cat off the plane. Thus, this is a natural example of a groupoid action.

In order to define groupoids and groupoid actions, it is useful to first review a precise definition of what they generalize - groups and group actions:

Definition 1. Group: a set G equipped with a binary operation $\cdot$ is a group iff:

1. G is closed under $\cdot$: $a, b \in G \implies a \cdot b \in G$
2. $\cdot$ is associative: $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
3. G has an identity element: $\exists e \in G$ such that $\forall g \in G$, $e \cdot g = g \cdot e = g$
4. Every element of G has an inverse: $\forall g \in G$, $\exists g^{-1}$ such that $g \cdot g^{-1} = g^{-1} \cdot g = e$

In their geometric interpretation, groups are synonymous with the concept of symmetry, as they describe the symmetries of some object by their action on that object:

Definition 2. Group action: given a group G and a set M, a (left) group action of G on M is a function:

1. $\hat{\cdot} : G \times M \to M \ni (g, s) \mapsto g \hat{\cdot} s$, where $g \hat{\cdot} s \in M$
2. Group identity is preserved: $\forall s \in M$, $e \hat{\cdot} s = s$.
3. Group multiplication is compatible with $\hat{\cdot}$: $\forall a, b \in G, \forall s \in M$: $(a \cdot b) \hat{\cdot} s = a \hat{\cdot} (b \hat{\cdot} s)$

Groupoids generalize groups, such that they retain invertibility of all operations, except that they no longer assume closure:

Definition 3. Groupoid: a set G equipped with a <u>partial operation</u> $\cdot$ is a groupoid iff:

1. $\cdot$ is associative: $\forall a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, assuming each multiplication is well-defined.
2. Every element of G has an inverse: $\forall g \in G$, $\exists g^{-1}$ such that $g \cdot g^{-1}$ and $g^{-1} \cdot g$ are well defined.
3. Inverses produce identities: $g \cdot g^{-1} \cdot a = a \cdot g^{-1} \cdot g = a$. (note: does <u>not</u> require that $g^{-1} \cdot a$ or $a \cdot g^{-1}$ are well-defined, since $g^{-1}g$ and $g^{-1}g$ <u>are</u> well-defined).

Every group trivially fulfills the axioms of a groupoid, and every finite groupoid can be realized by taking a group and deleting elements. For example, consider D_3, the group of rotations and flips on an equilateral triangle. If one allows rotation only 120° or 240° clockwise from the initial position, and then the ability to reverse these rotations, then clearly invertibility and associativity are fulfilled. But this is not a full group - attempting to first rotate 240° then 120° is not allowed - this is equivalent to deleting certain possible moves. It is important to note that some groupoids can exhibit awkward, unintuitive properties, like handed identities or disconnectedness:

Table 4: Disconnected vs Connected Groupoid

Connected	Disconnected
$\overset{b}{\circlearrowleft} \quad \underset{y^{-1}}{\overset{y}{\rightleftharpoons}} \quad \overset{a}{\circlearrowleft} \quad \underset{k^{-1}}{\overset{k}{\rightleftharpoons}} \quad \overset{c}{\circlearrowleft}$	$\overset{b}{\circlearrowleft} \quad \underset{y^{-1}}{\overset{y}{\rightleftharpoons}} \quad \overset{a}{\circlearrowleft} \qquad \overset{c}{\circlearrowleft}$

In the picture, “a”, “b”, and “c” are all identity elements. Notice that in the second diagram, there is no way to get to “c” by following the arrows starting at “a” or “b”. This is a textbook example of a disconnected groupoid, as “c” is not connected to the component with “a” and “b”. Formally:

Definition 4. Connected Groupoid: a set G equipped with a binary operation $\cdot$ is a connected groupoid iff:

1. G is a groupoid

2. Each element is "reachable" from any other element: $\forall a, c \in G, \exists k \in G \ni a \cdot k = c$

Note also that identities in groupoids can be "handed"; that is, one is not guaranteed that: $y^{-1}y = yy^{-1}$. In the above example, $y^{-1}y = a \neq b = yy^{-1}$. In this example, a is the "right-hand" identity of y and b is the "left-hand" identity of y. Once these awkward properties are accounted for, groupoid actions are exactly like group actions:

Definition 5. Groupoid action:

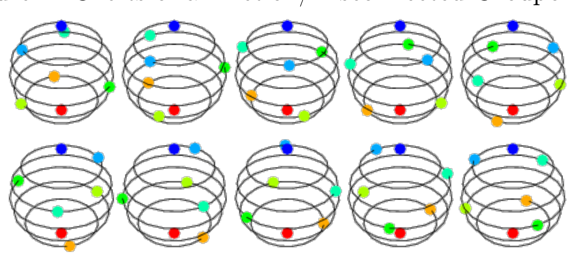
1. $\hat{\cdot} : G \times M \to M$ such that $(g, s) \mapsto g\hat{\cdot}s$, where $g\hat{\cdot}s \in M$. (notice that $G \times M$ is a total function but $G \times G$ isn't)
2. Groupoid identity is preserved: $\forall s \in M, \forall g \in G$, $(g \cdot g^{-1})\hat{\cdot}s = s = (g^{-1} \cdot g)\hat{\cdot}s$
3. Groupoid multiplication is compatible with $\hat{\cdot}$: $\forall a, b \in G, \forall s \in M$: $(a \cdot b)\hat{\cdot}s = a\hat{\cdot}(b\hat{\cdot}s)$, assuming $a \cdot b$ is well-defined.

It is important to note that groupoid actions describe invertible symmetries, so what's happening in Table 3 is not a groupoid action, as the image of the cat is not preserved. However, they do describe what's happening in Table 2, since all the operations done to the cat image are invertible and the cat is preserved. Table 2 is not an example of a group action, as translations cannot be performed without bound - that is; the groupoid isn't closed. For example, the action "move the cat to the right by one pixel" is a translation (therefore groupish) but can only be performed so many times before it no longer preserves they symmetry. In the cat example given above, every legal position and degree of rotation of the cat can be reached by every other legal position and degree of rotation, so it is a connected groupoid. This motivates the concept of "orbit of a group action". If a group G acts on a set M, then $m, n \in M$ share the same orbit iff $\exists g \in G$ such that $gm = n$. That is, if m can be transformed into n by way of G. Formally:

Definition 6. Group(oid) equivalence class under a group(oid) action: given group(oid) G, and sets M, M': $M \underset{G}{=} M' \iff \exists g \in G$ such that $gM = g\{s_1, s_2, s_3...\} = \{g\hat{\cdot}s_1, g\hat{\cdot}s_2, ...\} = M'$.

It is easy to check that this naturally partitions M into different orbits, all of which form groups (or groupoids iff G is a groupoid). Taken together, these different partitions form a disconnected groupoid (unless G acts trivially on M)! For example:

Figure 2: Orbits of an Action/Disconnected Groupoid [4]



Each circle with a rotating colored dot is an orbit, and each orbit forms a 2D-rotation group, and taken as a disconnected collection, they fit the axioms of a groupoid.

Toy example of a Perceptron

Now it is possible to motivate the concept of a perceptron. Perceptrons are defined by having the following properties:

They have an input space:

Definition 7. R: A collection of Boolean variables in a grid. Also called the "input space". Could be inputs from a some physical system, could be bits on a computer, etc. Assumed to be a family of possible input tuples for the purposes of this paper's proof and discussion.

They answer yes-or-no questions about properties of the specific input:

Definition 8. ψ: Given some $X \subseteq R$ (X is the specific input), $\psi_{\text{question}}(X)$ is a function that returns 1 if the yes-or-no question of ψ is true of input X and 0 if the question is false. Example: let $X = \{0,1,0,1,0\}$. Then: $\psi_{\text{has even number of 0 inputs}}(X) = 0$.

They answer the big yes-or-no question by first answering a number of smaller yes-or-no questions about the input:

Definition 9. ϕ: Given some $X \subseteq R$, $\phi(X)$ returns 1 or 0 based off of some property of X. Example: let $\phi(X) = \begin{cases} 1 & \forall x \in X, x > 0 \\ 0 & \text{otherwise} \end{cases}$.

They have a number of ϕ's that are summed and then compared:

Definition 10. Φ: A family of ϕ's. $\sum\limits_{s \in S} f(s)$: sum of all values of $f(s)$ for all elements s in set S. Example: suppose $f(s) = 2s$ and $S = \{4,7,9\}$. Then: $\sum\limits_{s \in S} f(s) = 2(4) + 2(7) + 2(9) = 40$.

These individual ϕ's usually have a bounded amount of the input that they actually compute on:

Definition 11. S_ϕ: The support of ϕ; the minimal set $S \ni \forall X \subseteq R$, $\phi(X \cap S) = \phi(X)$. That is, the smallest set S_ϕ such that everything in S_ϕ "actually matters" in the evaluation of ϕ.

The perceptron eventually returns 1 if $\psi(X)$ is true, or 0 if $\psi(X)$ is false. This is done by seeing if the result of a sum exceeds some value.

Definition 12. $\lceil\rceil$: Function that returns one if a proposition is true, zero otherwise. Example: $\lceil 16 > 0 \rceil = 1$, $\lceil \pi \in \mathbb{N} \rceil = 0$ etc.

Example. (This example is a near copy of the one on page 14 of M&P [1]):

Suppose you have:

an input space: $\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \end{bmatrix}$, with $a_i \in \{0,1\}$

an number of predicates, each $\phi_i(X) = \begin{cases} 1 & a_i = 1 \\ 0 & a_i = 0 \end{cases}$

an ordered set of predicates of the form $\Phi = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5\}$

Then in this case, $S_{\phi_1} = \{a_1\}$ and the ψ "there is more weight on the right side of the input space" is equivalent to:

$$\psi(X) = \lceil \phi_5(X) + \phi_4(X) - \phi_2(X) - \phi_1(X) > 0 \rceil$$

Group Invariance Theorem

Theorem. *Group Invariance Theorem. (Pages 48-50 of M&P [1])*
Suppose that:

1. G is a finite group of transformations of a finite space R.
2. Φ is a set of predicates on R closed under G;
3. ψ can be represented by a linear combination of elements of Φ and is invariant under G.

Then there exists a linear representation of ψ, $\psi = \left\lceil \sum_{\phi\in\Phi} \beta_\phi \phi > 0 \right\rceil$ for which the coefficients β_ϕ depend only on the G-equivalence class of ϕ, that is: $\phi \underset{G}{=} \phi' \implies \beta_\phi = \beta_{\phi'}$.

Extension of the Group Invariance Theorem

Theorem. *Groupoid Invariance Theorem.*
Suppose that:

1. G is a finite groupoid of transformations of a finite space R.
2. Φ is a set of predicates on R s.t. $\forall g \in G$, $\forall \phi \in \Phi$, $gS_\phi \subseteq R \implies \phi(gR) = (\phi g)(R)$ for some $\phi g = \phi' \in \Phi$.
3. ψ can be represented by a linear combination of elements of Φ, is invariant under G, and $\forall g \in G$, $gS_\psi \subseteq R$.

Then there exists a linear representation of ψ, $\psi = \left\lceil \sum_{\phi\in\Phi} \beta_\phi \phi > 0 \right\rceil$ for which the coefficients β_ϕ depend only on the G-equivalence class of ϕ, that is: $\phi \underset{G}{=} \phi' \implies \beta_\phi = \beta_{\phi'}$.

Note. this proof very closely resembles the proof given in Minsky & Papert (1969), by design.

Proof. Assume (1), (2), and (3), and let $\psi(X)$ have representation $\sum_{\phi\in\Phi} \alpha(\phi)\phi(X) > 0$.

Then: $\sum_{\phi\in\Phi} \alpha(\phi)\phi(X) = \sum_{\phi\in\Phi} \alpha(\phi g)\phi g(X)$, since ψ is $G-$invariant so the sums add up to exactly the same numbers.

Suppose $\psi(X) = 1$.

Since G is a groupoid, $g^{-1} \in G$.

Then $0 < 1 = \left\lceil \sum_{\phi\in\Phi} \alpha(\phi g)\phi g(g^{-1}X) > 0 \right\rceil$ and $\sum_{\phi\in\Phi} \alpha(\phi g)\phi g(g^{-1}X) = \sum_{\phi\in\Phi} \alpha(\phi g)\phi(X)$ thus $0 < 1 = \left\lceil \sum_{\phi\in\Phi} \alpha(\phi g)\phi(X) > 0 \right\rceil$.

Since this sum is positive for all $g \in G$, it follows that $\sum_{g\in G} \left(\sum_{\phi\in\Phi} \alpha(\phi g)\phi(X) \right) > 0$.

By the linearity of sums it follows that $0 < \sum_{g\in G}\left(\sum_{\phi\in\Phi}\alpha(\phi g)\phi(X)\right) = \sum_{\phi\in\Phi}\left(\sum_{g\in G}\alpha(\phi g)\right)\phi(X)$.
Then $\sum_{g\in G}\alpha(\phi g)$ can be rewritten as: β_ϕ.

The argument follows in exactly the same manner when $\psi(X) = 0$ for $\sum_{\phi\in\Phi}\beta(\phi)\phi(X) \leq 0$.

Combining both inequalities yields $\psi(X) = \left\lceil \sum_{\phi\in\Phi}\beta(\phi)\phi(X) > 0 \right\rceil$.

Now suppose $\phi \underset{G}{=} \phi'$.

By definition of $\phi \underset{G}{=} \phi'$, $\exists h \in G \ni \phi(X) = \phi' h(X)$.

Then $\phi' h = \phi g$ for some $g \in G$, thus $\phi(X) = \phi g(X)$.

$\beta(\phi) = \sum_{g\in G}\alpha(\phi g) = \sum_{g\in G}\alpha(\phi' hg) = \sum_{n\in G}\alpha(\phi' n) = \beta(\phi')$; i.e. $\beta(\phi')$ is a permutation of the sum of the values in $\beta(\phi)$. [1]

$\square$

Application

Now that it's proven, the expanded version of the group(oid) invariance theorem can be used to prove expanded versions of the theorems given in M&P, and possibly entirely new theorems in new, more generalized domains. As an example, an expansion of theorem 2.4 on page 53 of M&P[1] follows:

Theorem. Let G be a group(oid) acting on R that preserves some ψ, such that it partitions R into exactly one orbit. That is, for every pair of points, $(p, q) \in R$, $\exists g \in G$ such that $gp = q$. (note that this can only happen if G is connected or acts trivially! Disconnected components of a groupoid that act nontrivially on a set ensure there are at least two orbits!). Then either:

$$\textit{for some } m\text{:} \begin{cases} \psi(X) = & \lceil |X| > m \rceil \\ \psi(X) = & \lceil |X| \geq m \rceil \\ \psi(X) = & \lceil |X| < m \rceil \\ \psi(X) = & \lceil |X| \leq m \rceil \end{cases} \textit{(where } |X| \textit{ means "the number of nonzero entries in } X\textit{")}$$

- or -

for some $\phi \in \Phi$, S_ϕ is larger than one point

Proof. Since all ϕ_ps such that $\left|S_{\phi_p}\right| = 1$ are equivalent (where each p is a specific point $p \in X$), we can assume that:

[1] Note that if G is disconnected, then it is naturally partitioned into disjoint sets made up of its connected components. This means that if a disconnected groupoid is acting on R, then Φ is first partitioned by the disconnected components of G, and then those partitions are again partitioned by how the components act on R. This often leads to many more partitions in the disconnected case than in the original M&P cases, which often rely on the set having very few orbits or only one orbit. Thus, one needs to be careful when attempting to generalize those results to the disconnected case.

$\psi(X) = \left\lceil \sum_{p \in X} \alpha\phi_p(X) > \theta \right\rceil$

where the coefficients α are all independent of p (otherwise no group(oid) symmetry). Thus either:

$\psi(X) = \left\lceil \sum_{p \in X} \phi_p(X) > \frac{\theta}{\alpha} \right\rceil$ if $\alpha \geq 0$ -or- $\psi(X) = \left\lceil \sum_{p \in X} \phi_p(X) < \frac{\theta}{\alpha} \right\rceil$ if $\alpha < 0$

In either case:

$\sum_{p \in X} \phi_p(X) = |X|$

$\square$

Minsky and Papert then go on to comment that their version of the theorem applies to the example they give on page 46. This is not the case! The example on page 46 is a natural group<u>oid</u> action, not a group! This version, however, does apply.

Conclusion

The content of the Group Invariance theorem generalizes to groupoids, as has been shown above. This ensures that a version of M&P's group invariance theorem applies not just to cases where the ideal conditions are met, but to real-world examples such as using perceptrons on normal digital pictures, which have finite length and width, and edges. Additional work in this direction should focus on generalization to applications in the case where groupoids are disconnected, since each groupoid is made up of connected components and thus the groupoid invariance theorem applies to each disconnected component and thus has implications even for the wildly more misbehaved disconnected groupoids, and to the case where perceptrons might be multilayered.

Appendix: Category-theoretic Discussion of Groupoids

Note. This section is intended for readers who have some familiarity with basic category theory, but not necessarily for those who are already familiar with groupoids within category theory. Readers who are not familiar with category theory are encouraged to read the pages "Category Theory"[7], "Category"[6], and "Functor"[8] on nLab or an introduction to category theory of their choice, such as "Category Theory in Context"[11] before reading this section.

Many readers will most likely have first encountered groupoids in a category-theoretic framework. The arrow-theoretic version of groupoids was omitted so as not to rely on math that may be not familiar to some readers, but the definition of groupoid used in this paper was nearly identical to the arrow-theoretic version. This section will informally show the categorical equivalence between the arrow-theoretic definition of a (small) groupoid and the one used in the other sections of this paper.

Suppose G is a groupoid with operation $\cdot$, and F is a function such that:

$F : G \times G \to C$ is a bijection

F splits over multiplication: $\forall a, b \in G$, if $a \cdot b$ is well-defined, then $F(a \cdot b) = F(b) \circ F(a)$.

it is then simple to check that C naturally inherits the properties of a category from F (with all properties of C inherited from consequences of properties of F), that F is an invertible, contravariant functor, and that C is a groupoid with:

Inverses preserved over F: $F(a^{-1}ab) = F(b) = F(ba^{-1}a) \implies F(a) \circ F(a^{-1}) = F(a) \circ (F(a))^{-1}$

Inverses are *retained* through F: $\forall g \in G,\, g^{-1} \in G \implies F(g^{-1}) = (F(g))^{-1} \implies (F(g))^{-1} \in C$

Left identities in G as targets in C:
$F(a^{-1}ab) = F(b) = F(b) \circ F(a) \circ F(a^{-1}) \implies F(a) \circ F(a^{-1}) = t_C(b)$.

Right identities in G as sources in C:
$F(ba^{-1}a) = F(b) = F(a) \circ F(a^{-1}) \circ F(b) \implies F(a) \circ F(a^{-1}) = s_C(b)$.

Well-behaved composition in C is naturally inherited from $\cdot$ in G.

The identity morphisms of C are equivalent to the left/right identities elements in G, due to the compatibility assumption.

No size issues since we assumed that G is small enough to be a set (see “Category: Size Issues” [9]).

It is important to note that the normal direction of group(oid) multiplication is left-to-right, and that the normal direction of composition is right-to-left, thus F is contravariant and multiplications of the form abc in G are of the form $F(c) \circ F(b) \circ F(a)$ in C. If this causes confusion, C can easily be exchanged for its opposite category, C^{op}, which is categorically equivalent because all groupoids are self-dual (See Riehl, page 9 [11]), as can be shown by verifying that the following diagrams commute:

$$\begin{array}{ccc|ccc} t_C(a) & \overset{\text{op}}{\Rightarrow} & s_C(a) & a & \overset{\text{op}}{\Rightarrow} & a \\ \Downarrow M & & \Downarrow M & \Downarrow M & & \Downarrow M \\ t_C(a) & \overset{\text{op}}{\Rightarrow} & s_C(a) & a^{-1} & \overset{\text{op}}{\Rightarrow} & a^{-1} \end{array}$$

And that M is invertible.

References

[1] Minsky, David M and Papert, Seymour. 1987. “Perceptrons: An Introduction to Computational Geometry”, second edition. The MIT Press, Cambridge MA.

[2] Images of the cat on a globe generated with “Map to Globe”. Accessed Dec 2019. `https://www.maptoglobe.com/SkefU4XCH?key=SyegMIE7RS`

[3] Original image of a cat: Hodan, George. “Public Domain Pictures”. Accessed Dec 2019.`https://www.publicdomainpictures.net/en/view-image.php?image=167359&picture=cat-on-the-white`

[4] Orbits image: Rowland, Todd and Weisstein, Eric W. "Group Orbit." From MathWorld--A Wolfram Web Resource. Accessed Dec 2019. `http://mathworld.wolfram.com/GroupOrbit.html`

[5] nLab authors. “Groupoid”. Accessed Dec 2019. `http://ncatlab.org/nlab/show/groupoid`

[6] nLab authors. “Category”. Accessed Dec 2019. `https://ncatlab.org/nlab/show/category`

[7] nLab authors. “Category Theory”. Accessed Dec 2019. `https://ncatlab.org/nlab/show/category+theory`

[8] nLab authors. “Functor”. Accessed Dec 2019. `https://ncatlab.org/nlab/show/functor`

[9] nLab authors. “Category: Size Issues”. Accessed Dec 2019. `https://ncatlab.org/nlab/show/category#size`

[10] nLab authors. "Opposite Category". Accessed Dec 2019. https://ncatlab.org/nlab/show/opposite+category

[11] Riehl, Emily. "Category Theory in Context". 2016. Dover Press.

[12] Weinstein, Alan. 1996. "Groupoids: Unifying Internal and External Symmetry". https://arxiv.org/abs/math/9602220

[13] Brown, Robert. July 11, 2017. Accessed Dec 2019. "The Rotation Group". https://webhome.phy.duke.edu/~rgb/Class/Electrodynamics/Electrodynamics/node31.html

[14] Unknown Author. Open Source Geospatial Foundation. Accessed Dec 2019. "Equidistant Cylindrical (Plate Carrée)". https://proj.org/operations/projections/eqc.html