Manuscript submitted to
AIMS' Journals
Volume **X**, Number **0X**, XX **200X**

doi:10.3934/xx.xx.xx.xx

pp. **X–XX**

ASYMPTOTIC STRUCTURE OF THE SPECTRUM IN A DIRICHLET-STRIP WITH DOUBLE PERIODIC PERFORATIONS

Sergei A.Nazarov

Saint-Petersburg State University, Universitetskaya nab., 7-9, St. Petersburg, 199034, Russia,
& Institute of Problems of Mechanical Engineering RAS,
V.O., Bolshoj pr., 61, St. Petersburg, 199178, Russia

Rafael Orive-Illera[2], María-Eugenia Pérez-Martínez[3,*]

[2]Instituto de Ciencias Matemáticas, CSIC-UAM-UC3M-UCM,
Nicolás Cabrera 13-15, Campus de Cantoblanco-UAM, Madrid, 28049, Spain,
& Departamento de Matemáticas, Facultad de Ciencias,
Universidad Autónoma de Madrid, Spain
[3]Departamento de Matemática Aplicada y Ciencias de la Computación,
Universidad de Cantabria, Avenida de las Castros s/n, 39005 Santander, Spain

(Communicated by the associate editor name)

Abstract. We address a spectral problem for the Dirichlet-Laplace operator in a waveguide Π^ε. Π^ε is obtained from an unbounded two-dimensional strip Π which is periodically perforated by a family of holes, which are also periodically distributed along a line, the so-called "perforation string". We assume that the two periods are different, namely, $O(1)$ and $O(\varepsilon)$ respectively, where $0 < \varepsilon \ll 1$. We look at the band-gap structure of the spectrum σ^ε as $\varepsilon \to 0$. We derive asymptotic formulas for the endpoints of the spectral bands and show that σ^ε has a large number of short bands of length $O(\varepsilon)$ which alternate with wide gaps of width $O(1)$.

1. **Introduction.** In this paper we consider a spectral problem for the Laplace operator in an unbounded strip $\Pi \equiv (-\infty, \infty) \times (0, H) \subset \mathbb{R}^2$ periodically perforated by a family of holes, which are also periodically distributed along a line, the so-called "perforation string". The perforated domain Π^ε is obtained by removing the double periodic family of holes $\overline{\omega^\varepsilon}$ from the strip Π, cf. Figure 1,a), (4)-(6). The diameter of the holes and the distance between them in the string is $O(\varepsilon)$, while the distance between two perforation strings is 1. $\varepsilon \ll 1$ is a small positive parameter. A Dirichlet condition is prescribed on the whole boundary $\partial\Pi^\varepsilon$. We study the band-gap structure of the essential spectrum of the problem as $\varepsilon \to 0$.

2010 *Mathematics Subject Classification.* Primary: 35B27, 35P05, 47A55, 35J25, 47A10; Secondary: 35P10, 35P15, 47A75.

Key words and phrases. band-gap structure, spectral perturbations, homogenization, perforated media, double periodicity, Dirichlet-Laplace operator.

The first author is supported by Russian Foundation on Basic Research, grant 18-01-00325.

The second author is supported by the Spanish MINECO through the "Severo Ochoa Programme for Centres of Excellence in RaD" (SEV-2015-0554) and MTM2017-89976-P.

The third author is supported by the Spanish MINECO grant MTM2013- 44883-P and MICINN grant PGC2018-098178-B-I00.

* Corresponding author: María-Eugenia Pérez-Martínez.

We provide asymptotic formulas for the endpoints of the spectral bands and show that these bands collapse asymptotically at the points of the spectrum of the Dirichlet problem in a rectangle obtained by gluing the lateral sides of the periodicity cell. These formulas show that the spectrum has spectral bands of length $O(\varepsilon)$ that alternate with gaps of width $O(1)$. In fact, there is a large number of spectral gaps and their number grows indefinitely when $\varepsilon \to +0$.

It should be emphasized that waveguides with periodically perturbed boundaries have been the subject of research in the last decade: let us mention e.g. [34], [21], [22], [2] and [3] and the references therein. However the type of singular perturbation that we study in our paper has never been addressed. We consider a waveguide perforated by a periodic perforation string, which implies using a combination of homogenization methods and spectral perturbation theory.

As usual in waveguide theory, we first apply the Gelfand transform (cf. [6], [30], [33], [26], [11] and (11)) to convert the original problem, cf. (7), into a family of spectral problems depending on the Floquet-parameter $\eta \in [-\pi, \pi]$ posed in the periodicity cell ϖ^ε (cf. (13)-(16) and Fig. 1, b). Each one of these problems has a discrete spectrum, cf. (18), which describe the spectrum σ^ε as the union of the spectral bands, cf. (20) and (9). One of the main distinguishing features of this paper is that each problem constitutes itself a homogenization problem with one perforation string. As a consequence, in the stretched coordinates, cf. (30), there appears a boundary value problem in an unbounded strip Ξ which contains the unit hole ω (cf. (2), (31)-(33) and Fig. 2).

The above mentioned homogenization spectral problems have different boundary conditions from those considered in the literature (cf. [5], [14] and [16] for an extensive bibliography). Obtaining convergence for their spectra, correcting terms and precise bounds for discrepancies (cf. (10)), as $\varepsilon \to 0$, prove essential for our analysis. We use matched asymptotic expansions methods, homogenization theory and basic techniques from the spectral perturbation theory.

1.1. **Formulation of the problem.** Let

$$\Pi = \{x = (x_1, x_2) : \, x_1 \in \mathbb{R}, \, x_2 \in (0, H)\} \tag{1}$$

be a strip of width $H > 0$. Let ω be a domain in the plane $\mathbb{R}^2$ which is bounded by a simple closed contour $\partial\omega$ which, for simplicity, we assume to be of class C^∞, and that has the compact closure

$$\overline{\omega} = \omega \cup \partial\omega \subset \varpi^0, \tag{2}$$

where ϖ^0 is a rectangle, the "limit periodicity" cell in Π,

$$\varpi^0 = (-1/2, 1/2) \times (0, H) \subset \Pi. \tag{3}$$

We also introduce the strip Π^ε (see Figure 1,a) perforated by the holes

$$\omega^\varepsilon(j,k) = \left\{x : \, \varepsilon^{-1}(x_1 - j, x_2 - \varepsilon k H) \in \omega\right\} \quad \text{with } j \in \mathbb{Z}, \, k \in \{0, \dots, N-1\}, \tag{4}$$

where $\varepsilon = 1/N$ is a small positive parameter, and $N \in \mathbb{N}$ is a big natural number that we will send to ∞. The period of the perforation along the x_1-axis in the domain

$$\Pi^\varepsilon = \Pi \setminus \bigcup_{j \in \mathbb{Z}} \bigcup_{k=0}^{N-1} \overline{\omega^\varepsilon(j,k)} \tag{5}$$

is made equal to 1 by rescaling, and similarly, the period is made equal to εH in the x_2-direction. The periodicity cell in Π^ε takes the form

$$\varpi^\varepsilon = \varpi^0 \setminus \bigcup_{k=0}^{N-1} \overline{\omega^\varepsilon(0,k)},$$

(see b) in Figure 1). For brevity, we shall denote by ω^ε the union of all the holes in (4), namely,

$$\omega^\varepsilon = \bigcup_{j\in\mathbb{Z}} \bigcup_{k=0}^{N-1} \omega^\varepsilon(j,k), \tag{6}$$

while ω is referred to as the "unit hole", cf. (2).

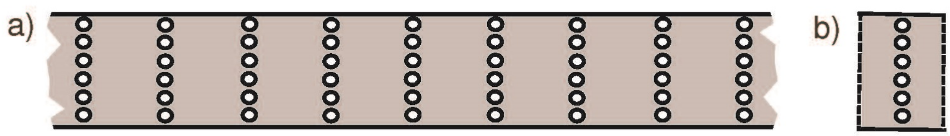

FIGURE 1. **a)** The perforated strip Π^ε is obtained by removing the double periodic family of holes $\overline{\omega^\varepsilon}$ from the strip $\Pi \equiv (-\infty,\infty)\times(0,H)$. The periodicities 1 and εH come from the width of he periodicity cell ϖ^ε and the distance between two consecutive holes in the perforation string. **b)** The periodicity cell ϖ^ε is obtained by removing a periodic family of holes of diameter $O(\varepsilon)$ from $\varpi^0 \equiv (-1/2,1/2)\times(0,H)$. It contains one perforation string.

In the domain (5) we consider the Dirichlet spectral problem

$$\begin{cases} -\Delta u^\varepsilon(x) = \lambda^\varepsilon u^\varepsilon(x), & x \in \Pi^\varepsilon, \\ u^\varepsilon(x) = 0, & x \in \partial\Pi^\varepsilon. \end{cases} \tag{7}$$

The variational formulation of problem (7) refers to the integral identity

$$(\nabla u^\varepsilon, \nabla v)_{\Pi^\varepsilon} = \lambda^\varepsilon\,(u^\varepsilon, v)_{\Pi^\varepsilon} \qquad \forall v \in H_0^1(\Pi^\varepsilon), \tag{8}$$

where $(\cdot,\cdot)_{\Pi^\varepsilon}$ is the scalar product in the space $L^2(\Pi^\varepsilon)$, and $H_0^1(\Pi^\varepsilon)$ denotes the completion, in the topology of $H^1(\Pi^\varepsilon)$, of the space of the infinitely differentiable functions which vanish on $\partial\Pi^\varepsilon$ and have a compact support in $\overline{\Pi^\varepsilon}$. Since the bi-linear form on the left of (8) is positive, symmetric and closed in $H_0^1(\Pi^\varepsilon)$, the problem (8) is associated with a positive self-adjoint unbounded operator $\mathcal{A}^\varepsilon$ in $L^2(\Pi^\varepsilon)$ with domain $H_0^1(\Pi^\varepsilon)\cap H^2(\Pi^\varepsilon)$ (see Ch. 10 in [1]).

Problem (7) gets a positive cutoff value $\lambda_\dagger^\varepsilon$ and, therefore, its spectrum $\sigma^\varepsilon \subset [\lambda_\dagger^\varepsilon,\infty)$ (cf. (20) and Remark 5). It is known, see e.g. [30], [33], [11] and [26], that σ^ε has the band-gap structure

$$\sigma^\varepsilon = \bigcup_{n\in\mathbb{N}} B_n^\varepsilon, \tag{9}$$

where B_n^ε are closed connected bounded segments in the real positive axis. The segments B_n^ε and B_{n+1}^ε may intersect but also they can be disjoint so that a spectral gap becomes open between them. Recall that a spectral gap is a non empty interval which is free of the spectrum but has both endpoints in the spectrum.

1.2. **On the results and structure of the paper.** In Section 2 we address the setting of the Floquet parametric family of problems (13)-(16), obtained by applying the Gelfand transform (11) to the original problem (7). They are homogenization spectral problems in a perforated domain, the periodicity cell ϖ^ε, with quasi-periodicity conditions (15)-(16) on the lateral sides of ϖ^ε. Obviously, each problem of the parametric family (13)-(16) depends on the Floquet-parameter η, cf. (11), (19) and (20). For a fixed $\eta \in [-\pi, \pi]$, the problem has the discrete spectrum $\Lambda_i^\varepsilon(\eta)$, $i = 1, 2, \cdots$, cf. (18). Section 2.2 contains a first approach to the eigenpairs (i.e., eigenvalues and eigenfunctions) of this problem via the homogenized problem, cf. (27). To get this homogenized problem, we use the energy method combined with techniques from the spectral perturbation theory. We show that its eigenvalues Λ_i^0, $i = 1, 2, \cdots$ do not depend on η, since they constitute the spectrum of the Dirichlet problem in $\upsilon = (0,1) \times (0, H)$, cf. (24). In particular, Theorem 2.1 shows that

$$\Lambda_i^\varepsilon(\eta) \to \Lambda_i^0 \quad \text{as} \quad \varepsilon \to 0, \quad \forall \eta \in [-\pi, \pi], \quad i = 1, 2, \cdots .$$

However, this result does not give information on the spectral gaps.

Using the method of matched asymptotic expansions for the eigenfunctions of the homogenization problems (cf. Section 4) we are led to the unit cell boundary value problem (31)-(33), the so-called local problem, that is, a problem to describe the boundary layer phenomenon. Section 3 is devoted to the study of this stationary problem for the Laplace operator, which is independent of η and it is posed in an unbounded strip Ξ which contains the unit hole ω. Its two solutions, with a polynomial growth at the infinity, play an important role when determining correctors for the eigenvalues $\Lambda_i^\varepsilon(\eta)$, $i = 1, 2, \cdots$. Further specifying, the definition and the properties of the so-called *polarization matrix* $p(\Xi)$, which depend on the "Dirichlet hole" ω, cf. (38) and Section 3.1, are related with the first term of the Fourier expansion of certain solutions of the unit cell problem (cf. (39) and (42)). The correctors $\varepsilon\Lambda_i^1(\eta)$ depend on the polarization matrix and the eigenfunctions of the homogenized problem, and we prove that for sufficiently small ε,

$$|\Lambda_i^\varepsilon(\eta) - \Lambda_i^0 - \varepsilon\Lambda_i^1(\eta)| \leq c_i \varepsilon^{3/2} \tag{10}$$

with some $c_i > 0$ independent of η. These bounds are obtained in Section 5, see Theorems 5.1 and 5.2 depending on the multiplicity of the eigenvalues of (24). $\Lambda_i^1(\eta)$ is a well determined function of η (see formulas (61), (62), (68), (69), (71) and Remarks 3 and 4); it is identified by means of matched asymptotic expansions in Section 4.

As a consequence, we deduce that the bands $B_i^\varepsilon = \{\Lambda_i^\varepsilon(\eta), \eta \in [-\pi, \pi]\}$ are contained in intervals

$$\left[\Lambda_i^0 + \varepsilon B_-^i - c_i \varepsilon^{3/2}, \, \Lambda_i^0 + \varepsilon B_+^i + c_i \varepsilon^{3/2}\right],$$

of length $O(\varepsilon)$, where B_-^i, B_+^i are also well determined values for each eigenvalue Λ_i^0 of (24) (cf. Corollaries 1 and 2 depending on the multiplicity). All of this together gives that for each i such that $\Lambda_i^0 < \Lambda_{i+1}^0$, cf. (23), the spectrum σ^ε opens a gap of width $O(1)$ between the corresponding spectral bands B_i^ε and B_{i+1}^ε.

Dealing with the precise length of the band, we note that the results rely on the fact that the elements of the antidiagonal of the polarization matrix do not vanish (cf. (70)-(75)), but this is a generic property for many geometries of the unit hole ω (see, e.g., (47) and (51)). Also note, that for simplicity, we have considered that

ω has a smooth boundary but most of the results hold in the case where ω has a Lipschitz boundary or even when ω is a vertical crack, cf. Section 3.1.

Summarizing, Section 2 addresses some asymptotics for the spectrum of the Floquet-parameter family of spectral problems; Section 3 considers the unit cell problem; Section 4 deals with the asymptotic expansions; in Section 5.1, we formulate the main asymptotic results of the paper, while the proofs are performed in Section 5.2.

2. **The Floquet-parameter family of spectral problems.**

In this section, we deal with the setting of the Floquet-parameter dependent spectral problems and the limit behavior of their spectra, cf. Sections 2.1 and 2.2, respectively.

2.1. **The model problem on the periodicity cell.**

The Floquet-Bloch-Gelfand transform (*FBG-transform*, in short)

$$u^\varepsilon(x) \to U^\varepsilon(x;\eta) = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} e^{-in\eta} u^\varepsilon(x_1+n, x_2) \tag{11}$$

see [6] and, e.g., [30], [33], [11], [26] and [4], converts problem (7) into a η-parametric family of spectral problems in the periodicity cell

$$\varpi^\varepsilon = \{x \in \Pi^\varepsilon : \quad |x_1| < 1/2\} \tag{12}$$

see Figure 1,b. Note that $x \in \Pi^\varepsilon$ on the left of (11), while $x \in \varpi^\varepsilon$ on the right. For each $\eta \in [-\pi, \pi]$, the spectral problem of the family is defined by the equations

$$-\Delta U^\varepsilon(x;\eta) = \Lambda^\varepsilon(\eta) U^\varepsilon(x;\eta), \qquad x \in \varpi^\varepsilon, \tag{13}$$

$$U^\varepsilon(x;\eta) = 0, \qquad x \in \Gamma^\varepsilon, \tag{14}$$

$$U^\varepsilon(1/2, x_2;\eta) = e^{i\eta} U^\varepsilon(-1/2, x_2;\eta), \quad x_2 \in (0, H), \tag{15}$$

$$\frac{\partial U^\varepsilon}{\partial x_1}(\frac{1}{2}, x_2;\eta) = e^{i\eta} \frac{\partial U^\varepsilon}{\partial x_1}(-\frac{1}{2}, x_2;\eta), \quad x_2 \in (0, H), \tag{16}$$

where $\Gamma^\varepsilon = \partial\varpi^\varepsilon \cap \partial\Pi^\varepsilon$, η is the dual variable, i.e., the Floquet-parameter, while $\Lambda^\varepsilon(\eta)$ and $U^\varepsilon(\cdot;\eta)$ denote the spectral parameter and an eigenfunction, respectively. If no confusion arises, they can be denoted by Λ^ε and U^ε, respectively. Conditions (15)-(16) are the quasi-periodicity conditions on the lateral sides $\{\pm\frac{1}{2}\} \times (0, H)$ of ϖ^ε.

The variational formulation of the spectral problem (13)-(16) reads:

$$(\nabla U^\varepsilon, \nabla V)_{\varpi^\varepsilon} = \Lambda^\varepsilon (U^\varepsilon, V)_{\varpi^\varepsilon} \qquad V \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon), \tag{17}$$

where $H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$ is a subspace of $H^1(\varpi^\varepsilon)$ of functions which satisfy the quasi-periodicity condition (15) and vanish on Γ^ε. In view of the compact embedding $H^1(\varpi^\varepsilon) \subset L^2(\varpi^\varepsilon)$, the positive, self-adjoint operator $\mathcal{A}^\varepsilon(\eta)$ associated with the problem (17) has the discrete spectrum constituting the monotone unbounded sequence of eigenvalues

$$0 < \Lambda_1^\varepsilon(\eta) \le \Lambda_2^\varepsilon(\eta) \le \cdots \le \Lambda_m^\varepsilon(\eta) \le \cdots \to \infty \tag{18}$$

which are repeated according to their multiplicities (see Ch. 10 in [1] and Ch. 13 in [30]). The eigenfunctions are assumed to form an orthonormal basis in $L^2(\varpi^\varepsilon)$.

The function

$$\eta \in [-\pi, \pi] \mapsto \Lambda_m^\varepsilon(\eta) \tag{19}$$

is continuous and 2π-periodic (see, e.g., Ch. 7 of [9]). Consequently, the sets

$$B_m^\varepsilon = \{\Lambda_m^\varepsilon(\eta) : \ \eta \in [-\pi, \pi]\} \tag{20}$$

are closed, connected and bounded intervals of the real positive axis $\overline{\mathbb{R}_+}$. Results (9) and (20) for the spectrum of the operator $\mathcal{A}^\varepsilon(\eta)$ and the boundary value problem (7) are well-known in the framework of the FBG-theory (see the above references). As a consequence of our results, we show that in our problem, depending on the geometry of the unit hole, and for certain lower frequency range of the spectrum, the spectral band (20) does not reduce to a point (cf. (72), (47), (70) and (74)).

2.2. **A homogenization result.** A first approach to the asymptotics for eigenpairs of (13)-(16) is given by the following convergence result, that we show adapting standard techniques in homogenization and spectral perturbation theory: see, e.g., Ch. 3 in [27] for a general framework and [14] for its application to spectral problems in perforated domains with different boundary conditions. Let us recall ϖ^0 which coincides with ϖ^ε at $\varepsilon = 0$ (cf. (12), and (3)) and contains the *perforation string*

$$\omega^\varepsilon(0,0), \dots, \omega^\varepsilon(0, N-1) \subset \varpi^0. \tag{21}$$

Theorem 2.1. *Let the spectral problem* (13)-(16) *and the sequence of eigenvalues* (18). *Then, for any* $\eta \in [-\pi, \pi]$, *we have the convergence*

$$\Lambda^\varepsilon_m(\eta) \to \Lambda^0_m, \qquad \text{as } \varepsilon \to 0, \tag{22}$$

where

$$0 < \Lambda^0_1 < \Lambda^0_2 \leq \dots \leq \Lambda^0_m \leq \dots \to \infty, \qquad \text{as } m \to \infty, \tag{23}$$

are the eigenvalues, repeated according to their multiplicities, of the Dirichlet problem

$$\begin{array}{l} -\Delta U^0(x) = \Lambda^0 U^0(x), \quad x \in \upsilon, \quad \upsilon \equiv (0,1) \times (0,H) \\ U^0(x) = 0, \quad x \in \partial\upsilon. \end{array} \tag{24}$$

Proof. First, for each fixed m, we show that there are two constants C, C_m such that

$$0 < C \leq \Lambda^\varepsilon_m(\eta) \leq C_m \qquad \forall \eta \in [-\pi, \pi]. \tag{25}$$

To obtain the lower bound in (25), it suffices to consider (17) for the eigenpair $(\Lambda^\varepsilon, U^\varepsilon)$ with $\Lambda^\varepsilon \equiv \Lambda^\varepsilon_1(\eta)$ and apply the the Poincaré inequality in $H^1(\varpi^0)$ once that U^ε is extended by zero in $\overline{\omega}^\varepsilon$. To get C_m in (25) we use the minimax principle,

$$\Lambda^\varepsilon_m(\eta) = \min_{E^\varepsilon_m \subset H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)} \ \max_{V \in E^\varepsilon_m, V \neq 0} \frac{(\nabla V, \nabla V)_{\varpi^\varepsilon}}{(V,V)_{\varpi^\varepsilon}},$$

where the minimum is computed over the set of subspaces E^ε_m of $H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$ with dimension m. Indeed, let us take a particular E^ε_m that we construct as follows. Consider the eigenfunctions corresponding to the m first eigenvalues of the mixed eigenvalue problem in the rectangle $(1/4, 1/2) \times (0, H)$, with Neumann condition on the part of the boundary $\{1/2\} \times (0, H)$, and Dirichlet condition on the rest of the boundary. Extend these eigenfunctions by zero for $x \in [0, 1/4] \times (0, H)$, and by symmetry for $x \in [-1/2, 0] \times (0, H)$. Finally, multiplying these eigenfunctions by $e^{i\eta x_1}$ gives E^ε_m and the rigth hand side of (25).

Hence, for each η and m, we can extract a subsequence, still denoted by ε such that

$$\Lambda^\varepsilon_m(\eta) \to \Lambda^0_m(\eta), \qquad U^\varepsilon_m(\cdot;\eta) \rightharpoonup U^0_m(\cdot;\eta) \text{ in } H^1(\varpi^0) - \textit{weak}, \text{ as } \varepsilon \to 0, \tag{26}$$

for a certain positive $\Lambda^0_m(\eta)$ and a certain function $U^0_m(\cdot;\eta) \in H^{1,\eta}_{per}(\varpi^0)$, both of which, in principle, can depend on η. Obviously, $U^0_m(\cdot;\eta)$ vanish on the lower and upper bases of ϖ^0. Also, we use the Poincaré inequality in $\varpi^0 \supset \omega$, cf. (3),

$$\|U; L^2(\varpi^0 \setminus \overline{\omega})\| \leq C \|\nabla U; L^2(\varpi^0 \setminus \overline{\omega})\| \quad \forall U \in H^1(\varpi^0 \setminus \overline{\omega}), \quad U = 0 \text{ on } \partial\omega,$$

and we deduce

$$\varepsilon^{-1}\|U_m^\varepsilon(\cdot;\eta);L^2(\{|x_1|\leq\varepsilon/2\}\cap\varpi^0)\|^2\leq C\varepsilon\|\nabla U_m^\varepsilon(\cdot;\eta);L^2(\{|x_1|\leq\varepsilon/2\}\cap\varpi^0)\|^2.$$

Now, taking limits as $\varepsilon\to0$, we get $U_m^0(\cdot,\eta)=0$ on $\{0\}\times(0,H)$ (cf., e.g., [16] and (25)). Hence, we identify $(\Lambda_m^0(\eta),\,U_m^0(\cdot;\eta))$ with an eigenpair of the following problem:

$$\begin{array}{l}-\Delta U_m^0(x;\eta)=\Lambda_m^0(\eta)U_m^0(x;\eta),\quad x_1\in\{(-1/2,0)\cup(0,1/2)\},\ x_2\in(0,H),\\ U_m^0(x;\eta)=0\quad \text{for } x_2\in\{0,H\},\ x_1\in(-1/2,1/2)\text{ and } x_1=0,\ x_2\in(0,H),\\ U_m^0(1/2,x_2;\eta)=e^{i\eta}U_m^0(-1/2,x_2;\eta),\quad x_2\in(0,H),\\ \dfrac{\partial U_m^0}{\partial x_1}(1/2,x_2;\eta)=e^{i\eta}\dfrac{\partial U_m^0}{\partial x_1}(-1/2,x_2;\eta),\quad x_2\in(0,H),\end{array}\tag{27}$$

where the differential equation has been obtained by taking limits in the variational formulation (17) for $V\in\mathcal{C}_0^\infty((-1/2,0)\times(0,H))$ and for $V\in\mathcal{C}_0^\infty((0,1/2)\times(0,H))$.

Now, from the orthonormality of $U_m^\varepsilon(\cdot;\eta)$ in $L^2(\varpi^\varepsilon)$, we get the orthonormality of $U_m^0(\cdot,\eta)$ in $L^2(\varpi^0)$. Also, an argument of diagonalization (cf., e.g., Ch. 3 in [27]) shows the convergence of the whole sequence of eigenvalues (18) towards those of (27) with conservation of the multiplicity, and that the set $\{U_m^0(\cdot;\eta)\}_{m=1}^\infty$ forms a basis of $L^2(\varpi^0)$.

In addition, extending by η-quasiperiodicity the eigenfunctions $U_m^0(\cdot;\eta)$,

$$u_m^0(x;\eta)=\left\{\begin{array}{ll}U_m^0(x;\eta), & x_1\in(0,1/2),\\ e^{i\eta}U_m^0(x_1-1,x_2;\eta), & x_1\in(1/2,1),\end{array}\right.\tag{28}$$

we obtain a smooth function in the rectangle υ, and moreover that the pair $(\Lambda_m^0(\eta)$, $U_m^0(\cdot,\eta))$ satisfies (24). In addition, the orthogonality of $\{U_m^0(\cdot;\eta)\}_{m=1}^\infty$ in $L^2(\varpi^0)$ implies that the extended functions $\{u_m^0(\cdot;\eta)\}_{m=1}^\infty$ in (28) form an orthogonal basis in $L^2(\upsilon)$, cf. also (55), and we have proved that $\Lambda_m^0(\eta)$ coincides with Λ_m^0 in the sequence (23) for any $\eta\in[-\pi,\pi]$. Consequently, the result of the theorem holds. □

Remark 1. Note that the eigenpairs of (24) can be computed explicitly

$$\Lambda_{np}^0=\pi^2\left(n^2+\frac{p^2}{H^2}\right),\quad U_{np}^0(x)=\frac{2}{\sqrt{H}}\sin(n\pi x_1)\sin(p\pi x_2/H),\ p,n\in\mathbb{N}.\tag{29}$$

The eigenvalues Λ_{np}^0 are numerated with two indexes and must be reordered in the sequence (23); the corresponding eigenfunctions U_{np}^0 are normalized in $L^2(\upsilon)$. Also, we note that if H^2 is an irrational number all the eigenvalues are simple.

3. **The unit cell problem and the polarization matrix.** In this section, we study the properties of certain solutions of the boundary value problem in the unbounded strip Ξ, cf. (31)-(33) and Figure 2. This problem, the so-called unit cell problem, is involved with the homogenization problem (13)-(16) and the periodical distribution of the openings in the periodicity cell ϖ^ε, but it remains independent of the Floquet-parameter.

In order to obtain a corrector for the approach to the eigenpairs of (13)-(16) given by Theorem 2.1, we introduce the stretched coordinates

$$\xi=(\xi_1,\xi_2)=\varepsilon^{-1}(x_1,x_2-\varepsilon kH).\tag{30}$$

which transforms each opening of the string $\omega^\varepsilon(0,k)$ into the unit opening ω. Then, we proceed as usual in two-scale homogenization when boundary layers arise (cf., e.g. [28], [18], [32] and [24]): assuming a periodic dependence of the eigenfunctions on the ξ_2-variable, cf. (34), we make the change (30) in (13)-(16), and take into

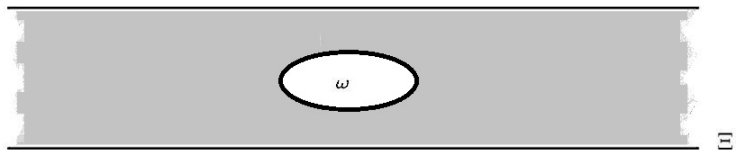

FIGURE 2. The strip Ξ with the hole ω. Ξ is involved with the unit cell for the homogenization problem (13)-(16).

account (22), to arrive at the unit cell problem. This problem consists of the Laplace equation

$$-\Delta_\xi W(\xi) = 0, \;\; \xi \in \Xi, \tag{31}$$

with the periodicity conditions

$$W(\xi_1, H) = W(\xi_1, 0), \;\; \frac{\partial W}{\partial \xi_2}(\xi_1, H) = \frac{\partial W}{\partial \xi_2}(\xi_1, 0), \;\; \xi_1 \in \mathbb{R}, \tag{32}$$

and the Dirichlet condition on the boundary of the hole $\overline{\omega}$

$$W(\xi) = 0, \;\; \xi \in \partial\omega. \tag{33}$$

Regarding (31)-(33), it should be noted that, for any $\Lambda^\varepsilon \leq C$, we have

$$\Delta_x + \Lambda^\varepsilon = \varepsilon^{-2}(\Delta_\xi + \varepsilon^2 \Lambda^\varepsilon),$$

and $\varepsilon^2 \Lambda^\varepsilon \leq C\varepsilon^2$ while the main part Δ_ξ is involved in (31). Also, the boundary condition (33) is directly inherited from (14), while the periodicity conditions (32) have no relation to the original quasi-periodicity conditions (15)-(16), but we need them to support the standard asymptotic ansatz

$$w(x_2) W(\varepsilon^{-1} x), \tag{34}$$

for the boundary layer. Here, w is a sufficiently smooth function in $x_2 \in [0, H]$ and W is H-periodic in $\xi_2 = \varepsilon^{-1} x_2$.

It is worth recalling that, according to the general theory of elliptic problems in domains with cylindrical outlets to infinity, cf., e.g., Ch. 5 in [26], problem (31)-(33) has just two solutions with a linear polynomial growth as $\xi_1 \to \pm\infty$. Here, we search for these two solutions $W^\pm(\xi)$ by setting ± 1 for the constants accompanying ξ_1 (cf. Proposition 1). In order to do it, let us consider a fixed positive R such that

$$\overline{\omega} \subset (-R, R) \times (0, H) \tag{35}$$

and define the cut-off functions $\chi_\pm \in C^\infty(\mathbb{R})$ as follows

$$\chi_\pm(y) = \begin{cases} 1, & \text{for } \pm y > 2R, \\ 0, & \text{for } \pm y < R, \end{cases} \tag{36}$$

where the subindex $\pm$ represent the support in $\pm\xi_1 \in [0, \infty)$.

Proposition 1. *There are two normalized solutions of* (31)-(33) *in the form*

$$W^\pm(\xi) = \pm\chi_\pm(\xi_1)\xi_1 + \sum_{\tau=\pm} \chi_\tau(\xi_1) p_{\tau\pm} + \widetilde{W}^\pm(\xi), \qquad \xi \in \Xi, \tag{37}$$

where the remainder $\widetilde{W}^{\pm}(\xi)$ *gets the exponential decay rate* $O(e^{-|\xi_1|2\pi/H})$*, and the coefficients* $p_{\tau\pm} \equiv p_{\tau\pm}(\Xi)$*, with* $\tau = \pm$*, which are independent of* R *and compose a* 2×2*-polarization matrix*

$$p(\Xi) = \begin{pmatrix} p_{++}(\Xi) & p_{+-}(\Xi) \\ p_{-+}(\Xi) & p_{--}(\Xi) \end{pmatrix}. \tag{38}$$

Proof. The existence of two linearly independent normalized solutions $W^{\pm}$ of (31)-(33) with a linear polynomial behavior $\pm\xi_1 + p_{\pm\pm}$, as $\pm\xi_1 \to \infty$, is a consequence of the Kondratiev theory [10] (cf. Ch. 5 in [26] and Sect. 3 [20]). Each solution has a linear growth in one direction and stabilizes towards a constant $p_{\mp\pm}$ in the other direction. In addition, it lives in an exponential weighted Sobolev space which guarantees that, substracting the linear part, the remaining functions have a gradient in $(L^2(\Xi))^2$.

Let us consider the functions

$$\widehat{W}^{\pm}(\xi) = W^{\pm}(\xi) \mp \chi_{\pm}(\xi_1)\xi_1, \tag{39}$$

which, obviously, satisfy (32), (33) and

$$-\Delta_\xi \widehat{W}^{\pm}(\xi) = F^{\pm}(\xi), \qquad \xi \in \Xi, \tag{40}$$

with $F^{\pm}(\xi) = F^{\pm}(\xi_1) = \pm\Delta(\chi_{\pm}(\xi_1)\xi_1) = \pm(\partial^2_{\xi_1}\chi_{\pm}\,\xi_1 + 2\partial_{\xi_1}\chi_{\pm})$. By construction, $F^{\pm}$ has a compact support in $\pm\xi_1 \in [R, 2R]$.

Let $C^{\infty}_{c\,per}(\overline{\Xi})$ be the space of the infinitely differentiable H-periodic functions, vanishing on $\partial\omega$, with compact support in $\overline{\Xi}$. Let us denote by $\mathcal{H}$ the completion of $C^{\infty}_{c\,per}(\overline{\Xi})$ in the norm

$$\|W, \mathcal{H}\| = \|\nabla_y W; L^2(\Xi)\|.$$

The variational formulation of (40), (32) and (33) reads: to find $\widehat{W}^{\pm} \in \mathcal{H}$ satisfying the integral identity

$$\Big(\nabla_y \widehat{W}^{\pm}, \nabla_y V\Big)_{\Xi} = \big(F^{\pm}, V\big)_{\Xi} \qquad \forall V \in \mathcal{H}. \tag{41}$$

Since $supp(F^{\pm})$ is compact, we can apply the Poincaré inequality to the elements of $\{V \in H^1([-2R, 2R]) \times (0, H)) : V|_{\partial\omega} = 0\}$, to derive that the right hand side of (41) defines a linear continuous functional on $\mathcal{H}$. In addition, the left-hand side of the integral identity (41) implies a norm in the Hilbert space $\mathcal{H}$, and consequently, the Riesz representation theorem assures that the problem (41) has a unique solution $\widehat{W} \in \mathcal{H}$ satisfying (41).

In addition, since for each τ, $\tau = \pm$, function $\widehat{W}^{\tau}(\xi)$ in (39) is harmonic for $|\xi_1| > 2R$ with gradient in $L^2((-\infty, -2R) \times (0, H)) \cap L^2((2R, +\infty) \times (0, H))$, the Fourier method (cf., e.g. [13] and [26]) ensures that

$$\widehat{W}^{\tau}(\xi) = c^{\tau}_{\pm} + O(e^{-(\pm\xi_1)2\pi/H}) \text{ as } \pm\xi_1 \to +\infty,$$

where the constants $c^{\tau}_{\pm}$ are defined by

$$c^{\tau}_{\pm} = \lim_{T\to\infty} \frac{1}{H}\int_0^H \widehat{W}^{\tau}(\pm T, \xi_2)d\xi_2 = \lim_{T\to\infty} \frac{1}{H}\int_0^H (W^{\tau}(\pm T, \xi_2) - \tau\delta_{\tau,\pm}T)d\xi_2. \tag{42}$$

Obviously, $c^{+}_{\pm}$ ($c^{-}_{\pm}$ respectively) are independent of R and they provide all the constants appearing in (37); namely, $c^{\tau}_{\pm} = p_{\tau\pm}(\Xi)$. Hence, the result of the proposition holds.

□

3.1. **Properties of the polarization matrix.** In this section, we detect certain properties of the matrix $p(\Xi)$. This matrix represent an integral characteristics of the "Dirichlet hole" $\overline{\omega}$ in the strip Π. Its definition is quite analogous to the classical polarization tensor in the exterior Dirichlet problem, see Appendix G in [29]. Let us refer to [23] for further properties of matrix $p(\Xi)$ as well as for examples on its dependence on the shape and dimensions of the hole.

Proposition 2. *The matrix $p(\Xi) + R\,\mathbb{I}$ is symmetric and positive, where $\mathbb{I}$ stands for the 2×2 unit matrix and R given in (35).*

Proof. We represent (37) in the form

$$W^{\pm}(\xi) = W_0^{\pm}(\xi) + \left\{ \begin{array}{cc} \pm\xi_1 - R & , \pm\xi_1 > R, \\ 0 & , \pm\xi_1 < R. \end{array} \right. \tag{43}$$

The function $W_0^{\pm}$ still satisfies the periodicity condition of (32) and the homogeneous Dirichlet condition (33) but remains harmonic in $\Xi \setminus \Upsilon^{\pm}(R)$, $\Upsilon^{\pm}(R) = \{\xi \in \Xi : \pm\xi_1 = R\}$, and its derivative has a jump on the segment $\Upsilon^{\pm}(R)$, namely

$$[W_0^{\pm}]_{\pm}(\xi_2) = 0, \quad \left[\frac{\partial W_0^{\pm}}{\partial|\xi_1|}\right]_{\pm}(\xi_2) = -1, \quad \xi_2 \in (0, H),$$

where $[\phi]_{\pm}(\xi_2) = \phi(\pm R \pm 0, \xi_2) - \phi(\pm R \mp 0, \xi_2)$.

In what follows, we write the equations for $\tau = \pm$. Since $\Delta W_0^{\pm} = 0$, we multiply it with W_0^{τ} and apply the Green formula in $(\Xi \setminus \Upsilon^{\pm}(R)) \cap \{|\xi_1| < T\}$. Finally, we send T to $+\infty$ and get

$$\int_0^H W_0^{\tau}(\pm R, \xi_2) d\xi_2 = -\int_0^H W_0^{\tau}(\pm R, \xi_2) \left[\frac{\partial W_0^{\pm}}{\partial|\xi_1|}\right]_{\pm}(\xi_2) d\xi_2 \\ = -\left(\nabla_\xi W_0^{\tau}, \nabla_\xi W_0^{\pm}\right)_{\Xi}. \tag{44}$$

On the other hand, on account of (43) and the definition of W^{τ}, we have

$$W_0^{\tau}(\pm R, \xi_2) = W^{\tau}(\pm R, \xi_2) \quad \text{and} \quad \left[\frac{\partial W^{\tau}}{\partial|\xi_1|}\right]_{\pm}(\xi_2) = 0.$$

Consequently, we can write

$$\int_0^H W_0^{\tau}(\pm R, \xi_2) d\xi_2 = -\int_0^H W^{\tau}(\pm R, \xi_2) \left[\frac{\partial W_0^{\pm}}{\partial|\xi_1|}\right]_{\pm}(\xi_2) d\xi_2 \\ = \int_0^H \left(W^{\tau}(\pm R, \xi_2) \left[\frac{\partial W_0^{\pm}}{\partial|\xi_1|}\right]_{\pm}(\xi_2) - W_0^{\pm}(\pm R, \xi_2) \left[\frac{\partial W^{\tau}}{\partial|\xi_1|}\right]_{\pm}(\xi_2) \right) d\xi_2,$$

and using again the Green formula for W^{τ} and $W_0^{\pm}$, in a similar way to (44) we get

$$\int_0^H W_0^{\tau}(\pm R, \xi_2) d\xi_2 \\ = + \lim_{T\to\infty} \int_0^H \left(W^{\tau}(\tau T, \xi_2) \frac{\partial W_0^{\pm}}{\partial|\xi_1|}(\tau T, \xi_2) - W_0^{\pm}(\tau T, \xi_2) \frac{\partial W^{\tau}}{\partial|\xi_1|}(\tau T, \xi_2) \right) d\xi_2 \\ = -H\left(p_{\tau\pm}(\Xi) + \delta_{\tau,\pm} R\right). \tag{45}$$

Here, we have used the following facts: $\partial/\partial|\xi_1|$ is the outward normal derivative at the end of the truncated domain $\{\xi \in \Xi : |\xi_1| < R\}$, the function W_0^{τ} is smooth near $\Upsilon^{\pm}(R)$, the derivative $\partial W_0^{\pm}/\partial|\xi_1|$ decays exponentially and, according to (37) and (43), the function $W_0^{\pm}$ admits the representation when $\pm\xi_1 > 2R$ (cf. (37))

$$W_0^{\pm}(\xi) = \chi_{\pm}(\xi_1)\,(p_{\pm\pm} + R) + \chi_{\mp}(\xi_1)p_{\mp\pm} + \widetilde{W}^{\pm}(\xi).$$

Considering (44) and (45) we have shown the equality for the Gram matrix

$$\left(\nabla_\xi W_0^{\tau}, \nabla_\xi W_0^{\pm}\right)_{\Xi} = H\,(p_{\tau\pm}(\Xi) + \delta_{\tau,\pm}R)\,,$$

which gives the symmetry and the positiveness of the matrix $p(\Xi) + R\,\mathbb{I}$. □

Let us note that our results above apply for Lipschitz domains or even cracks as it was pointed out in Section 2.1. Now, we get the following results in Propositions 3 and 4 depending on whether ω is an open domain in the plane with a positive measure $mes_2(\omega)$, or it is a crack with $mes_2(\omega) = 0$.

Proposition 3. *Let ω be such that $mes_2(\omega) > 0$. Then, the coefficients of the polarization matrix $p(\Xi)$ satisfy*

$$H\,(2p_{+-} - p_{++} - p_{--}) > mes_2(\omega).$$

Proof. We consider the linear combination

$$W_0(\xi) = W^+(\xi) - W^-(\xi) - \xi_1 = \chi_+(\xi_1)\,(p_{++} - p_{+-}) - \chi_-(\xi_1)\,(p_{--} - p_{-+}) + \widetilde{W}_0(\xi).$$

It satisfies

$$-\Delta_\xi W_0(\xi) = 0,\, \xi \in \Xi, \qquad W_0(\xi) = -\xi_1,\, \xi \in \partial\omega,$$

with the periodicity conditions in the strip, and $\widetilde{W}_0(\xi) = \widetilde{W}^+(\xi) - \widetilde{W}^-(\xi)$ gets the exponential decay rate $O(e^{-|\xi_1|2\pi/H})$. Considering the equations $\Delta W_0 = 0$ and $\Delta(W_0 + \xi_1) = 0$ in $\Xi \cap \{|\xi_1| < T\}$, and $\Delta\xi_1 = 0$ in ω, we apply the Green formula taking into account the boundary condition for W_0. Then, taking limits as $T \to \infty$, we have

$$\begin{aligned}
0 < \|\nabla W_0; L^2(\Xi)\|^2 + mes_2(\omega) &= -\int\limits_{\partial\omega} \xi_1 \partial_\nu(\xi_1) d\nu + \int\limits_{\partial\omega} W_0 \partial_\nu(W_0(\xi)) d\nu \\
= -\int\limits_{\partial\omega} \xi_1 \partial_\nu(\xi_1 + W_0(\xi)) d\nu &= \int\limits_{\partial\omega} \left(\partial_\nu \xi_1(\xi_1 + W_0(\xi)) - \xi_1 \partial_\nu(\xi_1 + W_0(\xi))\right) d\nu \\
&= -\lim_{T\to\infty} \sum_{\pm} \pm \int_0^H W_0(\pm T, \xi_2) d\xi_2 = -H\,(p_{++} + p_{--} - p_{+-} - p_{-+})\,.
\end{aligned}$$

□

Remark 2. We observe that for a hole ω, which is symmetric with respect to the x_1-axis, the matrix $p(\Xi)$ becomes symmetric with respect to the anti-diagonal, namely,

$$p_{++} = p_{--}\,. \tag{46}$$

Indeed, this is due to the fact that each one of the two normalized solutions in (37) are related with each other by symmetry. Also, we note that, on account of Proposition 2, the symmetry $p_{+-} = p_{-+}$ holds for any shape of the hole ω.

Proposition 4. *Let ω be the crack $\omega = \{\xi \in \mathbb{R}^2 : \xi_1 = 0,\, \xi_2 \in (h, H - h)\}$, where $h < H/2$. Then,*

$$p_{+-} = p_{-+} > 0. \tag{47}$$

In addition, $p_{--} = p_{++} = p_{-+} = p_{+-}$.

Proof. First, let us note that due to the symmetry $W^+(\xi_1,\xi_2) = W^-(-\xi_1,\xi_2)$, and the construction (43) when $R = 0$ reads

$$W^-(\xi_1,\xi_2) = \begin{cases} -\xi_1 + W^*(-\xi_1,\xi_2), & \xi_1 < 0, \\ W^*(\xi_1,\xi_2), & \xi_1 > 0. \end{cases} \tag{48}$$

where $W^*(\xi_1,\xi_2)$ is the function defined in $\Pi^+ = \{\xi : \xi_1 > 0,\ \xi_2 \in (0,H)\}$ satisfying the periodicity condition (32) and equations

$$\begin{array}{l} -\Delta_\xi W^*(\xi) = 0, \text{ for } \xi \in \Pi^+ \\ W^*(0,\xi_2) = 0, \text{ for } \xi_2 \in (h, H-h), \\ -\partial_{\xi_1} W^*(0,\xi_2) = 1/2, \text{ for } \xi_2 \in (0,h) \cup (H-h,H). \end{array} \tag{49}$$

Indeed, denoting by $\widetilde{W}^*$ the extension of W^* to $\Pi^- = \{\xi :\ \xi_1 < 0,\ \xi_2 \in (0,H)\}$, in order to verify the representation (48), it suffices to verify that the jump of $\widetilde{W}^*$ and its the derivative of through $\Upsilon(0) = \{\xi \in \Xi :\ \xi_1 = 0\}$ is given by

$$[\widetilde{W}^*](0,\xi_2) = 0, \qquad \left[\frac{\partial \widetilde{W}^*}{\partial \xi_1}\right](0,\xi_2) = -1,$$

and hence, the function on the right hand side of (48) is a harmonic function in Ξ.

Now, considering (49), integrating by parts on $(0,T) \times (0,H)$, and taking limits as $T \to +\infty$ provide

$$\int\limits_{\Upsilon(0)} W^*(0,\xi_2)d\xi_2 = \lim_{T\to\infty} \int\limits_0^H W^*(T,\xi_2)d\xi_2 = H p_{-+}(\Xi).$$

Similarly, from (49), we get

$$0 = -\int\limits_{\Pi^+} W^*(\xi)\Delta_\xi W^*(\xi)d\xi = \int\limits_{\Pi^+} |\nabla_\xi W^*(\xi)|^2 d\xi - \frac{1}{2}\int\limits_{\Upsilon(0)} W^*(0,\xi_2)d\xi_2.$$

Therefore, we deduce

$$\frac{H}{2} p_{-+}(\Xi) = \int\limits_{\Pi^+} |\nabla_\xi W^*(\xi)|^2 d\xi > 0 \tag{50}$$

and from the symmetry of $p(\Xi)$ (cf. Proposition 2), we obtain (47).

Also, from the definition (48), we have $p_{--}(\Xi) = p_{-+}(\Xi)$, and (cf. (46)) all the elements of the polarization matrix $p(\Xi)$ coincide. Thus, the proposition is proved. □

From Proposition 4, note that when ω is a vertical crack, the inequality in Proposition 3 must be replaced by $H\,(2p_{+-} - p_{++} - p_{--}) = mes_2(\omega) = 0$. Also, we observe that in order to get property (47) for a domain ω with a smooth boundary, we may apply asymptotic results on singular perturbation boundaries (cf. [7], Ch. 3 in [8] and Ch. 5 in [17]) which guarantee that for thin ellipses

$$\overline{\omega} = \{\xi :\ (\delta^{-2}\xi_1^2 + (\xi_2 - H/2)^2 \leq \tau^2\}, \quad \tau = H/2 - h, \tag{51}$$

(47) holds true, for a small $\delta > 0$.

4. **Asymptotic analysis in the periodicity cell ϖ^ε.** In this section we construct asymptotic expansions for the eigenpairs $(\Lambda_m^\varepsilon(\eta), U_m^\varepsilon(\cdot;\eta))$ of problem (13)-(16) on the periodicity cell ϖ^ε. The parameters $m \in \mathbb{N}$ and $\eta \in [-\pi, \pi]$ are fixed in this analysis. In Sections 4.1-4.2 we consider the case in which the eigenvalue Λ_m^0 of (24) is simple. Note that for many values of H, all the eigenvalues are simple (cf. Remark 1). Section 4.3 contains the asymptotic ansatz for the eigenpairs case where Λ_m^0 is an eigenvalue of (24) of multiplicity $\kappa_m \geq 2$.

4.1. **Asymptotic ansätze.** Let Λ_m^0 be a simple eigenvalue in sequence (23) and let U_m^0 be the corresponding eigenfunction of problem (24) normalized in $L^2(\upsilon)$. Then, on account of the Theorem 2.1, for the eigenvalue Λ_m^ε of problem (13)-(16) we consider the asymptotic ansätze

$$\Lambda_m^\varepsilon = \Lambda_m^0 + \varepsilon \Lambda_m^1(\eta) + \cdots . \tag{52}$$

To construct asymptotics of the corresponding eigenfunctions $U_m^\varepsilon(x;\eta)$, we employ the method of matched asymptotic expansions, see, e.g., the monographs [35] and [8], and the papers [32], [18] and [24] where this method has been applied to homogenization problems. Namely, we take

$$U_m^\varepsilon(x;\eta) = U_m^0(x;\eta) + \varepsilon U_m^1(x;\eta) + \cdots \tag{53}$$

as the outer expansion, and

$$U_m^\varepsilon(x;\eta) = \varepsilon \sum_{\pm} w_\pm^m(x_2;\eta) W^\pm\big(\frac{x}{\varepsilon}\big) + \cdots , \tag{54}$$

as the inner expansion near the perforation string, cf. (4) and (21),

Above, $U_m^0(x;\eta)$ is built from the eigenfunction U_m^0 of (24) by formula

$$U_m^0(x;\eta) = \begin{cases} U_m^0(x), & x_1 \in (0, 1/2), \\ e^{-i\eta} U_m^0(x_1+1, x_2), & x_1 \in (-1/2, 0), \end{cases} \tag{55}$$

$W^\pm$ are the solutions (37) to problem (31)-(33), while the functions U_m^1, $w_\pm^m$ and the number $\Lambda_m^1(\eta)$ are to be determined applying matching principles, cf. Section 4.2. Note that near the perforation string, cf. (4), (21), the Dirichlet condition satisfied by $U_m^0(x;\eta)$ implies that the term accompanying ε^0 in the inner expansion vanishes (see, e.g., [24]); this is why the first order function in (54) is ε. Also, above and in what follows, the ellipses stand for higher-order terms, inessential in our formal analysis.

4.2. **Matching procedure.** First, let us notice that $U_m^0 \in C^\infty(\overline{\upsilon})$, and the Taylor formula applied in the outer expansion (53) yields

$$\begin{aligned} U_m^\varepsilon(x;\eta) &= 0 + x_1 \frac{\partial U_m^0}{\partial x_1}(0, x_2) + \varepsilon U_m^1(+0, x_2;\eta) + \cdots , \; x_1 > 0, \\ U_m^\varepsilon(x;\eta) &= 0 + x_1 e^{-i\eta} \frac{\partial U_m^0}{\partial x_1}(1, x_2) + \varepsilon U_m^1(-0, x_2;\eta) + \cdots , \; x_1 < 0, \end{aligned} \tag{56}$$

where, for second formula (56), we have used (55).

The inner expansion (54) is processed by means of decompositions (37). We have

$$\begin{aligned} U_m^\varepsilon(x;\eta) &= \varepsilon w_+^m(x_2;\eta)(\xi_1 + p_{++}) + \varepsilon w_-^m(x_2;\eta) p_{-+} + \cdots , \; \xi_1 > 0, \\ U_m^\varepsilon(x;\eta) &= \varepsilon w_-^m(x_2;\eta)(-\xi_1 + p_{--}) + \varepsilon w_+^m(x_2;\eta) p_{+-} + \cdots , \; \xi_1 < 0. \end{aligned} \tag{57}$$

Recalling relationship between x_1 and ξ_1, we compare coefficients of ε and $x_1 = \varepsilon\xi_1$ on the right-hand sides of (56) and (57). As a result, we identify $w^m_\pm$ by

$$w^m_+(x_2;\eta) = \frac{\partial U^0_m}{\partial x_1}(0,x_2), \quad w^m_-(x_2;\eta) = -e^{-i\eta}\frac{\partial U^0_m}{\partial x_1}(1,x_2), \tag{58}$$

and also obtain the equalities

$$U^1_m(+0,x_2;\eta) = \sum_{\tau=\pm} w^m_\tau(x_2;\eta)p_{\tau+}, \quad U^1_m(-0,x_2;\eta) = \sum_{\tau=\pm} w^m_\tau(x_2;\eta)p_{\tau-}. \tag{59}$$

Formulas (58) define coefficients of the linear combination (54) while formulas (59) are the boundary conditions for the correction term in (53). Moreover, inserting ansätze (52) and (53) into (13)-(14), we derive that

$$\begin{cases} -\Delta_x U^1_m(x;\eta) - \Lambda^0_m U^1_m(x;\eta) = \Lambda^1_m(\eta)U^0_m(x;\eta),\ x\in\varpi^0,\ x_1\neq 0, \\ U^1_m(x_1,H;\eta) = U^1_m(x_1,0;\eta) = 0,\ x_1\in(-1/2,0)\cup(0,1/2), \end{cases} \tag{60}$$

and the quasi-periodic conditions with η (cf. (15)-(16)).

Since $U^0_m(x;\eta)$ is defined by (55), $(\Lambda^0_m, U^0_m(x))$ is an eigenpair of (24), and $\|U^0_m; L^2(\upsilon)\| = \|U^0_m(\cdot;\eta); L^2(\varpi^0)\| = 1$, we multiply by $U^0_m(x;\eta)$ in the differential equation of (60), integrate by parts and obtain

$$\int_{\varpi^0} \Lambda^1_m(\eta)U^0_m(x;\eta)\overline{U^0_m(x;\eta)}dx$$
$$= \int_0^H U^1_m(-0,x_2;\eta)\overline{\frac{\partial U^0_m}{\partial x_1}(-0,x_2;\eta)}dx_2 - \int_0^H U^1_m(+0,x_2;\eta)\overline{\frac{\partial U^0_m}{\partial x_1}(+0,x_2;\eta)}dx_2.$$

Thus, by (55) and (59), the only compatibility condition in (60) (recall that Λ^0_m is a simple eigenvalue) converts into

$$\Lambda^1_m(\eta) = -\int_0^H \overline{B_m(x_2;\eta)}\cdot p(\Xi)B_m(x_2;\eta)dx_2 \tag{61}$$

where

$$B_m(x_2;\eta) = \left(\frac{\partial U^0_m}{\partial x_1}(0,x_2), -e^{-i\eta}\frac{\partial U^0_m}{\partial x_1}(1,x_2)\right)^T \in \mathbb{C}^2, \tag{62}$$

and it determines uniquely the second term of the ansatz (52). Here and in what follows, the top index T indicates the transpose vector.

Also, from (53), (54) and (57) the composite expansion approaching $U^\varepsilon_m(x;\eta)$ in the whole domain ϖ_0 reads

$$U^\varepsilon_m(x;\eta) \approx U^0_m(x;\eta) + \varepsilon U^1_m(x;\eta) + \varepsilon\sum_{\tau=\pm} w^m_\tau(x_2;\eta)W^\tau\big(\frac{x}{\varepsilon}\big)$$
$$- \big(\varepsilon w^m_\pm(x_2;\eta)(\varepsilon^{-1}|x_1| + p_{\pm\pm}) + \varepsilon w^m_\mp(x_2;\eta)p_{\mp\pm}\big), \quad \pm x_1 \geq 0. \tag{63}$$

4.3. **The case of a multiple eigenvalue Λ^0_m.** We address the case where Λ^0_m is an eigenvalue of (24) with multiplicity $\kappa_m \geq 2$. Let us consider $\Lambda^0_m = \cdots = \Lambda^0_{m+\kappa_m-1}$ in the sequence (23) and the corresponding eigenfunctions $U^0_m, \cdots, U^0_{m+\kappa_m-1}$ which are orthonormal in $L^2(\upsilon)$. On account of Theorem 2.1 there are κ_m eigenvalues of problem (13)-(16), which we denote by $\Lambda^\varepsilon_{m+l}(\eta)$, $l = 0,\cdots,\kappa_m-1$, satisfying

$$\Lambda^\varepsilon_{m+l}(\eta) \to \Lambda^0_{m+l} \text{ as } \varepsilon\to 0, \qquad \text{for } l = 0,\cdots,\kappa_m-1. \tag{64}$$

Let $U^{\varepsilon}_{m+l}(\cdot;\eta)$, $l = 0, \cdots, \kappa_m - 1$, be the corresponding eigenfunctions among the set of the eigenfunctions which form an orthonormal basis in $L^2(\varpi^{\varepsilon})$, cf. (17).

Following Section 4.1, for each $l = 0, \cdots, \kappa_m - 1$, we take the ansatz for $\Lambda^{\varepsilon}_{m+l}(\eta)$

$$\Lambda^{\varepsilon}_{m+l} = \Lambda^0_m + \varepsilon \Lambda^1_{m+l}(\eta) + \cdots, \tag{65}$$

the outer expansion for $U^{\varepsilon}_{m+l}(\cdot;\eta)$

$$U^{\varepsilon}_{m+l}(x;\eta) = U^0_{m+l}(x;\eta) + \varepsilon U^1_{m+l}(x;\eta) + \cdots, \tag{66}$$

and the inner expansion

$$U^{\varepsilon}_{m}(x;\eta) = \varepsilon \sum_{\pm} w^{m+l}_{\pm}(x_2;\eta) W^{\pm}\big(\frac{x}{\varepsilon}\big) + \cdots, \tag{67}$$

where the terms $\Lambda^1_{m+l}(\eta)$, $U^1_{m+l}(x;\eta)$ and $w^{m+l}_{\pm}(x_2;\eta)$ have to be determined by the matching procedure, cf. Section 4.2, while $U^0_{m+l}(x;\eta)$ is constructed from $U^0_{m+l}(x)$ replacing U^0_m by U^0_{m+l} in formula (55), and $W^{\pm}$ are the solutions (37) to problem (31)-(33).

By repeating the reasoning in Section 4.2, we obtain formulas for the above mentioned terms in (65), (66) and (67) by replacing index m by $m + l$ in (56)-(62), while we realize that the compatibility condition for each $\Lambda^1_{m+l}(\eta)$ is satisfied. Indeed, multiplying by $U^0_{m+l'}(x;\eta)$, $l' = 0, \cdots, \kappa_m - 1$ in the partial differential equation satisfied by $U^1_{m+l'}(x;\eta)$ (cf. (60))

$$-\Delta_x U^1_{m+l}(x;\eta) - \Lambda^0_m U^1_{m+l}(x;\eta) = \Lambda^1_{m+l}(\eta) U^0_{m+l}(x;\eta), \quad x \in \varpi^0,\ x_1 \neq 0,$$

and integrating by parts, we obtain

$$\int_{\varpi^0} \Lambda^1_{m+l}(\eta) U^0_{m+l}(x;\eta) \overline{U^0_{m+l'}(x;\eta)} dx$$
$$= -\int_0^H \left(\frac{\partial U^0_{m+l'}}{\partial x_1}(0,x_2), -e^{i\eta} \frac{\partial U^0_{m+l'}}{\partial x_1}(1,x_2) \right) \cdot p(\Xi)$$
$$\times \left(\frac{\partial U^0_{m+l}}{\partial x_1}(0,x_2), -e^{-i\eta} \frac{\partial U^0_{m+l}}{\partial x_1}(1,x_2) \right)^T dx_2.$$

Since the eigenfunctions U^0_{m+l} and $U^0_{m+l'}$ have been computed (cf. the explicit formulas (29) in Remark 1), we conclude now that

$$\int_0^H \frac{\partial U^0_{m+l'}}{\partial x_1}(x_1^*, x_2) \frac{\partial U^0_{m+l}}{\partial x_1}(x_1^*, x_2) dx_2 = 0, \text{ with } x_1^* \in \{0,1\}, \quad l \neq l',$$

and, hence, for each $l = 0, \cdots, \kappa_m - 1$, the κ_m compatibility conditions to be satisfied by the pairs $(\Lambda^1_{m+l}(\eta), U^1_{m+l}(x;\eta))$, cf. (60), provide $\Lambda^1_{m+l}(\eta)$ given by

$$\Lambda^1_{m+l}(\eta) = -\int_0^H \overline{B_{m+l}(x_2;\eta)} \cdot p(\Xi) B_{m+l}(x_2;\eta) dx_2, \tag{68}$$

where $B_{m+l}(x_2;\eta)$ is defined by

$$B_{m+l}(x_2;\eta) = \left(\frac{\partial U^0_{m+l}}{\partial x_1}(0,x_2), -e^{-i\eta} \frac{\partial U^0_{m+l}}{\partial x_1}(1,x_2) \right)^T. \tag{69}$$

Therefore we have determined completely all the terms in the asymptotic ansätze (65), (66) and (67) for $l = 0, \cdots, \kappa_m - 1$.

5. **Justification of asymptotics.** In this section, we justify the results obtained by means of matched asymptotic expasions in Section 4. Since the case in which all the eigenvalues of the Dirichlet problem (24) are simple can be a generic property, we first consider this case, cf. Theorem 5.1 and Corollary 1, and then the case in which these eigenvalues have a multiplicity greater than 1, cf. Theorem 5.2 and Corollary 2. We state the results in Section 5.1 while we perform the proofs in Section 5.2.

5.1. **Asymptotics of eigenvalues: the results.**

Theorem 5.1. *Let $m \in \mathbb{N}$, let Λ_m^0 be a simple eigenvalue of the Dirichlet problem (24), and let $\Lambda_m^1(\eta)$ be defined in (61) and (62). There exist positive ε_m and c_m independent of η such that, for any $\varepsilon \in (0, \varepsilon_m]$, the eigenvalue $\Lambda_m^\varepsilon(\eta)$ of problem (13)-(16) meets the estimate*

$$|\Lambda_m^\varepsilon(\eta) - \Lambda_m^0 - \varepsilon\Lambda_m^1(\eta)| \leq c_m \varepsilon^{3/2} \tag{70}$$

and there are no other different eigenvalues in the sequence (18) satisfying (70).

Theorem 5.1 shows that $\varepsilon\Lambda_m^1(\eta)$ provides a correction term for $\Lambda_m^\varepsilon(\eta)$ improving the approach to λ_m^0 shown in Theorem 2.1. In particular, it justifies the asymptotic ansatz (52) and formula (61). This corrector depends on the polarization matrix $p(\Xi)$, which is given by the coefficients $p_{\tau\pm} \equiv p_{\tau\pm}(\Xi)$, with $\tau = \pm$, in the decomposition (37), and on the eigenfunction U_m^0 of problem (24), which corresponds to Λ_m^0 and is normalized in $L^2(\upsilon)$ (cf. (61) and (62)).

In order to detect the gaps between consecutive spectral bands (20) it is worthy writing formulas

$$\begin{aligned}
\Lambda_m^1(\eta) &= B_0(m) + B_1(m)\cos(\eta), \qquad \text{with} \\
B_0(m) &= \int_0^H \left(p_{++} \left| \frac{\partial U_m^0}{\partial x_1}(0, x_2) \right|^2 + p_{--} \left| \frac{\partial U_m^0}{\partial x_1}(1, x_2) \right|^2 \right) dx_2, \\
B_1(m) &= 2p_{+-} \int_0^H \frac{\partial U_m^0}{\partial x_1}(0, x_2) \frac{\partial U_m^0}{\partial x_1}(1, x_2) dx_2,
\end{aligned} \tag{71}$$

which are obtained from (61) and (62). Formula (29) demonstrates that

$$B_0(m) = (p_{++} + p_{--}) \int_0^H \left| \frac{\partial U_m^0}{\partial x_1}(0, x_2) \right|^2 dx_2,$$

and that the integral in $B_1(m)$ does not vanish. We note that $B_1(m) = 0$ only in the case when $p_{+-} = 0$; if so, $p(\Xi)$ is diagonal and the solutions of (37), $W^\pm$, decay exponentially when $\xi_1 \to \mp\infty$, respectively. However, we have given examples of cases where $p_{+-} \neq 0$ (cf. (47) and (51)).

Remark 3. Let us consider that the eigenvalue Λ_m^0 coincides with Λ_{nq}^0 in formula (29) for certain natural n and q. Then, we obtain

$$B_0(m) = 2\,(p_{++} + p_{--})\, n^2\pi^2, \quad B_1(m) = (-1)^n 4p_{+-} n^2\pi^2,$$

and, consequently,

$$\Lambda_m^1(\eta) = 2\,(p_{++} + p_{--})\, n^2\pi^2 + (-1)^n 4p_{+-} n^2\pi^2 \cos(\eta). \tag{72}$$

Corollary 1. *Under the hypothesis of Theorem 5.1, the endpoints $B^{\varepsilon\pm}(m)$ of the spectral band (20) satisfy the relation*

$$|B^{\varepsilon\pm}(m) - \Lambda_m^0 - \varepsilon(B_0(m)\pm|B_1(m)|)| \le c_m\varepsilon^{3/2}. \tag{73}$$

Hence, the length of the the band B_m^ε is $2\varepsilon|B_1(m)| + O(\varepsilon^{3/2})$.

Note that for the holes such that the polarization matrix (38) satisfies $p_{+-} \neq 0$, asymptotically, the bands B_m^ε have the precise length $2\varepsilon|B_1(m)| + O(\varepsilon^{3/2})$ and they cannot reduce to a point, namely to the point $\Lambda_m^0 + \varepsilon B_0(m)$ (cf. (71) and Remark 3). Also note that if $p_{+-} = 0$, Theorem 5.1 still provides a correction term for $\Lambda_m^\varepsilon(\eta)$ which however does not depend on η (cf. (70), (71) and Remark 3), the width of the band being $O(\varepsilon^{3/2})$. Although the length of the band is shorter than in the cases where $p_{+-} \neq 0$, bounds in Corollary 1 may not be optimal (cf. Remark 3) and further information on the corrector depending on η can be obtained by constructing higher-order terms in the asymptotic ansatz (53).

Theorem 5.2. *Let $m \in \mathbb{N}$, let Λ_m^0 be an eigenvalue of the Dirichlet problem (24) with multiplity $\kappa_m > 1$. Let $\Lambda_{m+l}^1(\eta)$ defined in (68) and (69) for $l = 0, \cdots, \kappa_m - 1$. There exist positive ε_m and c_m independent of η such that, for any $\varepsilon \in (0, \varepsilon_m]$, and for each $l = 0, \cdots, \kappa_m - 1$, at least one eigenvalue $\Lambda_{m+l_0}^\varepsilon(\eta)$ of problem (13)-(16) satisfying (64) meets the estimate*

$$|\Lambda_{m+l_0}^\varepsilon(\eta) - \Lambda_m^0 - \varepsilon\Lambda_{m+l}^1(\eta)| \le c_m\varepsilon^{3/2}. \tag{74}$$

In addition, when $l \in \{0, 1, \cdots, \kappa_m - 1\}$, the total multiplicity of the eigenvalues in (18) satisfying (74) is κ_m.

Corollary 2. *Under the hypothesis in Theorem 5.2, the spectral bands B_{m+l}^ε associated with $\Lambda_{m+l}^\varepsilon(\eta)$, for $l = 0, \cdots, \kappa_m - 1$, cf. (20), are contained in the interval*

$$[\Lambda_m^0 + \varepsilon \min_{\substack{0 \le l \le \kappa_m - 1 \\ \eta \in [-\pi, \pi]}} \Lambda_{m+l}^1(\eta) - c_m\varepsilon^{3/2}, \Lambda_m^0 + \varepsilon \max_{\substack{0 \le l \le \kappa_m - 1 \\ \eta \in [-\pi, \pi]}} \Lambda_{m+l}^1(\eta) + c_m\varepsilon^{3/2}]. \tag{75}$$

Hence, the length of the the bands B_{m+l}^ε are $O(\varepsilon)$ but they may not be disjoint.

Remark 4. Under the hypothesis of Theorem 5.2, it may happen that for $l = 0, \cdots, \kappa_m - 1$ only the eigenvalue $\Lambda_{m+l_0}^\varepsilon(\eta)$ in the sequence (18) satisfies (74). This depends on the polarization matrix $p(\Xi)$. As a matter of fact, it can be shown by contradiction under the assumption that for two different l the functions $\Lambda_{m+l}^1(\eta)$ do not intersect at any point η, cf. (71) and (72). For instance, this follows for ω with $p_{+-}(\Xi) = 0$.

Remark 5. Notice that the positive cutoff value $\lambda_\dagger^\varepsilon$ such that the spectrum $\sigma^\varepsilon \subset [\lambda_\dagger^\varepsilon, \infty)$ is bounded from above by a positive constant, cf. (9), (20) and (25). In addition, from Theorem 5.2 (cf. Remark 1), we have proved that $\lambda_\dagger^\varepsilon \to \pi^2(1 + H^{-2})$ as $\varepsilon \to 0$.

5.2. **The proofs.** In this section we prove the results of Theorems 5.1 and 5.2 and their respective corollaries.

Proof of Theorem 5.1. Let us fix η in $[-\pi,\pi]$. Let us endow the space $H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$, with the scalar product $\langle U^\varepsilon, V^\varepsilon\rangle = (\nabla U^\varepsilon, \nabla V^\varepsilon)_{\varpi^\varepsilon} + (U^\varepsilon, V^\varepsilon)_{\varpi^\varepsilon}$, and the positive, symmetric and compact operator $T^\varepsilon(\eta)$,

$$\langle T^\varepsilon(\eta)U^\varepsilon, V^\varepsilon\rangle = (U^\varepsilon, V^\varepsilon)_{\varpi^\varepsilon} \quad \forall U^\varepsilon, V^\varepsilon \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon). \tag{76}$$

The integral identity (17) for problem (13)-(16) can be rewritten as the abstract equation

$$T^\varepsilon(\eta)U^\varepsilon(\cdot;\eta) = \tau^\varepsilon(\eta)U^\varepsilon(\cdot;\eta), \quad \text{in } H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon),$$

with the new spectral parameter

$$\tau^\varepsilon(\eta) = (1+\Lambda^\varepsilon(\eta))^{-1}. \tag{77}$$

Since $T^\varepsilon(\eta)$ is compact (cf. e.g, Section I.4 in [31] and III.9 in [1]), its spectrum consists of the point $\tau = 0$, the essential spectrum, and of the discrete spectrum $\{\tau^\varepsilon_m(\eta)\}_{m\in\mathbb{N}}$ which, in view of (18) and (77), constitutes the infinitesimal sequence of positive eigenvalues

$$\left\{\tau^\varepsilon_m(\eta) = (1+\Lambda^\varepsilon_m(\eta))^{-1}\right\}_{m\in\mathbb{N}}.$$

For the point

$$t^\varepsilon_m(\eta) = (1+\Lambda^0_m + \varepsilon\Lambda^1_m(\eta))^{-1} \tag{78}$$

cf. (52) and (61), we construct a function $\mathcal{U}^\varepsilon_m \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$ such that

$$\|\mathcal{U}^\varepsilon_m; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\| \geq c_m, \tag{79}$$

$$\|T^\varepsilon(\eta)\mathcal{U}^\varepsilon_m - t^\varepsilon_m(\eta)\mathcal{U}^\varepsilon_m; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\| \leq C_m\varepsilon^{3/2}, \tag{80}$$

where c_m and C_m are some positive constants independent of $\varepsilon \in (0,\varepsilon_m]$, with $\varepsilon_m > 0$. These inequalities imply the estimate for the norm of the resolvent operator $(T^\varepsilon(\eta) - t^\varepsilon_m(\eta))^{-1}$

$$\|(T^\varepsilon(\eta) - t^\varepsilon_m(\eta))^{-1}; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon) \to H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\| \geq \mathbf{c}_m^{-1}\varepsilon^{-3/2},$$

with $\mathbf{c}_m = c_m^{-1}C_m > 0$. According to the well-known formula for self-adjoint operators

$$\operatorname{dist}(t^\varepsilon_m(\eta), \sigma(T^\varepsilon(\eta)) = \|(T^\varepsilon(\eta) - t^\varepsilon_m(\eta))^{-1}; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon) \to H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\|^{-1}$$

supported by the spectral decomposition of the resolvent (cf., e.g., Section V.5 in [9] and Ch. 6 in [1]), we deduce that the closed segment

$$[t^\varepsilon_m(\eta) - \mathbf{c}_m\varepsilon^{3/2}, t^\varepsilon_m(\eta) + \mathbf{c}_m\varepsilon^{3/2}]$$

contains at least one eigenvalue $\tau^\varepsilon_p(\eta)$ of the operator $T^\varepsilon(\eta)$. Since the eigenvalues of $T^\varepsilon(\eta)$ satisfy (77) and we get the definition (78), we derive that

$$\left|(1+\Lambda^\varepsilon_p(\eta))^{-1} - (1+\Lambda^0_m + \varepsilon\Lambda^1_m(\eta))^{-1}\right| \leq \mathbf{c}_m\varepsilon^{3/2}. \tag{81}$$

Then, simple algebraic calculations (cf. (81) and (25)) show that, for a $\varepsilon_m > 0$, the estimate

$$\left|\Lambda^\varepsilon_p(\eta) - \Lambda^0_m - \varepsilon\Lambda^1_m(\eta)\right| \leq \mathcal{C}_m\varepsilon^{3/2}, \tag{82}$$

is satisfied with a constant $\mathcal{C}_m$ independent of $\varepsilon \in (0,\varepsilon_m]$. Due to the convergence with conservation of the multiplicity (22), $p = m$ in (82) and this estimate becomes (70).

To conclude with the proof of Theorem 5.1, there remains to present a function $\mathcal{U}_m^\varepsilon \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$ enjoying restrictions (79) and (80). In what follows, we construct $\mathcal{U}_m^\varepsilon$ using (63) suitably modified with the help of cut-off functions with "overlapping supports", cf. [19], Ch. 2 in [17] and others. We define

$$\mathcal{V}^{\varepsilon m}_{out}(x;\eta) = U^0_m(x;\eta) + \varepsilon U^1_m(x;\eta), \tag{83}$$

with U^0_m satisfying (55) and U^1_m is the solution of (60) satisfying the boundary conditions (15)-(16) and (59). Similarly, we define

$$\mathcal{V}^{\varepsilon m}_{in}(x;\eta) = \varepsilon \sum_{\pm} w^m_\pm(x_2;\eta) W^\pm(\varepsilon^{-1}x), \tag{84}$$

and

$$\mathcal{V}^{\varepsilon m}_{mat}(x;\eta) = \varepsilon w^m_\pm(x_2;\eta)(\varepsilon^{-1}|x_1| + p_{\pm\pm}) + \varepsilon w^m_\mp(x_2;\eta) p_{\mp\pm}, \quad \pm x_1 > 0, \tag{85}$$

with $w^m_\pm$ defined in (58), and $W^\pm$ and matrix $p(\Xi)$ in Proposition 1. Finally, we set

$$\mathcal{U}^\varepsilon_m(x;\eta) = X^\varepsilon(x_1)\mathcal{V}^{\varepsilon m}_{out}(x;\eta) + \mathcal{X}(x_1)\mathcal{V}^{\varepsilon m}_{in}(x;\eta) - X^\varepsilon(x_1)\mathcal{X}(x_1)\mathcal{V}^{\varepsilon m}_{mat}(x;\eta), \tag{86}$$

where X^ε and $\mathcal{X}$ are two cut-off functions, both smoothly dependent on the x_1 variable, such that

$$X^\varepsilon(x_1) = \begin{cases} 1, & \text{for } |x_1| > 2R\varepsilon, \\ 0, & \text{for } |x_1| < R\varepsilon, \end{cases} \quad \text{and} \quad \mathcal{X}(x_1) = \begin{cases} 1, & \text{for } |x_1| < 1/6, \\ 0, & \text{for } |x_1| > 1/3. \end{cases} \tag{87}$$

Note that (85) takes into account components in both expressions (83) and (84), but the last subtrahend in $\mathcal{U}^\varepsilon_m$ compensates for this duplication. In further estimations, term (85) will be joined to either $\mathcal{V}^{\varepsilon m}_{out}$ or $\mathcal{V}^{\varepsilon m}_{in}$ in order to obtain suitable bounds.

First, let us show that $\mathcal{U}^\varepsilon_m \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$. Indeed, the function defined in (86) enjoys the conditions (15)-(16) and (14). This is due to the fact that $\mathcal{U}^\varepsilon_m = \mathcal{V}^{\varepsilon m}_{out}$ near the sides $\{x_1 = \pm 1/2,\, x_2 \in (0,H)\}$ and the quasi-periodicity conditions (15)-(16) are verified by both terms in (83). Also, $\mathcal{U}^\varepsilon_m = \mathcal{V}^{\varepsilon m}_{in}$ near the hole string (21) so that the Dirichlet condition are fulfilled on boundary of the perforation string $\Gamma^\varepsilon \cap \varpi^0$ because $W^\pm$ satisfy (33). Finally, formulas (58) and (29) assure that $w^m_\pm(H;\eta) = w^m_\pm(0;\eta) = 0$ and hence the Dirichlet condition is met on $\Gamma^\varepsilon \cap \partial\varpi^0$ as well.

First of all, we recall (83) and (87) to derive

$$\begin{aligned} &\|\mathcal{U}^\varepsilon_m; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\| \\ &\qquad \geq \big\|\mathcal{U}^\varepsilon_m; L^2(1/3,1/2)\times(0,H))\big\| = \|\mathcal{V}^{\varepsilon m}_{out}; L^2(1/3,1/2)\times(0,H))\| \\ &\qquad \geq \big\|U^0_m; L^2(1/3,1/2)\times(0,H))\big\| - \varepsilon\|U^1_m; L^2(1/3,1/2)\times(0,H))\| \geq c > 0, \end{aligned}$$

for a small $\varepsilon > 0$. Thus, (79) is fulfilled.

Using (76) and (78), we have

$$\begin{aligned} &\|T^\varepsilon(\eta)\mathcal{U}^\varepsilon_m - t^\varepsilon_m(\eta)\mathcal{U}^\varepsilon_m; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\| = \sup|\langle T^\varepsilon(\eta)\mathcal{U}^\varepsilon_m - t^\varepsilon_m(\eta)\mathcal{U}^\varepsilon_m, \mathcal{W}^\varepsilon\rangle| \\ &= (1 + \Lambda^0_m + \varepsilon\Lambda^1_m(\eta))^{-1} \sup\big|(\nabla\mathcal{U}^\varepsilon_m, \nabla\mathcal{W}^\varepsilon)_{\varpi^\varepsilon} - (\Lambda^0_m + \varepsilon\Lambda^1_m(\eta))(\mathcal{U}^\varepsilon_m, \mathcal{W}^\varepsilon)_{\varpi^\varepsilon}\big| \end{aligned} \tag{88}$$

where supremum is computed over all $\mathcal{W}^\varepsilon \in H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)$ such that

$$\|\mathcal{W}^\varepsilon; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\| \leq 1.$$

Taking into account the Dirichlet conditions on $\partial\omega^\varepsilon$ we use the Poincaré and Hardy inequalities, namely, for a fixed T such that $\omega \subset \Pi_T \equiv \Pi \cap \{y_1 < T\}$,

$$\int_{\Pi_T\setminus\overline{\omega}} |U|^2\,dy \le C_T \int_{\Pi_T\setminus\overline{\omega}} |\nabla_y U|^2 dy \quad \forall U \in H^1(\Pi_T \setminus \overline{\omega}),\, U = 0 \text{ on } \partial\omega,$$

where C_T is a constant independent of U, and

$$\int_0^\infty \frac{1}{t^2} z(t)^2\,dt \le 4\int_0^\infty \left|\frac{dz}{dt}(t)\right|^2 dt \quad \forall z \in C^1[0,\infty),\, z(0) = 0.$$

Then, we have

$$\|(\varepsilon + |x_1|)^{-1}\mathcal{W}^\varepsilon; L^2(\varpi^\varepsilon)\| \le c\|\nabla\mathcal{W}^\varepsilon; L^2(\varpi^\varepsilon)\| \le c. \tag{89}$$

Clearly, from (71), $(1+\Lambda_m^0 + \varepsilon\Lambda_m^1(\eta))^{-1} \le 1$ for a small $\varepsilon > 0$ independent of η, and there remains to estimate the last supremum in (88). We integrate by parts, take the Dirichlet ans quasi-periodic conditions into account and observe that

$$\begin{aligned}\big|(\nabla\mathcal{U}_m^\varepsilon, \nabla\mathcal{W}^\varepsilon)_{\varpi^\varepsilon} - (\Lambda_m^0 + \varepsilon\Lambda_m^1(\eta))(\mathcal{U}_m^\varepsilon, \mathcal{W}^\varepsilon)_{\varpi^\varepsilon}\big| \\ = \big|(\Delta\mathcal{U}_m^\varepsilon + (\Lambda_m^0 + \varepsilon\Lambda_m^1(\eta))\mathcal{U}_m^\varepsilon, \mathcal{W}^\varepsilon)_{\varpi^\varepsilon}\big|.\end{aligned}$$

On the basis of (83)-(86) we write

$$\begin{aligned}&\Delta\mathcal{U}_m^\varepsilon + (\Lambda_m^0 + \varepsilon\Lambda_m^1(\eta))\mathcal{U}_m^\varepsilon \\ &\quad = X^\varepsilon\left(\Delta U_m^0 + \Lambda_m^0 U_m^0 + \varepsilon(\Delta U_m^1 + \Lambda_m^0 U_m^1 + \Lambda_m^1 U_m^0) + \varepsilon^2\Lambda_m^1 U_m^1\right) \\ &\quad + [\Delta, X^\varepsilon](\mathcal{V}_{out}^{\varepsilon m} - \mathcal{V}_{mat}^{\varepsilon m}) + \mathcal{X}(\Delta\mathcal{V}_{in}^{\varepsilon m} - X^\varepsilon\Delta\mathcal{V}_{mat}^{\varepsilon m}) + [\Delta, \mathcal{X}](\mathcal{V}_{in}^{\varepsilon m} - \mathcal{V}_{mat}^{\varepsilon m}) \\ &\quad + (\Lambda_m^0 + \varepsilon\Lambda_m^1)\mathcal{X}(\mathcal{V}_{in}^{\varepsilon m} - X^\varepsilon\mathcal{V}_{mat}^{\varepsilon m}) =: S_1^\varepsilon + S_2^\varepsilon + S_3^\varepsilon + S_4^\varepsilon + S_5^\varepsilon.\end{aligned} \tag{90}$$

Here, $[\Delta, \chi] = 2\nabla\chi\cdot\nabla + \Delta\chi$ is the commutator of the Laplace operator with a function χ, and the equality $[\Delta, X^\varepsilon\mathcal{X}] = [\Delta, \mathcal{X}] + [\Delta, X^\varepsilon]$, which is valid due to the position of supports of functions in (87), is used when distributing terms originated by the last subtrahend in (86). Let us estimate the scalar products $(S_k^\varepsilon, \mathcal{W}^\varepsilon)_{\varpi^\varepsilon}$ for S_k^ε in (90).

Considering S_1^ε, because of (27), (60), (87) and (71), we have that in fact $S_1^\varepsilon = \varepsilon^2 X^\varepsilon\Lambda_m^1 U_m^1$, hence

$$|(S_1^\varepsilon, \mathcal{W}^\varepsilon)_{\varpi^\varepsilon}| \le \varepsilon^2\Lambda_m^1(\eta)\|U_m^1; L^2(\varpi^\varepsilon)\|\|\mathcal{W}^\varepsilon; L^2(\varpi^\varepsilon)\| \le C_m\varepsilon^2.$$

As regards S_2^ε, we take into account that the supports of the functions $\partial_{x_1}X^\varepsilon$ and ΔX^ε belong to the adherence of the thin domain $\varpi_{\varepsilon R}^\varepsilon = \{x \in \overline{\varpi^\varepsilon} : |x_1| \in (\varepsilon R, 2\varepsilon R)\}$, cf. (87). Thus, the error in the Taylor formula up to the second term, and relations (58), (59) and (85) provide

$$\begin{aligned}|\mathcal{V}_{out}^{\varepsilon m}(x;\eta) - \mathcal{V}_{mat}^{\varepsilon m}(x;\eta)| &\le c(|x_1|^2 + \varepsilon|x_1|), \\ \left|\frac{\partial\mathcal{V}_{out}^{\varepsilon m}}{\partial x_1}(x;\eta) - \frac{\partial\mathcal{V}_{mat}^{\varepsilon m}}{\partial x_1}(x;\eta)\right| &\le c(|x_1| + \varepsilon), \quad \pm x_1 \in [\varepsilon R, 2\varepsilon R].\end{aligned}$$

Above, we have also used the smoothness of the function U_m^1 which holds on account that $V = U_m^1 e^{-i\eta y_1}$ is a periodic function in the y_1 variable, solution of an elliptic problem with constant coefficients (cf. (60), (15)-(16) and (91)), and therefore it is smooth. Then, we make use of the weighted inequality (89) and write

$$\begin{aligned}|(S_2^\varepsilon,\mathcal{W}^\varepsilon)_{\varpi^\varepsilon}|\le\quad & \|S_2^\varepsilon;L^2(\varpi^\varepsilon_{\varepsilon R})\|\|\mathcal{W}^\varepsilon;L^2(\varpi^\varepsilon_{\varepsilon R})\|\le c\varepsilon\|(\varepsilon+|x_1|)^{-1}\mathcal{W}^\varepsilon;L^2(\varpi^\varepsilon)\|\\ &\times\left(\int\limits_0^H\int\limits_{\varepsilon R}^{2\varepsilon R}\left(\tfrac{1}{\varepsilon^2}\left|\tfrac{\mathcal{V}^{\varepsilon m}_{out}}{\partial x_1}-\tfrac{\mathcal{V}^{\varepsilon m}_{mat}}{\partial x_1}\right|^2+\tfrac{1}{\varepsilon^4}\left|\mathcal{V}^{\varepsilon m}_{out}-\mathcal{V}^{\varepsilon m}_{mat}\right|^2\right)d|x_1|dx_2\right)^{\frac12}\\ \le\quad & c\left(\tfrac{1}{\varepsilon^2}\varepsilon^2+\tfrac{1}{\varepsilon^4}\varepsilon^4\right)^{\frac12}(mes_2\varpi^\varepsilon_{\varepsilon R})^{\frac12}\,\varepsilon\|(\varepsilon+|x_1|)^{-1}\mathcal{W}^\varepsilon;L^2(\varpi^\varepsilon)\|\le c\varepsilon^{\frac32}.\end{aligned}$$

Dealing with S_3^ε, we match the definitions of the cut-off functions $\chi_\pm$ and X^ε such that $X^\varepsilon(x_1)=\chi_\pm(x_1/\varepsilon)$ for $\pm x_1>0$ (cf. (36)). Recalling formulas (37), (84) and (85), we write

$$\begin{aligned}&\Delta\mathcal{V}^{\varepsilon m}_{in}(x;\eta)-X^\varepsilon(x_1)\Delta\mathcal{V}^{\varepsilon m}_{mat}(x;\eta)\\ &\qquad=2\sum_\pm\frac{\partial w^m_\pm}{\partial x_2}(x_2;\eta)\frac{\partial W^\pm}{\partial\xi_2}(y)+\varepsilon\sum_\pm\frac{\partial^2 w^m_\pm}{\partial x_2^2}(x_2;\eta)\widetilde{W}^\pm(y),\end{aligned}$$

when $\pm x_1>0$, respectively. Note that $W^\pm$ are harmonics and both, $\partial W^\pm/\partial\xi_2$ and $\widetilde{W}^\pm$ decay exponentially as $|\xi_1|\to\infty$, see Proposition 1. Thus,

$$\begin{aligned}&|(S_3^\varepsilon,\mathcal{W}^\varepsilon)_{\varpi^\varepsilon}|\le\\ &c\left(\left\|(\varepsilon+|x_1|)\frac{\partial W^\pm}{\partial\xi_2};L^2(\varpi^\varepsilon)\right\|+\varepsilon\left\|(\varepsilon+|x_1|)\widetilde{W}^\pm;L^2(\varpi^\varepsilon)\right\|\right)\left\|\frac{1}{\varepsilon+|x_1|}\mathcal{W}^\varepsilon;L^2(\varpi^\varepsilon)\right\|\\ &\qquad\le c\left(\int\limits_0^{1/2}(\varepsilon+t)^2e^{-2\delta t/\varepsilon}dt\right)^{\frac12}\|\nabla\mathcal{W}^\varepsilon;L^2(\varpi^\varepsilon)\|\le c\varepsilon^{\frac32}.\end{aligned}$$

Above, obviously, we take the positive constant δ to be $2\pi/H$, cf. Proposition 1, and we note that the last integral has been computed to obtain the bound. With the same argument on the exponential decay of $\mathcal{V}^{\varepsilon m}_{in}-X^\varepsilon\mathcal{V}^{\varepsilon m}_{mat}$, one derives that

$$|(S_5^\varepsilon,\mathcal{W}^\varepsilon)_{\varpi^\varepsilon}|\le c\varepsilon^{\frac32}.$$

Moreover, the supports of the coefficients $\partial_{x_1}\mathcal{X}$ and $\Delta\mathcal{X}$ in the commutator $[\Delta,\mathcal{X}]$ are contained in the set $\overline{\varpi^\varepsilon}\cap\{1/6<|x_1|<1/3\}$ while the above-mentioned decay brings the estimate

$$|(S_4^\varepsilon,\mathcal{W}^\varepsilon)_{\varpi^\varepsilon}|\le ce^{-2\delta/(3\varepsilon)}.$$

Revisiting the obtained estimates we find the worst bound $c\varepsilon^{3/2}$, and this shows (80).

The fact that the constants ε_m and c_m of the statement of the theorem are independent of η follows from the independence of η of the above inequalities throughout the proof. Indeed, we use formulas (55) and (71) for the boundedness of U_m^0 and $\Lambda_m^1(\eta)$, while we note that $\|U_m^1;H^1(\varpi^\varepsilon)\|$ is bounded by a constant independent of η follows from the definition of the solution of (60) with the quasi-periodic boundary conditions (15)-(16). Further specifying, the change $V=U_m^1e^{-\mathrm{i}\eta y_1}$ converts the Laplacian into the differential operator

$$-\left(\frac{\partial}{\partial y_1}+\mathrm{i}\eta\right)\left(\frac{\partial}{\partial y_1}+\mathrm{i}\eta\right)-\frac{\partial^2}{\partial y_2^2},\tag{91}$$

and therefore, performing this change in (60), gives the solution $V\in H^1_{per}(\varpi^0)$. Then, as a consequence of the variational formulation of the problem for V in the set of spaces $L^2(\varpi_0) \subset H^1_{per}(\varpi^0)$, the bound of $\|U^1_m; H^1(\varpi^\varepsilon)\|$ independently of $\eta \in [-\pi, \pi]$ holds true. Hence, the proof of Theorem 5.1 is completed. □

Proof of Corollary 1. Due to the continuity of the function (19), the maximum and minimum of $\Lambda^\varepsilon_m(\eta)$ for $\eta \in [-\pi, \pi]$ are achieved at two points $\eta^\pm_{\varepsilon,m} \in [-\pi, \pi]$. Thus, the endpoints $B^{\varepsilon\pm}(m)$ of the spectral band (20) are given by $B^{\varepsilon\pm}(m) = \Lambda^\varepsilon_m(\eta^\pm_{\varepsilon,m})$.

In order to show (73) for the maximum $B^{\varepsilon+}(m)$, we consider $\eta = \eta^+$ to be π or $-\pi$ in such a way that $\Lambda^1(\eta^+) = B_0(m) + |B_1(m)|$. Since (70) is satisfied for $\eta = \eta^\pm_{\varepsilon,m}$ and for $\eta = \pm\pi$, we write

$$\Lambda^0_m + \varepsilon B_0(m) + \varepsilon|B_1(m)| - c_m\varepsilon^{3/2} \leq \Lambda^\varepsilon_m(\eta^+) \leq \Lambda^0_m + \varepsilon B_0(m) + \varepsilon|B_1(m)| + c_m\varepsilon^{3/2}$$

and

$$\Lambda^0_m + \varepsilon\Lambda^1_m(\eta^+_{\varepsilon,m}) - c_m\varepsilon^{3/2} \leq \Lambda^\varepsilon_m(\eta^+_{\varepsilon,m}) \leq \Lambda^0_m + \varepsilon\Lambda^1_m(\eta^+_{\varepsilon,m}) + c_m\varepsilon^{3/2}.$$

Consequently, from (71), we derive

$$\begin{aligned}\Lambda^0_m + \varepsilon B_0(m) + \varepsilon|B_1(m)| - c_m\varepsilon^{3/2} \leq \Lambda^\varepsilon_m(\eta^+) \leq \Lambda^\varepsilon_m(\eta^+_{\varepsilon,m})& \\ \leq \Lambda^0_m + \varepsilon B_0(m) + \varepsilon|B_1(m)| + c_m\varepsilon^{3/2}&,\end{aligned}$$

which gives (73) for $B^{\varepsilon+}(m)$.

We proceed in a similar way for the minimum $B^{\varepsilon-}(m)$ and $\eta = \eta^-$ such that $\Lambda^1(\eta^-) = B_0(m) - |B_1(m)|$ and we obtain (73). Obviously, this implies that $B^{\varepsilon\pm}(m)$ belong to the interval

$$\Big[\Lambda^0_m + \varepsilon B_0(m) - \varepsilon|B_1(m)| - c_m\varepsilon^{3/2}\,,\, \Lambda^0_m + \varepsilon B_0(m) + \varepsilon|B_1(m)| + c_m\varepsilon^{3/2}\Big]$$

Therefore, the whole band B^ε_m is contained in the interval above whose length is $2\varepsilon|B_1(m)| + O(\varepsilon^{3/2})$ and the corollary is proved. □

Proof of Theorem 5.2. This proof holds exactly the same scheme of Theorem 5.1. Indeed, for each $l = 0, \cdots, \kappa_m - 1$ we follow the reasoning in (76)-(82) and we deduce (cf. (81) and (25)) that for each l, and for a $\varepsilon_{m,l} > 0$, the estimate

$$\left|\Lambda^\varepsilon_p(\eta) - \Lambda^0_m - \varepsilon\Lambda^1_{m+l}(\eta)\right| \leq \mathcal{C}_{m,l}\varepsilon^{3/2} \tag{92}$$

is satisfied for a certain natural $p \equiv p(l)$ and $\mathcal{C}_{m,l}$ independent of $\varepsilon \in (0, \varepsilon_{m,l}]$. Due to the convergence with conservation of the multiplicity (64), the only possible eigenvalues $\Lambda^\varepsilon_p(\eta)$ of problem (13)-(16) satisfying (92) are the set $\{\Lambda^\varepsilon_{m+l}(\eta)\}_{l=0,\cdots,\kappa_m-1}$. Then, it suffices that there are κ_m linearly independent eigenfunctions associated with the eigenvalues $\{\Lambda^\varepsilon_{p(l)}(\eta)\}_{l=0,\cdots,\kappa_m-1}$ in (92), to deduce the result of the theorem.

We use a classical argument of contradiction (cf. [15] and [25]). We consider the set of functions $\{\mathcal{U}^\varepsilon_{m+l}(x;\eta)\}_{l=0,\cdots,\kappa_m-1}$, constructed in (86), and we verify that they satisfy almost orthogonality conditions

$$\|\mathcal{U}^\varepsilon_{m+l}; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\| \geq \widetilde{c}_m \quad \text{and} \quad \left|\langle\mathcal{U}^\varepsilon_{m+l}\,,\,\mathcal{U}^\varepsilon_{m+l'}\rangle\right| \leq \widetilde{C}_m\varepsilon^{1/2},\ \text{with } l \neq l'\,, \tag{93}$$

for certain constants $\widetilde{c}_m$ and $\widetilde{C}_m$. Indeed, the first inequality above is a consequence of (79), for $l = 0, \cdots, \kappa_m - 1$, while the second one follows from the orthogonality of the set of eigenfunctions $\{U^0_{m+l}(x,\eta)\}_{l=0,\cdots,\kappa_m-1}$ and the definitions (83)-(87).

Then, we define $\widetilde{\mathcal{U}}^\varepsilon_{m+l} = \mathcal{U}^\varepsilon_{m+l}\,\|\mathcal{U}^\varepsilon_{m+l}; H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\|^{-1}$ and consider $\mathcal{W}^\varepsilon_{m+l}$ the projection of $T^\varepsilon(\eta)\widetilde{\mathcal{U}}^\varepsilon_{m+l} - t^\varepsilon_m(\eta)\widetilde{\mathcal{U}}^\varepsilon_{m+l}$ in the space of the eigenfunctions of $T^\varepsilon(\eta)$

associated with all the eigenvalues $\{\Lambda^\varepsilon_{p(l)}(\eta)\}_{l=0,\cdots,\kappa_m-1}$ in (92) for certain constants $\mathcal{C}_{m,l}$, and more precisely, satisfying

$$\left|(1+\Lambda^\varepsilon_{p(l)}(\eta))^{-1}-(1+\Lambda^0_m+\varepsilon\Lambda^1_{m+l}(\eta))^{-1}\right|\leq \widetilde{\mathbf{c}}_m\varepsilon^{3/2}. \tag{94}$$

for a constant $\widetilde{\mathbf{c}}_m$ that we shall set later in the proof, cf. (81) and definitions (76)-(78) for the operator $T^\varepsilon(\eta)$ and the "almost eigenvalue" $t^\varepsilon_m(\eta)$. Then, we show

$$\left\|\widetilde{\mathcal{W}}^\varepsilon_{m+l}-\widetilde{\mathcal{U}}^\varepsilon_{m+l};H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\right\|\leq\widetilde{\mathcal{C}}_m, \tag{95}$$

where $\widetilde{\mathcal{W}}^\varepsilon_{m+l}=\mathcal{W}^\varepsilon_{m+l}\|\mathcal{W}^\varepsilon_{m+l};H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\|^{-1}$, and $\widetilde{\mathcal{C}}_m=2\widetilde{\mathbf{c}}_m^{-1}\max_{0\leq l\leq\kappa_m-1}\mathcal{C}_{l,m}$. This is due to the fact that

$$\left\|\widetilde{\mathcal{U}}^\varepsilon_{m+l}-\mathcal{W}^\varepsilon_{m+l};H^{1,\eta}_{per}(\varpi^\varepsilon;\Gamma^\varepsilon)\right\|\leq\widetilde{\mathbf{c}}_m^{-1}\max_{0\leq l\leq\kappa_m-1}\mathcal{C}_{l,m},$$

and some straightforward computation (cf., eg., Lemma 1 in Ch. 3 of [27]). Now, from (93) and (95) and straightforward computations we obtain

$$\left|\langle\widetilde{\mathcal{W}}^\varepsilon_{m+l}\,,\,\widetilde{\mathcal{W}}^\varepsilon_{m+l'}\rangle\right|\leq 5\widetilde{\mathcal{C}}_m\ \text{ with }\ l\neq l'\,, \tag{96}$$

and this allows us to assert that set $\{\widetilde{\mathcal{W}}^\varepsilon_{m+l}\}_{l=0,\cdots,\kappa_m-1}$ defines κ_m linearly independent functions. Indeed, to prove it, we proceed by contradiction, by assuming that there are constants α^ε_l different from zero such that

$$\sum_{l=0}^{\kappa_m-1}\alpha^\varepsilon_l\widetilde{\mathcal{W}}^\varepsilon_{m+l}=0.$$

Let us consider $\alpha^{*,\varepsilon}=\max_{0\leq l\leq\kappa_m-1}|\alpha^\varepsilon_l|$ and assume, without any restriction that $\alpha^{*,\varepsilon}=\alpha^\varepsilon_0$. Then, we write

$$\langle\widetilde{\mathcal{W}}^\varepsilon_m,\widetilde{\mathcal{W}}^\varepsilon_m\rangle\leq\sum_{l=1}^{\kappa_m-1}\left|\frac{\alpha^\varepsilon_l}{\alpha^\varepsilon_0}\right|\left|\langle\widetilde{\mathcal{W}}^\varepsilon_{m+l},\widetilde{\mathcal{W}}^\varepsilon_m\rangle\right|\leq(\kappa_m-1)5\widetilde{\mathcal{C}}_m.$$

Now, setting $(\kappa_m-1)5\widetilde{\mathcal{C}}_m<1$ gives a contradiction, since the left hand side takes the value 1, cf. (96). In this way, we also have fixed $\widetilde{\mathbf{c}}_m$ in (94).

Thus, $\{\widetilde{\mathcal{W}}^\varepsilon_{m+l}\}_{l=0,\cdots,\kappa_m-1}$ define κ_m linearly independent functions, which obviously implies that they are associated with κ_m eigenvalues; hence the set of eigenvalues $\{\Lambda^\varepsilon_{p(l)}(\eta)\}_{l=0,\cdots,\kappa_m-1}$ coincides with $\{\Lambda^\varepsilon_{m+l}(\eta)\}_{l=0,\cdots,\kappa_m-1}$ and this concludes the proof of the theorem. □

REFERENCES

[1] M.Sh. Birman and M.Z. Solomjak, *Spectral Theory of Selfadjoint Operators in Hilbert Spaces,* [Translated from the 1980 Russian original by S. Khrushchëv and V. Peller], Mathematics and its Applications (Soviet Series). D. Reidel Publishing Co., Dordrecht, 1987.

[2] D.I. Borisov and K.V. Pankrashkin, Gap opening and split band edges in waveguides coupled by a periodic system of small windows, *Math. Notes*, **93:5-6** (2013), 660-675.

[3] D.I. Borisov and K.V. Pankrashkin, Quantum waveguides with small periodic perturbations: gaps and edges of Brillouin zones, *Journal of Physics A: Mathematical and Theoretical*, **46:23** (2013), 235203,18.

[4] C. Conca, J. Planchard and M. Vanninathan, *Fluids and Periodic Structures,* RAM: Research in Applied Mathematics, 38. John Wiley & Sons, Ltd., Chichester; Masson, Paris, 1995.

[5] D. Cioranescu and F. Murat, A strange term coming from nowhere, in *Topics in the mathematical modelling of composite materials*, Progr. Nonlinear Differential Equations Appl., 31, Birkhäuser, Boston (1997), 4593.

[6] I.M. Gelfand, Expansion in characteristic functions of an equation with periodic coefficients (Russian), *Doklady Akad. Nauk SSSR*, **73** (1950), 1117-1120.

[7] A.M. Ilin, A boundary value problem for the elliptic equation of second order in a domain with a narrow slit. 1. The two-dimensional case, *Math. USSR-Sb.*, **28:4** (1976), 459-480.

[8] A.M. Ilin, *Matching of Asymptotic Expansions of Solutions of Boundary Value Problems,* Translations of Mathematical Monographs, 102. American Mathematical Society, Providence, RI, 1992.

[9] T. Kato, *Perturbation Theory for Linear Operators,* Springer-Verlag, New York, 1966.

[10] V.A. Kondratyev, Boundary value problems for elliptic equations in domains with conic and corner points, *Transactions Moscow Matem. Soc.*, **16** (1967), 227-313.

[11] P. Kuchment, *Floquet Theory for Partial Differential Equations,* Birkhäuser Verlag, Basel, 1993.

[12] N.S. Landkof, *Foundations of modern potential theory,* Springer-Verlag, New York-Heidelberg, 1972.

[13] D. Leguillon, E. Sanchez-Palencia, *Computations of singular solutions in elliptic problems and elasticity.* Masson, Paris, 1987.

[14] M. Lobo, O.A. Oleinik, E. Pérez and T.A. Shaposhnikova, On homogenization of solutions of boundary value problems in domains, perforated along manifolds, *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **25:3-4** (1998), 611-629.

[15] M. Lobo and E. Pérez, On the local vibrations for systems with many concentrated masses near the boundary, *C.R. Acad. Sci. Paris, Ser. IIb*, **324:5** (1997), 323-329.

[16] V.A. Marchenko, E.Ya. Khruslov, *Homogenization of Partial Differential Equations,* Progress in Mathematical Physics, 46. Birkhäuser Boston, Inc., Boston, MA, 2006.

[17] V. Maz'ya, S.A. Nazarov and B. Plamenevskij, *Asymptotic Theory of Elliptic Boundary Value Problems in Singularly Perturbed Domains, Vol. I*, Birkhäuser Verlag, 2000.

[18] G. Nguetseng, Problème d'écrans perforés pour l'équation de Laplace, *RAIRO. Modél. Math. Anal. Numér.*, **19:1** (1985), 33-63.

[19] S.A. Nazarov, Asymptotic conditions at a point, self-adjoint extensions of operators and the method of matched asymptotic expansions, *Trudy St.-Petersburg Mat. Obshch.*, **5** (1996), 112-183. English transl.: *Trans. Am. Math. Soc. Ser. 2*, **193** (1999), 77-126.

[20] S.A. Nazarov, The polynomial property of self-adjoint elliptic boundary-value problems and the algebraic description of their attributes, *Uspehi mat. nauk.* **54:5** (1999), 77-142. English transl.: *Russ. Math. Surveys,* **54:5** (1999), 947-1014.

[21] S.A. Nazarov, Opening of a gap in the continuous spectrum of a periodically perturbed waveguide, *Mathematical Notes*, **87:5** (2010), 738-756.

[22] S.A. Nazarov, Asymptotic behavior of spectral gaps in a regularly perturbed periodic waveguide, *Vestnik St. Petersburg Univ. Mathematics*, **46:2** (2013), 89-97.

[23] S.A. Nazarov, R. Orive-Illera and M.-E. Pérez-Martínez, On the polarization matrix for a perforated strip, in *Integral Methods in Science and Engineering: Analytic Treatment and Numerical Approximations* (Eds. C. Constanda and P. Harris), Birkhauser, N.Y., (2019 in print), Chapter 20.

[24] S. Nazarov and E. Pérez, New asymptotic effects for the spectrum of problems on concentrated masses near the boundary, *Comptes Rendues de Mecanique*, **337:8** (2009), 585-590.

[25] S. Nazarov and M.E. Pérez, On multi-scale asymptotic structure of eigenfunctions in a boundary value problem with concentrated masses near the boundary, *Rev. Mat. Complut.*, **31:1** (2018), 1-62.

[26] S.A. Nazarov and B.A. Plamenevskii, *Elliptic Problems in Domains with Piecewise Smooth Boundaries* [in Russian], Nauka, Moscow, 1991; English transl.: Walter de Gruyter, Berlin, 1994.

[27] O.A. Oleinik, A.S. Shamaev and G.A. Yosifia,.*Mathematical Problems in Elasticity and Homogenization,* North-Holland, Amsterdam, 1992

[28] G.P. Panasenko, Higher order asymptotics of solutions of problems on the contact of periodic structures, *Mat. Sb. (N.S.)*, **110(152):4(12)** (1979), 505-538; English transl: *Mathematics of the USSR-Sbornik,* **38:4** (1981), 465-494.

[29] G. Polya and G. Szego, *Isoperimetric Inequalities in Mathematical Physics,* Annals of Mathematics Studies, N. 27, Princeton University Press, Princeton, N. J., 1951.

[30] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. IV. Analysis of operators.* Academic Press, New York-London, 1978.

[31] J. Sanchez-Hubert and E. Sanchez-Palencia, *Vibration and Coupling of Continuous Systems. Asymptotic methods,* Springer-Verlag, Berlin, 1989.

[32] E. Sanchez-Palencia, Un problème d'ecoulement lent d'un fluide incompressible au travers d'une paroi finement perforée. In: *Homogenization Methods: Theory and Applications in Physics*, Collect. Dir. Études Rech. Élec. France, 57, Eyrolles, Paris, 1985, pp. 371-400.

[33] M.M. Skriganov, Geometric and Arithmetic Methods in the Spectral Sheory of Multidimensional Periodic Operators, A translation of Trudy Mat. Inst. Steklov., **171** (1985); Proc. Steklov Inst. Math., **171:2** (1987).

[34] K. Yoshitomi, Band spectrum of the Laplacian on a slab with the Dirichlet boundary condition on a grid, *Kyushu J. Math.*, **57:1** (2003), 87-116.

[35] M. Van Dyke, *Perturbation Methods in Fluid Mechanics,* Academic Press, New York, 1964.

Received xxxx 20xx; revised xxxx 20xx.

E-mail address: srgnazarov@yahoo.co.uk
E-mail address: rafael.orive@icmat.es
E-mail address: meperez@unican.es