Spectral study of $\{R, s+1, k\}$- and $\{R, s+1, k, *\}$-potent matrices

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Abstract

The $\{R, s+1, k\}$- and $\{R, s+1, k, *\}$-potent matrices have been studied in several recent papers. We continue these investigations from a spectral point of view. Specifically, a spectral study of $\{R, s+1, k\}$-potent matrices is developed using characterizations involving an associated matrix pencil (A, R). The corresponding spectral study for $\{R, s+1, k, *\}$-potent matrices involves the pencil (A^*, R). In order to present some properties, the relevance of the projector $I - AA^{\#}$ where $A^{\#}$ is the group inverse of A is highlighted. In addition, some applications and numerical examples are given, particularly involving Pauli matrices and the quaternions.

Keywords: k-involutory matrix; s-potent matrix; $\{R, s+1, k\}$-potent matrix; spectrum.
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1 Introduction

In recent years, investigations involving situations where a square matrix equals one of its powers ($A^s = A$ for some integer $s \geq 2$; such a matrix is called $\{s\}$-*potent*) have been approached from both theoretical and applications points of view [4, 7, 8, 14, 19, 20, 24]. We mention just a selection of these studies here. In [2], Baksalary and Trenkler investigated $\{s\}$-potent matrices, particularly focusing on obtaining new properties of tripotent matrices. In [9], Du and Li used quatripotent matrices to characterize *generalized projections* (i.e. the square coincides with the conjugate transpose of a matrix); while in [12], Groß and Trenkler used $\{s\}$-potent matrices to characterize *hypergeneralized projectors* (i.e., the square coincides with the Moore-Penrose inverse of a matrix). The extensions of generalized projections and hypergeneralized projectors to Hilbert spaces were investigated by Radosavljević and Djordjević in [28] and extensions to k-generalized projectors on Hilbert spaces were done by Lebtahi and Thome in [18]. Tosic, Cvetković-Ilić and Deng [32] applied these results to study linear combinations involving generalized and hypergeneralized projectors. Most methods to characterize several matrix partial orders require the use of powers of the involved matrices as can be extensively seen in [26].

A matrix $R \in \mathbb{C}^{n\times n}$ is called $\{k\}$-*involutory* if $R^k = I_n$ where I_n denotes the $n \times n$ identity matrix. The symbol Ω_ℓ will denote the set of all ℓ^{th} roots of unity where ℓ is a positive integer, that is, $\Omega_\ell = \{\omega_\ell^0, \omega_\ell^1, \dots, \omega_\ell^{\ell-1}\}$ where $\omega_\ell = e^{2\pi i/\ell}$.

When $A^{s+1} = A$ (i.e., A is $\{s+1\}$-*potent*) for some $s \in \{2, 3, \dots\}$, one has $A^\# = A^{s-1}$, where $A^\#$ represents the group inverse of the matrix $A \in \mathbb{C}^{n\times n}$ [1, 5]; such a matrix is also called $\{s\}$-*group involutory*. We recall that for a given matrix $A \in \mathbb{C}^{n\times n}$, the *group inverse of* A, denoted by $A^\#$, is the matrix satisfying the following conditions $AA^\#A = A$, $A^\#AA^\# = A^\#$, and $AA^\# = A^\#A$. The matrix $A^\#$ exists if and only if A and A^2 have the same rank. If it exists, $A^\#$ is unique.

In [24, 7], the authors generalized the concept of $\{s+1\}$-potent matrices by means of the following definition. For a given $\{k\}$- involutory matrix $R \in \mathbb{C}^{n\times n}$ and a fixed positive integer s, a matrix $A \in \mathbb{C}^{n\times n}$ is called $\{R, s+1, k\}$-*potent* if satisfies

$$RA = A^{s+1}R.$$

Clearly, this definition extends the classes mentioned above. Some properties concerning the $\{R, s+1, k\}$-potent matrices have been established in [24]. In particular, an $\{R, s+1, k\}$-potent matrix satisfies $A^{(s+1)^k} = A$.

Algorithms for constructing matrices in this class were presented in [21] for $k = 2$, and the inverse problem was solved in [22] where several algorithms were designed in order to compute all involutory matrices R such that $RA = A^{s+1}R$. With the necessary modifications, an interesting application of a class of matrices related to $\{R, s+1, k\}$-potent matrices was given in [23] to study image processing.

In [7], the projector $AA^{\#}$ was introduced as an important tool in the study of $\{R, s+1, k\}$-potent matrices. Specifically, from a given $\{R, s+1, k\}$-potent matrix A, a matrix group $\mathcal{G}$ was constructed such that the idempotent $AA^{\#}$ is the identity element of $\mathcal{G}$. Moreover, it was proved that $I - AA^{\#} \notin \mathcal{G}$, but the question of whether $I - AA^{\#}$ is an $\{R, s+1, k\}$-potent matrix or not was left unanswered. As an application, we mention that it is well known that in Markov chains, the projector $I - AA^{\#}$ is fundamental, among other things, in various types of limiting processes involving the transition matrix. For an n-state homogeneous Markov chain whose one-step transition matrix is T, the group inverse $A^{\#}$ of the matrix $A = I - T$ always exists, and contains relevant information about the Markov chain [6]; in particular, $\lim_{k \to +\infty} T^k = I - AA^{\#}$ whenever the limit exists [16]. Additionally, a representation of the second partial derivative of the Perron value function $r(A)$ at A, of a nonnegative irreducible matrix A, can be expressed in terms of $I - QQ^{\#}$ where $Q = r(A)I - A$ [17, p. 28].

Next, we state a spectral theorem that we mention and use in several places in this paper. We use the standard Kronecker delta notation, δ_{ij}.

Theorem 1 ([1]) *Let $A \in \mathbb{C}^{n \times n}$ with k distinct eigenvalues $\lambda_1, \dots, \lambda_k$. Then A is diagonalizable if and only if there exist disjoint projectors $P_1, P_2, \dots, P_k$, that is $P_i P_j = \delta_{ij} P_i$ for $i, j \in \{1, 2, \dots, k\}$, such that $A = \sum_{j=1}^{k} \lambda_j P_j$ and $I_n = \sum_{j=1}^{k} P_j$. Moreover, when A is a normal matrix, the projectors $P_1, P_2, \dots, P_k$ are orthogonal.*

The following is a characterization of $\{R, s+1, k\}$-potent matrices.

Theorem 2 ([24]) *Let $R \in \mathbb{C}^{n \times n}$ be a $\{k\}$-involutory matrix, $s \in \{1, 2, 3, \dots\}$, $n_{s,k} = (s+1)^k - 1$, and $A \in \mathbb{C}^{n \times n}$. Then the following conditions are equivalent:*

(a) A is $\{R, s+1, k\}$-potent.

(b) A is an $\{n_{s,k}\}$-group involutory matrix, and there exist disjoint projectors $P_0, P_1, \ldots, P_{n_{s,k}}$ with

$$A = \sum_{j=1}^{n_{s,k}} \omega^j P_j \qquad \text{and} \qquad \sum_{j=0}^{n_{s,k}} P_j = I_n,$$

where $\omega = e^{\frac{2\pi i}{n_{s,k}}}$, where $P_j = O$ when $\omega^j \notin \sigma(A)$ and $P_0 = O$ when $0 \notin \sigma(A)$, and where the projectors $P_0, P_1, \ldots, P_{n_{s,k}}$ satisfy

(i) For each $i \in \{1, \ldots, n_{s,k}-1\}$, there exists a unique $j \in \{1, \ldots, n_{s,k}-1\}$ such that $RP_iR^{-1} = P_j$,

(ii) $RP_{n_{s,k}}R^{-1} = P_{n_{s,k}}$, and

(iii) $RP_0R^{-1} = P_0$.

(c) A is diagonalizable and there exist disjoint projectors $P_0, P_1, \ldots, P_{n_{s,k}}$ satisfying conditions (i), (ii), and (iii) given in (b).

Motivated by the class introduced in [3], results on $\{R, s+1, k\}$-potent matrices, and their connections to important known classes of matrices, a further class of matrices was introduced and analyzed in [8]. For a given $\{k\}$-involutory matrix $R \in \mathbb{C}^{n\times n}$ and a fixed positive integer s, a matrix $A \in \mathbb{C}^{n\times n}$ is called *$\{R, s+1, k, *\}$-potent* if it satisfies

$$RA^* = A^{s+1}R.$$

In [8], various characterizations of the class of $\{R, s+1, k, *\}$-potent matrices were given, and relationships between these matrices and other classes of matrices were presented.

The main aim of this paper is to carry out a deeper study of properties of $\{R, s+1, k\}$-potent and $\{R, s+1, k, *\}$-potent matrices from a spectral point of view. The most important contribution of this paper concerns the computation of the spectrum of certain matrix pencils in terms of the spectrum of a specified single matrix, and the derivation of the corresponding eigenstructure relationships.

The paper is organized as follows. In section 2, we consider the matrix pencil (A, R) where A is a given $\{R, s+1, k\}$-potent matrix, and in section 3, we consider the pencil (A^*, R) where A is an $\{R, s+1, k, *\}$-potent matrix.

In both classes, the role of the projector $I - AA^{\#}$ is stated; in particular, we show that $I - AA^{\#}$ is $\{R, s+1, k\}$-potent (respectively, $\{R, s+1, k, *\}$-potent) whenever A is $\{R, s+1, k\}$-potent (respectively, $\{R, s+1, k, *\}$-potent). Finally, in section 4, we connect $\{R, s+1, k\}$-potent and $\{R, s+1, k, *\}$-potent matrices to the quaternions and Pauli matrices, which are important objects in quantum mechanics, and present some numerical examples. We conclude with constructions of smaller size $\{\tilde{R}, s+1, k\}$-potent or $\{\tilde{R}, s+1, k, *\}$-potent matrices using block decompositions of a given $\{R, s+1, k\}$-potent or $\{R, s+1, k, *\}$-potent matrix.

2 Spectral study of $\{R, s+1, k\}$-potent matrices

For given matrices $A, R \in \mathbb{C}^{n \times n}$ with R being $\{k\}$-involutory, $\lambda \in \mathbb{C}$ is called an *eigenvalue of the matrix pencil* (A, R) if $\det(\lambda R - A) = 0$. The set of all complex numbers λ satisfying the above condition is called the *spectrum of the pencil* and is denoted $\sigma(A, R)$. Furthermore, $\lambda \in \sigma(A, R)$ if and only if there exists a nonzero vector $x \in \mathbb{C}^{n \times 1}$ such that $Ax = \lambda Rx$. Since $R^k = I_n$, in this case this is equivalent to the existence of a nonzero vector $x \in \mathbb{C}^{n \times 1}$ such that $R^{k-1}Ax = \lambda x$.

We can compute the spectrum of the pencil by means of the spectrum of a (single) matrix as described in the following result.

Lemma 1 *Let $A \in \mathbb{C}^{n \times n}$ be an $\{R, s+1, k\}$-potent matrix. Then $\sigma(A, R) = \sigma(R^{k-1}A)$.*

The following "commutativity-type" property between A and R is needed for our next results.

Lemma 2 *Let $A \in \mathbb{C}^{n \times n}$ be an $\{R, s+1, k\}$-potent matrix. Then*

$$A^j R^{k-1} = R^{k-1}(A^{s+1})^j, \qquad \forall j \in \mathbb{N}.$$

Proof. We proceed by induction on j. The definition of an $\{R, s+1, k\}$-potent matrix gives that the property holds for $j = 1$.

Let $j \geq 1$ and assume that $A^j R^{k-1} = R^{k-1}(A^{s+1})^j$. Then,

$$A^{j+1}R^{k-1} = A(A^j R^{k-1}) = (AR^{k-1})(A^{s+1})^j = R^{k-1}A^{s+1}(A^{s+1})^j = R^{k-1}(A^{s+1})^{j+1}.$$

$\blacksquare$

The previous "commutativity-type" property allows us to give a formula for a power of $R^{k-1}A$ that is independent of R (that is, in terms of A only).

Lemma 3 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Then*

$$(R^{k-1}A)^k = A^{[(s+1)^{k-1}+\cdots+(s+1)+1]} = A^{[(s+1)^k-1]/s}.$$

Proof. We show by induction on j that $(R^{k-1}A)^j = (R^{k-1})^j A^{[(s+1)^{j-1}+\cdots+(s+1)+1]}$. The $j = 1$ case is trivial. Assume that the property is valid for some $j \geq 1$. Then, by Lemma 2,

$$\begin{aligned}(R^{k-1}A)^{j+1} &= (R^{k-1}A)^j R^{k-1}A = (R^{k-1})^j A^{[(s+1)^{j-1}+\cdots+(s+1)+1]} R^{k-1}A \\ &= (R^{k-1})^j R^{k-1}(A^{s+1})^{[(s+1)^{j-1}+\cdots+(s+1)+1]} A \\ &= (R^{k-1})^{j+1} A^{[(s+1)^j+\cdots+(s+1)+1]}.\end{aligned}$$

Setting $j = k$, we get the desired result. $\blacksquare$

We next prove that $R^{k-1}A$ and AR^{k-1} are both $\{ks+1\}$-potent matrices.

Lemma 4 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Then*

$$(R^{k-1}A)^{ks+1} = R^{k-1}A \qquad \textit{and} \qquad (AR^{k-1})^{ks+1} = AR^{k-1}. \tag{1}$$

Proof. Using Lemma 3, $(R^{k-1}A)^{ks+1} = [(R^{k-1}A)^k]^s (R^{k-1}A) = [A^{[(s+1)^k-1]/s}]^s (R^{k-1}A) = A^{(s+1)^k-1}(R^{k-1}A)$. By using Lemma 2 and Lemma 2 (a) and (c) in [7] we obtain

$$\begin{aligned}A^{(s+1)^k-1}(R^{k-1}A) &= R^{k-1}(A^{s+1})^{(s+1)^k-1}A = R^{k-1}(A^{(s+1)^k-1})^{s+1}A \\ &= R^{k-1}A^{(s+1)^k-1}A = R^{k-1}A^{(s+1)^k} = R^{k-1}A.\end{aligned}$$

Also, $(AR^{k-1})^{ks+1} = A(R^{k-1}A)^{ks}R^{k-1} = A[A^{[(s+1)^k-1]/s}]^s R^{k-1} = A^{(s+1)^k}R^{k-1} = AR^{k-1}$. $\blacksquare$

For a given matrix $M \in \mathbb{C}^{n\times n}$, we will denote by (λ, x) an *eigenvalue-eigenvector pair* for M, that is, $Mx = \lambda x$ where x is a nonzero vector. For two matrices K and S of the same size, it is well-known that KS and SK have exactly the same eigenvalues [13].

It then follows, using Lemma 1, that for an $\{R, s+1, k\}$-potent matrix A, we have $\sigma(A, R) = \sigma(R^{k-1}A) = \sigma(AR^{k-1})$. Furthermore, we can say the following.

Lemma 5 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Then $R^{k-1}A$ and AR^{k-1} are diagonalizable matrices and $\sigma(AR^{k-1}) = \sigma(R^{k-1}A)$. Moreover, (λ, x) is an eigenvalue-eigenvector pair for $R^{k-1}A$ (respectively, AR^{k-1}) if and only if (λ, Rx) is an eigenvalue-eigenvector pair for AR^{k-1} (respectively, $R^{k-1}A$).*

Proof. Lemma 4 establishes that $R^{k-1}A$ and AR^{k-1} are $\{ks+1\}$-potent and the characterization of $\{m\}$-potent matrices allows us to conclude that $R^{k-1}A$ and AR^{k-1} are both diagonalizable. Moreover, there exists a nonzero vector x such that $R^{k-1}Ax = \lambda x$ if and only if $AR^{k-1}(Rx) = \lambda(Rx)$ because R is $\{k\}$-involutory. Since R is nonsingular, it is clear that x is nonzero vector if and only if Rx is a nonzero vector. This completes the proof. ■

We arrive at the first main result of this section.

Theorem 3 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Then $\sigma(A, R) \subseteq \{0\} \cup \Omega_{ks}$.*

Proof. By Lemma 4, $R^{k-1}A$ is $\{ks+1\}$-potent. If λ is an eigenvalue of a $\{ks+1\}$-potent matrix, then $\lambda^{ks+1} = \lambda$, and hence, $\lambda \in \{0\} \cup \Omega_{ks}$. ■

If M is an $n \times n$ complex diagonalizable matrix whose distinct eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_r$, then it is well-known that the spectral projectors for M are given by

$$F_h(M) = \prod_{\ell \neq h} \frac{M - \lambda_\ell I_n}{\lambda_j - \lambda_\ell}$$

for $1 \leq h \leq r$, and thus,

$$I_n = \sum_{h=1}^{r} F_h(M)$$

and the spectral decomposition of M is given by

$$M = \sum_{h=1}^{r} \lambda_h F_h(M)$$

Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Since $R^{k-1}A$ and AR^{k-1} are $\{ks+1\}$-potent matrices, they are diagonalizable, and their spectra are contained in $\{0\} \cup \Omega_{ks}$. Let $M \in \left\{R^{k-1}A, AR^{k-1}\right\}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_r$

denote the distinct eigenvalues in $\sigma(M)$ listed in the order in which they appear in the sequence $0, \omega_{ks}, \omega_{ks}^2, \ldots, \omega_{ks}^{ks}$ where $\omega_{ks} = e^{\frac{2\pi i}{ks}}$. If $\lambda_1 = 0$, then let $E_0(M) = F_1(M)$; otherwise, let $E_0(M) = O$. For $1 \leq j \leq ks$, let $E_j(M) = F_h(M)$ when $\omega_{ks}^j = \lambda_h$ for some h, and let $E_j(M) = O$ otherwise. Then

$$I_n = \sum_{j=0}^{ks} E_j(M)$$

and M has spectral decomposition

$$M = \sum_{j=1}^{ks} \omega_{ks}^j E_j(M). \tag{2}$$

The next lemmas give results about the projector $I_n - AA^{\#}$ for a given $\{R, s+1, k\}$-potent matrix A.

Lemma 6 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix, and let $M \in \{AR^{k-1}, R^{k-1}A\}$. If $0 \in \sigma(M)$ then the projector $E_0(M)$ associated with the zero eigenvalue of M is independent of R and it has the following expression:*

$$E_0(M) = I_n - A^{(s+1)^k-1} = I_n - AA^{\#}.$$

Proof. We prove the result for $M = R^{k-1}A$; the proof for $M = AR^{k-1}$ is similar. Since $R^{k-1}A$ is $(ks+1)$-potent, it is $\{ks\}$-group involutory, i.e., $(R^{k-1}A)^{\#} = (R^{k-1}A)^{ks-1}$. Thus,

$$E_0(M) = I - MM^{\#} = I - (R^{k-1}A)(R^{k-1}A)^{\#} = I - (R^{k-1}A)(R^{k-1}A)^{ks-1} = I - (R^{k-1}A)^{ks}.$$

Using Lemma 3, $(R^{k-1}A)^{ks} = A^{(s+1)^k-1}$ and thus $E_0(M) = I_n - A^{(s+1)^k-1}$. In [7, Lemma 2 (b)], it was proved that $A^{\#} = A^{(s+1)^k-2}$. Hence, $E_0(M) = I_n - AA^{\#}$. ■

Lemma 7 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Then the matrix $A^{\#}$ and the projectors $AA^{\#}$ and $I_n - AA^{\#}$ are also $\{R, s+1, k\}$-potent matrices.*

Proof. Since $\{R, s+1, k\}$-potent matrices have index at most 1 (see [24, Theorem 1]), [7, Lemma 6] assures that there exist nonsingular matrices $P \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{r\times r}$ such that

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \qquad R = P \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} P^{-1},$$

for some $R_1 \in \mathbb{C}^{r\times r}$ and $R_2 \in \mathbb{C}^{(n-r)\times(n-r)}$ being $\{k\}$-involutory and for C being $\{R_1, s+1, k\}$-potent. It is clear that

$$A^{\#} = P \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} P^{-1}, \qquad I - AA^{\#} = P \begin{bmatrix} O & O \\ O & I \end{bmatrix} P^{-1}.$$

Then

$$(I - AA^{\#})^{s+1} R = P \begin{bmatrix} O & O \\ O & I \end{bmatrix} \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} P^{-1} = P \begin{bmatrix} O & O \\ O & R_2 \end{bmatrix} P^{-1}$$

and

$$R(I - AA^{\#}) = P \begin{bmatrix} R_1 & O \\ O & R_2 \end{bmatrix} \begin{bmatrix} O & O \\ O & I \end{bmatrix} P^{-1} = P \begin{bmatrix} O & O \\ O & R_2 \end{bmatrix} P^{-1}.$$

Hence, $I - AA^{\#}$ is an $\{R, s+1, k\}$-potent matrix. The matrix $A^{\#}$ and the projector $AA^{\#}$ belong to the set of $\{R, s+1, k\}$-potent matrices as proved in [7, Theorem 3]. ■

Lemma 5 states the equality $\sigma(AR^{k-1}) = \sigma(R^{k-1}A)$. Now, the following question arises in a natural way: In which cases is $AR^{k-1} = R^{k-1}A$ valid? The next lemma gives a sufficient condition.

Lemma 8 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix such that for some $\ell \in \{2, \ldots, ks-1\}$, the conditions*

$$\omega_{ks}^{j} \notin \sigma(A, R) \quad \text{hold for all } j \in \{1, 2, \ldots, \ell-1, \ell+1, \ldots, ks\}.$$

Then A commutes with R^{k-1} and $AA^{\#} = A^{ks}$. Moreover,

$$A = \begin{cases} \omega_{ks}^{\ell} R & \text{if } A \text{ is nonsingular} \\ \omega_{ks}^{\ell} A^{(s+1)^k - 1} R & \text{if } A \text{ is singular.} \end{cases}$$

Finally, $A^{-1} = \omega_{ks}^{-\ell} R^{k-1}$ whenever A is nonsingular, and $A^{\#} = \omega_{ks}^{\ell} A^{ks} R$ whenever A is singular.

Proof. By hypothesis, it is clear that for all $j = 1, 2, \dots, \ell-1, \ell+1, \dots, ks$ one has $E_j(R^{k-1}A) = O$. Now, the spectral decomposition given in (2) for $M = R^{k-1}A$ yields $R^{k-1}A = \omega_{ks}^{\ell} E_\ell(R^{k-1}A)$ and $I_n = E_0(R^{k-1}A) + E_\ell(R^{k-1}A)$. Then, by Lemma 6 we get

$$E_\ell(R^{k-1}A) = I_n - E_0(R^{k-1}A) = A^{(s+1)^k-1} \qquad \text{and so} \qquad R^{k-1}A = \omega_{ks}^{\ell} A^{(s+1)^k-1}.$$

In order to compute AR^{k-1} we use the two following properties (see [7] Lemma 2): $RA^j = A^{j(s+1)}R$ for all $j \in \{1, 2, \dots, (s+1)^k-1\}$ and $(A^{(s+1)^k-1})^h = A^{(s+1)^k-1}$ for all $h \in \mathbb{N}$. We obtain

$$AR^{k-1} = R(R^{k-1}A)R^{k-1} = \omega_{ks}^{\ell} RA^{(s+1)^k-1}R^{k-1} = \omega_{ks}^{\ell} A^{(s+1)[(s+1)^k-1]} = \omega_{ks}^{\ell} A^{(s+1)^k-1}.$$

Thus, A and R^{k-1} commute, which is equivalent to $AR = RA$. Now, by definition, since A is $\{R, s+1, k\}$-potent, it follows that $A^{s+1} = A$. Since $A^{\#}A = A^{(s+1)^k-1}$, the equality $(s+1)^k - 1 = s[(s+1)^{k-1} + (s+1)^{k-2} + \cdots + (s+1) + 1]$ yields $AA^{\#} = A^{ks}$ because $A^{(s+1)^j} = A$ for any positive integer j. If A is singular, premultiplying by $(A^{\#})^2$ the equality $A = \omega_{ks}^{\ell} A^{(s+1)^k-1}R = \omega_{ks}^{\ell} AA^{\#}R$ we get $A^{\#} = \omega_{ks}^{\ell} A^{ks} R$. ■

Note that, in general, the converse is not valid as is shown by the following counterexample:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \qquad \text{and} \qquad R = I_2.$$

It is clear that A is an $\{R, 5, 2\}$-potent matrix, and that $AR = RA$, however, $\omega_8^2 = i \in \sigma(RA)$ and $\omega_8^8 = 1 \in \sigma(RA)$.

The next result provides a more in-depth answer to the question posed prior to the statement of the previous lemma.

Theorem 4 *Let $A \in \mathbb{C}^{n \times n}$ be an $\{R, s+1, k\}$-potent matrix. Then the following statements are equivalent.*

(a) A commutes with R^{k-1} (equivalently, $AR = RA$).

(b) A is an $\{s+1\}$-potent matrix.

(c) Projectors $E_j(AR^{k-1})$ and $E_j(R^{k-1}A)$ are equal for all $j = 0, 1, 2 \dots, ks$.

Proof. The equivalence of (a) and (b) follows from the definition of an $\{R, s+1, k\}$-potent matrix (see also proof of Lemma 8). It is left to prove the equivalence of (a) and (c). From Theorem 3 and Lemma 5, AR^{k-1} and $R^{k-1}A$ are diagonalizable matrices and $\sigma(AR^{k-1}) = \sigma(R^{k-1}A)$. Then, the spectral theorem gives

$$AR^{k-1} = \sum_{j=1}^{ks} \omega_{ks}^j E_j(AR^{k-1}) \qquad \text{and} \qquad R^{k-1}A = \sum_{j=1}^{ks} \omega_{ks}^j E_j(R^{k-1}A).$$

Now, from Lemma 6 we get $E_0(AR^{k-1}) = E_0(R^{k-1}A)$, and the uniqueness established in the spectral theorem leads to the result. The converse is evident. ■

For an $\{R, s+1, k\}$-potent matrix A, we have $\sigma(A) \subseteq \{0\} \cup \Omega_{(s+1)^k-1}$ (Theorem 2) and $\sigma(A, R) \subseteq \{0\} \cup \Omega_{ks}$ (Theorem 3). The next theorem provides another main result in this section. It gives a nice formula that links $\sigma(R^{k-1}A)$ and $\sigma(A)$. For a finite multiset S of complex numbers and a positive integer p, $[S]^p$ denotes the multiset consisting of the p^{th} powers of the elements of S.

Theorem 5 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Then*

$$[\sigma(A, R)]^k = [\sigma(A)]^{[(s+1)^k-1]/s}. \tag{3}$$

Proof. The result follows from Lemma 1, Lemma 3, and the property $\sigma(B^m) = [\sigma(B)]^m$ for any square matrix B and any positive integer m. ■

Notice that it is easy to see that $0 \in \sigma(R^{k-1}A)$ if and only if $0 \in \sigma(A)$.

We need the following notation used by Yasuda in [31]. We write $K = \pm S$ if the elements of the set K are the same as those of the set S up to sign.

Corollary 1 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, 2\}$-potent matrix. Then*

$$\sigma(RA) = \pm[\sigma(A)]^{\frac{s+2}{2}}. \tag{4}$$

Note that the formula (4) remains valid for $s = 0$, which corresponds to the case of $\{R\}$-centrosymmetric matrices where R is an involution, as was shown in [30, 31].

Recall that for a square matrix R, a vector x is called $\{R\}$-*symmetric* when $Rx = x$ and it is called $\{R\}$-*skew-symmetric* when $Rx = -x$. The study of $\{R\}$-symmetric and $\{R\}$-skew-symmetric vectors for (complex) centrosymmetric matrices was done by Weaver in [33], the corresponding study for generalized real symmetric (skew-)centrosymmetric matrices by done by Tao and Yasuda in [31], and the corresponding study for mirror-symmetric matrices was done by Li and Feng in [25]. In what follows, we further extend these studies using $\{R, s+1, k\}$-potent matrices (Theorem 6 of this section) and $\{R, s+1, k, *\}$-potent matrices (Theorem 8 of Section 3).

For a fixed matrix $R \in \mathbb{C}^{n\times n}$, we recall that a matrix $A \in \mathbb{C}^{n\times n}$ is $\{R\}$-*centrosymmetric* if $AR = RA$ [29, 30]. When $R = J_n$, the $n \times n$ *exchange matrix* (that is, all entries of J_n are zero except for the ones on the main antidiagonal), the matrix A is called *centrosymmetric*. This class is used frequently in the construction of orthonormal wavelet bases [15, 35].

In order to generalize the results in the studies mentioned above, our next objective is to use Theorem 5 to obtain new information for special classes of matrices known in the literature. The next example involves R-centrosymmetric matrices and additionally presents $\{R\}$-symmetric and $\{R\}$-skew-symmetric vectors for this class [31, 33].

Example 1 *We consider the matrices*

$$A = \begin{bmatrix} i & 0 & 0 \\ 0 & 5 & -2 \\ 0 & 15 & -6 \end{bmatrix} \qquad \text{and} \qquad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

where A is R-centrosymmetric. It is easy to see that $\sigma(A) = \{-1, 0, i\}$ and the corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 0 \\ 0.3162 \\ 0.9487 \end{bmatrix}, \qquad v_2 = \begin{bmatrix} 0 \\ 0.3714 \\ 0.9285 \end{bmatrix}, \qquad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Moreover, $\sigma(RA) = \{1, 0, i\}$ and the corresponding eigenvectors are

$$u_1 = \begin{bmatrix} 0 \\ 0.3162 \\ 0.9487 \end{bmatrix}, \qquad u_2 = \begin{bmatrix} 0 \\ 0.3714 \\ 0.9285 \end{bmatrix}, \qquad u_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

We have $Rv_1 = -v_1$, $Rv_2 = -v_2$ and $Rv_3 = v_3$. That is, two eigenvectors are $\{R\}$-skew-symmetric and one is $\{R\}$-symmetric. In this case, we have $\sigma(A) = \pm\sigma(RA)$.

The next result corresponds to the more general case of $\{R\}$-centrosymmetric matrices.

Corollary 2 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R\}$-centrosymmetric matrix. Then*

$$\sigma(RA) = \pm\sigma(A).$$

On the other hand, for the matrix

$$R = \begin{bmatrix} & & J_k \\ & I_p & \\ J_k & & \end{bmatrix}, \tag{5}$$

where J_k is the $k \times k$ exchange matrix, we call $A \in \mathbb{C}^{n\times n}$ *(k,p)-mirror-symmetric* if $AR = RA$ [25]. A particular important case is obtained in the next result.

Corollary 3 *Let $A \in \mathbb{C}^{n\times n}$ be a (k,p)-mirror-symmetric matrix. Let R be given by (5). Then*

$$\sigma(RA) = \pm\sigma(A).$$

Persymmetric matrices are those matrices $A \in \mathbb{R}^{n\times n}$ such that $J_nAJ_n = A^T$, where A^T denotes the transpose of A; that is, those matrices that are symmetric with respect to the northeast-to-southwest diagonal (see [11], p. 208). For persymmetric matrices, however, the conclusion in Corollary 5 fails, as the following example shows. Let A be given by

$$A = \begin{bmatrix} 1 & 7 & 4 \\ 3 & 5 & 7 \\ 4 & 3 & 1 \end{bmatrix}.$$

Then A is persymmetric but not centrosymmetric, and $\sigma(A) = \{11.8061, -2.4030+1.9915i, -2.4030-1.9915i\}$ but $\sigma(J_3A) = \{-2.7949, 3.2905, 12.5044\} \neq \pm\sigma(A)$. The conclusion in Corollary 5 also fails for centrohermitian matrices. For example,

$$A = \begin{bmatrix} 1+i & 2-i & -i \\ 2-i & 1 & 2+i \\ i & 2+i & 1-i \end{bmatrix}$$

is a centrohermitian matrix with $\sigma(A) = \{-0.4757 + 0.7300i, -0.4757 - 0.7300i, 3.9514\}$ but $\sigma(J_3A) = \{-3, 0.2679, 3.7321\} \neq \pm\sigma(A)$.

Results related to the eigenstructure of an $\{R, s+1, k\}$-potent matrix A are given in the following theorem.

Theorem 6 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix. Then the following statements hold.*

(a) (λ, x) *is an eigenvalue-eigenvector pair for A if and only if (λ, Rx) is an eigenvalue-eigenvector pair for A^{s+1}.*

(b) Assume that (λ, x) is an eigenvalue-eigenvector pair for A.

  (i) Then $(\lambda^{s+1}, R^{k-1}x)$ is an eigenvalue-eigenvector pair for A.

  (ii) If $\lambda^{s+1} = \lambda$ and $\dim(\mathrm{Ker}(\lambda I_n - A^{s+1})) = 1$ then $Rx = \omega_k^j x$ for some $\omega_k^j \in \Omega_k$.

  (iii) If $Rx = \omega_k^j x$ for some $\omega_k^j \in \Omega_k$ then $\lambda^{s+1} = \lambda$.

Proof. Suppose that there is $x \in \mathbb{C}^n$ such that $Ax = \lambda x$, with $x \neq 0$. Then

$$A^{s+1}Rx = RAx = \lambda Rx,$$

that is Rx is an eigenvector of A^{s+1} associated with λ. The converse is similar, and thus, (a) is shown.

(b) (i) If $Ax = \lambda x$ for some $x \neq 0$ then $A^{s+1}x = \lambda^{s+1}x$. So, $RA(R^{k-1}x) = \lambda^{s+1}x$, and this implies $A(R^{k-1}x) = \lambda^{s+1}(R^{k-1}x)$, that is $(\lambda^{s+1}, R^{k-1}x)$ is an eigenvalue-eigenvector pair for A.

(ii) Since $A^{s+1}x = \lambda^{s+1}x = \lambda x$, we have $x \in \mathrm{Ker}(\lambda I_n - A^{s+1})$. Now, by (a), we also have $Rx \in \mathrm{Ker}(\lambda I_n - A^{s+1})$. As $\dim(\mathrm{Ker}(\lambda I_n - A^{s+1})) = 1$, it follows that $Rx = \mu x$ for some nonzero scalar μ. As the eigenvalues of R are in Ω_k then $\mu = \omega_k^j$ for some $j \in \{1, \ldots, k\}$.

(iii) From $Ax = \lambda x$, $x \neq 0$ we have $A^{s+1}x = \lambda^{s+1}x$. Moreover, from $Rx = \omega_k^j x$ we get

$$\omega_k^j A^{s+1}x = A^{s+1}Rx = RAx = \lambda Rx = \omega_k^j \lambda x.$$

That is, $A^{s+1}x = \lambda x$. Hence, from $x \neq 0$ we obtain $\lambda^{s+1} = \lambda$. This completes the proof. ∎

3 Spectral study of $\{R, s+1, k, *\}$-potent matrices

In [7] we have shown that $A^{\#}$ is a power of A when A is an $\{R, s+1, k\}$-potent matrix. Is this property also true for $\{R, s+1, k, *\}$-potent matrices? Consider the following example. The matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

is $\{R, 4, 4, *\}$-potent for

$$R = \begin{bmatrix} 1 & 2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & i \end{bmatrix}.$$

It is easy to check that

$$A^{\#} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad A^m = \begin{bmatrix} 1 & m & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ for all positive integer } m.$$

Clearly, $A^{\#}$ is not a power of A. That is, in general $\{R, s+1, k, *\}$-potent matrices are not $\{m\}$-group involutory for any $m \in \mathbb{N}$. This illustrates a big difference between the classes of $\{R, s+1, k\}$-potent and $\{R, s+1, k, *\}$-potent matrices.

The next result shows that it is not always true that an $\{R, s+1, k, *\}$-potent matrix A is normal, and characterizes the case when A is normal. Moreover, it is shown that $I - AA^{\#}$ belongs to the class of $\{R, s+1, k, *\}$-potent matrices (see [8]).

Lemma 9 *Let $A \in \mathbb{C}^{n \times n}$ be an $\{R, s+1, k, *\}$-potent matrix. Then*

(a) A is normal if and only if A^{s+1} commutes with RAR^{-1}.

(b) $I - AA^{\#}$ is an $\{R, s+1, k, *\}$-potent matrix.

Proof. (a) From definition, $A^{s+1}R = RA^*$, and then $A^* = R^{-1}A^{s+1}R$. Thus,

$$AA^* = A^*A \quad \Leftrightarrow \quad AR^{-1}A^{s+1}R = R^{-1}A^{s+1}RA \quad \Leftrightarrow \quad RAR^{-1}A^{s+1} = A^{s+1}RAR^{-1}.$$

(b) Since $\{R, s+1, k, *\}$-potent matrices have index at most 1, [8, Theorem 8] ensures that there exist nonsingular matrices P and C such that

$$A = P \begin{bmatrix} C & O \\ O & O \end{bmatrix} P^{-1}, \qquad R = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^*$$

for some X and T. It is clear that

$$A^{\#} = P \begin{bmatrix} C^{-1} & O \\ O & O \end{bmatrix} P^{-1}, \qquad I - AA^{\#} = P \begin{bmatrix} O & O \\ O & I \end{bmatrix} P^{-1}.$$

Then

$$(I - AA^{\#})^{s+1}R = P \begin{bmatrix} O & O \\ O & I \end{bmatrix} P^{-1}P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^* = P \begin{bmatrix} O & O \\ O & T \end{bmatrix} P^*$$

and

$$R(I - AA^{\#})^* = P \begin{bmatrix} X & O \\ O & T \end{bmatrix} P^*(P^{-1})^* \begin{bmatrix} O & O \\ O & I \end{bmatrix} P^* = P \begin{bmatrix} O & O \\ O & T \end{bmatrix} P^*.$$

Hence, $I - AA^{\#}$ is an $\{R, s+1, k, *\}$-potent matrix. ■

For the class of $\{R, s+1, k, *\}$-potent matrices, we consider the pencil (A^*, R) and provide results analogous to those in the previous section. In fact, $\lambda \in \sigma(A^*, R)$ if and only if there exists a nonzero vector $x \in \mathbb{C}^{n\times 1}$ such that $A^*x = \lambda Rx$. This is equivalent to the existence of a nonzero vector $x \in \mathbb{C}^{n\times 1}$ such that $R^{k-1}A^*x = \lambda x$.

We can compute the spectrum of the pencil $\sigma(A^*, R)$ by means of the spectrum of a (single) matrix as described in the following result.

Lemma 10 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k, *\}$-potent matrix. Then $\sigma(A^*, R) = \sigma(R^{k-1}A^*)$.*

The next result establishes that every power of an $\{R, s+1, k, *\}$-potent matrix is also an $\{R, s+1, k, *\}$-potent matrix.

Lemma 11 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k, *\}$-potent matrix. Then*

$$(A^*)^j R^{k-1} = R^{k-1}(A^{s+1})^j, \qquad \forall j \in \mathbb{N}.$$

Proof. We proceed by induction on j. The definition of an $\{R, s+1, k, *\}$-potent matrix states that the property holds for $j = 1$. Assume that $(A^*)^j R^{k-1} = R^{k-1}(A^{s+1})^j$ for some positive integer $j \geq 1$. Then,

$$(A^*)^{j+1}R^{k-1} = A^*[(A^*)^j R^{k-1}] = (A^*R^{k-1})(A^{s+1})^j = R^{k-1}A^{s+1}(A^{s+1})^j = R^{k-1}(A^{s+1})^{j+1}.$$

■

A relation like (1) does not hold for $\{R, s+1, k, *\}$-potent matrices. In fact, for small powers we have:

$$(R^{k-1}A^*)^2 = R^{k-1}A^*R^{k-1}A^* = R^{k-1}R^{k-1}A^{s+1}A^* = (R^{k-1})^2A^{s+1}A^*,$$

and

$$(R^{k-1}A^*)^3 = (R^{k-1}A^*)^2R^{k-1}A^* = (R^{k-1})^2A^{s+1}A^*R^{k-1}A^* = (R^{k-1})^2A^{s+1}R^{k-1}A^{s+1}A^*,$$

and, in general, it seems that a relation similar to that in Lemma 4, valid for $\{R, s+1, k\}$-potent matrices, does not hold for $\{R, s+1, k, *\}$-potent matrices.

The following two theorems, which are the main results of this section, allow us to analyze the eigenstructure of $\{R, s+1, k, *\}$-potent matrices.

Theorem 7 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k, *\}$-potent matrix.*

(a) $[\sigma(A)]^{s+1} = \sigma(\overline{A})$. In particular, if $\lambda \in \sigma(A)$, then $\lambda^{s+1} \in \sigma\left(\overline{A}\right)$.

(b) If $s = 0$, then all nonreal eigenvalues for A occur in conjugate pairs, and there is no restriction on the magnitudes of the eigenvalues.

(c) If $s \geq 1$, then for each nonzero eigenvalue $\lambda \in \sigma(A)$, there is a positive integer r with $1 \leq r \leq n$ such that λ is an h-root of unity where $h = (s+1)^r + (-1)^{r+1}$. Further, when λ is a primitive h-root of unity, then at least r of the eigenvalues of A are h-roots of unity.

Proof. From $RA^*R^{k-1} = A^{s+1}$, we have

$$[\sigma(A)]^{s+1} = \sigma(A^{s+1}) = \sigma(RA^*R^{k-1}) = \sigma(A^*) = \sigma(\overline{A}).$$

(a) Suppose $\lambda \in \sigma(A)$. Since $\sigma(A^{s+1}) = \sigma\left(\overline{A}\right)$, $\lambda^{s+1} = \overline{\mu}$ for some $\mu \in \sigma(A)$.

(b) Since $s = 0$, $\sigma(A) = \sigma\left(\overline{A}\right)$. This only implies that any nonreal eigenvalues of A must occur in conjugate pairs.

(c) Since $s \geq 1$, $\sigma(A^{s+1}) = \sigma\left(\overline{A}\right)$. If $\lambda \in \sigma(A)$, then $\lambda^{s+1} = \overline{\mu}$ for some $\mu \in \sigma(A)$, and hence, $|\lambda|^{s+1} = |\mu|$. Now choose $\widehat{\lambda} \in \sigma(A)$ of maximum modulus. Assume $\left|\widehat{\lambda}\right| > 1$. Then A has an eigenvalue with modulus $\left|\widehat{\lambda}\right|^{s+1} > \left|\widehat{\lambda}\right|$, a contradiction. If $\widehat{\lambda} \in \sigma(A) - \{0\}$ is chosen to be of minimum modulus, then $\left|\widehat{\lambda}\right| < 1$ leads to a similar contradiction. Thus, $|\lambda| = 1$ for every nonzero eigenvalue of A. If $\lambda^{s+1} = \overline{\lambda}$, then expressing $\lambda = e^{i\theta}$ for some real θ, we see that $(s+1)\theta \equiv -\theta \bmod 2\pi$, so that λ is an $(s+2)$-root of unity. Next, suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct nonzero eigenvalues of A such that $\lambda_1^{s+1} = \overline{\lambda_2}$, $\lambda_2^{s+1} = \overline{\lambda_3}$, $\ldots$, and $\lambda_r^{s+1} = \overline{\lambda_1}$. Then $r \leq n$. When r is odd, $(\lambda_j)^{(s+1)^r} = \overline{\lambda_j}$ for $1 \leq j \leq r$. Expressing $\lambda_j = e^{i\theta_j}$ for some real θ_j, we see that $(s+1)^r\theta_j \equiv -\theta_j \bmod 2\pi$, so that λ_j is an $((s+1)^r+1)$-root of unity for $1 \leq j \leq r$. When r is even, $(\lambda_j)^{(s+1)^r} = \lambda_j$ for $1 \leq j \leq r$. Expressing $\lambda_j = e^{i\theta_j}$ for some real θ_j, we see that $(s+1)^r\theta_j \equiv \theta_j \bmod 2\pi$, so that λ_j is an $((s+1)^r - 1)$-root of unity for $1 \leq j \leq r$. Thus, each nonzero eigenvalue of A is an h-root of unity where $h = (s+1)^r \pm 1$ for some positive integer $r \leq n$. ■

Example 2 *Let $s = 0$. Let $R = J_{11}$, the matrix obtained from I_{11} by reversing the order of the rows. Let A be the 11×11 diagonal matrix given by*

$$A = diag\left(3+5i, 3+5i, i, 0, \pi^\pi, 1, \pi^\pi, 0, -i, 3-5i, 3-5i\right).$$

Then $\sigma(A) = \{0, 0, 1, \pi^\pi, \pi^\pi, i, -i, 3+5i, 3+5i, 3-5i, 3-5i\}$, and $\sigma(A^{s+1}) = \sigma\left(\overline{A}\right)$. Further, A is an $\{R, 1, 2, *\}$-potent matrix since $R^{-1}AR = A^*$. If $\lambda \in \sigma(A)$, then $\lambda^{s+1} = \overline{\lambda}$ only when $\lambda \in \{0, 1, \pi^\pi\}$. Finally, A has nonzero eigenvalues that are not roots of unity.

Example 3 *Let $s = 1$. Let $r = 3$, and let $h = (s+1)^r + 1 = 9$. Let $\alpha = e^{\frac{2\pi i}{h}} = e^{\frac{2\pi i}{9}}$. Let $\beta = e^{\frac{-4\pi i}{9}}$ so $\overline{\beta} = \alpha^{s+1}$. Let $\gamma = e^{\frac{8\pi i}{9}}$ so $\overline{\gamma} = \beta^{s+1}$.*

Then $\gamma^{s+1} = e^{\frac{-2\pi i}{9}} = \overline{\alpha}$*. Let* $r = 2$*, and let* $h = (s+1)^r - 1 = 3$*. Let* $\delta = e^{\frac{2\pi i}{h}} = e^{\frac{2\pi i}{3}}$*. Let* $\varepsilon = e^{\frac{-4\pi i}{3}}$ *so* $\overline{\varepsilon} = \delta^{s+1}$*. Then* $\varepsilon^{s+1} = e^{\frac{-2\pi i}{3}} = \overline{\delta}$*. Let* A *be the* 12×12 *diagonal matrix given by*

$$A = diag\,(0, 0, 0, 1, \alpha, \beta, \gamma, \alpha, \beta, \gamma, \delta, \varepsilon)\,.$$

Then $\sigma(A) = \{0, 0, 0, 1, \alpha, \alpha, \beta, \beta, \gamma, \gamma, \delta, \varepsilon\}$ *and* $\sigma\left(A^{s+1}\right) = \sigma\left(\overline{A}\right)$*. The nonzero eigenvalues of* A *are either* 3*-roots of unity or* 9*-roots of unity. In particular, if* $R = I_4 \oplus P \oplus P \oplus I_2$ *where*

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix},$$

then $R^{-1}A^2R = A^*$*, so* A *is* $\{R, 2, 3, *\}$*-potent. For* $\lambda \in \{\alpha, \beta, \gamma\}$*,* $\lambda^{s+1} \neq \overline{\lambda}$*, while for* $\lambda \in \{0, 1, \delta, \varepsilon\}$*,* $\lambda^{s+1} = \overline{\lambda}$*.*

Notice that the previous two examples contradict Lemma 7(d) in [8], which incorrectly asserted that when $\sigma\left(A^{s+1}\right) = \sigma\left(\overline{A}\right)$, $\lambda^{s+1} = \overline{\lambda}$ for every $\lambda \in \sigma(A)$. Theorem 7 is the correct spectral characterization for $\{R, s+1, k, *\}$-matrices.

The next example, implied by Example 11 from [8], shows that for each $s \geq 0$, there is an $\{R, s+1, 2, *\}$-potent matrix that is not diagonalizable.

Example 4 *Let* s *be a nonnegative integer. Let* $a \in \mathbb{R}$*. Let* R *and* A *be given by*

$$R = \begin{bmatrix} 0 & \sqrt{s+1} \\ \frac{1}{\sqrt{s+1}} & 0 \end{bmatrix} \text{ and } A = \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix}.$$

Then $R^2 = I_2$ *for all* $s \geq 0$*. When* $s = 0$ *and* $a \in \mathbb{R}$*,* $R^{-1}AR = A^*$*, so* A *is an* $\{R, 1, 2, *\}$*-matrix that is not diagonalizable. When* $s \geq 1$ *and* $a = 1$*,* $R^{-1}A^{s+1}R = A^*$*, so* A *is an* $\{R, s+1, 2, *\}$*-matrix that is not diagonalizable.*

Theorem 8 *Let* $A \in \mathbb{C}^{n \times n}$ *be an* $\{R, s+1, k, *\}$*-potent matrix. Then the following statements hold:*

(a) If (λ, x) *is an eigenvalue-eigenvector pair for* A*, then* $(\lambda^{s+1}, R^{k-1}x)$ *is an eigenvalue-eigenvector pair for* A^**.*

(b) If (λ, Rx) *is an eigenvalue-eigenvector pair for* A*, then* (λ^{s+1}, x) *is an eigenvalue-eigenvector pair for* A^**.*

(c) If (τ, x) is an eigenvalue-eigenvector pair for A^*, then (τ, Rx) is an eigenvalue-eigenvector pair for A^{s+1}.

(d) If $(\tau, R^{k-1}x)$ is an eigenvalue-eigenvector pair for A^*, then (τ, x) is an eigenvalue-eigenvector pair for A^{s+1}.

(e) Suppose $s = 0$. Then, (λ, x) is an eigenvalue-eigenvector pair for A if and only if $(\lambda, R^{k-1}x)$ is an eigenvalue-eigenvector pair for A^*. Also (λ, Rx) is an eigenvalue-eigenvector pair for A if and only if (λ, x) is an eigenvalue-eigenvector pair for A^*.

Proof. (a) If $Ax = \lambda x$ for some $x \neq 0$ then $A^{s+1}x = \lambda^{s+1}x$. Thus, from $A^{s+1}R = RA^*$ we have

$$A^{s+1}RR^{-1}x = \lambda^{s+1}x \Rightarrow RA^*(R^{-1}x) = \lambda^{s+1}x \Rightarrow A^*(R^{k-1}x) = \lambda^{s+1}R^{k-1}x.$$

To obtain (b), replace x with Rx everywhere in (a).

(c) Suppose that there is a nonzero $x \in \mathbb{C}^n$ such that $A^*x = \tau x$. Then $RA^*x = \tau Rx$. Since A is $\{R, s+1, k, *\}$-potent,

$$A^{s+1}Rx = \tau Rx.$$

That is, Rx is an eigenvector of A^{s+1} associated with τ.

To obtain (d),replace x with $R^{k-1}x$ everywhere in (c). Finally, the first statement in (e) follows from (a) and (d), and the second follows from (b) and (c). ■

4 Applications and concluding discussions

The family of matrices known as the *quaternions* is important in the study of quantum physics. Recall that the quaternions form a four-dimensional vector space over $\mathbb{R}$ with an ordered basis, $\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}$, with defining relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -\mathbf{1}$, $\mathbf{ij} = \mathbf{k} = -\mathbf{ji}$, $\mathbf{jk} = \mathbf{i} = -\mathbf{kj}$, and $\mathbf{ki} = \mathbf{j} = -\mathbf{ik}$. Alternatively, the quaternions can be viewed as a subset of the ring of 2×2 matrices with complex entries (see [34]):

$$\left\{ \begin{bmatrix} a_0 & a_1 \\ -\bar{a}_1 & \bar{a}_0 \end{bmatrix} \mid a_0, a_1 \in \mathbb{C} \right\}.$$

In this case,

$$\mathbf{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{i} = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad \mathbf{j} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{k} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

Observe that

$$\mathbf{ici}^* = \mathbf{c}, \quad \mathbf{jcj}^* = \bar{\mathbf{c}} = \mathbf{c}^*, \quad \text{and } \mathbf{kck}^* = \bar{\mathbf{c}} = \mathbf{c}^*$$

for any complex number $\mathbf{c}$ (viewed as a quaternion) [34]. Thus the relations satisfied by the classes of matrices that we study in this paper generalize the relations given above.

Recall that the *Pauli spin matrices*, σ_x, σ_y, and σ_z, named after the Nobel Prize physicist Wolfgang Pauli, describe the interaction of a (spin $\frac{1}{2}$) particle with an external electromagnetic field, and are defined as follows (see [10]):

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Observe that the quaternions $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ can be written in terms of the Pauli matrices: $\mathbf{i} = i\sigma_z$, $\mathbf{j} = i\sigma_y$, $\mathbf{k} = i\sigma_x$, and that every quaternion can be represented as a matrix $M \in \mathbb{C}^{2\times 2}$ satisfying $\sigma_y M \sigma_y^* = \bar{M}$.

Pauli matrices appear prominently in the study of quantum computation and quantum information, which deal with information processing tasks that can be accomplished using quantum mechanical systems. A fundamental quantum mechanical system is described by a *qubit*, and the evolution of a qubit is described by unitary operators. The Pauli matrices are important examples of such operators; see [27] for reference and further details.

For a quantum system with a two-state space, the operators are described by 2×2 Hermitian matrices, and one of the reasons why the Pauli matrices are so useful in quantum physics is that any 2×2 Hermitian matrix can be written in terms of them [10]. More precisely, the set of Pauli matrices together with the 2×2 identity matrix form a basis for the real vector space of 2×2 Hermitian matrices.

Motivated by these applications, we next give examples of Hermitian $\{R, s{+}1, k\}$-potent and $\{R, s{+}1, k, *\}$-potent matrices and their relationships to the Pauli matrices.

Example 5 *For* $R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$*, the following are* $\{R, 2, 4\}$*-potent and* $\{R, 2, 4, *\}$*-potent matrices:*

$$\frac{1}{2}\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} = \frac{1}{2}I_2 - \frac{1}{2}\sigma_y$$

$$\frac{1}{2}\begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} = \frac{1}{2}I_2 + \frac{1}{2}\sigma_y,$$

and the following are $\{R, 3, 4\}$*-potent and* $\{R, 3, 4, *\}$*-potent matrices:*

$$\frac{1}{2}\begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} = -\frac{1}{2}I_2 - \frac{1}{2}\sigma_y$$

$$\frac{1}{2}\begin{bmatrix} -1 & -i \\ i & -1 \end{bmatrix} = -\frac{1}{2}I_2 + \frac{1}{2}\sigma_y.$$

The next example shows that both classes of matrices have infinitely many elements.

Example 6 *Using the same matrix* R *as in Example 5, the following matrices are* $\{R, 3, 4\}$*-potent:*

$$\begin{bmatrix} -i\sqrt{1+x^2} & x \\ x & i\sqrt{1+x^2} \end{bmatrix}, \qquad \text{for } x \in \mathbb{R},$$

and the following matrices are $\{R, 3, 4, *\}$*-potent:*

$$-\frac{1}{2}\begin{bmatrix} 1+i & -2y \\ \frac{i}{y} & 1+i \end{bmatrix},$$

for $y \in \mathbb{C}, y \neq 0$ *such that* $Im(y) = Re(y)$*.*

Below are some larger dimensional examples.

Example 7 *For*

$$R = \begin{bmatrix} 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & 0 & -\frac{1}{2}-\frac{i}{2} & 0 \\ 0 & \frac{1}{2}-\frac{i}{2} & 0 & \frac{1}{2}-\frac{i}{2} \\ -\frac{i}{\sqrt{2}} & 0 & -\frac{1}{2}-\frac{i}{2} & 0 \end{bmatrix},$$

the following are $\{R,3,4\}$-potent and $\{R,3,4,*\}$-potent matrices, respectively:

$$\begin{bmatrix} \frac{1}{2}\left(1-i\sqrt{x^2+1}\right) & 0 & -\frac{(1+i)\left(\sqrt{x^2+1}-i\right)}{2\sqrt{2}} & \frac{ix}{\sqrt{2}} \\ 0 & 1 & 0 & 0 \\ \frac{(1-i)\left(\sqrt{x^2+1}-i\right)}{2\sqrt{2}} & 0 & \frac{1}{2}\left(1-i\sqrt{x^2+1}\right) & \frac{1}{2}(-1+i)x \\ -\frac{ix}{\sqrt{2}} & 0 & -\frac{1}{2}(1+i)x & i\sqrt{x^2+1} \end{bmatrix}, \quad \textit{for } x \in \mathbb{C};$$

$$\begin{bmatrix} -\frac{1-i}{2\sqrt{2}} & 0 & -\frac{i}{2} & \frac{1}{2}(-1+i) \\ 0 & \frac{1-i}{\sqrt{2}} & 0 & 0 \\ \frac{1}{2} & 0 & -\frac{1-i}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2}(1-i) & 0 & -\frac{i}{\sqrt{2}} & 0 \end{bmatrix}.$$

We conclude by examining two questions that may be useful in a future study:

(a) Can we construct nonsingular $\{W,s+1,k\}$-potent matrices of smaller size from a given $\{R,s+1,k\}$-potent matrix?

(b) What special properties can be derived if both matrices A and $I-A$ are $\{R,s+1,k\}$-potent?

Motivated by the fact that the nonzero eigenvalues of an $\{R,s+1,k\}$-potent matrix have modulus 1, we consider a matrix $T \in \mathbb{C}^{n\times n}$ written as

$$T = S \begin{bmatrix} I & O \\ O & K \end{bmatrix} S^{-1}, \tag{6}$$

for some nonsingular matrix S and satisfying that $1 \notin \sigma(K)$. Note that the diagonal identity matrix block is vacuous if $1 \notin \sigma(T)$.

Proposition 1 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R,s+1,k\}$-potent matrix, and let $T = I - A$. Then:*

(a) $RTR^{-1} = I - (I-T)^{s+1}$.

(b) T can be written as in (7)*, and the diagonal block I is nonvacuous exactly when A is singular.*

(c) There exists a matrix W such that $T_K := I - K$ is a nonsingular $\{W, s+1, k\}$-potent matrix.

Proof. Since A is an $\{R, s+1, k\}$-matrix, $RTR^{-1} = I - RAR^{-1} = I - A^{s+1} = I - (I-T)^{s+1}$, so (a) holds. Since A is diagonalizable, (b) holds.

(c) By part (a) and (6) we have,

$$RS \begin{bmatrix} I & O \\ O & K \end{bmatrix} S^{-1}R^{-1} = S \begin{bmatrix} I & O \\ O & I-(I-K)^{s+1} \end{bmatrix} S^{-1}. \qquad (7)$$

We partition $S^{-1}RS$ conformally with T:

$$S^{-1}RS = \begin{bmatrix} X & Y \\ Z & W \end{bmatrix}.$$

Now, (7) yields

$$\begin{bmatrix} X & YK \\ Z & WK \end{bmatrix} = \begin{bmatrix} X & Y \\ (I-(I-K)^{s+1})Z & (I-(I-K)^{s+1})W \end{bmatrix}.$$

Since $I-K$ is nonsingular, $Y = O$ and $Z = O$. On the other hand, from $R^k = I$ we get $X^k = I$ and $W^k = I$. The equality $WK = (I-(I-K)^{s+1})W$ becomes $WKW^{-1} = I-(I-K)^{s+1}$, and then it is easy to see that $T_K := I-K$ is $\{W, s+1, k\}$-potent. ■

Proposition 2 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k\}$-potent matrix, and let $T = I - A$ be written as in (6) and let $\omega := e^{i\pi/3}$. If $T \in \mathbb{C}^{n\times n}$ is also $\{R, s+1, k\}$-potent then it follows that*

(a) $\sigma(A) - \{0,1\} \subseteq \{\omega, \bar{\omega}\}$ and $\sigma(T) - \{0,1\} \subseteq \{\omega, \bar{\omega}\}$.

(b) If $\sigma(A) \cap \{\omega, \bar{\omega}\} \neq \emptyset$ then $(s+1)^k - 1$ is a multiple of 6. Consequently, $\gcd(s+1, 6) = 1$.

(c) $I + K + K^2 + \cdots + K^s = (I-K)^s$.

Proof. (a) Since A and T are $\{R, s+1, k\}$-potent matrices, $\sigma(A), \sigma(T) \subseteq \{0\} \cup \Omega_{n_{s,k}}$. Thus, their nonzero eigenvalues have modulus 1, that is, $|\lambda| = 1 = |1-\lambda|$ for every $\lambda \in \sigma(A)$. It is easy to see that the only complex numbers λ satisfying these equalities are $\lambda \in \{\omega, \bar{\omega}\}$, where $\omega = e^{i\pi/3}$.

(b) It is clear that ω is a primitive root of unity of order 6. Assuming that $\sigma(A) \cap \{\omega, \bar{\omega}\} \neq \emptyset$ and recalling that $n_{s,k} := (s+1)^k - 1$, it is known that $\omega^{n_{s,k}} = 1$. It then follows that $n_{s,k}$ is a multiple of 6. Now, $(s+1)^k - 1 = 6t$, for some $t \in \mathbb{N}$, from which 1 is an integer linear combination of 6 and $s+1$.

(c) From Proposition 1 (a) we have

$$S \begin{bmatrix} I & O \\ O & K^{s+1} \end{bmatrix} S^{-1} = T^{s+1} = RTR^{-1} = S \begin{bmatrix} I & O \\ O & I-(I-K)^{s+1} \end{bmatrix} S^{-1}.$$

Equating, we get $K^{s+1} = I - (I-K)^{s+1}$, from which $(I-K)^{s+1} = I - K^{s+1} = (I-K)(I+K+K^2+\cdots+K^s)$. Since $I-K$ is nonsingular, $(I-K)^s = I + K + K^2 + \cdots + K^s$ holds. ■

Next, we find a result for $\{R, s+1, k, *\}$-matrices that is similar to Proposition 1. We do not expect to find a result for $\{R, s+1, k, *\}$-potent matrices that is similar to Proposition 2 since, in general, $\{R, s+1, k, *\}$-potent matrices are not diagonalizable. Making the obvious modifications in the proof of parts (a) and (c) of Proposition 1, we obtain

Proposition 3 *Let $A \in \mathbb{C}^{n\times n}$ be an $\{R, s+1, k, *\}$-potent matrix. Let $T = I - A$. Then:*

(a) $RT^*R^{-1} = I - (I-T)^{s+1}$.

(b) Suppose that T can be written as in (6) where S is unitary. Then the diagonal block I in (6) is nonvacuous exactly when A is singular. Further, there exists a matrix W such that $T_K := I - K$ is a nonsingular $\{W, s+1, k, *\}$-potent matrix.

To conclude, we would like to focus on our recent understanding of $\{R, s+1, k\}$-potent and $\{R, s+1, k, *\}$-potent matrices and future applications. At this moment, we have knowledge not only about the algebraic structures and eigenstructures of these matrices, but also about numerical constructions and real applications, as described in our algorithms for image blurring/deblurring; see, e.g., [7, 8, 19, 20, 21, 22, 23, 24]. This understanding leads us to think that there are further relationships between these classes of matrices and generalized Pauli matrices. Pauli matrices are constructed for particles of different spin values, giving rise to matrices of different sizes; they are also a basis for the Lie group $SU(2)$ (also related to Clifford algebras) which are widely used in quantum mechanics and quantum computing. We propose to relate these matrices, as well as quaternions, to $\{R, s+1, k\}$-potent and $\{R, s+1, k, *\}$-potent matrices to explain some symmetry properties that our approaches provide. A more in-depth study must be done in order to determine these links and relationships to the mentioned algebraic structures.

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References

[1] A. Ben-Israel, T. Greville, Generalized Inverses: Theory and Applications, John Wiley & Sons, 2nd Edition, 2003.

[2] O. Baksalary, G. Trenkler, *On k-potent matrices*. Electronic Journal of Linear Algebra, 26, 446–470, 2013.

[3] J. Benítez, N. Thome, *Characterizations and linear combinations of k-generalized projectors*. Linear Algebra and its Applications, 410, 150–159, 2005.

[4] J. Benítez, N. Thome, *$\{k\}$-group periodic matrices*. SIAM J. Matrix Anal. Appl. 28, 1, 9–25, 2006.

[5] R. Bru, N. Thome, *Group inverse and group involutory matrices*, Linear and Multilinear Algebra, 45 (2-3), 207–218, 1998.

[6] S.L. Campbell, C.D. Meyer Jr., Generalized Inverses of Linear Transformations, SIAM, Philadelphia, 2009.

[7] M. Catral, L. Lebtahi, J. Stuart, N. Thome, *On a matrix group constructed from an $\{R, s+1, k\}$-potent matrix*, Linear Algebra and its Applications, 461, 200–210, 2014.

[8] M. Catral, L. Lebtahi, J. Stuart, N. Thome, *Matrices A such that $A^{s+1}R = RA^*$ with $R^k = I$*, Linear Algebra and its Applications, 552, 85–104, 2018.

[9] H.K. Du, Y. Li, *The spectral characterization of generalized projections*, Linear Algebra and its Applications, 400, 313–318, 2005.

[10] R. Feynman. The Feynman Lectures on Physics, Volume 3, Addison Wesley, 1963.

[11] G.H. Golub, C.F. Van Loan, Matrix Computations, 4th edition, The Johns Hopkins University Press, Baltimore, MD, 2013.

[12] J. Groß, G. Trenkler. *Generalized and hypergeneralized projections*, Electronic Journal of Linear Algebra, 26, 446–470, 2013.

[13] R.A. Horn, C.R. Johnson, Matrix Analysis, Cambridge University Press, Cambridge, 1985.

[14] D. Ilišević, N. Thome, *When is the Hermitian/skew-Hermitian part of a matrix a potent matrix?*, Electronic Journal of Linear Algebra, 24, 95–112, 2012.

[15] J. Kautsky, R. Turcajová, *A matrix approach to discrete wavelets*, in Wavelets: Theory, Algorithms, and Applications, C.K. Chui, L. Montefusco, L. Puccio (eds.), 117–135, Academic Press, 1994.

[16] J.G. Kemeny, J.L. Snell, Finite Markov Chains, Van Nostrand, Princeton, 1960.

[17] S.J. Kirkland, M. Neumann, Group inverses of M-Matrices and their applications, CRC Press, New York, 2013.

[18] L. Lebtahi, N. Thome, *A note on k-generalized projections*, Linear Algebra and its Applications, 420, 2-3, 572–575, 2007.

[19] L. Lebtahi, N. Thome, *Properties of a matrix group associated to a $\{K, s+1\}$-potent matrix*, Electronic Journal of Linear Algebra, 24, 34–44, 2012.

[20] L. Lebtahi, O. Romero, N. Thome, *Characterizations of* $\{K, s+1\}$*-potent matrices and applications*, Linear Algebra and its Applications, 436, 293–306, 2012.

[21] L. Lebtahi, O. Romero, N. Thome, *Algorithms for* $\{K, s+1\}$ *-potent matrix constructions*, Journal of Computational and Applied Mathematics, 249, 157–162, 2013.

[22] L. Lebtahi, O. Romero, N. Thome, *Algorithms for solving the inverse problem associated with* $KAK = A^{s+1}$, Journal of Computational and Applied Mathematics, 309, 333–341, 2017.

[23] L. Lebtahi, O. Romero, N. Thome, *Generalized centro-invertible matrices with applications*, Applied Mathematics Letters, 38, 106–109, 2014.

[24] L. Lebtahi, J. Stuart, N. Thome, J.R. Weaver, *Matrices* A *such that* $RA = A^{s+1}R$ *when* $R^k = I$, Linear Algebra and its Applications, 439, 1017–1023, 2013.

[25] G.L. Li, Z.H. Feng. *Mirror-transformations of matrices and their application on odd/even modal decomposition of mirror-symmetric multiconductor transmission line equations.* IEEE Transactions on Advanced Packaging, 26, 2, 172–181, 2003.

[26] S.K. Mitra, P. Bhimasankaram, S. Malik. Matrix Partial Orders, Shorted Operators and Applications. World Scientific, New Jersey, 2010.

[27] M.A. Nielsen, I.L. Chuang, Quantum Computation and Quantum Information, Cambridge University Press, 2010.

[28] S. Radosavljević, D. Djordjević. *On pairs of generalized and hypergeneralized projections in a Hilbert space*, Functional Analysis, Approximation, and Computation, 5, 2, 76–75, 2013.

[29] M. Yasuda, *Spectral characterizations for hermitian centro-symmetric* K*-matrices and the hermitian skew-centrosymmetric* K*-matrices*, SIAM J. Matrix Analysis Applications, 25, 3, 601–605, 2003.

[30] M. Yasuda, *Some properties of commuting and anti-commuting m-involutions*, Acta Math. Sci. 32(2), 631–644, 2012.

[31] D. Tao, M. Yasuda, *A spectral characterization of generalized real symmetric centrosymmetric and generalized real symmetric skew-centrosymmetric matrices*, SIAM J. Matrix Analysis Applications, 23, 3, 885–895, 2002.

[32] M. Tosic, D.S. Cvetković-Ilić, C. Deng. *The Moore-Penrose inverse of a linear combination of commuting generalized and hypergeneralized projectors*, Electronic Journal of Linear Algebra, 22, 1129–1137, 2011.

[33] J. Weaver. *Centrosymmetric (cross-symmetric) matrices, their basic properties, eigenvalues and eigenvectors*, American Monthly, 2, 10, 711–717, 1985.

[34] F. Zhang. *Quaternions and Matrices of Quaternions*, Linear Algebra and its Applications, 251, 21–57, 1997.

[35] L. Zhong-Yun. *Some properties of centrosymmetric matrices and its applications*, Numerical Mathematics, 14, 2, 136–148, 2005.