Quantum local testability
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Symmetry, Phases of Matter, and Resources in Quantum Computing

We want:

a local Hamiltonian such that

- with degenerate ground space (quantum code)
- the energy of an error scales linearly with the size of the error

The current research on quantum error correction

mostly concerned with the goal of building a (large) quantum computer

desire for realistic constructions

- LDPC codes: the generators of the stabilizer group act on a small number of qubits
- spatial/geometrical locality: qubits on a 2D/3D lattice
- main contenders: surface codes, or 3D variants

A fairly reasonable and promising approach

- good performance for topological codes: efficient decoders, high threshold
- overhead still quite large for fault-tolerance (magic state distillation) but the numbers are improving regularly

Is this it?

Better quantum LDPC codes?

from a math/coding point of view, topological codes in 2D-3D are not that good

- 2D toric code $[\![n, k = O(1), d = O(\sqrt{n})]\!]$
- topological codes on 2D Euclidean manifold (Bravyi, Poulin, Terhal 2010)

$$kd^2 \leq cn$$

- topological codes on 2D hyperbolic manifold (Delfosse 2014)

$$kd^2 \leq c(\log k)^2 n$$

- things are better in 4D hyp. space: Guth-Lubotzky 2014 (also Londe-Leverrier 2018)

$$[\![n, k = \Theta(n), d = n^\alpha]\!], \quad \text{for} \quad \alpha \in [0.2, 0.3]$$

what can we get by relaxing geometric locality in 3D?

- we still want an LDPC construction, but allow for non local generators
- a nice mathematical topic with many frustrating open questions!

Classical LDPC codes are well understood

sparse parity-check matrix $\mathrm{H} \in \mathbb{F}_2^{\mathrm{m}\times\mathrm{n}}$:

$$\mathcal{C} = \ker \mathrm{H}$$

- good codes with $\mathrm{k} = \Theta(\mathrm{n}), \mathrm{d} = \Theta(\mathrm{n})$ can be found by picking H at random
- efficient decoding with belief propagation

quantum LDPC codes remain poorly understood

stabilizer group $\mathcal{S} = \langle g_1, \dots, g_m \rangle$ with $g_i \in \mathcal{P}_n$ (n-qubit Pauli group) such that $[g_i, g_j] = 0$

LDPC:

- $|g_i|$ small (constant or log)
- $\forall \ell \in [n], \quad \sharp\{i : \ell \in \mathbf{supp}(g_i)\}$ small

$$\mathcal{C} = \{|\psi\rangle \in (\mathbb{C}^2)^{\otimes n} : g_i|\psi\rangle = |\psi\rangle, \forall i \in [m]\}$$

The big questions (for me!)

- what kind of parameters are possible for qLDPC?
- efficient decoding??
- links with Hamiltonian complexity

quantum LDPC codes with large minimum distance

Beating the $\sqrt{n}$ of the toric code is very hard!

- Freedman, Meyer, Luo (2002): construction based on $S^1 \times S^2$

$$d \propto n^{1/2} \log^{1/4} n$$

- Kaufman, Kazhdan, Lubotzky (2016): construction based on Ramanujan complexes

$$d_X \propto n, \quad d_Z \propto \log n$$

  + balancing technique (Hastings 2017)

$$\implies d \propto n^{1/2} \log^{1/2} n$$

- construction by Hastings (2017) he conjectures could yield $d \propto n^{1-\varepsilon}$

quantum LDPC codes with large minimum distance

best minimum distance when asking for constant rate

hypergraph product codes (Tillich, Zémor 2009) of two good classical LDPC codes

$$[\![n, \Theta(n), \Theta(\sqrt{n})]\!]$$

- note 1: generalization of the toric code (product of 2 repetition codes)
- note 2: existence of codes with $d \propto n$ by relaxing the LDPC condition to $\sqrt{n}$-local generators (Bravyi, Hastings 2014)

Do good qLDPC codes exist?

Decoding quantum LDPC codes

- essentially solved for topological codes!

What about general codes?

- *belief propagation*: several issues (Poulin, Chung 2008)
  - lots of small cycles
  - many symmetric patterns (half generators) where the decoder gets stuck
  - how to deal with degenerescence??
- *greedy decoding in local balls*
  - for 4D hyperbolic codes (Hastings 2014)
  - small-set-flip for quantum expander codes (Leverrier, Tillich, Zémor 2015, Fawzi, Grospellier, Leverrier 2018)

Small-set-flip for quantum expander codes

- consider a classical expander code (Sipser, Spielman 1996), i.e. such that its factor graph is an expander
- hypergraph product code $\implies$ quantum expander code $[\![n, \Theta(n), \Theta(\sqrt{n})]\!]$

small-set-flip decoding

- for each g_i: consider all patterns of errors within g_i and apply the one that decreases the syndrome weight the most (if it exists)
- repeat while possible

- correct arbitrary errors of weight $O(\sqrt{n})$
- locality of SSF $\implies$ distant clusters of errors are also dealt with
- cst threshold for local stochastic errors on both qubits and syndrome measurements
- *reasonable performance in practice*: threshold around 6-7 % with noiseless syndrome measurement and $\approx 3\%$ for noisy syndrome measurement for phenomenological noise model (Grospellier, Krishna 2018, Grospellier, Grouès, Krishna, Leverrier 2019)

Soundness and local testability

The analysis of the bit-flip decoder for classical expander codes and SSF for quantum expander codes relies on the soundness of the codes:

soundness of quantum expander codes

for any error e such that $|e| := d(e, \mathcal{C}) \leq c\sqrt{n}$,

$$|s(e)| \geq \eta |e|$$

for $\eta = \mathrm{cst}$, and s(e) the syndrome

If true for any e, then *locally testable code*

$\implies$ easy to distinguish between codewords and words far from the code, making a constant number of queries to the word.

Many applications in the classical setting, mostly in theoretical CS, e.g. for PCP theorem

review paper by Goldreich (2006)

Quantum locally testable codes

- notion introduced by Aharonov, Eldar (2015)
- applications remain a bit unclear at the moment, essentially in Hamitlonian complexity
  - qLTC with linear minimum distance would establish the NLTS conjecture (Eldar, Harrow 2017)
  - existing qLTC (this talk) allows to prove an average-case version of NLTS (Eldar 2019)
  - strong form of confinement of errors (Stephen's talk yesterday)
  - link with *single-shot decoding* (Campbell 2018)
- definition requires to quantize notions of *distance to code* and *weight of the syndrome*

qLTC with soundness η

- q-local quantum code $\mathcal{C}$ $\Longrightarrow$ *Hamiltonian* $H_{\mathcal{C}} = \frac{1}{qm}\sum_{i=1}^{m}\frac{1}{2}(\mathbb{1} - g_i)$
- projector $\Pi_{\mathcal{C}_t}$ on *t-fattening of the code*

$$\mathcal{C}_t := \mathrm{Span}\{(A_1 \otimes \cdots \otimes A_n)|\psi\rangle \,:\, |\psi\rangle \in \mathcal{C}, |\{i \,:\, A_i \neq \mathbb{1}\}| \leq t\}$$

$$D_{\mathcal{C}} = \sum_t t(\Pi_{\mathcal{C}_t} - \Pi_{\mathcal{C}_{t-1}})$$

A quantum code is locally testable with soundness η if

$$H_{\mathcal{C}} \succeq \frac{\eta}{N} D_{\mathcal{C}} \quad (\text{energy} \geq \eta \times \text{distance})$$

2 known constructions

- Hastings (2017): $\eta = \frac{1}{\log^3 n}, k = 2$
- this work: $\eta = \frac{1}{\log^2 n}, k = 1$, possibly also for $k = \omega(1)$?

Examples of codes which are NOT locally testables

- 2D toric code: errors of weight $\Omega(\sqrt{\mathrm{n}})$ and constant energy
- D-dimensional toric code
- quantum expander codes: errors of weight $\Omega(\sqrt{\mathrm{n}})$ and constant energy

The hemicubic code construction

alternative name: the projective code (QIP'19)

Main properties of the hemicubic codes

almost LDPC: log-local

The simplest version: 1 logical qubit

$$[\![N, 1, d \geq \sqrt{N}/1.62]\!]$$

- locally testable with $\eta = \Omega\left(\frac{1}{\log^2 N}\right)$, open whether $\eta = \Theta\left(\frac{1}{\log N}\right)$?
- efficient decoder for adversarial errors of size $\frac{d}{\mathrm{polylog}(N)}$

The general case: $k = N^\alpha$

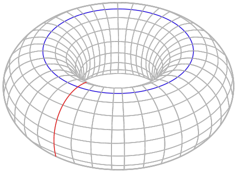
explicit parameters of the form: $[\![N, \mathrm{poly}(N), \mathrm{poly}(N)]\!]$

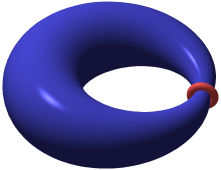
- conjectured local testability

Idea behind the construction: homological codes with large min distance?

Geometric interpretation of N and d for surfaces:

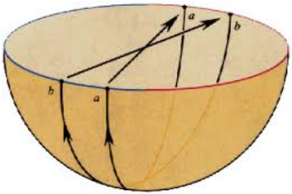
- N $\approx$ area of the surface
- d = *systole* of the surface, length of the shortest loop which is not the boundary of a 2D subregion of the surface
- idea: minimize N at fixed d
- work on surface with positive curvature $\Longrightarrow$ sphere (requires some identification to get a logical qubit)





The real projective plane

- identify antipodal points $\implies$ some loops are not boundaries: homology
- 1 logical qubit
  - systole $= \pi$
  - area $= 2\pi \implies$ systole $> \sqrt{\text{area}}$
  - $\overset{?}{\implies} D > \sqrt{N}$

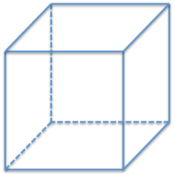


Not an infinite family of quantum codes...

Solution: increase the ambient dimension (similar to Hastings 2016)

A discrete real projective plane

- identify pairs of antipodal faces of the cube
- $N = 6$ (qubits on edges)
- $D_X = 3$ (smallest non trivial cycle)
- $D_Z = 2$ (smallest non trivial cocycle)
- $D = \min(D_X, D_Z) = 2$
- $N = D_X D_Z$



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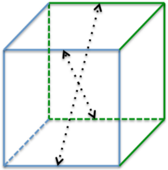

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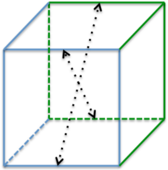

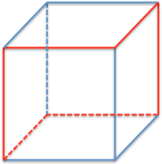

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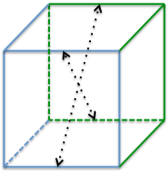
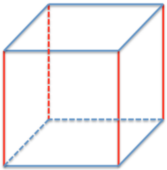




The hemicubic code: a discrete real projective n*-space*

- n-hemicube: antipodal quotient of the n-hypercube
- qubits on p-faces ($1 \leq p \leq n-1$), generators on ($p \pm 1$)-faces
- $N = \binom{n}{p} 2^{n-p-1}, \quad K = 1$
- $D_X = \binom{n}{p}$ (minimal nontrivial cycle has a p-face in every direction)
- $D_Z = 2^{n-p-1}$ (minimal nontrivial cocycle consists of all p-faces in a given direction)
- $N = D_X D_Z$
- $D_X \approx D_Z \approx \sqrt{N}$ for $p = \alpha n$ with $\alpha \approx 0.227$.

This code has already appeared in the literature in a completely different form relying on Khovanov homology (Audoux 2013).

Local testabilty of the hemicubic code

Recall that we want to prove that a lower bound on the syndrome weight:

$$\frac{1}{\mathrm{qm}}|\mathrm{s(e)}| \geq \frac{\eta}{\mathrm{N}} \mathrm{d}(\mathrm{e}, \mathcal{C}), \quad \forall \mathrm{e} \in \mathcal{P}_{\mathrm{N}}$$

Hemicubic code: $\mathrm{m} = \Theta(\mathrm{N})$, $\mathrm{q} = \Theta(\log \mathrm{N})$

We will prove $|\mathrm{s(e)}| = \Omega\left(\frac{\mathrm{d}(\mathrm{e},\mathcal{C})}{\log \mathrm{N}}\right)$, which implies $\eta = \Omega\left(\frac{1}{\log^2 \mathrm{N}}\right)$.

Geometric interpretation

for a homological code, $|\mathrm{s(e)}|$ is the weight of a boundary and $\mathrm{d}(\mathrm{e}, \mathcal{C})$ is the minimal weight of its filling.

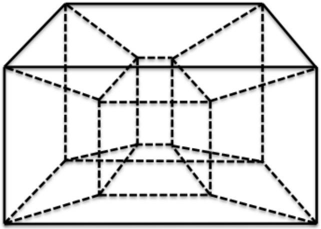
We are looking for filling inequalities.

A filling inequality for the n*-hypercube*

The syndrome is a boundary B. We are looking for a filling F of it of low weight.

Filling inequality by Dotterrer (2012)

- qubits on 2-faces, checks on edges
- send the syndrome to the left by filling with horizontal squares
- iterate
- choose the order of directions carefully



Dotterrer's bound

- Dotterrer's algorithm yields: $|B| \geq \mathrm{cst}\,|F| \quad \Longrightarrow \quad |S(e)| \geq \mathrm{cst}\,d(e, \mathcal{C})$
- here: no homology, but a similar approach works for the hemicube
- our current analysis loses a log factor compared to Dotterrer

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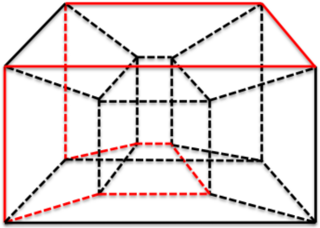

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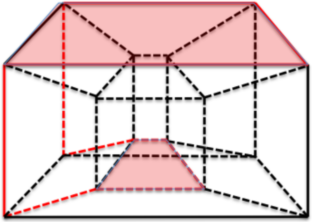

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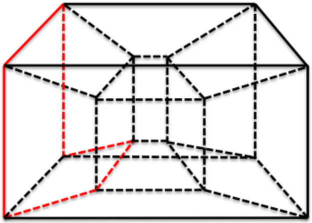

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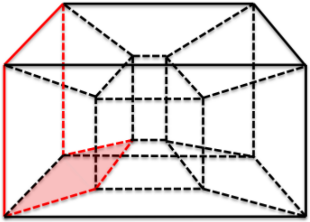

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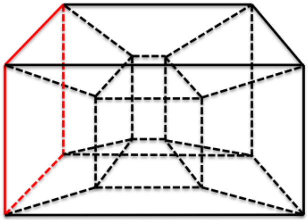

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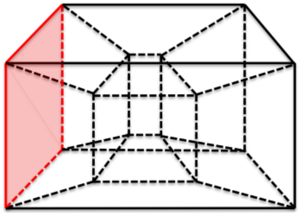

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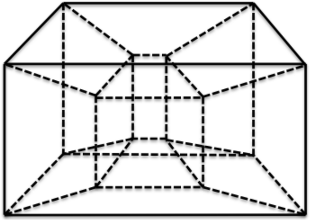

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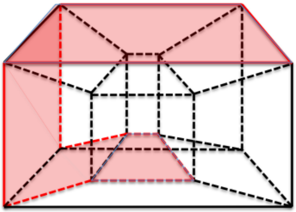

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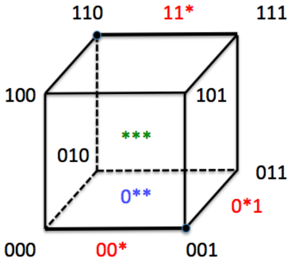
antipodal map = translation by the classical repetition code

codewords of the repetition code: {000,111}



$\tau_{111}(001) = 001 + 111 = 110$
$\tau_{111}(00*) = 00 * +111 = 11*$

τ_{111} is the antipodal map.

Generalization

Quotient of the n-cube by arbitrary linear codes?

The general construction

The simple construction identifies some cell x of the n-cube with $x + 111 \cdots 1$.
In other words, the faces are identified if they differ by an element of the repetition code.

- choose a classical linear code $\mathcal{C} = [n, k, d]$
- associate qubits with p-faces of the n-cube, where we identify elements of a given coset of $\mathcal{C}$:

$$x \sim y \quad \Longleftrightarrow \quad x + y \in \mathcal{C}$$

- many more logical qubits: $k = \binom{p+k-1}{p}$
- surprisingly, dimension and minimum distance only depend on the k and d from $\mathcal{C}$, not on H

Perspectives

hemicubic code

- simplest version: n-cube with identified antipodal faces
  - $d = \sqrt{N}$
  - locally testable
- general version
  - n-cube with identification of cosets of a linear code
  - explicit dimension and minimum distance
  - conjectured to be locally testable?

main open question

- what kind of length is possible for quantum LTC? exponential in k, polynomial?

Thanks!

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