UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA
doi: 10.4467/20843828AM.12.001.0453 FASCICULUS XLIX, 2011

INVARIANTS OF PLANE CURVE SINGULARITIES AND NEWTON DIAGRAMS

by Pierrette Cassou-Noguès and Arkadiusz Płoski

To Professor Kamil Rusek on his 65^{th} *birthday*

Abstract. We present an intersection-theoretical approach to the invariants of plane curve singularities μ, δ, r related by the Milnor formula $2\delta = \mu + r - 1$. Using Newton transformations we give formulae for μ, δ, r which imply planar versions of well-known theorems on nondegenerate singularities.

Introduction

The goal of this paper is to present an elementary, intersection-theoretical approach to the local invariants of plane curve singularities. We study in detail three invariants: the Milnor number μ, the number of double points δ and the number r of branches of a local plane curve. The technique of Newton diagrams plays an important part in the paper. It is well-known that Newton transformations which arise in a natural way when applying the Newton algorithm provide a useful tool for calculating invariants of singularities.

The formulae for the Milnor number in terms of Newton diagrams and Newton transformations presented in the paper grew out of our discussion on Eisenbud–Neumann diagrams. They have counterparts in toric geometry of plane curve singularities and in the case of two dimensions imply theorems due to Kouchnirenko, Bernstein and Khovanski.

2000 *Mathematics Subject Classification.* Primary 32S55; Secondary 14H20.

Key words and phrases. Plane local curve, intersection multiplicity, Milnor number, Newton's transformation, Newton diagram, equisingularity.

The first author is partially supported by grants MTM 2010-21740-C02-01 and MTM 2010-21740-C02-02.

The contents of the article are:

1. Plane local curves
2. The Milnor number: intersection theoretical approach
3. Newton diagrams and power series
4. Newton transformations and factorization of power series
5. Newton transformations, intersection multiplicity and the Milnor number
6. Nondegenerate singularities and equisingularity

1. Plane local curves

Let $\mathbb{C}\{X, Y\}$ be the ring of convergent complex power series in variables X, Y. For any nonzero power series $f = \sum c_{\alpha\beta} X^\alpha Y^\beta$ we put $\operatorname{supp} f = \{(\alpha, \beta) \in \mathbb{N}^2 : c_{\alpha\beta} \neq 0\}$, $\operatorname{ord} f = \inf\{\alpha + \beta : (\alpha, \beta) \in \operatorname{supp} f\}$ and $\operatorname{in} f = \sum c_{\alpha\beta} X^\alpha Y^\beta$ with summation over $(\alpha, \beta) \in \mathbb{N}^2$ such that $\alpha + \beta = \operatorname{ord} f$.

We put by convention $\operatorname{ord} 0 = +\infty$, $\operatorname{in} 0 = 0$. We call c_{00} the constant term of the power series f. The power series without constant term form the unique maximal ideal of $\mathbb{C}\{X, Y\}$. A power series is a unit if and only if its constant term is nonzero. We write $g = f \cdot \text{unit}$ if there is a unit u such that $g = fu$ in $\mathbb{C}\{X, Y\}$. We then also say that f and g are associated. Let $f \in \mathbb{C}\{X, Y\}$ be a nonzero power series without constant term. A *local (plane) curve* $f = 0$ is defined to be the ideal generated by f in $\mathbb{C}\{X, Y\}$. We say that a local curve $f = 0$ is *irreducible (reduced)* if $f \in \mathbb{C}\{X, Y\}$ is irreducible (f has no multiple factors). The irreducible curves are also called *branches.* If $f = f_1^{m_1} \dots f_r^{m_r}$ with non-associated irreducible factors f_i then we refer to $f_i = 0$ as the branches or components of $f = 0$. We say that a curve $f = 0$ is *singular (nonsingular)* if $\operatorname{ord} f > 1$ ($\operatorname{ord} f = 1$). We call $\operatorname{ord} f$ the *multiplicity of the curve* $f = 0$. The lines defined by the equation $\operatorname{in} f = 0$ are the *tangent lines* (in short: tangents) to the curve $f = 0$.

Let $\widetilde{X}, \widetilde{Y}$ be new variables. A local system of coordinates Φ is a pair of power series $\Phi(\widetilde{X}, \widetilde{Y}) = (a\widetilde{X} + b\widetilde{X} + \cdots, c\widetilde{X} + d\widetilde{Y} + \cdots)$ where $ad - bc \neq 0$ and the dots denote terms of order higher than 1 in $\widetilde{X}, \widetilde{Y}$. The map $f \to f \circ \Phi$ is an isomorphism of the rings $\mathbb{C}\{X, Y\}$ and $\mathbb{C}\{\widetilde{X}, \widetilde{Y}\}$.

For any power series $f, g \in \mathbb{C}\{X, Y\}$ we define the *intersection multiplicity* or *intersection number* $i_0(f, g)$ by putting

$$i_0(f, g) = \dim_{\mathbb{C}} \mathbb{C}\{X, Y\}/(f, g),$$

where (f, g) is the ideal of $\mathbb{C}\{X, Y\}$ generated by f and g. If f, g are nonzero power series without constant terms then $i_0(f, g) < +\infty$ if and only if the curves $f = 0$ and $g = 0$ have no common branch. The following properties are basic

1. $i_0(f,g)$ depends on the ideal (f,g) only. In particular, $i_0(f,g) = i_0(g,f)$ and $i_0(f, g+kf) = i_0(f,g)$.
2. If Φ is a local system of coodinates then $i_0(f\circ\Phi, g\circ\Phi) = i_0(f,g)$.
3. $i_0(f,gh) = i_\circ(f,g) + i_\circ(f,h)$.

Let t be a variable. A *parametrization* is a pair $(x(t), y(t)) \in \mathbb{C}\{t\}^2$ of power series without constant terms such that $x(t) \neq 0$ or $y(t) \neq 0$ in $\mathbb{C}\{t\}$. Two parametrizations $(x(t), y(t))$ and $(\widetilde{x}(\widetilde{t}), \widetilde{y}(\widetilde{t}))$ are equivalent if there is a power series $\tau(t) \in \mathbb{C}(t)$, $\operatorname{ord}\tau = 1$ such that $x(t) = \widetilde{x}(\tau(t))$, $y(t) = \widetilde{y}(\tau(t))$. A *parametrization* $(x(t), y(t)) \in \mathbb{C}\{t\}^2$ *is good* if there is no parametrization $(x_1(t_1), y_1(t_1)) \in \mathbb{C}\{t_1\}^2$ such that $x(t) = x_1(\tau_1(t))$, $y(t) = y_1(\tau_1(t))$ for a power series $\tau_1(t)$ such that $\operatorname{ord}\tau_1(t) > 1$.

A parametrization $(x(t), y(t))$ is a *Puiseux parametrization* if it is good and $x(t) = t^n$ for an integer $n > 0$. It may be proved that a parametrization $(t^n, y(t))$ is a Puiseux parametrization if and only if $\gcd(n, \operatorname{supp} y(t)) = 1$.

For any branch $f = 0$ there is a unique up to equivalence good parametrization $(x(t), y(t))$ such that $f(x(t), y(t)) = 0$. If $n = i_0(f,x) < +\infty$ then it is equivalent to a Puiseux' parametrization $(t^n, y(t))$. On the other hand, for any parametrization $(x(t), y(t))$ there is a unique branch $f = 0$ such that $f(x(t), y(t)) = 0$.

The following important property holds true:

4. If $(x(t), y(t))$ is a good parametrization of the branch $f = 0$ then $i_0(f,g) = \operatorname{ord} g(x(t), y(t))$.

This implies

5. Let $f = 0$ be a branch. Then for any power series $g, h \in \mathbb{C}\{X,Y\}$: $i_0(f, g+h) \geq \inf\{i_0(f,g), i_0(f,h)\}$ with equality if $i_0(f,g) \neq i_0(f,h)$.

Suppose that $f = 0$ is a branch and consider

$$\Gamma(f) = \{i_0(f,g) : g \in \mathbb{C}\{X,Y\} \text{ runs over all series such that } f \text{ does not divide } g\}.$$

Clearly $0 \in \Gamma(f)$ and $a, b \in \Gamma(f) \Rightarrow a+b \in \Gamma(f)$, since the intersection number is additive. We call $\Gamma(f)$ the semigroup of the branch $f = 0$. Note that $\Gamma(f) = \mathbb{N}$ if and only if the branch $f = 0$ is nonsingular.

Two reduced curves $f = 0$ and $g = 0$ are *equisingular* if and only if there are factorizations $f = f_1 \cdots f_r$ and $g = g_1 \cdots g_r$ with the same numbers $r > 0$ of irreducible factors f_i and g_i such that

- $\Gamma(f_i) = \Gamma(g_i)$ for all $i = 1, \ldots, r$,
- $i_0(f_i, f_j) = i_0(g_i, g_j)$ for $i, j = 1, \ldots, r$.

The bijection $f_i \mapsto g_i$ will be called equisingularity bijection. In particular, two branches are equisingular if and only if they have the same semigroup. A function defined on the set of reduced curves is an *invariant* if it is constant on equisingular curves. The multiplicity and the number of branches of a plane local curve are invariants.

Notes

The proofs omitted in this section are given in [**8**]. A beautiful introduction to the subject is given in [**34**]. The book [**2**] is very well written and contains historical information. For the systematic treatment of plane curve singularities see [**4**], [**15**, Chap. 5] and [**37**].

2. The Milnor number: intersection theoretical approach

For every power series $f \in \mathbb{C}\{X,Y\}$ without constant term we define the Milnor number $\mu_0(f)$ by putting

$$\mu_0(f) = i_0\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right).$$

Property 2.1. *There is $\mu_0(f) = +\infty$ if and only if f has a multiple factor in $\mathbb{C}\{X,Y\}$.*

Proof. If $f = h^2 g$ in $\mathbb{C}\{X,Y\}$, $\operatorname{ord} h > 0$, then $\frac{\partial f}{\partial X} = 2h\frac{\partial h}{\partial X}g + h^2\frac{\partial g}{\partial X}$ and $\frac{\partial f}{\partial Y} = 2h\frac{\partial h}{\partial Y}g + h^2\frac{\partial g}{\partial Y}$. Thus the derivatives $\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}$ have a common factor h of a positive order and $\mu_0(f) = i_0\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right) = +\infty$. Now suppose that $i_0\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right) = +\infty$. Then there exists an irreducible divisor h of the derivatives $\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}$. We claim that h divides f: if $(x(t), y(t))$ is a parametrization of the branch $h = 0$, then $\frac{d}{dt}f(x(t),y(t)) = \frac{\partial f}{\partial X}(x(t),y(t))x'(t) + \frac{\partial f}{\partial Y}(x(t),y(t))y'(t) = 0$ in $\mathbb{C}\{t\}$. Therefore, $f(x(t),y(t)) = 0$ and h divides f. From irreducibility of h there follows that $\operatorname{ord} h(X,0) = \operatorname{ord} h$ or $\operatorname{ord} h(0,Y) = \operatorname{ord} h$. Suppose that $\operatorname{ord} h(0,Y) = \operatorname{ord} h$. Thus $\operatorname{ord}\frac{\partial h}{\partial Y} = \operatorname{ord} h - 1$ and the power series h and $\frac{\partial h}{\partial Y}$ are coprime in $\mathbb{C}\{X,Y\}$. Write $f = hg$. Whence $\frac{\partial f}{\partial Y} = \frac{\partial h}{\partial Y}g + h\frac{\partial g}{\partial Y}$ and h divides $\frac{\partial h}{\partial Y}g$. Therefore, h divides g and h is a multiple factor of f. $\square$

Property 2.2. *For any local system of coordinates Φ, $\mu_0(f\circ\Phi) = \mu_0(f)$.*

Proof. Since $\mathrm{Jac}\Phi(0,0) \neq 0$, the ideals $\left(\frac{\partial}{\partial X}(f\circ\Phi), \frac{\partial}{\partial Y}(f\circ\Phi)\right)$ and $\left(\frac{\partial f}{\partial X}\circ\Phi, \frac{\partial f}{\partial Y}\circ\Phi\right)$ are equal. Thus we get $\mu_0(f\circ\Phi) = i_0\left(\frac{\partial f}{\partial X}\circ\Phi, \frac{\partial f}{\partial Y}\circ\Phi\right) = i_0\left(\frac{\partial f}{\partial X}, \frac{\partial f}{\partial Y}\right) = \mu_0(f)$. $\square$

The following lemma, due to Teissier ([**31**, Chap. II, Théorème 5]; [**32**, Chap. II, Prop. 1.2]) plays a crucial role in what follows. It is a particular case of a formula proved in [**31**] in the case of hypersurfaces.

Lemma 2.3 (Teissier's lemma). *Let $f \in \mathbb{C}\{X, Y\}$, $f(0,0) = 0$ be such that $f(0, Y) \neq 0$. Then*

$$i_0\left(f, \frac{\partial f}{\partial Y}\right) = \mu_0(f) + i_0(f, X) - 1.$$

Proof. Using Property 2.1, it is easy to check that $i_0\left(f, \frac{\partial f}{\partial Y}\right) = +\infty$ if and only if $\mu_0(f) = +\infty$. Suppose that $\mu_0(f) < +\infty$ and $\frac{\partial f}{\partial Y}(0,0) = 0$ (if $\frac{\partial f}{\partial Y}(0,0) \neq 0$ then the lemma is obvious). Write $\frac{\partial f}{\partial Y} = g_1 \cdots g_m$ with irreducible $g_i \in \mathbb{C}\{X, Y\}$. Let $(x_i(t_i), y_i(t_i)) \in \mathbb{C}\{t_i\}^2$ be a good parametrization of the branch $g_i = 0$. Differentiating and taking orders give $\operatorname{ord} f(x_i(t_i), y_i(t_i)) = \operatorname{ord} \frac{\partial f}{\partial X}(x_i(t_i), y_i(t_i)) + \operatorname{ord} x_i(t_i)$ that is $i_0(f, g_i) = i_0\left(\frac{\partial f}{\partial X}, g_i\right) + i_0(X, g_i)$ for $i = 1, \ldots, m$. Summing up the obtained equalities we get $i_0\left(f, \frac{\partial f}{\partial Y}\right) = \mu_0(f) + i_0\left(X, \frac{\partial f}{\partial Y}\right)$ and the lemma follows, since $i_0\left(X, \frac{\partial f}{\partial Y}\right) = i_0(X, f) - 1$. □

Property 2.4. *Let $f \in \mathbb{C}\{X, Y\}$, $f(0,0) = 0$ be a power series without multiple factors. Then*

(*i*) *$g = f \cdot$unit implies $\mu_0(g) = \mu_0(f)$,*

(*ii*) *if $f = f_1 \cdots f_m$, $f_i(0) = 0$ and f_i are pairwise coprime, then*

$$\mu_0(f) + m - 1 = \sum_{i=1}^{m} \mu_0(f_i) + 2 \sum_{1 \leq i < j \leq m} i_0(f_i, f_j).$$

Proof. We may assume that $f(0, Y) \neq 0$ in $\mathbb{C}\{Y\}$.
(*i*) It is easy to check that $i_0\left(g, \frac{\partial g}{\partial Y}\right) = i_0\left(f, \frac{\partial f}{\partial Y}\right)$ and $i_0(g, X) = i_0(f, X)$. Then $\mu_0(g) = \mu_0(f)$ by Teissier's lemma.
(*ii*) The basic properties of intersection multiplicity give

$$i_0\left(f, \frac{\partial f}{\partial Y}\right) = \sum_{i=1}^{m} i_0\left(f_i, \frac{\partial f_i}{\partial Y}\right) + 2 \sum_{1 \leq i < j \leq m} i_0(f_i, f_j),$$

$$i_0(f, X) = \sum_{i=1}^{m} i_0(f_i, X).$$

Then we use Teissier's lemma. □

In what follows, we need a lemma due to Jung ([**14**, Zehntes Kapitel, § 4, S. 181]).

Lemma 2.5 (Jung's lemma). *Let $f(X, Y) = Y^n + a_1(X)Y^{n-1} + \cdots + a_n(X) \in \mathbb{C}\{X\}[Y]$ be a distinguished irreducible polynomial of degree $n > 1$.*

Let $D(X) = \operatorname{disc}_Y f(X,Y)$ *be the discriminant of* f. *Then* $\operatorname{ord} D(X) \equiv n-1 \pmod 2$.

Proof. Let ε_0 be a primitive n-th root of unity. Then by the Puiseux Theorem $f(t^n, Y) = \prod_{k=0}^{n-1} (Y - y(\varepsilon_0^k t))$, where $y(t) \in \mathbb{C}\{t\}$.
Let $V_n(T_1, \dots, T_n) = \prod_{1 \le i < j \le n} (T_i - T_j)$.
Then $D(t^n) = \operatorname{disc}_Y f(t^n, Y) = v_n(t)^2$, where $v_n(t) = V_n(y(t), y(\varepsilon_0 t), \dots, y(\varepsilon_0^{n-1} t))$. It is easy to check that $v_n(\varepsilon_0 t) = (-1)^{n-1} v_n(t)$. Let us distinguish two cases.
Case 1. $n - 1 \equiv 0 \pmod 2$.
From $v_n(\varepsilon_0 t) = v_n(t)$ we get $v_n(t) \in \mathbb{C}\{t^n\}$ i.e. $v_n(t) = d(t^n)$, where $d(X) \in \mathbb{C}\{X\}$. Thus $D(X) = d(X)^2$ and we get $\operatorname{ord} D(X) \equiv 0 \pmod 2$.
Case 2. $n - 1 \equiv 1 \pmod 2$.
Then $v_n(\varepsilon_0 t) = -v_n(t)$ which implies $v_n(t) \in t^{\frac{n}{2}} \mathbb{C}\{t^n\}$ i.e. $v_n(t) = t^{\frac{n}{2}} d_1(t^n)$, where $d_1(X) \in \mathbb{C}\{X\}$. Thus $D(X) = X d_1(X)^2$ and $\operatorname{ord} D(X) \equiv 1 \pmod 2$.
Summing up we get $\operatorname{ord} D(X) \equiv n - 1 \pmod 2$. □

Now we can prove

Theorem 2.6. *Let* $r_0(f)$ *be the number of branches of the reduced local curve* $f = 0$. *Then*

$$\mu_0(f) + r_0(f) - 1 \equiv 0 \pmod 2.$$

Proof. Suppose that f is an irreducible power series. By the Weierstrass Preparation Theorem it suffices to consider the case where $f = Y^n + a_1(X) Y^{n-1} + \dots + a_n(X)$ is a distinguished polynomial. Let $D(X) = \operatorname{disc}_Y f(X,Y)$. By the classical formula for the intersection multiplicity, $i_0\left(f, \frac{\partial f}{\partial Y}\right) = \operatorname{ord} D(X)$. Thus by Jung's lemma $i_0\left(f, \frac{\partial f}{\partial Y}\right) \equiv n - 1 \pmod 2$ and by Teissier's lemma we get $\mu_0(f) = i_0\left(f, \frac{\partial f}{\partial Y}\right) - n + 1 \equiv 0 \pmod 2$.

We get the general case from Property 2.4 (ii) applied to the decomposition of f: $f = f_1 \cdots f_r$, $r = r_0(f)$ into irreducible factors f_i. □

For any reduced power series $f \in \mathbb{C}\{X, Y\}$ we put

$$\delta_0(f) = \frac{1}{2}(\mu_0(f) + r_0(f) - 1)$$

and call $\delta_0(f)$ the *double point number* of the local curve $f = 0$.
From the properties of the Milnor number we get

Proposition 2.7.

(i) $\delta_0(f) \geq 0$ *is an integer,* $\delta_0(f) = 0$ *if and only if* $f = 0$ *is nonsingular,*
(ii) $\delta_0(f \circ \Phi) = \delta_0(f)$ *for any local system of coordinates* Φ*,*
(iii) $\delta_0(\prod_{i=1}^m f_i) = \sum_{i=1}^m \delta_0(f_i) + \sum_{1 \leq i < j \leq m} i_0(f_i, f_j)$ *where* f_i *are coprime power series.*

Remark 2.8. The reduced curve $f = 0$ has an *ordinary* r*-fold singularity* if it has r branches, all nonsingular and intersecting each other with multiplicity 1. For such a curve we have $\mu_0 = (r-1)^2$ and $\delta_0 = \frac{1}{2} r(r-1)$.

Assume that $f \in \mathbb{C}\{X, Y\}$ is a power series with no multiple factors. If $f = f_1 \cdots f_r$ is a product of irreducible factors $f_i \in \mathbb{C}\{X, Y\}$, then we set

$$c_i(f) = \mu_0(f_i) + \sum_{j \neq i} i_0(f_i, f_j) \quad \text{for} \quad i = 1, \ldots, r.$$

A curve $\Psi = 0$ is said to be an *adjoint to* $f = 0$ if

$$i_0(f_i, \Psi) \geq c_i(f) \quad \text{for} \quad i = 1, \ldots, r.$$

Remark 2.9. Let $f = 0$ be an ordinary r-fold singularity. Then $\Psi = 0$ is an adjoint to $f = 0$ if and only if $\operatorname{ord} \Psi \geq r - 1$.

The following result is known as Noether's Theorem on the double-point divisor. Let $g, h \in \mathbb{C}\{X, Y\}$.

Theorem 2.10. *Suppose that the local curves* $f = 0$ *and* $g = 0$ *have no common component. If* h *satisfies Noether's conditions*

$$i_0(f_i, h) \geq i_0(f_i, g) + c_i(f) \quad \textit{for} \quad i = 1, \ldots, r$$

then h *belongs to the ideal generated by* f, g *in the ring* $\mathbb{C}\{X, Y\}$*.*

Let us write $h = \Phi f + \Psi g$ with $\Phi, \Psi \in \mathbb{C}\{X, Y\}$. Then Noether's conditions imply that $\Psi = 0$ is an adjoint to $f = 0$. In connection with Noether's Theorem let us note

Theorem 2.11. *Let* $f \in \mathbb{C}\{X, Y\}$ *be an irreducible power series. Then there does not exist* $\Psi \in \mathbb{C}\{X, Y\}$ *such that* $i_0(f, \Psi) = \mu_0(f) - 1$*. Let* $h \in \mathbb{C}\{X, Y\}$ *be such that* $i_0(f, h) = i_0(f, g) + \mu_0(f) - 1$*, then* $h \notin (f, g)\mathbb{C}\{X, Y\}$*.*

The second part of (2.11) follows easily from the first. Indeed, if we had $h = \Phi f + \Psi g$ with $\Phi, \Psi \in \mathbb{C}\{X, Y\}$ and $i_0(f, h) = i_0(f, g) + \mu_0(f) - 1$, then we would get $i_o(f, \Psi) = \mu_0(f) - 1$, a contradiction with the first part of (2.11).

Let us now pass to the proofs of Theorems (2.10) and (2.11).
Let $F(u, Y), G(u, Y), H(u, Y) \in \mathbb{C}\{u\}[Y]$ where u is a variable. Assume that $F(u, Y) = \prod_{i=1}^n (Y - y_i(u))$ in $\mathbb{C}\{u\}[Y]$ and $y_i(u) \neq y_j(u)$ for $i \neq j$.

Lemma 2.12. *If* $\operatorname{ord} H(u,y_i(u)) \geq \operatorname{ord} \frac{\partial F}{\partial Y}(u,y_i(u))G(u,y_i(u))$ *for* $i = 1,\dots,n$, *then* $H(u,Y) \in (F(u,Y),G(u,Y))\mathbb{C}\{u\}[Y]$.

Proof. Let

$$\Psi(u,Y) = \sum_{i=1}^{n} \frac{H(u,y_i(u))}{\frac{\partial F}{\partial Y}(u,y_i(u))G(u,y_i(u))} \frac{F(u,Y)}{(Y-y_i(u))}.$$

Then $\Psi(u,Y) \in \mathbb{C}\{u\}[Y]$ and $H(u,y_i(u)) = \Psi(u,y_i(u))G(u,y_i(u))$ for $i = 1,\dots,n$. Therefore, $H(u,Y) \equiv \Psi(u,Y)G(u,Y) \bmod (Y - y_i(u))$ for $i = 1,\dots,n$ and $H(u,Y) \equiv \Psi(u,Y)G(u,Y) \bmod F(u,Y)$ what implies $H(u,Y) \in (F(u,Y),G(u,Y))\mathbb{C}\{u\}[Y]$. □

Lemma 2.13. *If* $\Psi(u,Y) = \Psi_0(u)Y^{n-1} + \cdots + \Psi_{n-1}(u) \in \mathbb{C}\{u\}[Y]$, *then*

$$\sum_{i=1}^{n} \frac{\Psi(u,y_i(u))}{\frac{\partial F}{\partial Y}(u,y_i(u))} = \Psi_0(u).$$

Proof. The lemma follows immediately from the Lagrange interpolation formula. □

Proof of Theorem 2.10. (cf. [**38**, Achtes Kapitel]).
We may assume that $f_i = f_i(X,Y)$ are Y-distinguished polynomials and (after replacing g, h by the remainders of division by f) $g, h \in \mathbb{C}\{X\}[Y]$.
We have

$$\begin{aligned} i_0(f_i,g) + c_i(f) &= i_0(f_i,g) + \mu_0(f_i) + \sum_{j\neq i} i_0(f_i,f_j) \\ &= i_0(f_i,g) - i_0(f_i,X) + 1 + i_0\left(f_i, \frac{\partial f_i}{\partial Y}\right) + \sum_{j\neq i} i_0(f_i,f_j) \\ &= i_0(f_i,g) - i_0(f_i,X) + 1 + i_0\left(f_i, \frac{\partial f}{\partial Y}\right) \end{aligned}$$

by Teissier's lemma.
Let $n_i = i_0(f_i,X)$ for $i = 1,\dots,r$. The Noether conditions are equivalent to

$$i_0(f_i,h) \geq i_0(f_i,g) + i_0\left(f_i, \frac{\partial f}{\partial Y}\right) - n_i + 1 \quad \text{for} \quad i = 1,\dots,r. \tag{1}$$

By Puiseux' Theorem we can write

$$f_i(t^{n_i},Y) = (Y - y_{i1}(t))\cdots(Y - y_{in_i}(t)) \quad \text{in} \quad \mathbb{C}\{t\}[Y],$$

where $y_{i1}(t),\dots,y_{in_i}(t)$ are $\mathbb{C}\{t^{n_i}\}$-conjugate, i.e. $y_{ij}(t) = y_{i1}(\varepsilon_j t)$ for some ε_j such that $\varepsilon_j^{n_i} = 1$. Thus for every $h(X,Y) \in \mathbb{C}\{X,Y\}$:

$$\operatorname{ord} h(t^{n_i},y_{i1}(t)) = \cdots = \operatorname{ord} h(t^{n_i},y_{in_i}(t)) = i_0(f_i,h)$$

and we can rewrite (1) in the form

$$\text{(2)}\qquad \operatorname{ord} h(t^{n_i}, y_{ij}(t)) \geq \operatorname{ord} g(t^{n_i}, y_{ij}(t)) + \operatorname{ord} \frac{\partial f}{\partial Y}(t^{n_i}, y_{ij}(t)) - n_i + 1$$

or else

$$\text{(3)}\qquad \operatorname{ord}(t^{n_i-1} h(t^{n_i}, y_{ij}(t)) \geq \operatorname{ord} g(t^{n_i}, y_{ij}(t)) \frac{\partial f}{\partial Y}(t^{n}, y_{ij}(t)).$$

Let $N = n_1 \cdots n_r$ and $\overline{y}_{ij}(u) = y_{ij}(u^{N/n_i})$ for $i = 1, \ldots, r$. Obviously, $\frac{N}{n_i}(n_i - 1) \leq N - 1$. Therefore (3) implies

$$\text{(4)}\qquad \operatorname{ord}(u^{N-1} h(u^N, \overline{y}_{ij}(u))) \geq \operatorname{ord} g(u^N, \overline{y}_{ij}(u)) \frac{\partial f}{\partial Y}(u^N, \overline{y}_{ij}(u))$$

and we can apply Lemma 2.12 to the polynomials

$$F(u, Y) = f(u^N, Y) = \prod (Y - \overline{y}_{ij}(u)),\ G(u, Y) = g(u^N, Y)$$

and $H(u, Y) = u^{N-1} h(u^N, Y)$.
We get

$$u^{N-1} H(u^N, Y) \in (f(u^N, Y), g(u^N, Y))\mathbb{C}\{u\}[Y].$$

It is easy to check that $\mathbb{C}\{u\}[Y] = \sum_{i=0}^{N-1} \mathbb{C}\{u^N\} Y^i$ is a free $\mathbb{C}\{u^N\}[Y]$-module, so

$$h(u^N, Y) \in (f(u^N, Y), g(u^N, Y))\mathbb{C}\{u\}[Y]$$

and consequently $h(X, Y) \in (f(X, Y), g(X, Y)\mathbb{C}\{X\}[Y]$. □

Proof of Theorem 2.11. Suppose that there is a $\Psi = \Psi(X, Y) \in \mathbb{C}\{X, Y\}$ such that

$$\text{(5)}\qquad i_0(f, \Psi) = \mu_0(f) - 1.$$

We may assume that $f = f(X, Y)$ is a Y-distinguished polynomial of degree $n \geq 1$ and $\Psi \in \mathbb{C}\{X\}[Y]$ a polynomial of Y-degree $\leq n - 1$. By Teissier's lemma we can rewrite (5) in the form

$$\text{(6)}\qquad i_0(f, X\Psi) = i_0(f, \frac{\partial f}{\partial Y}).$$

By Puiseux Theorem, $f(u^n, Y) = \prod_{\varepsilon^n = 1} (Y - y(\varepsilon u))$.
Then (6) is equivalent to

$$\text{(7)}\qquad \operatorname{ord} u^n \Psi(u^n, y(u)) = \operatorname{ord} \frac{\partial f}{\partial Y}(u^n, y(u)).$$

By (7) we can write $\operatorname{in}(u^n \Psi(u^n, y(u))) = c_1 u^N$ $(c_1 \neq 0)$ and $\operatorname{in} \frac{\partial f}{\partial Y}(u^n, y(u)) = c_2 u^N$ $(c_2 \neq 0)$ where $N = i_0\left(f, \frac{\partial f}{\partial Y}\right)$. Therefore, we get

$$\text{(8)}\qquad \operatorname{in} \frac{u^n \Psi(u^n, y(\varepsilon u))}{\frac{\partial f}{\partial Y}(u^n, y(\varepsilon u))} = \frac{c_1 \varepsilon^N u^N}{c_2 \varepsilon^N u^N} = c, \quad c = \frac{c_1}{c_2}.$$

On the other hand, by Lemma 2.13 applied to $\Psi(u^n, Y)$ and $f(u^n, Y)$ we have

$$\sum_{\varepsilon^n=1} \frac{u^n \Psi(u^n, y(\varepsilon u))}{\frac{\partial f}{\partial Y}(u^n, y(\varepsilon u))} = u^n \Psi_0(u^n). \tag{9}$$

A contradiction, because the left-hand side of (9) is of order zero by (8). $\square$

In what follows, we will need

Lemma 2.14. *Let $f \in \mathbb{C}\{X, Y\}$ be an irreducible power series. Then for any integer $a \in \mathbb{Z}$ there exists power series $\phi, \psi \in \mathbb{C}\{X, Y\}$ such that $a = i_0(f, \phi) - i_0(f, \psi)$.*

Proof. Let $(x(t), y(t))$ be a good parametrization of the branch $f = 0$. Then the rings $\mathbb{C}\{x(t), y(t)\}$ and $\mathbb{C}\{t\}$ have the same field of fractions (see [**15**, Theorem 5.1.3.]). Then $t^a = \frac{\phi(x(t),y(t))}{\psi(x(t),y(t))}$ for some $\phi, \psi \in \mathbb{C}\{X, Y\}$ and taking orders gives $a = i_0(f, \phi) - i_0(f, \psi)$. $\square$

Theorem 2.15. *The semigroup $\Gamma(f)$ of the branch $f = 0$ contains all integers greater than or equal to the Milnor number $\mu_0(f)$. The number $\mu_0(f) - 1$ does not belong to $\Gamma(f)$.*

Proof. Let a be an integer such that $a \geq \mu_0(f)$. By Lemma 2.14 we can write $a = i_0(f, \phi) - i_0(f, \psi)$ for some $\phi, \psi \in \mathbb{C}\{X, Y\}$. Then $i_0(f, \phi) = i_0(f, \psi) + a \geq i_0(f, \psi) + \mu_0(f)$ and by Noether's Theorem $\phi = Af + B\psi$ for some $A, B \in \mathbb{C}\{X, Y\}$. Thus $a = i_0(f, Af + B\psi) - i_0(f, \psi) = i_0(f, B) \in \Gamma(f)$ and we are done.
The second part of 2.15 follows immediately from Theorem 2.11. $\square$

Using Theorem 2.15 and Property 2.4 (*ii*) we get

Theorem 2.16. *The Milnor number is an invariant of singularity.*

Notes
Milnor introduced and studied μ in the general case of isolated hipersurface singularities in his celebrated book [**26**]. A topological treatment of the Milnor number in the case of plane curve singularities is given in [**37**]. The invariant δ was defined in algebraical terms by Hironaka in [**13**]. The formula $2\delta = \mu + r - 1$, which in our approach served as the definition of δ, was proved in [**26**] by topological methods and in [**30**] in an algebraic way. The classical texts [**14**] and [**38**] where the Milnor number is implicit were very helpful when writing this article. Teissier's lemma has interesting generalizations involving the Jacobian (see the articles by Lê Dung Trang and Greuel quoted in [**33**]). For application of the Milnor number to singularities of plane algebraic curves see [**12, 24**] and the references given therein.

3. Newton diagrams and power series

Let $\mathbb{R}_+ = \{a \in \mathbb{R} \colon a \geq 0\}$. For any subsets $E, F \subset \mathbb{R}^2_+$, we consider the *Minkowski sum* $E + F = \{u + v \colon u \in E \text{ and } v \in F\}$. Let $E \subset \mathbb{N}^2$ and let us denote by $\Delta(E)$ the convex hull of the set $E + \mathbb{R}^2_+$. A subset $\Delta \subset \mathbb{R}^2_+$ is a *Newton diagram* (or *polygon*) if there is a set $E \subset \mathbb{N}^2$ such that $\Delta = \Delta(E)$. The smallest set $E_0 \subset \mathbb{N}^2$ such that $\Delta = \Delta(E_0)$ is called the set of *vertices* of the *Newton diagram* Δ. It is always finite and we can write $E_0 = \{v_0, v_1, \ldots, v_m\}$ where $v_i = (\alpha_i, \beta_i)$ and $\alpha_{i-1} < \alpha_i$, $\beta_{i-1} > \beta_i$ for all $i = 1, \ldots, m$. The Newton diagram with one vertex $v = (\alpha, \beta)$ is the quadrant $(\alpha, \beta) + \mathbb{R}^2_+$. After Teissier (see [**33, 34**]), for two positive integers a, b, by $\left\{\frac{a}{b}\right\}$ we denote the Newton diagram with vertices $(0, b)$ and $(a, 0)$. We also denote by $\left\{\frac{a}{\infty}\right\}$ and $\left\{\frac{\infty}{b}\right\}$ the quadrant with vertex $(a, 0)$ and $(0, b)$, respectively.

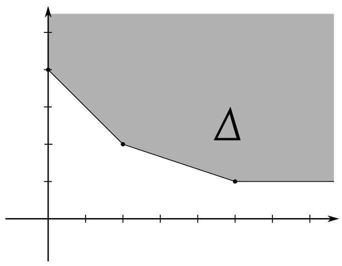

Figure 1.

We call a segment $E \subset \mathbb{R}^2_+$ a Newton edge if its vertices (α, β), (α', β') lie in $\mathbb{N}^2$ and $\alpha < \alpha'$, $\beta' < \beta$. We put $|E|_1 = \alpha' - \alpha$ and $|E|_2 = \beta - \beta'$ and call $|E|_1/|E|_2$ the inclination of E. We denote by $a(E)$ and $b(E)$ the distances of E to the vertical and horizontal axes respectively.

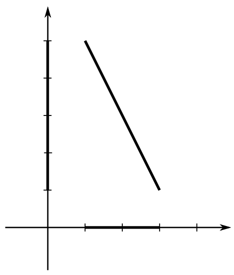

Figure 2.

The vertices of a Newton edge E are $(a(E), |E|_2+b(E))$ and $(a(E)+|E|_1, b(E))$. For any Newton diagram Δ we consider the set $n(\Delta)$ of 1-dimensional compact faces of the boundary of Δ. Note that $n(\Delta) = \emptyset$ if and only if Δ is a quadrant. If Δ has vertices $v_0, \dots, v_m$ $(m > 0)$ then $n(\Delta) = \{E_1, \dots, E_m\}$ where E_i is the edge with vertices v_{i-1}, v_i.

Let $a(\Delta)$ and $b(\Delta)$ denote the distances of Δ to the vertical and horizontal axes respectively. The diagram is *convenient* if $a(\Delta) = b(\Delta) = 0$. The reader will check the following two properties of Newton diagrams.

Property 3.1. *The Newton diagrams form a semigroup with respect to the Minkowski sum. For any Newton diagram Δ there is the* ***minimal decomposition***

$$(*) \qquad \Delta = \left\{\frac{a(\Delta)}{\infty}\right\} + \sum_{S \in n(\Delta)} \left\{\frac{|S|_1}{|S|_2}\right\} + \left\{\frac{\infty}{b(\Delta)}\right\}.$$

Property 3.2. *The line with the slope $-1/\theta$ $(\theta > 0)$ supporting the Newton diagram Δ with the minimal decomposition $(*)$ intersects the horizontal axis at the point with abscissa*

$$a(\Delta) + \sum_{S \in n(\Delta)} \inf\{|S|_1, \theta |S|_2\} + \theta b(\Delta).$$

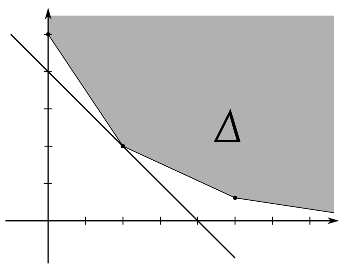

Figure 3. $\alpha + \theta\beta = \nu$

For any nonzero power series $f = \sum c_{\alpha\beta} X^\alpha Y^\beta \in \mathbb{C}\{X, Y\}$ we put $\Delta(f) = \Delta(\operatorname{supp} f)$ and $\mathcal{N}(f) = n(\Delta(f))$. We call $\Delta(f)$ the Newton diagram of f. The following property is of key importance.

Property 3.3. *For any nonzero power series $f, g \in \mathbb{C}\{X, Y\}$:*

$$\Delta(fg) = \Delta(f) + \Delta(g).$$

We refer the reader to [**22**] for a proof of 3.3.

In particular, if $g = f \cdot \text{unit}$, then $\Delta(g) = \Delta(f)$ since $\Delta(u) = \mathbb{R}^2_+$ (the zero of the semigroup of Newton's diagrams) if $u(0) \neq 0$ and we may speak about the Newton diagram of the local curve $f = 0$. A *power series* f is *convenient* if the diagram $\Delta(f)$ is convenient. Obviously, f is convenient if and only if the branches of the curve $f = 0$ are different from axes.

Observe that $\mathcal{N}(f) = \emptyset$ if and only if $f = X^{\alpha_0} Y^{\beta_0} \cdot \text{unit}$. Suppose that $\mathcal{N}(f) \neq \emptyset$. For any face $S \in \mathcal{N}(f)$ we consider the initial part $\text{in}(f, S)$ of f corresponding to S:

$$\text{in}(f, S) = \sum_{(\alpha,\beta)\in S} c_{\alpha\beta} X^\alpha Y^\beta.$$

Note that $\mathcal{N}(\text{in}(f, S)) = S$. If $a(S)$ and $b(S)$ are the distances of S to the axes, then $X^{a(S)} Y^{b(S)}$ is the monomial of maximal degree dividing $\text{in}(f, S)$. Let $r(S) = \gcd(|S|_1, |S|_2)$. Then $r(S) = \#(S \cap \mathbb{N}^2) - 1$. Let $m_S = |S|_1 / r(S)$, $n_S = |S|_2 / r(S)$. It is easy to check that

$$\text{in}(f, S) = X^{a(S)} Y^{b(S)} \Phi_S(X^{m_S}, Y^{n_S}),$$

where $\Phi_S(U, V) \in \mathbb{C}[U, V]$ is a homogeneous form of degree $r(S)$ such that $\Phi_S(U, 0)\Phi_S(0, V) \neq 0$ in $\mathbb{C}[U, V]$. Therefore, we may write

$$\text{in}(f, S) = c X^{a(S)} Y^{b(S)} \prod_{i=1}^{r} (Y^{n_S} - a_i X^{m_S})^{d_i},$$

where $a_i \neq a_j$ for $i \neq j$, $c \neq 0$ are constants.

We put $r(f, S) = r$. Since $r(S) = \sum_{i=1}^r d_i$, then $r(f, S) \leq r(S)$ with equality if and only if $d_1 = \ldots = d_r = 1$. We say that f is *nondegenerate on* S if $r(f, S) = r(S)$. A power series f is nondegenerate on S if and only if the system of equations $\frac{\partial}{\partial X}\text{in}(f, S) = \frac{\partial}{\partial Y}\text{in}(f, S) = 0$ has no solutions in $(\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\})$. The *power series* f is *nondegenerate* if it is nondegenerate on each $S \in \mathcal{N}(f)$. A binomial curve $Y^n - aX^m = 0$, $\gcd(n, m) = 1$, $a \neq 0$, will be called a *quasi-tangent to* $f = 0$ of (*tangential*) *multiplicity* d if $Y^n - aX^m$ is a factor of multiplicity d of the initial form $\text{in}(f, S)$. We say that $Y^{n_S} - a_i X^{m_S} = 0$, $i = 1, \ldots, r$, are quasi-tangents to $f = 0$ corresponding to the face $S \in \mathcal{N}(f)$.

Remark 3.4. *If* $\operatorname{ord} f(X, 0) = \operatorname{ord} f(0, Y) = \operatorname{ord} f$ *(this condition means that the axes* $Y = 0$ *and* $X = 0$ *are not tangent to the curve* $f = 0$*) then* $\Delta(f) = \left\{\frac{\operatorname{ord} f}{\operatorname{ord} f}\right\}$ *and the initial form corresponding to the unique face of* $\Delta(f)$ *is* $\text{in} f = c \prod_{i=1}^r (Y - a_i X)^{d_i}$ *where* $a_i \neq a_j$ *for* $i \neq j$*. In this case the quasi-tangents to* $f = 0$ *are ordinary tangents* $Y - a_i X = 0$*,* $i = 1, \ldots, r$*.*

Remark 3.5. *If* $f \in \mathbb{C}\{X,Y\}$ *is a convenient power series, then the local curve* $f=0$ *has exactly one quasi-tangent if and only if*

$$f = c(Y^n - aX^m)^d + \sum c_{\alpha\beta}X^\alpha Y^\beta, \quad gcd(n,m)=1,$$

where the summation is over (α,β) *such that* $\alpha n + \beta m > dmn$.
We have $\Delta(f) = \left\{\frac{i_0(f,Y)}{i_0(f,X)}\right\} = \left\{\frac{md}{nd}\right\}$.

Remark 3.6. *Let* $mult(f,\tau)$ *be the tangential multiplicity of the quasi-tangent* τ *to the local curve* $f=0$. *We put* $mult(f,\tau)=0$ *if a binomial curve* τ *is not a quasi-tangent to* $f=0$. *Then, for any nonzero power series* f,g: $mult(fg,\tau) = mult(f,\tau) + mult(g,\tau)$.

Notes
An interesting algebra of the Newton diagrams is developed in [**33**]. Newton introduced his diagrams to solve equations $f(X,Y)=0$ (see [**2**]). The notion of nondegeneracy appeared in a very general setting in [**16, 18, 35**]. The authors are responsible for the term "quasi-tangent."

4. Newton transformations and factorization of power series

Let $n,m>0$ be coprime integers and let $c \neq 0$ be a complex number. The Newton transformation (in short: the N-transformation) is defined by the following equations

$$\begin{aligned} X &= X_1^n, \\ Y &= (c+Y_1)X_1^m, \end{aligned} \tag{10}$$

where (X_1,Y_1) are new variables.
N-transformation (10) may be viewed as a deformation of the parametrization

$$\begin{aligned} X &= X_1^n, \\ Y &= cX_1^m \end{aligned} \tag{11}$$

of the binomial curve $Y^n - c^nX^m = 0$.
We omit the simple proof of the following

Lemma 4.1. *Let* $f = f(X,Y) \in \mathbb{C}\{X,Y\}$ *be a nonzero power series without constant term. Then there is a unique power series* $f_1 = f_1(X_1,Y_1) \in \mathbb{C}\{X_1,Y_1\}$ *and an integer* $k>0$ *such that*

$$f(X_1^n,(c+Y_1)X_1^m) = X_1^k f_1(X_1,Y_1), \ f_1(0,Y_1) \neq 0$$

in $\mathbb{C}\{X_1,Y_1\}$.

The line $\alpha n + \beta m = k$ *is a supporting line of* $\Delta(f)$. *Moreover, the series* f_1 *is without constant term if and only if the curve* $Y^n - c^nX^m = 0$ *is a* *quasi-tangent to curve $f=0$. Its tangential multiplicity equals $i_0(f_1,X_1)=\operatorname{ord}(f_1(0,Y_1))$.*

In what follows we call $f_1=f_1(X_1,Y_1)$ the strict transform of the series $f=f(X,Y)$ by N-transformation (10).

The following lemma gives a necessary condition for a power series to be irreducible.

LEMMA 4.2. *Let $f=f(X,Y)\in\mathbb{C}\{X,Y\}$ be a convenient irreducible power series. Then the local curve $f=0$ has exactly one quasi-tangent.*

PROOF. Let $N=\operatorname{ord} f(0,Y)$, $M=\operatorname{ord} f(X,0)$. By the Weierstrass Preparation Theorem, $f=(Y^N+a_1(X)Y^{N-1}+\cdots+a_N(X))\cdot$unit. By Puiseux Theorem $Y^N+a_1(t^N)Y^{N-1}+\cdots+a_N(t^N)=\prod_{\varepsilon^N=1}(Y-y(\varepsilon t))$, where $y(t)\in\mathbb{C}\{t\}$. A simple calculation shows that $\operatorname{ord} a_i(X)\geq i\frac{M}{N}$ with equality for $i=N$. Therefore, $\Delta(f)=\left\{\frac{M}{N}\right\}$. Let $I=\{i\in[1,N]\colon \operatorname{ord} a_i=i\frac{M}{N}\}$. Then the initial form of f corresponding to the unique face of $\Delta(f)$ is equal to const.$(Y^N+\sum_{i\in I}\operatorname{in} a_i(X)Y^{N-i})=$ const.$(Y^n-c^nX^m)^d$ where $N=nd$, $M=md$ and $\operatorname{in} y(t)=ct^M$. This proves the lemma. $\square$

We can use the N-transformations to verify in a finite number of steps if a power series is irreducible.

LEMMA 4.3. *Suppose that $f=f(X,Y)\in\mathbb{C}\{X,Y\}$ is a convenient power series such that the curve $f=0$ has exactly one quasi-tangent $Y^n-c^nX^m=0$. Let $f_1=f_1(X_1,Y_1)\in\mathbb{C}\{X_1,Y_1\}$ be the strict transform of $f=f(X,Y)$ by N-transformation (10). Then f is irreducible if and only if f_1 is irreducible.*

PROOF. Let d be the tangential multiplicity of the quasi-tangent $Y^n-c^nX^m=0$. Then $\operatorname{ord} f_1(0,Y_1)=d$. First assume that $f=f(X,Y)$ is an irreducible power series. Let $(t^e,\varphi(t))$ be a Puiseux parametrization of an irreducible factor of $f_1(X_1,Y_1)$. Then $e\leq\operatorname{ord} f_1(0,Y_1)=d$. On the other hand, by the definition of the strict transform, we get $f(t^{en},ct^{em}+t^{em}\varphi(t))=0$ in $\mathbb{C}\{t\}$. Since f is irreducible, we get $en\geq\operatorname{ord} f(0,Y)=dn$ and $e\geq d$. Thus $e=d$ and f_1 is an irreducible power series.

To check that the irreducibility of f_1 implies the irreducibility of f, assume that f_1 is irreducible. Then the branch $f_1=0$ has a Puiseux parametrization $(t^d,\varphi(t))$ where $d=\operatorname{ord} f_1(0,Y)$. By the definition of the strict transform we get $f(t^{dn},ct^{dm}+t^{dm}\varphi(t))=0$ in $\mathbb{C}\{t\}$. Since $\operatorname{ord} f(0,Y)=dn$, it suffices to check that $(t^{dn},ct^{dm}+t^{dm}\varphi(t))$ is a Puiseux parametrization. We have $\gcd(dn,\operatorname{supp}(ct^{dm}+t^{dm}\varphi(t)))=1$ since $\gcd(d,\operatorname{supp}\varphi(t))=1$. Therefore, the power series f is irreducible. $\square$

Corollary 4.4. *Every power series f of the form $f = Y^n - aX^m + \sum c_{\alpha\beta}X^\alpha Y^\beta$, $gcd(n,m) = 1$, where the summation is over (α,β) such that $\alpha n + \beta m > nm$, is irreducible.*

Proof. The strict transform f_1 of f by N-transformation (10) with c such that $c^n = a$ is of order 1, since $\frac{\partial f_1}{\partial Y_1}(0,0) \neq 0$. Therefore, by Lemma 4.3 the series f is irreducible. □

Example 4.5 (see [**19**]). Let $f = (X^2 - Y^3)^2 - Y^7$ and $g = (X^2 - Y^3)^2 - XY^5$. The both series have the unique quasi-tangent $Y^3 - X^2 = 0$ (of tangential multiplicity 2). The strict transforms of f and g by the N-transformation $X = X_1^3$, $Y = (1 + Y_1)X_1^2$ are $f_1(X_1,Y_1) = (3Y_1)^2 - X_1^2 +$ terms of order > 2 and $g_1(X_1,Y_1) = -X_1 +$ terms of order > 1. Thus by Lemma 4.3 the series f is reducible (since f_1 has two tangents) and the series g is irreducible.

With any binomial curve $\tau\colon Y^n - aX^m = 0$, $a \neq 0$, $\gcd(n,m) = 1$, we associate the N-transformation

$$\begin{aligned} X &= X_\tau^n, \\ Y &= (a^{1/n} + Y_\tau)X_\tau^m, \end{aligned}$$

where (X_τ, Y_τ) are new variables and $a^{1/n} = |a|^{1/n}\exp(i\frac{\alpha}{n})$ if $a = |a|\exp(i\alpha)$ with $0 \le \alpha < 2\pi$. We denote by $f_\tau = f_\tau(X_\tau, Y_\tau)$ the strict transform of $f = f(X,Y)$ by the N-transformation associated with τ.

The following property follows easily from the definitions.

Property 4.6. *Let f, g be nonzero power series without constant terms. Then*

(*i*) *a binomial curve τ is a quasi-tangent to the curve $f = 0$ if and only if $f_\tau(0) = 0$,*
(*ii*) *$(fg)_\tau = f_\tau g_\tau$ for any binomial curve τ,*
(*iii*) *if $f = f_1 \cdots f_r$ is a decomposition of f into irreducible factors, then for any binomial curve τ, τ is a quasi-tangent to the curve $f = 0$ if and only if τ is a quasi-tangent to a branch $f_i = 0$ for some $i \in \{1,\dots,r\}$.*

If $f = f_1 \cdots f_r$ is a decomposition of a nonzero power series f without constant term into irreducible factors, then we put $r_0(f) = r$, i.e. $r_0(f)$ is the number of irreducible factors of f counted with multiplicities.

Proposition 4.7. *If $f \in \mathbb{C}\{X,Y\}$ is a convenient power series, then $r_0(f) = \sum_\tau r_0(f_\tau)$, where the summation is over all quasi-tangents τ to the curve $f = 0$.*

PROOF. Let $f = f_1 \cdots f_r$ be a factorization of f into irreducible factors f_i. Let τ be a quasi-tangent to $f = 0$ and let

$$I_\tau = \{i \in [1,r] \colon \text{the branch } f_i = 0 \text{ has the quasi-tangent } \tau\}.$$

Thus the sets I_τ are nonempty, pairwise disjoint and $\bigcup_\tau I_\tau = [1,r]$.

Let $I_\tau^c = [1,r] \setminus I_\tau$. By Property 4.6 (ii) we get $f_\tau = \prod_{i\in I_\tau} (f_i)_\tau \cdot \text{unit}$, since $(f_i)_\tau(0) \neq 0$ for $i \in I_\tau^c$. Therefore, we obtain $r_0(f_\tau) = \sum_{i\in I_\tau} r_0((f_i)_\tau) = \# I_\tau$ since $r_0((f_i)_\tau) = 1$ for $i \in I_\tau$ by Lemma 4.2 and we have $\sum_\tau r_0(f_\tau) = \sum_\tau (\# I_\tau) = \#[1,r] = r = r_0(f)$. □

For any convenient power series $f \in \mathbb{C}\{X,Y\}$ we put

$$r(f,\Delta(f)) = \sum_{S\in \mathcal{N}(f)} r(f,S) = \text{the number of quasi-tangents to the curve } f=0$$

$$r(\Delta(f)) = \sum_{S\in \mathcal{N}(f)} r(S) = \begin{array}{l}\text{the number of quasi-tangents counted with}\\ \text{tangential multiplicities to the curve } f=0.\end{array}$$

Obviously, $r(f,\Delta(f)) \leq r(\Delta(f))$ with equality if and only if f is nondegenerate. Note also that $r(\Delta(f))$ = the number of integral points lying on $\bigcup \mathcal{N}(f) - 1$. Hence the integral points divide $\bigcup \mathcal{N}(f)$ into $r(\Delta(f))$ segments.

PROPOSITION 4.8. *For any convenient power series $f \in \mathbb{C}\{X,Y\}$ we have*

$$r(f,\Delta(f)) \leq r_0(f) \leq r(\Delta(f)).$$

If f is nondegenerate, then $r_0(f) = r(\Delta(f))$ and the quasi-tangents to the branches of the local curve $f = 0$ have tangential multiplicity equal to 1. Different branches have different quasi-tangents.

PROOF. By Proposition 4.7 we have $r_0(f) = \sum_\tau r_0(f_\tau)$. Therefore, $r_0(f,\Delta(f)) = \sum_\tau 1 \leq r_0(f) \leq \sum_\tau \operatorname{ord} f_\tau \leq \sum_\tau \operatorname{ord} f_\tau(0,Y_\tau) = r(\Delta(f))$ since $\operatorname{ord} f_\tau(0,Y_\tau)$ equals the tangential multiplicity of τ (by Lemma 4.1) and the number of quasi-tangents counted with multiplicities associated with the face S is equal to $r(S)$. Suppose that f is nondegenerate. Then $r(f,\Delta(f)) = r(\Delta(f))$ and $r(f) = r(\Delta(f))$ by the first part of the proposition. We have $\operatorname{mult}(f,\tau) = \prod_{i=1}^r \operatorname{mult}(f_i,\tau)$ by Remark 3.6 and the assertion about the branches of the local curve $f = 0$ follows. □

EXAMPLE 4.9. Let $f = X^7 + X^5Y + X^3Y^2 + 2X^2Y^3 + XY^4 + Y^6$. Then the local curve $f = 0$ has four quasi-tangents: $Y^2 + X = 0$, $Y - \varepsilon X^2 = 0$, $Y - \bar{\varepsilon} X^2 = 0$, $(\varepsilon^2 + \varepsilon + 1 = 0)$, $Y + X = 0$. The quasi-tangent $\tau \colon Y + X = 0$ is of tangential multiplicity 2, the remaining quasi-tangents are of tangential multiplicity 1. Then $4 \leq r_0(f) \leq 5$. By Proposition 4.8, $r_0(f) = 3 + r_0(f_\tau)$. To calculate $r_0(f_\tau)$ we use the N-transformation $X = X_\tau$, $Y = (-1 + Y_\tau)X_\tau$.

We get $f(X_\tau, (-1+Y_\tau)X_\tau) = X_\tau^5 f_\tau(X_\tau, Y_\tau)$, where $f_\tau = -X_\tau$+higher order terms. We have $\operatorname{ord} f_\tau = 1$ and f_τ is irreducible, i.e. $r_0(f_\tau) = 1$. Consequently, $r_0(f) = 3 + 1 = 4$.

Notes

Although the Newton transformations appear when using the Newton algorithm [**6,15,23**], a systematic treatment of this notion was given quite recently in [**5**].

5. Newton transformations, intersection multiplicity and the Milnor number

The Minkowski double area $[\Delta, \Delta'] \in \mathbb{N} \cup \{\infty\}$ of the pair Δ, Δ' of Newton diagrams is uniquely determined by the following conditions

(M1) $[\Delta_1 + \Delta_2, \Delta'] = [\Delta_1, \Delta'] + [\Delta_2, \Delta']$,

(M2) $[\Delta, \Delta'] = [\Delta', \Delta]$,

(M3) $\left[\left\{\frac{a}{b}\right\}, \left\{\frac{a'}{b'}\right\}\right] = \inf\{ab', a'b\}$.

Lemma 5.1. *If* $\Delta = \sum_{S \in n(\Delta)} \left\{\frac{|S|_1}{|S|_2}\right\}$ *is a convenient Newton diagram then*

(*i*) $[\Delta, \Delta] = 2$ *area* $(\mathbb{R}^2_+ \setminus \Delta)$

(*ii*) $[\Delta, \Delta] = \sum_{S \in n(\Delta)} (|S|_1 |S|_2 + a(S)|S|_2 + b(S)|S|_1)$.

Proof. By (M1) and (M3) we get

$$[\Delta, \Delta] = \sum_{S,T \in n(\Delta)} \inf\{|S|_1|T|_2, |S|_2|T|_1)\},$$

which implies (*i*).
To check (*ii*) observe that $|S|_1|S|_2 + a(S)|S|_2 + b(S)|S|_1$ equals to the double area of triangle with vertices $(0,0)$, $(a(S), |S|_2 + b(S))$ and $(a(S) + |S|_1, b(S))$ and use (*i*). □

Lemma 5.2. *Let* Δ *be a Newton diagram. Then for every of Newton's edge* E *the supporting line of* Δ *parallel to* E *is described by the equation*

$$|E|_2\alpha + |E|_1\beta = \left[\left\{\frac{|E|_1}{|E|_2}\right\}, \Delta\right].$$

Proof. The lemma follows from Property 3.2 by putting $\theta = \frac{|E|_1}{|E|_2}$ into the formula for the abscissa of the point at which the supporting line intersects the axis $\beta = 0$. □

THEOREM 5.3. *Let f be a nonzero power series without constant term. Then for every convenient power series h:*

$$i_0(f,h) = [\Delta(f),\Delta(h)] + \sum_\tau i_0(f_\tau,h_\tau),$$

where the summation is over all quasi-tangents τ to the curve $h=0$.

PROOF. Fix a nonzero power series f without constant term. It is easy to check that if the theorem is true for two power series h_1, h_2, then it is true for their product h_1h_2. Thus it suffices to prove the theorem for irreducible power series h.

Let h be a convenient irreducible power series and let $\tau\colon Y^n - aX^m = 0$ be the unique quasi-tangent to the branch $h=0$ of tangential multiplicity d. Let $c = a^{1/n}$. Then $\Delta(h) = \left\{\frac{dn}{dm}\right\}$, $h(X_\tau^n,(c+Y_\tau)X_\tau^m) = X_\tau^{dmn}h_\tau(X_\tau,Y_\tau)$, $\operatorname{ord} h_\tau(0,Y_\tau) = d$ and $f(X_\tau^n,(c+Y_\tau)X_\tau^m) = X_\tau^k f_\tau(X_\tau,Y_\tau)$, $f_\tau(0,Y_\tau)\neq 0$ in $\mathbb{C}\{Y_\tau\}$.

Let $(t^d,\varphi(t))$ be a Puiseux parametrization of $h_\tau(X_\tau,Y_\tau)=0$.
Then $(t^{dn},(c+\varphi(t)t^{dm})$ is a Puiseux parametrization of $h(X,Y)=0$ and $i_0(f,h) = \operatorname{ord} f(t^{dn},(c+\varphi(t))t^{dm}) = dk + \operatorname{ord} f_\tau(t^d,\varphi(t)) = dk + i_0(f_\tau,h_\tau) = [\Delta(h),\Delta(f)] + i_0(f_\tau,h_\tau)$ by Lemma 5.2, since $\alpha dn + \beta dm = dk$ is the supporting line of $\Delta(f)$ parallel to the unique face of $\Delta(h)$. □

EXAMPLE 5.4. Let $f = Y^3+X^4Y-X^7$, $g = XY-(X^2+Y^2)^2$. Then $\Delta(f) = \left\{\frac{4}{2}\right\} + \left\{\frac{3}{1}\right\}$, $\Delta(g) = \left\{\frac{1}{3}\right\} + \left\{\frac{3}{1}\right\}$ and $[\Delta(f),\Delta(g)] = \inf\{4\cdot 3, 1\cdot 2\} + \inf\{4\cdot 1, 2\cdot 3\} + \inf\{3\cdot 3, 1\cdot 1\} + \inf\{3\cdot 1, 1\cdot 3\} = 10$. The local curves $f=0$ and $g=0$ have exactly one common quasi-tangent $\tau\colon Y - X^3 = 0$. The N-transformation associated with τ is $X = X_\tau$, $Y = (1+Y_\tau)X_\tau^3$. A simple calculation shows that $f_\tau = Y_\tau + (1+Y_\tau)^3X_\tau^2$ and $g_\tau = Y_\tau - 2X_\tau^4(1+Y_\tau)^2 - (1+Y_\tau)^3X_\tau^5$. Thus, $\Delta(f_\tau) = \left\{\frac{2}{1}\right\}$, $\Delta(g_\tau) = \left\{\frac{4}{1}\right\}$, the local curves $f_\tau = 0$, $g_\tau = 0$ have no common quasi-tangent and $i_0(f_\tau,g_\tau) = [\Delta(f_\tau),\Delta(g_\tau)] = \inf\{2\cdot 1, 1\cdot 4\} = 2$. By Theorem 5.3 we get $i_0(f,g) = [\Delta(f),\Delta(g)] + [\Delta(f_\tau),\Delta(g_\tau)] = 10+2 = 12$.

A pair of power series f,h is *nondegenerate* if the local curves $f=0$, $h=0$ have no common quasi-tangent. It is easy to check that the pair f,h is nondegenerate if and only if for $S\in\mathcal{N}(f)$ and $T\in\mathcal{N}(h)$ there is:

(a) either S and T are not parallel, i.e. $|S|_1|T|_2 \neq |S|_2|T|_1$, or
(b) the faces S and T are parallel and the system of equations $\operatorname{in}(f,S)(X,Y) = 0$, $\operatorname{in}(h,T)(X,Y) = 0$ has no solutions in $(\mathbb{C}\setminus\{0\})\times(\mathbb{C}\setminus\{0\})$.

Corollary 5.5 (see [**1**,**17**]). *Let f, h be nonzero power series without constant terms. Suppose that f or h is convenient. Then $i_0(f,h) \geq [\Delta(f), \Delta(h)]$ with equality if and only if the pair (f,h) is nondegenerate.*

For any convenient Newton diagram $\Delta = \sum_{S \in n(\Delta)} \left\{ \frac{|S|_1}{|S|_2} \right\}$ we put $|\Delta|_1 = \sum_{S \in n(\Delta)} |S|_1$, $|\Delta|_2 = \sum_{S \in n(\Delta)} |S|_2$. Then Δ intersects the axes at the points $(0, |\Delta|_2)$ and $(|\Delta|_1, 0)$.

Theorem 5.6. *For any convenient power series $h \in \mathbb{C}\{X,Y\}$:*

$$i_0\left(h, \frac{\partial h}{\partial Y}\right) = [\Delta(h), \Delta(h)] - |\Delta(h)|_1 + \sum_\tau i_0\left(h_\tau, \frac{\partial h_\tau}{\partial Y_\tau}\right),$$

where the summation is over all quasi-tangents τ to the local curve $h = 0$.

Proof. We may assume that $\frac{\partial h}{\partial Y}(0,0) = 0$. We will check that

(i) $[\Delta(h), \Delta\left(\frac{\partial h}{\partial Y}\right)] = [\Delta(h), \Delta(h)] - |\Delta(h)|_1$,

(ii) if τ is a quasi-tangent to the curve $h = 0$ then $\left(\frac{\partial h}{\partial Y}\right)_\tau = \frac{\partial h_\tau}{\partial Y_\tau}$.

Proof of (i). We have $[\Delta(h), \Delta\left(\frac{\partial h}{\partial Y}\right)] = \sum_{S \in n(\Delta)} \left[\left\{\frac{|S|_1}{|S|_2}\right\}, \Delta\left(\frac{\partial h}{\partial Y}\right)\right]$. The line $\alpha|S|_2 + \beta|S|_1 = \left[\left\{\frac{|S|_1}{|S|_2}\right\}, \Delta\left(\frac{\partial h}{\partial Y}\right)\right]$ supporting the diagram $\Delta\left(\frac{\partial h}{\partial Y}\right)$ and parallel to S passes through the point $(a(S), |S|_2 + b(S) - 1)$. Thus, $a(S)|S|_2 + (|S|_2 + b(S) - 1)|S|_1 = \left[\left\{\frac{|S|_1}{|S|_2}\right\}, \Delta\left(\frac{\partial h}{\partial Y}\right)\right]$ and we get $\left[\Delta(h), \Delta\left(\frac{\partial h}{\partial Y}\right)\right] = \sum_S (a(S)|S|_2 + |S|_2|S|_1 + b(S)|S|_1 - |S|_1) = [\Delta(h), \Delta(h)] - |\Delta(h)|_1$ by Lemma 5.1 (ii).
Proof of (ii). Let $\tau\colon Y^n - aX^m = 0$ be a quasi-tangent to the curve $h = 0$. Let $c = a^{1/n}$. There is

$$(\ ) \qquad h(X_\tau^n, (c + Y_\tau)X_\tau^m) = X_\tau^k h_\tau(X_\tau, Y_\tau) \quad \text{in} \quad \mathbb{C}\{X_\tau, Y_\tau\},$$

where $\alpha n + \beta m = k$ is a supporting line of $\Delta(h)$.
Then $k > m$ and the line $\alpha n + \beta m = k - m$ supports the diagram $\Delta\left(\frac{\partial h}{\partial Y}\right)$. Differentiating ( ) with respect to Y_τ, we get $\frac{\partial h}{\partial Y}(X_\tau^n, (c + Y_\tau)X_\tau^m)X_\tau^m = X_\tau^k \frac{\partial h_\tau}{\partial Y_\tau}(X_\tau, Y_\tau)$ and $\frac{\partial h}{\partial Y}(X_\tau^n, (c + Y_\tau)X_\tau^m) = X_\tau^{k-m}\frac{\partial h_\tau}{\partial Y_\tau}(X_\tau, Y_\tau)$. Therefore, $\left(\frac{\partial h}{\partial Y}\right)_\tau = \frac{\partial h_\tau}{\partial Y_\tau}$ and (ii) follows.
Now from Theorem 5.3 and properties (i), (ii) there follows:

$$\begin{aligned} i_0\left(h, \frac{\partial h}{\partial Y}\right) &= \left[\Delta(h), \Delta\left(\frac{\partial h}{\partial Y}\right)\right] + \sum_\tau i_0\left(h_\tau, \left(\frac{\partial h}{\partial Y}\right)_\tau\right) \\ &= [\Delta(h), \Delta(h)] - |\Delta(h)|_1 + \sum_\tau i_0\left(h_\tau, \frac{\partial h_\tau}{\partial Y_\tau}\right). \end{aligned} \qquad \square$$

For any convenient Newton diagram Δ we put

$$\mu(\Delta) = [\Delta, \Delta] - |\Delta|_1 - |\Delta|_2 + 1,$$
$$\delta(\Delta) = \frac{1}{2}(\mu(\Delta) + r(\Delta) - 1).$$

THEOREM 5.7. *Let $f \in \mathbb{C}\{X, Y\}$ be a convenient power series. Then*

(*i*) $\mu_0(f) = \mu_0(\Delta(f)) + r(\Delta(f)) + \sum_\tau (\mu_0(f_\tau) - 1)$,

(*ii*) $\delta_0(f) = \delta(\Delta(f)) + \sum_\tau \delta_0(f_\tau)$,

where the summation is over all quasi-tangents τ to the local curve $f = 0$.

PROOF. (*i*) By Teissier's lemma and Theorem 5.6 applied to the power series f we get

$$\begin{aligned}
\mu_0(f) &= i_0\left(f, \frac{\partial f}{\partial Y}\right) - i_0(f, X) + 1 \\
&= [\Delta(f), \Delta(f)] - |\Delta(f)|_1 - |\Delta(f)|_2 + 1 + \sum_\tau i_0\left(f_\tau, \frac{\partial f_\tau}{\partial Y_\tau}\right) \\
&= \mu(\Delta(f)) + \sum_\tau i_0\left(f_\tau, \frac{\partial f_\tau}{\partial Y_\tau}\right) \\
&= \mu(\Delta(f)) + \sum (\mu_0(f_\tau) + i_0(f_\tau, X) - 1) \\
&= \mu(\Delta(f)) + r(\Delta(f)) + \sum_\tau (\mu_0(f_\tau) - 1).
\end{aligned}$$

(*ii*)
$$\begin{aligned}
2\delta_0(f) &= \mu_0(f) + r_0(f) - 1 \\
&= \mu(\Delta(f)) + r(\Delta(f)) + \sum_\tau (\mu_0(f_\tau) - 1) + \sum_\tau r_0(f_\tau) - 1 \\
&= \mu(\Delta(f)) + r(\Delta(f)) - 1 + \sum_\tau (\mu_0(f_\tau) + r_0(f_\tau) - 1) \\
&= 2\delta(\Delta(f)) + 2\sum_\tau \delta_0(f_\tau)
\end{aligned}$$

and (*ii*) follows. □

We can rewrite formula 5.7 (*i*) for the Milnor number in the form

$$\mu_0(f) = \mu(\Delta(f)) + r(\Delta(f)) - r(f, \Delta(f)) + \sum_\tau \mu_0(f_\tau).$$

Then we get

Corollary 5.8. *For any convenient power series $f \in \mathbb{C}\{X,Y\}$:*

(*i*) $\mu_0(f) \geq \mu(\Delta(f)) + r(\Delta(f)) - r(f,\Delta(f))$ *with equality if and only if all strict transforms f_τ corresponding to the quasi-tangents τ of the curve $f=0$ are nonsingular,*

(*ii*) $\delta_0(f) \geq \delta(\Delta(f))$ *with equality if and only if* $\mu_0(f) = \mu(\Delta(f)) + r(\Delta(f)) - r(f,\Delta(f))$.

Corollary 5.9 (Kouchnirenko's Planar Theorem, see [**18**]).
For any convenient power series $f \in \mathbb{C}\{X,Y\}$:
$\mu_0(f) \geq \mu(\Delta(f))$ *with equality if and only if f is nondegenerate.*

Example 5.10. (cf. Example 4.9)
Let $f = X^7 + X^5Y + X^3Y^2 + 2X^2Y^3 + XY^4 + X^6$ and $\Delta = \Delta(f)$.
Then $\mu(\Delta) = 18$, $r(\Delta) = 5$, $r(f,\Delta) = 4$, $\delta(\Delta) = 11$. All strict transforms of f by N-transformations corresponding to the quasi-tangents to $f=0$ are nonsingular. Therefore, $\mu_0(f) = \mu(\Delta) + r(\Delta) - r(f,\Delta) = 19$ and $\delta_0(f) = \delta(\Delta) = 11$.

Example 5.11. (cf. Example 4.5)
Let $f = (X^2 - Y^3)^2 - Y^7$ and $g = (X^2-Y^3)^2 - XY^5$. Then $\Delta(f) = \Delta(g) = \Delta = = \left\{\frac{6}{4}\right\}$, $\mu(\Delta) = 15$, $r(f,\Delta) = r(g,\Delta) = 1$ and $r(\Delta) = 2$.
The strict transforms f_1 and g_1 of f and g by the N-transformation corresponding to the unique quasi-tangent $Y^3 - X^2 = 0$ of f and g are $f_1 = (3Y_1)^2 - X_1^2 + \cdots$ and $g_1 = -X_1 + \cdots$. Therefore we get $\mu_0(f) = \mu(\Delta) + r(\Delta) - r(f,\Delta) + \mu_0(f_1) = 16 + \mu_0(f_1) = 16+1 = 17$ and $\mu_0(g) = \mu(\Delta) + r(\Delta) - r(f,\Delta) + \mu_0(g_1) = 16 + \mu_0(g_1) = 16 + 0 = 16$.

We say that a local curve $f=0$ is in a general position with respect to coordinates (X,Y) if the axes $Y=0$ and $X=0$ are not tangent to $f=0$ (see Remark 3.4). Let $t_0(f)$ be the number of tangents to the curve $f=0$.

Corollary 5.12 (see [**4**] and appendix to [**29**]).
Suppose that the local curves $f=0$ and $g=0$ are in a general position with respect to (X,Y). Let $n = \mathrm{ord} f$ and $m = \mathrm{ord} g$.
Then

(*i*) $i_0(f,g) = nm + \sum_\tau i_0(f_\tau, g_\tau)$,

(*ii*) $\delta_0(f) = \frac{1}{2}n(n-1) + \sum_\tau \delta_0(f_\tau)$,

(*iii*) $i_0(f, \frac{\partial f}{\partial Y}) = n(n-1) + \sum_\tau i_0(f_\tau, \frac{\partial f_\tau}{\partial Y_\tau})$,

(*iv*) $\mu_0(f) + t_0(f) - 1 = n(n-1) + \sum_\tau \mu_0(f_\tau)$.

Notes
The formulae for the local invariants in terms of the Newton diagrams and Newton transformations (Theorems 5.3, 5.6, 5.7) are very close to Gwoździewicz's

formulae [**10**] in toric geometry of plane curve singularities (see also [**27**]) and like Newton trees and Newton process developed by Pi. Cassou-Noguès and Veys in [**5**] provide an effective method of calculations. The Newton number $\mu(\Delta)$ can be defined for all Newton diagrams Δ in such a way that Kouchnirenko's theorem holds for any reduced power series (see [**7, 21, 36**]. Corollary 5.8 provides a new characterization of weakly Newton nondegenerate singularities (see [**9**], Theorem 3.3).

6. Nondegenerate singularities and equisingularity

In this section we will prove that the equisingularity class of the curve $f = 0$ can be recovered form the Newton diagram $\Delta(f)$, provided that f is a convenient and nondegenerate power series.

Lemma 6.1. *Let $f \in \mathbb{C}\{X, Y\}$ be a convenient power series such that the curve $f = 0$ has exactly one quasi-tangent. If its tangential multiplicity is equal to 1, then f is irreducible and $\Gamma(f) = i_0(f, X)\mathbb{N} + i_0(f, Y)\mathbb{N}$.*

Proof. Let $m = i_0(f, Y)$, $n = i_0(f, X)$. Then $\gcd(m, n) = 1$ and after multiplying f by a constant, we may assume (see Remark 3.5) that $f = Y^n + aX^m + \sum c_{\alpha\beta} X^\alpha Y^\beta$ where $a \neq 0$ and the summation is over (α, β) such that $\alpha n + \beta m > mn$. The power series f is irreducible by Corollary 4.4.

To prove that $\Gamma(f) = \mathbb{N}n + \mathbb{N}m$, we follow [**39**] (proof of Theorem 3.9). Consider the intersection number $i_0(f, g)$ where $g \in \mathbb{C}\{X, Y\}$ is not a multiple of f. By the Weierstrass Division Theorem we may assume that $g = g_0(X) + g_1(X) + \cdots + g_{n-1}(X)Y^{n-1} \in \mathbb{C}\{X\}[Y]$. We have $i_0(f, g_k(X)Y^{n-k}) = (\operatorname{ord} g_k)n + (n - k)m \equiv (n - k)m \pmod{n}$. If $k, l < n$ and $k \neq l$, then $(k - l)m \not\equiv 0 \pmod{n}$. Thus $i_0(f, g_k(X)Y^{n-k}) \neq i_0(f, g_l(X)Y^{n-l})$ for $k \neq l$ and by Property 5 of intersection multiplicity (Section 1) we get $i_0(f, g) = i_0(f, g_k(X)Y^{n-k})$ for a $k \in [0, n - 1]$, which implies $\Gamma(f) \subset \mathbb{N}n + \mathbb{N}m$. Since $n, m \in \Gamma(f)$, there is $\Gamma(f) = \mathbb{N}n + \mathbb{N}m$. $\square$

Lemma 6.2. *Let $f \in \mathbb{C}\{X, Y\}$ be a convenient, nondegenerate power series and let $f = \prod_{i=1}^{r} f_i$ with $f_i \in \mathbb{C}\{X, Y\}$ irreducible. For any $S \in \mathcal{N}(f)$ we put $I(S) = \{i \in [1, r] \colon \frac{i_0(f_i, Y)}{i_0(f_i, X)} = \frac{m_S}{n_S}\}$. Then*

(1) $\Gamma(f_i) = n_S\mathbb{N} + m_S\mathbb{N}$ *for* $i \in I(S)$,

(2) $i_0(f_i, f_j) = \inf\{m_S n_T, m_T n_S\}$ *for* $(i, j) \in I(S) \times I(T)$, $i \neq j$.

Proof. By Proposition 4.8 the irreducible factors f_i, $i = 1, \ldots, r$ satisfy the assumptions of Lemma 6.1. Thus $\Gamma(f_i) = i_0(f_i, X)\mathbb{N} + i_0(f_i, Y)\mathbb{N} = n_S\mathbb{N} + m_S\mathbb{N}$ (we have $i_0(f_i, X) = m_S$, $i_0(f_i, Y) = n_S$ since $i_0(f, X)$, $i(f_i, X)$ and $m_S = |S|_1/r(S)$, $n_S = |S|_2/r(S)$ are coprime) and we get (1). By Proposition 4.8 the pairs f_i, f_j, $i \neq j$ are nondegenerate, then (2) follows from Corollary 5.5. $\square$

Remark 6.3. The quasi-tangents to the branches $f_i = 0$, $i \in I(S)$ are exactly the quasi-tangents to the curve $f = 0$ corresponding to the face $S \in n(f)$. Therefore, $\#I(S) = r(S)$ and $\bigcup_{S \in n(f)} I(S) = [1, r]$.

Theorem 6.4 (see [**21**]).
Let $f, g \in \mathbb{C}\{X, Y\}$ be convenient power series such that $\Delta(f) = \Delta(g)$. Then

- (i) *if f, g are nondegenerate, then the curves $f = 0$ and $g = 0$ are equisingular,*
- (ii) *if f is nondegenerate, but g is degenerate, then the curves $f = 0$ and $g = 0$ are not equisingular.*

Proof. Let $\Delta = \Delta(f) = \Delta(g)$ and $r = r(\Delta)$.
(i) We have $r(f) = r(g) = r$ by Proposition 4.8. Moreover, we can label the irreducible factors f_i of f and g_i of g ($i = 1, \ldots, r$) in such a way that

$$\frac{i_0(f_i, Y)}{i_0(f_i, X)} = \frac{i_0(g_i, Y)}{i_0(g_i, X)} \quad \text{for} \quad i = 1, \ldots, r.$$

Therefore, $\Gamma(f_i) = \Gamma(g_i)$ for all $i = 1, \ldots, r$ and $i_0(f_i, f_j) = i_0(g_i, g_j)$ for $i, j \in \{1, \ldots, r\}$ by Lemma 6.2, that is $f_i \mapsto g_i$ is an equisingularity bijection and the curves $f = 0$ and $g = 0$ are equisingular.
(ii) By Kouchnirenko's theorem (Corollary 5.9) we have $\mu_0(f) = \mu(\Delta)$ and $\mu_0(g) > \mu(\Delta)$. Therefore, $\mu_0(f) \neq \mu_0(g)$ and the curves $f = 0$ and $g = 0$ are not equisingular by Theorem 2.16. □

Remark 6.5. We have proved that if f and g are nondegenerate, then the equisingularity bijection $f_i \mapsto g_i$ preserves the intersection multiplicities of the branches with the axes: $i_0(f_i, X) = i_0(g_i, X)$, $i_0(f_i, Y) = i_0(g_i, Y)$ for $i = 1, \ldots, r$.

Remark 6.6. We can weaken the assumption "f, g convenient" of Theorem 6.4 by assuming only that f, g have no multiple factors. To prove this it suffices to use Theorem 6.4 and Remark 6.5.

Example 6.7. Let $\Delta \in \mathbb{R}^2_+$ be a convenient Newton diagram with vertices $(\alpha_0, \beta_0), \ldots, (\alpha_m, \beta_m)$ where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_m$ and $\beta_0 > \beta_1 > \cdots > \beta_m = 0$.
Then the series $f_0 = X^{\alpha_0} Y^{\beta_0} + \cdots + X^{\alpha_m} Y^{\beta_m}$ is nondegenerate and $\Delta(f_0) = \Delta$.

Let us consider an invariant I of equisingularity. For every convenient Newton diagram Δ, we put $I(\Delta) = I(\Delta(f))$ where f is a nondegenerate power series. According to Theorem 6.4, $I(\Delta)$ is defined correctly (does not depend on f). There is a natural problem: calculate $I(\Delta)$ effectively in terms of Δ. The best known result of this kind is due to Kouchnirenko, see [**18**] and Corollary 5.9 in this note.

Notes

The nondegenerate plane curve singularities may be characterized without refering to the coordinates [**7**]. An unexpected example of degeneracy is discussed in [**3**]. A lot of invariants of nondegenerate singularities are computed in terms of their Newton diagrams: see survey articles [**28**] and [**11**]. A description of the adjoints to the local nondegenerate hypersurface is given in [**25**]. The Newton diagrams and the notion of non-degeneracy are also useful in real analytic geometry [**20**].

References

1. Bernstein D. N., *The number of roots of a system of equations*, Funct. Anal. Appl., **9**, No. **3** (1975), 183–185.
2. Brieskorn E., Knoerr H., *Plane algebraic curves*, Birkhäuser, 1986.
3. Brzostowski Sz., *Degenerate singularities and their Milnor numbers*, Univ. Iagel. Acta Math., **49** (2011), 37–44.
4. Casas-Alvero E., *Singularities of plane curves*, London Math. Soc. Lecture Notes, **276**, Cambridge University Press, 2000.
5. Cassou-Noguès Pi., Veys W., *Newton trees for ideals in two variables and applications*, arXiv: 1201.0467 v 1 [math.AG] 2 Jan 2012.
6. Duval D., *Rational Puiseux expansions*, Composito Math., **70** (1989), 119–154.
7. García Barroso E., Lenarcik A., Płoski A., *Characterization of non-degenerate plane curve singularities*, Univ. Iagel. Acta Math., **45** (2007), 27–36. Erratum, ibidem. **47** (2009), 321–322.
8. Greuel G. M., Lossen Ch., Shustin E., *Introduction to singularities and deformations*, Springer Verlag, 2006.
9. Greuel G. M., Nguyen Hong Duc, *Some remarks on the planar Kouchnirenko's theorem*, arXiv: 1009.4889 v 1 [math. AG] 24 Sep. 2010.
10. Gwoździewicz J., *Generalized Noether's Formulas for plane curve singularities*, Univ. Iagel. Acta Math., **48** (2010), 55–62.
11. Gwoździewicz J., Lenarcik A., Płoski A., *Polar invariants of plane curve singularities: intersection theoretical approach*, Demonstratio Math., **43**, N° **2** (2010), 303–323.
12. Gwoździewicz J., Płoski A., *Formulae for the singularities at infinity of plane algebraic curves*, Univ. Iagel. Acta Math., **39** (2001), 109–133.
13. Hironaka H., *On the arithmetic genera and the effective genera of algebraic curves*, Mem. Coll. Sci. Univ. Kyoto. Ser. A. Math., **30** (1957), 177–195.
14. Jung H. W. E., *Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen*, Walter de Gruyter & Co, Berlin und Leipzig, 1923.
15. de Jong T., Pfister G., *Local analytic geometry. Basic theory and applications*, Vieweg, 2000.
16. Khovanski A. G., *Newton polyhedra and toric varieties*, Funkt. Anal. Prloz., **11**(**4**) (1977), 56–67 and **12**(**1**) (1978), 51–61.
17. Khovanski A. G., *The index of polynomial vector field*, (Russian), Funkt. Anal. Prloz., **13**(**1**) (1979), 49–58.
18. Kouchnirenko A. G., *Polyèdres de Newton et nombres de Milnor*, Invent. Math., **32** (1976), 1–31.

19. Kuo T. C., *Generalized Newton-Puiseux theory and Hensel's lemma in* $\mathbb{C}[[x, y]]$, Canad. J. Math., **41**, No. **6** (1989), 1101–1116.
20. Kucharz W., *Newton polygons and topological determinancy of analytic germs*, Periodica Math. Hungarica, **22**(**2**) (1991), 129–132.
21. Lenarcik A., *On the Jacobian Newton polygon of plane curve singularities*, Manuscripta Math., **125** (2008), 309–324.
22. Lipovski A., *Newton polyhedra and irreducibility*, Math. Z., **199** (1988), 119–127.
23. Maurer J., *Puiseux expansion for space curves*, Manuscripta Math., **32** (1980), 91–100.
24. Masternak M., *Invariants of singularities of polynomials in two complex variables and the Newton diagrams*, Univ. Iagel. Acta Math., **39** (2001), 179–188.
25. Merle M., Teissier B., *Conditions d'adjonction d'après du Val*, Séminaire sur les Singularités des Surfaces, Palaiseau, France 1976–77, Lecture Notes in Math. Springer, **777**, 230–245.
26. Milnor J., *Singular Points of Complex Hypersurfaces*, Princeton University Press, 1968.
27. Oka M., *Geometry of plane curves via toroidal resolution*, Algebraic Geometry and Singularities (eds. A. Campillo López and Norrváez Macarro), Progr. Math., **134**, Birkhäuser, Basel (1996), 95–121.
28. Pfister G., *Invariants of singularities and Newton polyhedron*, Proceedings of the Week of Algebraic Geometry, Bucharest 1980, Teuber-Texte zur Mathematik, Band **40**, 123–138.
29. Pham E., *Courbes discriminantes des singularités planes d'ordre 3*, Astérisque, **7–8** (1973), 363–391.
30. Risler J. J., *Sur l'idéal jacobian d'une courbe planes*, Bull. Soc. Math. Fr., **99**(**4**) (1971), 305–311.
31. Teissier B., Thése, 2° partie, Paris 1973.
32. Teissier B., *Cycles évanescents, section planes et conditions de Whitney*, Astérisque, No **7–8** (1973), 285–362.
33. Teissier B., *The hunting of invariants in the geometry of discriminants*, Nordic Summer School/NAVF Symposium in Mathematics, Oslo, August 5–25, 1976.
34. Teissier B., *Introduction to Curve Singularities*, Singularity Theory, Eitors D. T. Lê, K. Saito, B. Teissier, World Scientific 1991.
35. Varchenko A. N., *Zeta function of monodromy and Newton's diagram*, Invent. Math., **37** (1976), 253–262.
36. Wall C. T. C., *Newton polytopes and nondegeneracy*, J. reine angew. Math., **509** (1999).
37. Wall C. T. C., *Singular Points of Plane Curves*, London Mathematical Society, Student Text **63**, Cambridge University Press, 2004.
38. van der Waerden B. L., *Einführung in die algebraische Geometrie*, Springer, Berlin, 1939.
39. Zariski O., Le problème des modules pour les branches planes. Centre de Mathématiques de l'École Polytechnique, Paris, 1973.

Received December 31, 2011

Institut de Matématiques de Bordeaux
Université Bordeaux I, 350, Cours de la Libération
33 405, Talence Cedex 05, France
e-mail: `picassou@math.u-bordeaux1.fr`

Department of Mathematics
Kielce University of Technology
Al. 1000L PP 7, 25-314 Kielce, Poland
e-mail: `matap@tu.kielce.pl`