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Classification in Finite Model Theory[1]

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Given a class $\mathcal{C}$ of structures two basic questions about the class $\mathcal{C}$ are:

- what are the cardinalities of structures in $\mathcal{C}$,
- how many non-isomorphic structures of a given cardinality are in $\mathcal{C}$.

The first question is a problem of determining the *spectrum* $\mathrm{Spec}(\mathcal{C})$ of the class $\mathcal{C}$, that is, the class of cardinalities that occur as sizes of structures in $\mathcal{C}$. This problem appears in all sorts of contexts in mathematics. For example, if $\mathcal{C}$ is the class of models of a first order theory in a countable language, then by the Löwenheim–Skolem theorems, every infinite cardinality is in $\mathrm{Spec}(\mathcal{C})$. Thus, $\mathrm{Spec}(\mathcal{C})$ is usually defined to be the set of sizes of finite models in $\mathcal{C}$. Let us here mention a result of R. Fagin [11] that characterizes those sets of integers that can be expressed as $\mathrm{Spec}(\mathcal{C})$, where $\mathcal{C}$ is a class of all models of a first order sentence in a first order language. They are exactly the sets that can be recognized in Nondeterministic Exponential time. He also proved a similar connection between Nondeterministic Polynomial time and what are called generalized spectra.

The second basic question is about the *fine spectrum* of theclass $\mathcal{C}$, i.e., about the function that assigns to each cardinal k the number of non-isomorphic k-element structures in $\mathcal{C}$. The problem of determining the fine spectrum of a class, either exactly or asymptotically, is a natural one, and is a problem that has been considered for various classes $\mathcal{C}$ and in various contexts over the years. Combinatorial enumeration problems such as finding the number of k-vertex graphs in some specific class of graphs are familiar examples of such fine spectra problems.

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In Model Theory one of the fundamental topics is Shelah's classification theory. Given a first order theory T we are interested in the number of non-isomorphic models of T of a given (infinite) cardinality k. Moreover, we want to classify theories with the number of non-isomorphic models of infinite cardinality k given by a prescribed function $f(k)$. Some of the results here say that not every function f can be realized. For example, an early result of M. Morley says that if T is a complete theory in a countable language and T is κ-categorical for some uncountable cardinal, then T is categorical in all uncountable powers.

When we are interested in finite models only, i.e., restricting k to be finite, and in integer valued functions f, then the situation changes dramatically. For any function $f : \omega \longrightarrow \omega$ there is a first order theory T having exactly $f(k)$ models with k elements. Simply let T be axiomatized by sentences expressing:

> *if a model has k elements, then it is isomorphic with one of the $f(k)$ choices given by their diagrams.*

Even restricting the language to be finite leaves a lot of room for possible fine spectra. A function $f : \omega \longrightarrow \omega$ counts non-isomorphic models of a first order theory in a finite language if and only if $\log f(k)$ is bounded from above by a polynomial in k.

Therefore, the finite part of a fine spectrum problem is much more interesting in case of restricted theories, e.g. in equational theories of algebras. Actually, the terminology "fine spectrum" was introduced by W. Taylor in [27] in the case that $\mathcal{C}$ is a variety, that is, a class of algebraic structures or algebras closed under the formation of homomorphisms, subalgebras and direct products. Among the results of Taylor's paper is a characterization of those varieties that have exactly one model of size 2^k for each finite k and no other finite models. All such varieties must be generated by a 2-element algebra. Subsequent papers by R. Quackenbush [24] and D. Clark and P. Krauss [9] extended Taylor's work to n-element algebras that generate varieties having minimal fine spectra.

Despite this work on minimal fine spectra there has been relatively little success on general problems involving fine spectra of varieties. In [28] Taylor writes of fine spectra functions that "Characterizations of such functions seems hopeless" and Quackenbush states in his survey on enumeration problems for ordered structures [25] that the fine spectrum problem here "is usually hopeless".

There are many reasons for the difficulties in determining the fine spectra of varieties. One reason that interests us is that algebras are often described by means of a set of generators. Once we know the generators of an algebra $\mathbf{A}$ in a variety $\mathcal{V}$ and some of the conditions that these generators satisfy, our freedom in building the rest of the model is heavily restricted. This effect is widely used in group theory where a group is usually presented by a (finite) set of generators and a set of relations the generators must obey. The constraints put on the behavior of the generators place restrictions on the structure of the entire algebra **A**. However, in general there is no obvious or transparent way to determine the cardinality of **A**. This makes the counting of all n-element or at most n-element algebras difficult, if at all possible, even if we content ourselves with an asymptotic estimate.

Another area of research in which fine spectra appear is in finite model theory. When investigating asymptotic probabilities in finite model theory, some of the results rely on counting all finite models, up to isomorphism, for a given theory T. This usually is not hard when T has no axioms. However, there are only a few results on zero–one (or more generally on limit) laws for specific theories T. One reason is that for such counting a deep insight into the structure of finite models of T is required. The counting is even more difficult if the language of T contains function symbols. Except for unary functions [21] (in which case the models behave much more like relational structures than algebras) and Abelian groups [10] (where we completely understand the structure) there are only a few other results on limit laws for algebras to report. The reader may wish to consult [4, 5, 6, 7, 8, 20] for related work.

Again, there are various reasons for the difficulties here with fine spectra for classes arising from theories in a language containing function symbols. Some important techniques for asymptotic probabilities rely on extension axioms. These techniques work perfectly well for purely relational structures when often the resulting random structure is model complete. However (randomly), adding a new element a to the universe of an algebra **A** in a variety $\mathcal{V}$ and keeping the resulting extended algebra in $\mathcal{V}$ requires a much bigger extension $\mathbf{A}^*$ of **A**. The number and the behavior of all those new elements in $\mathbf{A}^*$ is fully determined by the old algebra **A** and the interaction of the new element a with the elements of A. Thus, in vector spaces by adding a new element we actually increase the dimension.

One possible way to overcome these difficulties is to count k-generated models instead of k-element ones. This is a more tractable problem when the classes are varieties of algebras, and we believe it is the proper setting for asymptotic probabilities in algebra. Note, however, that these numbers are the same for purely relational languages.

Thus, we introduce the *G-spectrum*, or *generative complexity*, of a class $\mathcal{C}$, which is the function $\mathrm{G}_{\mathcal{C}}(k)$ that counts the number of non-isomorphic (at most) k-generated models in $\mathcal{C}$.

We concentrate on the case that $\mathcal{C}$ is a variety of algebras and restrict ourselves to finite k. Even in this new setting the counting remains hard.

It requires an understanding of the 'generating power' in a given variety $\mathcal{V}$. This is related to the old problem of determining the *free spectrum* of $\mathcal{V}$, that is, the sizes of free algebras $\mathbf{F}_{\mathcal{V}}(k)$ in $\mathcal{V}$ with $k = 1, 2, \ldots$ free generators. This is because every k-generated algebra in $\mathcal{V}$ is isomorphic to a quotient of $\mathbf{F}_{\mathcal{V}}(k)$ by a congruence relation. However, two different congruences may give rise to isomorphic quotients. Thus, the second problem we meet here is to measure the amount of homogeneity in $\mathbf{F}_{\mathcal{V}}(k)$. One cannot hope to solve these two problems without understanding the structure of algebras from $\mathcal{V}$.

As we have already mentioned the infinite counterpart of the problem of counting non-isomorphic models is widely studied in Model Theory, and is one of the fundamental topics for Shelah's classification theory and for stability theory. Note that in the infinite realm (and for a countable language) being κ-generated and having κ elements are the same.

For example, the celebrated Vaught conjecture says that the number of non-isomorphic countable models of any first order theory in a countable language is either countable or 2^ω. In [15] and [16] B. Hart, S. Starchenko and M. Valeriote have been able to prove this conjecture for varieties of algebras. They actually determined the possible infinite fine spectra (which, in this case, are the same as infinite G-spectra) of varieties and correlated them with algebraic properties of varieties. A characterization of locally finite varieties with strictly less than 2^ω non-isomorphic countable models can be easily inferred from a deep work on decidability done by R. McKenzie and M. Valeriote [23].

However, one cannot easily (if at all) transfer infinite methods and results into the finite world. Thus, we focus on the G-spectrum of a variety rather than its fine spectrum. We further restrict ourselves to locally finite varieties, i.e., varieties in which all finitely generated algebras are finite. This gives that the G-spectrum of the variety $\mathcal{V}$ is integer-valued. Even with this finiteness restriction G-spectra can be arbitrarily large: for any sequence $(p_k)_{k\in\omega}$ of integers there is a locally finite variety $\mathcal{V}$ of groupoids such that $G_{\mathcal{V}}(k) \geq p_k$ for all k (Example 5.9 in [2]). On the other hand, if $\mathcal{V}$ is a finitely generated variety of finite type then easily established upper bounds for the G-spectrum, free spectrum and fine spectrum of $\mathcal{V}$ are $2^{2^{2^{ck}}}$, $2^{2^{ck}}$ and 2^{ck^s}, respectively, for some constants c and s.

The main problems that stimulate our research in this area are:

- *How does the growth of the G-spectrum of a locally finite variety $\mathcal{V}$ affect the structure of algebras in $\mathcal{V}$?*
- *In what way do algebraic properties of $\mathcal{V}$ influence the behavior of* $\mathrm{G}_{\mathcal{V}}(k)$*?*
- *For a given variety $\mathcal{V}$, can an explicit formula for* $\mathrm{G}_{\mathcal{V}}(k)$ *be found?*

- *Can the asymptotic behavior of* $\mathrm{G}_{\mathcal{V}}(k)$ *be determined?*

For some $\mathcal{V}$, the value of $\mathrm{G}_{\mathcal{V}}(k)$ is easily determined as the following examples show.

Example 1. For the variety $\mathcal{V}$ of sets, that is, algebras with no fundamental operations, $\mathrm{G}_{\mathcal{V}}(k) = k$. In Section 3 of [2] a full description of G-spectra for varieties generated by arbitrary multi-unary algebras is given.

Example 2. Let $\mathcal{V}$ be the variety of vector spaces over a fixed field. A k-generated algebra here is a vector space of dimension at most k. So $\mathrm{G}_{\mathcal{V}}(k) = k + 1$. If $\mathcal{A}_p$ is the variety generated by the p-element group for a prime p, then a k-generated group in $\mathcal{A}_p$ has size p^m, for $0 \leq m \leq k$. So $\mathrm{G}_{\mathcal{A}_p}(k) = k+1$. In Example 6.7 of [2] the function $\mathrm{G}_{\mathcal{V}}(k)$ for $\mathcal{V}$ an arbitrary, finitely generated variety of Abelian groups is determined.

Example 3. For the variety $\mathcal{B}$ of Boolean algebras we have $|\mathrm{F}_{\mathcal{B}}(k)| = 2^{2^k}$. Every k-generated member of $\mathcal{B}$ is a Boolean algebra with m atoms, $0 \leq m \leq 2^k$. Thus, $\mathrm{G}_{\mathcal{B}}(k) = 1 + 2^k$.

Example 4. In contrast to these varieties that have small G-spectra, Section 4 of [2] shows that semilattices and distributive lattices have $\mathrm{G}_{\mathcal{V}}(k)$ that are doubly exponential functions of k. Actually, it is shown that $\mathrm{G}_{\mathcal{S}}(k) = 2^{\binom{k}{\lfloor k/2 \rfloor}(1+o(1))}$ for the variety $\mathcal{S}$ of semilattices and $\mathrm{G}_{\mathcal{D}}(k) = 2^{2^k(1+o(1))}$ for the variety $\mathcal{D}$ of distributive lattices.

Moreover, Section 5 of [2] contains examples of locally finite varieties with arbitrarily large G-spectra.

To report the results on G-spectra we introduce some classes of functions that are helpful in describing the possible behavior of the $\mathrm{G}_{\mathcal{C}}(k)$.

Definition 1. Let f be a real-valued function of the positive integers.

- We say f is *at most 0-fold exponential* if there exists a polynomial p such that $f(k) \leq p(k)$ for all k.
- For $m > 0$ the function f is *at most m-fold exponential* if there is an at most $(m-1)$-fold exponential function g such that $f(k) \leq 2^{g(k)}$ for all k.
- The function f is called *at least 0-fold exponential* if there exists a constant $c > 0$ such that $f(k) \geq ck$ for all but finitely many k.

- For $m > 0$ the function f is *at least m-fold exponential* if there is an at least $(m-1)$-fold exponential function g such that $f(k) \geq 2^{g(k)}$ for all but finitely many k.
- The function f is of *m-fold exponential complexity* if f is both at most and at least m-fold exponential.
- A class $\mathcal{C}$ of structures has m-fold exponential generative complexity if the function $\mathrm{G}_{\mathcal{C}}(k)$ is of m-fold exponential complexity.

We usually write polynomial, exponential, doubly exponential, and triply exponential in place of 0-fold exponential, 1-fold exponential, 2-fold exponential, and 3-fold exponential.

DEFINITION 2. Let $\mathcal{V}$ be a locally finite variety.

- $\mathcal{V}$ has *many models* if $\mathrm{G}_{\mathcal{V}}(k)$ is at least doubly exponential.
- $\mathcal{V}$ has *few models* if $\mathrm{G}_{\mathcal{V}}(k)$ is at most exponential.
- $\mathcal{V}$ has *very few models* if $\mathrm{G}_{\mathcal{V}}(k)$ is at most polynomial.

The research reported here was motivated by a desire to know which locally finite varieties have few and very few models, respectively. Although we have not managed to solve these problems in the most general setting we have obtained such a characterization for a very broad class of varieties including most known and well-studied types of algebras, such as groups, rings, modules, lattices.

The proofs of the results give a deep insight into the structure of locally finite varieties with few and very few models. The analysis relies heavily on two major developments of the late 70's and early 80's. One of them is *modular commutator theory.* The theory had been introduced by J.D.H. Smith [26] for congruence permutable varieties. It was further developed by J. Hagemann and Ch. Herrmann [14], H.P. Gumm [13] and R. Freese and R. McKenzie [12]. The book of Freese and McKenzie contains several important results and techniques that are extremely useful when studying congruence modular varieties. A binary operation on congruences that simultaneously generalizes the concept of a commutator$[H, K]$ of two normal subgroups H, K of a group as well as the ideal multiplication in rings is defined. The theory shows how some information about algebras or varieties can be recovered from congruence lattices endowed with this binary operation. Moreover, the concept of the commutator allows us to speak about a solvable, nilpotent or Abelian congruence (or algebra) as well as about the center of an algebra or the centralizer of a congruence relation.

The second big development in the universal algebra that we use is *tame congruence theory*, which was created and described in D. Hobby and R. McKenzie [17]. Tame congruence theory is a tool for studying the local structure of finite algebras. Instead of considering the whole algebra and all of its operations at once, tame congruence theory allows us to localize small subsets on which the structure is much simpler to understand and to handle. According to this theory there are only five possible ways a finite algebra can behave locally. The local behavior must be one of the following:

1. a finite set with a group action on it,

2. a finite vector space over a finite field,

3. a two element Boolean algebra,

4. a two element lattice,

5. a two element semilattice.

Now, if from our point of view a local behavior of an algebra is 'bad', then we can often show that the algebra itself behaves 'badly'. For example, since the varieties of distributive lattices or semilattices have many models (see Example 4), then one can argue that in any locally finite variety with few models structures of type **4** or **5** cannot occur.

On the other hand, it is not true that if the local behavior is 'good' then the global one is as well. Several kinds of interactions between these small sets can produce a fairly messy global behavior. Such interactions often contribute to produce many models. Also the relative 'geographical layout' of those small sets can result in unpredictable phenomena.

The main results of the work in this area are stated in the following theorems. The first one gives a full characterization of locally finite varieties omitting type **1** with polynomially many models:

THEOREM 1. **(P. Idziak and R. McKenzie [18])**
A locally finite variety omitting type **1** *has very few models if and only if it is an affine variety over a ring of finite representation type.*

Very recently this characterization was extended to all locally finite varieties:

THEOREM 2. **(P. Idziak, R. McKenzie and M. Valeriote [19])**
A locally finite variety has very few models if and only if it decomposes into a varietal product of an affine variety over a ring of finite representation type, and a sequence of strongly Abelian varieties equivalent to matrix powers of varieties of G-sets, with constants, for various finite groups G.

The work on characterizing locally finite varieties with at most singly exponentially many models is still in progress. However, for the case of finitely generated varieties omitting type **1** the following characterization (with more than 100 pages proof) was obtained:

THEOREM 3. **(J. Berman and P. Idziak [2])**
Let $\mathcal{V}$ be a finitely generated variety omitting type **1**. *Then $\mathcal{V}$ has few models if and only if the following conditions hold:*

- **(1)** *$\mathcal{V}$ is congruence permutable,*
- **(2)** *for any finite subdirectly irreducible algebra* **A** *in $\mathcal{V}$ with monolith μ and its centralizer $\nu = (0 : \mu)$ we have:*
  - **(2.1)** *ν is the solvable radical of* **A**,
  - **(2.2)** *ν is comparable to all congruences of* **A**,
  - **(2.3)** *ν is Abelian or* **A** *is nilpotent,*
  - **(2.4)** *the quotient* $\mathbf{A}/\nu$ *is either trivial or simple non-Abelian,*
- **(3)** *the variety $\mathcal{N}$ of all nilpotent algebras in $\mathcal{V}$ has a finitely generated clone and $\mathcal{N}$ itself is generated by finitely many finite algebras each being of prime power order,*
- **(4)** *for any finite simple non-Abelian algebra* **S** *in $\mathcal{V}$ the ring* $\mathbf{R}^{\mathcal{V}}_{\mathbf{S}}$ (connected with the Abelian part of subdirectly irreducible algebras **A** in $\mathcal{V}$ with the quotient $\mathbf{A}/\nu$ isomorphic to **S**) *is of finite representation type.*

The necessity of the conditions in the above theorems was shown by detecting more than two dozens different ways in which a 'bad' local behavior can occur in an algebra **A**. In each such situation we are able to produce many non-isomorphic k–generated algebras in the variety generated by **A**. After detecting all those instances of 'bad' local behavior we formulate global algebraic conditions that forbid such undesirable behavior. More surprisingly we are able to show the list of conditions we obtain is actually complete, at least for finitely generated varieties, i.e., these conditions taken together are sufficient for a variety to have few (or very few) models.

The conditions given in Theorems 1 and 2 are very simple and easily stated. The conditions involved in the second characterization (Theorem 3) are more complicated, although they also have a natural algebraic meaning. In both cases we know that the bound for the number of algebras implies a very transparent and manageable structure. For example, when specializing our results to groups we get the following:

- every finitely generated variety of groups has at most doubly exponentially many models,
- a finitely generated variety of groups has few models if and only if it is nilpotent,
- a finitely generated variety of groups has very few models if and only if it is Abelian.

while for commutative rings with unit our characterization reduces to:

- every finitely generated variety of rings has at most doubly exponentially many models,
- a finitely generated variety of commutative rings with unit has few models iff the Jacobson radical in the generating ring squares to 0,
- no nontrivial variety of rings with unit has very few models.

The work on this project had started in the fall of 1996 during the semester on Algebraic Model Theory in the Fields Institute in Toronto.

Since then several researchers have also worked on the topic. Here are the results on G-spectra we wish to report:

- J. Berman and P. Idziak [2] contains most of the material described in this talk. It introduces and motivates the notion of generative complexity, provides many examples and contains a proof of Theorem 3.
- J. Berman and P. Idziaks [1] manuscript described G-spectra of all Post varieties, i.e., varieties generated by a single two-element algebra.
- M. Bilski [3] characterized finitely generated varieties of semigroups with very few models.
- P. Idziak and R. McKenzie [18] succeeded in characterizing locally finite varieties omitting type **1** with very few models.
- P. Idziak, R. McKenzie and M. Valeriote [19] extended the above characterization of varieties with very few models to all locally finite varieties.
- R. McKenzie [22] produced several examples of locally finite varieties with arbitrarily large free spectra and G-spectra.

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