UNIVERSITATIS IAGELLONICAE ACTA MATHEMATICA, FASCICULUS XL
2002

THE WU METRIC IN ELEMENTARY REINHARDT DOMAINS

by Piotr Jucha

Abstract. In the paper we give the explicit formula for the Wu metric in elementary Reinhardt domains.

Introduction. A new holomorphically invariant Hermitian pseudometric was introduced by H. Wu in paper [**4**]. We will call it *the Wu pseudometric*. Theorem 0 gives a short description of the pseudometric.

First let us introduce some notions concerning invariant metrics. Let $\mathbb{B}^n$ be the unit ball in $\mathbb{C}^n$ and $\Delta := \mathbb{B}^1$. Denote by β_0 the normalized Poincaré–Bergman metric of $\mathbb{B}^n$ at the point 0, i.e.

$$\beta_0 = \sum_{j,k=1}^{n} dz_j \otimes d\bar{z}_k.$$

Define the Kobayashi–Royden pseudometric on a domain $D \subset \mathbb{C}^n$ at a point $x \in D$, for a vector $v \in \mathbb{C}^n$,

$$\kappa_M(x;v) := \inf\{t>0 \,|\, \exists\, \phi : U \to M \text{ holomorphic}, U \supset \overline{\Delta},\ \phi(0)=x,\ t\phi'(0)=v\};$$

and *the integrated form* of κ_D, for $x, y \in D$,

$$(\textstyle\int \kappa_D)(x,y) := \inf\{L_{\kappa_D}(\alpha) \,|\, \alpha \text{ is a piecewise } C^1\text{-curve in } D \text{ from } x \text{ to } y\},$$

where $L_{\kappa_D}(\alpha) := \int_0^1 \kappa_D(\alpha(t);\alpha'(t))\,dt$ is the κ_D-length of a curve α.

A domain $D \subset \mathbb{C}^n$ is called *hyperbolic*, if for any $x_0 \in D$ there exist a neighborhood $U \subset D$ of x_0 and a positive real number C such that $\kappa_D(x;v) \geq C\|v\|$ for all $x \in U$ and $v \in \mathbb{C}^n$. We say that D is *complete* if any $(\int \kappa_D)$-Cauchy sequence $(x_\nu)_{\nu=1}^{\infty} \subset D$ converges to a point $x_0 \in D$ in the natural topology.

THEOREM 0. ([**4**]) *The Wu pseudometric h, which is an (upper semicontinuous) semi-definite Hermitian metric, can be defined on every domain $D \subset \mathbb{C}^n$ and has the following properties:*

(a) *For $0 \in \mathbb{B}^n$, $h_{\mathbb{B}^n,0} = \beta_0$.*
(b) *If the Kobayashi–Royden pseudometric κ_D vanishes identically on D, then so does h_D.*
(c) *If D is hyperbolic, then h_D is a Hermitian metric.*
(d) *If D is complete hyperbolic, then h_D is a continuous Hermitian metric.*
(e) *If $G \subset \mathbb{C}^m$ is a domain and $f : D \to G$ is a holomorphic mapping, then $h_{G,f(x)}(f'(x)(u), f'(x)(v)) \le \sqrt{n}\, h_{D,x}(u,v)$, for $x \in D, u, v \in \mathbb{C}^n$. If κ_D is already a Hermitian pseudometric, then $h_{G,f(x)}(f'(x)(u), f'(x)(v)) \le h_{D,x}(u,v)$.*

Due to the relation between the Wu and Kobayashi–Royden pseudometrics it was possible to find the effective formulas for the Wu metric. It was studied by C. K. Cheung and Kang–Tae Kim in [**1**] and [**2**] on Thullen domains. W. Zwonek computed the formulas for the Kobayashi–Royden pseudometric on elementary Reinhardt domains (in [**5**]). Using them, we give here the formula for the Wu metric (Theorem 7).

1. The Wu pseudometric. The idea and the sketch of the construction of the Wu pseudometric on complex manifolds was first given in [**4**] (see also [**1**]). We recall here the details of the construction on domains in $\mathbb{C}^n$ as well as simple properties for the sake of completeness.

For a domain $D \subset \mathbb{C}^n$ let us introduce the family of sets $(\mathcal{F}_x)_{x\in D}$, where

$$\mathcal{F}_x := \{f \colon \mathbb{B}^n \to D \text{ holomorphic} \mid f(0) = x,\ f'(0)\colon \mathbb{C}^n \to \mathbb{C}^n \text{ is an isomorphism}\}.$$

For any $f \in \mathcal{F}_x$ we can consider the Hermitian inner product on $\mathbb{C}^n$ induced by the normalized Bergman metric:

$$f_!\beta_0(u,v) := \beta_0\big(f'(0)^{-1}(u), f'(0)^{-1}(v)\big), \quad u, v \in \mathbb{C}^n.$$

Let $\Psi_x := \{f_!\beta_0 \mid f \in \mathcal{F}_x\}$. Then the family $\{\Psi_x\}_{x\in D}$ is a biholomorphic invariant of D.

Let $\mathcal{Q}$ be the space of all positive semi-definite Hermitian inner products on $\mathbb{C}^n$ with the natural topology of the vector space of all sesquilinear forms on $\mathbb{C}^n$. Define the partial ordering $\ll$ in $\mathcal{Q}$:

$$\alpha \ll \beta \qquad \text{if} \quad \forall\, v \in \mathbb{C}^n : \alpha(v,v) \le \beta(v,v),$$

and also the set of lower bounds of Ψ_x:

$$l(\Psi_x) := \{\alpha \in \mathcal{Q} \mid \forall\, \beta \in \Psi_x : \alpha \ll \beta\}.$$

We are going to pick up an element of this set which will be the Wu pseudometric of the domain D at the point x.

First, let us make some remarks:

LEMMA 1. *The set $l(\Psi_x)$ is a compact subset of $\mathcal{Q}$.*

PROOF. Fix any $\beta \in \Psi_x$. Let us define the set $l(\beta)$ of lower bounds of β:

$$l(\beta) := \{\alpha \in \mathcal{Q} \,|\, \alpha \ll \beta\}.$$

It suffices to prove that the set $l(\beta)$ is compact since $l(\Psi_x) = \bigcap_{\beta \in \Psi_x} l(\beta)$. Choose a basis $\{e_1, \dots, e_n\}$ of $\mathbb{C}^n$ such that the matrix $[\beta(e_j, e_k)]_{j,k=1,\dots,n}$ is diagonal.

Then, for any $\alpha \in l(\beta)$, we have $\alpha(e_j, e_j) \le \beta(e_j, e_j)$ and also $|\alpha(e_j, e_k)|^2 \le \alpha(e_j, e_j)\alpha(e_k, e_k) \le \beta(e_j, e_j)\beta(e_k, e_k)$. This implies the boundedness of $l(\beta)$. The closedness is obvious. □

Let K_x be the Kobayashi indicatrix at a point x:

$$K_x := \{v \in \mathbb{C}^n \,|\, \kappa_D(x; v) < 1\}$$

and for any $\alpha \in \mathcal{Q}$

$$B_\alpha := \{v \in \mathbb{C}^n \,|\, \alpha(v, v) < 1\}$$

be the "α-ball". Notice that $B_\alpha \supset B_\beta$ if $\alpha \ll \beta$. Moreover, the following holds:

LEMMA 2.

$$K_x = \bigcup_{\alpha \in \Psi_x} B_\alpha$$

PROOF. Notice that, for a holomorphic mapping $f \in \mathcal{F}_x$ such that $f'(0)(w) = v$, we have:

$$\kappa_D(x; v)^2 = \kappa_D\big(f(0); f'(0)(w)\big)^2 \le \kappa_{\mathbb{B}^n}(0; w)^2 = \beta_0(w, w) = f_!\beta_0(v, v).$$

In consequence, for any $\alpha \in \Psi_x$ we have $K_x \supset B_\alpha$.

To prove the other inclusion, fix $v \in \mathbb{C}^n \setminus \{0\}$ such that $\kappa_D(x; v) = \delta < 1$. Choose $\phi : U \to D$ from the definition of $\kappa_D(x; v)$ and $t \in (\delta, 1)$ such that $t\phi'(0) = v$. We may assume that $\phi_1'(0) \ne 0$. Define the mapping $f : \mathbb{B}^n \to \mathbb{C}^n$, $f(z_1, \dots, z_n) := \phi(z_1) + \epsilon(0, z_2, \dots, z_n)$. For $\epsilon > 0$ small enough we obtain $f \in \mathcal{F}_x$. Moreover, we can check that $f'(0)^{-1}(v) = (t, 0, \dots, 0)$. Thus $f_!\beta_0(v, v) = t^2 < 1$, which finishes the proof. □

Let

$$V_x := \bigcap_{\gamma \in l(\Psi_x)} \{v \in \mathbb{C}^n \,|\, \gamma(v, v) = 0\}$$

and take the orthogonal supplement I_x of the space V_x in $\mathbb{C}^n$:

$$I_x \oplus V_x = \mathbb{C}^n.$$

By the Cauchy–Schwarz inequality, any $\gamma \in l(\Psi_x)$ is of the form

$$\gamma = \gamma_1 \oplus \gamma_2, \text{ where } \gamma_1 = \gamma|_{I_x \times I_x},\ \gamma_2 = \gamma|_{V_x \times V_x} \equiv 0,$$

i.e. $\gamma(u_1 + v_1, u_2 + v_2) = \gamma(u_1, u_2)$ for $u_1, u_2 \in I_x, v_1, v_2 \in V_x$.

LEMMA 3.

(a) *The set* $\bigcup_{\alpha\in\Psi_x} B_\alpha \cap I_x$ *is bounded in* I_x.

(b) $\mathrm{SPAN}\{v \in \mathbb{C}^n |\, \kappa_D(x; v) = 0\} \subset V_x$.

PROOF. Suppose that the set $\bigcup_{\alpha\in\Psi_x} B_\alpha \cap I_x$ is unbounded. Then there exist sequences $(x_\nu)_{\nu=1}^\infty \subset \partial\mathbb{B}^n \cap I_x$ and $(\alpha_\nu)_{\nu=1}^\infty \subset \Psi_x$ such that $\lim_{\nu\to\infty} x_\nu = x_0 \in \partial\mathbb{B}^n \cap I_x$ and $\alpha_\nu(x_\nu, x_\nu) < \frac{1}{\nu}$.

Hence we have $0 \le \gamma(x_0, x_0) = \lim_{\nu\to\infty} \gamma(x_\nu, x_\nu) \le \lim_{\nu\to\infty} \alpha_\nu(x_\nu, x_\nu) = 0$, for any $\gamma \in l(\Psi_x)$. But this implies that $x_0 \in V_x$ which cannot hold.

To prove the other part, it suffices to show that $\{v|\, \kappa_D(x; v) = 0\} \subset \{v|\, \gamma(v, v) = 0\}$, for any $\gamma \in l(\Psi_x)$. Fix $v \neq 0$ satisfying $\kappa_D(x; v) = 0$. Reasoning in the same way as in the proof of the previous lemma, for arbitrarily small $t > 0$, we can find an element $\alpha \in \Psi_x$ such that $\alpha(v, v) < t$. But $\gamma \ll \alpha$ for any $\alpha \in \Psi_x$, hence $\gamma(v, v) = 0$. $\square$

LEMMA 4. *There exists the unique element h_x of the set $l(\Psi_x)$ such that for any choice of a basis $\{e_1, \dots, e_m\}$ of the space I_x*

$$\det[\gamma(e_j, e_k)]_{j,k=1,\dots,m} \le \det[h_x(e_j, e_k)]_{j,k=1,\dots,m}, \qquad \textit{for any } \gamma \in l(\Psi_x).$$

PROOF. It is easy to see the existence of such an element (by Lemma 1) and independence of the choice of the basis. Suppose that $h_1 \neq h_2$ are two elements satisfying the above condition. Let $h_0 := \frac{1}{2}h_1 + \frac{1}{2}h_2$. We have $h_0 \in l(\Psi_x)$.

Let us fix a basis $\{e_1, \dots, e_m\}$ of the space I_x. Notice that there must be $\det[h_l(e_j, e_k)]_{j,k=1,\dots,m} \neq 0$ for $l = 1, 2$. Indeed, define the Hermitian metric $h_r(e_j, e_k) := 0$ for $j \neq k$, $h_r(e_j, e_j) := r > 0$ and put $h_r|_{V_x\times V_x} \equiv 0$. By Lemma 3(a), we can find $r > 0$ so small that $\bigcup_{\beta\in\Psi_x} B_\beta \cap I_x \subset B_{h_r} \cap I_x$. Of course, $h_r \in l(\Psi_x)$ and $\det[h_r(e_j, e_k)]_{j,k=1,\dots,m} = r^m > 0$.

Let A and B be the matrices of h_1 and h_2, respectively, in this basis. However, since BA^{-1} is Hermitian, we could choose the basis $\{e_1, \dots, e_m\}$ in such a way that the matrix BA^{-1} is diagonal with strictly positive coefficients $\lambda_1, \dots, \lambda_m$ on the diagonal.

Hence

$$\begin{aligned}\det[h_0(e_j, e_k)] = \frac{1}{2^m}\det(A + B) &= \frac{1}{2^m}\det((I + BA^{-1})A) \\ &= \frac{1}{2^m}(\lambda_1 + 1)\dots(\lambda_m + 1)\det A \\ &\ge \sqrt{\lambda_1 \dots \lambda_m}\det A = \sqrt{\det(BA^{-1})}\det A = \det A.\end{aligned}$$

The equality holds only if $\lambda_j = 1$, $j = 1, \dots, m$ but that is impossible when $A \neq B$ (in our case). Thus, we obtain $\det[h_1(e_j, e_k)] = \det[h_2(e_j, e_k)] < \det[h_0(e_j, e_k)]$ which cannot hold. $\square$

We call this unique element h_x *the Wu pseudometric* of the domain D at the point x. (We will also write $h_x = h_{D,x}$.)

LEMMA 5. *If the Kobayashi–Royden pseudometric $\kappa_D(x;\cdot)$ is Hermitian at the point $x \in D$, then $h_{D,x}(v,v) = \kappa_D(x;v)^2$. In particular, this is the case when $D \subset \mathbb{C}$.*

PROOF. We will show that $K_x = B_{h_x}$, because each Hermitian metric is determined by its unit ball.

Let $\widetilde{\kappa}_x \in \mathcal{Q}$ be such that $\widetilde{\kappa}_x(v,v) = \kappa_D(x;v)^2$. We get $\widetilde{\kappa}_x \in l(\Psi_x)$ directly from the proof of Lemma 2. We have $\widetilde{\kappa}_x(v,v) = h_x(v,v) = 0$ for $v \in V_x$ by the definition of V_x. So, we only need to show that $K_x = B_{h_x}$ in I_x.

Choose any basis $\{e_1, \dots, e_m\}$ of I_x. By the change of coordinates, we obtain the general property:

$$\begin{aligned}\det[\alpha(e_j, e_k)]_{j,k=1,\dots,m}\, dL^{2m}(B_\alpha \cap I_x) &= C dL^{2m}(\mathbb{B}^m)\\ &= \det[\beta(e_j, e_k)]_{j,k=1,\dots,m}\, dL^{2m}(B_\beta \cap I_x), \qquad \alpha, \beta \in l(\Psi_x),\end{aligned}$$

where the constant $C > 0$ depends only on the choice of the basis and dL^{2m} is $2m$-dimensional Lebesgue measure.

Lemma 2 implies $K_x \cap I_x \subset B_{h_x} \cap I_x$. Suppose that these unit balls are not equal, i.e. $L^{2m}(K_x \cap I_x) < L^{2m}(B_{h_x} \cap I_x)$. Then, due to the above property, there must be $\det[h_x(e_j, e_k)]_{j,k=1,\dots,m} < \det[\widetilde{\kappa}_x(e_j, e_k)]_{j,k=1,\dots,m}$ but that contradicts Lemma 4. $\square$

Geometrically speaking, "h_x-ball" is the smallest posible ellipsoid containing the Kobayashi indicatrix.

2. Elementary Reinhardt domains. Let us introduce some notations concerning elementary Reinhardt domains. For $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n \setminus \{0\}$ define

$$D_\mu := \{z \in \mathbb{C}^n : |z_1|^{\mu_1} \dots |z_n|^{\mu_n} < 1, \text{ if } \mu_j < 0 \text{ then } z_j \neq 0\}.$$

We say that μ is of rational type if there exist $t > 0$ and $\nu \in \mathbb{Z}^n_+$ such that $\mu = t\nu$; otherwise μ is said to be of irrational type.

Without loss of generality we may assume that $\mu_1, \dots, \mu_l < 0$, $\mu_{l+1}, \dots, \mu_n > 0$, where $l \in \{0, \dots, n\}$.

We remind the formulas for the Kobayashi–Royden pseudometric κ_{D_μ} in elementary Reinhardt domain D_μ (see [**5**]). Below, $\boldsymbol{\gamma}$ denotes

$$\boldsymbol{\gamma}(x;v) = \frac{|v|}{1-|x|^2}, \qquad \text{for } x \in \Delta,\ v \in \mathbb{C}$$

and κ_{Δ_*} (where $\Delta_* := \Delta \setminus \{0\}$) is given by the formula (see [**3**])

$$\kappa_{\Delta_*}(x;v) = -\frac{|v|}{2|x|\log|x|}, \qquad \text{for } x \in \Delta_*, v \in \mathbb{C}.$$

THEOREM 6. *Let $(x,v) \in D_\mu \times \mathbb{C}^n$ and $\mathcal{J} := \big\{j \in \{1,\dots,n\} : x_j = 0\big\} = \{j_1,\dots,j_k\}$. Define $\tilde{\mu}_{l+1} := min\{\mu_{l+1},\dots,\mu_n\}$.*

(i) *Assume that μ is of rational type and $l < n$, then*

$$\kappa_{D_\mu}(x;v) = \begin{cases} \boldsymbol{\gamma}\Big((x^\mu)^{\frac{1}{\tilde{\mu}_{l+1}}}; (x^\mu)^{\frac{1}{\tilde{\mu}_{l+1}}} \dfrac{1}{\tilde{\mu}_{l+1}} \displaystyle\sum_{j=1}^n \frac{\mu_j v_j}{x_j}\Big) & \text{if } \mathcal{J} = \varnothing \\ \big(|x_1|^{\mu_1} \dots |v_{j_1}|^{\mu_{j_1}} \dots |v_{j_k}|^{\mu_{j_k}} \dots |x_n|^{\mu_n}\big)^{\frac{1}{\mu_{j_1}+\dots+\mu_{j_k}}} & \text{if } \mathcal{J} \neq \varnothing. \end{cases}$$

(ii) *Assume that μ is of irrational type and $l < n$, then*

$$\kappa_{D_\mu}(x;v) = \begin{cases} \boldsymbol{\gamma}\Big(\Big(\displaystyle\prod_{j=1}^n |x_j|^{\mu_j}\Big)^{\frac{1}{\tilde{\mu}_{l+1}}}; \Big(\displaystyle\prod_{j=1}^n |x_j|^{\mu_j}\Big)^{\frac{1}{\tilde{\mu}_{l+1}}} \dfrac{1}{\tilde{\mu}_{l+1}} \displaystyle\sum_{j=1}^n \frac{\mu_j v_j}{x_j}\Big) & \text{if } \mathcal{J} = \varnothing \\ \big(|x_1|^{\mu_1} \dots |v_{j_1}|^{\mu_{j_1}} \dots |v_{j_k}|^{\mu_{j_k}} \dots |x_n|^{\mu_n}\big)^{\frac{1}{\mu_{j_1}+\dots+\mu_{j_k}}} & \text{if } \mathcal{J} \neq \varnothing. \end{cases}$$

(iii) *Assume that μ is of rational type and $l = n$, then*

$$\kappa_{D_\mu}(x;v) = \kappa_{\Delta_*}\Big(x^\mu; x^\mu \sum_{j=1}^n \frac{\mu_j v_j}{x_j}\Big).$$

(iv) *Assume that μ is of irrational type and $l = n$, then*

$$\kappa_{D_\mu}(x;v) = \kappa_{\Delta_*}\Big(|x_1|^{\mu_1} \dots |x_n|^{\mu_n}; |x_1|^{\mu_1} \dots |x_n|^{\mu_n} \sum_{j=1}^n \frac{\mu_j v_j}{x_j}\Big).$$

Now, we state the main result:

THEOREM 7. *Let $(x,v) \in D_\mu \times \mathbb{C}^n$ and $\mathcal{J} := \big\{j \in \{1,\dots,n\} : x_j = 0\big\}$. Then*

$$h_{D_\mu,x}(v,v) = \begin{cases} \kappa_{D_\mu}(x;v)^2 & \text{if } \#\mathcal{J} \le 1 \\ 0 & \text{if } \#\mathcal{J} \ge 2, \end{cases}$$

where $\#\mathcal{J}$ denotes the cardinality of $\mathcal{J}$.

Proof. Notice that whenever $\#\mathcal{J} \leq 1$, the Kobayashi–Royden pseudometric is Hermitian (both $\boldsymbol{\gamma}$ and κ_{Δ_*} are Hermitian) and use Lemma 5. Otherwise, the linear span of the set of zeros of the Kobayashi–Royden pseudometric coincides with $\mathbb{C}^n$. Thus, by Lemma 3(b), the Wu pseudometric is zero. $\square$

The Wu metric h_D is not necessarily continuous with respect to the variable $x \in D$ which is illustrated by the following example.

Example. For the domain $D_{(2,1)} = \{(x_1, x_2) \in \mathbb{C}^2 \,|\, |x_1|^2|x_2| < 1\}$, we have

$$h_{D_{(2,1)},(x_1,x_2)}(1,1) = \frac{|2x_1x_2 + x_1^2|^2}{\big(1 - |x_1^2x_2|^2\big)^2}, \qquad \text{if } x_1, x_2 \neq 0,$$

while $h_{D_{(2,1)},(0,x_2)}(1,1) = |x_2|^2 \neq \lim_{x_1\to 0} h_{D_{(2,1)},(x_1,x_2)}(1,1) = 0$, if $x_2 \neq 0$.

However, the following is true:

Theorem 8. *The Wu metric $(x,u,v) \mapsto h_{D_\mu,x}(u,v)$ is real analytic in the domain $\widetilde{D}_\mu \times \mathbb{C}^{2n}$, where $\widetilde{D}_\mu := D_\mu \setminus \{(x_1,\dots,x_n) \in \mathbb{C}^n \,|\, x_1 \dots x_n = 0\}$.*

Proof. The formulas in Theorem 6 and 7 imply the fact directly. $\square$

References

1. Cheung C. K., Kim Kang–Tae, *Analysis of the Wu metric I: The case of convex Thullen domains,* Trans. Amer. Math. Soc., **328** (1996), 1429–1457.
2. Cheung C. K., Kim Kang–Tae, *Analysis of the Wu metric II: The case of non-convex Thullen domains,* Proc. Amer. Math. Soc., **125** (1997), 1131–1142.
3. Jarnicki M., Pflug P., *Invariant Distances and Metrics in Complex Analysis,* de Gruyter, 1993.
4. Wu H., *Old and new invariant metrics,* Several complex variables: Proc. of Mittag–Leffler Inst. 1987–88, (J. E. Fornaess ed.) Math. Notes, Princeton Univ. Press, **38** (1993), 640–682.
5. Zwonek W., *Completeness, Reinhardt domains and the method of complex geodesics in the theory of invariant functions,* Dissertationes Math., **388 (CCCLXXXVIII)**, 2000.

Received April 11, 2002

Jagiellonian University
Institute of Mathematics
Reymonta 4
30–059 Kraków, Poland
e-mail: jucha@im.uj.edu.pl