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ITERATED FUNCTION SYSTEMS WITH CONTINUOUS PLACE DEPENDENT PROBABILITIES

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Abstract. We study the asymptotic behaviour of iterated function systems built of a finite number of contractions and positive, continuous, place dependent probabilities. We prove that these systems have some special property which is called quasistability. We also establish the existence of an invariant distribution for such systems.

1. Introduction. We study asymptotic properties of iterated function systems (IFSs) defined by a finite number of contractions and positive, continuous, place dependent probabilities. A question concerning the asymptotic behaviour of these systems arose from attempts at completing a gap in Karlin's proof of Theorem 36 in [**6**], where Karlin postulated the asymptotic stability of IFSs built of two affine contractions and positive, continuous probabilities. The problem of the asymptotic stability of IFSs was treated by a number of authors and solved under some additional assumptions (in comparison with the Karlin's theorem) see e.g. [**2**], [**10**]. Recently Ö. Stenflo showed incorrectness of Karlin's theorem by constructing an example of an IFS, which is built from two affine contractions and positive probabilities and has two stationary distributions. In particular, this system is not asymptotically stable. Now new questions appear. What are the limiting properties of an IFS consisting of contractions and positive continuous probabilities? Are we able to associate with such an IFS any invariant distributions? Is an invariant distribution corresponding to a given IFS unique? The purpose of this paper is to answer these questions.

The organization of the paper is as follows. Section 2 contains basic definitions and facts concerning Barnsley operators and attractors defined for finite families of continuous transformations of X into itself. In Section 3 we recall some notation and definitions from the theory of Markov operators. In Section

4 we introduce iterated function systems and prove the main stability result, Theorem 1, which assures the quasistability of iterated function systems with contractive transformations and positive continuous probabilities. Section 5, which concludes the paper, is devoted to the problem of the existence and non-uniqueness of an invariant distribution.

2. Barnsley operators. Let (X, ρ) be a complete metric space. By $\mathcal{P}(X)$ we denote the space of all nonempty subsets of X. For $r > 0$, $B(x, r)$ stands for the open ball with center at x and radius r and $B(K, r)$ stands for the union of all open balls with radii r and with centers in K. By $\mathbb{N}$ we denote the set of positive integers.

Let I be a nonempty finite set. Given a family $\{S_i : i \in I\}$ of continuous transformations from X into itself, we define the corresponding *Barnsley operator* $F : \mathcal{P}(X) \to \mathcal{P}(X)$ by the formula

$$F(A) = \bigcup_{i \in I} S_i(A) \quad \text{for } A \in \mathcal{P}(X).$$

For $A, B \in \mathcal{P}(X)$, let

$$h(A, B) = \max\left\{\sup_{a \in A} \rho(a, B), \sup_{b \in B} \rho(b, A)\right\}. \tag{1}$$

Obviously, if $A, B \in \mathcal{P}(X)$ are bounded then $h(A, B) < \infty$ and $h(A, B)$ equals the Hausdorff distance between A and B. Furthermore, for $A, B \in \mathcal{P}(X)$ we have

$$h(A, B) = h(\operatorname{cl} A, B). \tag{2}$$

In the proof of Theorem 1, we will use the following well-known fact (for a proof see [**1**, page 82] or [**5**, page 30]):

LEMMA 1. *If $\{S_i : i \in I\}$ is a finite family of contractions with respective contractivity constants L_i and F is the corresponding Barnsley operator, then*

$$h(F(A), F(B)) \le \left(\max_{i \in I} L_i\right) h(A, B) \quad \text{for } A, B \in \mathcal{P}(X). \tag{3}$$

Moreover, there exists the unique nonempty compact set $K \subset X$ which is invariant with respect to the family $\{S_i : i \in I\}$, i.e.

$$K = \bigcup_{i \in I} S_i(K). \tag{4}$$

The set K mentioned in the statement of the above theorem is called the *attractor* of the family $\{S_i : i \in I\}$.

We finish this section with a simple lemma, which we provide with the detailed proof for the reader's convenience.

LEMMA 2. *If a family $\{S_i : i \in I\}$ consists of contractions and K is the attractor corresponding to that family, then for each $\{i_n\} \in I^{\mathbb{N}}$ the limit*

$$\lim_{n\to\infty} S_{i_1} \circ ... \circ S_{i_n}(x) \tag{5}$$

exists, is independent of $x \in X$ and belongs to K. Moreover, for every $\varepsilon > 0$ the convergence in (5) is uniform over $x \in B(K, \varepsilon)$. Furthermore, the function $\varphi : I^{\mathbb{N}} \to K$ defined by the formula

$$\varphi(\{i_n\}) = \lim_{n\to\infty} S_{i_1} \circ ... \circ S_{i_n}(x)$$

is onto.

PROOF. Fix $\{i_n\} \in I^{\mathbb{N}}$. To see that limit (5) exists, choose $x \in X$ and $k, l \in \mathbb{N}$, $k < l$. Let $\varepsilon > 0$ be such a number that $x \in B(K, \varepsilon)$. Find $c \in K$ such that $\rho(x, c) < \varepsilon$ and set $y = S_{i_{k+1}} \circ ... \circ S_{i_l}(c)$. Observe that $y \in K$ and

$$\begin{aligned}
&\rho(S_{i_1} \circ ... \circ S_{i_k}(x), S_{i_1} \circ ... \circ S_{i_l}(x)) \\
&\quad\leq \rho(S_{i_1} \circ ... \circ S_{i_k}(x), S_{i_1} \circ ... \circ S_{i_k}(c)) \\
&\quad+ \rho(S_{i_1} \circ ... \circ S_{i_k}(c), S_{i_1} \circ ... \circ S_{i_l}(c)) \\
&\quad+ \rho(S_{i_1} \circ ... \circ S_{i_l}(c), S_{i_1} \circ ... \circ S_{i_l}(x)) \\
&\quad\leq L^k \rho(x, c) + L^k \rho(c, y) + L^l \rho(c, x) \\
&\quad\leq L^k (2\varepsilon + \operatorname{diam}(K)).
\end{aligned} \tag{6}$$

Thus $\{S_{i_1} \circ ... \circ S_{i_n}(x)\}$ is a Cauchy sequence. Additionally, since the estimate in (6) is uniform for $x \in B(K, \varepsilon)$, the convergence in (5) is also uniform. Next, from the contractivity of transformations S_i $(i \in I)$ we obtain that the limit (5) does not depend on $x \in X$, that is the function φ is well defined. From the invariance of the attractor it follows that the limit (5) belongs to K. We are left with the task of proving that φ is a surjection. Fix $c \in K$. Observe that (4) implies the existence of $i_1 \in I$ such that $c \in S_{i_1}(K)$. Proceeding by induction we choose a sequence $\{i_n\} \in I^{\mathbb{N}}$ such that $c \in S_{i_1} \circ ... \circ S_{i_n}(K)$ for all $n \in \mathbb{N}$. Since K is compact and $\{S_{i_1} \circ ... \circ S_{i_n}(K)\}$ is a decreasing sequence of sets containing c and diameters of these sets decrease to 0, we obtain $\{c\} = \bigcap_{n=1}^{\infty} S_{i_1} \circ ... \circ S_{i_n}(K)$. Thus, if $x \in K$, then $c = \lim_{n\to\infty} S_{i_1} \circ ... \circ S_{i_n}(x)$. This completes the proof. $\square$

3. Markov operators and their asymptotic behaviour. In what follows we assume that (X, ρ) is a Polish space. By $\mathcal{B}_X$ we denote the σ-algebra of Borel subsets of X, by $\mathcal{M}(X)$ the family of all finite Borel measures on X and by $\mathcal{M}_1(X)$ the family of all $\mu \in \mathcal{M}(X)$ such that $\mu(X) = 1$. The elements of $\mathcal{M}_1(X)$ are called *distributions*.

We say that $\mu \in \mathcal{M}(X)$ is *concentrated* on a set $A \in \mathcal{B}_X$ if $\mu(X \setminus A) = 0$. By $\mathcal{M}_1^A(X)$ we denote the set of all distributions concentrated on A. The *support* of $\mu \in \mathcal{M}(X)$, which is defined by the formula

$$\operatorname{supp} \mu = \{x \in X : \mu(B(x,r)) > 0 \text{ for } r > 0\},$$

turns out to be the smallest closed set on which a measure μ is concentrated.

As usually, by $B(X)$ we denote the space of all bounded Borel measurable functions $f : X \to \mathbb{R}$ and by $C(X)$ the subspace of all bounded continuous functions. For $f \in B(X)$ and $\mu \in \mathcal{M}(X)$ we write

$$\langle f, \mu \rangle = \int_X f(x)\mu(dx).$$

Let $\mathcal{M}_s(X) = \{\mu_1 - \mu_2 : \mu_1, \mu_2 \in \mathcal{M}(X)\}$ be the space of finite signed measures. In $\mathcal{M}_s(X)$ we introduce the Fortet-Mourier norm given by

$$\|\mu\| = \sup\{|\langle f, \mu\rangle| : f \in \mathcal{L}(X)\},$$

where

$$\mathcal{L}(X) = \{f \in C(X) : \|f\| \le 1, |f(x) - f(y)| \le \rho(x,y) \text{ for } x, y \in X\}.$$

An operator $P : \mathcal{M}(X) \to \mathcal{M}(X)$ is called a *Markov operator* if it satisfies the following two conditions:

(i) *positive linearity*:

$$P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P(\mu_1) + \lambda_2 P(\mu_2)$$

for $\lambda_1, \lambda_2 \ge 0$ and $\mu_1, \mu_2 \in \mathcal{M}(X)$;

(ii) *preservation of the norm*:

$$P\mu(X) = \mu(X) \quad \text{for } \mu \in \mathcal{M}(X).$$

It is easy to show that a Markov operator on $\mathcal{M}(X)$ can be uniquely extended to a linear operator on $\mathcal{M}_s(X)$, which transforms $\mathcal{M}(X)$ into itself.

We say that a Markov operator P is a *Feller operator* if there is a linear operator $U : C(X) \to C(X)$ dual to P, that is

$$\langle Uf, \mu\rangle = \langle f, P\mu\rangle \quad \text{for } f \in C(X) \text{ and } \mu \in \mathcal{M}(X).$$

A measure $\mu_* \in \mathcal{M}(X)$ is called *stationary* (or *invariant*) with respect to a Markov operator P if $P\mu_* = \mu_*$. A Markov operator P is called *asymptotically stable* if there exists a stationary distribution μ_* such that

$$\lim_{n\to\infty} \|P^n\mu - \mu_*\| = 0 \quad \text{for } \mu \in \mathcal{M}_1(X). \tag{7}$$

Clearly the distribution μ_* satisfying (7) is unique.

We say that a Markov operator P is *quasistable on a set* K, if K is nonempty closed subset of X and if

(i) for every $c \in K$ and for every $\varepsilon > 0$ there exists a number $\alpha > 0$ such that

$$\liminf_{n\to\infty} P^n\mu\,(B(c,\varepsilon)) \geq \alpha \quad \text{for } \mu \in \mathcal{M}_1(X); \tag{8}$$

(ii) for every $c \notin K$ there exists a number $\varepsilon > 0$ such that

$$\lim_{n\to\infty} P^n\mu\,(B(c,\varepsilon)) = 0 \quad \text{for } \mu \in \mathcal{M}_1(X). \tag{9}$$

The operator P is called *quasistable* if there exists a set K such that P is quasistable on K. The definition of quasistability of Markov operators is quite close to the notion of the asymptotic positivity, defined in [**8**, page 193]. Observe, that an asymptotically stable Markov operator P with the stationary distribution μ_* is quasistable on the set $\operatorname{supp}\mu_*$, which is a consequence of the Alexandrov theorem (see [**3**, page 11]). The converse implication is not true (which follows from Theorem 1 and Theorem 4).

4. Quasistability of iterated function systems. Assume that I is a nonempty finite set (this assumption will not be repeated). By an *iterated function system* (shortly IFS) $\{(S_i, p_i) : i \in I\}$ we mean a family of pairs consisting of continuous maps $S_i : X \to X$ and $p_i : X \to [0,1]$, defined for each $i \in I$ and such that $\sum_{i\in I} p_i(x) = 1$ for all $x \in X$.

Given an IFS $\{(S_i, p_i) : i \in I\}$ we define the corresponding Markov operator $P : \mathcal{M}(X) \to \mathcal{M}(X)$ by the formula

$$P\mu(A) = \sum_{i\in I} \int_X 1_A\,(S_i(x))\,p_i(x)\mu(dx) \quad \text{for } A \in \mathcal{B}_X,\ \mu \in \mathcal{M}(X). \tag{10}$$

Observe that P is a Feller operator and the dual operator U is given by

$$Uf = \sum_{i\in I} p_i\,(f \circ S_i) \quad \text{for } f \in C(X).$$

To simplify the language we will say that an IFS $\{(S_i, p_i) : i \in I\}$ is *asymptotically stable*, *quasistable* or has *invariant distribution* if the Markov operator (10) has the corresponding property.

In order to prove the main result of this section we need two simple lemmas. The first easily follows from [**9**, Lemma 6.2] (compare also with [**7**, Proposition 12.8.2]) and allows us to construct supports of iterations of a given measure.

Lemma 3. *Consider an IFS* $\{(S_i, p_i) : i \in I\}$ *with* $p_i > 0$ *for* $i \in I$. *Let* P *be the corresponding Markov operator and let* F *be the corresponding Barnsley operator. Then*

$$\operatorname{supp} P^n\mu = \operatorname{cl} F^n\,(\operatorname{supp}\mu) \ \ \textit{for } \mu \in \mathcal{M}(X),\ n \in \mathbb{N}.$$

Lemma 4. *If $f : X \to (0, \infty)$ is a continuous function and a set $K \subset X$ is compact, then there exist numbers $\beta > 0$ and $\delta > 0$ such that $f(x) > \beta$ for all $x \in B(K, \delta)$.*

Proof. The above statement is an obvious conclusion of the compactness of K. For every $x \in K$ define numbers $\beta_x, \delta_x > 0$ such that $f(y) > \beta_x$ for $y \in B(x, 2\delta_x)$. Let $\left\{B\left(x_j, \delta_{x_j}\right)\right\}_{j=1}^{q}$ be a finite subcover of K chosen from the cover $\{B(x, \delta_x)\}_{x \in K}$ of K. Set $\beta = \min_{j=1,\ldots,q} \beta_{x_j}$ and $\delta = \min_{j=1,\ldots,q} \delta_{x_j}$. From the triangle inequality it follows that $B(K, \delta) \subset \bigcup_{j=1}^{q} B\left(x_j, 2\delta_{x_j}\right)$, so if $x \in B(K, \delta)$ then $f(x) > \beta$, which completes the proof. □

Theorem 1. *Consider an IFS $\{(S_i, p_i) : i \in I\}$ such that for each $i \in I$ the transformation S_i is a contraction and the function p_i is positive. Then $\{(S_i, p_i) : i \in I\}$ is quasistable.*

Proof. Let P and U denote the Markov operator and the dual operator corresponding to a given IFS $\{(S_i, p_i) : i \in I\}$, respectively. Let L_i be a Lipschitz constant for S_i, $i \in I$. Since $L = \max_{i \in I} L_i < 1$, we can apply Lemma 1, by virtue of which there exists a nonempty compact set $K \subset X$, invariant with respect to the family $\{S_i : i \in I\}$. We will prove that the IFS $\{(S_i, p_i) : i \in I\}$ is quasistable with respect to K. The proof falls naturally into two parts, due to the form of the definition of the quasistability.

To begin with the first part, fix $c \in K$ and $\varepsilon > 0$. According to the properties of φ defined in the statement of Lemma 2, there exists $\{i_n\} \in I^{\mathbb{N}}$ such that $c = \lim_{n \to \infty} S_{i_1} \circ \ldots \circ S_{i_n}(x)$ for all $x \in B(K, \varepsilon)$. The convergence to the limit is uniform over $x \in B(K, \varepsilon)$, so we can choose a number $m \in \mathbb{N}$ such that

$$S_{i_1} \circ \ldots \circ S_{i_m}(B(K, \varepsilon)) \subset B(c, \varepsilon). \tag{11}$$

Further, since K is compact, we can apply Lemma 4 to the set K and to each of the functions $p_{i_k} \circ S_{i_{k+1}} \circ \ldots \circ S_{i_m}$ in turn ($k = 1, \ldots, m$). We obtain constants $\beta_k > 0$ and $\delta_k > 0$ such that $p_{i_k} \circ S_{i_{k+1}} \circ \ldots \circ S_{i_m}(x) > \beta_k$ for $x \in B(K, \delta_k)$, $k \in \{1, \ldots, m\}$. Now let

$$\delta = \min\{\varepsilon, \delta_1, \ldots, \delta_m\} \quad \text{and} \quad \alpha = \beta_1 \cdot \ldots \cdot \beta_m / 2.$$

Next, fix a distribution $\mu \in \mathcal{M}_1$ concentrated on a bounded subset of X, that is with the bounded $\operatorname{supp} \mu$. The inequality (3) gives

$$h\left(F^n(\operatorname{supp} \mu), K\right) \leq L^n h(\operatorname{supp} \mu, K) \quad \text{for } n \in \mathbb{N}.$$

Therefore, from the finiteness of $h(\operatorname{supp} \mu, K)$, Lemma 3 and property (2) we conclude that there exists $n_0 \in \mathbb{N}$ such that

$$h(\operatorname{supp} P^n \mu, K) \leq \delta \quad \text{for } n \geq n_0.$$

According to formula (1) we then have

(12) $$\operatorname{supp} P^n\mu \subset B(K,\delta) \quad \text{for } n \geq n_0.$$

We are now in a position to show (8). Let $n \geq m + n_0$. We have

$$\begin{aligned} P^n\mu(B(c,\varepsilon)) &= \langle U^m 1_{B(c,\varepsilon)}, P^{n-m}\mu\rangle \\ &= \sum_{k_1,\dots,k_m \in I} \int_X (p_{k_m}) \dots (p_{k_1} \circ S_{k_2} \circ \dots \circ S_{k_m}) \left(1_{B(c,\varepsilon)} \circ S_{k_1} \circ \dots \circ S_{k_m}\right) dP^{n-m}\mu \\ &\geq \int_X (p_{i_m}) \dots (p_{i_1} \circ S_{i_2} \circ \dots \circ S_{i_m}) \left(1_{B(c,\varepsilon)} \circ S_{i_1} \circ \dots \circ S_{i_m}\right) dP^{n-m}\mu. \end{aligned}$$

The definition of δ and inclusion (11) yield

$$B(K,\delta) \subset (S_{i_1} \circ \dots \circ S_{i_m})^{-1}(B(c,\varepsilon)),$$

and consequently

$$P^n\mu(B(c,\varepsilon)) \geq \int_{B(K,\delta)} (p_{i_m}) \dots (p_{i_1} \circ S_{i_2} \circ \dots \circ S_{i_m}) \, dP^{n-m}\mu.$$

From the definition of α and (12) it follows that

(13) $$P^n\mu(B(c,\varepsilon)) \geq 2\alpha,$$

where $n \geq m + n_0$. Letting $n \to \infty$ we obtain

$$\liminf_{n\to\infty} P^n\mu(B(c,\varepsilon)) \geq 2\alpha > \alpha,$$

so (8) is proved for distributions with bounded supports. Now consider a distribution $\mu \in \mathcal{M}_1$ with an arbitrary support. Choose a bounded Borel set $S \subset X$ such that $\mu(S) \geq 1/2$. We have then $\mu \geq \mu^S/2$, where $\mu^S \in \mathcal{M}_1^S$ is of the form

$$\mu^S(A) = \frac{\mu(A \cap S)}{\mu(S)} \text{ for } A \in \mathcal{B}(X).$$

Applying (13) to μ^S leads to inequalities

$$P^n\mu(B(c,\varepsilon)) \geq P^n\mu^S(B(c,\varepsilon))/2 \geq \alpha,$$

which are true for n large enough. This completes the first part of the proof. In order to deal with the second, fix $c \notin K$ and choose $\varepsilon > 0$ such that

(14) $$B(K,\varepsilon) \cap B(c,\varepsilon) = \emptyset.$$

Consider a distribution $\mu \in \mathcal{M}_1$ with a bounded support and proceeding as before find a number $n_0 \in \mathbb{N}$ such that

$$\operatorname{supp} P^n\mu \subset B(K,\varepsilon) \text{ for } n \geq n_0.$$

Combining this with (14) we can assert that

$$\lim_{n\to\infty} P^n\mu\left(B\left(c,\varepsilon\right)\right)=0,$$

which implies (9) for measures with bounded supports. Passing to the case of a distribution with unbounded support, fix such a distribution $\mu \in \mathcal{M}_1$ and a number $\eta > 0$. Next, choose such a bounded Borel set $S \subset X$ that $\mu\left(S\right) \geq 1-\eta$ and define two distributions, $\mu^S \in \mathcal{M}_1^S$ and $\gamma \in \mathcal{M}_1$, by the formulae

$$\begin{aligned} \mu^S(A) &= \frac{\mu\left(A\cap S\right)}{\mu\left(S\right)} \quad \text{for } A\in\mathcal{B}(X); \\ \gamma(A) &= \frac{1}{\eta}\left(\mu(A)-\left(1-\eta\right)\mu^S(A)\right) \quad \text{for } A\in\mathcal{B}(X). \end{aligned}$$

Obviously

$$\mu=\left(1-\eta\right)\mu^S+\eta\gamma$$

and consequently

$$\begin{aligned} &\limsup_{n\to\infty} P^n\mu\left(B\left(c,\varepsilon\right)\right) \\ &\qquad\leq \left(1-\eta\right)\lim_{n\to\infty} P^n\mu^S\left(B\left(c,\varepsilon\right)\right)+\eta\limsup_{n\to\infty} P^n\gamma\left(B\left(c,\varepsilon\right)\right)\leq\eta. \end{aligned}$$

Since the number η is arbitrary, the last inequality completes the proof. □

5. The existence and non-uniqueness of stationary distribution. In this section we will discuss the problem of the existence and non-uniqueness of an invariant distribution. We start from quoting a result proved by A. Lasota and J. A. Yorke (see [**10**], Theorem 3.1).

Theorem 2. *[Lasota–Yorke Theorem] Let (K,ρ) be a metric space in which closed balls are compact and let $P_K:\mathcal{M}(K)\to\mathcal{M}(K)$ be a Feller operator. Assume that there is a compact set $Y\subset K$ and a distribution $\upsilon_0\in\mathcal{M}_1(K)$ such that*

$$\limsup_{n\to\infty}\left(\frac{1}{n}\sum_{m=1}^{n} P_K^m\upsilon_0\left(Y\right)\right)>0. \tag{15}$$

Then P_K has a stationary distribution.

Using the above theorem we may prove the following sufficient condition for the existence of a stationary distribution for iterated function systems.

Theorem 3. *Consider an IFS $\{(S_i,p_i):i\in I\}$ such that for every $i\in I$ the transformation S_i is a contraction and p_i is positive. Then $\{(S_i,p_i):i\in I\}$ has an invariant distribution.*

PROOF. Let P denote the Markov operator corresponding to $\{(S_i, p_i) : i \in I\}$. Let K be the attractor of the family $\{S_i : i \in I\}$, existing by virtue of Lemma 1. Consider transformations $\mathcal{M}(K) \ni \upsilon \longmapsto \upsilon^X \in \mathcal{M}(X)$ and $\mathcal{M}(X) \ni \mu \longmapsto \mu^K \in \mathcal{M}(K)$ defined by the following formulae

$$\upsilon^X(A) = \upsilon(A \cap K) \text{ for } \upsilon \in \mathcal{M}(K) \text{ and } A \in \mathcal{B}_X;$$

$$\mu^K = \mu|_{\mathcal{B}_K} \text{ for } \mu \in \mathcal{M}(X).$$

Next, examine an operator $P_K : \mathcal{M}(K) \to \mathcal{M}(K)$ given by

$$P_K \upsilon = \left(P\left(\upsilon^X\right)\right)^K \text{ for } \upsilon \in \mathcal{M}(K).$$

Lemma 3 and the compactness of K yield that

$$\text{supp}\, P\mu \subset K \text{ for } \mu \in \mathcal{M}(X) \text{ such that } \text{supp}\, \mu \subset K. \tag{16}$$

This implies that P_K is a Markov operator and a Feller operator. Fix an arbitrary $x_0 \in K$. Since K is compact, from (16) it also follows that an operator P_K, a set $Y = K$ and a distribution $\upsilon_0 = \delta_{x_0}$ satisfy condition (15). Thus the assumptions of Theorem 2 are fulfilled. According to this theorem there exists a distribution $\upsilon_* \in \mathcal{M}_1(K)$ which is invariant for P_K. The measure $(\upsilon_*)^X \in \mathcal{M}(X)$ is a stationary distribution for P. □

Observe that if an IFS consists of contractions and positive probabilities then it is globally and locally concentrating (for definitions see [**12**]), which is a consequence of the Chebyshev inequality and can be proved following the main idea of the proof of Theorem 3.2 from [**8**]. If a given IFS is additionally nonexpansive, then it is also asymptotically stable, which follows from Theorem 3.1 from [**12**]. This implies the existence and uniqueness of an invariant distribution. However, in general, the assumptions of the contractivity of transformations and the positivity and continuity of probabilities of a given IFS imply neither asymptotic stability of this IFS nor the uniqueness of an invariant distribution. The counter-example illustrating the above statement is contained in Theorem 1 from [**11**] and is an application of the results from [**4**]. We quote this theorem below.

THEOREM 4. *Let S_1 and S_2 be maps from $[0,1]$ into itself defined by the formulae*

$$S_1(x) = \frac{x}{3},\ S_2(x) = \frac{x}{3} + \frac{2}{3} \text{ for } x \in [0,1];$$

Then there exists a continuous function $p : [0,1] \to (0,1)$ such that the IFS $\{(S_1, p), (S_2, 1-p)\}$ admits two different invariant distributions.

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