Vol. 50 (2019) Acta Physica Polonica B No 12

RELATIVISTIC QUANTUM MECHANICS OF THE MAJORANA PARTICLE*

H. Arodź

M. Smoluchowski Institute of Physics, Jagiellonian University, Kraków, Poland

(Received November 4, 2019)

This article is a pedagogical introduction to relativistic quantum mechanics of the free Majorana particle. This relatively simple theory differs from the well-known quantum mechanics of the Dirac particle in several important aspects. First, we present its three equivalent formulations. Next, a so-called axial momentum observable is introduced, and the general solution of the Dirac equation is discussed in terms of eigenfunctions of that operator. We also present pertinent irreducible representations of the Poincaré group. Finally, we show that in the case of massless Majorana particle, the quantum mechanics can be reformulated as a spinorial gauge theory.

DOI:10.5506/APhysPolB.50.2165

1. Introduction

The concept of Majorana particles is very popular in particle physics nowadays. The physical object of interest is a spin 1/2, electrically neutral, fermionic particle, which does not have its anti-particle. The common spin-1/2 particles such as electrons or quarks do possess anti-particles. The Majorana particles are hypothetical objects as yet, but it is not excluded that there exist neutrinos of this kind [1, 2]. Anyway, theory of such particles is interesting on its own right. The Majorana (quasi-)particles are intensely studied also in condensed matter physics, but we shall not touch upon this line of research.

The theory of Majorana particles can be developed on two levels: as a quantum field theory or relativistic quantum mechanics. Of course, the state-of-the-art approach is the field theoretic one. Nevertheless, the relativistic quantum mechanics also offers some advantages, in particular, it is

---

* Based on lecture delivered at the LIX Cracow School of Theoretical Physics "Probing the Violent Universe with Multimessenger Eyes: Gravitational Waves, High-energy Neutrinos, Gamma Rays, and Cosmic Rays", Zakopane, Poland, June 14–22, 2019.

much simpler — the field theory is hard to use except some rather narrow range of problems like scattering processes at energies lying within a perturbative regime. The usefulness of the relativistic quantum mechanics is well-documented in the theory of electrons, either bound in atoms or traveling in space. The electron is the example of Dirac particle. Relativistic quantum mechanical theories of the Majorana and the Dirac particle are significantly different. The Dirac particle is a well-understood textbook item, as opposed to the Majorana particle where several subtleties are present.

In this paper, we attempt to give a pedagogical introduction to relativistic quantum mechanics of the free Majorana particle. It is not comprehensive, we rather focus on selected topics: the problem of momentum observable; the general solution of the Dirac equation for the Majorana bispinor; and relativistic invariance in terms of representations of the Poincaré group. We also describe in detail the path from quantum mechanics of the Dirac particle to quantum mechanics of the Majorana particle. We emphasize the fact that in the case of Majorana particle, the quantum mechanics employs only the algebraic field of real numbers $\mathbb{R}$, while in the Dirac case, the complex numbers are essential. Such real quantum mechanics is less known, but it is thoroughly discussed in literature, see, *e.g.*, [3–5]. There is an interesting aspect of the theory of massless Majorana particle, namely a local gauge invariance in the momentum representation for bispinors, presented in Section 5.2. To the best of our knowledge, such gauge invariance has not been discussed in literature.

Few words about our conventions. We use the natural units $c = \hbar = 1$. Metric tensor $(\eta_{\mu\nu})$ in the Minkowski space-time is diagonal with the entries $(1, -1, -1, -1)$. Summation over repeated indices is understood. Four-vectors and three-component vectors have components with upper indices, for example $p = (p^0, p^1, p^2, p^3)^{\mathrm{T}}$ or $\boldsymbol{x} = (x^1, x^2, x^3)^{\mathrm{T}}$, unless stated otherwise. Three-component vectors are denoted by the boldface. T denotes the matrix transposition. In the matrix notation, $\boldsymbol{x}$ is a column with three, and p with four elements. Bispinors are columns with four elements. For convenience, we do not avoid complex numbers when it is natural to use them. For example, we stick to the standard notation for the Dirac matrices γ^μ. In the Majorana quantum mechanics they are imaginary, hence we use the real matrices $i\gamma^\mu$, where i is the imaginary unit. Of course, we could get rid of the complex numbers completely at the price of introducing a new notation.

The paper is organized as follows. In Section 2, we introduce charge conjugation and we define the Majorana bispinors. Section 3 is devoted to the momentum observable for the Majorana particle. In Section 4, we study general solution of the Dirac equation in the case of Majorana particle. Relativistic invariance and pertinent representations of the Poincaré group are discussed in Section 5. Remarks are collected in Section 6.

2. The Majorana bispinors and the Majorana mass term

Let us begin from the Dirac equation for complex four-component bispinor $\psi(x)$

$$i\gamma_{\rm D}^{\mu}(\partial_{\mu} + iqA_{\mu}(x))\psi(x) - m\psi(x) = 0\,, \tag{2.1}$$

where m and q are real constants, the index μ takes values 0, 1, 2 and 3, $A_{\mu}(x)$ is a fixed four-potential of the electromagnetic field. The argument of ψ, $x = (t, \boldsymbol{x})$, denotes points in the Minkowski space-time.

The matrices $\gamma_{\rm D}^{\mu}$ have the following form:

$$\gamma_{\rm D}^{0} = \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}, \qquad \gamma_{\rm D}^{i} = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{2.2}$$

known as the Dirac representation. Here σ_i are the Pauli matrices, the index i takes values 1, 2 and 3, σ_0 denotes the two-by-two unit matrix. The matrices $\gamma_{\rm D}^{\mu}$ obey the condition

$$\gamma_{\rm D}^{\mu}\gamma_{\rm D}^{\nu} + \gamma_{\rm D}^{\nu}\gamma_{\rm D}^{\mu} = 2\eta^{\mu\nu} I\,, \tag{2.3}$$

where I denotes four-by-four unit matrix. Note that $\gamma_{\rm D}^{2}$ is imaginary, while the remaining matrices are real.

The information given above is purely mathematical. The physical meaning of it is established by interpreting $\psi(x)$ as the wave function of certain particle. The constants m and q then give, respectively, the rest mass and the electric charge of this particle (in fact, the rest mass is given by $|m|$, not by m). Equation (2.1) and such interpretation of ψ are the basic ingredients of the theory called the quantum mechanics of the Dirac particle. It is the most important example of relativistic quantum mechanics. Scalar product of two wave functions ψ_1, ψ_2 — necessary in quantum mechanics — has the form of

$$\langle\psi_1|\psi_2\rangle = \int {\rm d}^3x\, \bar{\psi}_1(t,\boldsymbol{x})\gamma_{\rm D}^{0}\psi_2(t,\boldsymbol{x}) = \int {\rm d}^3x\, \psi_1^{\dagger}(t,\boldsymbol{x})\psi_2(t,\boldsymbol{x})\,, \tag{2.4}$$

where $\bar{\psi} = \psi^{\dagger}\gamma_{\rm D}^{0}$ and $\dagger$ denotes the Hermitian conjugation. The bispinors ψ are columns with four elements, and $\bar{\psi}$'s are one-row matrices with four elements. One can prove that the scalar product (2.4) does not depend on time t provided that ψ_1, ψ_2 are solutions of the Dirac equation (2.1).

Alternative interpretation of $\psi(x)$, which is not used here, is that it is a classical field known as the Dirac field.

Charge conjugate bispinor $\psi_{\rm c}(x)$ is defined as follows:

$$\psi_{\rm c}(x) = i\gamma_{\rm D}^{2}\psi^{*}(x)\,, \tag{2.5}$$

where * denotes the complex conjugation. Taking complex conjugation of Eq. (2.1) and using the relation $\gamma^2_{\rm D}(\gamma^\mu_{\rm D})^*\gamma^2_{\rm D} = \gamma^\mu_{\rm D}$, we obtain the equation

$$i\gamma^\mu_{\rm D}(\partial_\mu - iqA_\mu(x))\psi_{\rm c}(x) - m\psi_{\rm c}(x) = 0\,, \tag{2.6}$$

which differs from Eq. (2.1) by the sign in the first term. In consequence, $\psi_{\rm c}(x)$ is the wave function of another Dirac particle which has the electric charge $-q$.

Let us now consider the Poincaré transformations of the Cartesian coordinates in the Minkowski space-time, $x' = Lx + a$. The corresponding transformations of the bispinor $\psi(x)$ have the form of

$$\psi'(x) = S(L)\psi\left(L^{-1}(x-a)\right)\,, \tag{2.7}$$

where $S(L) = \exp(\omega_{\mu\nu}[\gamma^\mu_{\rm D}, \gamma^\nu_{\rm D}]/8)$. The bracket [,] denotes the commutator of the matrices. The real numbers $\omega_{\mu\nu} = -\omega_{\nu\mu}$ parameterize the proper orthochronous Lorentz group in a vicinity of the unit element I, namely $L = \exp(\omega^\mu{}_\nu)$, where $\omega^\mu{}_\nu = \eta^{\mu\lambda}\omega_{\lambda\nu}$. Using definition (2.5) and formula (2.7), we find that $\psi_{\rm c}$ has the same transformation law as ψ

$$\psi'_{\rm c}(x) = S(L)\psi_{\rm c}\left(L^{-1}(x-a)\right)\,. \tag{2.8}$$

This fact inspired Majorana [6] to proposing an interesting modification of the quantum mechanics of the Dirac particle.

The modification consists in generalizing equation (2.1) by including the term $m_{\rm M}\psi_{\rm c}(x)$, often called the Majorana mass term,

$$i\gamma^\mu_{\rm D}(\partial_\mu + iqA_\mu(x))\psi(x) - m\psi(x) - m_{\rm M}\psi_{\rm c}(x) = 0\,,$$

where we assume for simplicity that the constant $m_{\rm M}$ is a real. Such a modification, however, cannot be done without a price. We know from classical electrodynamics that all four-potentials which differ by a gauge transformation are physically equivalent, that is, $A_\mu(x)$ is equivalent to $A'_\mu(x) = A_\mu(x) + \partial_\mu\chi(x)$, where $\chi(x)$ is an arbitrary smooth real function which vanishes quickly when $x \to \infty$ (in mathematical terms, it is a test function of the Schwartz class). Thus, let us write Eq. (2.1) with $A'_\mu(x)$

$$i\gamma^\mu_{\rm D}(\partial_\mu + iqA'_\mu(x))\psi'(x) - m\psi'(x) = 0\,, \tag{2.9}$$

where $\psi'(x)$ denotes solutions of this new equation. It is clear that this equation is equivalent to (2.1) — it suffices to substitute $\psi'(x) = \exp(-iq\chi(x))\psi(x)$ and to divide both sides of equation (2.9) by $\exp(-iq\chi(x))$. We say that Eq. (2.1) is gauge invariant. The gauge invariance is lost when we include the Majorana mass term. The reason is that $\psi'_{\rm c}(x) = \exp(iq\chi(x))\psi_{\rm c}(x)$, as follows from definition (2.5) and, therefore, the exponential factors cannot be removed. The Majorana mass term breaks the gauge invariance. Therefore, such a mass term can be considered only if $q = 0$, that is, when the particle is electrically neutral. For such an electrically neutral particle, we may consider the equation

$$i\gamma_{\mathrm{D}}^{\mu}\partial_{\mu}\psi(x) - m\psi(x) - m_{\mathrm{M}}\psi_{\mathrm{c}}(x) = 0 \tag{2.10}$$

known as the Dirac equation with the Majorana mass term.

The inclusion of the Majorana mass term has a deep implication for the structure of the theory — it partially breaks the superposition principle of quantum mechanics of the Dirac particle. The original Dirac equation (2.1) is linear over $\mathbb{C}$, that is, any complex linear combination of its solutions also is a solution. Because ψ_{c} involves the complex conjugation, Eq. (2.10) allows for linear combinations with real coefficients only. On the other hand, the bispinor ψ is still complex, that is the Hilbert space of the wave functions is linear over $\mathbb{C}$. It is clear that one can avoid this discrepancy by taking a smaller Hilbert space in which only real linear combinations are allowed. The crucial condition for such restriction is that it should be compatible with the Poincaré invariance.

Equation (2.10) can be transformed into equivalent equation for ψ_{c}, namely

$$i\gamma_{\mathrm{D}}^{\mu}\partial_{\mu}\psi_{\mathrm{c}}(x) - m\psi_{\mathrm{c}}(x) - m_{\mathrm{M}}\psi(x) = 0\,. \tag{2.11}$$

Let us introduce new bispinors $\psi_{\pm}(x) = \frac{1}{2}(\psi(x) \pm \psi_{\mathrm{c}}(x))$. It follows from Eqs. (2.10), (2.11) that

$$i\gamma_{\mathrm{D}}^{\mu}\partial_{\mu}\psi_{\pm}(x) - (m \pm m_{\mathrm{M}})\psi_{\pm}(x) = 0\,. \tag{2.12}$$

These equations for $\psi_{\pm}$ have the Dirac form (2.1) (with $q = 0$), but the rest masses are different if $m \neq 0$ and $m_{\mathrm{M}} \neq 0$, namely $m_{+} = |m + m_{\mathrm{M}}|$, $m_{-} = |m - m_{\mathrm{M}}|$. Thus, instead of single equation (2.10), we now have two independent equations (2.12). The bispinor ψ is split into the $\psi_{\pm}$ components: $\psi(x) = \psi_{+}(x) + \psi_{-}(x)$. It turns out that also scalar product (2.4) is split,

$$\langle\psi_1|\psi_2\rangle = \langle\psi_{1+}|\psi_{2+}\rangle + \langle\psi_{1-}|\psi_{2-}\rangle\,.$$

The Poincaré transformations of $\psi_{\pm}$ have the same form as for ψ or ψ_{c}, *cf.* formulas (2.7), (2.8). To summarize, quantum mechanics of the Dirac particle with the Majorana mass term has been split into two independent sectors. The splitting is preserved by the Poincaré transformations.

The components $\psi_{\pm}$ are characterized by their behavior under the charge conjugation. The component ψ_{+} is charge conjugation even while ψ_{-} is odd, namely

$$(\psi_{+})_{\mathrm{c}}(x) = \psi_{+}(x), \qquad (\psi_{-})_{\mathrm{c}}(x) = -\psi_{-}(x)\,. \tag{2.13}$$

The decomposition of ψ into even and odd components is unique: if $\psi = \chi_+ + \chi_-$, where χ_+ is even and χ_- odd, then one can easily show that $\chi_+ = \psi_+$, $\chi_- = \psi_-$. Conditions (2.13) define two subspaces of bispinors which are linear spaces over $\mathbb{R}$, not over $\mathbb{C}$. For example, let us consider a linear combination of two charge conjugation even bispinors $c_1\psi_+ + c_2\chi_+$. The charge conjugation acting on it gives

$$(c_1\psi_+ + c_2\chi_+)_{\rm c} = c_1^*\psi_+ + c_2^*\chi_+ \,.$$

Thus, the linear combination is charge conjugation even only if c_1, c_2 are real numbers.

The relativistic quantum mechanics of the Majorana particle is obtained by taking only the charge conjugation even sector. In accordance with considerations presented above, the Hilbert space consists of (in general) complex bispinors — we now denote them by ψ instead of ψ_+ — which obey the condition

$$\psi_{\rm c}(x) = \psi(x)\,. \tag{2.14}$$

This Hilbert space is linear over $\mathbb{R}$. The scalar product still has the form of (2.4). Time evolution of ψ is governed by the Dirac equation (2.12), in which we rename $m + m_{\rm M}$ to m. Here, we consider only the free Majorana particle. More general theory can be obtained by including a certain fixed potential in the Dirac equation.

Condition (2.14) can be solved. To this end, we write $\psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}$, where ξ, ζ are two-component spinors. Using definition (2.5) and the explicit form of $\gamma_{\rm D}^2$ given by (2.2), we find that $\zeta = -i\sigma_2\xi^*$. Therefore,

$$\psi = \begin{pmatrix} \xi \\ -i\sigma_2\xi^* \end{pmatrix}, \tag{2.15}$$

where ξ is arbitrary complex spinor. The scalar product of ψ and $\chi = (\eta, -i\sigma_2\eta^*)^{\rm T}$ is expressed by ξ and η,

$$\langle\psi|\chi\rangle = \int {\rm d}^3x \, \left(\xi^\dagger\eta + \eta^\dagger\xi\right)\,. \tag{2.16}$$

The Dirac equation is equivalent to the following equation for the spinor ξ:

$$i\partial_0\xi(x) + \sigma_i\sigma_2\partial_i\xi^*(x) - m\xi(x) = 0\,. \tag{2.17}$$

Formulas (2.15), (2.16) and Eq. (2.17) constitute the so-called two-component formulation of the quantum mechanics of the Majorana particle. It is used, for example, in [7].

Yet another formulation is obtained by decomposing the spinor ξ into real and imaginary parts, $\xi = (\xi' + i\xi'')/\sqrt{2}$, and rewriting formula (2.15) in the following form:

$$\psi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & i\sigma_0 \\ -i\sigma_2 & -\sigma_2 \end{pmatrix} \begin{pmatrix} \xi' \\ \xi'' \end{pmatrix} . \tag{2.18}$$

The coefficient $1/\sqrt{2}$ is introduced for convenience. The four-by-four matrix on the r.h.s. of formula (2.18) is nonsingular — in fact, it is unitary. Therefore, the Dirac equation for ψ can be equivalently rewritten as equation for the real bispinor $\mathit{\Xi} = (\xi', \xi'')^{\mathrm{T}}$. This new equation also has the form of Dirac equation

$$i\gamma_{\mathrm{M}}^{\mu}\partial_\mu \mathit{\Xi}(x) - m\mathit{\Xi}(x) = 0\,, \tag{2.19}$$

with the following matrices $\gamma_{\mathrm{M}}^{\mu}$ in place of $\gamma_{\mathrm{D}}^{\mu}$:

$$\begin{aligned} \gamma_{\mathrm{M}}^{0} &= i\begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \qquad \gamma_{\mathrm{M}}^{1} = -i\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \\ \gamma_{\mathrm{M}}^{2} &= i\begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \qquad \gamma_{\mathrm{M}}^{3} = i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \end{aligned} \tag{2.20}$$

These matrices are unitarily equivalent to the matrices $\gamma_{\mathrm{D}}^{\mu}$. Note that all matrices $\gamma_{\mathrm{D}}^{\mu}$ are purely imaginary[1]. They, of course, satisfy the Dirac condition (2.3). For the scalar product, we obtain

$$\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^3x \left(\xi_1'^{\mathrm{T}}(x)\xi_2'(x) + \xi_1''^{\mathrm{T}}(x)\xi_2''(x)\right) = \int \mathrm{d}^3x\, \mathit{\Xi}_1^{\mathrm{T}}(x)\mathit{\Xi}_2(x)\,, \tag{2.21}$$

where $\mathit{\Xi}_1$ ($\mathit{\Xi}_2$) corresponds to ψ_1 (ψ_2). In the remaining part of this article, we will use this last formulation.

Quantum mechanics with (bi)spinorial wave functions is also used in theory of the Weyl particle. Relations between the Dirac, Majorana, and Weyl quantum particles are elucidated in, *e.g.*, [8].

3. The axial momentum

Let us dig a bit deeper into the relativistic quantum mechanics of the Majorana particle. We will use the third formulation presented above. In order to facilitate the considerations, we now adjust the notation and list the basic tenets of the theory. From now on, the Majorana real bispinor is

[1] In such a case, we say that we have the Majorana representation for γ^μ matrices.

denoted by ψ instead of Ξ. As the Dirac matrices in the Majorana representation, we take (for a personal reason) the following matrices:

$$\begin{aligned}\gamma^0 &= \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}, \quad \gamma^1 = i\begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}, \quad \gamma^2 = i\begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}, \\ \gamma^3 &= -i\begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix}, \qquad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i\begin{pmatrix} 0 & \sigma_0 \\ -\sigma_0 & 0 \end{pmatrix}, \end{aligned} \tag{3.1}$$

which are unitarily equivalent to the matrices γ^μ_{M}. The matrices γ^0, γ_5 are Hermitian and anti-symmetric, γ^i are anti-Hermitian and symmetric. The pertinent Hilbert space $\mathcal{H}$ consists of all normalizable real bispinors. It is linear space over $\mathbb{R}$, not over $\mathbb{C}$. The scalar product is defined as follows:

$$\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^3x\ \psi_1^{\mathrm{T}}(t,\boldsymbol{x})\psi_2(t,\boldsymbol{x})\,. \tag{3.2}$$

Observables are represented by linear operators which are Hermitian with respect to this scalar product. Time evolution of the real bispinors is governed by the Dirac equation

$$i\gamma^\mu\partial_\mu\psi(x) - m\psi(x) = 0 \tag{3.3}$$

with imaginary γ^μ matrices (3.1). It is convenient to rewrite this equation in the Hamiltonian form

$$\partial_t\psi = \hat{h}\psi\,, \tag{3.4}$$

where

$$\hat{h} = -\gamma^0\gamma^k\partial_k - im\gamma^0\,.$$

This operator is real, but it is not Hermitian. Nevertheless, the scalar product turns out to be constant in time because $\hat{h}$ is anti-symmetric as operator in $\mathcal{H}$, that is,

$$\left\langle\psi_1|\hat{h}\psi_2\right\rangle = -\left\langle\hat{h}\psi_1|\psi_2\right\rangle.$$

We shall study solutions of Eq. (3.4) in the next section.

The quantum mechanical framework described above has certain unusual features. First, the Hamiltonian $\hat{h}$ is not Hermitian, hence it is not an observable. Let us stress that it is not a disaster for the quantum mechanics — what really matters is constant in time scalar product. Simple calculation shows that scalar product (3.2) is constant in time provided that ψ_1, ψ_2 obey equation (3.4). Of course, the question arises whether there is a certain Hermitian energy operator. The form of general solution of Eq. (3.4) presented in the next section, see formula (4.5), suggests the operator

$$\hat{E} = \sqrt{m^2 - \nabla^2}\,.$$

In the present section, we focus on another peculiarity: the standard momentum operator $\hat{\boldsymbol{p}} = -i\nabla$ turns real bispinors into imaginary ones, hence it is not operator in the Hilbert space $\mathcal{H}$. To the best of our knowledge, this problem was noticed first in [9] and later readdressed in [10]. Is there a replacement for $\hat{\boldsymbol{p}}$? The momentum operator is usually associated with transformation of the wave function ψ under spatial translations, $\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}-\boldsymbol{a})$, where $\boldsymbol{a}$ is a constant vector. For infinitesimal translations,

$$\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) - (\boldsymbol{a}\nabla)\psi(\boldsymbol{x}) + \mathcal{O}\left(\boldsymbol{a}^2\right).$$

Thus, the actual generator of translations is just the ∇ operator, but it is not Hermitian. When complex numbers are allowed, we multiply ∇ by $-i$ in order to obtain the Hermitian operator $\hat{\boldsymbol{p}}$. Then we have

$$\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) - i(\boldsymbol{a}\hat{\boldsymbol{p}})\psi(\boldsymbol{x}) + \mathcal{O}\left(\boldsymbol{a}^2\right) .$$

Below we give an argument that in the Majorana case, the natural choice is to multiply ∇ by the matrix $-i\gamma_5$. This gives the Hermitian operator $\hat{\boldsymbol{p}}_5 = -i\gamma_5\nabla$, called by us the axial momentum. In this case,

$$\psi'(\boldsymbol{x}) = \psi(\boldsymbol{x}) - i\gamma_5(\boldsymbol{a}\hat{\boldsymbol{p}}_5)\psi(\boldsymbol{x}) + \mathcal{O}\left(\boldsymbol{a}^2\right),$$

because $\gamma_5^2 = I$.

The argument for $\hat{\boldsymbol{p}}_5$ is as follows. There exists a mapping between the Majorana bispinors ψ and right-handed (or left-handed) Weyl bispinors ϕ, namely $\psi = \phi + \phi^*$. By the definition of right-handed bispinors, $\gamma_5\phi = \phi$. It follows that $\gamma_5\phi^* = -\phi^*$. Therefore, $\gamma_5\psi = \phi - \phi^*$ and $\phi = (I+\gamma_5)\psi/2$, $\phi^* = (I-\gamma_5)\psi/2$. We see that the mapping is invertible. Now, the momentum operator $\hat{\boldsymbol{p}} = -i\nabla$ is well-defined for the Weyl bispinors because they are complex. Moreover, because $\hat{\boldsymbol{p}}$ commutes with γ_5, also $\hat{\boldsymbol{p}}\phi$ is the right-handed Weyl bispinor. Let us find the Majorana bispinor that corresponds to $\hat{\boldsymbol{p}}\phi$:

$$\hat{\boldsymbol{p}}\phi + (\hat{\boldsymbol{p}}\phi)^* = -i\nabla(\phi - \phi^*) = -i\nabla\gamma_5(\phi + \phi^*) = \hat{\boldsymbol{p}}_5\psi .$$

Thus, the axial momentum operator in the space of Majorana bispinors corresponds to the standard momentum operator in the space of right-handed Weyl bispinors.

Normalized eigenfunctions $\psi_{\boldsymbol{p}}(\boldsymbol{x})$ of the axial momentum obey the equations

$$\hat{\boldsymbol{p}}_5\psi_{\boldsymbol{p}}(\boldsymbol{x}) = \boldsymbol{p}\,\psi_{\boldsymbol{p}}(\boldsymbol{x}), \qquad \int \mathrm{d}^3x\, \psi^{\mathrm{T}}_{\boldsymbol{p}}(\boldsymbol{x})\,\psi_{\boldsymbol{q}}(\boldsymbol{x}) = \delta(\boldsymbol{p}-\boldsymbol{q}),$$

and they have the following form:

$$\psi_{\boldsymbol{p}}(\boldsymbol{x}) = (2\pi)^{-3/2}\exp(i\gamma_5\boldsymbol{p}\boldsymbol{x})\,v . \tag{3.5}$$

Here, v an arbitrary real, constant, normalized ($v^{\mathrm{T}}v = 1$) bispinor. For the exponential, we may use the formula

$$\exp(i\gamma_5 \boldsymbol{p}\boldsymbol{x}) = \cos(\boldsymbol{p}\boldsymbol{x})I + i\gamma_5 \sin(\boldsymbol{p}\boldsymbol{x})\,.$$

The eigenvalues $\boldsymbol{p}$ take arbitrary real values.

The axial momentum is not constant in time in the Heisenberg picture when $m \neq 0$. This is a rather unexpected feature, recall that we consider a free particle. Let us first introduce the Heisenberg picture. Equation (3.4) has the formal solution

$$|t\rangle = \exp\left(t\hat{h}\right)\,|t_0\rangle\,,$$

where $|t_0\rangle$ is an initial state. Time-dependent expectation value of an observable $\hat{\mathcal{O}}$ is given by

$$\langle t|\hat{\mathcal{O}}|t\rangle = \langle t_0|\exp\left(-t\hat{h}\right)\,\hat{\mathcal{O}}\,\exp\left(t\hat{h}\right)|t_0\rangle\,.$$

Therefore, we define the Heisenberg picture version of $\hat{\mathcal{O}}$ as

$$\hat{\mathcal{O}}(t) = \exp\left(-t\hat{h}\right)\,\hat{\mathcal{O}}\,\exp\left(t\hat{h}\right)\,.$$

In consequence,

$$\frac{\mathrm{d}\hat{\mathcal{O}}(t)}{\mathrm{d}t} = \left[\hat{\mathcal{O}}(t), \hat{h}\right] + \left(\partial_t \hat{\mathcal{O}}\right)(t)\,, \tag{3.6}$$

where the last term on the r.h.s. appears when $\hat{\mathcal{O}}$ is time-dependent in the Schroedinger picture. In the case of axial momentum, the r.h.s. of Eq. (3.6) does not vanish when $m \neq 0$,

$$\left[\hat{\boldsymbol{p}}_5, \hat{h}\right] = 2im\gamma^0 \hat{\boldsymbol{p}}_5\,.$$

The solution of the Heisenberg equation (3.6) reads [10]

$$\hat{\boldsymbol{p}}_5(t) = -i\gamma_5(t)\nabla\,, \tag{3.7}$$

where

$$\gamma_5(t) = \gamma_5 + im\hat{E}^{-1}\gamma^0\gamma_5\left[\sin\left(2\hat{E}t\right) + \hat{J}\left(1 - \cos\left(2\hat{E}t\right)\right)\right]\,, \tag{3.8}$$

and $\hat{J} = \hat{h}/\hat{E}$. Since $\hat{J}^2 = -I$, the two oscillating terms on the r.h.s of formula (3.8) are of the same order $m/\hat{E}$.

Notice that $\hat{\boldsymbol{p}}_5^2 = -\nabla^2$ commutes with $\hat{h}$. Therefore, the energy $\hat{E}$ as well as $|\hat{\boldsymbol{p}}_5|$ are constant in time. The evolution of $\hat{\boldsymbol{p}}_5(t)$ reminds a precession.

Matrix elements of the axial momentum can depend on time, for example,

$$\int d^3x\, \psi^{\mathrm{T}}_{\boldsymbol{p}}(\boldsymbol{x})\hat{\boldsymbol{p}}_5(t)\psi_{\boldsymbol{q}}(\boldsymbol{x}) = \boldsymbol{p}\left[1+\frac{m^2}{E_p^2}(\cos(2E_pt)-1)\right]\left(v^{\mathrm{T}}w\right)\ \delta(\boldsymbol{p}-\boldsymbol{q})$$
$$-\boldsymbol{p}\frac{m}{E_p}\left[i\sin(2E_pt)\left(v^{\mathrm{T}}\gamma^0w\right)+(1-\cos(2E_pt))\left(v^{\mathrm{T}}\gamma_5\frac{\gamma^jp^j}{E_p}w\right)\right]\delta(\boldsymbol{p}+\boldsymbol{q})\,.$$

Here, v and w are the constant bispinors present in, respectively, $\psi_{\boldsymbol{p}}$ and $\psi_{\boldsymbol{q}}$, see formula (3.5), and $E_p=\sqrt{m^2+\boldsymbol{p}^2}$.

The Heisenberg uncertainty relation for the position and the axial momentum has the same form as with the standard momentum [11],

$$\langle\psi|\left(\Delta\hat{x}^j\right)^2|\psi\rangle\langle\psi|\left(\Delta\hat{p}_5^k\right)^2|\psi\rangle \geq \frac{1}{4}\delta_{jk}\,,$$

where $\qquad \Delta\hat{x}^j=\hat{x}^j-\langle\psi|\hat{x}^j|\psi\rangle, \qquad \Delta\hat{p}_5^k=\hat{p}_5^k-\langle\psi|\hat{p}_5^k|\psi\rangle\,.$

4. General solution of the Dirac equation

From a mathematical viewpoint, the Dirac equation (3.3), or equivalently Eq. (3.4), is rather a simple linear partial differential equation with constant coefficients. It can be solved by the Fourier transform method. The standard Fourier transform uses the functions $\exp(i\boldsymbol{p}\boldsymbol{x})$ which are eigenfunctions of the standard momentum $\hat{\boldsymbol{p}}$. In view of the inadequacy of this momentum for the Majorana particle, we prefer an expansion into the eigenfunctions of the axial momentum with the exponential orthogonal matrices $\exp(i\gamma_5\boldsymbol{p}\boldsymbol{x})$.

Eigenfunctions (3.5) contain arbitrary real bispinors v. At each fixed eigenvalue $\boldsymbol{p}$, they form real four-dimensional space. We choose as the basis in this space eigenvectors of the real and Hermitian matrix $\gamma^0\gamma^kp^k$, *i.e.*, such v that

$$\gamma^0\gamma^kp^k\,v=E_0\,v\,, \tag{4.1}$$

where the matrices γ^μ have the form given by (3.1). It turns out that the eigenvalues $E_0=\pm|\boldsymbol{p}|$. The eigenvectors have the following form: for $E_0=|\boldsymbol{p}|$,

$$v_1^{(+)}(\boldsymbol{p})=\frac{1}{\sqrt{2|\boldsymbol{p}|(|\boldsymbol{p}|-p^2)}}\begin{pmatrix}-p^3\\ p^2-|\boldsymbol{p}|\\ p^1\\ 0\end{pmatrix},\qquad v_2^{(+)}(\boldsymbol{p})=i\gamma_5\,v_1^{(+)}(\boldsymbol{p})\,, \tag{4.2}$$

and for $E_0=-|\boldsymbol{p}|$

$$v_1^{(-)}(\boldsymbol{p})=i\gamma^0\,v_1^{(+)}(\boldsymbol{p})\,,\qquad v_2^{(-)}(\boldsymbol{p})=i\gamma_5\,v_1^{(-)}(\boldsymbol{p})=-\gamma_5\gamma^0v_1^{(+)}(\boldsymbol{p})\,. \tag{4.3}$$

These bispinors are real and orthonormal

$$\left(v_j^{(\epsilon)}\right)^{\mathrm{T}}(\boldsymbol{p})\, v_k^{(\epsilon')}(\boldsymbol{p}) = \delta_{\epsilon\epsilon'}\delta_{jk}\,,$$

where $\epsilon, \epsilon' = +, -$, and $j, k = 1, 2$.

Equation (4.1) is equivalent to[2]

$$\hat{E}_0\, \psi_{\boldsymbol{p}}(\boldsymbol{x}) = E_0\, \psi_{\boldsymbol{p}}(\boldsymbol{x})\,,$$

where

$$\hat{E}_0 = \gamma^0\gamma^k\hat{p}_5^k\,.$$

As shown in [10], $\hat{E}_0$ is related to the standard helicity operator $\hat{\lambda} = S^i\hat{p}^i/|\hat{\boldsymbol{p}}|$, namely

$$\hat{E}_0 = 2|\hat{\boldsymbol{p}}|\hat{\lambda}\,,$$

where $S^j = i\epsilon_{jkl}[\gamma^k, \gamma^l]/8$ are spin matrices, and $|\hat{\boldsymbol{p}}| = \sqrt{\hat{\boldsymbol{p}}^2} = \sqrt{\hat{\boldsymbol{p}}_5^2} = |\hat{\boldsymbol{p}}_5|$. Both $\hat{E}_0$ and $\hat{\lambda}$ are observables (they are real and Hermitian), as opposed to S^i and $\hat{\boldsymbol{p}}$ which are not real. Thus, $\hat{E}_0$ is essentially equivalent to the helicity. The plus sign in (4.2) and the minus in (4.3) correspond to the helicities $+1/2$ and $-1/2$, respectively.

The expansion of the wave function we start from reads

$$\psi(t, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}}\sum_{\alpha=1}^{2}\int \mathrm{d}^3p\, \mathrm{e}^{i\gamma_5\boldsymbol{p}\boldsymbol{x}}\left(v_\alpha^{(+)}(\boldsymbol{p})c_\alpha(\boldsymbol{p}, t) + v_\alpha^{(-)}(\boldsymbol{p})d_\alpha(\boldsymbol{p}, t)\right)\,. \tag{4.4}$$

The time dependence of the axial momentum amplitudes $c_\alpha(\boldsymbol{p}, t)$, $d_\alpha(\boldsymbol{p}, t)$ is determined by the Dirac equation (3.3). A series of mathematical steps described in [11] leads to the following result:

$$\begin{aligned}\psi(t, \boldsymbol{x}) \;=\; & \frac{1}{2(2\pi)^{3/2}}\int \mathrm{d}^3p\, \left[\cos(\boldsymbol{p}\boldsymbol{x} - E_pt)\, A_+(\boldsymbol{p}) + \cos(\boldsymbol{p}\boldsymbol{x} + E_pt)\, A_-(\boldsymbol{p})\right. \\ & \left. + \sin(\boldsymbol{p}\boldsymbol{x} - E_pt)\, B_+(\boldsymbol{p}) + \sin(\boldsymbol{p}\boldsymbol{x} + E_pt)\, B_-(\boldsymbol{p})\right]\,,\end{aligned} \tag{4.5}$$

where

$$A_\pm(\boldsymbol{p}) = v_1^{(+)}(\boldsymbol{p})A_\pm^1(\boldsymbol{p}) + v_2^{(+)}(\boldsymbol{p})A_\pm^2(\boldsymbol{p}) + v_1^{(-)}(\boldsymbol{p})A_\pm^3(\boldsymbol{p}) + v_2^{(-)}(\boldsymbol{p})A_\pm^4(\boldsymbol{p})\,,$$

$$B_\pm(\boldsymbol{p}) = v_1^{(+)}(\boldsymbol{p})B_\pm^1(\boldsymbol{p}) + v_2^{(+)}(\boldsymbol{p})B_\pm^2(\boldsymbol{p}) + v_1^{(-)}(\boldsymbol{p})B_\pm^3(\boldsymbol{p}) + v_2^{(-)}(\boldsymbol{p})B_\pm^4(\boldsymbol{p})\,,$$

[2] $\hat{E}_0$ should not be confused with the energy operator $\hat{E} = \sqrt{m^2 - \nabla^2}$. We keep here the notation introduced in [10].

and

$$\begin{aligned}
A^1_{\pm} &= \left(1 \pm \frac{p}{E_p}\right) c_1(\boldsymbol{p},0) \mp \frac{m}{E_p} d_2(\boldsymbol{p},0)\,,\\
A^2_{\pm} &= \left(1 \pm \frac{p}{E_p}\right) c_2(\boldsymbol{p},0) \mp \frac{m}{E_p} d_1(\boldsymbol{p},0)\,,\\
A^3_{\pm} &= \left(1 \mp \frac{p}{E_p}\right) d_1(\boldsymbol{p},0) \pm \frac{m}{E_p} c_2(\boldsymbol{p},0)\,,\\
A^4_{\pm} &= \left(1 \mp \frac{p}{E_p}\right) d_2(\boldsymbol{p},0) \pm \frac{m}{E_p} c_1(\boldsymbol{p},0)\,,\\
B^1_{\pm} &= -\left(1 \pm \frac{p}{E_p}\right) c_2(\boldsymbol{p},0) \mp \frac{m}{E_p} d_1(\boldsymbol{p},0)\,,\\
B^2_{\pm} &= \left(1 \pm \frac{p}{E_p}\right) c_1(\boldsymbol{p},0) \pm \frac{m}{E_p} d_2(\boldsymbol{p},0)\,,\\
B^3_{\pm} &= -\left(1 \mp \frac{p}{E_p}\right) d_2(\boldsymbol{p},0) \pm \frac{m}{E_p} c_1(\boldsymbol{p},0)\,,\\
B^4_{\pm} &= \left(1 \mp \frac{p}{E_p}\right) d_1(\boldsymbol{p},0) \mp \frac{m}{E_p} c_2(\boldsymbol{p},0)\,.
\end{aligned}$$

In these formulas, $p \equiv |\boldsymbol{p}|$, $E_p = \sqrt{\boldsymbol{p}^2 + m^2}$, and $c_\alpha(\boldsymbol{p},0), d_\alpha(\boldsymbol{p},0)$ are the initial values of the amplitudes given at $t=0$. Let us remind that $\boldsymbol{p}$ is the eigenvalue of the axial momentum.

Let us return to the question of energy operator raised in the previous section. Because the Hamiltonian $\hat{h}$ is not observable when $m \neq 0$, we have to look for another operator. Heuristically, energy in quantum physics is related to frequency. This idea can be embodied in the formula

$$\partial_t^2 \psi(t,\boldsymbol{x}) = -\hat{E}^2 \psi(t,\boldsymbol{x})\,.$$

Inserting here $\psi(t,\boldsymbol{x})$ given by formula (4.5), we obtain the condition $\hat{E}^2 = -\nabla^2 + m^2$, from which we would like to determine the energy operator $\hat{E}$. The simplest real and Hermitian solution is $\hat{E} = \sqrt{m^2 - \nabla^2}$. The square root can be a multivalued operation — in order to avoid misunderstandings let us specify that by $\sqrt{m^2 - \nabla^2}$ we mean the operator such that

$$\sqrt{m^2 - \nabla^2}\, \psi_{\boldsymbol{p}}(\boldsymbol{x}) = \sqrt{m^2 + \boldsymbol{p}^2}\, \psi_{\boldsymbol{p}}(\boldsymbol{x})$$

for all eigenfunctions (3.5) of $\hat{\boldsymbol{p}}_5$. The square root on the r.h.s. has only non-negative values by assumption.

Single mode with fixed value $\boldsymbol{q}$ of the axial momentum is obtained by putting in the formulas above

$$c_\alpha(\boldsymbol{p},0) = c_\alpha\,\delta(\boldsymbol{p}-\boldsymbol{q})\,, \qquad d_\alpha(\boldsymbol{p},0) = d_\alpha\,\delta(\boldsymbol{p}-\boldsymbol{q})\,,$$

where c_α, d_α, $\alpha = 1,2$, are constants. Then, in the massless case,

$$A^1_+ = 2c_1,\ A^2_+ = 2c_2,\ A^3_+ = A^4_+ = A^1_- = A^2_- = 0,\ A^3_- = 2d_1,\ A^4_- = 2d_2\,,$$
$$B^1_+ = -2c_2,\ B^2_+ = 2c_1,\ B^3_+ = B^4_+ = B^1_- = B^2_- = 0,\ B^3_- = -2d_2,\ B^4_- = 2d_1\,.$$

It is clear that the A_+, B_+ part on the r.h.s. of formula (4.5) is independent of the A_-, B_- part. In particular, we can put one of them to zero in order to obtain a plane wave propagating in the direction of $\boldsymbol{q}$ or $-\boldsymbol{q}$. The massive case is very different — always two components propagating in the opposite directions, $\boldsymbol{q}$ and $-\boldsymbol{q}$, are present. If we assume that $A_- = B_- = 0$, a simple calculation shows that then also $A_+ = B_+ = 0$, and *vice versa*. Such a pairing of traveling plane waves is one more peculiarity of quantum mechanics of the massive Majorana particle.

Continuing the analysis of the single mode, let us put $d_1 = d_2 = 0$ and keep c_1 and c_2 finite. In the massless case, we obtain plane wave moving in the direction $\boldsymbol{q}$, namely

$$\begin{aligned}\psi(\boldsymbol{x},t) \;=\; & \frac{1}{(2\pi)^{3/2}}\Big(\cos(\boldsymbol{q}\boldsymbol{x}-E_q t)\,\Big(c_1 v_1^{(+)}(\boldsymbol{q}) + c_2 v_2^{(+)}(\boldsymbol{q})\Big)\\ & + \sin(\boldsymbol{q}\boldsymbol{x}-E_q t)\,\Big(-c_2 v_1^{(+)}(\boldsymbol{q}) + c_1 v_2^{(+)}(\boldsymbol{q})\Big)\Big)\,. \end{aligned} \tag{4.6}$$

In the massive case, all four components in (4.5) do not vanish. However, the amplitudes A_- and B_- of the $-\boldsymbol{q}$ components are negligibly small in the high-energy limit ($m/E_q \ll 1$). In this limit,

$$\begin{aligned} A^1_+ &\approx 2c_1\,, & A^2_+ &\approx 2c_2\,, & A^3_+ &= \frac{m}{E_q}c_2\,, & A^4_+ &= \frac{m}{E_q}c_1\,,\\ B^1_+ &\approx -2c_2\,, & B^2_+ &\approx 2c_1\,, & B^3_+ &= \frac{m}{E_q}c_1\,, & B^4_+ &= -\frac{m}{E_q}c_2\,, \end{aligned}$$

and

$$\begin{aligned} A^1_- &\approx \frac{m^2}{2E_q^2}c_1\,, & A^2_- &\approx \frac{m^2}{2E_q^2}c_2\,, & A^3_- &= -\frac{m}{E_q}c_2\,, & A^4_- &= \frac{m}{E_q}c_1\,,\\ B^1_- &\approx -\frac{m^2}{2E_q^2}c_2\,, & B^2_- &\approx \frac{m^2}{2E_q^2}c_1\,, & B^3_- &= -\frac{m}{E_q}c_1\,, & B^4_- &= \frac{m}{E_q}c_2\,. \end{aligned}$$

On the other hand, in the limit of small energies ($E_q \approx m$), magnitudes of the $\boldsymbol{q}$ and $-\boldsymbol{q}$ components are approximately equal,

$$A^1_\pm \approx c_1 \,, \qquad A^2_\pm \approx c_2 \,, \qquad A^3_\pm \approx \pm c_2 \,, \qquad A^4_\pm \approx \pm c_1 \,,$$

and

$$B^1_\pm \approx -c_2 \,, \qquad B^2_\pm \approx c_1 \,, \qquad B^3_\pm \approx \pm c_1 \,, \qquad B^4_\pm \approx \mp c_2 \,.$$

5. The relativistic invariance

Relativistic transformations of the Majorana bispinor have the form of (2.7), where now $S(L) = \exp(\omega_{\mu\nu}[\gamma^\mu, \gamma^\nu]/8)$, where the matrices γ^μ have the form given by (3.1). Our goal is to check which irreducible representations of the Poincaré group are hidden in the space of real solutions of the Dirac equation (3.3), if any.

Instead of $\psi(t, \boldsymbol{x})$, we shall consider its counterpart in the axial momentum representation — the real bispinor $v(\boldsymbol{p}, t)$ introduced as follows:

$$\psi(t, \boldsymbol{x}) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{E_p}\, \mathrm{e}^{i\gamma_5 \boldsymbol{p}\boldsymbol{x}}\, v(\boldsymbol{p}, t) \,, \tag{5.1}$$

where $E_p = \sqrt{m^2 + \boldsymbol{p}^2}$. Equation (3.4) gives time evolution equation for v

$$\dot{v}(\boldsymbol{p}, t) = -i\gamma^0\gamma^k\gamma_5 p^k v(\boldsymbol{p}, t) - im\gamma^0 v(-\boldsymbol{p}, t) \,. \tag{5.2}$$

We have $v(-\boldsymbol{p}, t)$ in the last term on the r.h.s. because $\gamma^0 \exp(i\gamma_5 \boldsymbol{p}\boldsymbol{x}) = \exp(-i\gamma_5 \boldsymbol{p}\boldsymbol{x})\gamma^0$. From Eq. (5.2), we obtain equation

$$\ddot{v}(\boldsymbol{p}, t) = -E_p^2 v(\boldsymbol{p}, t) \,,$$

which has the general solution in the form of

$$v(\boldsymbol{p}, t) = \exp(-i\gamma_5 E_p t) v_+(\boldsymbol{p}) + \exp(i\gamma_5 E_p t) v_-(-\boldsymbol{p}) \,, \tag{5.3}$$

where $v_\pm$ are arbitrary real bispinors (we write $v_-(-\boldsymbol{p})$ for later convenience). Formulas (5.3) and (5.1) give

$$\psi(\boldsymbol{x}, t) = \frac{1}{(2\pi)^{3/2}} \int \frac{\mathrm{d}^3 p}{E_p} \left(\mathrm{e}^{i\gamma_5 (\boldsymbol{p}\boldsymbol{x} - E_p t)}\, v_+(\boldsymbol{p}) + \mathrm{e}^{-i\gamma_5 (\boldsymbol{p}\boldsymbol{x} - E_p t)}\, v_-(\boldsymbol{p}) \right) . \tag{5.4}$$

We have changed the integration variable to $-\boldsymbol{p}$ in the v_- term. Furthermore, Eq. (5.2) implies the following relations:

$$E_p \gamma_5 v_\pm(\boldsymbol{p}) = \gamma^0\gamma^k p^k \gamma_5 v_\pm(\boldsymbol{p}) \pm m\gamma^0 v_\mp(\boldsymbol{p}) \,. \tag{5.5}$$

Transformation law (2.7) with $a = 0$ applied to solution (5.4) gives Lorentz transformation of the bispinors $v_\pm(\boldsymbol{p})$,

$$v'_\pm(p) = S(L)\, v_\pm\left(L^{-1}p\right) . \tag{5.6}$$

Here, we use the four-vector p instead of $\boldsymbol{p}$ in order to simplify notation $v_+(p) \equiv v_+(\boldsymbol{p})$, where $p^0 = E_p$. In the case of space-time translations $x' = x + a$, we obtain

$$v'_\pm(\boldsymbol{p}) = \mathrm{e}^{\pm i\gamma_5 pa} v_\pm(\boldsymbol{p}) . \tag{5.7}$$

Further steps depend on whether the particle is massive or massless. The presented below discussion of the massive case is based on Section 4 of [11], where all missing details can be found. The massless case is not covered in that paper — it is presented below for the first time.

5.1. The massive Majorana particle

In this case, $v_-(\boldsymbol{p})$ can be expressed by $v_+(\boldsymbol{p})$, see (5.5). Using formula (5.4) we find that the scalar product $\langle\psi_1|\psi_2\rangle = \int \mathrm{d}^3x\ \psi_1^{\mathrm{T}}(t,\boldsymbol{x})\psi_2(t,\boldsymbol{x})$ is equal to

$$\langle\psi_1|\psi_2\rangle = \frac{2}{m^2}\int \frac{\mathrm{d}^3p}{E_p}\,\overline{v_{1+}(\boldsymbol{p})}\,\left(\gamma^0 E_p - \gamma^k p^k\right)\, v_{2+}(\boldsymbol{p}) , \tag{5.8}$$

where $\overline{v_{1+}(\boldsymbol{p})} = v_{1+}^{\mathrm{T}}(\boldsymbol{p})\gamma^0$, and v_{1+} (v_{2+}) corresponds to ψ_1 (ψ_2) by formula (5.4). The form (5.8) of the scalar product is explicitly Poincaré-invariant and time-independent.

Transformations (5.6), (5.7) are unitary with respect to scalar product (5.8). Thus, we have found certain real unitary, *i.e.*, orthogonal, representation of the Poincaré group. In order to determine the spin quantum number for this representation, we recast it to the standard form with the Wigner rotations [12]. First, we choose the standard momentum $\overset{(0)}{p} = (m,0,0,0)^{\mathrm{T}}$, where $m > 0$, and a Lorentz boost $H(p)$, $H(p)\,\overset{(0)}{p} = p$. At each p, we introduce the basis of real bispinors

$$v_i(p) = S(H(p))v_i(\overset{(0)}{p}) , \tag{5.9}$$

where $i = 1,2,3,4$. Here, $v_i(\overset{(0)}{p})$ is a basis at $\overset{(0)}{p}$ such that $v_i^{\mathrm{T}}(\overset{(0)}{p})v_k(\overset{(0)}{p}) = \delta_{ik}/m$. Actually, we assume that this basis has the Kronecker form, *i.e.*, the i^{th} component of the bispinor $v_k(\overset{(0)}{p})$ is equal to $\delta_{ik}/\sqrt{m}$. The factor m

is present for dimensional reason. In (5.9) the four-momentum notation is used, as in (5.6). We write $v_+(p)$ in this basis

$$v_+(p) = a^i(p) v_i(p)\,.$$

The amplitudes $a^i(p)$, $i = 1, 2, 3, 4$, are real and dimensionless. The scalar product (5.8) is equal to

$$\langle\psi_1|\psi_2\rangle = \frac{2}{m^2}\int \frac{\mathrm{d}^3p}{E_p}\, a_1^i(p) a_2^i(p)\,, \tag{5.10}$$

where a_1^i, a_2^i correspond to ψ_1, ψ_2, respectively. The remaining steps are rather technical. For detailed description of them, we refer the reader to paper [11]. Below, we cite the main results.

It turns out that Lorentz transformations (5.6) imply the following transformation of the amplitudes a^i

$$a'^k(p) = S_{ki}(\mathcal{R}(L,p)) a^i\left(L^{-1}p\right)\,, \tag{5.11}$$

where $\mathcal{R}(L,p) = H^{-1}(p) L H(L^{-1}p)$ is the Wigner rotation, and S_{ki} are the matrix elements of the matrix $S(L)$ introduced in formula (2.7). In the case of translations,

$$a'^k(p) = (\mathrm{e}^{i\gamma_5 pa})_{ki} a^i(p)\,. \tag{5.12}$$

For an arbitrary rotation R, including the Wigner rotation, the matrix $S(R)$ has the form of

$$S(R) = \exp\left(\tfrac{1}{2}\left(\omega_{12}\gamma^1\gamma^2 + \omega_{31}\gamma^3\gamma^1 + \omega_{23}\gamma^2\gamma^3\right)\right).$$

It can be shown that there exists a real orthogonal matrix $\mathcal{O}$ such that

$$\mathcal{O} S(R) \mathcal{O}^{-1} = \hat{T}, \tag{5.13}$$

where the four-by-four real matrix $\hat{T}$ has the form of

$$\hat{T} = \begin{pmatrix} \alpha' & -\alpha'' & -\beta' & \beta'' \\ \alpha'' & \alpha' & -\beta'' & -\beta' \\ \beta' & \beta'' & \alpha' & \alpha'' \\ -\beta'' & \beta' & -\alpha'' & \alpha' \end{pmatrix}. \tag{5.14}$$

The parameters $\alpha', \alpha'', \beta', \beta''$ are certain functions of ω_{12}, ω_{31}, ω_{23}.

In the last step, we recognize in the matrix $\hat{T}$ the real form of the spin-1/2 representation $T(u)$ of SU(2) group. This representation is given by the transformations $T(u)\xi = u\xi$, where $u \in \mathrm{SU}(2)$ and ξ is a two-component spinor, in general complex. Its real form is obtained simply by using the real and imaginary parts. Let us take

$$u = \begin{pmatrix} \alpha & -\beta \\ \beta^* & \alpha^* \end{pmatrix}, \qquad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix},$$

where $\alpha = \alpha' + i\alpha''$, $\beta = \beta' + i\beta''$, $\xi_1 = \xi_1' + i\xi_1''$, $\xi_2 = \xi_2' + i\xi_2''$, and $\alpha\alpha^* + \beta\beta^* = (\alpha')^2 + (\alpha'')^2 + (\beta')^2 + (\beta'')^2 = 1$. The real forms of ξ and u read

$$\vec{\xi} = \begin{pmatrix} \xi_1' \\ \xi_1'' \\ \xi_2' \\ \xi_2'' \end{pmatrix}, \qquad \hat{T}(u) = \begin{pmatrix} \alpha' & -\alpha'' & -\beta' & \beta'' \\ \alpha'' & \alpha' & -\beta'' & -\beta' \\ \beta' & \beta'' & \alpha' & \alpha'' \\ -\beta'' & \beta' & -\alpha'' & \alpha' \end{pmatrix}.$$

The real form of the spinor $u\xi$ is equal to $\hat{T}(u)\vec{\xi}$.

We conclude that representation (5.11) is equivalent to the real form of the spin-1/2 representation $T(u)$ of SU(2) group. Thus, the unveiled representation of the Poincaré group is the spin-1/2, $m > 0$, representation. Let us emphasize that we have obtained just one such representation. For comparison, in the case of Dirac particle, two spin-1/2 representations are present. The representations usually reappear in quantum field theory. Single representation in the Majorana case would correspond to a single spin-1/2 particle. In the Dirac case, there are two representations because there is particle and its anti-particle.

5.2. *The massless Majorana particle*

We again use formula (5.4) and transformations (5.6), (5.7). The difference with the massive case is that now the bispinors v_+, v_- are independent. Relations (5.5) with $m = 0$ become constraints for them, namely

$$\left(\gamma^0 E_p - \gamma^k p^k\right) v_\pm(\boldsymbol{p}) = 0\,, \tag{5.15}$$

where $E_p = |\boldsymbol{p}|$. Linear conditions (5.15) define two subspaces of real bispinors v_+, v_- which are two-dimensional. Each subspace spans the same representation (5.6), (5.7). It turns out that these representations are irreducible, orthogonal, and characterized by the helicities $\pm 1/2$. The reason for the opposite signs of the helicities in spite of the same transformation law is that the axial momenta corresponding to $v_+(\boldsymbol{p})$ and $v_-(\boldsymbol{p})$ are $\boldsymbol{p}$ and $-\boldsymbol{p}$, respectively, because of the opposite signs in the two exponents in formula (5.4).

One can easily show that general solution of conditions (5.15) has the form of

$$v_{\pm}(\boldsymbol{p}) = i\left(\gamma^0|\boldsymbol{p}| - \gamma^k p^k\right)\, w_{\pm}(\boldsymbol{p})\,, \tag{5.16}$$

where real bispinors $w_{\pm}(\boldsymbol{p})$) are arbitrary. The crucial fact here is nilpotency of the matrix on the l.h.s. of conditions (5.15)

$$\left(\gamma^0|\boldsymbol{p}| - \gamma^k p^k\right)^2 = 0\,.$$

For a given $v_{\pm}(\boldsymbol{p})$, formula (5.16) determines $w_{\pm}(\boldsymbol{p})$ up to a gauge transformation of the form

$$w'_{\pm}(\boldsymbol{p}) = w_{\pm}(\boldsymbol{p}) + i\left(\gamma^0|\boldsymbol{p}| - \gamma^k p^k\right)\, \chi_{\pm}(\boldsymbol{p}) \tag{5.17}$$

with arbitrary real bispinors $\chi_{\pm}(\boldsymbol{p})$.

Inserting (5.16) into formula (5.4), we obtain the following formula for the scalar product (3.2):

$$\begin{aligned}\langle\psi_1|\psi_2\rangle \;=\; 2\int \frac{\mathrm{d}^3p}{|\boldsymbol{p}|}\, &\left[\overline{w_{1+}(\boldsymbol{p})}\,\left(\gamma^0|\boldsymbol{p}| - \gamma^k p^k\right)\, w_{2+}(\boldsymbol{p})\right.\\ &\left.+\,\overline{w_{1-}(\boldsymbol{p})}\,\left(\gamma^0|\boldsymbol{p}| - \gamma^k p^k\right)\, w_{2-}(\boldsymbol{p})\right]\,. \end{aligned}\tag{5.18}$$

Notice that the scalar product is invariant with respect to gauge transformations (5.17).

There is a caveat concerning the r.h.s. of formula (5.18). Namely, it should not be considered as scalar product of w's, but rather as scalar product of equivalence classes of which the concrete w's are mere representatives. The equivalence class contains all bispinors $w_+(\boldsymbol{p})$ (or $w_-(\boldsymbol{p})$) related to each other by gauge transformations (5.17). All they give the same $v_{\pm}(\boldsymbol{p})$ and $\psi(t,\boldsymbol{x})$. The r.h.s. of formula (5.18) does not fulfill the requirement that for $w_{2\pm}(\boldsymbol{p}) = w_{1\pm}(\boldsymbol{p})$ it vanishes if and only if $w_{1\pm}(\boldsymbol{p}) = 0$ — the property of any true scalar product. The r.h.s. of formula (5.18) vanishes for any $w_{\pm}$ of the form of $w_{\pm}(\boldsymbol{p}) = i(\gamma^0|\boldsymbol{p}| - \gamma^k p^k)\,\chi_{\pm}(\boldsymbol{p})$. All such $w_{\pm}$ are gauge equivalent to $w_{\pm} = 0$ and they give $\psi(t,\boldsymbol{x}) = 0$.

We assume that Lorentz transformation of $w_{\pm}$ has the following form:

$$w'_{\pm}(p) = S(L)\, w_{\pm}\left(L^{-1}p\right)\,. \tag{5.19}$$

It implies transformation law (5.6) for $v_{\pm}$ given by formula (5.16). In the case of translations,

$$w'_{\pm}(p) = \mathrm{e}^{\mp i\gamma_5 pa} w_{\pm}(\boldsymbol{p})\,. \tag{5.20}$$

Notice that we could allow for certain gauge transformations on the r.h.s.'s of these formulas.

Scalar product (5.18) is invariant with respect to transformations (5.19), (5.20). Therefore, we have two independent unitary (*i.e.*, orthogonal) representations of the Poincaré group. In order to identify these representations, we check the related representations of the so-called little group [12]. In the massless case, the standard momentum is $\overset{(0)}{p} = (\kappa, 0, 0, \kappa)^{\mathrm{T}}$, where $\kappa > 0$ is fixed. The pertinent little group, called $E(2)$, is the maximal subgroup of the Lorentz group which leaves the standard momentum invariant. It is three-dimensional, and it includes spatial rotations around $\overset{(0)}{\boldsymbol{p}} = (0, 0, \kappa)^{\mathrm{T}}$ as well as certain combinations of Lorentz boosts and rotations[3]. Unitary irreducible representations of E(2) are either infinite-dimensional or one-dimensional (over complex numbers) [12].

In the considerations presented below, we concentrate on the bispinors w_+. Parallel considerations for w_- are essentially identical. Let us introduce a basis $w_i(\overset{(0)}{p})$, $i = 1, 2, 3, 4$, of bispinors at $\overset{(0)}{p}$. Applying Lorentz boosts $H_0(p)$, which transform $\overset{(0)}{p}$ into p, $H_0(p)\overset{(0)}{p} = p$, where $(p^0)^2 - \boldsymbol{p}^2 = 0$ and $p^0 > 0$, we obtain a basis $w_i(p)$ at each p belonging to the upper light-cone

$$w_i(p) = S(H_0(p))w_i(\overset{(0)}{p})\,. \tag{5.21}$$

We decompose $w_+(p)$ in this basis,

$$w_+(p) = w_i(p)c^i(p)\,.$$

Lorentz transformations (5.19) of bispinors are equivalent to certain transformations of the amplitudes $c^i(p)$ which give a representation of the little group $E(2)$. The form of these transformations is deduced from (5.19) as follows. First,

$$\begin{aligned} w'_+(p) &= w_k(p){c'}^k(p) = S(L)w_+\left(L^{-1}p\right) = c^i\left(L^{-1}p\right)\, S(L)\, w_i\left(L^{-1}p\right) \\ &= c^i\left(L^{-1}p\right)\, S(H_0(p))\, S\left(H_0^{-1}(p)LH_0(L^{-1}p)\right)\, w_i(\overset{(0)}{p})\,. \end{aligned}$$

Next, we notice that the Lorentz transformation $H_0^{-1}(p)LH_0(L^{-1}p)$ — let us denote it by $\mathcal{E}(L,p)$ — leaves the standard momentum $\overset{(0)}{p}$ invariant, hence

[3] In the case of massive Majorana particle, the little group is the SO(3) subgroup of the Lorentz group.

it belongs to the little group $E(2)$. We decompose bispinor $S(\mathcal{E}(L,p))w_i(\overset{(0)}{p})$ in the basis $w_k(\overset{(0)}{p})$,

$$S(\mathcal{E}(L,p))w_i(\overset{(0)}{p}) = D_{ki}(\mathcal{E}(L,p))w_k(\overset{(0)}{p})\,, \tag{5.22}$$

and write

$$w_k(p){c'}^k(p) = c^i(L^{-1}p)D_{ki}(\mathcal{E}(L,p))w_k(p)\,.$$

We see from this formula that

$${c'}^k(p) = D_{ki}(\mathcal{E}(L,p))\,c^i(L^{-1}p)\,. \tag{5.23}$$

It is clear that the Lorentz transformation (2.7) of the Majorana bispinors $\psi(x)$ follows from transformation (5.23) of the amplitudes $c^i(p)$ (in the massless case, of course).

Let us consider transformation (5.23) when $p = \overset{(0)}{p}$, and $L = R(\theta)$ is a rotation around the vector $\overset{(0)}{\boldsymbol{p}} = (0,0,\kappa)^{\mathrm{T}}$. Such L belongs to the E(2) group. In this case, (5.22) and (5.23) read

$$S(R(\theta))w_i(\overset{(0)}{p}) \;=\; D_{ki}(R(\theta))w_k(\overset{(0)}{p})\,, \tag{5.24}$$

$${c'}^k(\overset{(0)}{p}) \;=\; D_{ki}(R(\theta))\,c^i(\overset{(0)}{p})\,. \tag{5.25}$$

For the rotations around the third axis,

$$S(R(\theta)) = \exp\left(\gamma^1\gamma^2\theta/2\right) = \cos(\theta/2)I + \gamma^1\gamma^2\sin(\theta/2)\,,$$

where θ is the angle, and γ^1, γ^2 are given by (3.1). We obtain

$$S(R(\theta)) = \begin{pmatrix} \cos(\theta/2) & 0 & 0 & \sin(\theta/2) \\ 0 & \cos(\theta/2) & \sin(\theta/2) & 0 \\ 0 & -\sin(\theta/2) & \cos(\theta/2) & 0 \\ -\sin(\theta/2) & 0 & 0 & \cos(\theta/2) \end{pmatrix}. \tag{5.26}$$

It remains to specify the basis $w_k(\overset{(0)}{p})$. When doing this, we should take into account the fact that not all directions in the bispinor space are relevant for physics because of gauge transformations (5.17). In the case at hand, they have the following form:

$$w'_+(\overset{(0)}{p}) = w_+(\overset{(0)}{p}) + \kappa\left(\chi^4 - \chi^3\right)e_3 + \left(\chi^2 - \chi^1\right)e_4\,, \tag{5.27}$$

where χ^i are arbitrary real numbers (components of the bispinor $\chi_+(\overset{(0)}{p})$ present in (5.17)), and $e_3 = (1,1,0,0)^{\mathrm{T}}$, $e_4 = (0,0,1,1)^{\mathrm{T}}$. It is clear that e_3 and e_4 give the nonphysical directions in the space of bispinors w_+ at $\overset{(0)}{p}$. The remaining two directions — the physical ones — are given by $e_1 = (1,-1,0,0)^{\mathrm{T}}$ and $e_2 = (0,0,1,-1)^{\mathrm{T}}$. Thus, there is a natural choice for the basis $w_k(\overset{(0)}{p})$, namely

$$w_k(\overset{(0)}{p}) = e_k \,.$$

Only the amplitudes $c^1(\overset{(0)}{p})$, $c^2(\overset{(0)}{p})$ are physically interesting.

Using (5.26) and (5.24), we easily compute $D_{ki}(R(\theta))$ for $i,k = 1,2$. The transformations (5.25) have now the form of

$$\begin{aligned} c'^1(\overset{(0)}{p}) &= \cos(\theta/2)c^1(\overset{(0)}{p}) - \sin(\theta/2)c^2(\overset{(0)}{p}) \,, \\ c'^2(\overset{(0)}{p}) &= \sin(\theta/2)c^1(\overset{(0)}{p}) + \cos(\theta/2)c^2(\overset{(0)}{p}) \,. \end{aligned}$$

These transformations are the real version of the following transformations of the complex amplitude $z(\overset{(0)}{p}) = c^1(\overset{(0)}{p}) + ic^2(\overset{(0)}{p})$:

$$z'(\overset{(0)}{p}) = \exp(i\theta/2) z(\overset{(0)}{p}) \,.$$

Such transformations are characteristic for a massless particle with helicity 1/2.

Calculations for the w_- bispinors give the same formulas for transformations, but the helicity is equal to $-1/2$, because the standard vector $\overset{(0)}{\boldsymbol{p}} = (0,0,\kappa)^{\mathrm{T}}$ corresponds to the axial momentum $-\overset{(0)}{\boldsymbol{p}}$, recall $v_-(-\boldsymbol{p})$ in formula (5.3).

6. Remarks

1. We have outlined the basic structure of the relativistic quantum mechanics of the Majorana particle. List of its specific elements includes: the Hilbert space over $\mathbb{R}$, not over $\mathbb{C}$ as for other particles; the lack of the standard momentum operator and the appearance of the axial momentum with its peculiarities present in the case of massive particle; multiplicities of real orthogonal irreducible representations of the Poincaré group consistent with the expected lack of anti-particle in quantum theory of the Majorana field. One more item, not discussed here, is a relation with quaternions [11]. It is clear that it is a very interesting theory, worth further studies.

2. Formula (4.5) can be used for studying time evolution of wave packets with specified initial content of the axial momentum [13]. This is a rather interesting problem because the axial momentum is not constant in time when $m \neq 0$, hence we do not have any simple intuitions about the time evolution. Another topic one would like to know more about is behavior of the Majorana particle in external potentials, which can also be studied with use of the axial momentum.

3. The results of the analysis of relativistic invariance illustrate the well-known fact that the theory of massless particle is not a simple $m \to 0$ limit of the theory of massive particle. In particular, in the massive case, there is a single orthogonal irreducible representation of the Poincaré group, while for the massless particle, we have two representations. Moreover, the case of massless Majorana particle is distinguished by the presence of the gauge structure, as shown in Section 5.2. Gauge structures behind massless photons and gluons are well-known, but its presence also in the case of Majorana particle is a surprise.

The author acknowledges a support from the Marian Smoluchowski Institute of Physics, Jagiellonian University, Contract No. 337.1104.112.2019.

REFERENCES

[1] C. Giunti, Chung W. Kim, *Fundamentals of Neutrino Physics and Astrophysics*, Oxford University Press, Oxford 2007.

[2] S. Bilenky, *Introduction to the Physics of Massive and Mixed Neutrinos*, Lecture Notes in Physics 947, Springer Int. Publ. AG, Cham, 2018.

[3] V. Varadarajan, *Geometry of Quantum Theory*, D. van Nostrand Comp., Inc. Princeton, New Jersey, 1968.

[4] W.K. Wootters, in *Quantum Theory: Informational Foundations and Foils*, G. Chiribella and R.W. Speckens (Eds.), Springer, 2016, pp 21–43 [`arXiv:1301.2018 [quant-ph]`].

[5] J. Myrheim, `arXiv:quant-ph/9905037`.

[6] E. Majorana, *Nuovo Cim.* **14**, 171 (1937).

[7] M.H. Al-Hashimi, A.M. Shalaby, U.-J. Wiese, *Phys. Rev. D* **95**, 065007 (2017).

[8] Palash B. Pal, *Am. J. Phys.* **79**, 458 (2011).

[9] L. Pedro, `arXiv:1212.5465 [math-ph]`.

[10] H. Arodź, *Phys. Lett. A* **383**, 1242 (2019) [`arXiv:1805.03016 [physics.gen-ph]`].

[11] H. Arodź, Z. Świerczyński, submitted for publication, `arXiv:1910.13920 [quant-ph]`.

[12] See, *e.g.*, A.O. Barut, R. Raczka, *Theory of Group Representations and Applications*, PWN-Polish Scientific Publishers, Warsaw, 1977, Chapter 17.

[13] H. Arodź, Z. Świerczyński, work in progress.