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Evolutionary variational–hemivariational inequalities with applications to dynamic viscoelastic contact mechanics

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Abstract. The purpose of this work is to introduce and investigate a complicated variational–hemivariational inequality of parabolic type with history-dependent operators. First, we establish an existence and uniqueness theorem for a first-order nonlinear evolution inclusion problem, which is driven by a convex subdifferential operator for a proper convex function and a generalized Clarke subdifferential operator for a locally Lipschitz superpotential. Then, we employ the fixed point principle for history-dependent operators to deliver the unique solvability of the parabolic variational–hemivariational inequality. Finally, a dynamic viscoelastic contact problem with the nonlinear constitutive law involving a convex subdifferential inclusion is considered as an illustrative application, where normal contact and friction are described, respectively, by two nonconvex and nonsmooth multi-valued terms.

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1. Introduction

The contact processes between deformable bodies around in industry and our real-life and, for this reason, a considerable effort for modeling, mathematical analysis, numerical simulation and optimal control of various frictional contact problems are quite interesting and important.

The theory of variational inequalities can be used to describe the principles of virtual work and power which was initially proposed by Fourier in 1823. The prototypes, which lead to a class of variational inequalities, are the problems of Signorini–Fichera and frictional contact in elasticity. However, the first complete proof of unique solvability to Signorini Problem was provided by Signorini's student Fichera in 1964. The solution of the Signorini Problem coincides with the birth of the field of variational inequalities. For more on the initial developments of elasticity theory and variational inequalities, cf. e.g., [1]. With the gradual improvement of the theory of variational inequalities, there are numerous monographs dedicated to solving various complex phenomena in contact problems with different bodies and foundations, see for instance [7,8,12,32] and others. As the generalization of variational inequalities, the theory of hemivariational inequalities was first introduced and studied by Panagiotopoulos in [30]. The mathematical theory of hemivariational inequalities has been of great interest recently, which is due to the intensive development of applications of hemivariational inequalities in contact mechanics, control theory, games and so forth. Some comprehensive references are [4,13,15–19,21,24–27,29,31].

Recently, Han–Migórski–Sofonea [11], Migórski–Ogorzaly [22] and Migórski–Bai [23] studied the history-dependent variational–hemivariational inequality of parabolic type as follows

$$\begin{cases} \text{find } w \in \mathcal{W} \text{ such that for a.e. } t \in [0,T] \text{ and all } v \in V, \\ \big\langle w'(t) + \mathcal{A}\big(t, w(t)\big) + (\mathcal{S}_1 w)(t) - f(t), v - w(t)\big\rangle_{V^*\times V} + \phi^0\big(t, (\mathcal{S}_3 w)(t), w(t); v - w(t)\big) \\ \quad + \varphi(t, (\mathcal{S}_2 w)(t), v) - \varphi(t, (\mathcal{S}_2 w)(t) w(t)) \geq 0, \\ w(0) = w_0. \end{cases} \tag{1.1}$$

It is worth mentioning that problem (1.1) cannot able to be a mathematical model to handle with the following problem with constraints

$$\begin{cases} \text{find } w \in \mathcal{W} \text{ with } w(t) \in K \text{ for a.e. } t \in [0,T] \text{ such that for a.e. } t \in [0,T] \text{ and all } v \in K, \\ \big\langle w'(t) + \mathcal{A}\big(t, w(t)\big) + (\mathcal{S}_1 w)(t) - f(t), v - w(t)\big\rangle_{V^* \times V} + \varphi(v) - \varphi(w(t)) \\ \quad + \phi^0\big(t, (\mathcal{S}_2 w)(t), w(t); v - w(t)\big) \geq 0, \\ w(0) = w_0. \end{cases} \tag{1.2}$$

However, we know that the problem (1.2) can be used as a powerful mathematical tool to describe precisely various mechanical contact phenomena, such as unilateral constraint models. Based on this motivation, in the present paper, we are interested in the study of Problem 4.1, which expresses a generalized formulation of problem (1.2).

More precisely, the intention of the current work contains twofold. The first goal of the paper is to explore a generalized existence and uniqueness theorem to Problem 4.1. Our approach is based on the surjectivity theorem for the sum of operators together with the theory of nonsmooth and nonconvex analysis. However, the second purpose of the work is to apply the theoretical results established previously to investigate a complicated and new dynamic viscoelastic contact problem, in which the nonlinear constitutive law is characterized by a convex subdifferential inclusion. Also, the boundary conditions are described by two Clarke subdifferential terms for two locally Lipschitz potentials, which are nonconvex in general.

We arrange our paper in the following way. In Sect. 2, some preliminary materials of mathematics and mechanics are provided. Section 3 is devoted to treat a first-order nonlinear evolution inclusion problem involving a convex subdifferential operator and a generalized Clarke subgradient term within the framework of an evolution triple of spaces, and to prove a new existence and uniqueness result. Section 4 explores the unique solvability of the variational–hemivariational inequality of parabolic type under consideration by using the fixed point principle for history-dependent operators. In Sect. 5, a dynamic viscoelastic contact problem with the nonlinear constitutive law involving a convex subdifferential inclusion is considered as an illustrative application, where normal contact and friction are described, respectively, by two nonconvex and nonsmooth multi-valued terms.

2. Preliminaries

In this section, we briefly review basic notation and some results which are needed in the sequel. For more details, we refer to monographs [4–6,21].

Throughout the paper, we denote by $\langle \cdot, \cdot \rangle_{X^* \times X}$ the duality pairing between a Banach space X and its dual X^*. A single-valued mapping $A\colon X \to X^*$ is called to be demicontinuous, if for all $w \in X$, the functional $u \to \langle Au, w \rangle_{X^* \times X}$ is continuous. Let K be a nonempty subset of X. In what follows, by the notation 2^K, we represent the so-called power set of K, i.e., the set of all of its subsets. The domain, image and graph of a multi-valued operator $B\colon X \to 2^{X^*}$ are defined by $D(B) = \{x \in X \,|\, Bx \neq \emptyset\}$,

$$R(B) = \bigcup_{x \in X} Bx \quad \text{and} \quad Gr(B) = \big\{(x, x^*) \in X \times X^* \,|\, x^* \in Bx\big\},$$

respectively. Recall that a multi-valued mapping $B\colon X \to 2^{X^*}$ is said to be

(i) bounded, if it maps bounded sets of X into bounded sets of X^*.

(ii) strongly quasi-bounded, if for each $M > 0$, there exists $K_M > 0$ satisfying if $u \in D(B)$ and $u^* \in Bu$ are such that

$$\langle u^*, u \rangle_{X^* \times X} \leq M \quad \text{and} \quad \|u\|_X \leq M,$$

then we have $\|u^*\|_{X^*} \leq K_M$.

(iii) maximal monotone, if it is monotone such that $(y, y^*) \in X \times X^*$ satisfying

$$\langle x^* - y^*, x - y\rangle_{X^*\times X} \geq 0 \quad \text{for all } (x, x^*) \in Gr(B),$$

implies that $x_2^* \in B(x_2)$.

(iv) coercive, if it holds

$$\lim_{\|u\|_X\to\infty, u\in D(B)} \frac{\inf\left\{\langle u^*, u\rangle_{X^*\times X} \mid u^* \in Bu\right\}}{\|u\|_X} = +\infty.$$

We now recall the definition of L-pseudomonotonicity of multi-valued operators.

Definition 2.1. Let $L\colon D(L) \subset X \to X^*$ be a linear maximal monotone operator and $B\colon X \to 2^{X^*}$. We say that B is L-pseudomonotone (or B is pseudomonotone with respect to L), if the following conditions are satisfied

(a) for each $u \in X$, Bu is nonempty, bounded, convex and closed in X;
(b) B is upper semicontinuous from each finite-dimensional subspace of X to X^* endowed with the weak* topology;
(c) $\{u_n\} \subset D(L)$, $u_n^* \in Bu_n$ with $u_n \to u$ weakly in X, $Lu_n \to Lu$ weakly in X^*, $u_n^* \to u^*$ weakly in X^* and

$$\limsup_{n\to\infty}\langle u_n^*, u_n - u\rangle_{X^*\times X} \leq 0,$$

entail that $u^* \in Bu$ and $\langle u_n^*, u_n\rangle_{X^*\times X} \to \langle u^*, u\rangle_{X^*\times X}$.

In general, it is difficult to verify that an operator is strongly quasi-bounded by using its definition. Fortunately, the following proposition provides a useful criterion to guarantee an operator is strongly quasi-bounded, where its proof can be found in [2, Proposition 14].

Proposition 2.2. *Assume that $B\colon D(B) \subset X \to 2^{X^*}$ is a monotone operator such that $0 \in \operatorname{int}(D(B))$, then B is strongly quasi-bounded.*

Let $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. We denote the (convex) subdifferential operator $\partial_c\varphi\colon X \to 2^{X^*}$ of φ by

$$\partial_c\varphi(u) := \left\{u^* \in X^* \mid \varphi(v) - \varphi(u) \geq \langle u^*, v - u\rangle_{X^*\times X} \quad \text{for all} \quad v \in X\right\}$$

for all $u \in D(\varphi)$.

Proposition 2.3. *Let $\varphi\colon X \to \mathbb{R}\cup\{+\infty\}$ be a proper, convex and lower semicontinuous function. Then, $\partial_c\varphi\colon X \to 2^{X^*}$ is a maximal monotone operator.*

Let $h\colon X \to \mathbb{R}$ be a locally Lipschitz function. The (Clarke) generalized directional derivative of h at $u \in X$ in the direction $v \in X$ is defined by

$$h^0(u; v) = \limsup_{y\to u,\, \lambda\downarrow 0} \frac{h(y + \lambda v) - h(y)}{\lambda}.$$

In the meantime, the Clarke subdifferential operator $\partial h\colon X \to 2^{X^*}$ of h is given by

$$\partial h(u) = \{\, \zeta \in X^* \mid h^0(u; v) \geq \langle\zeta, v\rangle_{X^*\times X} \text{ for all } v \in X \,\} \text{ for all } u \in X.$$

The generalized gradient and generalized directional derivative of a locally Lipschitz function enjoy many nice properties and rich calculus. Here we just collect below some basic and crucial results, see for instance, [21, Proposition 3.23].

Proposition 2.4. *Let $h\colon X \to \mathbb{R}$ be a locally Lipschitz function, then the following statements are true*

(i) *for each $x \in X$, $\partial h(x)$ is nonempty, convex and weakly compact in X^*.*

(ii) *The graph of ∂h is closed in $X \times (w^* - X^*)$ topology, i.e., if $\{x_n\} \subset X$ and $\{\xi_n\} \subset X^*$ are such that $\xi_n \in \partial h(x_n)$ and $x_n \to x$ in X, $\xi_n \to \xi$ weakly* in X^*, then it holds $\xi \in \partial h(x)$.*

(iii) *The multi-valued mapping $X \ni x \mapsto \partial h(x) \subseteq X^*$ is upper semicontinuous from X into $w^* - X^*$.*

Furthermore, we shall review the well-known surjectivity result for L-pseudomonotone multi-valued operators, which will play a significant role in the proof of the main theorem in Sect. 4. For more details concerning the surjectivity theorem, one can find in [9, Theorem 3.1].

Theorem 2.5. *Let X be a reflexive Banach space, and $L\colon X \subset D(L) \to X^*$ be a linear maximal monotone operator. If $A\colon X \to 2^{X^*}$ is coercive, bounded and L-pseudomonotone, and $B\colon X \to 2^{X^*}$ is maximal monotone and strongly quasi-bounded with $0 \in B(0)$, then the mapping $L + A + B$ is surjective, i.e., $R(L + A + B) = X^*$.*

At the end, we shall introduce the usual notation, symbols, and function spaces, which will be used in the study of the dynamic viscoelastic contact problem in Sect. 5.

Let Ω be a bounded and connected domain in $\mathbb{R}^d$, where $(d = 2, 3)$, such that the boundary $\Gamma = \partial\Omega$ is Lipschitz continuous. The normal and tangential components of a vector field $\boldsymbol{\xi}$ on the boundary are given by $\xi_\nu = \boldsymbol{\xi} \cdot \boldsymbol{\nu}$ and $\boldsymbol{\xi}_\tau = \boldsymbol{\xi} - \xi_\nu \boldsymbol{\nu}$, respectively, where $\boldsymbol{\nu} = (\nu_i)$ denotes the outward unit normal at the boundary. Likewise, the notation σ_ν and $\boldsymbol{\sigma}_\tau$ represents the normal and tangential components of the stress field $\boldsymbol{\sigma}$ on the boundary, that is, $\sigma_\nu = (\boldsymbol{\sigma}\boldsymbol{\nu}) \cdot \boldsymbol{\nu}$ and $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\boldsymbol{\nu} - \sigma_\nu \boldsymbol{\nu}$. Furthermore, $\mathbb{S}^d$ denotes the space of real symmetric $d \times d$ matrices. On $\mathbb{R}^d$ and $\mathbb{S}^d$ we use the standard notation for inner products and norms which are defined by

$$\begin{aligned}
\boldsymbol{\xi} \cdot \boldsymbol{\eta} = \xi_i \eta_i, \|\boldsymbol{\xi}\| = (\boldsymbol{\xi} \cdot \boldsymbol{\xi})^{1/2} \quad &\text{for } \boldsymbol{\xi} = (\xi_i), \boldsymbol{\eta} = (\eta_i) \in \mathbb{R}^d, \\
\boldsymbol{\sigma} \cdot \boldsymbol{\tau} = \sigma_{ij}\tau_{ij}, \|\boldsymbol{\sigma}\| = (\boldsymbol{\sigma} \cdot \boldsymbol{\sigma})^{1/2} \quad &\text{for } \boldsymbol{\sigma} = (\sigma_{ij}), \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d.
\end{aligned}$$

Here, $i, j, k, l \in \{1, \ldots, d\}$ and the summation convention over repeated indices is used.

We also consider the following function spaces

$$H = L^2(\Omega; \mathbb{R}^d), \quad \mathcal{H} = L^2(\Omega; \mathbb{S}^d), \quad H_1 = \{v \in H \mid \boldsymbol{\varepsilon}(\boldsymbol{u}) \in \mathcal{H}\},$$

and $\mathcal{H}_1 = \{\boldsymbol{\tau} \in \mathcal{H} \mid \operatorname{Div} \boldsymbol{\tau} \in H\}$, where $\boldsymbol{\varepsilon}$ and Div, respectively, stand for the deformation and divergence operators given by

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(u)), \ \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j}), \quad i, j = 1, \ldots, d,$$

and the index following a comma indicates a partial derivative. By defining the following inner products

$$\begin{aligned}
\langle \boldsymbol{u}, \boldsymbol{v} \rangle_H &= \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v}\, dx, \qquad \langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{\mathcal{H}} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau}\, dx, \\
\langle \boldsymbol{u}, \boldsymbol{v} \rangle_{H_1} &= \langle \boldsymbol{u}, \boldsymbol{v} \rangle_H + \langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}) \rangle_{\mathcal{H}}, \\
\langle \boldsymbol{\sigma}, \tau \rangle_{\mathcal{H}_1} &= \langle \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle_{\mathcal{H}} + \langle \operatorname{Div} \boldsymbol{\sigma}, \operatorname{Div} \boldsymbol{\tau} \rangle_H,
\end{aligned}$$

it is obvious that the spaces $H, \mathcal{H}, H_1$ and $\mathcal{H}_1$ are Hilbert spaces.

3. First-order nonlinear evolution inclusion problems with nonsmooth and nonconvex potentials

This section is devoted to explore the existence and uniqueness for a generalized first-order evolution inclusion problem, which is driven by a generalized Clarke subdifferential of a locally Lipschitz function and a subdifferential operator of a convex potential, within the framework of an evolution triple of spaces $V \subset H \subset V^*$ (see, e.g., [21, Definition 1.52]).

Given $0 < T < +\infty$, in what follows, we adopt the following function spaces in the evolution triple of spaces $V \subset H \subset V^*$

$$\mathcal{V} = L^2(0,T;V),\ \hat{\mathcal{H}} = L^2(0,T;H),\ \mathcal{V}^* = L^2(0,T;V^*)$$

and $\mathcal{W} = \{v \in \mathcal{V} \,|\, v' \in \mathcal{V}^*\}$, where the time derivative $v' = \partial v/\partial t$ is understood in the sense of vector-valued distributions. It is not difficult to prove that the space $\mathcal{W}$ endowed with the norm

$$\|v\|_{\mathcal{W}} = \|v\|_{\mathcal{V}} + \|v'\|_{\mathcal{V}^*}$$

is a separable and reflexive Banach space, and the embeddings $\mathcal{W} \subset \mathcal{V} \subset \hat{\mathcal{H}} \subset \mathcal{V}^*$ are continuous. Besides, it follows from [21, Proposition 2.54(ii)] that the embedding $\mathcal{W} \subset C(0,T;H)$ is continuous as well. Throughout the paper, we denote by

$$\langle u^*, u\rangle_{\mathcal{V}^*\times\mathcal{V}} = \int_0^T \langle u^*(t), u(t)\rangle_{V^*\times V}\,\mathrm{d}t \quad \text{for all} \quad (u^*, u) \in \mathcal{V}^* \times \mathcal{V},$$

the duality pairing of $\mathcal{V}^*$ and $\mathcal{V}$.

Before proving the main problem, it should be mentioned that all of the convex and Clarke subdifferentials which are appeared in the sequel of the present paper are always understood with respect to the last variable of the corresponding functions.

The abstract evolution inclusion problem of parabolic type under the consideration is formulated as follows.

Problem 3.1. *Find $w \in \mathcal{W}$ such that*

$$\begin{cases} w'(t) + A(t, w(t)) + \zeta(t) + \xi(t) = f(t) & \text{for a.e. } t \in [0,T], \\ \zeta(t) \in \partial\varphi(t, w(t)) \quad \text{and} \quad \xi(t) \in \partial_c\psi(w(t)) & \text{for a.e. } t \in [0,T], \\ w(0) = w_0, \end{cases} \tag{3.1}$$

where the function f and initial data w_0 are assumed to satisfy the following regularities

$$f \in \mathcal{V}^*, \quad w_0 \in V. \tag{3.2}$$

To deliver the existence and uniqueness of solution to Problem 3.1, we make the following assumptions. The nonlinear function $A\colon [0,T] \times V \to V^*$ satisfies the following conditions.

$$H(A)\colon \begin{cases} \text{(i) } t \mapsto A(t,w) \text{ is measurable on } [0,T] \text{ for all } w \in V. \\ \text{(ii) } A(t,\cdot) \text{ is demicontinuous on } V \text{ for a.e. } t \in [0,T]. \\ \text{(iii) there exist a function } a_1 \in L^2_+(0,T) \text{ and a positive constant } a_2 > 0 \text{ such that} \\ \qquad \|A(t,w)\|_{V^*} \le a_1(t) + a_2\|w\|_V \text{ for all } w \in V \text{ and a.e. } t \in [0,T]. \\ \text{(iv) for a.e. } t \in [0,T], u \mapsto A(t,u) \text{ is strongly monotone, i.e.,} \\ \qquad \langle A(t,w_1) - A(t,w_2), w_1 - w_2\rangle_{V^*\times V} \ge \alpha\|w_1 - w_2\|_V^2 \\ \qquad \text{with } \alpha > 0 \text{ for all } w_1, w_2 \in V. \end{cases}$$

The functions $\varphi\colon [0,T]\times V\to\mathbb{R}$ and $\psi\colon V\to\mathbb{R}\cup\{+\infty\}$, respectively, read the next assumptions $H(\varphi)$ and $H(\psi)$.

$$H(\varphi)\colon\begin{cases}\text{(i) } \varphi(\cdot,w) \text{ is measurable on } [0,T] \text{ for all } w\in V.\\ \text{(ii) } \varphi(t,\cdot) \text{ is locally Lipschitz continuous on } V \text{ for a.e. } t\in[0,T].\\ \text{(iii) there exist a function } c_1\in L^2_+(0,T) \text{ and a constant } c_2>0\\ \qquad \text{such that for all } w\in V,\ \text{a.e. } t\in[0,T],\ \text{and all } \zeta(t)\in\partial\varphi(t,w)\\ \qquad\quad \|\zeta(t)\|_{V^*}\le c_1(t)+c_2\|w\|_V.\\ \text{(iv) there exists a constant } \beta\ge 0 \text{ such that}\\ \qquad \left\langle \zeta_1(t)-\zeta_2(t),w_1-w_2\right\rangle_{V^*\times V}\ge-\beta\|w_1-w_2\|_V^2\\ \qquad \text{for all } w_1,w_2\in V,\ \text{a.e. } t\in[0,T] \text{ with } \zeta_i(t)\in\partial\varphi(t,w_i), i=1,2.\end{cases}$$

$$H(\psi)\colon\begin{cases}\text{(i) } \psi(\cdot) \text{ is proper, convex and l.s.c. on } V.\\ \text{(ii) } w_0\in\operatorname{int}D(\psi) \text{ and } 0\in\partial_c\psi(w_0).\end{cases}$$

The main result of the section concerning the existence and uniqueness for Problem 3.1 is provided as follows.

Theorem 3.2. *Under the assumptions of $H(A), H(\varphi), H(\psi)$ and (3.2), and if, in addition, the inequality*

$$\max\{\beta,2c_2\}<\alpha \tag{3.3}$$

holds, then Problem 3.1 admits a unique solution $w\in\mathcal{W}$.

We shall employ the surjectivity result, Theorem 2.5, to obtain the desired conclusion in Theorem 3.2, by formulating Problem 3.1 to an abstract operator inclusion problem. To the end, we define an operator $\mathcal{A}\colon\mathcal{V}\to\mathcal{V}^*$ by

$$\langle\mathcal{A}w,v\rangle_{\mathcal{V}^*\times\mathcal{V}}=\int_0^T\langle A(t,w(t)),v(t)\rangle_{V^*\times V}\,\mathrm{d}t\quad\text{for all } w,v\in\mathcal{V},$$

and introduce a convex function $\Phi\colon\mathcal{V}\to\mathbb{R}\cup\{+\infty\}$ by

$$\Phi(v)=\int_0^T\psi(v(t))\,\mathrm{d}t\quad\text{for all } v\in\mathcal{V}. \tag{3.4}$$

For any $v\in\mathcal{V}$ and $t\in[0,T]$ fixed, we may restate Problem 3.1 to the inequality problem, by multiplying the first equation of (3.1) with $v(t)-w(t)$ and integrating the resulting over $[0,T]$

$$\begin{cases}\text{find } w\in\mathcal{W} \text{ such that}\\ \langle w'+\mathcal{A}w+\zeta-f,v-w\rangle_{\mathcal{V}^*\times\mathcal{V}}+\Phi(v)-\Phi(w)\ge 0\ \text{ for all } v\in\mathcal{V},\\ \zeta(t)\in\partial\varphi(t,w(t))\quad \text{ for a.e. } t\in[0,T],\\ w(0)=w_0.\end{cases} \tag{3.5}$$

In the meantime, consider the functions $\mathcal{A}_{w_0}\colon\mathcal{V}\to\mathcal{V}^*$, $\mathcal{F}_{w_0}\colon\mathcal{V}\to 2^{\mathcal{V}^*}$, and $\Phi_{w_0}\colon\mathcal{V}\to\mathbb{R}\cup\{+\infty\}$ by

$$\begin{cases}\mathcal{A}_{w_0}w=\mathcal{A}(w+w_0),\\ (\mathcal{F}_{w_0}w)(t)=\left\{\hat\zeta\in\mathcal{V}^*\,|\hat\zeta(t)\in\partial\varphi(t,w(t)+w_0)\right\}\quad\text{for a.e. } t\in[0,T],\\ \Phi_{w_0}(w)=\Phi(w+w_0)\end{cases} \tag{3.6}$$

for all $w \in \mathcal{V}$, and introduce the operator $L\colon D(L) \subset \mathcal{V} \to \mathcal{V}^*$ by

$$Lw = w' \quad \text{for all } \ w \in D(L) := \big\{ w \in \mathcal{W} \ \big| \ w(0) = 0 \big\}. \tag{3.7}$$

Then, under the above definitions, it is easy to see that $u \in \mathcal{W}$ is a solution to problem (3.5), if and only if, $z := u - w_0 \in D(L)$ solves the following operator inclusion problem

$$\begin{cases} \text{find } z \in D(L) \text{ such that} \\ Lz + \mathcal{A}_{w_0} z + \mathcal{F}_{w_0} z + \partial_c \Phi_{w_0}(z) \ni f. \end{cases} \tag{3.8}$$

Proof of Theorem 3.2. With respect to the existence of solutions to Problem 3.1, the proof will be based on Theorem 2.5.

Invoking [21, Lemma 3.64], it is well-known that the operator L defined in (3.7) is densely defined, linear, and maximal monotone. We assert that the mapping $\mathcal{Q}_{w_0}\colon \mathcal{V} \to 2^{\mathcal{V}^*}$ defined by

$$\mathcal{Q}_{w_0} z = \mathcal{A}_{w_0} z + \mathcal{F}_{w_0} z \quad \text{for } z \in \mathcal{V}$$

is coercive and bounded.

By virtue of hypotheses $H(A)$(iii), (iv), Hölder inequality and the element inequality $(a+b)^2 \geq (a^2/2) - b^2 (a, b \in \mathbb{R})$, we have that, for all $z \in \mathcal{V}$,

$$\begin{aligned} \langle \mathcal{A}_{w_0} z, z \rangle_{\mathcal{V}^* \times \mathcal{V}} &= \int_0^T \big\langle A(t, z(t) + w_0) - A(t, 0), z(t) + w_0 \big\rangle_{V^* \times V} \, \mathrm{d}t \\ &\quad + \int_0^T \big\langle A(t,0), z(t) + w_0 \big\rangle_{V^* \times V} \, \mathrm{d}t - \int_0^T \langle A(t, z(t) + w_0), w_0 \rangle_{V^* \times V} \, \mathrm{d}t \\ &\geq \alpha \int_0^T \|z(t) + w_0\|_V^2 \, \mathrm{d}t - \int_0^T a_1(t) \|z(t) + w_0\|_V \, \mathrm{d}t - \int_0^T a_1(t) \|w_0\|_V \, \mathrm{d}t \\ &\quad - a_2 \int_0^T \|z(t) + w_0\|_V \|w_0\|_V \, \mathrm{d}t \geq \frac{\alpha}{2} \|z\|_{\mathcal{V}}^2 - (\alpha + a_2) T \|w_0\|_V^2 \\ &\quad - (\|a_1\|_{L^2(0,T)} + a_2 \sqrt{T} \|w_0\|_V) \|z\|_{\mathcal{V}} - 2\sqrt{T} \|a_1\|_{L^2(0,T)} \|w_0\|_V. \end{aligned}$$

On the other hand, $H(\varphi)$(iii) and Hölder inequality deduce

$$\begin{aligned} \big| \langle \mathcal{F}_{w_0} z, z \rangle_{\mathcal{V}^* \times \mathcal{V}} \big| &\leq \int_0^T \big| \langle \partial \varphi(t, z(t) + w_0), z(t) \rangle_{V^* \times V} \big| \, \mathrm{d}t \\ &\leq \int_0^T \big(c_1(t) + c_2 \|z(t) + w_0\|_V \big) \|z(t)\|_V \, \mathrm{d}t \\ &\leq \int_0^T \Big(c_2 \|z(t)\|_V^2 + \big(c_1(t) + c_2 \|w_0\|_V \big) \|z(t)\|_V \Big) \, \mathrm{d}t \\ &\leq c_2 \|z\|_{\mathcal{V}}^2 + (\|c_1\|_{L^2(0,T)} + c_2 \sqrt{T} \|w_0\|_V) \|z\|_{\mathcal{V}} \end{aligned}$$

for all $z \in \mathcal{V}$, hence,

$$\langle \mathcal{F}_{w_0} z, z \rangle_{\mathcal{V}^* \times \mathcal{V}} \geq -c_2 \|z\|_{\mathcal{V}}^2 - (\|c_1\|_{L^2(0,T)} + c_2 \sqrt{T} \|w_0\|_V) \|z\|_{\mathcal{V}}$$

for all $z \in \mathcal{V}$. Notice that

$$\begin{aligned}\langle \mathcal{Q}_{w_0} z, z\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \langle \mathcal{A}_{w_0} z + \mathcal{F}_{w_0} z, z\rangle_{\mathcal{V}^*\times\mathcal{V}} \geq \Big(\frac{\alpha}{2} - c_2\Big)\|z\|_{\mathcal{V}}^2 \\ &\quad - (\|a_1\|_{L^2(0,T)} + \|c_1\|_{L^2(0,T)} + a_2\sqrt{T}\|w_0\|_V + c_2\sqrt{T}\|w_0\|_V)\|z\|_{\mathcal{V}} \\ &\quad - (\alpha + a_2)T\|w_0\|_V^2 - 2\sqrt{T}\|a_1\|_{L^2(0,T)}\|w_0\|_V \quad \text{for all } z \in \mathcal{V},\end{aligned}$$

we are now in a position to utilize the smallness condition (3.3) to conclude that $\mathcal{Q}_{w_0}$ is coercive.

Applying conditions $H(A)$(iii), $H(\varphi)$(iii), it yields that, for all $z \in \mathcal{V}$,

$$\begin{aligned}\|\mathcal{Q}_{w_0} z\|_{\mathcal{V}^*}^2 &\leq 2\|\mathcal{A}_{w_0} z\|_{\mathcal{V}^*}^2 + 2\|\mathcal{F}_{w_0} z\|_{\mathcal{V}^*}^2 \leq 2\int_0^T \|A(t, z(t) + w_0)\|_{V^*}^2 \,\mathrm{d}t \\ &\quad + 2\int_0^T \|\partial\varphi(t, z(t) + w_0)\|_{V^*}^2 \,\mathrm{d}t \leq 2(a_2^2 + c_2^2)\|z\|_{\mathcal{V}}^2 \\ &\quad + 4(a_2\|a_1\|_{L^2(0,T)} + c_2\|c_1\|_{L^2(0,T)} + a_2^2\sqrt{T}\|w_0\|_V + c_2^2\sqrt{T}\|w_0\|_V)\|z\|_{\mathcal{V}} \\ &\quad + 2T(a_2^2 + c_2^2)\|w_0\|_V^2 + 4\sqrt{T}\|w_0\|_V(a_2\|a_1\|_{L^2(0,T)} + c_2\|c_1\|_{L^2(0,T)}) \\ &\quad + 2(\|a_1\|_{L^2(0,T)}^2 + \|c_1\|_{L^2(0,T)}^2),\end{aligned}$$

i.e.,

$$\|\mathcal{Q}_{w_0} z\|_{\mathcal{V}^*} \leq r_1\|z\|_{\mathcal{V}} + r_2\sqrt{\|z\|_{\mathcal{V}}} + r_3 \quad \text{for all } z \in \mathcal{V}, \tag{3.9}$$

where the constants $r_1, r_2, r_3 \geq 0$ are all independent of z. Therefore, $\mathcal{Q}_{w_0}$ is a bounded mapping.

Next, we shall demonstrate that $\mathcal{Q}_{w_0}$ is L-pseudomonotone in the sense of Definition 2.1. To the end of this, we make the following three claims.

Claim 1. *The set $\mathcal{Q}_{w_0} z$ is nonempty, bounded, closed and convex in $\mathcal{V}^*$ for every $z \in \mathcal{V}$.*

Let $z \in \mathcal{V}$ be fixed. Proposition 2.4(i) implies that the set $\mathcal{F}_{w_0} z$ is a nonempty and convex in $\mathcal{V}^*$, so does $\mathcal{Q}_{w_0} z$. However, the inequality (3.9) guarantees the boundedness of $\mathcal{Q}_{w_0} z$. To illustrate that the set of $\mathcal{Q}_{w_0} z$ is closed, let $\{\eta_n\} \subset \mathcal{Q}_{w_0} z$ be such that $\eta_n \to \eta$ in $\mathcal{V}^*$, as $n \to \infty$. So, there exists a sequence $\{\zeta_n\} \subset \mathcal{F}_{w_0} z$ such that $\eta_n = \zeta_n + \mathcal{A}_{w_0} z$ and $\zeta_n \to \eta - \mathcal{A}_{w_0} z$ in $\mathcal{V}^*$, as $n \to \infty$. Then, passing to a subsequence if necessary, we assume that $\zeta_n(t) \to \eta(t) - \mathcal{A}_{w_0}(z)(t)$ in V^* for a.e. $t \in [0, T]$. In accordance with Proposition 2.4(ii), it finds that the set of $\eta - \mathcal{A}_{w_0} z \in \mathcal{F}_{w_0} z$. Therefore, the set $\mathcal{Q}_{w_0} z$ is also closed.

Claim 2. *$\mathcal{Q}_{w_0}$ is upper semicontinuous from $\mathcal{V}$ to $\mathcal{V}^*$ endowed with the weak topology.*

From [21, Proposition 3.8], it is enough to verify that for each weakly closed set C in $\mathcal{V}^*$, the set

$$\mathcal{Q}_{w_0}^-(C) = \big\{z \in \mathcal{V} \,|\, \mathcal{Q}_{w_0} z \cap C \neq \emptyset\big\}$$

is closed in $\mathcal{V}$.

We now show that $\mathcal{A}_{w_0} \colon \mathcal{V} \to \mathcal{V}^*$ is demicontinuous. Let $\{z_n\} \subset \mathcal{V}$ be such that $z_n \to z$ in $\mathcal{V}$, as $n \to \infty$. By passing to a subsequence if necessary, we may say

$$z_n(t) \to z(t) \text{ in } V \quad \text{for a.e. } t \in [0, T]. \tag{3.10}$$

In view of the condition $H(A)$(ii), it reads

$$\langle A(t, z_n(t) + w_0), w(t)\rangle_{V^*\times V} \to \langle A(t, z(t) + w_0), w(t)\rangle_{V^*\times V}$$

for all $w \in \mathcal{V}$ and a.e. $t \in [0, T]$. The latter combined with hypothesis $H(A)$(ii) and Lebesgue-dominated convergence theorem implies

$$\lim_{n\to\infty} \int_0^T \langle A(t, z_n(t) + w_0), w(t)\rangle_{V^*\times V}\,\mathrm{d}t \to \int_0^T \langle A(t, z(t) + w_0), w(t)\rangle_{V^*\times V}\,\mathrm{d}t$$

for all $w \in \mathcal{V}$, which means

$$\mathcal{A}_{w_0} z_n \to \mathcal{A}_{w_0} z \quad \text{weakly in } \mathcal{V}^*,$$

so, $\mathcal{A}_{w_0}$ is demicontinuous. In addition, we shall prove that $\mathcal{F}_{w_0} \colon \mathcal{V} \to 2^{\mathcal{V}^*}$ has a closed graph in $\mathcal{V} \times (w - \mathcal{V}^*)$. Let $z_n \to z$ in $\mathcal{V}$ and $\hat{\zeta}_n \to \hat{\zeta}$ weakly in $\mathcal{V}^*$ with

$$\hat{\zeta}_n(t) \in \partial\varphi(t, z_n(t) + w_0) \quad \text{for a.e. } t \in [0, T].$$

Invoking Proposition 2.4 and [21, Theorem 3.13] indicates

$$\hat{\zeta}(t) \in \partial\varphi(t, z(t) + w_0) \quad \text{for a.e. } t \in [0, T],$$

hence $\hat{\zeta} \in \mathcal{F}_{w_0} z$. Therefore, $\mathcal{F}_{w_0}$ is closed in the topology of $\mathcal{V} \times (w - \mathcal{V}^*)$.

Let $\{w_n\} \subset \mathcal{Q}^-_{w_0}(C)$ be a sequence such that $w_n \to w$ in $\mathcal{V}$, as $n \to \infty$; thus, there is $\bar{\zeta}_n = \mathcal{A}_{w_0} w_n + \hat{\zeta}_n$ with $\hat{\zeta}_n \in \mathcal{F}_{w_0} w_n$. From the boundedness of $\mathcal{F}_{w_0}$, we may assume that $\hat{\zeta}_n \to \hat{\zeta}$ weakly in $\mathcal{V}$, as $n \to \infty$, whereas by the demicontinuity of $\mathcal{A}_{w_0}$ and the fact, operator $\mathcal{F}_{w_0}$ is closed in $\mathcal{V} \times (w - \mathcal{V}^*)$ topology, it finds

$$\bar{\zeta} = \mathcal{A}_{w_0} w + \hat{\zeta} \in \mathcal{Q}_{w_0} w \ \text{ with } \hat{\zeta} \in \mathcal{F}_{w_0} w. \tag{3.11}$$

Furthermore, recall that the subset $C \subset \mathcal{V}^*$ is weakly closed, so it holds $w \in C$. Therefore, we have that $w \in \mathcal{Q}^-_{w_0}(C)$. This proves that $\mathcal{Q}^-_{w_0}(C)$ is closed in $\mathcal{V}$. Consequently, $\mathcal{Q}_{w_0}$ is u.s.c. from $\mathcal{V}$ to $\mathcal{V}^*$ endowed with the weak topology.

Claim 3. *$\mathcal{Q}_{w_0}$ is L-pseudomonotone.*

Let $\{z_n\} \subset D(L)$, $\bar{\zeta}_n \in \mathcal{Q}_{w_0} z_n$ with $z_n \to z$ weakly in $\mathcal{V}$, $Lz_n \to Lz$ and $\bar{\zeta}_n \to \bar{\zeta}$ both weakly in $\mathcal{V}^*$, be such that

$$\limsup_{n\to\infty} \langle \bar{\zeta}_n, z_n - z\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0. \tag{3.12}$$

We are going to show $\bar{\zeta} \in \mathcal{Q}_{w_0} z$ and

$$\langle \bar{\zeta}_n, z_n\rangle_{\mathcal{V}^*\times\mathcal{V}} \to \langle \bar{\zeta}, z\rangle_{\mathcal{V}^*\times\mathcal{V}}. \tag{3.13}$$

We now assert the convergence holds

$$z_n \to z \quad \text{strongly in } \mathcal{V}. \tag{3.14}$$

Let $\hat{\zeta}_n \in \mathcal{F}_{w_0} z_n$ be such that $\bar{\zeta}_n = \hat{\zeta}_n + \mathcal{A}_{w_0} z_n$. For any $\hat{\zeta} \in \mathcal{F}_{w_0} z$, $H(\varphi)$(iv) turns out

$$\begin{aligned}\langle \hat{\zeta}_n - \hat{\zeta}, z_n - z\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \left\langle \hat{\zeta}_n(t) - \hat{\zeta}(t), z_n(t) - z(t)\right\rangle_{V^*\times V}\mathrm{d}t \\ &\ge -\beta\|z_n - z\|^2_{\mathcal{V}}.\end{aligned}$$

The latter together with the strongly monotonicity of A (see $H(A)$(iv)) deduces

$$\begin{aligned}\langle \bar{\zeta}_n - \bar{\zeta}, z_n - z\rangle_{\mathcal{V}^*\times\mathcal{V}} &= \int_0^T \left\langle \bar{\zeta}_n(t) - \bar{\zeta}(t), z_n(t) - z(t)\right\rangle_{V^*\times V}\mathrm{d}t \\ &\ge (\alpha - \beta)\|z_n - z\|^2_{\mathcal{V}}\end{aligned}$$

for all $\bar{\zeta} \in \mathcal{Q}_{w_0} z$ and all $z \in \mathcal{V}$. Then, if for the above inequality, passing to the upper limit, as $n \to \infty$, and using (3.12), we derive

$$\begin{aligned}(\alpha - \beta) \limsup_{n\to\infty} \|z_n - z\|_{\mathcal{V}}^2 &\le \limsup_{n\to\infty} \langle \bar{\zeta}_n - \bar{\zeta}, z_n - z\rangle_{\mathcal{V}^*\times\mathcal{V}} \\ &= \limsup_{n\to\infty} \langle \bar{\zeta}_n, z_n - z\rangle_{\mathcal{V}^*\times\mathcal{V}} - \lim_{n\to\infty} \langle \bar{\zeta}, z_n - z\rangle_{\mathcal{V}^*\times\mathcal{V}} \le 0.\end{aligned}$$

But, the smallness condition (3.3) indicates that $\alpha - \beta > 0$, namely (3.14) is valid. In the meanwhile, employing the demicontinuity of $\mathcal{A}_{w_0}$ and the closedness of $\mathcal{F}_{w_0}$ (see the proof of Claim 2), it yields $\bar{\zeta} \in \mathcal{Q}_{w_0} z$. This means that (3.13) is satisfied.

Moreover, we also admit that $\Phi_{w_0} : \mathcal{V} \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous. The result $\Phi_{w_0} \not\equiv +\infty$ is a direct consequence of hypothesis $H(\psi)$(i). Also, the convexity of Φ_{w_0} can be obtained by applying the convexity of ψ. Let $z_n \to z$ in $\mathcal{V}$, as $n \to \infty$. Passing to a subsequence, if necessary, one has

$$z_n(t) + w_0 \to z(t) + w_0 \text{ in } V \text{ for a.e. } t \in [0, T].$$

However, from [28, Lemma 2.5(2)], we are able to find a function $h \in L^1(0,T)$ such that $h(t) \le \psi(z_n(t) + w_0)$ for all $t \in [0,T]$. Notice that (see the lower semicontinuity of ψ)

$$\psi(z(t) + w_0) \le \liminf_{n\to\infty} \psi(z_n(t) + w_0) \quad \text{for a.e. } \ t \in [0,T],$$

we can utilize Fatou's lemma to find

$$\begin{aligned}\int_0^T \psi(z(t) + w_0)\,\mathrm{d}t &\le \int_0^T \liminf_{n\to\infty} \psi(z_n(t) + w_0)\,\mathrm{d}t \\ &\le \liminf_{n\to\infty} \int_0^T \psi(z_n(t) + w_0)\,\mathrm{d}t.\end{aligned}$$

Therefore, Φ_{w_0} is lower semicontinuous on $\mathcal{V}$. Invoking Proposition 2.3 indicates that $\partial_c \Phi_{w_0} : \mathcal{V} \to 2^{\mathcal{V}^*}$ is maximal monotone.

Additionally, we shall demonstrate that $\partial_c \Phi_{w_0}$ is strongly quasi-bounded on $\mathcal{V}$ with $0 \in \partial_c \Phi_{w_0}(0)$. For any $M > 0$ fixed, let $z \in D(\partial_c \Phi_{w_0})$ and $\xi \in \partial_c \Phi_{w_0}(z)$ be such that

$$\|z\|_{\mathcal{V}} \le M, \quad \langle \xi, z\rangle_{\mathcal{V}^*\times\mathcal{V}} \le M. \tag{3.15}$$

Recall that $w_0 \in \mathrm{int} D(\psi)$, there exist an $\varepsilon > 0$ and $K_\varepsilon \in \mathbb{R}$ such that $\psi(y) \le K_\varepsilon < +\infty$ for all $y \in \{x \in V \mid \|x - w_0\|_V < \varepsilon\}$ (since ψ is locally Lipschitz continuous in $\mathrm{int} D(\psi)$). Define the open neighborhood $O_\varepsilon := \{z^* \in \mathcal{V} \mid \|z^*(t) - w_0\|_V < \varepsilon \text{ for a.e. } t \in [0,T]\}$ of $\mathcal{V}$. It is obvious that $\Phi(u) \le T|K_\varepsilon| < +\infty$ for all $u \in O_\varepsilon$, namely $0 \in \mathrm{int} D(\Phi_{w_0})$. The latter together with the fact $\mathrm{int} D(\Phi_{w_0}) \subset D(\partial_c \Phi_{w_0})$ implies $0 \in \mathrm{int} D(\partial_c \Phi_{w_0})$. Therefore, by using Proposition 2.2, we conclude that $\partial_c \Phi_{w_0}$ is strongly quasi-bounded on $\mathcal{V}$. On the other hand, the estimates

$$\Phi_{w_0}(u) - \Phi_{w_0}(0) = \int_0^T \psi(u(t) + w_0) - \psi(w_0)\,\mathrm{d}t \ge \int_0^T \langle \xi, u(t)\rangle_{V^*\times V}\,\mathrm{d}t$$

for all $\xi \in \partial_c \psi(w_0)$ and all $u \in \mathcal{V}$. But, condition $H(\psi)$(ii) ensures $0 \in \partial_c \Phi_{w_0}(0)$.

To conclude, we have verified all conditions of Theorem 2.5. Using this theorem, we conclude that $L + \mathcal{Q}_{w_0} + \partial_c \Phi_{w_0}$ is onto; thus, the inclusion (3.8) has a solution $z \in D(L)$. Consequently, $w = z + w_0 \in \mathcal{W}$ solves Problem 3.1.

We illustrate that Problem 3.1 is unique solvability. Let $w_1, w_2 \in \mathcal{W}$ be two solutions to Problem 3.1, i.e., for $i = 1, 2$,

$$\begin{cases} w_i'(t) + A(t, w_i(t)) + \zeta_i(t) + \xi_i(t) = f(t) & \text{for a.e. } t \in [0,T], \\ \zeta_i(t) \in \partial\varphi(t, w_i(t)),\ \xi_i(t) \in \partial_c\psi(w_i(t)) & \text{for a.e. } t \in [0,T], \\ w_i(0) = w_0. \end{cases}$$

A simple calculation gives

$$\begin{aligned} &\langle w_1'(t) - w_2'(t) + A(t, w_1(t)) - A(t, w_2(t)), w_1(t) - w_2(t)\rangle_{V^*\times V} \\ &\quad + \langle \zeta_1(t) - \zeta_2(t) + \xi_1(t) - \xi_2(t), w_1(t) - w_2(t)\rangle_{V^*\times V} = 0 \end{aligned}$$

for a.e. $t \in [0,T]$. Then, integrating the above equality over on $[0,t]$ with $t \in [0,T]$, and using $H(A)$(iv), $H(\varphi)$(iv) and $H(\psi)$, we have

$$\frac{1}{2}\|w_1(t) - w_2(t)\|_H^2 + (\alpha - \beta)\int_0^t \|w_1(s) - w_2(s)\|_V^2\,\mathrm{d}s \le 0$$

for all $t \in [0,T]$, whereas the smallness condition (3.3) indicates $w_1 = w_2$. This concludes the proof of the theorem. □

4. History-dependent variational–hemivariational inequalities

In this section, we are interesting in the study of existence and uniqueness of solution to a generalized variational–hemivariational inequality involving history-dependent operators, in which the history-dependent operators are, respectively, acted on the elastic operator and locally Lipschitz function. In what follows, let Y_i for $i = 1, 2, 3$ be Banach spaces. The problem under investigation reads as follows.

Problem 4.1. *Find $w \in \mathcal{W}$ such that for a.e. $t \in [0,T]$ and all $v \in V$,*

$$\begin{cases} \big\langle w'(t) + \mathcal{A}\big(t, (\mathcal{S}_1 w)(t), w(t)\big) - f(t), v - w(t)\big\rangle_{V^*\times V} \\ \quad + \phi^0\big(t, (\mathcal{S}_2 w)(t), (\mathcal{S}_3 w)(t), w(t); v - w(t)\big) + \psi(v) - \psi(w(t)) \ge 0, \\ w(0) = w_0. \end{cases} \tag{4.1}$$

To establish main results on Problem 4.1, we now impose the following assumptions on its data.

<u>$H(\mathcal{S})$</u>: $\mathcal{S}_1 \colon \mathcal{V} \to L^2(0,T;Y_1)$, $\mathcal{S}_2 \colon \mathcal{V} \to L^2(0,T;Y_2)$, and $\mathcal{S}_3 \colon \mathcal{V} \to L^2(0,T;Y_3)$ are three history-dependent operators, i.e., there exist constants $L_{\mathcal{S}_1}, L_{\mathcal{S}_2}, L_{\mathcal{S}_3} > 0$ such that for all $w_1, w_2 \in \mathcal{V}$ and a.e. $t \in [0,T]$,

$$\begin{cases} \text{(a) } \|(\mathcal{S}_1 w_1)(t) - (\mathcal{S}_1 w_2)(t)\|_{Y_1} \le L_{\mathcal{S}_1}\displaystyle\int_0^t \|w_1(s) - w_2(s)\|_V\,\mathrm{d}s, \\ \text{(b) } \|(\mathcal{S}_2 w_1)(t) - (\mathcal{S}_2 w_2)(t)\|_{Y_2} \le L_{\mathcal{S}_2}\displaystyle\int_0^t \|w_1(s) - w_2(s)\|_V\,\mathrm{d}s, \\ \text{(c) } \|(\mathcal{S}_3 w_1)(t) - (\mathcal{S}_3 w_2)(t)\|_{Y_3} \le L_{\mathcal{S}_3}\displaystyle\int_0^t \|w_1(s) - w_2(s)\|_V\,\mathrm{d}s. \end{cases} \tag{4.2}$$

$\underline{H(\mathcal{A})}$: $\mathcal{A}\colon [0,T]\times Y_1\times V\to V^*$ satisfies the conditions

$$\begin{cases} (a)\ \mathcal{A}(\cdot,y,w) \text{ is measurable on } [0,T] \text{ for each } (y,w)\in Y_1\times V.\\ (b)\ y\mapsto \mathcal{A}(t,y,w) \text{ is continuous for a.e. } t\in[0,T] \text{ and all } w\in V.\\ (c)\ w\mapsto \mathcal{A}(t,y,w) \text{ is demicontinuous for a.e. } t\in[0,T] \text{ and all } y\in Y_1.\\ (d)\ \text{there are } a_2>0 \text{ and } a_1\in L^2_+(0,T) \text{ such that}\\ \qquad \|\mathcal{A}(t,y,w)\|_{V^*}\le a_1(t)+a_2(\|y\|_{Y_1}+\|w\|_V)\\ \quad \text{for all } (y,w)\in Y_1\times V \text{ and a.e. } t\in[0,T].\\ (e)\ \text{there exists a constant } \alpha>0 \text{ such that}\\ \qquad \langle \mathcal{A}(t,y_1,w_1)-\mathcal{A}(t,y_2,w_2),w_1-w_2\rangle_{V^*\times V}\\ \qquad \ge \alpha(\|w_1-w_2\|_V-\|y_1-y_2\|_{Y_1})\|w_1-w_2\|_V\\ \quad \text{for all } (y_1,w_1),(y_2,w_2)\in Y_1\times V \text{ and a.e. } t\in[0,T]. \end{cases} \tag{4.3}$$

$\underline{H(\phi)}$: $\phi\colon [0,T]\times Y_2\times Y_3\times V\to\mathbb{R}$ reads the conditions

$$\begin{cases} (a)\ \phi(\cdot,z,q,w) \text{ is measurable on } [0,T] \text{ for all } (z,q,w)\in Y_2\times Y_3\times V.\\ (b)\ (z,q)\mapsto \phi(t,z,q,w) \text{ is continuous for a.e. } t\in[0,T] \text{ and all } w\in V.\\ (c)\ w\mapsto\phi(t,z,q,w) \text{ is locally Lipschitz continuous on for a.e. } t\in[0,T] \text{ and all } (z,q)\in Y_2\times Y_3.\\ (d)\ \text{there exist } c_1\in L^2_+(0,T) \text{ and } c_2>0 \text{ such that}\\ \qquad \|\zeta(t)\|_{V^*}\le c_1(t)+c_2(\|z\|_{Y_2}+\|q\|_{Y_3}+\|w\|_V)\\ \quad \text{for all } \zeta(t)\in\partial\phi(t,z,q,w),\ \text{all } (z,q,w)\in Y_2\times Y_3\times V \text{ and a.e. } t\in[0,T].\\ (e)\ \text{there is a constant } \beta>0 \text{ such that}\\ \quad \phi^0(t,z_1,q_1,w_1;w_2-w_1)+\phi^0(t,z_2,q_2,w_2;w_1-w_2)\\ \quad \le\beta(\|z_1-z_2\|_{Y_2}+\|q_1-q_2\|_{Y_3}+\|w_1-w_2\|_V)\|w_1-w_2\|_V\\ \quad \text{for all } (z_1,q_1,w_1),(z_2,q_2,w_2)\in Y_2\times Y_3\times V \text{ and a.e. } t\in[0,T]. \end{cases} \tag{4.4}$$

The main theorem of the section is delivered as follows.

Theorem 4.2. *Assume that (3.2), $H(\psi)$, $H(\mathcal{S})$, $H(\mathcal{A})$, and $H(\phi)$ hold. If, in addition, the smallness condition (3.3) is fulfilled, then Problem 4.1 has a unique solution $w\in\mathcal{W}$.*

Proof. For $(\eta,\theta,\varsigma)\in L^2(0,T;Y_1\times Y_2\times Y_3)$ fixed, first, we consider the intermediate problem:

$$\begin{cases} \text{find } w_{\eta\theta\varsigma}\in\mathcal{W} \text{ such that}\\ w'_{\eta\theta\varsigma}(t)+\mathcal{A}(t,\eta(t),w_{\eta\theta\varsigma}(t))+\zeta(t)+\xi(t)=f(t),\\ \zeta(t)\in\partial\phi(t,\theta(t),\varsigma(t),w_{\eta\theta\varsigma}(t)),\ \xi(t)\in\partial_c\psi(w_{\eta\theta\varsigma}(t)),\\ w_{\eta\theta\varsigma}(0)=w_0, \end{cases} \tag{4.5}$$

for a.e. $t\in[0,T]$. We shall use Theorem 3.2 to show that problem (4.5) has a unique solution.

Consider the functions $A\colon [0,T]\times V\to V^*$ and the function $\varphi\colon[0,T]\times V\to\mathbb{R}$ defined by

$$A(t,w)=\mathcal{A}(t,\eta(t),w)\ \text{ and }\ \varphi(t,w)=\phi(t,\theta(t),\varsigma(t),w), \tag{4.6}$$

respectively, for all $w\in V$ and a.e. $t\in[0,T]$. By virtue of hypotheses (4.3)(a)–(e) and (4.4)(a)–(d), it is not difficult to verify that the operator A and the function φ defined in (4.6) enjoy the conditions of $H(A)$ and $H(\varphi)$(i)–(iii), respectively. Besides, hypothesis (4.4)(e) and the following estimates

$$
\begin{aligned}
&\langle \zeta_1(t)-\zeta_2(t), w_1(t)-w_2(t)\rangle_{V^*\times V}\\
&\quad \ge -\varphi^0\big(t, w_1(t); w_2(t)-w_1(t)\big)-\varphi^0\big(t, w_2(t); w_1(t)-w_2(t)\big)\\
&\quad = -\phi^0\big(t,\theta(t),\varsigma(t), w_1(t); w_2(t)-w_1(t)\big)-\phi^0\big(t,\theta(t),\varsigma(t), w_2(t); w_1(t)-w_2(t)\big)\\
&\quad \ge -\beta\|w_1(t)-w_2(t)\|_V^2
\end{aligned}
$$

for all $\zeta_1(t)\in\partial\varphi(t,w_1(t))$, all $\zeta_2(t)\in\partial\varphi(t,w_2(t))$, and a.e. $t\in[0,T]$, ensure the validity of $H(\varphi)$(iv). Therefore, from the conditions (3.2) and (3.3), we are able to employ Theorem 3.2 to obtain that for each $(\eta,\theta,\varsigma)\in L^2(0,T;Y_1\times Y_2\times Y_3)$ fixed, problem (4.5) admits a unique solution $w_{\eta\theta\varsigma}\in\mathcal{W}$.

Observe that for every $(\eta,\theta,\varsigma)\in L^2(0,T;Y_1\times Y_2\times Y_3)$ fixed, if $w_{\eta\theta\varsigma}\in\mathcal{W}$ is a solution to problem (4.5), then it solves the following problem too

$$
\begin{cases}
\text{find } w_{\eta\theta\varsigma}\in\mathcal{W} \text{ such that}\\
\langle w'_{\eta\theta\varsigma}(t)+\mathcal{A}(t,\eta(t),w_{\eta\theta\varsigma}(t))-f(t), v-w_{\eta\theta\varsigma}(t)\rangle_{V^*\times V}\\
\;+\phi^0\big(t,\theta(t),\varsigma(t),w_{\eta\theta\varsigma}(t); v-w_{\eta\theta\varsigma}(t)\big)+\psi(v)-\psi(w_{\eta\theta\varsigma}(t))\ge 0,\\
w_{\eta\theta\varsigma}(0)=w_0,
\end{cases}
\tag{4.7}
$$

for all $v\in V$ and a.e. $t\in[0,T]$. We assert that problem (4.7) is unique solvability. Let $w_{\eta\theta\varsigma 1}, w_{\eta\theta\varsigma 2}\in\mathcal{W}$ be two solutions to problem (4.7). A simple calculation gives

$$
\begin{aligned}
&\langle w'_{\eta\theta\varsigma 1}(t)-w'_{\eta\theta\varsigma 2}(t), w_{\eta\theta\varsigma 1}(t)-w_{\eta\theta\varsigma 2}(t)\rangle_{V^*\times V}\\
&\quad+\langle \mathcal{A}(t,\eta(t),w_{\eta\theta\varsigma 1}(t))-\mathcal{A}(t,\eta(t),w_{\eta\theta\varsigma 2}(t)), w_{\eta\theta\varsigma 1}(t)-w_{\eta\theta\varsigma 2}(t)\rangle_{V^*\times V}\\
&\le \phi^0\big(t,\theta(t),w_{\eta\theta\varsigma 1}(t); w_{\eta\theta\varsigma 2}(t)-w_{\eta\theta\varsigma 1}(t)\big)+\phi^0\big(t,\theta(t),w_{\eta\theta\varsigma 2}(t); w_{\eta\theta\varsigma 1}(t)-w_{\eta\theta\varsigma 2}(t)\big)
\end{aligned}
$$

for a.e. $t\in[0,T]$. Integrating the above inequality over $[0,t]$ for $t\in[0,T]$ and using hypotheses $H(\mathcal{A})$(e) and $H(\phi)$(e), it reads

$$
\frac{1}{2}\|w_{\eta\theta\varsigma 1}(t)-w_{\eta\theta\varsigma 2}(t)\|_H^2+(\alpha-\beta)\int_0^t\|w_{\eta\theta\varsigma 1}(s)-w_{\eta\theta\varsigma 2}(s)\|_V^2\,\mathrm{d}s\le 0
$$

for all $t\in[0,T]$. But, the smallness condition (3.3) indicates $w_{\eta\theta\varsigma 1}=w_{\eta\theta\varsigma 2}$. So, (4.7) is unique solvability.

Additionally, let us introduce the mapping $\Upsilon\colon L^2(0,T;Y_1\times Y_2\times Y_3)\to L^2(0,T;Y_1\times Y_2\times Y_3)$ by

$$
\Upsilon(\eta,\theta,\varsigma)=(\mathcal{S}_1 w_{\eta\theta\varsigma},\mathcal{S}_2 w_{\eta\theta\varsigma},\mathcal{S}_3 w_{\eta\theta\varsigma})
\tag{4.8}
$$

for all $(\eta,\theta,\varsigma)\in L^2(0,T;Y_1\times Y_2\times Y_3)$, in which $w_{\eta\theta\varsigma}$ is the unique solution to problem (4.7) corresponding to (η,θ,ς). Indeed, Υ has a unique fixed point in $L^2(0,T;Y_1\times Y_2\times Y_3)$.

For any $(\eta_1,\theta_1,\varsigma_1),(\eta_2,\theta_2,\varsigma_2)\in L^2(0,T;Y_1\times Y_2\times Y_3)$, let $w_1=w_{\eta_1\theta_1\varsigma_1}$ and $w_2=w_{\eta_2\theta_2\varsigma_2}$ be the unique solutions of (4.7) associated with $(\eta_1,\theta_1,\varsigma_1)$ and $(\eta_2,\theta_2,\varsigma_2)$, respectively. Carrying out a analogous procedure as the proof of the uniqueness of solution to problem (4.7), one has

$$
\begin{aligned}
&\frac{1}{2}\|w_1(t)-w_2(t)\|_H^2+(\alpha-\beta)\|w_1-w_2\|_{L^2(0,t;V)}^2-\alpha\|\eta_1-\eta_2\|_{L^2(0,t;Y_1)}\|w_1-w_2\|_{L^2(0,t;V)}\\
&\quad\le\int_0^t\phi^0\big(s,\theta_1(s),\varsigma_1(s),w_1(s);w_2(s)-w_1(s)\big)+\phi^0\big(s,\theta_2(s),\varsigma_2(s),w_2(s);w_1(s)-w_2(s)\big)\,\mathrm{d}s
\end{aligned}
$$

for all $t\in[0,T]$. The latter combined with hypothesis $H(\phi)$(e) and the Hölder inequality finds

$$
\begin{aligned}
&\frac{1}{2}\|w_1(t)-w_2(t)\|_H^2+(\alpha-\beta)\|w_1-w_2\|_{L^2(0,t;V)}^2\\
&\quad\le\alpha\|\eta_1-\eta_2\|_{L^2(0,t;Y_1)}\|w_1-w_2\|_{L^2(0,t;V)}\\
&\qquad+\beta(\|\theta_1-\theta_2\|_{L^2(0,t;Y_2)}+\|\varsigma_1-\varsigma_2\|_{L^2(0,t;Y_3)})\|w_1-w_2\|_{L^2(0,t;V)}.
\end{aligned}
$$

Hence,

$$\|w_1 - w_2\|_{L^2(0,t;V)} \le c\|\eta_1 - \eta_2\|_{L^2(0,t;Y_1)} + c(\|\theta_1 - \theta_2\|_{L^2(0,t;Y_2)} + \|\varsigma_1 - \varsigma_2\|_{L^2(0,t;Y_3)}) \tag{4.9}$$

for all $t \in [0,T]$ with $c = \max\{\alpha/(\alpha-\beta), \beta/(\alpha-\beta)\} > 0$. Combining the definition of Υ with hypotheses $H(\mathcal{S})$, inequality (4.9) and Hölder inequality, we conclude

$$\begin{aligned}
&\|\Upsilon(\eta_1,\theta_1,\varsigma_1)(t) - \Upsilon(\eta_2,\theta_2,\varsigma_2)(t)\|^2_{Y_1\times Y_2\times Y_3} \\
&\quad\le \|(\mathcal{S}_1 w_1)(t) - (\mathcal{S}_1 w_2)(t)\|^2_{Y_1} + \|(\mathcal{S}_2 w_1)(t) - (\mathcal{S}_2 w_2)(t)\|^2_{Y_2} + \|(\mathcal{S}_3 w_1)(t) - (\mathcal{S}_3 w_2)(t)\|^2_{Y_3} \\
&\quad\le (L^2_{\mathcal{S}_1} + L^2_{\mathcal{S}_2} + L^2_{\mathcal{S}_3})\Big(\int_0^t \|w_1(s) - w_2(s)\|_V \,\mathrm{d}s\Big)^2 \\
&\quad\le 3c^2 T(L^2_{\mathcal{S}_1} + L^2_{\mathcal{S}_2} + L^2_{\mathcal{S}_3}) \int_0^t \|(\eta_1,\theta_1,\varsigma_1)(s) - (\eta_2,\theta_2,\varsigma_2)(s)\|^2_{Y_1\times Y_2\times Y_3} \,\mathrm{d}s
\end{aligned}$$

for all $t \in [0,T]$, where we have used the elementary inequality $(a+b+d)^2 \le 3(a^2+b^2+d^2)$ for all $a,b,d \in \mathbb{R}$. We are now in a position to invoke fixed point theorem [14, Lemma 7] to get that Υ has a unique fixed point in $L^2(0,T;Y_1\times Y_2\times Y_3)$.

Let $(\eta^*,\theta^*,\varsigma^*)$ be the unique fixed point of Υ, and $w_{\eta^*\theta^*\varsigma^*}$ be the unique solution to problem (4.7) corresponding to $(\eta^*,\theta^*,\varsigma^*)$. It is obvious that $w := w_{\eta^*\theta^*\varsigma^*} \in \mathcal{W}$ is the unique solution to Problem 4.1. □

We end the section by providing the following particular cases of Problem 4.1. Let K be a nonempty, closed and convex subset of V such that $w_0 \in \mathrm{int}(K) \neq \emptyset$, consider the function $\psi\colon V \to \mathbb{R}\cup\{+\infty\}$ by

$$\psi(v) = \varphi(v) + I_K(v) \ \text{ for all } v \in V, \tag{4.10}$$

where $\varphi\colon V \to \mathbb{R}$ is a convex and lower semicontinuous function, and $I_K\colon V \to \mathbb{R}\cup\{+\infty\}$ is the indicator function of K given by

$$I_K(v) = \begin{cases} 0, & \text{if } v \in K, \\ +\infty, & \text{if } v \notin K. \end{cases}$$

Obviously, we can see that the function ψ defined in (4.10) satisfies conditions $H(\psi)$. In this case, we have the following corollary.

Corollary 4.3. *Let K be a nonempty, closed and convex subset of V such that $w_0 \in \mathrm{int}(K) \neq \emptyset$. Assume that (3.2), (3.3), (4.2), $H(\mathcal{S})$, $H(\mathcal{A})$, and $H(\phi)$ hold. If $\varphi\colon V \to \mathbb{R}$ is a convex and l.s.c. function, then the problem*

$$\begin{cases}
\text{find } w \in \mathcal{W} \text{ with } w(t) \in K \text{ for a.e. } t \in [0,T] \text{ such that for a.e. } t \in [0,T] \text{ and all } v \in K, \\
\big\langle w'(t) + \mathcal{A}\big(t, (\mathcal{S}_1 w)(t), w(t)\big) - f(t), v - w(t)\big\rangle_{V^*\times V} + \varphi(v) - \varphi(w(t)) \\
+ \phi^0\big(t, (\mathcal{S}_2 w)(t), (\mathcal{S}_3 w)(t), w(t); v - w(t)\big) \ge 0, \\
w(0) = w_0,
\end{cases} \tag{4.11}$$

has a unique solution $w \in \mathcal{W}$.

Indeed, under the suitable assumptions, this corollary, Corollary 4.3, can imply that problem (1.2) has a unique solution.

5. A dynamic viscoelastic contact problem

In the present section, we are concerned with the applicability of the results obtained in Sect. 4 to a new dynamic contact model for a viscoelastic material with the constitutive law involving a convex subdifferential inclusion, and multi-valued boundary conditions with nonconvex contact and friction potentials.

The physical setting of the model is described as follows. Assume a viscoelastic body occupies a bounded and connected domain Ω in $\mathbb{R}^d$ ($d = 2, 3$) such that its boundary $\Gamma = \partial\Omega$ is Lipschitz continuous. The boundary also is considered to be composed of three mutually disjoint and measurable parts Γ_D, Γ_N and Γ_C with $\text{meas}(\Gamma_D) > 0$ (i.e., the measure of Γ_D is positive). In the meanwhile, we adopt the standard notation and function spaces H, $\mathcal{H}$ and H_1, which are mentioned in the end of Sect. 2. We set $\mathcal{Q} = \Omega \times [0, T]$, $\Sigma = \Gamma \times [0, T]$, $\Sigma_D = \Gamma_D \times [0, T]$, $\Sigma_N = \Gamma_N \times [0, T]$ and $\Sigma_C = \Gamma_C \times [0, T]$.

The classical formulation of the contact problem is stated as follows.

Problem 5.1. *Find a displacement field $\boldsymbol{u}\colon \mathcal{Q} \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma}\colon \mathcal{Q} \to \mathbb{S}^d$ such that*

$$\boldsymbol{\sigma}(t) \in \mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))\big) + \partial_c\varphi(\boldsymbol{\varepsilon}(\boldsymbol{u}'(t))) \qquad \text{in} \quad \mathcal{Q}, \tag{5.1}$$

$$\boldsymbol{u}''(t) = \mathit{Div}\, \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) \qquad \text{in} \quad \mathcal{Q}, \tag{5.2}$$

$$\boldsymbol{u}(t) = \boldsymbol{0} \qquad \text{on} \quad \Sigma_D, \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\nu = \boldsymbol{f}_N(t) \qquad \text{on} \quad \Sigma_N, \tag{5.4}$$

$$\begin{cases} -\boldsymbol{\sigma}_\nu(t) \in \partial j_\nu(t, u_\nu(t), u'_\nu(t)), \\ -\boldsymbol{\sigma}_\tau(t) \in \partial j_\tau\Big(t, \displaystyle\int_0^t \|\boldsymbol{u}_\tau(s)\|_{\mathbb{R}^d}\, ds, \boldsymbol{u}'_\tau(t)\Big) \end{cases} \qquad \text{on} \quad \Sigma_C, \tag{5.5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0,\ \boldsymbol{u}'(0) = \boldsymbol{w}_0 \qquad \text{in} \quad \Omega. \tag{5.6}$$

We now provide a brief description on the equations, conditions and relations appeared in Problem 5.1. Inclusion (5.1) is a nonlinear viscoelastic constitutive law, where $\varphi\colon \mathbb{S}^d \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower semicontinuous function, and $\mathcal{A}\colon \mathcal{Q} \times \mathbb{S}^d \times \mathbb{S}^d \to \mathbb{S}^d$ presents a viscoelasticity operator (see for example, [34]), which is considered to read the following conditions.

$$H(\mathcal{A})\colon \begin{cases} (a)\ \mathcal{A}(\cdot,\cdot,\boldsymbol{\varepsilon},\boldsymbol{\eta}) \text{ is measurable on } \mathcal{Q}, \text{ for all } \boldsymbol{\varepsilon},\boldsymbol{\eta} \in \mathbb{S}^d. \\ (b)\ \text{there exists } L_\mathcal{A} > 0 \text{ such that} \\ \qquad \|\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1,\boldsymbol{\eta}_1) - \mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2,\boldsymbol{\eta}_2)\|_{\mathbb{S}^d} \le L_\mathcal{A}(\|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d} + \|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathbb{S}^d}) \\ \qquad \text{for a.e. } (\boldsymbol{x},t) \in \mathcal{Q} \text{ and all } \boldsymbol{\varepsilon}_i, \boldsymbol{\eta}_i \in \mathbb{S}^d \text{ for } i = 1,2. \\ (c)\ \text{there exists } \alpha_\mathcal{A} > 0 \text{ such that} \\ \qquad \big(\mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_1,\boldsymbol{\eta}_1) - \mathcal{A}(\boldsymbol{x},t,\boldsymbol{\varepsilon}_2,\boldsymbol{\eta}_2)\big) : (\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2) \\ \qquad \ge \alpha_\mathcal{A}(\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathbb{S}^d} - \|\boldsymbol{\varepsilon}_1 - \boldsymbol{\varepsilon}_2\|_{\mathbb{S}^d})\|\boldsymbol{\eta}_1 - \boldsymbol{\eta}_2\|_{\mathbb{S}^d} \\ \qquad \text{for all } \boldsymbol{\varepsilon}_1, \boldsymbol{\varepsilon}_2, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2 \in \mathbb{S}^d \text{ and a.e. } (\boldsymbol{x},t) \in \mathcal{Q}. \\ (d)\ \mathcal{A}(\boldsymbol{x},t,0,0) = \boldsymbol{0} \text{ for a.e. } (\boldsymbol{x},t) \in \mathcal{Q}. \end{cases}$$

As a special case, $\mathcal{A}$ can be specialized by the sum of a viscosity operator $\mathcal{P}$ and an elasticity operator $\mathcal{B}$, i.e., $\mathcal{A}(\boldsymbol{x}, t, \boldsymbol{\varepsilon}, \boldsymbol{\eta}) := \mathcal{P}\big(t, \boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))\big) + \mathcal{B}\big(t, \boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big)$. In this moment, when $\partial_c\varphi \equiv \boldsymbol{0}$, the constitutive law (5.1) reduces to the nonlinear Kelvin–Voigt constitutive law, thus,

$$\boldsymbol{\sigma}(t) = \mathcal{P}\big(t, \boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))\big) + \mathcal{B}\big(t, \boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{u}(t))\big) \quad \text{for a.e. } (\boldsymbol{x}, t) \in \mathcal{Q},$$

which has been frequently used to the study of various dynamic or quasi-static contact problems, see for instance, [23,35,36].

Equation (5.2) is derived directly by the fundamental principle of momentum conservation which describes the evolution of the mechanical state of the viscoelastic body; without loss of generality, the mass density of the body is assumed to be one in (5.2), where the time-dependent volume forces of density $\boldsymbol{f}_0$ act in $\mathcal{Q}$ and fulfills the following regularity

$$\boldsymbol{f}_0 \in L^2(0,T;H). \tag{5.7}$$

Conditions (5.3) and (5.4) reveal the phenomena that the body is clamped on Γ_D, but it is subjected to the surface tractions of density $\boldsymbol{f}_N$ on Γ_N, where the function $\boldsymbol{f}_N$ satisfies the condition

$$\boldsymbol{f}_N \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d)). \tag{5.8}$$

The multi-valued relations (5.5) characterize a generalized normal contact condition and a frictional law, where the superpotential functions $j_\nu\colon \Sigma_C \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $j_\tau\colon \Sigma_C \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ are locally Lipschitz, which are nonconvex in general, and fulfill the assumptions

$$H(j_\nu)\colon \begin{cases} \text{(a) } j_\nu(\cdot,\cdot,r,s) \text{ is measurable on } \Sigma_C \text{ for all } r,s\in\mathbb{R} \text{ and there exists a function} \\ \quad e\in L^2(\Sigma_C) \text{ such that for all } w\in L^2(\Sigma_C), \text{ it holds } j_\nu(\cdot,\cdot,w,e)\in L^1\big(\Sigma_C\big). \\ \text{(b) } j_\nu(\boldsymbol{x},t,\cdot,s) \text{ is continuous on } \mathbb{R} \text{ for a.e } (\boldsymbol{x},t)\in\Sigma_C \text{ and all } s\in\mathbb{R}. \\ \text{(c) } j_\nu(\boldsymbol{x},t,r,\cdot) \text{ is locally Lipschitz for a.e. } (\boldsymbol{x},t)\in\Sigma_C \text{ and all } r\in\mathbb{R}. \\ \text{(d) there exist } c_{0\nu}\in L^2_+(\Sigma_C) \text{ and a constant } c_{1\nu}\ge 0 \text{ such that} \\ \qquad |\partial j_\nu(\boldsymbol{x},t,r,s)| \le c_{0\nu}(\boldsymbol{x},t) + c_{1\nu}(|r|+|s|) \\ \quad \text{for a.e. } (\boldsymbol{x},t)\in\Sigma_C \text{ and all } r,s\in\mathbb{R}. \\ \text{(e) there is a constant } \beta_{j_\nu}\ge 0 \text{ such that} \\ \qquad j_\nu^0(\boldsymbol{x},t,r_1,s_1;s_2-s_1) + j_\nu^0(\boldsymbol{x},t,r_2,s_2;s_1-s_2) \\ \qquad \le \beta_{j_\nu}(|r_1-r_2|+|s_1-s_2|)|s_1-s_2| \\ \quad \text{for all } r_1,r_2,s_1,s_2\in\mathbb{R} \text{ and a.e. } (t,\boldsymbol{x})\in\Sigma_C. \end{cases}$$

$$H(j_\tau)\colon \begin{cases} \text{(a) } j_\tau(\cdot,\cdot,q,\boldsymbol{z}) \text{ is measurable on } \Sigma_C \text{ for all } q\in\mathbb{R}_+ \text{ and all } \boldsymbol{z}\in\mathbb{R}^d \\ \quad \text{and there exists a function } \boldsymbol{e}\in L^2(\Sigma_C;\mathbb{R}^d) \text{ such that for all} \\ \quad \eta\in L^2(\Sigma_C), \text{ we have } j_\tau(\cdot,\cdot,\eta(\cdot),\boldsymbol{e}(\cdot))\in L^1\big(\Sigma_C\big). \\ \text{(b) } j_\tau(\boldsymbol{x},t,\cdot,\boldsymbol{z}) \text{ is continuous on } \mathbb{R} \text{ for a.e } (\boldsymbol{x},t)\in\Sigma_C \text{ and all } \boldsymbol{z}\in\mathbb{R}^d. \\ \text{(c) } j_\tau(\boldsymbol{x},t,q,\cdot) \text{ is locally Lipschitz for a.e. } (\boldsymbol{x},t)\in\Sigma_C \text{ and all } q\in\mathbb{R}. \\ \text{(d) there exist } c_{0\tau}\in L^2_+\big(\Sigma_C\big) \text{ and } c_{1\tau}\ge 0 \text{ such that} \\ \qquad \|\partial j_\tau(\boldsymbol{x},t,q,\boldsymbol{z})\|_{\mathbb{R}^d} \le c_{0\tau}(\boldsymbol{x},t) + c_{1\tau}(|q|+\|\boldsymbol{z}\|_{\mathbb{R}^d}) \\ \quad \text{for all } q\in\mathbb{R}, \text{ all } \boldsymbol{z}\in\mathbb{R}^d, \text{ and a.e. } (\boldsymbol{x},t)\in\Sigma_C. \\ \text{(e) there exists } \beta_{j_\tau}\ge 0 \text{ such that} \\ \qquad j_\tau^0(\boldsymbol{x},t,q_1,\boldsymbol{z}_1;\boldsymbol{z}_2-\boldsymbol{z}_1) + j_\tau^0(\boldsymbol{x},t,q_2,\boldsymbol{z}_2;\boldsymbol{z}_1-\boldsymbol{z}_2) \\ \qquad \le \beta_{j_\tau}(|q_1-q_2|+\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbb{R}^d})\|\boldsymbol{z}_1-\boldsymbol{z}_2\|_{\mathbb{R}^d} \\ \quad \text{for all } (q_1,\boldsymbol{z}_1),(q_2,\boldsymbol{z}_2)\in\mathbb{R}\times\mathbb{R}^d \text{ and a.e. } (\boldsymbol{x},t)\in\Sigma_C. \end{cases}$$

In fact, as we know, many typical laws in various mechanical contact phenomena could be formulated by the special forms of (5.5); for the detailed explanation, one can refer the monographs [21, Chapter 6.3] and [10,11,33].

Condition (5.6) presents the initial displacement and velocity fields, which entail the following condition

$$\boldsymbol{u}_0,\, \boldsymbol{w}_0 \in V \quad \text{with } \boldsymbol{w}_0 \in \operatorname{int} D(\psi) \text{ and } 0_{\mathbb{S}^d} \in \partial_c \varphi(\boldsymbol{\varepsilon}(\boldsymbol{w}_0(\boldsymbol{x}))) \text{ for a.e. } \boldsymbol{x} \in \Omega, \tag{5.9}$$

where $\psi\colon V \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\psi(\boldsymbol{v}) := \int_\Omega \varphi(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})))\, d\boldsymbol{x} \;\text{ for all } \boldsymbol{v} \in V, \tag{5.10}$$

and V is a closed subspace of H_1 given by

$$V = \left\{\, v \in H_1 \;\middle|\; \boldsymbol{v} = \boldsymbol{0} \text{ on } \Gamma_D \,\right\}.$$

Let V^* be the dual space of V. Recall that $\operatorname{meas}(\Gamma_D) > 0$, it follows from Korn's inequality that the space V is a real Hilbert space equipped with the inner product

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{V^*\times V} = \langle \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle_{\mathcal{H}} \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V$$

and the associated norm $\|\cdot\|_V$. However, by the Sobolev trace theorem, we have

$$\|\boldsymbol{v}\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le C_0 \|\boldsymbol{v}\|_V \;\text{ for all } \boldsymbol{v} \in V \tag{5.11}$$

for some $C_0 > 0$, which only depends on the domain Ω, Γ_D and Γ_C.

To deliver the variational formulation of Problem 5.1, we now assume that there are the displacement field $\boldsymbol{u}$ and the stress field $\boldsymbol{\sigma}$ sufficiently smooth which satisfy (5.1)–(5.6). Denote $\boldsymbol{w} = \boldsymbol{u}'$ the velocity field. Also, we introduce the operator $\mathcal{S}\colon L^2(0,T;V) \to L^2(0,T;V)$ defined by

$$(\mathcal{S}\boldsymbol{w})(t) := \boldsymbol{u}_0 + \int_0^t \boldsymbol{w}(s)\, \mathrm{d}s \text{ for all } t \in [0,T] \text{ and all } \boldsymbol{w} \in L^2(0,T;V). \tag{5.12}$$

Employing the Green's formula (see for example, [21, Theorem 2.25]), it is not difficult to obtain the following variational formulation of Problem 5.1 in terms of velocity.

Problem 5.2. *Find a velocity field $\boldsymbol{w}\colon [0,T] \to V$ such that for a.e. $t \in [0,T]$ and all $\boldsymbol{v} \in V$,*

$$\begin{aligned}
&\big\langle \boldsymbol{w}'(t), \boldsymbol{v} - \boldsymbol{w}(t)\big\rangle_{V^*\times V} + \big\langle \mathcal{A}\big(t, \boldsymbol{\varepsilon}((\mathcal{S}\boldsymbol{w})(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}(t))\big), \boldsymbol{\varepsilon}(\boldsymbol{v}) - \boldsymbol{\varepsilon}(\boldsymbol{w}(t))\big\rangle_{\mathcal{H}} + \int_\Omega \varphi(\boldsymbol{v})\, d\boldsymbol{x} \\
&\quad - \int_\Omega \varphi(\boldsymbol{w}(t))\, d\boldsymbol{x} + \int_{\Gamma_C} j_\nu^0\big(t, (\mathcal{S}\boldsymbol{w})_\nu(t), w_\nu(t); v_\nu - w_\nu(t)\big)\, d\Gamma \\
&\quad + \int_{\Gamma_C} j_\tau^0\Big(t, \int_0^t \|(\mathcal{S}\boldsymbol{w})_\tau(s)\|_{\mathbb{R}^d}\, ds, \boldsymbol{w}_\tau(t); \boldsymbol{v}_\tau - \boldsymbol{w}_\tau(t)\Big)\, d\Gamma \ge \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle_{V^*\times V}
\end{aligned} \tag{5.13}$$

with $\boldsymbol{w}(0) = \boldsymbol{w}_0$, where $\boldsymbol{f}\colon [0,T] \to V^*$ is such that

$$\langle \boldsymbol{f}(t), \boldsymbol{v}\rangle_{V^*\times V} = \langle \boldsymbol{f}_0(t), \boldsymbol{v}\rangle_H + \langle \boldsymbol{f}_N(t), \boldsymbol{v}\rangle_{L^2(\Gamma_N;\mathbb{R}^d)} \;\textit{ for all } \boldsymbol{v} \in V. \tag{5.14}$$

Remark 5.3. It should be underlined that if $\boldsymbol{w}$ is a solution to Problem 5.2, then by using the equality $\boldsymbol{u} = \mathcal{S}\boldsymbol{w}$ and taking a suitable function $\boldsymbol{\eta}\colon \mathcal{Q} \to \mathbb{S}^d$ with $\boldsymbol{\eta}(t) \in \partial_c \varphi(\boldsymbol{\varepsilon}(\boldsymbol{u}'(t)))$ for a.e. $(\boldsymbol{x}, t) \in \mathcal{Q}$, such that $\boldsymbol{\sigma}(t) = \mathcal{A}\big(t, \boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{u}'(t))\big) + \boldsymbol{\eta}(t)$ for a.e. $(\boldsymbol{x}, t) \in \mathcal{Q}$ (see the viscoelastic constitutive law (5.1)), we can see that the couple of functions $(\boldsymbol{u}, \boldsymbol{\sigma})$ also solves problem (5.1)–(5.6), which is called a weak solution to problem (5.1)–(5.6). In the meantime, it is easy to see that

$$\boldsymbol{u} \in \mathcal{V},\ \boldsymbol{u}' \in \mathcal{W},\ \boldsymbol{u}'' \in \mathcal{V}^*,\ \boldsymbol{\sigma} \in L^2(0,T;\mathcal{H}) \text{ and } \operatorname{Div} \boldsymbol{\sigma} \in \mathcal{V}^*.$$

The existence and uniqueness theorem to Problem 5.2 is given as follows.

Theorem 5.4. *Let $\varphi\colon \mathbb{S}^d \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Assume that $H(\mathcal{A})$, $H(j_\nu)$, $H(j_\tau)$, (5.7), (5.8), (5.9) hold. If, in addition, the inequality holds*

$$\alpha_{\mathcal{A}} > \max\big\{(\beta_{j_\nu} + \beta_{j_\tau})\max\{C_0, C_0^2\}, 2C_0\mathrm{meas}(\Gamma_\mathrm{C})(1 + C_0)(c_{1\nu} + c_{1\tau})\big\}, \tag{5.15}$$

then Problem 5.2 has a unique solution $w \in \mathcal{W}$.

We define the operators $\mathcal{S}_1\colon \mathcal{V} \to \mathcal{V}$, $\mathcal{S}_2, \mathcal{S}_3\colon \mathcal{V} \to L^2\big(0, T; L^2(\Gamma_C)\big)$ and $\mathcal{A}\colon [0, T] \times V \times V \to V^*$ by

$$(\mathcal{S}_1\boldsymbol{w})(t) = (\mathcal{S}\boldsymbol{w})(t),\ (\mathcal{S}_2\boldsymbol{w})(t) = (\mathcal{S}\boldsymbol{w})_\nu(t),\ (\mathcal{S}_3\boldsymbol{w})(t) = \int_0^t \|(\mathcal{S}\boldsymbol{w})_\tau(s)\|_{\mathbb{R}^d}\, \mathrm{d}s, \tag{5.16}$$

$$\big\langle \mathcal{A}(t, \boldsymbol{u}, \boldsymbol{w}), \boldsymbol{v}\big\rangle_{V^*\times V} = \big\langle \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{w})), \boldsymbol{\varepsilon}(\boldsymbol{v})\big\rangle_{\mathcal{H}} \tag{5.17}$$

for all $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V$ and a.e. $t \in [0, T]$. Also, consider the function $\phi\colon [0, T] \times L^2(\Gamma_C) \times L^2(\Gamma_C) \times V \to \mathbb{R}$ as follows

$$\phi(t, z, q, \boldsymbol{w}) = \int_{\Gamma_C} \Big(j_\nu\big(t, z(t), w_\nu(t)\big) + j_\tau\big(t, q(t), \boldsymbol{w}_\tau(t)\big)\Big)\, d\Gamma \tag{5.18}$$

for all $z, q \in L^2(\Gamma_C)$, all $\boldsymbol{w} \in V$ and a.e. $t \in [0, T]$. Next, we shall prove that the problem: find $\boldsymbol{w} \in \mathcal{W}$ such that

$$\begin{aligned} &\big\langle \boldsymbol{w}'(t) + \mathcal{A}\big(t, (\mathcal{S}\boldsymbol{w})(t), \boldsymbol{w}(t)\big), \boldsymbol{v} - \boldsymbol{w}(t)\big\rangle_{V^*\times V} + \psi(\boldsymbol{v}) - \psi(\boldsymbol{w}(t)) \\ &\quad + \phi^0(t, (\mathcal{S}_2\boldsymbol{w})(t), (\mathcal{S}_3\boldsymbol{w})(t), \boldsymbol{w}(t); \boldsymbol{v} - \boldsymbol{w}(t)) \geq \langle \boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{w}(t)\rangle_{V^*\times V} \end{aligned} \tag{5.19}$$

for all $\boldsymbol{v} \in V$ and a.e. $t \in [0, T]$ with $\boldsymbol{w}(0) = \boldsymbol{w}_0$, has a unique solution. The proof of the assertion is mainly based on the theoretical result, Theorem 4.2. Hence, the current goal is to illustrate that all of conditions presented in Theorem 5.4 are valid.

Let $Y_1 = V$, $Y_2 = Y_3 = L^2(\Gamma_C)$. From the formulations of $\mathcal{S}_i$, $i = 1, 2, 3$, we have the lemma.

Lemma 5.5. *The operators $\mathcal{S}_i (i = 1, 2, 3)$ defined in (5.16) are history-dependent, i.e., condition (4.2) is satisfied with*

$$L_{\mathcal{S}_1} = 1, \quad L_{\mathcal{S}_2} = C_0, \quad \textit{and} \quad L_{\mathcal{S}_3} = C_0T.$$

The following lemma indicates that $\mathcal{A}$ defined in (5.17) reads conditions $H(\mathcal{A})$.

Lemma 5.6. *If the hypotheses $H(\mathcal{A})$ hold, then the operator $\mathcal{A}$ defined in (5.17) satisfies the conditions (4.3) with $\alpha = \alpha_{\mathcal{A}}$.*

Proof. It follows from hypotheses $H(\mathcal{A})$(a), (b), (d), Hölder's inequality, Fubini's theorem and Pettis measurability theorem that condition (4.3)(a) is fulfilled (for more details, one may also refer to the proof of [21, Theorem 8.3]).

By virtue of hypotheses $H(\mathcal{A})$(b), (d), and Hölder inequality, we obtain

$$\begin{aligned} \big|\langle \mathcal{A}(t, \boldsymbol{y}, \boldsymbol{w}), \boldsymbol{v}\rangle_{V^*\times V}\big| &\leq \int_\Omega \|\mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}), \boldsymbol{\varepsilon}(\boldsymbol{w}))\|\|\boldsymbol{\varepsilon}(v)\|\, d\boldsymbol{x} \\ &\leq L_{\mathcal{A}}\Big(\int_\Omega \big(\|\boldsymbol{\varepsilon}(\boldsymbol{y})\| + \|\boldsymbol{\varepsilon}(\boldsymbol{w})\|\big)^2 d\boldsymbol{x}\Big)^{1/2}\|\boldsymbol{v}\|_V \\ &\leq \sqrt{2}L_{\mathcal{A}}(\|\boldsymbol{y}\|_V + \|\boldsymbol{w}\|_V)\|\boldsymbol{v}\|_V \end{aligned}$$

for all $\boldsymbol{y}, \boldsymbol{w}, \boldsymbol{v} \in V$ and a.e. $t \in [0, T]$, namely

$$\|\mathcal{A}(t, \boldsymbol{y}, \boldsymbol{w})\|_{V^*} \leq \sqrt{2}L_{\mathcal{A}}(\|\boldsymbol{y}\|_V + \|\boldsymbol{w}\|_V) \quad \text{for all } \boldsymbol{y}, \boldsymbol{w} \in V \text{ and a.e. } t \in [0, T].$$

This means that the condition (4.3)(d) holds with $a_1 = 0$ and $a_2 = \sqrt{2}L_{\mathcal{A}}$.

To verify the conditions (4.3)(b) and (4.3)(c), let sequences $\{\boldsymbol{y}_n\}$ and $\{\boldsymbol{w}_n\}$ be such that $\boldsymbol{y}_n \to \boldsymbol{y}$ and $\boldsymbol{w}_n \to \boldsymbol{w}$ in V, as $n \to \infty$. Then, it may say

$$\boldsymbol{\varepsilon}(\boldsymbol{y}_n) \to \boldsymbol{\varepsilon}(\boldsymbol{y}), \quad \boldsymbol{\varepsilon}(\boldsymbol{w}_n) \to \boldsymbol{\varepsilon}(\boldsymbol{w}) \text{ in } L^2(\Omega; \mathbb{S}^d), \text{ as } n \to \infty.$$

By converse-Lebesgue-dominated convergence theorem, we are able to find two subsequences $\{\boldsymbol{y}_{n_k}\}$, $\{\boldsymbol{w}_{n_k}\}$ of $\{\boldsymbol{y}_n\}$, $\{\boldsymbol{w}_n\}$ satisfying

$$\boldsymbol{\varepsilon}(\boldsymbol{y}_{n_k})(\boldsymbol{x}) \to \boldsymbol{\varepsilon}(\boldsymbol{y})(\boldsymbol{x}), \quad \boldsymbol{\varepsilon}(\boldsymbol{w}_{n_k})(\boldsymbol{x}) \to \boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x}) \text{ in } \mathbb{S}^d \text{ as } n_k \to \infty$$

for a.e. $\boldsymbol{x} \in \Omega$. The latter combined with the continuity of $\mathcal{A}$, see $H(\mathcal{A})$(b), implies

$$\|\mathcal{A}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{y}_{n_k})(\boldsymbol{x}), \boldsymbol{\varepsilon}(\boldsymbol{w}_{n_k})(\boldsymbol{x})) - \mathcal{A}(t, \boldsymbol{x}, \boldsymbol{\varepsilon}(\boldsymbol{y})(\boldsymbol{x}), \boldsymbol{\varepsilon}(\boldsymbol{w})(\boldsymbol{x}))\|_{\mathbb{S}^d} \to 0$$

as $n_k \to \infty$ for a.e. $(\boldsymbol{x}, t) \in \mathcal{Q}$. However, from assumptions $H(\mathcal{A})$ (b), (d), and Lebesgue-dominated convergence theorem, it yields

$$\|\mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}_{n_k}), \boldsymbol{\varepsilon}(\boldsymbol{w}_{n_k})) - \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}), \boldsymbol{\varepsilon}(\boldsymbol{w}))\|_{\mathcal{H}} \to 0$$

as $n_k \to \infty$, for a.e. $t \in [0, T]$.

Notice that

$$\begin{aligned}
&\left|\left\langle \mathcal{A}(t, \boldsymbol{y}_{n_k}, \boldsymbol{w}_{n_k}) - \mathcal{A}(t, \boldsymbol{y}, \boldsymbol{w}), \boldsymbol{v}\right\rangle_{V^* \times V}\right| \\
&\quad = \left|\left\langle \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}_{n_k}), \boldsymbol{\varepsilon}(\boldsymbol{w}_{n_k})) - \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}), \boldsymbol{\varepsilon}(\boldsymbol{w})), \boldsymbol{\varepsilon}(\boldsymbol{v})\right\rangle_{\mathcal{H}}\right| \\
&\quad \le \|\mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}_{n_k}), \boldsymbol{\varepsilon}(\boldsymbol{w}_{n_k})) - \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}), \boldsymbol{\varepsilon}(\boldsymbol{w}))\|_{\mathcal{H}} \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{\mathcal{H}} \\
&\quad = \|\mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}_{n_k}), \boldsymbol{\varepsilon}(\boldsymbol{w}_{n_k})) - \mathcal{A}(t, \boldsymbol{\varepsilon}(\boldsymbol{y}), \boldsymbol{\varepsilon}(\boldsymbol{w}))\|_{\mathcal{H}} \|\boldsymbol{v}\|_V \text{ for all } \boldsymbol{v} \in V,
\end{aligned}$$

it deduces $\mathcal{A}(t, \boldsymbol{y}_{n_k}, \boldsymbol{w}_{n_k}) \to \mathcal{A}(t, \boldsymbol{y}, \boldsymbol{w})$ in V^* for a.e. $t \in [0, T]$. In accordance with [21, Proposition 1.14], we directly obtain $\mathcal{A}(t, \boldsymbol{y}_n, \boldsymbol{w}_n) \to \mathcal{A}(t, \boldsymbol{y}, \boldsymbol{w})$ in V^* for a.e. $t \in [0, T]$, which implies (4.3)(b) and (c).

Finally, condition (4.3)(e) is ensured directly by applying hypothesis $H(\mathcal{A})$(c), which ends the proof. □

Lemma 5.7. *Assume that $H(j_\nu)$ and $H(j_\tau)$ hold. Then, the function ϕ defined in (5.18) satisfies conditions (4.4) and*

$$\phi^0(t, z, q, \boldsymbol{w}; \boldsymbol{v}) \le \int_{\Gamma_C} j_\nu^0(t, z, w_\nu; v_\nu)\, d\Gamma + \int_{\Gamma_C} j_\tau^0(t, q, \boldsymbol{w}_\tau; \boldsymbol{v}_\tau)\, d\Gamma \tag{5.20}$$

for all $z, q \in L^2(\Gamma_C)$ all $\boldsymbol{w}, \boldsymbol{v} \in V$ and a.e. $t \in [0, T]$.

Proof. From the assumptions, the conditions (4.4)(a)–(c) are the direct consequences of [20, Lemma 5] (or [21, Corollary 4.18]).

We show the condition (4.4)(d) by using the hypotheses $H(j_\nu)(d)$ and $H(j_\tau)(d)$. Invoking [20, Lemma 5], it has

$$\partial j(\boldsymbol{x}, t, z, q, \boldsymbol{\xi}) \subseteq \partial j_\nu(\boldsymbol{x}, t, z, \xi_\nu)\nu + \partial j_\tau(\boldsymbol{x}, t, q, \boldsymbol{\xi}_\tau) \quad \text{for all } z, q \in \mathbb{R}, \boldsymbol{\xi} \in \mathbb{R}^d \text{ and a.e. } (\boldsymbol{x}, t) \in \Sigma_C,$$

where j is defined by

$$j(x, t, z, q, w) = j_\nu(\boldsymbol{x}, t, z, \xi_\nu) + j_\tau(\boldsymbol{x}, t, q, \boldsymbol{\xi}_\tau) \quad \text{for all } z, q \in \mathbb{R}, \boldsymbol{\xi} \in \mathbb{R}^d \text{ and a.e. } (\boldsymbol{x}, t) \in \Sigma_C. \tag{5.21}$$

But, hypotheses $H(j_\nu)$(d) and $H(j_\tau)$(d) ensure

$$\begin{aligned}
\|\partial j(\boldsymbol{x}, t, z, q, \boldsymbol{\xi})\|_{\mathbb{R}^d} &\le |\partial j_\nu(\boldsymbol{x}, t, z, \xi_\nu)| + \|\partial j_\tau(\boldsymbol{x}, t, q, \boldsymbol{\xi}_\tau)\|_{\mathbb{R}^d} \\
&\le c_{0\nu}(\boldsymbol{x}, t) + c_{0\tau}(\boldsymbol{x}, t) + c_{1\nu}(|z| + |\xi_\nu|) + c_{1\tau}(|q| + \|\boldsymbol{\xi}_\tau\|_{\mathbb{R}^d})
\end{aligned}$$

for all $z, q \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^d$ and a.e. $(\boldsymbol{x}, t) \in \Sigma_C$. The above inequality together with [21, Theorem 3.47 (v)], (5.18), and Hölder inequality deduces

$$\begin{aligned}\|\partial\phi(t,z,q,\boldsymbol{w})\|_{V^*} &\leq C_0 \int_{\Gamma_C} \|\partial j(\boldsymbol{x},t,z,q,\boldsymbol{w})\|_{\mathbb{R}^d} d\Gamma \\ &\leq C_0 \text{meas}(\Gamma_{\text{C}})(c_{0\nu}(t)+c_{0\tau}(t)) + C_0 c_{1\nu}\sqrt{\text{meas}(\Gamma_{\text{C}})}\|z\|_{L^2(\Gamma_C)} \\ &\quad + C_0 c_{1\tau}\sqrt{\text{meas}(\Gamma_{\text{C}})}\|q\|_{L^2(\Gamma_C)} + C_0^2(c_{1\nu}+c_{1\tau})\sqrt{\text{meas}(\Gamma_{\text{C}})}\|\boldsymbol{w}\|_V\end{aligned}$$

for all $z, q \in L^2(\Gamma_C)$, $\boldsymbol{w} \in V$ and a.e. $t \in [0, T]$. This means that (4.4)(d) is valid with

$$c_1(t) = C_0 \text{meas}(\Gamma_{\text{C}})(c_{0\nu}(t)+c_{0\tau}(t)) \text{ and } c_2 = C_0 \text{meas}(\Gamma_{\text{C}})(1+C_0)(c_{1\nu}+c_{1\tau}) \tag{5.22}$$

for a.e. $t \in [0, T]$.

It remains to verify the validity of (4.4)(e) and (5.20). Employing [20, Proposition 2] and conditions $H(j_\nu)$(e) as well as $H(j_\tau)$(e), we find

$$j^0(\boldsymbol{x},t,z,q,\boldsymbol{\xi};\boldsymbol{\eta}) \leq j_\nu^0(\boldsymbol{x},t,z,\xi_\nu;\eta_\nu) + j_\tau^0(\boldsymbol{x},t,q,\boldsymbol{\xi}_\tau;\boldsymbol{\eta}_\tau) \tag{5.23}$$

for all $z, q \in \mathbb{R}$, $\boldsymbol{\xi} \in \mathbb{R}^d$ and a.e. $(\boldsymbol{x}, t) \in \Sigma_C$, and

$$\begin{aligned}&j^0(\boldsymbol{x},t,z_1,q_1,\boldsymbol{\xi}_1;\boldsymbol{\xi}_2-\boldsymbol{\xi}_1) + j^0(\boldsymbol{x},t,z_2,q_2,\boldsymbol{\xi}_2;\boldsymbol{\xi}_1-\boldsymbol{\xi}_2) \\ &\quad\leq j_\nu^0(\boldsymbol{x},t,z_1,\xi_{1\nu};\xi_{2\nu}-\xi_{1\nu}) + j_\nu^0(\boldsymbol{x},t,z_2,\xi_{2\nu};\xi_{1\nu}-\xi_{2\nu}) \\ &\qquad + j_\tau^0(\boldsymbol{x},t,q_1,\boldsymbol{\xi}_{1\tau};\boldsymbol{\xi}_{2\nu}-\boldsymbol{\xi}_{1\nu}) + j_\tau^0(\boldsymbol{x},t,q_2,\boldsymbol{\xi}_{2\tau};\boldsymbol{\xi}_{1\nu}-\boldsymbol{\xi}_{2\nu}) \\ &\quad\leq (\beta_{j_\nu}+\beta_{j_\tau})(|z_1-z_2|+|q_1-q_2|+\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathbb{R}^d})\|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\|_{\mathbb{R}^d}\end{aligned}$$

for all $z_1, z_2, q_1, q_2 \in \mathbb{R}$, $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2 \in \mathbb{R}^d$ and a.e. $(\boldsymbol{x}, t) \in \Sigma_C$. So, we conclude the inequality (5.20), see (5.23) and [21, Theorem 3.47 (iv)]. Nevertheless, from [21, Theorem 3.47 (iv)], we immediately get

$$\begin{aligned}&\phi^0(t,z_1,q_1,\boldsymbol{w}_1;\boldsymbol{w}_2-\boldsymbol{w}_1) + \phi^0(t,z_2,q_2,\boldsymbol{w}_2;\boldsymbol{w}_1-\boldsymbol{w}_2) \\ &\quad\leq (\beta_{j_\nu}+\beta_{j_\tau})\int_{\Gamma_C}\big(|z_1-z_2|+|q_1-q_2|+\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{\mathbb{R}^d}\big)\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{\mathbb{R}^d} d\Gamma \\ &\quad\leq C_0(\beta_{j_\nu}+\beta_{j_\tau})(\|z_1-z_2\|_{L^2(\Gamma_C)}+\|q_1-q_2\|_{L^2(\Gamma_C)})\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V \\ &\qquad + C_0^2(\beta_{j_\nu}+\beta_{j_\tau})\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_V^2\end{aligned}$$

and all $(z_1, q_1, \boldsymbol{w}_1), (z_2, q_2, \boldsymbol{w}_2) \in L^2(\Gamma_C) \times L^2(\Gamma_C) \times V$ and a.e. $t \in [0, T]$. Therefore, the condition (4.4)(e) holds with

$$\beta = (\beta_{j_\nu}+\beta_{j_\tau})\max\{C_0, C_0^2\}. \tag{5.24}$$

This completes the proof of the lemma. □

Under the above analysis, we are now in a position to apply Theorem 4.2 to prove Theorem 5.4.

Proof of Theorem 5.4. In fact, Lemmas 5.5–5.7 guarantee the validity of the conditions (4.2), (4.3), (4.4). Besides, from regularity conditions (5.9), it is not difficult to prove that the function ψ defined in (5.10) reads $H(\psi)$ (see [3, p. 875]). The smallness condition (3.3) and condition (3.2) can be obtained directly by using (5.15), (5.7), (5.8) and (5.9).

So, Theorem 4.2 is applicable. Employing the theorem, we conclude that Problem 5.19 has a unique solution $\boldsymbol{w} \in \mathcal{W}$. However, inequality (5.20) implies that Problem 5.2 has at least one solution in $\mathcal{W}$.

Let $\boldsymbol{w}_1, \boldsymbol{w}_2 \in \mathcal{W}$ be two solutions to Problem 5.2. A simple computing finds a constant $M_0 > 0$ such that

$$\begin{aligned}
&\frac{1}{2}\|\boldsymbol{w}_1(t)-\boldsymbol{w}_2(t)\|_H^2+\alpha\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^2(0,t;V)}^2-\alpha\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^2(0,t;V)}\|(\mathcal{S}\boldsymbol{w}_1)-(\mathcal{S}\boldsymbol{w}_2)\|_{L^2(0,t;Y)}\\
&\quad\le\int_0^t\int_{\Gamma_C} j_\nu^0\big(t,(\mathcal{S}\boldsymbol{w}_1)_\nu(s),w_{1,\nu}(s);w_{2,\nu}(s)-w_{1,\nu}(s)\big)\,d\Gamma\,\mathrm{d}t\\
&\qquad+\int_0^t\int_{\Gamma_C} j_\nu^0\big(t,(\mathcal{S}\boldsymbol{w}_2)_\nu(s),w_{2,\nu}(s);w_{1,\nu}(s)-w_{2,\nu}(s)\big)\,d\Gamma\,\mathrm{d}t\\
&\qquad+\int_0^t\int_{\Gamma_C} j_\tau^0\Big(s,\int_0^s\|(\mathcal{S}\boldsymbol{w}_1)_\tau(\eta)\|_{\mathbb{R}^d}\,d\eta,\boldsymbol{w}_{1,\tau}(s);\boldsymbol{w}_{2,\tau}(s)-\boldsymbol{w}_{1,\tau}(s)\Big)\,d\Gamma\,\mathrm{d}s\\
&\qquad+\int_0^t\int_{\Gamma_C} j_\tau^0\Big(s,\int_0^s\|(\mathcal{S}\boldsymbol{w}_2)_\tau(\eta)\|_{\mathbb{R}^d}\,d\eta,\boldsymbol{w}_{2,\tau}(s);\boldsymbol{w}_{1,\tau}(s)-\boldsymbol{w}_{2,\tau}(s)\Big)\,d\Gamma\,\mathrm{d}s\\
&\quad\le\beta\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^2(0,t;V)}^2+M_0\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^2(0,t;V)}\|(\mathcal{S}\boldsymbol{w}_1)-(\mathcal{S}\boldsymbol{w}_2)\|_{L^2(0,t;Y)}
\end{aligned}$$

for all $t \in [0, T]$. Hence, we have

$$\|\boldsymbol{w}_1-\boldsymbol{w}_2\|_{L^2(0,t;V)}\le\frac{M_0+\alpha}{\alpha-\beta}\|(\mathcal{S}\boldsymbol{w}_1)-(\mathcal{S}\boldsymbol{w}_1)\|_{L^2(0,t;Y)}$$

for all $t \in [0, T]$. However, from the Gronwall's inequality, we conclude that $\boldsymbol{w}_1 = \boldsymbol{w}_2$, so Problem 4.1 has a unique solution $\boldsymbol{w} \in \mathcal{W}$. □

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