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# ON $k$-TREES AND SPECIAL CLASSES OF $k$-TREES 

A Thesis<br>presented for the Doctorate of Philosophy<br>Department of Mathematics<br>University of Mississippi<br>JOHN WHELESS ESTES

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#### Abstract

A tree is a connected graph with no cycles. In 1968 Beineke and Pippet introduced the class of generalized trees known as $k$-trees [3]. In this dissertation, we classify a subclass of $k$-trees known as tree-like $k$-trees and show that tree-like $k$-trees are a common generalization of paths, maximal outerplanar graphs, and chordal planar graphs with toughness exceeding one.

A set $I$ of vertices in a graph $G$ is said to be independent if no pair of vertices of $I$ are incident in $G$. Let $f_{s}=f_{s}(G)$ be the number of independent sets of cardinality $s$ of $G$. Then the polynomial $I(G ; x)=\sum_{s \geq 0}^{\alpha(G)} f_{s}(G) x^{s}$ is called the independence polynomial of the graph $G$. [21]. In this dissertation, all rational roots of the independence polynomials of paths are found, and the exact paths whose independence polynomials have these roots are characterized. Additionally, trees are characterized that have $-1 / q$ as a root of their independence polynomials for $1 \leq q \leq 4$. The well known vertex and edge reduction identities for independence polynomials are generalized, and the independence polynomials of $k$-trees are investigated. Additionally, sharp upper and lower bounds for $f_{s}$ of maximal outerplanar graphs, i.e. tree-like 2-trees, are shown along with characterizations of the unique maximal outerplanar graphs that obtain these bounds respectively. These results are extensions of the works of Wingard, Song et al., and Alameddine [1].

The first Zagreb index $M_{1}(G)$ and the second Zagreb index $M_{2}(G)$ of the graph $G$ are given by: $M_{1}(G)=\sum_{u \in V(G)} d(u)^{2}$, and $M_{2}(G)=\sum_{u v \in E(G)} d(u) d(v)$. The study of the


Zagreb indices $M_{1}$ and $M_{2}$ have been an active area of research since the report of Gutman and Trinajstić in computational chemistry [23] in 1972. The minimum and maximum $M_{1}$ and $M_{2}$ values for $k$-trees are determined, and the unique $k$-trees that obtain these minimum and maximum values respectively are characterized.

In 2011, Hou, Li, Song, and Wei characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum $M_{1}$ and $M_{2}$ values, respectively, as well as maximum $M_{1}$ and $M_{2}$ values respectively [29]. Select works of Hou et al. are extended to all tree-like $k$-trees. That is, the maximum $M_{1}$ value for tree-like $k$-trees is determined, and the unique tree-like $k$-tree that obtains this maximum values respectively is characterized. Additionally, a partial result for the maximum $M_{2}$ value for tree-like $k$-trees is determined, and a conjecture for a full result is presented.

## DEDICATION

To Ethan, my son, who is my motivation to become a better person.

## LIST OF SYMBOLS

$G \quad$ a graph
$V \quad$ the vertex set of a graph $G$
$E \quad$ the edge set of a graph $G$
$G[S] \quad$ the subgraph induced by $S$
$G-v \quad G[V-v]$
$G-S \quad G[V-S]$
$G-e \quad G$ remove an edge $e$
$G-F \quad G$ remove a set of edges $F$
$G \cup\{u v\} \quad G$ add the edge $u v$
$G \cup H \quad$ the union of $G$ and $H$
$|G| \quad$ the cardinality of the vertex set of $G$
$\|G\| \quad$ the cardinatlity of the edge set of $G$
$N(v) \quad$ the neighborhood of vertex $v$
$N[v] \quad$ the closed neighborhood of vertex $v$
$N(e) \quad$ the neighborhood of edge $e$
$d(v) \quad$ the degree of $v$
$\delta(G) \quad$ the minimum degree of $G$
$\Delta(G) \quad$ the maximum degree of $G$
$T \quad$ a tree
$d(v, u) \quad$ the minimum distance from $v$ to $u$ in a graph
$K_{n} \quad$ the complete graph on $n$ vertices
$P_{n} \quad$ the path on $n$ vertices
$S_{n} \quad$ the star on $n$ vertices
$K_{n_{1}, n_{2}} \quad$ the complete bipartite graph on $n_{1}+n_{2}$ vertices
$T_{n}^{k} \quad$ a $k$-tree on $n$ vertices
$S_{1}(G) \quad$ the set of simplicial vertices in $G$
$P_{n}^{k} \quad$ the $k$-path on $n$ vertices
$S_{k, n-k} \quad$ the $k$-star on $n$ vertices
$S_{n}^{k} \quad$ the $k$-spiral on $n$ vertices
$D_{n}^{k} \quad$ the $k$-diamond on $n$ vertices
$\operatorname{Sh}\left(T_{n}^{k}\right) \quad$ the shell of a $k$-tree
$\tau(G) \quad$ the toughness of $G$
$\alpha(G) \quad$ the independence number of $G$
$f(G)$ the fibonacci number of $G$
$f_{s}(G) \quad$ the number of indepedent sets of cardinality $s$ of $G$
$I(G ; x) \quad$ the independence polynomial of $G$
$M_{1}(G) \quad$ the first Zagreb index
$M_{2}(G) \quad$ the second Zagreb index
$T(r) \quad$ a $p$-descendant or $p$-ancestor of $T$
$\left\{T_{i}\right\}_{i=0}^{\beta}$
$\beta\left(T_{n}^{k}\right) \quad$ the branching number of $T_{n}^{k}$

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## 1. Introduction

A graph is chordal if it does not have an induced cycle of length greater than three. A graph is said to be $k$-degenerate if all of its subgraphs have minimum degree at most $k$, a concept introduced by Lick and White in 1970 [32], and a graph is maximally $k$-degenerate if it is $k$-degenerate and not a spanning subgraph of any other $k$-degenerate graph. In 1968 Beineke and Pippet introduced the class of generalized trees known as $k$-trees [3], and these graphs have attracted considerable research as well as many applications [4, 16, 33, 37, 38]. A purpose of this dissertation is to investigate the class of graphs that are both chordal and maximally $k$-degenerate, and it is shown that a graph is chordal and maximally $k$-degenerate if and only if it is a $k$-tree.

A major emphasis of this dissertation is to classify a subclass of $k$-trees based on a "new" parameter known as the shell of a $k$-tree, which is a reformation of the $(k+1)$-line graph first introduced in 2006 by Markenzon et al. [33]. The shell gives a way to distinguish $k$-trees with a particular underlying structure. In particular, two $k$-tree subclasses known as path-like and tree-like $k$-trees hold interest. These concepts are introduced in Chapter 2 along with a survey of facts and propositions about $k$-trees including path-like and tree-like $k$-trees.

Path-like and tree-like $k$-trees generalize several commonly studied graph classes including paths, maximal outerplanar graphs, and chordal planar graphs with toughness exceeding one. Thus many results about $k$-trees may be expanded for tree-like $k$-trees and the previously
listed graph classes. Likewise, results on paths, maximal outerplanar graphs, and chordal planar graphs with toughness exceeding one may generalize to all tree-like $k$-trees.

A set $I$ of vertices in a graph $G$ is said to be independent if no pair of vertices of $I$ are incident in $G$. Let $f_{s}=f_{s}(G)$ be the number of independent sets of cardinality $s$ of $G$. Then the polynomial $I(G ; x)=\sum_{s \geq 0}^{\alpha(G)} f_{s}(G) x^{s}$ is called the independence polynomial (Gutman and Harary [21]), the independent set polynomial (Hoede and Li [26]), or Fibonacci polynomial (Hopkins and Staton [27]) of G. In 1995, Wingard investigated the number of independent sets in trees along with the independence polynomials of trees [44].

In Chapter 3, the works of Wingard are extended by investigating rational roots of the independence polynomials of paths and trees. All rational roots of the independence polynomials of paths are found, and the exact paths whose independence polynomials have these roots are characterized. Additionally trees are characterized that have $-1 / q$ as a root of their independence polynomials for $1 \leq q \leq 4$.

Chapter 4 investigates the independence polynomials of $k$-trees. In 2010, Song, Staton, and Wei generalized select results of Wingard presented in Chapter 3 to $k$-trees [41], and following their lead a result of Wingard is extended to $k$-trees in Chapter 4. This result is proven using generalizations of the well known vertex and edge reduction identities for independence polynomials which are also introduced in Chapter 4. Additionally, sharp upper and lower bounds for $f_{s}$ of maximal outerplanar graphs, i.e. tree-like 2-trees, are shown along with characterizations of the unique maximal outerplanar graphs that obtain these bounds respectively. These results are extensions of the works of Wingard, Song et al., and Alameddine [1].

In 1975, Randić introduced the branching index which later became known as the Randić connectivity index [36]. The Randić connectivity index has been generalized as the general Randić connectivity index and the general zeroth-order Randić connectivity index, where the Zagreb indices appeared as a special case [8]. The first Zagreb index $M_{1}(G)$ and the second Zagreb index $M_{2}(G)$ of the graph $G$ are given by:

$$
M_{1}(G)=\sum_{u \in V(G)} d(u)^{2}, \quad M_{2}(G)=\sum_{u v \in E(G)} d(u) d(v) .
$$

The Zagreb indices $M_{1}$ and $M_{2}$ have been an active area of research going back to 1972 in the report of Gutman and Trinajstić in computational chemistry [23].

In particular, Das and Gutman in 2004 characterized the Zagreb indices for trees and determined the unique tree that obtains minimum $M_{1}$ and $M_{2}$ values respectively, as well as maximum $M_{1}$ and $M_{2}$ values respectively [12, 20]. In Chapter 5 , the results of Das and Gutman are generalized to $k$-trees. That is, the minimum and maximum $M_{1}$ and $M_{2}$ values for $k$-trees are determined, and the unique $k$-trees that obtain these minimum and maximum values respectively are characterized.

In 2011, Hou, Li, Song, and Wei characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum $M_{1}$ and $M_{2}$ values respectively, as well as maximum $M_{1}$ and $M_{2}$ values respectively [29]. In Chapter 6, select works of Hou et al. are extended to all tree-like $k$-trees. That is, the maximum $M_{1}$ value for tree-like $k$-trees is determined, and the unique tree-like $k$-tree that obtains this maximum value is characterized. Additionally, a partial result for the maximum $M_{2}$ value for tree-like $k$-trees is determined, and a conjecture for a full result is presented.

Wingard determined that for $s \geq 0, f_{s}$ is minimized among trees by the path, and $f_{s}$ is maximized by the star. Similarly, Das and Gutman determined that $M_{i}$ is minimized among
trees by the path, and $M_{i}$ is maximized among trees by the star for $i \in\{1,2\}$. In Chapter 7, it is shown that for a given tree, it is possible to create a sequence of trees such that $f_{s}$ (respectively $M_{1}$ ) of a given tree in this sequence is greater than or equal to $f_{s}$ (respectively $M_{1}$ ) of any previous tree in the sequence for $s \geq 0$.

### 1.1. Definitions and Notation.

The following definitions will be used throughout this dissertation. For definitions not presented here, we refer the reader to Diestel [14].

Definition 1.1. A graph $G$ is an ordered pair $G=(V, E)$, where $V$ is a non-empty finite set and $E$ is a collection of unorderd pairs from $V$. Each element of $V$ is called a vertex and each element of $E$ is called an edge.

Note that in the above definition, graphs are simple and undirected. That is, no edge joins a vertex to itself, no two edges join the same pair of vertices, and edges are not given a direction.

The subgraph $G[S]$ induced by the vertex set $S \subset V(G)$ is the subgraph with vertex set $S$ and edge set $\{u v \mid u \in S, v \in S, u v \in E(G)\}$. In particular, $G-v$ denotes the induced subgraph $G[V(G) \backslash\{v\}]$ and $G-S$ denotes the induced subgraph $G[V(G) \backslash S]$ for $S \subseteq V(G)$. The graph $G-e$ is the graph resulting from deleting the edge $e$ from $G$, and $G-F$ is the graph resulting from deleting $F \subseteq E(G)$. Let $G$ and $H$ be two graphs with no common vertex. Then $G \cup H$ is the subgraph with $V(G \cup H)=V(G) \cup V(H)$ and $E(G \cup H)=E(G) \cup E(H)$. Let $u, v \in V(G)$ such that $u v \notin E(G)$. Then $G \cup\{u v\}$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup\{u v\}$. Let $\|G\|$ denote $|E(G)|$.

Let $v \in V(G)$. Then the neighborhood of $v$ is the set $N(v)=\{u \mid u v \in E(G)\}$, and $N_{H}(v)$ denotes the neighborhood of $v$ in the subgraph $H$. The set $N[v]=\{v\} \cup N(v)$ is
called the closed neighborhood of $v$. The degree of $v$ is defined as $d(v)=|N(v)|$, similarly for a subgraph $H, d_{H}(v)=\left|N_{H}(v)\right|$. For a graph $G, \delta(G)$ (respectively $\left.\Delta(G)\right)$ denotes the minimum (respectively maximum) degree of $G$. Let $u, v \in V(G)$. Then $d(u, v)$ is the length of a shortest path connecting $u$ to $v$ in $G$. Let $K_{n}, P_{n}$, and $S_{n}$ denote the complete graph, the path, and the star respectively on $n$ vertices, and let $K_{n_{1}, n_{2}}$ be the complete bipartite graph on $n_{1}+n_{2}$ vertices.

## 1.2. $k$-degenerate Graphs and $k$-trees.

Definition 1.2. A graph $G$ is called $k$-degenerate if every subgraph $H$ of $G$ is such that $\delta(H) \leq k$.

Note that if $G$ is $k$-degenerate, then $G$ is $(k+1)$-degenerate. Likewise, if $G$ is $k$-degenerate and $H$ is any subgraph of $G$, then $H$ is also $k$-degenerate. We say that $G$ is maximally $k$ degenerate if $G$ is $k$-degenerate and $G$ is not a spanning proper subgraph of any $k$-degenerate graph.

A vertex in a graph is simplicial if the subgraph induced by its neighborhood is a clique. We say that a vertex $v$ is $k$-simplicial if $G[N(v)] \cong K_{k}$. It is commonly known that a chordal graph on at least two vertices contains a simplicial vertex $v$. Let $G=G_{0}$, and let $G_{i}=G_{i-1}-v_{i}$ for $i \geq 1$. If each $v_{i}$ is a simplicial vertex in $G_{i-1}$, then $\left\{v_{1}, \ldots, v_{n}\right\}$ is a simplicial elimination ordering of the $n$-vertex graph $G$. With these defintions, I will define the concept of a $k$-tree, an idea first introduced by Beineke and Pippet in 1968 [3] and the subject of emphasis for this dissertation.

Definition 1.3. Let $T_{n}^{k}$ denote a $k$-tree on $n$ vertices.
(i) The smallest $k$-tree is the $k$-clique $K_{k}$.
(ii) If $T_{n}^{k}$ is a $k$-tree with $n$ vertices and a new vertex $v$ of degree $k$ is added and joined to the vertices of a $k$-clique in $G$, then the larger graph is a $k$-tree with $n+1$ vertices $T_{n+1}^{k}$.


Figure 1. A 4-tree on 10 vertices

By the definition of $k$-trees, it is clear that $k$-trees are a direct generalization of trees. In fact, trees are $k$-trees with $k=1$. The simplicial vertices of a tree are said to be"leaves", and the unique neighbor of a leaf in a tree is said to be the "support vertex" of the leaf.

It was noted by Song in 2010 that $k$-trees are $k$-degenerate [40], and clearly $k$-trees are chordal as well. However, through use of the Principle of Mathematical Induction, we may make a stronger statement.

Theorem 1.4. Let $G$ be a graph on $n \geq k$ vertices. Then $G$ is $a k$-tree if and only if $G$ is chordal and maximally $k$-degenerate.

Proof. It is clear that if $G$ is a $k$-tree, then $G$ is chordal and maximally $k$-degenerate. Suppose that $G$ is chordal and maximally $k$-degenerate, and suppose that $G$ is smallest such graph that is not a $k$-tree. Then $n \geq k+2$. As $G$ is a chordal graph on $n \geq 2$, there is a simplicial vertex $v \in V(G)$. Since $G$ is maximally $k$-degenerate, $\delta(G)=k$. If $|N(v)| \geq k+1$, then $G[N[v]] \cong K_{k+2}$, and so $G$ has a subgraph with $\delta(G) \geq k+1$; a contradiction. Thus $|N(v)|=k$, and $G-v$ is a chordal maximally $k$-degenerate graph. Hence $G-v$ is a $k$-tree, and $G$ is formed by attaching a vertex of degree $k$ to a $k$-clique of $G-v$. Thus $G$ is a $k$-tree contradicting the assumption that $G$ is not a $k$-tree. Hence $G$ is a $k$-tree.

| $d\left(v_{i}\right)$ for the $k$-path on $k+4 \leq n \leq 2 k$ vertices |  |  |  |
| :---: | :---: | :---: | :---: |
| $i$ | $1 \leq i \leq n-k-1$ | $n-k \leq i \leq k+1$ | $k+2 \leq i \leq n$ |
| $d\left(v_{i}\right)$ | $k+i-1$ | $n-1$ | $k+n-i$ |
| $d\left(v_{i}\right)$ for the $k$-path on $n \geq 2 k+1$ vertices |  |  |  |
| $i$ | $1 \leq i \leq k$ | $k+1 \leq i \leq n-k$ | $n-k+1 \leq i \leq n$ |
| $d\left(v_{i}\right)$ | $k+i-1$ | $2 k$ | $k+n-i$ |

TABLE 1. $d\left(v_{i}\right)$ for the $k$-path on $n$ vertices

Let $T_{n}^{k}$ be a $k$-tree. If $n \geq k+2, T_{n}^{k}$ has at least two simplicial vertices. If $n=k+1$, then by definition every vertex is $k$-simplicial. For convention, we say that $T_{k+1}^{k}$ has one simplicial vertex.

Definition 1.5. Let $G_{1}$ be a $k$-tree, and let $S_{1}$ be the set $k$-simplicial vertices of $G_{1}$. For $i \geq 2$, let $G_{i}=G_{i-1}-S_{1}\left(G_{i-1}\right)$. Then $S_{i}$ denotes the set of $k$-simplicial vertices of $G_{i}$.

Many results throughout this dissertation depend on several particular $k$-trees. These graphs will now be defined.

Definition 1.6. The $k$-path, $P_{n}^{k}$, has vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$ where $G\left[\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}\right] \cong K_{k}$. For $k+1 \leq i \leq n$, let vertex $v_{i}$ be adjacent to vertices $\left\{v_{i-1}, v_{i-2}, \ldots, v_{i-k}\right\}$.

A helpful characteristic of the $k$-path $P_{n}^{k}$ is that we may order the vertices $v_{1}, v_{2}, \ldots, v_{n}$ such that $P_{n}^{k}-\left\{v_{1}, \ldots, v_{i}\right\}$ is a $k$-path on $n-i$ vertices for $1 \leq i \leq n-k-1$.

Additionally, the degree of vertex $v_{i}$ for the $k$-path may be characterized as follows: for $k+4 \leq n \leq 2 k$ and $k \geq 4, d\left(v_{i}\right)=\min (k+i-1, n-1, k+n-i)$ and for $n \geq 2 k+1$, $d\left(v_{i}\right)=\min (k+i-1,2 k, k+n-i)$. Table 1 shows when these values are reached.

Definition 1.7. The $k$-star, $S_{k, n-k}$, has vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$ where $G\left[\left\{v_{1}, v_{2}, \ldots, v_{k}\right\}\right] \cong$ $K_{k}$ and $N\left(v_{i}\right)=\left\{v_{1}, \ldots, v_{k}\right\}$ for $k+1 \leq i \leq n$.

Definition 1.8. The $k$-spiral, $S_{n}^{k}$, has vertex set $\left\{v_{1}, \ldots, v_{n}\right\}$ where $G\left[\left\{v_{1}, v_{2}, \ldots, v_{k-1}\right\}\right] \cong$ $K_{k-1}, N\left(\left\{v_{1}, \ldots, v_{k-1}\right\} \subseteq N\left(v_{i}\right)\right.$ for $k \leq i \leq n$, and $\left\{v_{i-1} v_{i}, v_{i} v_{i+1}\right\} \subseteq E\left(S_{n}^{k}\right)$ for $k+1 \leq i \leq$ $n-1$.

Definition 1.9. A $k$-diamond $D_{n}^{k}$ has vertex set $V\left(D_{n}^{k}\right)=\left\{v_{1}, v_{2}, \ldots, v_{k+1}\right\} \cup\left\{u_{1}, \ldots, u_{i}\right\}$ for $1 \leq i \leq k+1$ such that $G\left[\left\{v_{1}, v_{2}, \ldots, v_{k+1}\right\}\right] \cong K_{k+1}$ and $N\left(u_{i}\right)=\left\{v_{1}, v_{2}, \ldots, v_{k+1}\right\}-\left\{v_{i}\right\}$ for all $i$.


Figure 2. The 3-path, 3-star, 3-spiral, and 3-diamond on 7 vertices

## 2. Tree-Like $k$-Trees

A major focal point of the research in this dissertation stems from the ideas that will now be presented. A $k$-clique in a chordal graph is said to be "bound" if it is contained in more than one $(k+1)$-clique. A $k$-clique in a $k$-tree that is not bound is said to be "unbound". The bound and unbound $k$-cliques of a $k$-tree help determine the underlying structure of the $k$-tree, and this structure is referred to as the shell of a $k$-tree.

Definition 2.1. Let $T_{n}^{k}$ be a $k$-tree. Then shell of $T_{n}^{k}, S h\left(T_{n}^{k}\right)$, is the graph defined as follows:
(i) If $X$ is a $(k+1)$-clique in $T_{n}^{k}$, then $X$ is a vertex in $\operatorname{Sh}\left(T_{n}^{k}\right)$. Hence $V\left(\operatorname{Sh}\left(T_{n}^{k}\right)\right)$ is the set of $(k+1)$-cliques in $T_{n}^{k}$.
(ii) If $X$ and $Y$ are $(k+1)$-cliques in $T_{n}^{k}$ such that $|V(X) \cap V(Y)|=k$, then $X Y \in$ $E\left(S h\left(T_{n}^{k}\right)\right)$.


Figure 3. A 2-tree and its shell

We see from this definition that two $(k+1)$-cliques $X$ and $Y$ are adjacent in $S h\left(T_{n}^{k}\right)$ if and only if the intersection of $X$ and $Y$ is a bound $k$-clique.

From the shell of the $k$-tree, special subclasses of $k$-trees emerge that may now be defined.

Definition 2.2. The $k$-tree $T_{n}^{k}$ is called path-like if $S h\left(T_{n}^{k}\right) \cong P_{n-k}$, the path on $n-k$ vertices.

Definition 2.3. The $k$-tree $T_{n}^{k}$ is called tree-like if $\operatorname{Sh}\left(T_{n}^{k}\right) \cong T$ where $T$ is a tree.

In 2005, Markenzon, Justel, and Paciornik defined simple-clique $k$-trees. A $k$-tree is defined to be a simple-clique $k$-tree if any bound $k$-clique is bound by exactly two $(k+1)$-cliques. From this definition, Markenzon et al. introduced the $(k+1)$-line graph for $k$-trees which is analagous to the shell of the $k$-tree [33], and they showed that if a $k$-tree is a simple-clique $k$-tree, then its $(k+1)$-line graph is a tree. Hence, the simple-clique $k$-trees of Markenzon et al. are synonymous with tree-like $k$-trees.

### 2.1. Facts and Propositions of $k$-trees and Tree-like $k$-trees.

In this chapter, several facts and propositions about $k$-trees, in particular path-like and tree-like $k$-trees, will be noted. These ideas will be used throughout the dissertation and are integral to the study of path-like and tree-like $k$-trees.

Fact 2.4. Let $T_{n}^{k}$ be a $k$-tree on $n$ vertices. Then
(i) $S_{1}\left(T_{n}^{k}\right) \neq \emptyset$ for $n \geq k+1$,
(i) $S_{1}\left(T_{n}^{k}\right)$ is an independent set for $n \geq k+2$,
(ii) $S_{2}\left(T_{n}^{k}\right) \neq \emptyset$ for $n \geq k+3$,
(iii) every $k$-clique is contained in a $(k+1)$-clique,
(iv) $T_{n}^{k}$ is $K_{k+2}-$ free.

Proposition 2.5. A $k$-tree on $n$ vertices has $\binom{k}{2}+(n-k) k$ edges.

Proof. Let $v_{1}, \ldots, v_{n}$ be a simplicial elimination ordering, and $G_{i}=G\left[\left\{v_{i}, \ldots, v_{n}\right\}\right]$ for $1 \leq$ $i \leq n$. Then $G_{n-k+1}$ is a $k$-clique with $\binom{k}{2}$ edges. For $1 \leq i \leq n-k, d_{G_{i}}\left(v_{i}\right)=k$, and $\left\|G_{i}\right\|=\left\|G_{i+1}\right\|+k$. Hence $\left\|T_{n}^{k}\right\|=\left\|G_{1}\right\|=\left\|G_{n-k}\right\|+(n-k) k=\binom{k}{2}+(n-k) k$.

In 1992, Fröberg generalized Proposition 2.5 to determine the number of $i$-cliques in a $k$-tree for $0 \leq i \leq k$. Define the $g$-vector of a graph $G$ as $g=\left(g_{1}, \ldots, g_{k+1}\right)$ where $g_{i}$ is the number of $i$-cliques in $G$ for $1 \leq i \leq k+1$. Fröberg deteremined the $g$-vector for $k$-trees.

Theorem 2.6. [16] For $n \geq 0$, the $g$-vector of $a k$-tree on $n$ vertices is as follows:

$$
\left(\binom{k}{1},\binom{k}{2}, \ldots,\binom{k}{k}, 0\right)+(n-k)\left(\binom{k}{0},\binom{k}{1}, \ldots,\binom{k}{k-1},\binom{k}{k}\right) .
$$

Fröberg also determined that a $k$-tree may be characterized by the neighborhoods of its vertices.

Theorem 2.7. [16] Let $G$ be a connected graph on $n$ vertices and $\binom{k}{2}+(n-k) k$ edges. Then $G$ is a $k$-tree if and only if $G[N(v)]$ is a $(k-1)$-tree for each $v \in V(G)$.

From Theorem 2.7, if $T_{n}^{k}$ is a $k$-tree with a vertex $v$ such that $d(v)=n-1$, then $T_{n}^{k}-v$ is a $(k-1)$-tree. Thus as an extentsion of Theorem 2.7, we state the following theorem.

Theorem 2.8. Let $T_{n}^{k}$ be a $k$-tree on $n$ vertices and $R=\{v \mid d(v)=n-1\}$ such that $|R|=r$. Then $T_{n}^{k}-R$ is a $(k-r)$-tree.

Proof. Let $R=\left\{v_{1}, \ldots, v_{r}\right\}$. Clearly $r \leq k$ as otherwise $T_{n}^{k}$ has a $K_{k+2}$ subgraph and is not $k$-degenerate. From Theorem 2.7, it is clear that $T_{n}^{k}-v_{1}$ is a $(k-1)$-tree and $d_{T_{n}^{k}-v_{1}}\left(v_{i}\right)=n-2$ for $2 \leq i \leq r$. Hence $T_{n}^{k}-v_{1}-v_{2}$ is a $(k-2)$-tree. Clearly, $T_{n}^{k}-R$ is a $(k-r)$-tree.

Proposition 2.9. Let $T_{n}^{k}$ be a $k$-tree, and let $X$ be a $k$-clique of $T_{n}^{k}$. Then $X$ is bound if and only if $X$ is a cut set.

Proof. Suppose that $X$ is a bound $k$-clique, but not a cut set. There are at least two vertices $v_{1}$ and $v_{2}$ such that $X \subseteq N\left(v_{1}\right) \cap N\left(v_{2}\right)$. If $v_{1} v_{2} \in E\left(T_{n}^{k}\right)$, then $T_{n}^{k}$ has a $(k+2)$-clique. Let $P$ be a shortest $v_{1}, v_{2}$-path in $T_{n}^{k}-X$ and $v_{3}$ be the vertex on $P$ closest to $v_{1}$ such that $N\left(v_{3}\right) \cap X \neq \emptyset$.

If $v_{3}=v_{2}$, then $v_{1} P v_{3} x v_{1}$ is an induced cycle of length at least four where $x \in V(X)$. If $v_{3} \neq v_{2}$, then $\left|v_{1} P v_{3}\right| \geq 3$. Let $x \in N\left(v_{3}\right) \cap X$. Then $v_{1} P v_{3} x v_{1}$ is an induced cycle of length at least four. This contradicts the fact that $T_{n}^{k}$ is chordal. Hence $T_{n}^{k}$ is disconnected.

Let $X$ be a cut set, then $T_{n}^{k}-X$ has at least two components $H_{1}$ and $H_{2}$. We may assume that there exists a vertex $v \in V\left(H_{2}\right)$ such that $X \subseteq N(v)$.

Suppose $X$ is not bound, then there is no vertex $u \in V\left(H_{1}\right)$ such that $X \subseteq N(u)$. However $T_{n}^{k}$ is $k$-connected, so there are at least $k$-edges from $H_{1}$ to $X$. Hence $\left|H_{1}\right| \geq 2$, and there exists $\left\{u_{1}^{\prime}, u_{2}^{\prime}\right\} \subseteq V\left(H_{1}\right)$ such that $1 \leq\left|N\left(u_{1}^{\prime}\right) \cap N\left(u_{2}^{\prime}\right) \cap V(X)\right| \leq k-2$. That is, there exists $x_{1} \in N\left(u_{1}^{\prime}\right) \cap V(X)$ and $x_{2} \in N\left(u_{2}^{\prime}\right) \cap V(X)$ such that $x_{1} \notin N\left(u_{2}^{\prime}\right), x_{2} \notin N\left(u_{1}^{\prime}\right)$. Of all pairs $\left\{u_{1}^{\prime}, u_{2}^{\prime}\right\}$ of $V\left(H_{1}\right)$ meeting these conditions, choose $\left\{u_{1}, u_{2}\right\}$ such that the smallest $u_{1}, u_{2}$-path $P$ is minimal. Then $x_{1} u_{1} P u_{2} x_{2} x_{1}$ is an induced cycle of at least four. This contradicts the fact that $T_{n}^{k}$ is chordal. Thus $X$ is bound.

In 1974, Rose gave several characterizations of $k$-trees.

Theorem 2.10. [37] A graph $G$ is a $k$-tree if and only if
(i) $G$ is connected,
(ii) $G$ has a $k$-clique but no $(k+2)$-clique, and
(iii) every minimal $x, y$ separator of $G$ is a $k$-clique.

Theorem 2.11. [37] Let $G$ be a graph on $n \geq k$ vertices such that $G$ has a $k$-clique but no $(k+2)$-clique and every minimal $x, y$ separator of $G$ is a clique. Then $|E(G)| \leq k n-\binom{k}{2}$ with equality holding if and only if $G$ is a $k$-tree.

Theorem 2.12. [37] $A$ graph $G$ is a $k$-tree if and only if
(i) $G$ is connected,
(ii) every minimal $x$, $y$ separator of $G$ is a $k$-clique, and (iii) $|E(G)|=\binom{k}{2}+(n-k) k$.

Theorem 2.13. [37] $A$ graph $G$ is a $k$-tree if and only if
(i) $G$ has a $k$-clique but no $(k+2)$-clique,
(ii) every minimal $x$, $y$ separator of $G$ is a $k$-clique, and
(iii) for all distinct nonadjacent pairs $x, y \in V(G)$, there exists exactly $k$ vertex-disjoint $x, y$-paths.

### 2.2. Propositions about Path-like and Tree-like $k$-trees.

Proposition 2.14. Let $T_{n}^{k}$ be a $k$-tree, then $\operatorname{Sh}\left(T_{n}^{k}\right)$ is chordal.

Proof. Suppose $S h\left(T_{n}^{k}\right)$ has an induced cycle of length at least four. Then there are at least four ( $k+1$ )-cliques $X_{1}, X_{2}, X_{3}, X_{4}$ such that $\left|X_{i} \cap X_{i+1}\right|=k$ for $1 \leq i \leq 4$ with arithmetic on the indices is modulo 4. Then $X_{1} \cap X_{2}=Y$ is a bound $k$-clique and $T_{n}^{k}-Y$ is connected; a contradiction. Hence $S h\left(T_{n}^{k}\right)$ is chordal.

Proposition 2.15. Let $T_{n}^{k}$ be a $k$-tree, then $S h\left(T_{n}^{k}\right)$ has $n-k$ vertices.

Proof. By Theorem 2.6, $T_{n}^{k}$ has $n-k(k+1)$-cliques. Hence $S h\left(T_{n}^{k}\right)$ has $n-k$ vertices.

Fact 2.16. $A k$-tree $T_{n}^{k}$ on $n \geq k_{2}$ vertices is path-like if and only if $\left|S_{1}\left(T_{n}^{k}\right)\right|=2$.

Fact 2.17. A $k$-tree $T_{n}^{k}$ with $n \geq k+2$ is path-like if and only if, its vertices may be arranged $v_{1}, v_{2}, \ldots, v_{n}$ so that
(i) The vertices $v_{1}, v_{2}, \ldots, v_{k+1}$ induce a $(k+1)$-clique.
(ii) For each $i \geq k+2$, the vertices $v_{1}, v_{2}, \ldots, v_{i}$ form a path-like $k$-tree with simplicial vertices $v_{1}$ and $v_{i}$.

Such an arrangement of the vertices of a path-like $k$-tree is called a presentation.

Fact 2.18. In a presentation of a path-like $k$-tree $T_{n}^{k}, v_{i} v_{i+1} \in E\left(T_{n}^{k}\right)$ for each $i<n$. It follows that each path-like $k$-tree has a spanning path.

Fact 2.19. There is a unique tree-like $k$-tree on $n$ vertices for $k \leq n \leq k+3$.

Fact 2.20. $A k$-tree $T_{n}^{k}$ is tree-like if and only if, every bound $k$-clique is the intersection of exactly two $(k+1)$-cliques.

Fact 2.20 states that the simple-clique $k$-trees defined by Markenzon et al. are in fact tree-like $k$-trees.

Fact 2.21. Let $T_{n}^{k}$ be a tree-like $k$-tree on $n \geq k+2$ vertices with $v \in S_{1}\left(T_{n}^{k}\right)$ and $N(v)=$ $\left\{u_{1}, \ldots, u_{k}\right\}$. Then $\left|\cap_{i=1}^{k} N\left(u_{i}\right)\right|=2$.

Proposition 2.22. If $T_{n}^{k}$ is a tree-like $k$-tree, then $\Delta\left(S h\left(T_{n}^{k}\right)\right) \leq k+1$.

Proof. Suppose that $T_{n}^{k}$ is a $k$-tree such that $\Delta\left(S h\left(T_{n}^{k}\right)\right) \geq k+2$. Then there is a $(k+1)$-clique $X$ and $r(k+1)$-cliques $X_{1}, X_{2}, \ldots, X_{r}$ such that $r=\Delta\left(S h\left(T_{n}^{k}\right)\right) \geq k+2$ and $\left|X \cap X_{i}\right|=k$
for all $i$. Hence there are at least two $(k+1)$-cliques $Y_{1}, Y_{2} \in\left\{X_{1}, \ldots, X_{r}\right\}$ such that $\left(Y_{1} \cap X\right)=\left(Y_{2} \cap X\right)$. Thus $\left\{X Y_{1}, Y_{1} Y_{2}, Y_{2} X\right\} \subseteq E\left(S h\left(T_{n}^{k}\right)\right)$, and so $S h\left(T_{n}^{k}\right)$ is not a tree.

Proposition 2.23. If $T_{n}^{k}$ is a tree-like $k$-tree, then $S h\left(T_{n}^{k}\right)$ has $n-k-1$ edges.

Proof. By Fact 2.15 and the fact that $\operatorname{Sh}\left(T_{n}^{k}\right)$ is a tree, it is clear that $\operatorname{Sh}\left(T_{n}^{k}\right)$ has $n-k-1$ edges.

Proposition 2.24. Let $T_{n}^{k}$ be a tree-like $k$-tree, then $T_{n}^{k}$ has $n k-(k-1)(k+1) k$-cliques, where $n-k-1$ are bound and $(k-1) n-(k-2)(k+1)$ are unbound.

Proof. Let $v_{1}, \ldots, v_{n}$ be a simplicial elimination ordering, and $G_{i}=G\left[\left\{v_{i}, \ldots, v_{n}\right\}\right]$ for $1 \leq$ $i \leq n$. Then $G_{n-k-1}$ is a $(k+1)$-clique with $k+1$ unbound $k$-cliques and no bound $k$-cliques. Let $\|G\|_{k}$ (respectively $\|G\|_{k}^{\prime}$ ) be the number of unbound (respectively bound) $k$-cliques in $G$ for a graph $G$. For $1 \leq i \leq n-k-1,\left\|G_{i}\right\|_{k}=\left\|G_{i-1}\right\|_{k}+k-1$ and $\left\|G_{i}\right\|_{k}^{\prime}=\left\|G_{i-1}\right\|_{k}^{\prime}+1$. Hence $\left\|T_{n}^{k}\right\|_{k}=\left\|G_{n}\right\|_{k}=\left\|G_{n-k-1}\right\|_{k}+(n-k-1)(k-1)=(k-1) n-(k-2)(k+1)$, and $\left\|T_{n}^{k}\right\|_{k}^{\prime}=\left\|G_{n}\right\|_{k}^{\prime}=\left\|G_{n-k-1}\right\|_{k}^{\prime}+(n-k-1)=n-k-1$. As every $k$-clique is either bound or unbound there are $n-k-1+(k-1) n-(k-2)(k+1)=n k-(k-1)(k+1) k$-cliques in $T_{n}^{k}$.

Theorem 2.25. Let $T_{n}^{k}$ be a tree-like $k$-tree on $n \equiv(j+1) \bmod k$ vertices for $2 \leq j \leq k+1$. Then $\left|S_{1}\left(T_{n}^{k}\right)\right| \leq \frac{k-1}{k}(n-k-1)+\frac{j}{k}$.

Proof. There is a one-to-one correspondence between simplicial vertices of $T_{n}^{k}$ and the leaves of the shell of $T_{n}^{k}$, and we will count the number of simplicial vertices by counting the leaves of the shell of $T_{n}^{k}$. Consider $T=S h\left(T_{n}^{k}\right)$, and let $S_{i}$ be the set of vertice of degree $i$ for
$1 \leq i \leq k+1$ and $S=\cup_{i=2}^{k}\left(S_{i}\right)$. Now $|V(T)|=n-k$, and so

$$
n-k=\left|S_{1}\right|+\left|S_{k+1}\right|+|S| .
$$

Suppose $|S| \geq 2$. Then there exists another tree $T^{\prime}$ such that $\left|S_{1}(T)\right| \leq\left|S_{1}\left(T^{\prime}\right)\right|$ and $\left|S\left(T^{\prime}\right)\right| \leq 1$. We may assume $0 \leq|S(T)| \leq 1$.

If $|S|=0$ note that $T$ may be formed by starting with $K_{1, k+1}$ and recursively attaching $k$ leaves to a leaf in the previous tree. Thus $n-k \equiv k+2 \equiv 2 \bmod k$. Then as $\sum_{v \in V(G)} d(v)=$ $2|E(G)|$,

$$
\begin{aligned}
& \left|S_{1}\right|+(k+1)\left|S_{k+1}\right|=2(n-k-1) \\
& \begin{aligned}
\left|S_{1}\right| & +(k+1)\left(n-k-\left|S_{1}\right|\right)=2(n-k-1) \\
\left|S_{1}\right| & =\frac{k-1}{k}(n-k)+\frac{2}{k} \\
& =\frac{k-1}{k}(n-k)-\frac{k-1}{k}+\frac{k+1}{k} \\
\quad & =\frac{k-1}{k}(n-k-1)+\frac{j}{k}
\end{aligned}
\end{aligned}
$$

where $j=k+1$.
Suppose $|S|=1$, and let $v \in S$ such that $d(v)=j$ for some $2 \leq j \leq k$. Then $n-k \equiv$ $k+2+j-1 \equiv(j+1) \bmod k$. Then

$$
\begin{aligned}
& \left|S_{1}\right|+(k+1)\left|S_{k+1}\right|+j|S|=2(n-k-1) \\
& \begin{array}{l}
\left|S_{1}\right|+(k+1)\left(n-k-\left|S_{1}\right|-1\right)+j=2(n-k-1) \\
\left|S_{1}\right|=\frac{k-1}{k}(n-k)-\frac{k-1-j}{k} \\
\quad=\frac{k-1}{k}(n-k-1)+\frac{j}{k} .
\end{array}
\end{aligned}
$$

As $\left|S_{1}(T)\right|=\left|S_{1}\left(T_{n}^{k}\right)\right|,\left|S_{1}\left(T_{n}^{k}\right)\right| \leq \frac{k-1}{k}(n-k-1)+\frac{j}{k}$ where $n \equiv(j+1) \bmod k$ for $2 \leq j \leq k+1$.

### 2.3. Particular Classes of $k$-trees.

Fixing $k$ to be 1,2 or 3 , it becomes clear that tree-like $k$-trees are particular classes of graphs.

Fact 2.26. The only tree-like tree (a 1-tree) on $n$ vertices is $P_{n}$, the path.

Markenzon et al. verified the following about tree-like $k$-trees.

Theorem 2.27. [33] Let $G$ be a graph. Then $G$ is maximal outerplanar if and only if, $G$ is a tree-like 2-tree.

Theorem 2.28. [33] Let $G$ be a graph with $n>3$. Then $G$ is a planar 3 -tree if and only if $G$ is a tree-like 3-tree.

Additionally, for 3-trees, Markenzon et al. found the following.

Theorem 2.29. [33] Let $G$ be a graph with $n \geq 3$. Then $G$ is a planar 3 -tree if and only if $G$ is a chordal and maximal planar graph.

Let $\omega(G)$ denote the number of components of a graph $G$. A graph $G$ is $t$-tough if $t \leq \frac{|S|}{\omega(G-S)}$ for every subset $S$ of the vertex set $V(G)$ with $\omega(G-S)>1$. The toughness of $G$, denoted $\tau(G)$, is the maximum value for $t$ for which $G$ is $t$-tough.

With adding the condition of toughness exceeding 1 to a tree-like 3-tree, we may restate Theorem 2.29 to all chordal planar graphs.

Theorem 2.30. Let $G$ be a graph with $\tau(G)>1$. Then $G$ is chordal planar if and only if, $G$ is a tree-like 3-tree.

Proof. We need to only show that if $G$ is a chordal planar graph with $\tau(G)>1$, then $G$ is a tree-like 3 -tree. We will proceed by induction on the number of vertices $n$. If $n=3$, then $G \cong K_{3}$. If $n=4$, then $G \cong K_{4}$. In both of these cases, $G$ is a tree-like 3-tree.

Suppose that the theorem is true for smaller $n$, and consider $G$, a chordal planar graph with $\tau(G)>1$ on $n$ vertices. Since $G$ is a chordal graph, there is a simplicial vertex $v$, and since $\tau(G)>1 d(v)=3$. Let $N(v)=\left\{u_{1}, u_{2}, u_{3}\right\}$, and $G[N(v)]=X$ which is a triangle. By induction, $G-v$ is a tree-like 3 -tree.

Suppose $X$ is a bound $k$-clique in $G-v$. Then there are two vertices $x_{1}$ and $x_{2}$ such that $X \subseteq N\left(x_{i}\right)$ for $i \in\{1,2\}$. Then $G$ contains a $K_{3,3}$ subgraph with vertex set $\left\{u_{1}, u_{2}, u_{3}, x_{1}, x_{2}, v\right\}$. Hence $G$ is not planar.

Then $X$ is unbound in $G-v$, and $G$ is a tree-like 3-tree. Thus the theorem holds by the Principle of Mathematical Induction.

## 3. Independent Sets of Trees

An independent set in a graph $G$ is a set of pairwise non-adjacent vertices, and the independence number $\alpha(G)$ is the size of a maximum independent set of $G$. The idea of counting independent sets in graphs was introduced by Prodinger and Tichy in 1982 [35], where they defined, for a graph $G$, the Fibonacci number $f(G)$ to be the total number of independent sets of $G$. The Fibonacci number is a parameter of interest to chemists who call it the Merrifield-Simmons index $[18,22,42]$. Let $f_{s}=f_{s}(G)$ be the number of independent sets of cardinality $s$ of $G$. Then the polynomial

$$
I(G ; x)=\sum_{s \geq 0}^{\alpha(G)} f_{s}(G) x^{s}
$$

is called the independence polynomial (Gutman and Harary [21]), the independent set polynomial (Hoede and Li [26]), or Fibonacci polynomial (Hopkins and Staton [27]) of G. There are numerous results calculating the Fibonacci number and independence polynomial of classes of graphs $[9,10,15,27,28]$. Not only have independence polynomials been related to interesting theoretical problems in graph theory and combinatorics, they have been used in studying statistical physics and combinatorial chemistry. As an example, see [19, 24].

In general, finding the independence polynomial of a graph is a very difficult problem. Most of the literature consists of inequalities and asymptotic results. For more results not given here, we refer to the reader to a thorough survey paper by Levit and Mandrescu [30].

The following propositions are commonly known and are very useful in calculating independence polynomials of graphs.

Proposition 3.1. Let $G=(V, E)$ with $|V|=n$ and $|E|=m$. Then
(i) $f_{0}(G)=1$,
(ii) $f_{1}(G)=n$,
(iii) $f_{2}(G)=\binom{n}{2}-m$,
(iv) $f(G)=f(G-v)+f(G-N[v])$,
(v) $f_{s}(G)=f_{s}(G-v)+f_{s-1}(G-N[v])$,
(vi) $I(G ; x)=I(G-v ; x)+x I(G-N[v] ; x)$,
(vii) $f(G)=f(G-e)-f(G-N(e))$,
(viii) $f_{s}(G)=f_{s}(G-e)-f_{s-2}(G-N(e))$,
(ix) $I(G ; x)=I(G-e ; x)-x^{2} I(G-N(e) ; x)$, and
(x) If $G$ is the empty graph, then $I(G ; x)=1$.

Proposition 3.2. Let $G, H$, and $J$ be graphs such that $G=H \cup J$. Then $I(G ; x)=$ $I(H ; x) I(J ; x)$.

Proposition 3.3. Let $G$ and $H$ be graphs such that $H$ is a spanning subgraph of $G$. Then $f_{s}(G) \leq f_{s}(H)$.

### 3.1. Results of Wingard.

In 1995, Wingard researched independence polynomials of trees with an emphasis on roots of the independence polynomial. Among his results are the following:

Theorem 3.4. [44] Let $T$ be a tree. Then $|I(T ;-1)| \leq 1$.

Lemma 3.5. [44] If $G$ is a graph with $A$ independent sets of even cardinality and $B$ independent sets of odd cardinality, then $A-B=I(G ;-1)$.

From Lemma 3.5, we see that for a graph $G$, if $I(G ;-1)=0$, then $G$ has the same number of independent sets of even cardinality as independent sets of odd cardinality. In fact, Wingard characterized exactly when $I(T ;-1)=0$ for trees. First, we state the following definition.

Definition 3.6. If $T$ is a tree, and $P$ is a path in $T$, then for every vertex $v$ of $T$, the unique vertex of $P$ of minimal distance from $v$ is called the nearpoint of $v$, denoted $n(v, P)$.

Theorem 3.7. [44] Let $G$ be a forest. Then $I(G ;-1)=0$ if and only if there is a path $P=\left\{v_{1}, \ldots, v_{n}\right\}$ in some component $T$ of $G$ where:
(i) $d\left(v_{1}\right)=d\left(v_{n}\right)=1$
(ii) $n \equiv 1 \bmod 3$
(iii) for every leaf $v \in V(T)-P$, if $n(v, P)=v_{i}$ for $i \equiv 1 \bmod 3$, then $d\left(v, v_{i}\right) \equiv 0$ $\bmod 3$.

Wingard also classified which forests have independence polynomials that do not have -1 as a root. The result is determined by a sequence of "reductions". A reduction is carried out by choosing a vertex $v$ which is the neighbor of an end vertex, and removing $v$ and its neighbors from $G$. The sequence terminates when every remaining component is a star.

Theorem 3.8. [44] Let $G$ be a forest. If $I(G ;-1) \neq 0$ and if $k$ reductions by neighbors of end vertices leaves $c$ components of the type $K_{1, t_{i}}, t_{i} \geq 1$ for $1 \leq i \leq c$, then $I(G ;-1)=(-1)^{k+c}$.

### 3.2. Roots of Independence Polynomials of Paths and Trees.

Continuing the work of Wingard, we will now investigate rational roots of independence polynomials of trees. In 1984, Hopkins and Staton gave a characterization of the independence polynomial of the path which is now presented.

Theorem 3.9. [27] Let $P_{n}$ be a path on $n$ vertices and $l=\frac{1}{2}(1+\sqrt{1+4 x})$. Then $I\left(P_{n} ; x\right)=$ $(2 l-1)^{-1}\left(l^{n+2}-(1-l)^{n+2}\right)$.

From the characterization given by Hopkins and Staton, we may determine all possible rational roots for the independence polynomial of the path. Additionally, we are able to determine which paths have these rational roots for their respective independence polynomials.

Theorem 3.10. Let $P_{n}$ be the path on $n$ vertices and $c$ a rational number such that $I\left(P_{n} ; c\right)=$ 0. Then $c \in\left\{-1,-\frac{1}{2},-\frac{1}{3}\right\}$.

Proof. The coefficients of $I\left(P_{n} ; x\right)$ are all positive. Thus $c<0$. According to the Rational Root Theorem, $c=-\frac{1}{q}$ for some $q \geq 1$, and so $c \in[-1,0)$. Let $p=\sqrt{1+4 x}$. Then according to Theorem 3.9, $I\left(P_{n} ; x\right)=\left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left((1+p)^{n+2}-(1-p)^{n+2}\right)$.

Thus

$$
\begin{aligned}
I\left(P_{n} ; x\right) & =\left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left((1+p)^{n+2}-(1-p)^{n+2}\right) \\
& =\left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k} 1^{n-k} p^{k}-\sum_{k=0}^{n+2}\binom{n+2}{k} 1^{n-k}(-1)^{k} p^{k}\right) \\
& =\left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k}\left(p^{k}-(-1)^{k} p^{k}\right)\right)
\end{aligned}
$$

Suppose $n$ is even.

$$
\begin{align*}
I\left(P_{n} ; x\right) & =\left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k}\left(p^{k}-(-1)^{k} p^{k}\right)\right) \\
& =\left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left((n+2)(2 p)+\binom{n+2}{3}\left(2 p^{3}\right)+\ldots+\binom{n+2}{n+1}\left(2 p^{n+1}\right)\right) \\
& =\left(\frac{1}{2}\right)^{n+1}\left((n+2)+\binom{n+2}{3} p^{2}+\ldots+\binom{n+2}{n+1} p^{n}\right) \\
& =\left(\frac{1}{2}\right)^{n+1}\left((n+2)+\binom{n+2}{3}(1+4 x)+\ldots+\binom{n+2}{n+1}(1+4 x)^{\frac{n}{2}}\right) . \tag{1}
\end{align*}
$$

Suppose $n$ is odd.

$$
\begin{align*}
I\left(P_{n} ; x\right)= & \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k}\left(p^{k}-(-1)^{k} p^{k}\right)\right) \\
= & \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left((n+2)(2 p)+\binom{n+2}{3}\left(2 p^{3}\right)+\ldots+\binom{n+2}{n}\left(2 p^{n}\right)+\left(2 p^{n+2}\right)\right) \\
= & \left(\frac{1}{2}\right)^{n+1}\left((n+2)+\binom{n+2}{3} p^{2}+\ldots+\binom{n+2}{n} p^{n-1}+p^{n+1}\right) \\
& =\left(\frac{1}{2}\right)^{n+1}\left((n+2)+\binom{n+2}{3}(1+4 x)+\ldots+(1+4 x)^{\frac{n+1}{2}}\right) . \tag{2}
\end{align*}
$$

For expressions (1) and (2) to be equal to zero, there must be summands of (1) and
(2) that are negative. Clearly, this is only possible if $1+4 x<0$. Hence $x<-\frac{1}{4}$. Now $I\left(P_{2} ; x\right)=1+2 x$ and $I\left(P_{4} ; x\right)=(1+x)(1+3 x)$. Hence $I\left(P_{4} ;-1\right)=0, I\left(P_{2} ;-\frac{1}{2}\right)=0$, and $I\left(P_{4} ;-\frac{1}{3}\right)=0$. Thus if $c$ is a rational root of the independence polynomial of $P_{n}$, then $c \in\left\{-1,-\frac{1}{2},-\frac{1}{3}\right\}$.

As the previous theorem states, we have found that the only possible rational roots for independence polynomials of paths are $-1,-\frac{1}{2}$, and $-\frac{1}{3}$. Now, we will demonstrate which paths have these roots for their respective independence polynomials.

Theorem 3.11. Let $P_{n}$ be the path on $n$ vertices. Then

$$
I\left(P_{n} ;-1\right)= \begin{cases}-1 & \text { if } n \equiv\{2,3\} \quad \bmod 6 \\ 0 & \text { if } n \equiv 1 \quad \bmod 3 \\ 1 & \text { if } n \equiv\{5,0\} \quad \bmod 6\end{cases}
$$

Proof. Proceed by induction on $n$. Suppose $1 \leq n \leq 6$. Then by Table 3.2, the theorem holds.

| $T$ | $I(T ; x)$ | $I(T ;-1)$ | $I\left(T ;-\frac{1}{2}\right)$ | $I\left(T ;-\frac{1}{3}\right)$ |
| :--- | :--- | :---: | :---: | :---: |
| $P_{1}$ | $1+x$ | 0 | $1 / 2$ | $2 / 3$ |
| $P_{2}$ | $1+2 x$ | -1 | 0 | $1 / 3$ |
| $P_{3}$ | $1+3 x+x^{2}$ | -1 | $-1 / 4$ | $1 / 9$ |
| $P_{4}$ | $1+4 x+3 x^{2}$ | 0 | $-1 / 4$ | 0 |
| $P_{5}$ | $1+5 x+6 x^{2}+x^{3}$ | 1 | $-1 / 8$ | $-1 / 27$ |
| $P_{6}$ | $1+6 x+10 x^{2}+4 x^{3}$ | 1 | 0 | $-1 / 27$ |
| $P_{7}$ | $1+7 x+15 x^{2}+10 x^{3}+4 x^{4}$ | 0 | $1 / 16$ | $-2 / 81$ |
| $P_{8}$ | $1+8 x+21 x^{2}+20 x^{3}+5 x^{4}$ | -1 | $1 / 16$ | $-1 / 81$ |
| $P_{9}$ | $1+9 x+28 x^{2}+35 x^{3}+15 x^{4}+x^{5}$ | -1 | $1 / 32$ | $-1 / 243$ |
| $P_{10}$ | $1+10 x+36 x^{2}+56 x^{3}+21 x^{4}+6 x^{5}$ | 0 | 0 | 0 |

Table 2. Independence Polynomials of $P_{n}$ for $n \leq 10$

Suppose the theorem is true for paths on $1 \leq n^{\prime}<n$ vertices, and consider $P_{n}$ and let $v$ be a leaf such that $N(v)=u$. Then by Proposition 3.1,

$$
I\left(P_{n} ; x\right)=I\left(P_{n}-u ; x\right)+x I\left(P_{n}-N[u] ; x\right)
$$

Now $P_{n}-u$ has two components $P_{1}$ and $P_{n-2}$. Hence

$$
I\left(P_{n} ; x\right)=I\left(P_{1} ; x\right) I\left(P_{n-2} ; x\right)+x I\left(P_{n-3} ; x\right),
$$

and thus $I\left(P_{n} ;-1\right)=0+(-1) I\left(P_{n-3} ;-1\right)=-I\left(P_{n-3} ;-1\right)$.
Suppose $n \equiv 1 \bmod 3$. Then $n-3 \equiv 1 \bmod 3$, and so by induction $I\left(P_{n} ;-1\right)=$ $-I\left(P_{n-3} ;-1\right)=0$.

Suppose $n \equiv 2($ or 3$) \bmod 6$. Then $n-3 \equiv 5($ or 0$) \bmod 6$. Thus by induction $I\left(P_{n} ;-1\right)=$ $-I\left(P_{n-3} ;-1\right)=-(1)=-1$.

Suppose $n \equiv 5($ or 0$) \bmod 6$. Then $n-3 \equiv 2($ or 3$) \bmod 6$. Thus by induction $I\left(P_{n} ;-1\right)=$ $-I\left(P_{n-3} ;-1\right)=-(-1)=1$.

Hence by the Principle of Mathematical Induction, the theorem holds for all $n$.

Theorem 3.12. Let $P_{n}$ be the path on $n$ vertices. Then

$$
I\left(P_{n} ;-\frac{1}{2}\right)=\left\{\begin{array}{ll}
(-1)\left(\frac{1}{2}^{\left\lceil\frac{n}{2}\right\rceil}\right) & \text { if } n \equiv\{3,4,5\} \quad \bmod 8 \\
0 & \text { if } n \equiv 2 \bmod 4 \\
\frac{1}{2}\left\lceil\frac{n}{2}\right\rceil & \text { if } n \equiv\{7,0,1\} \quad \bmod 8
\end{array} .\right.
$$

Proof. Proceed by induction on $n$. Suppose $1 \leq n \leq 8$. Then by Table 3.2, the theorem holds.

Suppose the theorem is true for paths on $1 \leq n^{\prime}<n$ vertices, and consider $P_{n}$ and let $v$ be a leaf and $u \in V(T)$ such that $d(v, u)=2$. Then by Proposition 3.1,

$$
I\left(P_{n} ; x\right)=I\left(P_{n}-u ; x\right)+x I\left(P_{n}-N[u] ; x\right)
$$

Now $P_{n}-u$ has two components $P_{2}$ and $P_{n-3}$, and $P_{n}-N[u]$ has two components $P_{1}$ and $P_{n-4}$. Hence

$$
I\left(P_{n} ; x\right)=I\left(P_{2} ; x\right) I\left(P_{n-3} ; x\right)+x I\left(P_{1} ; x\right) I\left(P_{n-4} ; x\right)
$$

and thus $I\left(P_{n} ;-\frac{1}{2}\right)=0+-\frac{1}{2}\left(\frac{1}{2}\right) I\left(P_{n-4} ;-\frac{1}{2}\right)=-\frac{1}{4} I\left(P_{n-4} ;-\frac{1}{2}\right)$.
Suppose $n \equiv 2 \bmod 4$. Then $n-4 \equiv 2 \bmod 4$, and so by induction $I\left(P_{n} ;-\frac{1}{2}\right)=$ $-\frac{1}{4} I\left(P_{n-4} ;-\frac{1}{2}\right)=0$.

Suppose $n \equiv 3($ or 4,5$) \bmod 8$. Then $n-4 \equiv 7($ or 0,1$) \bmod 8$. Thus by induction $I\left(P_{n} ;-\frac{1}{2}\right)=-\frac{1}{4} I\left(P_{n-4} ;-\frac{1}{2}\right)=-\frac{1}{4}\left(\frac{1}{2}\right)^{\left\lceil\frac{n-4}{2}\right\rceil}=-\frac{1}{2}\left\lceil^{\left.\frac{n-4}{2}\right\rceil+2}=-\frac{1}{2}\left\lceil^{\left.\frac{n}{2}\right\rceil}\right.\right.$.

Suppose $n \equiv 7($ or 0,1$) \bmod 8$. Then $n-4 \equiv 3($ or 4,5$) \bmod 6$. Thus by induction $I\left(P_{n} ;-\frac{1}{2}\right)=-\frac{1}{4} I\left(P_{n-4} ;-\frac{1}{2}\right)=\left(-\frac{1}{4}\right)(-1)\left(\frac{1}{2}\right)^{\left\lceil\frac{n-4}{2}\right\rceil}=\frac{1}{2}^{\left\lceil\frac{n-4}{2}\right\rceil+2}=-\frac{1}{2}\left\lceil\frac{n}{2}\right\rceil$.

Hence by the Principle of Mathematical Induction, the theorem holds for all $n$.

Theorem 3.13. Let $P_{n}$ be the path on $n$ vertices. Then

$$
I\left(P_{n} ;-\frac{1}{3}\right)=\left\{\begin{array}{ll}
(-1)\left(\frac{2}{3}\left\lceil^{\left.\frac{n}{2}\right\rceil}\right)\right. & \text { if } n \equiv 7 \quad \bmod 12 \\
(-1)\left(\frac{1}{3}^{\left\lceil\frac{n}{2}\right\rceil}\right) & \text { if } n \equiv\{5,6,8,9\} \quad \bmod 12 \\
0 & \text { if } n \equiv 4 \bmod 6 \\
\frac{1}{3}^{\left\lceil\frac{n}{2}\right\rceil} & \text { if } n \equiv\{11,0,2,3\} \quad \bmod 12 \\
\frac{2}{3}\left\lceil\frac{n}{2}\right\rceil & \text { if } n \equiv 1 \bmod 12
\end{array} .\right.
$$

Proof. Proceed by induction on $n$. Suppose $1 \leq n \leq 10$. Then by Table 3.2, the theorem holds. Now $I\left(P_{11}\right)=1+11 x+45 x^{2}+84 x^{3}+70 x^{4}+21 x^{5}+x^{6}$, and $I\left(P_{12} ; x\right)=1+12 x+$ $55 x^{2}+120 x^{3}+126 x^{4}+56 x^{5}+7 x^{6}$. Thus $I\left(P_{i} ;-\frac{1}{3}\right)=1 / 729$ for $11 \leq i \leq 12$, and so the theorem holds.

Suppose the theorem is true for paths on $1 \leq n^{\prime}<n$ vertices, and consider $P_{n}$ and let $v$ be a leaf and $u \in V(T)$ such that $d(v, u)=4$. Then by Proposition 3.1,

$$
I\left(P_{n} ; x\right)=I\left(P_{n}-u ; x\right)+x I\left(P_{n}-N[u] ; x\right)
$$

Now $P_{n}-u$ has two components $P_{4}$ and $P_{n-5}$, and $P_{n}-N[u]$ has two components $P_{3}$ and $P_{n-6}$. Hence

$$
I\left(P_{n} ; x\right)=I\left(P_{4} ; x\right) I\left(P_{n-5} ; x\right)+x I\left(P_{3} ; x\right) I\left(P_{n-6} ; x\right),
$$

and thus $I\left(P_{n} ;-\frac{1}{3}\right)=0+-\frac{1}{3}\left(\frac{1}{9}\right) I\left(P_{n-6} ;-\frac{1}{3}\right)=-\frac{1}{27} I\left(P_{n-6} ;-\frac{1}{3}\right)$.
Suppose $n \equiv 1 \bmod 12$. Then $n-6 \equiv 7 \bmod 12$, and so by induction $I\left(P_{n} ;-\frac{1}{3}\right)=$ $-\frac{1}{27} I\left(P_{n-6} ;-\frac{1}{3}\right)=\left(-\frac{1}{27}\right)(-1)^{\frac{2}{3}\left\lceil\frac{n-6}{2}\right\rceil}=\frac{2}{3}\left\lceil\frac{n-6}{2}\right\rceil+3=\frac{2}{3}^{\left\lceil\frac{n}{2}\right\rceil}$.

Suppose $n \equiv 7 \bmod 12$. Then $n-6 \equiv 1 \bmod 12$, and so by induction $I\left(P_{n} ;-\frac{1}{3}\right)=$ $-\frac{1}{27} I\left(P_{n-6} ;-\frac{1}{3}\right)=\left(-\frac{1}{27}\right)^{\frac{2}{3}\left\lceil\frac{n-6}{2}\right\rceil}=-\left(\frac{2}{3}^{\left\lceil\frac{n-6}{2}\right\rceil+3}\right)=-\left(\frac{2}{3}\left\lceil\frac{n}{2}\right\rceil\right)$.

Suppose $n \equiv 5($ or $6,8,9) \bmod 12$. Then $n-6 \equiv 11($ or $0,2,3) \bmod 12$. Thus by induction $I\left(P_{n} ;-\frac{1}{3}\right)=-\frac{1}{27} I\left(P_{n-6} ;-\frac{1}{3}\right)=-\frac{1}{27} \frac{1}{3}\left\lceil^{\left.\frac{n-6}{2}\right\rceil}=-\left(\frac{1}{3}^{\left\lceil\frac{n-6}{2}\right\rceil+3}\right)=-\left(\frac{1}{3}^{\left\lceil\frac{n}{2}\right\rceil}\right)\right.$.

Suppose $n \equiv 11$ (or $0,2,3) \bmod 12$. Then $n-6 \equiv 5($ or $6,8,9) \bmod 12$. Thus by induction $I\left(P_{n} ;-\frac{1}{3}\right)=-\frac{1}{27} I\left(P_{n-6} ;-\frac{1}{3}\right)=\left(-\frac{1}{27}\right)(-1)^{\left\lceil\frac{n-6}{2}\right\rceil}=\frac{1}{3}^{\left\lceil\frac{n-6}{2}\right\rceil+3}=\frac{1}{3}^{\left\lceil\frac{n}{2}\right\rceil}$.

Suppose $n \equiv 4 \bmod 6$. Then $n-6 \equiv 4 \bmod 6$, and so by induction $I\left(P_{n} ;-\frac{1}{3}\right)=$ $-\frac{1}{27} I\left(P_{n-6} ;-\frac{1}{3}\right)=0$.

Hence by the Principle of Mathematical Induction, the theorem holds for all $n$.

Let $\mathcal{A}_{-1}$ be the family of trees defined as follows:
(i) $P_{1} \in \mathcal{A}_{-1}$.
(ii) Let $T^{\prime}, T_{1}, T_{2}$ be trees such that $u v \in E\left(T^{\prime}\right), T_{1}, T_{2} \in \mathcal{A}_{-1}$, and $v_{i} \in V\left(T_{i}\right)$ for $i \in\{1,2\}$, and let $T$ be a tree with $V(T)=V\left(T^{\prime}\right) \cup V\left(T_{1}\right) \cup V\left(T_{2}\right)$ and $E(T)=$ $E\left(T^{\prime}\right) \cup E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup\left\{v_{1} u\right\} \cup\left\{v v_{2}\right\}$. Then $T \in \mathcal{A}_{-1}$.

In Theorem 3.7, Wingard gave a necessary induced subgraph to guarantee that the independence polynomial of a forest has -1 as a root. With the defintion of $\mathcal{A}_{-1}$, Theorem 3.7 may be restated as follows.

Theorem 3.14. Let $F$ be a forrest. Then $I(F ;-1)=0$ if and only if $F$ has a component $T$ such that $T \in \mathcal{A}_{-1}$.

Proof. Let $F$ be a forrest with component $T$ such that $T \in \mathcal{A}_{-1}$ with $V(T)$ and $E(T)$ defined as above. By Proposition 3.2, $I(F ; x)=I(H ; x) I(T ; x)$ where $H \cong F-V(T)$. By Proposition 3.1 $I(T ;-1)=I(T-v ;-1)-I(T-N[v] ;-1)$. Now $T-v$ has $T_{2}$ as a component, and $T-N[v]$ has $T_{1}$ as a component. As $T_{1}, T_{2} \in \mathcal{A}_{-1}, I(T ;-1)=0$, and thus $I(F ;-1)=0$.

Suppose that $I(F ;-1)=0$. By induction, we will show that $F$ has a component $T$ such that $T \in \mathcal{A}_{-1}$. If $n \in\{1,2,3,4\}$, then it is routine to check that $F$ has a component $T \in\left\{P_{1}, P_{4}\right\}$, and thus $T \in \mathcal{A}_{-1}$. Suppose that if $I(F ;-1)=0$ for a forest $F$ on $1 \leq n^{\prime}<n$ vertices, then $F$ has a component $T$ such that $T \in \mathcal{A}_{-1}$, and let $F$ be a tree on $n$ vertices such that $I(F ;-1)=0$.

Let $x$ be a vertex of degree 1 such that $N(x)=y$. Once again, by Proposition 3.1 $I(F ;-1)=I(F-u ;-1)-I(F-N[u] ;-1)$. Now $F-u$ has $P_{1}$ as a component, and thus $I(F-u ;-1)=0$. Hence $I(F-N[v] ;-1)=0$. By induction, $F-N[v]$ has a component $T$ such that $T \in \mathcal{A}_{-1}$. If $T$ is a component of $F$, then the theorem is verified. Suppose then that $T$ is not a component of $F$. Then there is a vertex $z \in V(T)$ such that $z w \in E(F)$ for some $w \in N(y)$. Hence there is a component of $F, T^{\prime \prime}$, such that $V\left(T^{\prime \prime}\right)=V\left(T^{\prime}\right) \cup V\left(T_{1}\right) \cup V\left(T_{2}\right)$ and $E\left(T^{\prime \prime}\right)=E\left(T^{\prime}\right) \cup E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup\{x y\} \cup\{w z\}$ where $T_{1} \cong P_{1}, T_{2} \cong T$, and $T^{\prime} \cong$ $G[N[u]-v]$. By definition, $T^{\prime \prime} \in \mathcal{A}_{-1}$. By the Principle of Mathematical Induction, the theorem is verified.

In a similar manner, other rational roots for independence polynomials of trees may be found.

Let $\mathcal{A}_{c}$ be the family of trees defined as follows:
(i) Let $T_{c}$ be a smallest tree such that $I\left(T_{c} ; c\right)=0$. Then $T_{c} \in \mathcal{A}_{c}$.
(ii) Let $T^{\prime}, T_{1}, T_{2}$ be trees such that $u v \in E\left(T^{\prime}\right), T_{1}, T_{2} \in \mathcal{A}_{c}$, and $v_{i} \in V\left(T_{i}\right)$ for $i \in\{1,2\}$, and let $T$ be a tree with $V(T)=V\left(T^{\prime}\right) \cup V\left(T_{1}\right) \cup V\left(T_{2}\right)$ and $E(T)=E\left(T^{\prime}\right) \cup E\left(T_{1}\right) \cup$ $E\left(T_{2}\right) \cup\left\{v_{1} u\right\} \cup\left\{v v_{2}\right\}$. Then $T \in \mathcal{A}_{c}$.

Theorem 3.15. Let $T \in \mathcal{A}_{c}$. Then $I(T ; c)=0$.

Proof. Proceed by induction on $|V(T)|$. The smallest tree in $\mathcal{A}_{c}$ is $T_{c}$. In this case, the theorem holds. Let $T \in \mathcal{A}_{c}$, and suppose that for trees in $\mathcal{A}_{c}$ on fewer than $|V(T)|$ the theorem holds. As $T \in \mathcal{A}_{c}$, there exists trees $T^{\prime}, T_{1}$, and $T_{2}$ such that $V(T)=V\left(T^{\prime}\right) \cup$ $V\left(T_{1}\right) \cup V\left(T_{2}\right)$, and $E(T)=E\left(T^{\prime}\right) \cup E\left(T_{1}\right) \cup E\left(T_{2}\right) \cup\left\{v_{1} u\right\} \cup\left\{v v_{2}\right\}$ where $u v \in E\left(T^{\prime}\right)$, $T_{1}, T_{2} \in \mathcal{A}_{c}$, and $v_{i} \in V\left(T_{i}\right)$ for $i \in\{1,2\}$. By Proposition 3.1

$$
I(T ; x)=I(T-v ; x)+x I(T-N[v] ; x) .
$$

Now $T-v$ has $T_{2}$ as a component, and $T-N[v]$ has $T_{1}$ as a component. By induction, as $T_{1}, T_{2} \in \mathcal{A}_{c}, I(T-v ; c)=I(T-N[v] ; c)=0$. Thus $I(T ; c)=I(T-v ; c)+c I(T-N[v] ; c)=0$. Thus by the Principle of Mathematical Induction, if $T \in \mathcal{A}_{c}$, then $I(T ; c)=0$.

By Theorem 3.15, to classify a family of trees whose independence polynomials have $c$ as a root one simply has to find a minimal example of a tree that has $c$ as a root of its independence polynomial. In this sense, $\mathcal{A}_{c}$ is characterized by $T_{c}$. Thus by Theorem 3.12 and Theorem 3.13, $\mathcal{A}_{-\frac{1}{2}}$ and $\mathcal{A}_{-\frac{1}{3}}$ quickly follow.

Theorem 3.16. Let $T_{c}$ be a smallest tree such that $I\left(T_{c} ; c\right)=0$. Then
(i) $T_{-\frac{1}{2}} \cong P_{2}$,
(ii) $T_{-\frac{1}{3}} \cong P_{4}$.

As stated, the only rational roots for the independence polynomials of the path are $-1,-\frac{1}{2}$, and $-\frac{1}{3}$. The smallest examples of a tree in the families of trees $\mathcal{A}_{-1}, \mathcal{A}_{-\frac{1}{2}}$, and $\mathcal{A}_{-\frac{1}{3}}$ are all paths. It may seem that $-1,-\frac{1}{2}$, and $-\frac{1}{3}$ are the only possible rational roots for independence polynomials of trees. However, that is not the case as we will now demonstrate. Let $T_{7,7}$ be the tree in Figure 4.

Theorem 3.17. Let $T_{c}$ be a smallest tree such that $I\left(T_{c} ; c\right)=0$. Then $T_{-\frac{1}{4}} \cong T_{7,7}$.

Proof. The independence polynomial of $T_{7,7}$ is $I\left(T_{7,7} ; x\right)=(1+2 x)^{2}(1+x)^{2}+x(1+x)^{2}=$ $(1+x)^{2}\left(1+5 x+4 x^{2}\right)=(1+x)^{3}(1+4 x)$. Hence $I\left(T_{7,7} ;-\frac{1}{4}\right)=0$. It is easy to verify that for a tree $T \not \approx T_{7,7}$ on $1 \leq n \leq 7$ vertices that $I\left(T ;-\frac{1}{4}\right) \neq 0$.


Figure 4. $T_{7,7}$

The question of what possible rational roots exist for the independence polynomials of paths is closed. However, Theorem 3.17 raises the question as to what are the possible rational roots of independence polynomials of trees. If $T_{-\frac{1}{q}}$ exists for $q \geq 5$ exists, it would be interesting to determine such trees.

Additionally, $I(T ;-1)=0$ if and only if $T \in \mathcal{A}_{-1}$. It would be interesting to determine whether or not "if and only if" statements can be made for $\mathcal{A}_{-\frac{1}{2}}, \mathcal{A}_{-\frac{1}{3}}$, and $\mathcal{A}_{-\frac{1}{4}}$ as well.

## 4. Independent Sets of $k$-Trees and tree-Like $k$-Trees

As discussed in Chapter 3, Wingard determined numerous results in regards to independence polynomials of trees. It is then a natural train of thought to generalize the results of Wingard to independence polynomials of $k$-trees.

### 4.1. The Results of Song et al.

In 2010, Song, Staton, and Wei characterized independence polynomials for certain classes of $k$-trees and $k$-tree related graphs. Among their results, they found the following.

Theorem 4.1. [41] For the $k$-path $P_{n}^{k}$, the following are true:
(i) $\alpha\left(P_{n}^{k}\right)=\left\lfloor\frac{n+1}{k+1}\right\rfloor ;$
(ii) If $1 \leq s \leq \alpha\left(P_{n}^{k}\right)$, then $f_{s}\left(P_{n}^{k}\right)=f_{s}\left(P_{n-1}^{k}\right)+f_{s-1}\left(P_{n-k-1}^{k}\right)$;
(iii) If $0 \leq s \leq \alpha\left(P_{n}^{k}\right)$, then $f_{s}\left(P_{n}^{k}\right)=\binom{n-k(s-1)}{s}$;
(iv) $I\left(P_{n}^{k} ; x\right)=\sum_{s=0}^{\alpha\left(P_{n}^{k}\right)}\binom{n-k(s-1)}{s} x^{s}$.

Theorem 4.2. [41] For the $k$-star $S_{k, n-k}$, the following are true:
(i) $\alpha\left(S_{k, n-k}\right)=n-k$;
(ii) $f_{s}\left(S_{k, n-k}\right)=\binom{n-k}{s}, s \geq 2$;
(iii) $I\left(S_{k, n-k} ; x\right)=k x+(1+x)^{n-k}$.

Theorem 4.3. [40] For the $k$-spiral $S_{n}^{k}$, the following are true:
(i) $\alpha\left(S_{n}^{k}\right)=\left\lfloor\frac{n-k+2}{2}\right\rfloor$;
(ii) $f_{s}\left(S_{n}^{k}\right)=\binom{n+2-k-s}{s}, s \geq 2$;


Theorem 4.4. [41] Let $G$ be a $k$-degenerate graph on $n$ vertices. For $2 \leq s \leq \alpha(G)$, the following are true:
(i) $\binom{n-k(s-1)}{s} \leq f_{s}(G)$;
(ii) $f_{s}(G) \leq\binom{ n-k}{s}$ if $G$ is maximum $k$-degenerate.

Theorem 4.5. [41] If $I(G ; x)=I\left(S_{k, n-k} ; x\right)$ for a graph $G$ of order $n \geq k+1$, then $G \cong$ $S_{k, n-k}$.

Theorem 4.6. [41] If $T_{n}^{k}$ is a $k$-tree with $I\left(T_{n}^{k} ; x\right)=I\left(P_{n}^{k} ; x\right)$ and $\alpha\left(T_{n}^{k}\right) \geq 3$, then $T_{n}^{k} \cong P_{n}^{k}$.

As an extension of the previous results of Song et al., it is not difficult to obtain a similar result for the $k$-diamond.

Theorem 4.7. Let $D_{n}^{k}$ be the $k$-diamond on $n$ vertices. Then
(i) $\alpha\left(D_{n}^{k}\right)=n-k-1$;
(ii) $f_{2}\left(D_{n}^{k}\right)=\binom{n-k-1}{2}+(2 k+2-n)$;
(iii) $f_{s}\left(D_{n}^{k}\right)=\binom{n-k-1}{s}$ for $s \geq 3$;
(iv) $I\left(D_{n}^{k} ; x\right)=1+n x+\left(\binom{n-k-1}{2}+(2 k+2-n)\right) x^{2}+\sum_{s=3}^{n-k-1}\binom{n-k-1}{s} x^{s}$.

Along with the results of Wingard listed in Chapter 3, Wingard determined, in what we refer to as Wingard's bound, sharp bounds of the function values of independence polynomials of trees obtained at $x=-1$, which is now presented.

Theorem 4.8 (Wingard's Bound). [44] Let $T$ be a tree. Then $|I(T ;-1)| \leq 1$.

We seek to generalize Wingard's Bound to $k$-degenerate graphs and thus $k$-trees. We will give Lemma 4.10 which generalizes Proposition 3.1(vi) to vertex sets. This formula may be useful for the study of independence polynomials. As an application, we use Lemma 4.10 to give Theorem 4.11 which generalizes Wingard's Bound to the $k$-path. In Section 4.3, we give Lemma 4.13 which generalizes Proposition 3.1(ix) to edge sets. Through use of Lemma 4.13 we give Theorem 4.14 which generalizes Wingard's Bound to all $k$-degenerate graphs. Though the result of Theorem 4.14 covers the result of Theorem 4.11, both approaches are useful.

### 4.2. Wingard's Bound for the $k$-path and $k$-star.

As mentioned, Song et al. demonstrated that, for a $k$-degenerate graph $G$, the lower bound for $f_{s}(G)$ is obtained uniquely for the class of $k$-trees by the $k$-path, and the upper bound for $f_{s}(G)$ is obtained uniquely for maximal $k$-degenerate graphs by the $k$-star [41]. In this sense, the $k$-path and $k$-star are extremal cases for the number of independent sets among $k$-trees.

We will now generalize Wingard's Bound to the $k$-path and $k$-star. First, we introduce some lemmas.

Lemma 4.9. Let $K_{n}$ be the clique on $n$ vertices. Then $\left|I\left(K_{n} ;-\frac{1}{k}\right)\right|<1$ for $n \leq k$.

Proof. $I\left(K_{n} ; x\right)=1+n x$. Hence $I\left(K_{n} ;-\frac{1}{k}\right)=1-\frac{n}{k}<1$.

Lemma 4.10. Let $G$ be a graph on $n$ vertices, and let $j$ be an integer, $1 \leq j \leq n$. Let $S_{j}=\left\{u_{1}, \ldots, u_{j}\right\} \subseteq V(G)$, and define $S_{i}=\left\{u_{1}, \ldots, u_{i}\right\}$ and $G_{i}=G-S_{i}$ for $1 \leq i \leq j$.

Then

$$
\begin{aligned}
I(G ; x)= & I\left(G_{j} ; x\right)+x I\left(G_{j-1}-N\left[u_{j}\right] ; x\right)+x I\left(G_{j-2}-N\left[u_{j-1}\right] ; x\right) \\
& +\ldots+x I\left(G_{1}-N\left[u_{2}\right] ; x\right)+x I\left(G-N\left[u_{1}\right] ; x\right)
\end{aligned}
$$

Proof. We will proceed by induction on $\left|S_{j}\right|$. If $j=1$, then by Proposition 3.1(vi) $I(G ; x)=$ $I\left(G-v_{1} ; x\right)+x I\left(G-N\left[v_{1}\right] ; x\right)$. Suppose the statement is true for vertex sets with cardinality less than $j$. Then by induction,

$$
I(G ; x)=I\left(G_{j-1} ; x\right)+x I\left(G_{j-2}-N\left[v_{j-1}\right] ; x\right)+\ldots+x I\left(G-N\left[v_{1}\right] ; x\right)
$$

By Proposition 3.1(vi) $I\left(G_{j-1} ; x\right)=I\left(G_{j-1}-u_{j} ; x\right)+x I\left(G_{j-1}-N_{G_{j-1}}\left[u_{j}\right] ; x\right)$, and note that $N_{G_{j-1}}\left[u_{j}\right] \subseteq N\left[u_{j}\right]$. Hence $G_{j-1}-N_{G_{j-1}}\left[u_{j}\right] \cong G_{j-1}-N_{G}\left[u_{j}\right]$. Thus,

$$
\begin{aligned}
I(G ; x)= & I\left(G_{j-1} ; x\right)+x I\left(G_{j-2}-N\left[v_{j-1}\right] ; x\right) \\
& +x I\left(G_{j-3}-N\left[v_{j-2}\right] ; x\right)+\ldots+x I\left(G-N\left[v_{1}\right] ; x\right) \\
= & I\left(G_{j-1}-u_{j} ; x\right)+x I\left(G_{j-1}-N\left[u_{j}\right] ; x\right)+x I\left(G_{j-2}-N\left[u_{j-1}\right] ; x\right) \\
& +\ldots+x I\left(G_{1}-N\left[v_{2}\right] ; x\right)+x I\left(G-N\left[v_{1}\right] ; x\right) \\
= & I\left(G_{j} ; x\right)+x I\left(G_{j-1}-N\left[v_{j}\right] ; x\right)+x I\left(G_{j-2}-N\left[v_{j-1}\right] ; x\right) \\
& +\ldots+x I\left(G_{1}-N\left[v_{2}\right] ; x\right)+x I\left(G-N\left[v_{1}\right] ; x\right) .
\end{aligned}
$$

Hence, the lemma is true for vertex sets of cardinality $j$ for $1 \leq j \leq n$.

As an application of Lemma 4.10, we will now generalize Wingard's Bound to the $k$-path.

Theorem 4.11. Let $P_{n}^{k}$ be the $k$-path on $n \geq k$ vertices and $k \geq 2$. Then

$$
\left|I\left(P_{n}^{k} ;-\frac{1}{k}\right)\right|<1 .
$$

Proof. We will proceed by induction on $n$. If $n=k, P_{n}^{k}$ is a $k$-clique, and the theorem is true by Lemma 4.9. If $n=k+1$, then $P_{n}^{k} \cong K_{k+1}$. Thus $I\left(P_{n}^{k} ;-\frac{1}{k}\right)=1-(k+1) / k=1-1-\frac{1}{k}$.

Suppose the theorem is true for $k$-paths with less than $n \geq k+2$ vertices, and consider $P_{n}^{k}$ on $n$ vertices with $v_{1}, v_{2}, \ldots, v_{n}$ ordered according to a presentation. Let $u_{i}=v_{k+i}$ for $1 \leq i \leq r$ where $r=\min (k, n-k)$, and define $U_{i}=\left\{u_{1}, \ldots, u_{i}\right\}, G_{i}=P_{n}^{k}-U_{i}$ for $1 \leq i \leq r$, and $G_{0}=P_{n}^{k}$. Then by Proposition 3.1(x) and Lemma 4.10,

$$
\begin{align*}
I\left(P_{n}^{k} ; x\right)= & I\left(G_{r} ; x\right)+x I\left(G_{r-1}-N\left[u_{r}\right] ; x\right)+x I\left(G_{r-2}-N\left[u_{r-1}\right] ; x\right)  \tag{3}\\
& +\ldots+x I\left(G_{1}-N\left[u_{2}\right] ; x\right)+x I\left(P_{n}^{k}-N\left[u_{1}\right] ; x\right),
\end{align*}
$$

and this summation has $r+1$ summands on the right hand side.
According to the definition of a $k$-path $G\left[U_{r}\right] \cong K_{r}$, and hence $U_{r} \subseteq N\left[u_{i}\right]$ for $1 \leq i \leq r$. Now $G\left[\left\{v_{1}, \ldots, v_{k}\right\}\right] \cong K_{k}$ is a component of $G_{r}$, and so $I\left(G_{r} ;-\frac{1}{k}\right)=0$. Also, by the structure of the $k$-path, each graph $G_{i}-N\left[u_{i+1}\right]$ for $0 \leq i \leq r-1$ has at most two components $J_{i}$ and $H_{i}$ such that $V\left(J_{i}\right) \subseteq\left\{v_{1}, \ldots, v_{k}\right\}$ and $V\left(H_{i}\right) \subseteq V\left(P_{n}^{k}\right)-\left(\left\{v_{1}, \ldots, v_{k}\right\} \cup U_{r}\right)$. Hence each component $J_{i}$ and $H_{i}$ are congruent to either cliques of smaller size than $k$, the empty graph, or a $k$-path on fewer than $n$ vertices. Hence by Lemma 4.9, Proposition 3.1, and induction, $\left|I\left(G_{i}-N\left[u_{i+1}\right] ;-\frac{1}{k}\right)\right| \leq 1$ for $0 \leq i \leq r-1$ with equality holding if and only if $G_{i}-N\left[u_{i+1}\right]$ is the empty graph. However, if $G_{i}-N\left[u_{i+1}\right]$ is the empty graph for all $i, 0 \leq i \leq r-1$, then $\left\{v_{1}, \ldots, v_{k}\right\} \subseteq N\left[u_{i+1}\right]$ for $0 \leq i \leq r-1$. Then $r=1$, as $u_{1} \notin N\left[u_{2}\right]$. However, $r \geq 2$, so there is a $j$ such that $G_{j}-N\left[u_{j+1}\right]$ is not the empty graph. Thus

$$
\begin{aligned}
\left|I\left(P_{n}^{k} ;-\frac{1}{k}\right)\right| \leq & \left|I\left(G_{r} ;-\frac{1}{k}\right)\right|+\left|\frac{1}{k}\right|\left|I\left(G_{r-1}-N\left[u_{r}\right] ;-\frac{1}{k}\right)\right| \\
& +\ldots\left|\frac{1}{k}\right|\left|I\left(G_{1}-N\left[u_{2}\right] ;-\frac{1}{k}\right)\right|+\left|\frac{1}{k}\right|\left|I\left(G-N\left[u_{1}\right] ;-\frac{1}{k}\right)\right| \\
& <0+r\left(\frac{1}{k}\right) \leq 1 .
\end{aligned}
$$

Therefore, by the Principle of Mathematical Induction, $\left|I\left(P_{n}^{k} ;-\frac{1}{k}\right)\right|<1$.

Using Song's characterization of the independence polynomial of the $k$-star given in Theorem 4.2, we can easily verify the following theorem.

Theorem 4.12. Let $S_{k, n-k}$ be the $k$-star on $n$ vertices and $k \geq 2$. Then

$$
\left|I\left(S_{k, n-k} ;-\frac{1}{k}\right)\right|<1
$$

### 4.3. Wingard's Bound for $k$-degenerate Graphs.

In this section, we seek to generalize Theorem 4.11 and Theorem 4.12 by investigating Wingard's Bound to $k$-degenerate graphs. Though Lemma 4.10 is not sufficient to do so, we will introduce Lemma 4.13, a generalization of Proposition 3.1(ix) that will be useful to generalize Wingard's Bound to $k$-degenerate graphs.

Lemma 4.13. Let $G$ be a graph with $v \in V(G)$ where $\left\{u_{1}, \ldots, u_{r}\right\} \subseteq N(v)$ for some $1 \leq$ $r \leq d(v)$. Let $e_{i}=v u_{i}, E_{i}=\left\{e_{1}, \ldots, e_{i}\right\}$, and $G_{i}^{\prime}=G-E_{i}$ for $1 \leq i \leq r$. Then

$$
\begin{aligned}
I(G ; x)= & I\left(G_{r}^{\prime} ; x\right)-x^{2} I\left(G_{r-1}^{\prime}-N_{G_{r-1}^{\prime}}\left(e_{r}\right) ; x\right)-x^{2} I\left(G_{r-2}^{\prime}-N_{G_{r-2}^{\prime}}\left(e_{r-1}\right) ; x\right) \\
& -\ldots-x^{2} I\left(G_{1}^{\prime}-N_{G_{1}^{\prime}}\left(e_{2}\right) ; x\right)-x^{2} I\left(G-N_{G}\left(e_{1}\right) ; x\right) .
\end{aligned}
$$

Proof. We will proceed by induction on $r$. If $r=1$, by Proposition 3.1 we have the identity $I(G ; x)=I\left(G-e_{1} ; x\right)-x^{2} I\left(G-N\left(e_{1}\right) ; x\right)$. Suppose the statement is true for $1 \leq r^{\prime}<r$. By induction,

$$
\begin{aligned}
I(G ; x)= & I\left(G_{r-1}^{\prime} ; x\right)-x^{2} I\left(G_{r-2}^{\prime}-N_{G_{r-2}^{\prime}}\left(e_{r-1}\right) ; x\right) \\
& -x^{2} I\left(G_{r-3}^{\prime}-N_{G_{r-3}^{\prime}}\left(e_{r-2}\right) ; x\right)-\ldots-x^{2} I\left(G_{1}^{\prime}-N_{G_{1}^{\prime}}\left(e_{2}\right) ; x\right) \\
& -x^{2} I\left(G-N_{G}\left(e_{1}\right) ; x\right) .
\end{aligned}
$$

Now by Proposition 3.1, $I\left(G_{r-1}^{\prime} ; x\right)=I\left(G_{r-1}^{\prime}-e_{r} ; x\right)-x^{2} I\left(G_{r-1}^{\prime}-N_{G_{r-1}^{\prime}}\left(e_{r}\right) ; x\right)$. Thus

$$
\begin{aligned}
I(G ; x)= & I\left(G_{r-1}^{\prime}-e_{r} ; x\right)-x^{2} I\left(G_{r-1}^{\prime}-N_{G_{r-1}^{\prime}}\left(e_{r}\right) ; x\right) \\
& -\ldots-x^{2} I\left(G_{1}^{\prime}-N_{G_{1}^{\prime}}\left(e_{2}\right) ; x\right)-x^{2} I\left(G-N_{G}\left(e_{1}\right) ; x\right) \\
= & I\left(G_{r}^{\prime} ; x\right)-x^{2} I\left(G_{r-1}^{\prime}-N_{G_{r-1}^{\prime}}\left(e_{r}\right) ; x\right)-x^{2} I\left(G_{r-2}^{\prime}-N_{G_{r-2}^{\prime}}\left(e_{r-1}\right) ; x\right) \\
& -\ldots-x^{2} I\left(G_{1}^{\prime}-N_{G_{1}^{\prime}}\left(e_{2}\right) ; x\right)-x^{2} I\left(G-N_{G}\left(e_{1}\right) ; x\right) .
\end{aligned}
$$

Hence, the lemma is true by the Principle of Mathematical Induction.

We will now, with the help of Lemma 4.13, generalize Wingard's Bound to $k$-degenerate graphs and thus $k$-trees.

Theorem 4.14. Let $G$ be a $k$-degenerate graph on $n \geq 1$ vertices with $k \geq 2$. Then $\left|I\left(G ;-\frac{1}{k}\right)\right|<1$.

Proof. We will proceed by induction on $n$. If $n=1$, then $I(G ; x)=1+x$. We see, then, that $I\left(G ;-\frac{1}{k}\right)=\frac{k-1}{k}$.

Suppose the theorem is true for $k$-degenerate graphs of order less than $n$, and consider $G$, a $k$-degenerate graph on $n$ vertices. As $G$ is $k$-degenerate, $\delta \leq k$. Choose $v \in V(G)$ such that $d(v)=\delta$ and $v$ is incident to edges $e_{1}, e_{2}, \ldots, e_{\delta}$. Let $E_{i}=\left\{e_{1}, \ldots, e_{i}\right\}$ and $G_{i}^{\prime}=G-E_{i}$ for $1 \leq i \leq \delta$.

Then by Lemma 4.13,

$$
\begin{align*}
I(G ; x)= & I\left(G_{\delta}^{\prime} ; x\right)-x^{2} I\left(G_{\delta-1}^{\prime}-N_{G_{\delta-1}^{\prime}}\left(e_{\delta}\right) ; x\right)-x^{2} I\left(G_{\delta-2}^{\prime}-N_{G_{\delta-2}^{\prime}}\left(e_{\delta-1} ; x\right)\right.  \tag{4}\\
& -\ldots-x^{2} I\left(G_{1}^{\prime}-N_{G_{1}^{\prime}}\left(e_{2}\right) ; x\right)-x^{2} I\left(G-N\left(e_{1}\right) ; x\right),
\end{align*}
$$

and the right hand side has $\delta+1 \leq k+1$ summands.

Now $G_{\delta}^{\prime}$ has a component of order one, and the other components of $G_{\delta}^{\prime}$ are $k$-degenerate graphs. So $\left|I\left(G_{\delta}^{\prime} ;-\frac{1}{k}\right)\right|<\left(1-\frac{1}{k}\right)(1)=\frac{k-1}{k}$. Also each of the components of $G_{i}^{\prime}-N_{G_{i}^{\prime}}\left(e_{i+1}\right)$ for $1 \leq i \leq \delta-1$ and $G-N\left(e_{1}\right)$ is either the empty graph or a $k$-degenerate graph on at least one vertex and on fewer than $n$ vertices.

Hence, by applying the induction hypothesis and (4),

$$
\begin{aligned}
\left|I\left(G ;-\frac{1}{k}\right)\right| \leq & \left|I\left(G_{\delta}^{\prime} ;-\frac{1}{k}\right)\right|+\left|\left(-\frac{1}{k}\right)^{2}\right|\left|I\left(G_{\delta-1}^{\prime}-N_{G_{\delta-1}^{\prime}}\left(e_{\delta}\right) ;-\frac{1}{k}\right)\right| \\
& +\left|\left(-\frac{1}{k}\right)^{2}\right|\left|I\left(G_{\delta-2}^{\prime}-N_{G_{\delta-2}^{\prime}}\left(e_{\delta-1}\right) ;-\frac{1}{k}\right)\right|+\ldots \\
& +\left|\left(-\frac{1}{k}\right)^{2}\right|\left|I\left(G_{1}^{\prime}-N_{G^{\prime}-1}\left(e_{2}\right) ;-\frac{1}{k}\right)\right|+\left|\left(-\frac{1}{k}\right)^{2}\right|\left|I\left(G-N_{G}\left(e_{1}\right) ;-\frac{1}{k}\right)\right| \\
< & \frac{k-1}{k}+\left|\frac{1}{k^{2}}\right||1|+\ldots+\left|\frac{1}{k^{2}}\right||1| \\
= & \frac{k-1}{k}+\delta \frac{1}{k^{2}} \leq \frac{k-1}{k}+k \frac{1}{k^{2}}=1 .
\end{aligned}
$$

Therefore, by the Principle of Mathematical Induction, $\left|I\left(G ;-\frac{1}{k}\right)\right|<1$ for $G$ a $k$-degenerate graph.

Corollary 4.15. Let $T_{n}^{k}$ be a $k$-tree on $n$ vertices and $k \geq 2$. Then $\left|I\left(T_{n}^{k} ;-\frac{1}{k}\right)\right|<1$.

Proof. All $k$-trees are $k$-degenerate.

We note that Wingard's bound is achieved when $k=1$. In particular, there are examples of trees such that $|I(T ;-1)|=1$; for example $S_{1, n-1}$. However, for $k$-degenerate graphs with $k \geq 2$, Wingard's Bound is strict.

### 4.4. The Fibonacci Number of Maximal Outerplanar Graphs.

As has been mentioned, Song et al. determined that $f_{s}\left(S_{k, n-k}\right)$ is a strict upper bound among $k$-trees for $s \geq 3$ that is uniquely obtained by $S_{k, n-k}$. For $n \geq k+3, S_{k, n-k}$ is not
tree-like. Thus for tree-like $k$-trees, we seek a stricter upper bound of $f_{s}$ for $s \geq 0$ than the one provided by Song et al..

In 1998, Alameddine determined sharp bounds of the Fibonacci number of maximal outerplar graphs and characterized the unique maximal outerplar graphs that obtained these bounds. He found the following:

Theorem 4.16. [1] Let $G$ be a maximal outerplanar graph on $n \geq 3$ vertices. Then $f(G) \geq$ $f\left(P_{n}^{2}\right)$, and equality is reached if and only if $G \cong P_{n}^{2}$.

Theorem 4.17. [1] Let $G$ be a maximal outerplanar graph on $n \geq 3$ vertices. Then $f(G) \leq$ $f\left(S_{n}^{2}\right)$, and equality is reached if and only if $G \cong S_{n}^{2}$.

We note for $n=6, f\left(S_{6}^{2}\right)=f\left(D_{6}^{2}\right)=14$, and thus Theorem 4.17 is not complete. We will demonstrate a revision of the results of Alameddine including this special case through investigating lower and upper bounds of the coefficients of $I(G ; x), f_{s}(G)$ for $s \geq 0$. Additionally, we will classify the unique graphs that obtain these bounds.

As the $k$-path is a tree-like $k$-tree, it is clear by Theorem 4.4 and Theorem 4.6 that the lower bound of $f_{s}$ for maximal outerplar graphs for $s \geq 3$ immediately follows by the results of Song et al.. with $k=2$. We only need to consider the upper bound.


Figure 5. Maximal outerplanar graphs on $n=7$ vertices

Theorem 4.18. Let $G$ be a maximal outerplanar graph on $n \geq 6$ vertices. Then for all $s \geq 3$,

$$
f_{s}(G) \leq\binom{ n-s}{s}
$$

and equality holds if and only if $G \in\left\{S_{6}^{2}, D_{6}^{2}\right\}$ for $n=6$ and $G \cong S_{n}^{2}$ for $n \geq 7$.

Proof. Suppose $n=6$. Then $G \in\left\{P_{6}^{2}, S_{6}^{2}, D_{6}^{2}\right\}$. Let $n=7$. Then $G \in\left\{P_{7}^{2}, G_{1}, G_{2}, S_{7}^{2}\right\}$ as pictured in Figure 5. Routine calculations show that for $n \in\{6,7\}, \alpha(G) \leq 3, f_{3}\left(P_{6}^{2}\right)=0$, $f_{3}\left(D_{6}^{2}\right)=f_{3}\left(S_{6}^{2}\right)=1=\binom{6-3}{3}, f_{3}\left(P_{7}^{2}\right)=1, f_{3}\left(G_{1}\right)=2, f_{3}\left(G_{2}\right)=3$, and $f_{3}\left(S_{7}^{2}\right)=4=\binom{7-3}{3}$. Thus the theorem holds for $n \in\{6,7\}$.

Suppose that for maximal outerplanar graphs on $7 \leq n^{\prime}<n$ vertices the theorem holds, and let $G$ be a maximal outerplanar graph on $n \geq 8$ vertices. Let $v \in V(G)$ such that $d(v)=2$ and $N(v)=\left\{u_{1}, u_{2}\right\}$. By Proposition 3.1(iv)

$$
\begin{equation*}
f_{s}(G)=f_{s}(G-v)+f_{s-1}(G-N[v]), \tag{5}
\end{equation*}
$$

and as $G-v$ is a maximal outerplanar graph by induction, $f_{s}(G-v) \leq f_{s}\left(S_{n-1}^{2}\right)=\binom{n-1-s}{s}$.
Now $G$ has a hamiltonian cycle $C$ that passes through all of the unbound edges of $G$. Thus $u_{1} v u_{2}$ is a segment of $C$, and so $G-N[v]$ has a spanning path on $n-3$ vertices, namely $C-N[v]$. By Proposition 3.1, $f_{s-1}(G-N[v]) \leq f_{s-1}\left(P_{n-3}\right)=\binom{n-3+1-(s-1)}{s-1}=\binom{n-1-s}{s-1}$.

Thus by induction and (5),

$$
\begin{aligned}
f_{s}(G) & =f_{s}(G-v)+f_{s-1}(G-N[v]) \\
& \leq f_{s}\left(S_{n-1}^{2}\right)+f_{s-1}\left(P_{n-3}\right) \\
& =\binom{n-1-s}{s}+\binom{n-1-s}{s-1} \\
& =\binom{n-s}{s},
\end{aligned}
$$

and for $s \geq 3$ equality holds if and only if $G-v \cong S_{n-1}^{2}$ and $G-N[v] \cong P_{n-3}$, i.e. $G \cong S_{n}^{2}$.

As a corollary, we obtain the following modified result of Alameddine.

Corollary 4.19. Let $G$ be a maximal outerplanar graph on $n \geq 6$ vertices such that $I(G ; x)=$ $I\left(S_{n}^{2} ; x\right)$. Then, if $n=6, G \in\left\{D_{6}^{2}, S_{6}^{2}\right\}$, and if $n \geq 7, G \cong S_{n}^{2}$.

Corollary 4.20. Let $G$ be a maximal outerplanar graph on $n \geq 6$ vertices. Then $f(G) \leq$ $f\left(S_{n}^{2}\right)$. Equality is reached if and only if $G \in\left\{D_{6}^{2}, S_{6}^{2}\right\}$ when $n=6$ and if and only if $G \cong S_{n}^{2}$ when $n \geq 7$.


Figure 6. A tree-like 3-tree on 11 vertices $G$

### 4.5. Independent Sets of Cardinality $s$ in Chordal Planar Graphs with Toughness

## Exceeding 1.

It should be noted that for the general $k \geq 3$, there is a tree-like $k$-tree $T_{n}^{k}$ such that $f_{s}\left(T_{n}^{k}\right)>\binom{n-k(s-1)}{s}$ for some $s \geq 0$. As an example, $f_{6}(G)=1$ for the tree-like 3 -tree in Figure 6 whereas $f_{6}\left(S_{11}^{3}\right)=0$.

As mentioned in Chapter 2, a graph is a tree-like 3-trees with toughness exceeding 1 if and only if it is a chordal planar graph with toughness exceeding 1 . Let $\mathcal{L}$ be the set of chordal planar graphs with toughness exceeding 1 . Though there are tree-like $k$-trees with
$k \geq 3$ with $f_{s}$ greater than $\binom{n-k(s-1)}{s}$ for some $s \geq 0$, for the class $\mathcal{L}$ we state the following theorem.

Theorem 4.21. Let $G \in \mathcal{L}$ on $n \geq 7$ vertices. Then for all $s \geq 3$,

$$
f_{s}(G) \leq\binom{ n-1-s}{s}
$$

and equality holds if and only if $G \in\left\{S_{7}^{3}, D_{7}^{3}\right\}$ for $n=7$ and $G \cong S_{n}^{3}$ for $n \geq 8$.

In order to prove Theorem 4.21, we must first show that for $G \in \mathcal{L}$ with $v \in S_{1}(T)$, $G-N[v]$ has a spanning path which will now be presented.

Lemma 4.22. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k$ vertices such that $e=x y \in E\left(T_{n}^{k}\right)$. Then there exists a simplicial elimination ordering $v_{1}, \ldots, v_{n}$ of $T_{n}^{k}$ such that $x=v_{n}$ and $y=v_{n-1}$.

Proof. The vertices $x$ and $y$ are in a $k$-clique $D$ in $G$. Let $V(D)=\left\{x_{1}, \ldots, x_{k}\right\}$, and let $x=x_{k}$ and $y=x_{k-1}$. For $n \geq 5$, there exists a simplicial vertex $v \notin V(D)$. Let $v_{1} \in S_{1}\left(T_{n}^{k}\right)-V(D)$, and $v_{i} \in S_{1}\left(T_{n}^{k}-\left\{v_{1}, \ldots, v_{i-1}\right\}\right)-V(D)$ for $1 \leq i \leq n-k$. Then $V\left(T_{n}^{k}-\left\{v_{1}, \ldots, v_{n-k}\right\}\right)=V(D)$. Then without loss of generality $x_{i}=v_{n-k-i}$ for $1 \leq i \leq k$. Hence $x=v_{n}$ and $y=v_{n-1}$.

Theorem 4.23. Let $G \in \mathcal{L}$ on $n \geq 4$ vertices, and let $e \in E(G)$. Then $G$ has a hamiltonian cycle passing through e.

Proof. Let $e=x y$, and let $v_{1}, v_{2}, \ldots, v_{n}$ be a simplicial elimination ordering such that $x=v_{n}$ and $y=v_{n-1}$. Let $G_{0} \cong T_{n}^{k}, G_{i} \cong G-\left\{v_{1}, \ldots, v_{i}\right\}$ for $1 \leq i \leq n-3$. Note then that $e \in E\left(G_{i}\right)$ for $0 \leq i \leq n-3$. Then $v_{i} \in S_{1}\left(G_{i-1}\right)$, and so $D_{i}=G\left[N_{G_{i-1}}\left(v_{i}\right)\right] \cong K_{3}$ for $1 \leq i \leq n-3$. We say that the 3 -clique $D$ is active in $G_{i}$ if $D$ is unbound in $G_{i}$ but bound in
$G$. As $G$ is tree-like, it is clear that if $D$ is bound, there are exactly two vertices $u_{1}$ and $u_{2}$ such that $V(D) \subseteq N\left(u_{i}\right)$ for $1 \leq i \leq 2$. Then clearly $D_{i}$ is active in $G_{i-1}$ for $1 \leq i \leq n-3$. A good cycle pair of $G_{i}$ is an ordered pair $(C, f)$ where $C$ is a hamiltonian cycle of $G_{i}$, and $f$ is an injection that maps every active 3-clique $D$ of $G_{i}$ onto an edge $f(D)$ such that $f(D) \in E(C \cap D)-e$. We claim that for every $i \in\{0, \ldots, n-4\}$, there is a good cycle pair $\left(C_{i}, f_{i}\right)$ of $G_{i}$. We will proceed by induction on $n-i$.

Suppose $i=4$. Then $G_{n-4} \cong K_{4}$ with vertex set $\left\{v_{n}, v_{n-1}, v_{n-2}, v_{n-3}\right\}$. Then $G_{n-4}$ has four 3-cliques $D_{0}^{\prime}, D_{1}^{\prime}, D_{2}^{\prime}$, and $D_{3}^{\prime}$ such that $V\left(D_{j}^{\prime}\right)=\left\{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\right\}-\left\{v_{n-j}\right\}$ for $0 \leq j \leq 3$. At most three of these cliques are active as otherwise $G$ has toughness at most 1. Let $C_{n-4}=v_{n} v_{n-1} v_{n-2} v_{n-3} v_{n}$, and define $f_{n-4}$ as in Table 4.5. Then $f_{n-4}$ is an injection between all active 3 -cliques of $G_{n-4}$ and edges of $C_{n-4}$. Thus $G_{n-4}$ has a good cycle pair $\left(C_{n-4}, f_{n-4}\right)$.

Suppose that $G_{n-i}$ has a good cycle pair $\left(C_{n-i}, f_{n-i}\right)$ for $4 \leq i<r \leq n$, and consider $G_{n-r}$. Suppose $D_{n-r}$ is bound in $G_{n-r}$. Then there are two vertices $\left\{u_{1}, u_{2}\right\} \in V\left(G_{n-r}\right)$ such that $V\left(D_{n-r}\right) \subseteq N_{G_{n-r}}\left(u_{i}\right)$ for $1 \leq i \leq 2$. Then $V\left(D_{n-r}\right) \subseteq\left(N_{G_{n-r-1}}\left(u_{1}\right) \cap N_{G_{n-r-1}}\left(u_{2}\right) \cap\right.$

| Suppose $D_{j}^{\prime}$ is not active | $f_{n-4}\left(D_{j+1}^{\prime}\right)$ | $f_{n-4}\left(D_{j+2}^{\prime}\right)$ | $f_{n-4}\left(D_{j+3}^{\prime}\right)$ |
| :---: | :---: | :---: | :---: |
| $D_{0}^{\prime}$ | $x_{n-2} x_{n-3}$ | $x_{n-3} x_{n}$ | $x_{n-1} x_{n-2}$ |
| $D_{1}^{\prime}$ | $x_{n-3} x_{n}$ | $x_{n-1} x_{n-2}$ | $x_{n-2} x_{n-3}$ |
| $D_{2}^{\prime}$ | $x_{n-1} x_{n-2}$ | $x_{n-2} x_{n-3}$ | $x_{n-3} x_{n}$ |
| $D_{3}^{\prime}$ | $x_{n-1} x_{n-2}$ | $x_{n-2} x_{n-3}$ | $x_{n-3} x_{n}$ |

TABLE 3. $f_{n-4}$
$N_{G_{n-r-1}}\left(v_{n-r}\right)$ ), and so $G$ is not tree-like. Thus $D_{n-r}$ is unbound in $G_{n-r}$ and bound in $G_{n-r-1}$. Hence $D_{n-r}$ is active in $G_{n-r}$.

By the inductive hypothesis there is a hamiltonian cycle pair $C_{n-r}$ passing through $e$ and a injection $f_{n-r}$ of $G_{n-r}$ mapping all active 3-cliques of $G_{n-r}$ to an edge of $C_{n-r}-e$. Let $V\left(D_{n-r}\right)=\{a, b, c\}$ and $f_{n-r}\left(D_{n-r}\right)=a b$. Then, by induction $a b \neq e$, and as $D_{n-r}$ is bound in $G_{n-r-1} D_{n-r}$ is not active in $G_{n-r-1}$. There are exactly three new 3-cliques in $G_{n-r-1}: C_{a}$, $C_{b}$, and $C_{c}$ such that $V\left(C_{a}\right)=\left\{v_{n-r}, b, c,\right\}, V\left(C_{b}\right)=\left\{v_{n-r}, c, a\right\}$, and $V\left(C_{c}\right)=\left\{v_{n-r}, a, b\right\}$. If all three of these 3-cliques are active in $G_{n-r-1}$, then $G-\left\{v_{n-r}, a, b, c\right\}$ has four components. Thus, in this case $G$ is at most 1-tough. Thus at most two of $C_{a}, C_{b}$, and $C_{c}$ are active in $G_{n-r-1}$. We may assume that $C_{c}$ is not active in $G_{n-r-1}$.

Let $C_{n-r-1}$ be the hamiltonian cycle of $G_{n-r-1}$ obtained from $C_{n-r}$ by replacing the edge $a b$ by the path $a v_{n-r} b$, and define $f_{n-r-1}$ as follows. Let $D$ be an active 3 -clique of $G_{n-r-1}$. If $D$ is a subgraph of $G_{n-r}$, then $D$ is an active 3-clique of $G_{n-r}$. Let $f_{n-r-1}(D)=f_{n-r}(D)$. For $z \in\{a, b\}$ and $z^{\prime} \in\{a, b\}-z$, let $f_{n-r-1}\left(C_{z}\right)=v_{n-r} z^{\prime}$ if $C_{z}$ is an active 3-clique of $G_{n-r-1}$. Thus $G_{n-r-1}$ has a hamiltonian cycle $C_{n-r-1}$ that passes through $e$ and an injection $f_{n-r-1}$ that maps every active 3-clique to an edge $f_{n-r-1}(D)$ such that $f_{n-r-1}(D) \in E\left(C_{n-r-1}\right)-e$. That is, $G_{n-r-1}$ has a good cycle pair.

Thus by the Principle of Mathematical Induction, $G$ has a hamiltonian cycle that passes through $e$.

With use of Theorem 4.23, we are now able to prove Theorem 4.21.

Proof. If $n=7$, then $G \in\left\{P_{7}^{3}, S_{7}^{3}, D_{7}^{3}\right\}$. If $n=8$, then $G \in\left\{P_{8}^{3}, G_{1}, G_{2}, G_{3}, G_{4}, S_{8}^{3}\right\}$ where $G_{1}, G_{2}, G_{3}$, and $G_{4}$ are the graphs in Figure 7. It is routine to deduce that if $n=8$, then $\alpha(G) \leq 3$ and $f_{3}\left(P_{8}^{3}\right)=0, f_{3}\left(G_{1}\right)=2, f_{3}\left(G_{2}\right)=3, f_{3}\left(G_{3}\right)=2, f_{3}\left(G_{4}\right)=1$, and $f_{3}\left(S_{8}^{3}\right)=4$.


Figure 7. Tree-like 3 -trees on 8 vertices with $\tau>1$

Proceed by induction on $n$. Suppose that the theorem holds for chordal planar graphs with toughness exceeding 1 on $8 \leq n^{\prime}<n$ vertices, and consider $G \in \mathcal{L}$ on $n$ vertices. As $G$ is a tree-like 3-tree, there exists $v \in S_{1}(G)$. Then by Proposition 3.1, $f_{s}(G)=$ $f_{s}(G-v)+f_{s-1}(G-N[v])$ for $s \geq 1$.

Now $G-v \in \mathcal{L}$ on $n-1$ vertices, and so by induction $f_{s}(G-v) \leq\binom{ n-1+2-3-s}{s}=\binom{n-2-s}{s}$ for $s \geq 0$ with equality holding if and only if $G-v \cong S_{n-1}^{3}$. As $G[N(v)]$ is a bound $k$-clique, $G-N(v)$ has two components; one being $v$. Hence $G-N[v]$ is connected. By Proposition 3.3, $f_{s}(G-N[v])$ for $s \geq 0$ is maximized when $E(G-N[v])$ is minimal i.e. $G-N[v]$ is a tree.

Suppose that $G-N[v]$ is a tree with at least 3 leaves. Then there is a vertex $u \in$ $V(G-N[v])$ such that $d_{G-N[v]}(u) \geq 3$. However, in this case, $G-(N(v) \cup\{u\})$ has at least four components. Thus $\tau(G) \leq 1$, a contradiction. Hence if $G-N[v]$ is a tree, then $G-N[v] \cong P_{n-4}$. Thus $f_{s}(G-N[v]) \leq f_{s}\left(P_{n-4}\right)=\binom{n-4+1-s}{s}=\binom{n-3-s}{s}$ for $s \geq 0$.

Thus

$$
\begin{aligned}
f_{s}(G) & =f_{s}(G-v)+f_{s-1}(G-N[v]) \\
& \leq\binom{ n-2-s}{s}+\binom{n-3-(s-1)}{s-1}
\end{aligned}
$$

$$
\begin{aligned}
& =\binom{n-2-s}{s}+\binom{n-2-s}{s-1} \\
& =\binom{n-1-s}{s}
\end{aligned}
$$

for $s \geq 0$ with equality holding if and only if $G-v \cong S_{n-1}^{3}$ and $G-N[v] \cong P_{n-4}$. Hence equality holds if and only if $G \cong S_{n}^{3}$.


Figure 8. A tree-like 4-tree on 10 vertices with toughness exceeding 1

Now Theorem 4.21 can not be directly generalized to tree-like $k$-trees for $k \geq 4$. For example, the 4 -tree in Figure 8 is tree-like with toughness exceeding 1 on 10 vertices, and $f_{4}$ of this 4 -tree is 2 while $f_{4}\left(S_{10}^{4}\right)=1$.

### 4.6. Independent Sets of Cardinality $s$ of Path-like $k$-trees.

Though is is not clear how to generalize Theorem 4.4 to tree-like $k$-trees for the general $k$, we may demonstrate a stricter upper bound of $f_{s}$ for $s \geq 3$ for path-like $k$-trees. We will now demonstrate this strict upper bound in a similar manner to Theorem 4.21.

Lemma 4.24. Let $T_{n}^{k}$ be a path-like $k$-tree on $n$ vertices and $v \in S_{1}\left(T_{n}^{k}\right)$. Then $T_{n}^{k}-N[v]$ has a spanning path.

Proof. Let $\left\{v_{1}, v_{2}, \ldots, v_{n}\right\}$ be a simplicial elmination ordering such that $v=v_{n}$ and $\left\{v_{n-1}, \ldots\right.$, $\left.v_{n-k-1}\right\}=N(v)$. As $G[N[v]] \cong K_{k+1}$, such an ordering exists. Let $G_{0} \cong T_{n}^{k}$ and $G_{i} \cong$
$T_{n}^{k}-\left\{v_{1}, \ldots, v_{i}\right\}$ for $1 \leq i \leq n$. Clearly $v_{i+1}$ is simplicial in $G_{i}$ for $1 \leq i \leq n-k-2$. If $v_{i+1} \notin N\left(v_{i}\right)$, then $S_{1}\left(G_{i}\right)=\left\{v, v_{i+1}, v_{i}\right\}$. Hence $v_{i+1} v_{i} \in E\left(T_{n}^{k}\right)$ for $1 \leq i \leq n-k-2$. It immediately follows that $T_{n}^{k}-N[v]$ has a spanning path.

Theorem 4.25. Let $T_{n}^{k}$ be a path-like $k$-tree on $n$ vertices. Then for $s \geq 3$,

$$
\binom{n-k(s-1)}{s} \leq f_{s}\left(T_{n}^{k}\right) \leq\binom{ n+2-k-s}{s}
$$

with the left inequality holding if and only if $T_{n}^{k} \cong P_{n}^{k}$, and the right inequality holding if and only $T_{n}^{k} \cong S_{n}^{k}$.

Proof. The lower bound was shown to be true by Song et. al [41]. Thus, we only need to show the upper bound. Proceed by induction on $n$. If $k \leq n \leq k+3$, then $T_{n}^{k}$ is unique, and the theorem is true. Suppose that for path-like $k$-trees on $k \leq n^{\prime}<n$ vertices, the theorem holds, and consider $T_{n}^{k}$, a path-like $k$-tree on $n$ vertices.

Let $v_{1}, v_{2}, \ldots, v_{n}$ be a presentation of $T_{n}^{k}$. Then $v_{n} \in S_{1}\left(T_{n}^{k}\right)$. By Proposition 3.1, for $s \geq 1$

$$
f_{s}\left(T_{n}^{k}\right)=f_{s}\left(T_{n}^{k}-v_{n}\right)+f_{s-1}\left(T_{n}^{k}-N\left[v_{n}\right]\right) .
$$

Now $T_{n}^{k}-v_{n}$ is a path-like $k$-tree on $n-1$ vertices. Hence, by induction $f_{s}\left(T_{n}^{k}-v_{n}\right) \leq\binom{ n+1-k-s}{s}$ for $s \geq 0$ with equality holding if and only if $T_{n}^{k} \cong S_{n}^{k}$.

Now by Lemma 4.24, $T_{n}^{k}-N\left[v_{n}\right]$ has a spanning path. As $G\left[N\left[v_{n}\right]\right]$ is a bound $(k+1)$ clique and $v_{n} \in S_{1}\left(T_{n}^{k}\right)$, it is clear that $T_{n}^{k}-N\left[v_{n}\right]$ is connected. Thus by Proposition 3.3 $f_{s}\left(T_{n}^{k}-N\left[v_{n}\right]\right) \leq\binom{ n-k+1-s}{s}$ for $s \geq 0$ with equality holding if and only if $T_{n}^{k}-N\left[v_{n}\right] \cong P_{n-k-1}$.

Thus,

$$
\begin{aligned}
f_{s}\left(T_{n}^{k}\right) & =f_{s}\left(T_{n}^{k}-v_{n}\right)+f_{s-1}\left(T_{n}^{k}-N\left[v_{n}\right]\right) \\
& \leq\binom{ n+1-k-s}{s}+\binom{n-k+1-(s-1)}{s-1}
\end{aligned}
$$

$$
=\binom{n-k+2-s}{s}
$$

with equality holding if and only if $T_{n}^{k}-v_{n} \cong S_{n-1}^{k}$ and $T_{n}^{k}-N\left[v_{n}\right] \cong P_{n-k-1}$. Thus equality holds if and only if $T_{n}^{k} \cong S_{n}^{k}$, and by the Principle of Mathematical Induction the theorem holds.

## 5. The Zagreb Indices of $k$-trees

In 1975, Randić introduced the branching index which later became known as the Randić connectivity index [36]. The Randić connectivity index is mostly used as a molecular discriptor in computational chemistry describing nonempirical quantitative structure-property relationships and quantitative structure-activity relationships [17]. However, mathematicians have also expressed interest in the Randić connectivity index [6].

The Randić connectivity index has been generalized as the general Randić connectivity index and the general zeroth-order Randić connectivity index, where the Zagreb indices appeared as a special case [8]. The first Zagreb index $M_{1}(G)$ and the second Zagreb index $M_{2}(G)$ of the graph $G$ are given by:

$$
M_{1}(G)=\sum_{u \in V(G)} d(u)^{2}, \quad M_{2}(G)=\sum_{u v \in E(G)} d(u) d(v) .
$$

The Zagreb indices $M_{1}$ and $M_{2}$ have been an active area of research going back to 1972 in the report of Gutman and Trinajstić in computational chemistry [23].

In regards to the Zagreb indices, there are two classical problems which have attracted the attention of researchers for some time:
(i) How $M_{1}(G)$ (respectively $M_{2}(G)$ ) depends on the structure of $G$.
(ii) Given a set of graphs $\mathcal{G}$, find upper and lower bounds for $M_{1}(G)$ and $M_{2}(G)$ of graphs in $\mathcal{G}$ and characterize the graphs in which the maximal (respectively minimal) $M_{1}$ and $M_{2}$ values are attained.

There have been numerous studies in the literature of the properties of Zagreb indices of given graph classes [11, 13, 29, 31]. In particular, Das and Gutman in 2004 characterized the Zagreb indices for trees and determined the unique tree that obtains minimum $M_{1}$ and $M_{2}$ values respectively, as well as maximum $M_{1}$ and $M_{2}$ values respectively.

Theorem 5.1. [12, 20] Let $T$ be any tree of order $n$. Then
(i) $4 n-6 \leq M_{1}(T) \leq n^{2}-n$, the left equality holds if and only if $T \cong P_{n}$, and the right equality holds if and only if $T \cong S_{n}$.
(ii) $4 n-8 \leq M_{2}(T) \leq n^{2}-2 n+1$, the left equality holds if and only if $T \cong P_{n}$ and the right equality holds if and only if $T \cong S_{n}$.


Figure 9. The path and star on 7 vertices

In 2011, Hou et al. characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum $M_{1}$ and $M_{2}$ values respectively, as well as maximum $M_{1}$ and $M_{2}$ values respectively. Hou et al. found the following:

Theorem 5.2. [29] Let $G$ be a maximal outerplaner graph on $n \geq 4$ vertices. Then
(i) $M_{1}(G) \geq 16 n-38$, with equality holding if and only if $G \cong P_{n}^{2}$.
(ii) $M_{2}(G) \geq 32 n-100$, with equality holding if and only if $G \cong P_{n}^{2}$.

Theorem 5.3. [29] Let $G$ be a maximal outerplanar graph on $n \geq 4$ vertices. Then
(i) When $n=6, M_{1}(G) \leq 60$ with equality if and only if $G \cong S_{6}^{2}$ or $D_{6}^{2}$.
(ii) When $n \neq 6, M_{1}(G) \leq n^{2}+7 n-18$ with equality if and only if $G \cong S_{n}^{2}$.

Theorem 5.4. [29] Let $G$ be a maximal outerplanar graph on $n \geq 4$ vertices.
(i) When $n=6, M_{2}(G) \leq 96$ with equality if and only if $G \cong D_{6}^{2}$.
(ii) When $n \neq 6, M_{2}(G) \leq 3 n^{2}+n-19$ with equality if and only if $G \cong S_{n}^{2}$.

As $k$-trees are a generalization of trees and maximal outerplanar graphs, it is a natural connection to generalize the results of Das and Gutman, as well as the results of Hou et al., to the broader class of $k$-trees.

### 5.1. Some Lemmas.

In this section, we give some lemmas that are critical in subsequent sections. For the remainder of this chapter, let $T_{n}^{k}$ be a $k$-tree on $n$ vertices, and let $v \in S_{1}\left(T_{n}^{k}\right)$ such that $N(v)=U=\left\{u_{1}, \ldots, u_{k}\right\}$. Then $T_{n}^{k}-v$ is a $k$-tree. Let $V\left(T_{n}^{k}\right)=\{v\} \cup U \cup X \cup Y$ where $X=N(U)-N[v]$ and $Y=V\left(T_{n}^{k}\right)-X-N[v]$. Let $|X|=l$ and $X=\left\{x_{1}, \ldots, x_{l}\right\}$. Then $l \geq \min (n-k-1, k)$. Arrange the vertices of $X$ such that $x_{i} \in U$ for $1 \leq i \leq j$ and $\left|N\left(x_{i}\right) \cap U\right| \geq\left|N\left(x_{i+1}\right) \cap U\right|$ for $j+1 \leq i \leq l-1$.

If $n \geq k+2$, then $\left|S_{1}\left(T_{n}^{k}\right)\right| \geq 2$. Thus if $n \geq k+2$, there exists $v^{\prime} \in S_{1}\left(T_{n}^{k}\right)-v$. Choose $v^{\prime}$ such that $\left|N\left(v^{\prime}\right) \cap U\right|=t$ is as small as possible, and let $N\left(v^{\prime}\right)=U^{\prime}=\left\{u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right\}$. Arrange the vertices of $U^{\prime}$ such that $u_{i}^{\prime} \in U$ for $1 \leq i \leq t$ and $\left|N\left(u_{i}^{\prime}\right) \cap U\right| \geq\left|N\left(u_{i+1}^{\prime}\right) \cap U\right|$ for $t+1 \leq i \leq k$. Let $f: U^{\prime} \rightarrow \mathbb{N}$ where $f\left(u_{i}^{\prime}\right)=\left\{\begin{array}{ll}0 & \text { if } u_{i}^{\prime} \in U \\ \left|N\left(u_{i}^{\prime}\right) \cap U\right| & \text { if } u_{i}^{\prime} \notin U\end{array}\right.$. Let $d^{*}\left(v_{j}\right)$
(respectively $d_{*}\left(v_{j}\right)$ ) be the degree obtained by vertex $v_{j}$ of a presentation of $P_{n}^{k}$ (respectively $\left.P_{n-1}^{k}\right)$.

Then we may state the following lemmas.

Lemma 5.5. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k+3$ vertices, and let $v \in S_{1}$ where $N(v)=$ $\left\{u_{1}, \ldots, u_{k}\right\}$. Then:
(i) $\sum_{i=1}^{k} d\left(u_{i}\right) \geq 2 k n-\frac{1}{2}(k(k+5))-\frac{1}{2}((n-1)(n-2))$ for $k+3 \leq n \leq 2 k$;
(ii) $\sum_{i=1}^{k} d\left(u_{i}\right) \geq k^{2}+\frac{1}{2}(k(k+1))$ for $n \geq 2 k+1$.

Equality is reached if and only if $G\left[\cup_{i=1}^{k} N\left[u_{i}\right]\right] \cong P_{r}^{k}, r=\min (n, 2 k+1)$.

Proof. We will proceed by induction on $n$. There are two $k$-trees on $k+3$ vertices: $P_{k+3}^{k}$ and $S_{k, 3}$. If $T_{n}^{k} \cong P_{k+3}^{k}$, then $\sum_{i=1}^{k} d\left(u_{i}\right)=k^{2}+2 k-1$. If $T_{n}^{k} \cong S_{k, 3}$, then $\sum_{i=1}^{k} d\left(u_{i}\right)=k^{2}+2 k$, and so the lemma holds. Suppose the lemma is true for $T_{n^{\prime}}^{k}$ with $k+3<n^{\prime}<n$, and consider $T_{n}^{k}$. Clearly for the simplicial vertex $v^{\prime} \neq v$,

$$
\begin{equation*}
\sum_{i=1}^{k} d\left(u_{i}\right)=\sum_{i=1}^{k} d_{T_{n}^{k}-v^{\prime}}\left(u_{i}\right)+\left|U^{\prime} \cap U\right| \tag{6}
\end{equation*}
$$

Suppose $k+4 \leq n \leq 2 k$ which implies $k \geq 4$. Then $l=n-k-1$ and $|Y|=0$. Hence $v^{\prime} \in X$, and without loss of generality let $v^{\prime}=x_{l}$. Thus $\left|N\left(x_{l}\right) \cap U\right| \neq \emptyset$. As $k+4 \leq n \leq 2 k$, $3 \leq l \leq k-1$ and so $k-(l-1) \leq\left|N\left(x_{l}\right) \cap U\right| \leq k$. Thus $2 k-n+2 \leq\left|N\left(x_{l}\right) \cap U\right| \leq k$. By induction and (6),

$$
\begin{aligned}
\sum_{i=1}^{k} d\left(u_{i}\right) & =\sum_{i=1}^{k} d_{T_{n}^{k}-x_{l}}\left(u_{i}\right)+\left|N\left(x_{l}\right) \cap U\right| \\
& \geq 2 k(n-1)-\frac{1}{2}(k(k+5))-\frac{1}{2}(n-2)(n-3)+2 k-n+2 \\
& =2 k n-\frac{1}{2}(k(k+5))-\frac{1}{2}(n-1)(n-2)
\end{aligned}
$$

with equality holding if and only if $T_{n}^{k}-x_{l} \cong G\left[\cup_{i=1}^{k} N_{T_{n}^{k}-x_{l}}\left[u_{i}\right]\right] \cong P_{n-1}^{k}$ and $\left|N\left(x_{l}\right) \cap U\right|=$ $2 k-n+2=k-(l-1)$. In this case $X-x_{l} \subseteq N\left(x_{l}\right), x_{l-1} \in S_{2}\left(T_{n}^{k}\right)$, and so $N\left(x_{l}\right) \subseteq N\left[x_{l-1}\right]$. Thus $\left(N\left(x_{l}\right) \cap U\right) \subseteq\left(N\left(x_{l-1}\right) \cap U\right)$. Hence $T_{n}^{k} \cong G\left[\cup_{i=1}^{k} N\left[u_{i}\right]\right] \cong P_{n}^{k}$.

Suppose $n=2 k+1>k+3$, then $l=k$ and $|Y|=0$. Thus without loss of generality let $v^{\prime}=x_{k}$. Hence $1 \leq\left|N\left(x_{k}\right) \cap U\right| \leq k$. Then by induction and (6)

$$
\begin{aligned}
\sum_{i=1}^{k} d\left(u_{i}\right) & =\sum_{i=1}^{k} d_{T_{n}^{k}-x_{k}}\left(u_{i}\right)+\left|N\left(x_{k}\right) \cap U\right| \\
& \geq 2 k(n-1)-\frac{1}{2}(k(k+5))-\frac{1}{2}(n-2)(n-3)+1 \\
& =4 k^{2}-\frac{1}{2}(k(k+5))-\frac{1}{2}\left(4 k^{2}-6 k+2\right)+1 \\
& =k^{2}+\frac{1}{2}\left(k^{2}+k\right)
\end{aligned}
$$

with equality holding if and only if $T_{n}^{k}-x_{k} \cong G\left[\cup_{i=1}^{k} N_{T_{n}^{k}-x_{k}}\left[u_{i}\right]\right] \cong P_{2 k}^{k}$ and $\left|N\left(x_{k}\right) \cap U\right|=1$, i.e. $T_{2 k+1}^{k} \cong G\left[\cup_{i=1}^{k} N\left[u_{i}\right]\right] \cong P_{2 k+1}^{k}$.

Suppose $n \geq 2 k+2>k+3$. Then $l \geq k$ and $|Y| \geq 0$. If $|Y|=0$, then $\left|N\left(v^{\prime}\right) \cap U\right| \geq 1$. If $|Y| \geq 1$, then $\left|N\left(v^{\prime}\right) \cap U\right|=0$, and so by induction and (6)

$$
\begin{aligned}
\sum_{i=1}^{k} d\left(u_{i}\right) & =\sum_{i=1}^{k} d_{T_{n}^{k}-v^{\prime}}\left(u_{i}\right)+\left|N\left(v^{\prime}\right) \cap U\right| \\
& \geq k^{2}+\frac{1}{2}\left(k^{2}+k\right)
\end{aligned}
$$

with equality holding if and only if $G\left[\cup_{i=1}^{k} N_{T_{n}^{k-v^{\prime}}}\left[u_{i}\right]\right] \cong P_{2 k+1}^{k}$ and $\left|N\left(v^{\prime}\right) \cap U\right|=0$. Hence $G\left[\cup_{i=1}^{k} N\left[u_{i}\right]\right] \cong P_{2 k+1}^{k}$.

Hence by the Principle of Mathematical Induction, the lemma holds.

Lemma 5.6. Let $G$ be a $k$-degenerate graph on $n \geq k+1$ vertices, and let $v \in V(G)$ such that $d(v)=\delta(G)$ and $N(v)=\left\{u_{1}, \ldots, u_{\delta(G)}\right\}$. Then $\sum_{i=1}^{\delta(G)} d\left(u_{i}\right) \leq k(n-1)$, and equality holds if and only if $G \cong S_{k, n-k}$.

Proof. Clearly $\delta(G) \leq k$ and $d\left(u_{i}\right) \leq n-1$ for $1 \leq i \leq \delta(G)$. Thus $\sum_{i=1}^{\delta(G)} d\left(u_{i}\right) \leq k(n-1)$ with equality reached if and only if $\delta(G)=k$ and $d\left(u_{i}\right)=n-1$ for $1 \leq i \leq k$. Furthermore, equality is reached if and only if $V(G)-N[v]$ is independent as otherwise $G$ has a $K_{k+2}$ subgraph and thus not $k$-degenerate. That is, for $x \in V(G)-N[v], x$ is a $k$-simplicial vertex. Thus equality is reached if and only if $G \cong S_{k, n-k}$.

Let $N_{0}\left(u_{i}\right)=N\left(u_{i}\right)-N[v]$, and let

$$
\Psi\left(T_{n}^{k} ; v\right)=\sum_{x \in N_{0}\left(u_{1}\right)} d(x)+\sum_{x \in N_{0}\left(u_{2}\right)} d(x)+\ldots+\sum_{x \in N_{0}\left(u_{k}\right)} d(x) .
$$

Then for $n \geq k+2$ and $v^{\prime} \in S_{1}\left(T_{n}^{k}\right)-v$,

$$
\begin{align*}
\Psi\left(T_{n}^{k} ; v\right) & =\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=1}^{k} f\left(u_{i}^{\prime}\right) \\
& =\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| . \tag{7}
\end{align*}
$$

With this in mind, we may state the following lemmas:

Lemma 5.7. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k+5$ vertices and $v \in S_{1}\left(T_{n}^{k}\right)$. Then $\Psi\left(T_{n}^{k} ; v\right) \geq$ $\sum_{i=1}^{l}(k+1-i) d\left(x_{i}\right)$ where $d\left(x_{i}\right)=d^{*}\left(v_{i+k+1}\right)$ with respect to a presentation of $P_{n}^{k}$. Furthermore equality holds if and only if $G\left[N\left(N_{0}\left(u_{1}\right)\right) \cup N\left(N_{0}\left(u_{2}\right)\right) \cup \ldots \cup N\left(N_{0}\left(u_{k}\right)\right)\right] \cong P_{r}^{k}$ where $r=\min (n, 3 k+1)$.

Proof. We will proceed by induction on $n$. Note that

$$
\sum_{i=1}^{l}(k+1-i) d^{*}\left(v_{i+k+1}\right)=\sum_{i=k+2}^{l+k+1}(2 k+2-i) d^{*}\left(v_{i}\right)
$$

Now $d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|$ is a summand with $k$ summands with at least $t$ summands of value $k$ and at most $k-t$ summands of value at most $k-1$. It is clear then that $d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|$ is minimized when $t$ is minimized.

Suppose $n=k+5$. Then $t \geq \max (0, k-3)$, and by $(7) \Psi\left(T_{k+5}^{k} ; v\right)=\Psi\left(T_{k+5}^{k}-v^{\prime} ; v\right)+$ $k t+\sum_{i=1}^{k} f\left(u_{i}^{\prime}\right)$. Suppose $k \in\{1,2\}$. Then $l \geq k$ and $t \geq 0$, and $\Psi\left(T_{k+5}^{k} ; v\right)$ is minimized when $t=0$, and so clearly $\sum_{i=1}^{k} f\left(u_{i}^{\prime}\right) \geq k-1$. Hence by Table 1 ,

$$
\begin{aligned}
\Psi\left(T_{k+5}^{k} ; v\right) & =\Psi\left(T_{k+5}^{k}-v^{\prime} ; v\right)+k t+\sum_{i=1}^{k} f\left(u_{i}^{\prime}\right) \\
& \geq 3 k^{2}-1+0+k-1 \\
& =\sum_{i=k+2}^{2 k+1}(2 k+2-i) d^{*}\left(v_{i}\right) .
\end{aligned}
$$

Suppose $n=k+5$ and $k \geq 3$. Then $t \geq k-3$. If $t=k-3$, then $T_{k+5}^{k} \cong P_{k+5}^{k}$. That is, $\Psi\left(T_{n}^{k} ; v\right)$ is minimized when $T_{k+5}^{k} \cong P_{k+5}^{k}$. Hence $\Psi\left(T_{k+5}^{k} ; v\right) \geq \Psi\left(P_{k+5}^{k} ; v\right)=\sum_{i=k+2}^{l+k+1}(2 k+$ $2-i) d^{*}\left(v_{i}\right)$ with equality holding if and only if $T_{k+5}^{k} \cong P_{k+5}^{k}$.

Suppose that the lemma is true for $T_{m}^{k}$ where $k+5 \leq m<n$ and consider $T_{n}^{k}$. Let $\left|X-v^{\prime}\right|=l^{\prime}$. Now $k t+\sum_{i=1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|$ is minimized when $\left|N\left(v^{\prime}\right) \cap Y\right|$ is as large as possible and $t$ is as small as possible.

Suppose $k+6 \leq n \leq 2 k+1$. Then $l=n-k-1,|Y|=0$, and $l^{\prime}=l-1$. As $n \leq 2 k+1$, $d_{*}\left(v_{i+k+1}\right)=d^{*}\left(v_{i+k+1}\right)-1$ for $1 \leq i \leq l^{\prime}$. Now $t \geq k-(n-1-(k+1))=2 k+2-n$. If $t=2 k+2-n$, then $\left|N\left(u_{i}^{\prime}\right) \cap U\right|$ is minimized when $u_{j}^{\prime} \in N\left(u_{i}^{\prime}\right)$ for $t+1 \leq j<i$. Hence $\left|N\left(u_{i}^{\prime}\right) \cap U\right| \geq k+t+1-i$ for $t+1 \leq i \leq k$. Thus

$$
\begin{aligned}
\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| & \geq \sum_{i=t+1}^{k}(k+t-1-i)=\sum_{i=2 k+3-n}^{k}(3 k+3-n-i) \\
& =\sum_{i=k+2}^{n-1}(2 k+2-i)
\end{aligned}
$$

with equality holding if and only if $T_{n}^{k} \cong P_{n}^{k}$. Hence by induction and (7),

$$
\Psi\left(T_{n}^{k} ; v\right)=\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|
$$

$$
\begin{aligned}
& \geq \Psi\left(P_{n-1}^{k} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& =\sum_{i=1}^{l-1}(k+1-i) d_{*}\left(v_{i+k+1}\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& =\sum_{i=k+2}^{n-1}(2 k+2-i)\left(d^{*}\left(v_{i}\right)-1\right)+d^{*}\left(v_{n}\right)(2 k+2-n) \\
& \quad+\sum_{i=k+2}^{n-1}(2 k+2-i) \\
& =\sum_{i=k+2}^{n}(2 k+2-i) d^{*}\left(v_{i}\right)
\end{aligned}
$$

with equality holding if and only if $T_{n}^{k} \cong P_{n}^{k}$.
Suppose $k+6 \leq 2 k+2 \leq n \leq 3 k+1$. Then $k \leq l^{\prime} \leq l \leq l^{\prime}+1$. As $Y \cap(\{v\} \cup U \cup X)=\emptyset$, $\left|N\left(v^{\prime}\right) \cap Y\right| \leq n-1-(2 k+1)=n-(2 k+2)$ and $t \geq 0$ with equality if and only if $T_{n}^{k} \cong P_{n}^{k}$. Hence $\Psi\left(T_{n}^{k} ; v\right)$ is minimized when $\left|N\left(v^{\prime}\right) \cap Y\right|=n-(2 k+2)$ and $t=0$. That is, $u_{i}^{\prime} \in Y$ for $k-(n-(2 k+2))=3 k+3-n \leq i \leq k$ and $\left|N\left(u_{i}^{\prime}\right) \cap U\right| \geq 3 k+3-n-i$ for $1 \leq i \leq 3 k+2-n$.

In this case, $d^{*}\left(v_{i}\right)=d_{*}\left(v_{i}\right)$ for $k+2 \leq i \leq n-k$, and

$$
\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|=\sum_{i=1}^{3 k+2-n}(3 k+3-n-i)=\sum_{i=n-k+1}^{2 k+2}(2 k+2-i)
$$

Hence by induction and (7),

$$
\begin{aligned}
\Psi\left(T_{n}^{k} ; v\right) & =\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& \geq \Psi\left(P_{n-1}^{k} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& =\sum_{i=1}^{l^{\prime}}(k+1-i) d_{*}\left(v_{i+k+1}\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& \geq \sum_{i=k+2}^{n-k}(2 k+2-i) d^{*}\left(v_{i}\right)+\sum_{i=n-k+1}^{2 k+2}(2 k+2-i)\left(d^{*}\left(v_{i}\right)-1\right)
\end{aligned}
$$

$$
\begin{aligned}
& +d^{*}\left(v_{n}\right)(2 k+2-n)+\sum_{i=n-k-1}^{2 k+2}(2 k+2-i) \\
= & \sum_{i=k+2}^{n}(2 k+2-i) d^{*}\left(v_{i}\right)
\end{aligned}
$$

with equality holding if and only if $T_{n}^{k} \cong P_{n}^{k}$.
Suppose $k+6 \leq 3 k+2 \leq n$. Let $G^{\prime}=T_{n}^{k}-v^{\prime}$. Then by induction and (7),

$$
\begin{aligned}
\Psi\left(T_{n}^{k} ; v\right) & =\Psi\left(G^{\prime} ; v\right)+k t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& \geq \sum_{i=k+2}^{2 k+1}(2 k+2-i) d_{*}\left(v_{i}\right)+k t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|
\end{aligned}
$$

with equality holding if and only if $G\left[N\left(N_{o}\left(u_{1}\right)\right) \cup N\left(N_{o}\left(u_{2}\right)\right) \cup \ldots\right.$
$\left.\cup N\left(N_{o}\left(u_{k}\right)\right)\right] \cong G\left[N_{G^{\prime}}\left(N_{o}\left(u_{1}\right)\right) \cup N_{G^{\prime}}\left(N_{o}\left(u_{2}\right)\right) \cup \ldots \cup N_{G^{\prime}}\left(N_{o}\left(u_{k}\right)\right)\right] \cong P_{3 k+1}^{k}$. Note that $t \geq 0$ and $\left|N\left(v^{\prime}\right) \cap Y\right| \leq k$ with both equalities holding if and only if $G\left[N\left(N_{o}\left(u_{1}\right)\right) \cup N\left(N_{o}\left(u_{2}\right)\right) \cup\right.$ $\left.\ldots \cup N\left(N_{o}\left(u_{k}\right)\right)\right] \cong P_{3 k+1}^{k}$, and note that if $\left|N\left(v^{\prime}\right) \cap Y\right|=k$ then $\sum_{i=1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|=0$. Hence

$$
\Psi\left(T_{n}^{k} ; v\right) \geq \sum_{i=k+2}^{2 k+1}(2 k+2-i) d^{*}\left(v_{i}\right)
$$

with equality holding if and only if $G\left[N\left(N_{o}\left(u_{1}\right)\right) \cup N\left(N_{o}\left(u_{2}\right)\right) \cup \ldots \cup N\left(N_{o}\left(u_{k}\right)\right)\right] \cong P_{3 k+1}^{k}$.
Hence, by the Principle of Mathematical Induction, $\Psi\left(T_{n}^{k} ; v\right) \geq \sum_{i=1}^{l}(k+1-i) d^{*}\left(v_{i+k+1}\right)$, and equality is reached if and only if $G\left[N\left(N_{0}\left(u_{1}\right)\right) \cup N\left(N_{0}\left(u_{2}\right)\right) \cup \ldots \cup N\left(N_{0}\left(u_{k}\right)\right)\right] \cong P_{r}^{k}$ where $r=\min (n, 3 k+1)$.

Lemma 5.8. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k+4$ vertices and $v \in S_{1}\left(T_{n}^{k}\right)$ with $N(v)=U=$ $\left\{u_{1}, \ldots, u_{k}\right\}$. Then:
(i) $\Psi\left(T_{n}^{k} ; v\right) \geq \frac{1}{6}(n-k-1)\left(2 n k+5 n-n^{2}+5 k^{2}-5 k-6\right)$ for $k+4 \leq n \leq 2 k$
(ii) $\Psi\left(T_{n}^{k} ; v\right) \geq \frac{1}{6}\left(n^{3}-9 n^{2}-6 n^{2}+27 n k^{2}+36 n k+6 n-21 k^{3}-24 k^{2}-33 k-6\right)$ for $2 k+1 \leq n \leq 3 k$
(iii) $\Psi\left(T_{n}^{k} ; v\right) \geq k^{3}+k^{2}$ for $n \geq \max (5,3 k+1)$.

And for $n \geq k+5$, equality is reached if and only if $G\left[N\left(N_{0}\left(u_{1}\right)\right) \cup N\left(N_{0}\left(u_{2}\right)\right) \cup \ldots \cup\right.$ $\left.N\left(N_{0}\left(u_{k}\right)\right)\right] \cong P_{r}^{k}$ where $r=\min (n, 3 k+1)$.

Proof. First consider the two $k$-trees on $k+3$ vertices: $P_{k+3}^{k}$ and $S_{k, 3}$. It is routine to determine that $\Psi\left(P_{k+3}^{k} ; v\right)=\Psi\left(S_{k, 3} ; v\right)=2 k^{2}$.

Suppose $n=k+4$, and suppose $k \in\{1,2\}$. Then $T_{k+4}^{k}-v^{\prime} \in\left\{P_{k+3}^{k}, S_{k, 3}\right\}, l \geq k, t \geq 0$, and $\sum_{i=1}^{k} f\left(u_{i}^{\prime}\right) \leq k^{2}-1$. Hence by (7)

$$
\begin{aligned}
\Psi\left(T_{k+4}^{k} ; v\right) & =\Psi\left(T_{k+4}^{k}-v^{\prime} ; v\right)+k t+\sum_{i=1}^{k} f\left(u_{i}^{\prime}\right) \\
& \geq 2 k^{2}+k^{2}-1=2 k^{2}+k^{2}-k+1
\end{aligned}
$$

Note that when $k=1$ and $n=k+4,3 k^{2}-1=k^{3}+k^{2}$, and when $k=2$ and $n=k+4$, $3 k^{2}-1=\frac{1}{6}\left(n^{3}-9 n^{2}-6 n^{2}+27 n k^{2}+36 n k+6 n-21 k^{3}-24 k^{2}-33 k-6\right)$, and so the lemma holds.

Suppose that $n=k+4$ and $k \geq 3$. Then $l=3$ and $|Y|=0$. Hence there exists $v^{\prime} \in S_{1}\left(T_{n}^{k}\right) \cap X$, and $k-2 \leq t \leq k$. If $t=k-2$ (respectively $t=k$ ), then $T_{k+4}^{k} \cong P_{k+4}^{k}$ $\left(T_{k+4}^{k} \cong S_{k, 4}\right)$ and $\Psi\left(T_{k+4}^{k} ; v\right)=3 k^{2}-1\left(\Psi\left(T_{k+4}^{k} ; v\right)=3 k^{2}\right)$. Suppose that $t=k-1$. Then $T_{k+4}^{k} \in\left\{G_{1}, G_{2}, G_{3}\right\}$ where $G_{1}, G_{2}$, and $G_{3}$ are defined as follows. Let $G_{1}$ and $G_{2}$ be $k$-trees such that $G_{i}-v \cong P_{k+3}^{k}$ for $1 \leq i \leq 2$, and let $x \in S_{1}\left(G_{i}-v^{\prime}\right)$ for $1 \leq i \leq 2$ such that $x \in S_{1}\left(G_{1}\right)$ and $x \notin S_{1}\left(G_{2}\right)$. Let $G_{3}$ be a $k$-tree such that $G_{3}-v^{\prime} \cong S_{k, 3}$, but $G_{3} \nsubseteq S_{k, 4}$. Then $\Psi\left(G_{1} ; v\right)=3 k^{2}+k-1, \Psi\left(G_{2} ; v\right)=3 k^{2}-1$, and $\Psi\left(G_{3} ; v\right)=3 k^{2}+k-1$. By these calculations, we see that $\Psi\left(T_{k+4}^{k} ; v\right) \geq 3 k^{2}-1$. Note if $k=3$, then $n=7=2 k+1$ and
$3 k^{2}-1=k(n-2 k-1)(4 k+2-n)+\frac{1}{6}(n-3 k-1)\left(n^{2}-5 n-9 k^{2}+3 k+6\right)$. If $k \geq 4$, then $3 k^{2}-1=\frac{1}{6}(n-k-1)\left(2 n k+5 n-n^{2}+5 k^{2}-5 k-6\right)$.

Suppose that $n \geq k+5$. Then by Lemma 5.7, $\Psi\left(T_{n}^{k} ; v\right) \geq \sum_{i=k+2}^{l+k+1}(2 k+2-i) d^{*}\left(v_{i}\right)$.
According to Table 1,
(i) $\Psi\left(T_{n}^{k} ; v\right) \geq \frac{1}{6}(n-k-1)\left(2 n k+5 n-n^{2}+5 k^{2}-5 k-6\right)$ for $k+4 \leq n \leq 2 k$
(ii) $\Psi\left(T_{n}^{k} ; v\right) \geq \frac{1}{6}\left(n^{3}-9 n^{2}-6 n^{2}+27 n k^{2}+36 n k+6 n-21 k^{3}-24 k^{2}-33 k-6\right)$ for $2 k+1 \leq n \leq 3 k$
(iii) $\Psi\left(T_{n}^{k} ; v\right) \geq k^{3}+k^{2}$ for $n \geq 3 k+1$.

Furthermore, by Lemma 5.7, for $n \geq k+5$ equality is reached if and only if $G\left[N\left(N_{0}\left(u_{1}\right)\right) \cup\right.$ $\left.N\left(N_{0}\left(u_{2}\right)\right) \cup \ldots \cup N\left(N_{0}\left(u_{k}\right)\right)\right] \cong P_{r}^{k}$ where $r=\min (n, 3 k+1)$.

Let $\mathcal{J}^{k}$ be the set of $k$-trees $T_{n}^{k}$ with a vertex $v \in S_{1}\left(T_{n}^{k}\right)$ and vertex set $P=\{p \mid p \in$ $\left.V\left(T_{n}^{k}\right),|N(p) \cap N(v)|=k\right\}$ such that $V\left(T_{n}^{k}\right)-N[v]-P$ is an independent set. Then we may state the following lemma.

Lemma 5.9. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k+1$ vertices. Then $\Psi\left(T_{n}^{k} ; v\right) \leq$ $(n-k-1) k^{2}$ with equality holding if and only if $T_{n}^{k} \in \mathcal{J}^{k}$.

Proof. Let $P=\left\{p_{1}, \ldots, p_{r}\right\}=\left\{p \in V\left(T_{n}^{k}\right)| | N(p) \cap U \mid=k\right\}$ and $Q=\left\{q_{1}, \ldots, q_{s}\right\}=$ $V\left(T_{n}^{k}\right)-P-N[v]$. Order the vertices of $Q$ such that $\left|N\left(q_{i}\right) \cap P\right| \geq\left|N\left(q_{i+1}\right) \cap P\right|$ for $1 \leq i \leq t-1$. Then $|P|+|Q|=r+s=n-k-1$.

Proceed by induction on $|Q|=s$. If $s=0$, then $\Psi\left(T_{n}^{k} ; v\right)=r k^{2}=(n-k-1) k^{2}$ as for any $p \in P, p \in N\left(u_{i}\right)$ for $1 \leq i \leq k$. Suppose that for $k$-trees with $|Q|=s^{\prime}$ such that $0<s^{\prime}<s$, $\Psi\left(T_{n}^{k} ; v\right) \leq(n-k-1) k^{2}$ with equality holding if and only if $T_{n}^{k} \in \mathcal{J}^{k} ;$ consider $T_{n}^{k}$ with $|Q|=s$.

As $|Q| \neq 0$, there exists $v^{\prime} \in S_{1}\left(T_{n}^{k}\right) \cap Q$. Let $N\left(v^{\prime}\right)=U^{\prime}=\left\{u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right\}$. Arrange the vertices of $U^{\prime}$ such that $u_{i}^{\prime} \in U$ for $1 \leq i \leq t$ and $\left|N\left(u_{i}^{\prime}\right) \cap U\right| \geq\left|N\left(u_{i+1}^{\prime}\right) \cap U\right|$ for $t+1 \leq i \leq k$. Then $T_{n}^{k}-v^{\prime}$ is a $k$-tree with $\left|V\left(T_{n}^{k}-v^{\prime}\right)-P-N[v]\right|=s-1$. By induction, $\Psi\left(T_{n}^{k}-v^{\prime} ; v\right) \leq(n-k-2) k^{2}$ with equality holding if and only if $T_{n}^{k}-v^{\prime} \in \mathcal{J}^{k}$.

Now $\left|N\left(u_{i}^{\prime}\right) \cap U\right| \leq k-1$ for $t+2 \leq i \leq k$. Hence $k\left|U^{\prime} \cap U\right|+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|$ is maximized when $\left|U^{\prime} \cap U\right|$ is maximized. By (7),

$$
\Psi\left(T_{n}^{k} ; v\right)=\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+k\left|U^{\prime} \cap U\right|+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| .
$$

Suppose $T_{n}^{k} \notin \mathcal{J}^{k}$, then $\left|U^{\prime} \cap U\right| \leq k-1$. If $\left|U^{\prime} \cap U\right|=k-1$, then $\left|N\left(u_{k}^{\prime}\right) \cap U\right| \leq k-1$ otherwise $T_{n}^{k} \in \mathcal{J}^{k}$, and so if $\left|U^{\prime} \cap U\right|$, then $k\left|U^{\prime} \cap U\right|+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \leq k(k-1)+k-1=$ $k^{2}-1$. Hence

$$
\begin{aligned}
\Psi\left(T_{n}^{k} ; v\right) & =\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+k\left|U^{\prime} \cap U\right|+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& \leq(n-k-2) k^{2}+k(k-1)+k-1=(n-k-1) k^{2}-1 .
\end{aligned}
$$

Suppose then that $T_{n}^{k} \in \mathcal{J}^{k}$, then $T_{n}^{k}-v^{\prime} \in \mathcal{J}^{k},\left|U^{\prime} \cap U\right|=k$, and $\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|=0$ as $t=k$. Hence

$$
\begin{aligned}
\Psi\left(T_{n}^{k} ; v\right) & =\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+k\left|U^{\prime} \cap U\right| \\
& \leq(n-k-2) k^{2}+k^{2}=(n-k-1) k^{2} .
\end{aligned}
$$

Hence by induction, the lemma holds.

### 5.2. The Zagreb Indices of the $k$-path and the $k$-star.

The following lemmas may be deduced through fairly routine calculations by induction on $n$.

Lemma 5.10. Let $P_{n}^{k}$ be the $k$-path on $n \geq k+3$ vertices. Then

$$
\begin{aligned}
& M_{1}\left(P_{n}^{k}\right)=2 n k(n-2)-\frac{1}{3}(n(n-1)(n-2))-\frac{1}{3}(k(k+1)(2 k-5)) \\
& \quad \text { for } k+3 \leq n \leq 2 k \text { and } k \geq 3 \\
& M_{1}\left(P_{n}^{k}\right)=4 n k^{2}-\frac{1}{3}(k(10 k-1)(k+1)) \text { for } n \geq \max (4,2 k+1) .
\end{aligned}
$$

Lemma 5.11. Let $S_{k, n-k}$ be the $k$-star on $n \geq k+1$ vertices. Then $M_{1}\left(S_{k, n-k}\right)=n^{2} k+$ $\left(k^{2}-2 k\right) n-k^{3}+1$.

For the second Zagreb indices, we have the following:

Lemma 5.12. Let $P_{n}^{k}$ be the $k$-path on $n \geq k+3$ vertices. Then

$$
\begin{aligned}
M_{2}\left(P_{n}^{k}\right)= & \frac{1}{2}\left(k^{4}+9 k^{3}+12 k^{2}-8 k+2\right), \text { for } n=k+3 \\
M_{2}\left(P_{n}^{k}\right)= & \frac{1}{24}\left((10-4 k) n^{3}-n^{4}+\left(54 k^{2}-18 k-23\right) n^{2}-\right. \\
& \left.\left(44 k^{3}+66 k^{2}-54 k-14\right) n+7 k^{4}+38 k^{3}+5 k^{2}-26 k\right) \\
& \text { for } k+4 \leq n \leq 2 k, \\
M_{2}\left(P_{n}^{k}\right)= & \frac{1}{24}\left(n^{4}-(12 k+6) n^{3}+\left(54 k^{2}+54 k+11\right) n^{2}-\right. \\
& \left.\left(12 k^{3}+162 k^{2}+66 k+6\right) n-\left(25 k^{4}-70 k^{3}-109 k^{2}-14 k\right)\right) \\
& \text { for } 2 k+1 \leq n \leq 3 k-1, \\
M_{2}\left(P_{n}^{k}\right)= & \frac{1}{12}\left(48 n k^{3}-53 k^{4}-46 k^{3}+5 k^{2}-2 k\right) \text { for } n \geq \max (5,3 k)
\end{aligned}
$$

Proof. We will proceed by induction on $n$. By simple calculations, $M_{2}\left(P_{k+3}^{k}\right)=\frac{1}{2}\left(k^{4}+9 k^{3}+\right.$ $\left.12 k^{2}-8 k+2\right)$, and the lemma holds true. Suppose that for $k$-paths of an order smaller than $n \geq k+4$ the lemma holds, and consider $P_{n}^{k}$. Let $T_{n}^{k}$ be a $k$-tree on $n$ vertices, and
let $v \in S_{1}\left(T_{n}^{k}\right)$ with $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}$. Let $G^{\prime} \cong T_{n}^{k}-v$, which is a $k$-tree. Note that $d_{G^{\prime}}\left(u_{i}\right)=d\left(u_{i}\right)-1$ for $1 \leq i \leq k$ and $\sum_{u_{i} u_{j}, i \neq j}\left[\left(d_{G^{\prime}}\left(u_{i}\right)+1\right)\left(d_{G^{\prime}}\left(u_{j}\right)+1\right)-d_{G^{\prime}}\left(u_{i}\right) d_{G^{\prime}}\left(u_{j}\right)\right]=$ $\sum_{u_{i} u_{j}, i \neq j}\left(d_{G^{\prime}}\left(u_{i}\right)+d_{G^{\prime}}\left(u_{j}\right)+1\right)=\sum_{u_{i} u_{j}, i \neq j}\left(d\left(u_{i}\right)+d\left(u_{j}\right)-1\right)=\sum_{u_{i} u_{j}, i \neq j}\left(d\left(u_{i}\right)+d\left(u_{j}\right)\right)-\binom{k}{2}$.

Thus

$$
\begin{align*}
& M_{2}\left(T_{n}^{k}\right)=\left.M_{2}\left(G^{\prime}\right)+d(v)\left(\sum_{i=1}^{k} d\left(u_{i}\right)\right)+\left(\sum_{i=1}^{k}\left(d\left(u_{i}\right)-d_{G^{\prime}}\left(u_{i}\right)\right)\right)\left(\sum_{x \in N_{o}\left(u_{i}\right)} d(x)\right)\right) \\
&+\sum_{u_{i} u_{j}, i \neq j}\left[\left(d_{G^{\prime}}\left(u_{i}\right)+1\right)\left(d_{G^{\prime}}\left(u_{j}\right)+1\right)-d_{G^{\prime}}\left(u_{i}\right) d_{G^{\prime}}\left(u_{j}\right)\right] \\
&= M_{2}\left(G^{\prime}\right)+k \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)+\sum_{u_{i} u_{j}, i \neq j}\left(d\left(u_{i}\right)+d\left(u_{j}\right)\right)-\binom{k}{2} \\
&= M_{2}\left(G^{\prime}\right)+k \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)+(k-1) \sum_{i=1}^{k} d\left(u_{i}\right)-\binom{k}{2} \\
& \quad=M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} \tag{8}
\end{align*}
$$

and so for $P_{n}^{k}$,

$$
\begin{equation*}
M_{2}\left(P_{n}^{k}\right)=M_{2}\left(P_{n-1}^{k}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(P_{n}^{k} ; v\right)-\binom{k}{2} . \tag{9}
\end{equation*}
$$

As a special case, consider when $k=1$ and $n=5$. In this case, clearly $M_{2}\left(P_{n}^{k}\right)=12=$ $\frac{1}{12}\left(48 n k^{3}-53 k^{4}-46 k^{3}+5 k^{2}-2 k\right)$.

Let $f_{1}=4 n^{3}+12 n^{2} k-36 n^{2}-108 n k^{2}+24 n k+80 n+44 k^{3}+120 k^{2}-68 k-48$ and $f_{2}=-4 n^{3}+36 n^{2} k+24 n^{2}-108 n k^{2}-144 n k-44 n-12 k^{3}-216 k^{2}-132 k-24$.

Suppose that $k+4 \leq n \leq 2 k$ which implies $k \geq 4$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
M_{2}\left(P_{n}^{k}\right)=M_{2}\left(P_{n-1}^{k}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(P_{n}^{k} ; v\right)-\binom{k}{2}
$$

$$
\begin{aligned}
= & M_{2}\left(P_{n-1}^{k}\right)+(2 k-1)\left(2 k n-\frac{1}{2}(k(k+5)+(n-1)(n-2))\right. \\
& +\frac{1}{6}(n-k-1)\left(2 n k+5 n-n^{2}+5 k^{2}-5 k-6\right)+\frac{1}{2}(k(k-1)) \\
= & M_{2}\left(P_{n-1}^{k}\right)-\frac{1}{24} f_{1} \\
= & \frac{1}{24}\left(-(n-1)^{4}-(4 k-10)(n-1)^{3}+\left(54 k^{2}-18 k-23\right)(n-1)^{2}\right. \\
& \left.-\left(44 k^{3}+66 k^{2}-54 k-14\right)(n-1)+7 k^{4}+38 k^{3}+5 k^{2}-26 k\right) \\
& -\frac{1}{24} f_{1} \\
= & \frac{1}{24}\left(-\left(n^{4}-4 n^{3}+6 n^{2}-4 n+1\right)-(4 k-10)\left(n^{3}-3 n^{2}+3 n-1\right)\right. \\
& +\left(54 k^{2}-18 k-23\right)\left(n^{2}-2 n+1\right)-\left(44 k^{3}+66 k^{2}-54 k\right. \\
& \left.-14)(n-1)+7 k^{4}+38 k^{3}+5 k^{2}-26 k\right)-\frac{1}{24} f_{1} \\
= & \frac{1}{24}\left(-n^{4}-(4 k-10) n^{3}+\left(54 k^{2}-18 k-23\right) n^{2}-\left(44 k^{3}+66 k^{2}\right.\right. \\
& \left.-54 k-14) n+7 k^{4}+38 k^{3}+5 k^{2}-26 k\right)+\frac{1}{24} f_{1}-\frac{1}{24} f_{1} \\
= & \frac{1}{24}\left((10-4 k) n^{3}-n^{4}+\left(54 k^{2}-18 k-23\right) n^{2}-\right. \\
& \left.\left(44 k^{3}+66 k^{2}-54 k-14\right) n+7 k^{4}+38 k^{3}+5 k^{2}-26 k\right) .
\end{aligned}
$$

Suppose that $n=2 k+1 \geq k+4$ which implies $k \geq 3$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
\begin{aligned}
M_{2}\left(P_{2 k+1}^{k}\right)= & M_{2}\left(P_{2 k}^{k}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(P_{2 k+1}^{k} ; v\right)-\binom{k}{2} \\
= & M_{2}\left(P_{2 k}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2}(k(k+1))\right. \\
& -\frac{1}{6}\left(5 k+6 k^{2}+k^{3}-\left(3 k+3 k^{2}\right)(2 k+1)\right)-\frac{1}{2}(k(k-1))
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{24}\left(-(2 k)^{4}-(4 k-10)(2 k)^{3}+\left(54 k^{2}-18 k-23\right)(2 k)^{2}-\left(44 k^{3}+66 k^{2}\right.\right. \\
& \left.-54 k-14) 2 k+7 k^{4}+38 k^{3}+5 k^{2}-26 k\right)+\frac{1}{24}\left(72 k^{3}-24 k^{2}\right) \\
& -\frac{1}{24}\left(20 k+24 k^{2}+4 k^{3}\right)+\frac{1}{24}\left(24 k^{3}+36 k^{2}+12 k\right) \\
= & \frac{1}{8}\left(29 k^{4}+2 k^{3}+3 k^{2}-2 k\right) \\
= & \frac{1}{24}\left(n^{4}-(12 k+6) n^{3}+\left(54 k^{2}+54 k+11\right) n^{2}-\right. \\
& \left.\left(12 k^{3}+162 k^{2}+66 k+6\right) n-\left(25 k^{4}-70 k^{3}-109 k^{2}-14 k\right)\right) .
\end{aligned}
$$

Suppose that $2 k+2 \leq n \leq 3 k-1$ which implies $k \geq 3$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
\begin{aligned}
M_{2}\left(P_{n}^{k}\right)= & M_{2}\left(P_{n-1}^{k}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(P_{n}^{k} ; v\right)-\binom{k}{2} \\
= & M_{2}\left(P_{n-1}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2} k(k+1)\right)+\frac{1}{6}\left(n^{3}-9 n^{2}\right. \\
& \left.-6 n^{2}+27 n k^{2}+36 n k+6 n-21 k^{3}-24 k^{2}-33 k-6\right) \\
& -\frac{1}{2}(k(k-1)) \\
= & M_{2}\left(P_{n-1}^{k}\right)-\frac{1}{24} f_{2} \\
= & \frac{1}{24}\left((n-1)^{4}-(12 k+6)(n-1)^{3}+\left(54 k^{2}+54 k+11\right)(n-1)^{2}\right. \\
& \left.-\left(12 k^{3}+162 k^{2}+66 k+6\right)(n-1)-25 k^{4}+70 k^{3}+109 k^{2}+14 k\right) \\
& -\frac{1}{24} f_{2} \\
= & \frac{1}{24}\left(\left(n^{4}-4 n^{3}+6 n^{2}-4 n+1\right)-(12 k+6)\left(n^{3}-3 n^{2}+3 n-1\right)\right. \\
& +\left(54 k^{2}+54 k+11\right)\left(n^{2}-2 n+1\right)-\left(12 k^{3}+162 k^{2}+66 k\right. \\
& \left.+6)(n-1)-25 k^{4}+70 k^{3}+109 k^{2}+14 k\right)-\frac{1}{24} f_{2}
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{24}\left(n^{4}-(12 k+6) n^{3}+\left(54 k^{2}+54 k+11\right) n^{2}-\left(12 k^{3}\right.\right. \\
& \left.\left.+162 k^{2}+66 k+6\right) n-25 k^{4}+70 k^{3}+109 k^{2}+14 k\right)+\frac{1}{24} f_{2}-\frac{1}{24} f_{2} \\
= & \frac{1}{24}\left(n^{4}-(12 k+6) n^{3}+\left(54 k^{2}+54 k+11\right) n^{2}-\right. \\
& \left.\left(12 k^{3}+162 k^{2}+66 k+6\right) n-\left(25 k^{4}-70 k^{3}-109 k^{2}-14 k\right)\right) .
\end{aligned}
$$

Suppose that $n=3 k \geq k+4$ which implies $k \geq 2$. Then, by (9), Lemma 5.5 , and Lemma 5.8,

$$
\begin{aligned}
M_{2}\left(P_{3 k+1}^{k}\right)= & M_{2}\left(P_{3 k}^{k}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(P_{3 k+1}^{k} ; v\right)-\binom{k}{2} \\
= & M_{2}\left(P_{3 k-1}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2} k(k+1)\right)+-k(1-k)(2+k) \\
& +\left(\frac{1}{6}\left(-6-3 k+9 k^{2}+15 k-9 k^{2}\right)-\frac{1}{2}(k(k-1))\right. \\
= & M_{2}\left(P_{3 k-1}^{k}\right)+4 k^{3}-1 \\
= & \frac{1}{24}\left((3 k-1)^{4}-(12 k+6)(3 k-1)^{3}+\left(54 k^{2}+54 k+11\right)(3 k-1)^{2}\right. \\
& -\left(12 k^{3}+162 k^{2}+66 k+6\right)(3 k-1)-\left(25 k^{4}-70 k^{3}-109 k^{2}\right. \\
& -14 k))+4 k^{3}-1 \\
= & \frac{1}{12}\left(91 k^{4}-46 k^{3}+5 k^{2}-2 k\right) \\
= & \frac{1}{12}\left(48 n k^{3}-53 k^{4}-46 k^{3}+5 k^{2}-2 k\right) .
\end{aligned}
$$

Suppose $n \geq \max (6,3 k+1)$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
\begin{aligned}
M_{2}\left(P_{n}^{k}\right) & =M_{2}\left(P_{n-1}^{k}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(P_{n}^{k} ; v\right)-\binom{k}{2} \\
& =M_{2}\left(P_{n-1}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2} k(k+1)\right)+k^{3}+k^{2}-\frac{1}{2}(k(k-1))
\end{aligned}
$$

$$
\begin{aligned}
& =M_{2}\left(P_{n-1}^{k}\right)+4 k^{3} \\
& =\frac{1}{12}\left(48(n-1) k^{3}-53 k^{4}-46 k^{3}+5 k^{2}-2 k\right)+4 k^{3} \\
& =\frac{1}{12}\left(48 n k^{3}-53 k^{4}-46 k^{3}+5 k^{2}-2 k\right) .
\end{aligned}
$$

Thus by the principle of mathmatical induction, the lemma is verified.

The following lemma follows from direct calculation and can be easily verified through induction.

Lemma 5.13. Let $S_{k, n-k}$ be the $k$-star on $n \geq k+1$ vertices. Then

$$
M_{2}\left(S_{k, n-k}\right)=\frac{1}{2}\left(\left(3 k^{2}-k\right) n^{2}-\left(2 k^{3}+4 k^{2}-2 k\right) n+k(2 k-1)(k+1)\right) .
$$

### 5.3. Sharp Upper and Lower Bounds for $M_{1}$ of $k$-trees.

In this section, we determine the upper and lower bounds of $M_{1}$ of $k$-trees, and the corresponding extremal graphs are characterized.

Theorem 5.14. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k$ vertices. Then $M_{1}\left(P_{n}^{k}\right) \leq M_{1}\left(T_{n}^{k}\right)$, and equality is reached if and only if $T_{n}^{k} \cong P_{n}^{k}$.

Proof. For $k \leq n \leq k+1, T_{n}^{k} \cong K_{n}$, and $M_{1}\left(K_{n}\right)=n(n-1)^{2}$. Note that in this case, $K_{n} \cong P_{n}^{k}$. If $n=k+2$, then $T_{n}^{k} \cong P_{k+2}^{k}$, which is a $k$-clique bound by two simplicial vertices. Hence $M_{1}\left(P_{k+2}^{k}\right)=M_{1}\left(T_{k+2}^{k}\right)$. Suppose $n=k+3$. Then $T_{k+3}^{k} \in\left\{P_{k+3}^{k}, S_{k, 3}\right\}$. By routine calculations, $M_{1}\left(P_{k+3}^{k}\right)=k^{3}+7 k^{2}+4 k-2$ and $M_{1}\left(S_{k, 3}\right)=k^{3}+7 k^{2}+4 k$, and so the lemma holds.

We now use induction on $n \geq k+4$. If $T_{n}^{k} \cong P_{n}^{k}$, we are done. Suppose, then, that $T_{n}^{k} \not \neq P_{n}^{k}$, and let $v \in S_{1}\left(T_{n}^{k}\right)$ be such that $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}$ and $d\left(u_{1}\right)+\ldots+d\left(u_{k}\right)$ is as
small as possible. Consider $G^{\prime}=T_{n}^{k}-v$. By the choice of $v$, if $T_{n}^{k} \not \equiv P_{n}^{k}$ then $G^{\prime} \not \equiv P_{n-1}^{k}$.
Now,

$$
\begin{aligned}
M_{1}\left(T_{n}^{k}\right)= & M_{1}\left(G^{\prime}\right)+(d(v))^{2}+\left(\left(d\left(u_{1}\right)^{2}-\left(d_{G^{\prime}}\left(u_{1}\right)\right)^{2}\right)+\ldots\right. \\
& +\left(\left(d\left(u_{k}\right)^{2}-\left(d_{G^{\prime}}\left(u_{k}\right)\right)^{2}\right)\right. \\
= & M_{1}\left(G^{\prime}\right)+k^{2}+\left(\left(d\left(u_{1}\right)^{2}-\left(d\left(u_{1}\right)-1\right)^{2}\right)+\ldots\right. \\
& +\left(\left(d\left(u_{k}\right)^{2}-\left(d\left(u_{k}\right)-1\right)^{2}\right) .\right.
\end{aligned}
$$

Thus,

$$
\begin{equation*}
M_{1}\left(T_{n}^{k}\right)=M_{1}\left(G^{\prime}\right)+k^{2}+2 \sum_{i=1}^{k} d\left(u_{i}\right)-k . \tag{10}
\end{equation*}
$$

Suppose $k+4 \leq n \leq 2 k$ which implies $k \geq 4$. Then from (10), Lemma 5.5, and Lemma 5.10,

$$
\begin{aligned}
M_{1}\left(T_{n}^{k}\right)= & M_{1}\left(G^{\prime}\right)+k^{2}+2 \sum_{i=1}^{k} d\left(u_{i}\right)-k \\
> & M_{1}\left(P_{n-1}^{k}\right)+k^{2}+4 k n-k(k+5)-(n-1)(n-2)-k \\
= & 2 k(n-1)(n-3)-\frac{1}{3}((n-1)(n-2)(n-3)+3(n-1)(n-2)) \\
& -\frac{1}{3}(k(k+1)(2 k-5))+k^{2}+4 k n-k(k+5)-k \\
= & 2 n k(n-2)-\frac{1}{3}(n(n-1)(n-2))-\frac{1}{3}(k(k+1)(2 k-5))=M_{1}\left(P_{n}^{k}\right)
\end{aligned}
$$

Suppose $n=2 k+1>k+3$, which implies $k \geq 3$. Then from (10), Lemma 5.5, and Lemma 5.10,

$$
\begin{aligned}
M_{1}\left(T_{n}^{k}\right) & =M_{1}\left(G^{\prime}\right)+k^{2}+2 \sum_{i=1}^{k} d\left(u_{i}\right)-k \\
& >M_{1}\left(P_{n-1}^{k}\right)+k^{2}+2 k^{2}+k(k+1)-k
\end{aligned}
$$

$$
\begin{aligned}
= & 2 k(n-1)(n-3)-\frac{1}{3}(n-1)(n-2)(n-3)-\frac{1}{3} k(k+1)(2 k-5) \\
& +4 k^{2} \\
= & 2 k(2 k)(2 k-2)-\frac{1}{3}(2 k(2 k-1)(2 k-2))-\frac{1}{3}\left(2 k^{3}-3 k^{2}-5 k\right)+4 k^{2} \\
= & 8 k^{3}+4 k^{2}-\frac{1}{3}\left(10 k^{3}+9 k^{2}+k\right) \\
= & 4 n k^{2}-\frac{1}{3} k(10 k-1)(k+1)=M_{1}\left(P_{n}^{k}\right) .
\end{aligned}
$$

Suppose $n \geq 2 k+2$. Then from (10), Lemma 5.5, and Lemma 5.10,

$$
\begin{aligned}
M_{1}\left(T_{n}^{k}\right) & =M_{1}\left(G^{\prime}\right)+k^{2}+2 \sum_{i=1}^{k} d\left(u_{i}\right)-k \\
& >M_{1}\left(P_{n-1}^{k}\right)+k^{2}+2 k^{2}+k(k+1)-k \\
& =4(n-1) k^{2}-\frac{1}{3}(k(10 k-1)(k+1))+4 k^{2} \\
& =4 n k^{2}-\frac{1}{3} k(10 k-1)(k+1)=M_{1}\left(P_{n}^{k}\right) .
\end{aligned}
$$

Thus, the theorem is true for all $n \geq k$ by the Principle of Mathematical Induction.

Theorem 5.15. Let $G$ be $k$-degenerate on $n \geq k$ vertices. Then $M_{1}(G) \leq M_{1}\left(S_{k, n-k}\right)$ with equality holding if and only if $G \cong S_{k, n-k}$.

Proof. We will proceed by induction on $n$. If $n \in\{k, k+1\}$, then $M_{1}(G) \leq M_{1}\left(K_{n}\right)$ with equality holding if and only if $G \cong K_{n}$. Note $K_{n} \cong S_{k, n-k}$ in this case. Suppose that the theorem holds for $k$-degenerate graphs of order smaller than $n$ and consider $G$, a $k$ degenerate graph on $n$ vertices. Let $v \in V(G)$ such that $d(v)=\delta$ with $N(v)=\left\{u_{1}, \ldots, u_{\delta}\right\}$ and $G^{\prime}=G-v$. Then $G^{\prime}$ is $k$-degenerate. Hence by induction, (10), Lemma 5.6, and

Lemma 5.11,

$$
\begin{aligned}
M_{1}(G) & =M_{1}\left(G^{\prime}\right)+k^{2}+2 \sum_{i=1}^{\delta} d\left(u_{i}\right)-k \\
& \leq M_{1}\left(S_{k, n-1-k}\right)+k^{2}+2 \sum_{i=1}^{\delta} d\left(u_{i}\right)-k \\
& \leq(n-1)^{2} k+\left(k^{2}-2 k\right)(n-1)-k^{3}+1+k^{2}+2 k(n-1)-k \\
& =n^{2} k+\left(k^{2}-2 k\right) n-k^{3}+1=M_{1}\left(S_{k, n-k}\right) .
\end{aligned}
$$

Here equality holds if and only if $\delta(G)=k, \sum_{i=1}^{\delta(G)} d\left(u_{i}\right)=k(n-1)$ and $G^{\prime} \cong S_{k, n-1-k}$ i.e. $G \cong S_{k, n-k}$.

Since all $k$-trees are $k$-degenerate, the following corollary is immediate.

Corollary 5.16. Let $T_{n}^{k}$ be a $k$-tree on $n$ vertices. Then $M_{1}\left(T_{n}^{k}\right) \leq M_{1}\left(S_{k, n-k}\right)$ with equality holding if and only if $T_{n}^{k} \cong S_{k, n-k}$.

### 5.4. Sharp Upper and Lower Bounds for $M_{2}$ for $k$-trees.

In this section, we determine upper and lower bounds of $M_{2}$ for $k$-trees. Also, the corresponding extremal graphs will be characterized.

Theorem 5.17. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k+3$ vertices. Then $M_{2}\left(P_{n}^{k}\right) \leq M_{2}\left(T_{n}^{k}\right)$, and equality is reached if and only if $T_{n}^{k} \cong P_{n}^{k}$.

Proof. We will proceed by induction on $n$. There are just two $k$-trees on $k+3$ vertices, which are $P_{k+3}^{k}$ and $S_{k, n-k}$. By simple calculations, $M_{2}\left(P_{k+3}^{k}\right)=\frac{1}{2}\left(k^{4}+9 k^{3}+12 k^{2}-8 k+2\right)$ and $M_{2}\left(S_{k, n-k}\right)=\frac{1}{2}\left(k^{4}+9 k^{3}+12 k^{2}-4 k\right)$. Thus, the theorem holds true. Suppose that for $k$-trees of an order smaller than $n$ the theorem holds, and consider $T_{n}^{k}$. We may assume that
$T_{n}^{k} \not \equiv P_{n}^{k}$. Choose $v \in S_{1}\left(T_{n}^{k}\right)$ with $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}$ such that $\sum_{i=1}^{k} d\left(u_{i}\right)$ is minimal. Then $G^{\prime} \cong T_{n}^{k}-v$, a $k$-tree, is not isomorphic to $P_{n-1}^{k}$ by choice of $v$. Hence by (8),

$$
M_{2}\left(T_{n}^{k}\right)=M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2}
$$

Suppose as a special case that $n=5$ and $k=1$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12

$$
\begin{aligned}
M_{2}\left(T_{5}^{1}\right) & =M_{2}\left(G^{\prime}\right)+d\left(u_{1}\right)+\Psi\left(T_{5}^{1} ; v\right) \\
& >M_{2}\left(P_{4}^{1}\right)+2+2=12=M_{2}\left(P_{5}^{1}\right),
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.
Suppose then that $k+4 \leq n \leq 2 k$ which implies $k \geq 4$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$
\begin{aligned}
M_{2}\left(T_{n}^{k}\right)= & M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} \\
> & M_{2}\left(P_{n-1}^{k}\right)+(2 k-1)\left(2 k n-\frac{1}{2}(k(k+5)+(n-1)(n-2))\right. \\
& +\frac{1}{6}(n-k-1)\left(2 n k+5 n-n^{2}+5 k^{2}-5 k-6\right)+\frac{1}{2}(k(k-1)) \\
= & M_{2}\left(P_{n}^{k}\right)
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.
Suppose that $n=2 k+1 \geq k+4$ which implies $k \geq 3$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$
\begin{aligned}
M_{2}\left(T_{n}^{k}\right) & =M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} \\
& >M_{2}\left(P_{2 k}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2} k(k+1)\right)
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{1}{6}\left(5 k+6 k^{2}+k^{3}-\left(3 k+3 k^{2}\right)(2 k+1)\right)-\frac{1}{2}(k(k-1)) \\
= & \frac{1}{8}\left(29 k^{4}+2 k^{3}+3 k^{2}-2 k\right)=M_{2}\left(P_{2 k+1}^{k}\right),
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.
Suppose that $2 k+2 \leq n \leq 3 k-1$ which implies $k \geq 3$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$
\begin{aligned}
M_{2}\left(T_{n}^{k}\right)= & M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} \\
> & M_{2}\left(P_{n-1}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2} k(k+1)\right)+\frac{1}{6}\left(n^{3}-9 n^{2}\right. \\
& \left.-6 n^{2}+27 n k^{2}+36 n k+6 n-21 k^{3}-24 k^{2}-33 k-6\right) \\
& -\frac{1}{2}(k(k-1)) \\
= & M_{2}\left(P_{n}^{k}\right)
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.
Suppose that $n=3 k \geq k+4$ which implies $k \geq 2$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$
\begin{aligned}
M_{2}\left(T_{n}^{k}\right)= & M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} \\
> & M_{2}\left(P_{3 k-1}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2}(k(k+1))+-k(1-k)(2+k)\right. \\
& +\left(\frac{1}{6}\left(-6-3 k+9 k^{2}+15 k-9 k^{2}\right)-\frac{1}{2}(k(k-1))\right. \\
= & M_{2}\left(P_{3 k}^{k}\right)
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.

Suppose $n \geq \max (6,3 k+1)$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$
\begin{aligned}
M_{2}\left(T_{n}^{k}\right) & =M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} \\
& >M_{2}\left(P_{n-1}^{k}\right)+(2 k-1)\left(k^{2}+\frac{1}{2} k(k+1)\right)+k^{3}+k^{2}-\frac{1}{2}(k(k-1)) \\
& =M_{2}\left(P_{n}^{k}\right)
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.
Thus by the Principle of Mathematical Induction, $M_{2}\left(P_{n}^{k}\right) \leq M_{2}\left(T_{n}^{k}\right)$ with equality holding if and only if $T_{n}^{k} \cong P_{n}^{k}$.

Theorem 5.18. Let $T_{n}^{k}$ be a $k$-tree on $n \geq k$ vertices. Then $M_{2}\left(T_{n}^{k}\right) \leq M_{2}\left(S_{k, n-k}\right)$ with equality holding if and only if $T_{n}^{k} \cong S_{k, n-k}$.

Proof. We will proceed by induction on $n$. If $n \in\{k, k+1\}$, then $M_{2}\left(T_{n}^{k}\right) \leq M_{2}\left(K_{n}\right)$ with equality holding if and only if $G \cong K_{n}$. Note $K_{n} \cong S_{k, n-k}$ in this case. Suppose that the theorem holds for $k$-trees of smaller order and consider $T_{n}^{k}$, a $k$-tree on $n$ vertices. Let $v \in S_{1}\left(T_{n}^{k}\right)$ with $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}$ and $G^{\prime}=T_{n}^{k}-v$, which is a $k$-tree. By (8),

$$
M_{2}\left(T_{n}^{k}\right)=M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} .
$$

Thus by induction, Lemma 5.6, Lemma 5.9, and Lemma 5.13, we have

$$
\begin{aligned}
M_{2}\left(T_{n}^{k}\right) & =M_{2}\left(G^{\prime}\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} \\
& \leq M_{2}\left(S_{k, n-1-k}\right)+(2 k-1)(n k-k)+(n-k-1) k^{2}-\frac{1}{2}(k-1) k
\end{aligned}
$$

$$
\begin{aligned}
= & \frac{1}{2}\left(\left(3 k^{2}-k\right)(n-1)^{2}-\left(2 k^{3}+4 k^{2}-2 k\right)(n-1)+k(2 k-1)(k+1)\right) \\
& +(2 k-1)(n k-k)+(n-k-1) k^{2}-\frac{1}{2}(k-1) k \\
= & M_{2}\left(S_{k, n-k}\right)+2 n k^{2}-2 k^{2}-n k+k+\frac{1}{2}\left(-6 n k^{2}+3 k^{2}+2 n k-k\right) \\
& +\frac{1}{2}\left(2 k^{3}+4 k^{2}-2 k\right)+n k^{2}-k^{3}-k^{2}+\frac{1}{2}\left(k-k^{2}\right) \\
= & M_{2}\left(S_{k, n-k}\right)
\end{aligned}
$$

Here equality is obtained if and only if $G^{\prime} \cong S_{k, n-1-k}$ and $T_{n}^{k} \in \mathcal{J}^{k}$. Hence equality holds when $T_{n}^{k} \cong S_{k, n-k}$. Thus the theorem holds by the Principle of Mathematical Induction.

The upper bound for $M_{1}$ values given in Theorem 5.15 applies to $k$-degenerate graphs, a generalization of $k$-trees. However the proof techniques presented here are not sufficient to demonstrate similar results for a lower bound of $M_{1}$ values of maximally $k$-degenerate graphs and an upper bound of $M_{2}$ values of $k$-degenerate graphs. It may be interesting to show that for a maximally $k$-degenerate graph $G$ and a $k$-degenerate graph $G^{\prime}, M_{i}\left(P_{n}^{k}\right) \leq M_{i}(G)$ for $1 \leq i \leq 2$ and $M_{2}\left(G^{\prime}\right) \leq M_{2}\left(S_{k, n-k}\right)$.

## 6. The Zagreb Indices of Tree-Like $k$-trees

In 2010 Hou et al. characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum $M_{1}, M_{2}$ values respectively, as well as maximum $M_{1}, M_{2}$ values respectively. As mentioned in Chapter 5, they determined the following:

Theorem 6.1. [29] Let $G$ be a maximal outerplaner graph on $n \geq 4$ vertices. Then
(i) $M_{1}(G) \geq 16 n-38$, with equality holding if and only if $G \cong P_{n}^{2}$.
(ii) $M_{2}(G) \geq 32 n-100$, with equality holding if and only if $G \cong P_{n}^{2}$.

Theorem 6.2. [29] Let $G$ be a maximal outerplanar graph on $n \geq 4$ vertices. Then
(i) When $n=6, M_{1}(G) \leq 60$ with equality if and only if $G \cong S_{6}^{2}$ or $D_{6}^{2}$.
(ii) When $n \neq 6, M_{1}(G) \leq n^{2}+7 n-18$ with equality if and only if $G \cong S_{n}^{2}$.

Theorem 6.3. [29] Let $G$ be a maximal outerplanar graph on $n \geq 4$ vertices.
(i) When $n=6, M_{2}(G) \leq 96$ with equality if and only if $G \cong D_{6}^{2}$.
(ii) When $n \neq 6, M_{2}(G) \leq 3 n^{2}+n-19$ with equality if and only if $G \cong S_{n}^{2}$.

It has been shown that a graph $G$ is a maximal outerplanar graph if and only if $G$ is a tree-like 2-tree. By making this connection, it is a natural question to generalize the works of Hou et al. to tree-like $k$-trees. In Chapter 5, we deduced sharp upper and lower bounds of $M_{1}$ and $M_{2}$ for $k$-trees and showed that the $k$-path (respectively the $k$-star) uniquely obtains the sharp lower bound (respectively the sharp upper bound) of $M_{1}$ and $M_{2}$.


Figure 10. The 2-path, 2-diamond, and 2-star on 6 vertices

As the $k$-path is tree-like, it is clear that the sharp lower bounds of $M_{1}$ and $M_{2}$ for tree-like $k$-trees are obtained uniquely by the $k$-path. Hence, to generalize the results of Hou et al., we need to only consider upper bounds of $M_{1}$ of $M_{2}$ for tree-like $k$-trees.

### 6.1. Some Lemmas.

In this section, we give some lemmas that will be relied upon in subsequent sections.
Define $\mathcal{G}_{n}^{k}$ to be the class of tree-like $k$-trees as follows: Let $T_{n}^{k} \in \mathcal{G}_{n}^{k}$. Then there exists a vertex $v \in S_{1}\left(T_{n}^{k}\right)$ such that for any vertex $x \in V\left(T_{n}^{k}\right)-v,|N(x) \cap N(v)| \geq k-1$.

Lemma 6.4. Let $T_{n}^{k}$ be a tree-like $k$-tree on $n \geq k+2$ vertices and $v \in S_{1}\left(T_{n}^{k}\right)$ with $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}$. Then $\sum_{i=1}^{k} d\left(u_{i}\right) \leq(k-1)(n-1)+(k+1)$, with equality holding when $T_{n}^{k} \in \mathcal{G}_{n}^{k}$.

Proof. As $v \in S_{1}\left(T_{n}^{k}\right), G[N(v)] \cong K_{k}$. By Fact $2.21\left|\cap_{i=1}^{k} N\left(u_{i}\right)\right|=2$, so we may assume that $\{v, x\}=\cap_{i=1}^{k} N\left(u_{i}\right)$ where $x \neq v$. There are $n-(k+2)$ vertices in $V^{\prime}=V\left(T_{n}^{k}\right)-$ $\left\{v, x, u_{1}, u_{2}, \ldots, u_{k}\right\}$. It is clear to see that $\sum_{i=1}^{k} d\left(u_{i}\right)$ attains maximality if and only if for every $y \in V^{\prime},\left|N(y) \cap\left\{v_{1}, \ldots, v_{k}\right\}\right|=k-1$.

$$
\begin{gathered}
\text { Now } \sum_{i=1}^{k} d\left(u_{i}\right)=\sum_{u \in V\left(T_{n}^{k}\right)}|N(v) \cap N(u)| \text {. Thus } \sum_{i=1}^{k} d\left(u_{i}\right)=|N(v)|+|N(v) \cap N(x)|+ \\
\sum_{i=1}^{k}\left|N(v) \cap N\left(u_{i}\right)\right|+\sum_{y \in V^{\prime}}|N(v) \cap N(y)| \leq 2 k+k(k-1)+(n-k-2)(k-1)=(k-
\end{gathered}
$$

1) $(n-1)+(k+1)$, and equality holds if and only if $|N(v) \cap N(y)|=k-1$ for each $y \in V^{\prime}$. In other words, equality holds if and only if $T_{n}^{k} \in \mathcal{G}_{n}^{k}$.

The following lemmas demonstrate the Zagreb indices for the specific tree-like $k$-trees, the $k$-star and $k$-diamond and may be deduced through routine calculations.

Lemma 6.5. Let $S_{n}^{k}$ the $k$-spiral on $n$ vertices. Then

$$
M_{1}\left(S_{n}^{k}\right)=(k-1) n^{2}+\left(k^{2}+3\right) n-\left(k^{3}+k^{2}+2 k+2\right) .
$$

Lemma 6.6. For $k+1 \leq n \leq 2 k+2, M_{1}\left(D_{n}^{k}\right)=M_{1}\left(S_{n}^{k}\right)$.

Proof. The $k$-diamond $D_{n}^{k}$ is only defined for $k+1 \leq n \leq 2 k+2$. Let $V\left(D_{n}^{k}\right)=\left\{v_{1}, \ldots, v_{k+1}\right\}$ $\cup\left\{u_{1}, \ldots, u_{j}\right\}$ for some $j \in\{1, \ldots, k+1\}$ where $n=k+1+j$, and $G\left[\left\{v_{1}, \ldots, v_{k+1}\right\}\right] \cong K_{k+1}$. For any $u \in S_{1}\left(D_{n}^{k}\right)$, there exists a unique vertex $v \in V\left(\left\{v_{1}, \ldots, v_{k}\right\}\right)$ such that $v \notin N(u)$. Without loss of generality, $v_{i} \notin N\left(u_{i}\right)$ for $1 \leq i \leq j$. Thus, $d\left(v_{i}\right)=n-2$ for $1 \leq i \leq j$. That is, there are $j=n-(k+1)$ simplicial vertices, $n-(k+1)$ vertices of $\left\{v_{1}, \ldots, v_{k}\right\}$ of degree $n-2$, and $k+1-(n-(k+1))=2 k+2-n$ vertices of $\left\{v_{1}, \ldots, v_{k}\right\}$ of degree $n-1$. Hence

$$
\begin{aligned}
M_{1}\left(D_{n}^{k}\right) & =(n-(k+1)) k^{2}+(n-(k+1))(n-2)^{2}+(2 k+2-n)(n-1)^{2} \\
& =k n^{2}-n^{2}+n k^{2}+3 n-k^{3}-k^{2}-2 k-2 \\
& =(k-1) n^{2}+\left(k^{2}+3\right) n-\left(k^{2}+2\right)(k+1)=M_{1}\left(S_{n}^{k}\right) .
\end{aligned}
$$

Lemma 6.7. Let $S_{n}^{k}$ be the $k$-spiral on $n$ vertices. Then

$$
M_{2}\left(S_{n}^{k}\right)=\left(k^{2}-1\right) n^{2}-\left(k^{3}-k^{2}-k-3\right) n-\left(3 k^{2}+2 k+3\right)+\binom{k-1}{2}(n-1)^{2} .
$$

Lemma 6.8. Let $D_{n}^{k}$ be the $k$-diamond on $n$ vertices. Then

$$
M_{2}\left(D_{n}^{k}\right)=\frac{1}{2}\left(\left(3 k^{2}-3 k+1\right) n^{2}-\left(2 k^{3}-6 k+3\right) n-4 k^{2}-2 k+2\right) .
$$

For the remainder of the chapter, let $T_{n}^{k}$ be a tree-like $k$-tree such that $v \in S_{1}\left(T_{n}^{k}\right)$ and $N(v)=U=\left\{u_{1}, \ldots, u_{k}\right\}$. Let $N_{0}\left(u_{i}\right)=N\left(u_{i}\right)-N[v]$, and let

$$
\Psi\left(T_{n}^{k} ; v\right)=\sum_{x \in N_{0}\left(u_{1}\right)} d(x)+\sum_{x \in N_{0}\left(u_{2}\right)} d(x)+\ldots+\sum_{x \in N_{0}\left(u_{k}\right)} d(x) .
$$

Let $v^{\prime} \in S_{1}\left(T_{n}^{k}\right)-v$ and $N\left(v^{\prime}\right)=\left\{u_{1}^{\prime}, \ldots, u_{k}^{\prime}\right\}$. Arrange the vertices of $N\left(v^{\prime}\right)$ such that $u_{i}^{\prime} \in N(v)$ for $1 \leq i \leq t$ and $\mid N(v) \cap N\left(u_{i-1}^{\prime}\left|\geq\left|N(v) \cap N\left(u_{i}^{\prime}\right)\right|\right.\right.$ for $t+1 \leq i \leq k$. Then for $n \geq k+2$ and $v^{\prime} \in S_{1}\left(T_{n}^{k}\right)-v$,

$$
\begin{equation*}
\Psi\left(T_{n}^{k} ; v\right)=\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \tag{11}
\end{equation*}
$$

As mentioned in Chapter $5, d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|$ is a summand with $k$ summands with at least $t$ summands of value $k$ and at most $k-t$ summands of value at most $k-1$. It is clear then that $d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|$ is maximized when $t$ is maximized.

Lemma 6.9. Let $T_{n}^{k}$ be a tree-like $k$-tree on $k+1 \leq n \leq 2 k+2$ vertices. Then $\Psi\left(T_{n}^{k} ; v\right) \leq$ $k^{2}(n-k-1)$ with equality holding if and only if $T_{n}^{k} \cong D_{n}^{k}$.

Proof. Proceed by induction on the number of vertices. Suppose $n=k+1$, then $T_{n}^{k} \cong K_{k+1}$. Then for $v \in S_{1}\left(T_{n}^{k}\right), \Psi\left(T_{n}^{k} ; v\right)=0$, and thus the theorem holds. Suppose that the theorem is true for tree-like $k$-trees on $k+1 \leq n^{\prime}<n$ vertices, and consider $T_{n}^{k}$, a tree-like $k$-tree on $n$ vertices.

Let $v^{\prime} \in S_{1}\left(T_{n}^{k}\right)-v$. Then by (11),

$$
\Psi\left(T_{n}^{k} ; v\right)=\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|
$$

As $T_{n}^{k}$ is tree-like $t \leq k-1$. If $t=k-1$, then $\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right|=\left|N\left(u_{k}^{\prime}\right) \cap U\right| \leq k$. Also by induction, $\Psi\left(T_{n}^{k}-v^{\prime} ; v\right) \leq \Psi\left(D_{n-1}^{k} ; v\right)=k^{2}(n-k-2)$. Hence

$$
\begin{aligned}
\Psi\left(T_{n}^{k} ; v\right) & =\Psi\left(T_{n}^{k}-v^{\prime} ; v\right)+d\left(v^{\prime}\right) t+\sum_{i=t+1}^{k}\left|N\left(u_{i}^{\prime}\right) \cap U\right| \\
& \leq \Psi\left(D_{n-1}^{k} ; v\right)+k(k-1)+k \\
& =k^{2}(n-k-2)+k^{2} \\
& =k^{2}(n-k-1)
\end{aligned}
$$

with equality holding if and only if $T_{n}^{k}-v^{\prime} \cong D_{n-1}^{k}, t=k-1$, and $\left|N\left(u_{k}^{\prime}\right) \cap N(v)\right|=k$. Hence equality holds if and only if $T_{n}^{k} \cong D_{n}^{k}$. By the Principle of Mathematical Induction, the theorem holds.

### 6.2. Sharp Upper Bounds of $M_{1}$ of Tree-like $k$-trees.

In this section, we determine the upper bounds of $M_{1}$ of tree-like $k$-trees, and the corresponding extremal graphs are characterized.

Theorem 6.10. Let $T_{n}^{k}$ be a tree-like $k$-tree on $n \geq k+1$ vertices. Then $M_{1}\left(T_{n}^{k}\right) \leq(k-$ 1) $n^{2}+\left(k^{2}+3\right) n-\left(k^{2}+2\right)(k+1)$. Equality is reached if and only if $T_{n}^{k} \in\left\{D_{n}^{k}, S_{n}^{k}\right\}$. In particular, if $n \geq 2 k+3$, then equality is reached if and only if $T_{n}^{k} \cong S_{n}^{k}$.

Proof. Proceed by induction on $n$. If $n=k+1$, then $T_{n}^{k} \cong K_{k+1}$, and $M_{1}\left(K_{k+1}\right)=(k+1) k^{2}=$ $2 k^{2}+k^{3}-k^{2}=2 k^{2}+(k-1) k^{2}=2 k^{2}+(k-1)(n-1)^{2}+(n-(k+1))(k+1)^{2}$. Suppose $n=k+2$. Then $T_{n}^{k}$ is a $k$-clique bound by two simplicial vertices. Hence $M_{1}\left(T_{n}^{k}\right)=2 k^{2}+k(k+1)^{2}$, and it may be verified that $(k-1) n^{2}+\left(k^{2}+3\right) n-\left(k^{3}+k^{2}+2 k+2\right)=k^{3}+2 k^{2}+2 k^{2}+k=2 k^{2}+k(k+1)^{2}$.

Suppose the theorem is true for tree-like $k$-trees on $k+1 \leq n^{\prime}<n$ vertices, and consider $T_{n}^{k}$ vertices. Let $v \in S_{1}\left(T_{n}^{k}\right)$ where $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}$. Let $G^{\prime}=T_{n}^{k}-v$. By Lemma 6.5
and the inductive hypothesis,

$$
\begin{aligned}
M_{1}\left(G^{\prime}\right) & \leq M_{1}\left(S_{n-1}^{k}\right) \\
& =(k-1)(n-1)^{2}+\left(k^{2}+2\right)(k+1) \\
& =(k-1) n^{2}+\left(5-2 k+k^{2}\right) n-\left(k^{3}+2 k^{2}+k+6\right) .
\end{aligned}
$$

Now $d_{G^{\prime}}\left(u_{i}\right)=d\left(u_{i}\right)-1$ for $1 \leq i \leq k$ and $d_{G^{\prime}}(x)=d(x)$ for all $x \in V-\left\{u_{1}, \ldots, u_{k}\right\}$. Then by Lemma 6.4,

$$
\begin{aligned}
M_{1}\left(T_{n}^{k}\right) & =M_{1}\left(G^{\prime}\right)+(d(v))^{2}+\left(\left(d\left(u_{1}\right)\right)^{2}-\left(d_{G^{\prime}}\left(u_{1}\right)\right)^{2}\right)+\ldots+\left(\left(d\left(u_{k}\right)\right)^{2}-\left(d_{G^{\prime}}\left(u_{k}\right)\right)^{2}\right) \\
& =M_{1}\left(G^{\prime}\right)+\left(\left(d\left(u_{1}\right)\right)^{2}-\left(d\left(u_{1}\right)-1\right)^{2}\right)+\ldots+\left(\left(d\left(u_{k}\right)\right)^{2}-\left(d\left(u_{k}\right)-1\right)^{2}\right) \\
& =M_{1}\left(G^{\prime}\right)+k^{2}+2\left[d\left(u_{1}\right)+\ldots+d\left(u_{k}\right)\right]-k \\
& \leq M_{1}\left(S_{n-1}^{k}\right)+k^{2}-k+2((k-1)(n-1)+(k+1)) \\
& =(k-1) n^{2}+\left(5-2 k+k^{2}\right) n-\left(k^{3}+2 k^{2}+k+6\right)+k^{2}-k+2 k n-2 n+4 \\
& =(k-1) n^{2}+\left(k^{2}+3\right) n-\left(k^{3}+k^{2}+2 k+2\right)
\end{aligned}
$$

Here, equality is reached only when $G^{\prime} \cong S_{n-1}^{k}$ or $G^{\prime} \cong D_{n-1}^{k}$, and $T_{n}^{k} \in \mathcal{G}_{n}^{k}$. If $G^{\prime} \cong S_{n-1}^{k}$ and $T_{n}^{k} \in \mathcal{G}_{n}^{k}$, then clearly $T_{n}^{k} \cong S_{n}^{k}$.

Suppose $G^{\prime} \cong D_{n-1}^{k}$. If $G^{\prime} \cong D_{2 k+2}^{k}$ and $T_{n}^{k} \in \mathcal{G}_{n}^{k}$, then $T_{n}^{k}$ is not tree-like. Hence $n \leq 2 k+2$, and clearly $T_{n}^{k} \cong D_{n}^{k}$.

### 6.3. Sharp Upper Bounds of $M_{2}$ for Tree-like $k$-trees.

In this section, we determine upper bounds of $M_{2}$ for tree-like $k$-trees on less than $2 k+$ 2 vertices and characterize the corresponding extremal graphs. Additionally, we state a conjecture for sharp upper bounds of $M_{2}$ for tree-like $k$-trees on at least $2 k+3$ vertices.

Theorem 6.11. Let $T_{n}^{k}$ be a tree-like $k$-tree on $k+1 \leq n \leq 2 k+2$ vertices. Then $M_{2}\left(T_{n}^{k}\right) \leq$ $M_{2}\left(D_{n}^{k}\right)$ with equality holding if and only if $T_{n}^{k} \cong D_{n}^{k}$.

Proof. Proceed by induction on $n$. If $n=k+1, k+2$. Suppose that for tree-like $k$-trees on $k+1 \leq n^{\prime}<n \leq 2 k+2$ vertices, and let $T_{n}^{k}$ be a tree-like $k$-tree on $n \leq 2 k+2$ vertices. Let $v \in S_{1}\left(T_{n}^{k}\right)$ and $N(v)=\left\{u_{1}, \ldots, u_{k}\right\}$. From Chapter 5 , we know that

$$
\begin{equation*}
M_{2}\left(T_{n}^{k}\right)=M_{2}\left(T_{n}^{k}-v\right)+(2 k-1) \sum_{i=1}^{k} d\left(u_{i}\right)+\Psi\left(T_{n}^{k} ; v\right)-\binom{k}{2} . \tag{12}
\end{equation*}
$$

Thus by Lemma 6.8, Lemma 6.4, and Lemma 6.9

$$
\begin{aligned}
M_{2}\left(T_{n}^{k}\right) & \leq M_{2}\left(D_{n-1}^{k}\right)+(2 k-1)((k-1)(n-1)+k+1)+k^{2}(n-k-1)-\binom{k}{2} \\
& =M_{2}\left(D_{n-1}^{k}\right)+\frac{1}{2}\left(6 n k^{2}-6 n k+2 n-2 k^{3}-3 k^{2}+9 k-4\right) \\
& \left.=\frac{1}{2}\left(3 k^{2}-3 k+1\right)\left(n^{2}-2 n+1\right)-\left(2 k^{3}-6 k+3\right)(n-1)-4 k^{2}-2 k+2\right) \\
& +\frac{1}{2}\left(6 n k^{2}-6 n k+2 n-2 k^{3}-3 k^{2}+9 k-4\right) \\
& \left.=\frac{1}{2}\left(3 k^{2}-3 k+1\right) n^{2}-\left(2 k^{3}-6 k+3\right) n-4 k^{2}-2 k+2\right) \\
& +\frac{1}{2}\left(\left(3 k^{2}-3 k+1\right)(-2 n+1)+\left(2 k^{3}-6 k+3\right)\right)+ \\
& +\frac{1}{2}\left(6 n k^{2}-6 n k+2 n-2 k^{3}-3 k^{2}+9 k-4\right) \\
& =M_{2}\left(D_{n}^{k}\right)-\frac{1}{2}\left(6 n k^{2}-6 n k+2 n-2 k^{3}-3 k^{2}+9 k-4\right) \\
& +\frac{1}{2}\left(6 n k^{2}-6 n k+2 n-2 k^{3}-3 k^{2}+9 k-4\right) \\
& =M_{2}\left(D_{n}^{k}\right) .
\end{aligned}
$$

Hence, by the Principle of Mathematical Induction, the theorem holds.

The $k$-diamond is only defined for $k \leq n \leq 2 k+2$ vertices. We strongly believe that for $n \geq 2 k+3$, the $k$-spiral uniquely obtains the strong upper bound for $M_{2}$ among tree-like
$k$-trees. However, the techniques presented here are not sufficient to prove that such is the case. Instead we state the following conjecture.

Conjecture 6.12. Let $T_{n}^{k}$ be a tree-like $k$-tree on $n$ vertices such that $T_{n}^{k} \not \neq D_{n}^{k}$. Then $M_{2}\left(T_{n}^{k}\right) \leq M_{2}\left(S_{n}^{k}\right)$ with equality holding if and only if $T_{n}^{k} \cong S_{n}^{k}$.

We believe that the ideas presented in Chapter 7 , once generalized to $k$-trees, will provide a framework to prove Conjecture 6.12.

## 7. Tree Genealogies

Continuing his investigation of the independence polynomials of trees in 1995, Wingard determined sharp lower and upper bounds of $f_{s}$ for trees for $s \geq 0$ on $n$ vertices and characterized the unique trees that obtain these bounds.

Theorem 7.1. [44] Let $T_{n}$ be a tree with $n$ vertices. Then for any $s \geq 2,\binom{n-s+1}{s} \leq f_{s}\left(T_{n}\right) \leq$ $\binom{n-1}{s}$.

Theorem 7.2. [44] Let $T_{n}$ be a tree with $n \geq 2 s$ vertices for $s \geq 3$. If $f_{s}\left(T_{n}\right)=\binom{n-s+1}{s}$, then $T_{n} \cong P_{n}$.

Theorem 7.3. [44] Let $T_{n}$ be a tree with $n$ vertices. If $f_{s}\left(T_{n}\right)=\binom{n-1}{s}$ for $3 \leq s \leq n-1$, then $T_{n} \cong S_{n}$.

$P_{7}$

$S_{7}$

Figure 11. The path and star on 7 vertices

As stated in Chapter 5, Das and Gutman characterized the Zagreb indices for trees and determined the unique tree that obtains minimum $M_{1}$ and $M_{2}$ values respectively, as well as maximum $M_{1}$ and $M_{2}$ values respectively in 2004.

Theorem 7.4. [12, 20] Let $T$ be any tree of order $n$. Then
(i) $4 n-6 \leq M_{1}(T) \leq n^{2}-n$, the left equality holds if and only if $T \cong P_{n}$, and the right equality holds if and only if $T \cong S_{n}$.
(ii) $4 n-8 \leq M_{2}(T) \leq n^{2}-2 n+1$, the left equality holds if and only if $T \cong P_{n}$ and the right equality holds if and only if $T \cong S_{n}$.

From Wingard and Das and Gutman we deduce that $P_{n}$ and $S_{n}$ can be thought of as the extremal trees in regards to $f_{s}, M_{1}$, and $M_{2}$.

Let $T$ be a tree. Define a starring triple $r$ to be $r=\{v, u, x\}$ where $v \in S_{1}(T), u \in N(v)$, and $x \in V(T)-\{v, u\}$. Let $R_{1}(T)$ be the set of starring triples of $T$. For $r \in R_{1}(T), T(r)$ is the tree with $V(T(r))=V(T)$ and $E(T(r))=(E(T) \cup\{v x\})-\{v u\}$. Let $g_{1}: R_{1} \rightarrow \mathbb{Z}$ be such that for $r=\{v, u, x\} \in R_{1}(T), g_{1}(r)=d_{T}(x)-d_{T}(u)$. If $g_{1}(r) \geq 0$, then $T(r)$ is said to be a 1 -descendant of $T$. If $g_{1}(r) \leq-2$, then $T(r)$ is said to be a 1 -ancestor of $T$.


Figure 12. A tree $T$ and $T(r)$

Theorem 7.5. Let $T$ and $T^{\prime}$ be trees. Then $T$ is an 1-ancestor of $T^{\prime}$ if and only if $T^{\prime}$ is a 1-descendant of $T$.

Proof. Let $r=\{v, u, x\}$ be a starring triple of $T$, and suppose $g_{1}(r) \geq 0$. Then $T(r)$ is a 1-descendant of $T$, and $r^{\prime}=\{v, x, u\}$ is a starring triple of $T(r)$. Note that $T(r)\left(r^{\prime}\right) \cong T$. Now $d_{T(r)}(u)=d_{T}(u)-1$ and $d_{T(r)}(x)=d_{T}(x)+1$. Thus $g_{1}\left(r^{\prime}\right)=d_{T(r)}(u)-d_{T(r)}(x)=$
$d_{T}(u)-d_{T}(x)-2 \leq-2$. Thus $T$ is a 1 -ancestor of $T(r)$. The argument is reversible to show that if $T(r)$ is a 1-ancestor of $T$, then $T$ is a 1-descendant of $T(r)$.

It is easy to see that any tree $T \not \not S_{n}$ has a 1-descendant. Hence there is a sequence of trees $\left\{T_{i}\right\}_{i=0}^{\beta}$ such that $T_{0} \cong T, T_{\beta} \cong S_{n}$, and $T_{i+1}$ is a 1-descendant of $T_{i}$ for $i \leq 0 \leq \beta-1$. Then we say that $T_{i}$ is a $1^{i}$-descendant of $T$ for $1 \leq i \leq \beta$.

Now there are trees that have no 1-ancestor as in Figure 13. Hence there is a sequence of trees $\left\{T_{i}\right\}_{i=0}^{\beta_{2}}$ such that $T_{0} \cong T, T_{\beta_{2}} \cong T^{\prime}$ where $T^{\prime}$ is a tree that has no 1-ancestor, and $T_{i+1}$ is a 1 -ancestor of $T_{i}$ for $0 \leq i \leq \beta_{2}$.


Figure 13. A tree with no 1 -ancestor

For the tree $T$, we may generalize the starring triple as follows: Let $r=\{v, u, x\}$ be such that
(i) $v \in S_{1}(T)$
(ii) the $v u$-path $P$ is of length $p$,
(iii) for any $y \in V(P)-\{v, u\}, d(y)=2$,
(iv) $x \in V(T)-V(P)$.

Let $R_{p}(T)$ be the set of such triples. For $r \in R_{p}(T), T(r)$ is the tree with $V(T(r))=V(T)$ and $E(T(r))=\left(E(T) \cup\left\{y^{\prime} x\right\}\right)-\left\{y^{\prime} u\right\}$ where $y^{\prime} \in N(u) \cap V(P)$. Define $g_{p}: R_{p} \rightarrow \mathbb{Z}$ such that $g_{p}(r)=d_{T}(x)-d_{T}(u)$. If $g_{p}(r) \geq 0$, then $T(r)$ is said to be a $p$-descendant of $T$. If $g_{p}(r) \leq-2$, then $T(r)$ is said to be a $p$-ancestor of $T$.

Theorem 7.6. Let $T$ and $T^{\prime}$ be trees. Then $T$ is a $p$-ancestor of $T^{\prime}$ if and only if $T^{\prime}$ is a p-descendant of $T$.

Proof. Let $r=\{v, u, x\} \in R_{p}(T)$, and suppose $g_{p}(r) \geq 0$. Then $T(r)$ is a $p$-descendant of $T$, and $r^{\prime}=\{v, x, u\} \in R_{p}(T(r))$. Note that $T(r)\left(r^{\prime}\right) \cong T$. Now $d_{T(r)}(u)=d_{T}(u)-1$ and $d_{T(r)}(x)=d_{T}(x)+1$. Thus $g_{p}\left(r^{\prime}\right)=d_{T(r)}(u)-d_{T(r)}(x)=d_{T}(u)-d_{T}(x)-2 \leq-2$. Thus $T$ is a $p$-ancestor of $T(r)$. The argument is reversable to show that if $T(r)$ is a $p$-ancestor of $T$, then $T$ is a $p$-descendant of $T(r)$.

Let $T$ and $T^{\prime}$ be trees, and suppose that $T^{\prime}$ is the $p_{1}$-descendant of a $p_{2}$-descendant of $T$. Then we say that $T^{\prime}$ is a $p_{1}, p_{2}$-descendant of $T$, and $T$ is a $p_{2}, p_{1}$-ancestor of $T^{\prime}$. Suppose that $p_{1}=p_{2}$. Then $T^{\prime}$ is a $p_{1}^{2}$-descendant of $T$, and $T$ is a $p_{1}^{2}$-ancestor of $T^{\prime}$.

Theorem 7.7. Let $T \not \not P_{n}$ be a tree on $n$ vertices. Then $P_{n}$ is a $p_{1}^{i_{1}}, p_{2}^{i_{2}}, \ldots, p_{j}^{i_{j}}$-ancestor of $T$ for some $j \geq 1$.

Proof. All we must show is that for any tree with more than two leaves, there exists a $p$ ancestor of $T$ for some $p \geq 1$ such that this $p$-ancestor has fewer leaves than $T$. Let $v \in S_{1}(T)$. As $T \not \not P_{n}$, there is a vertex $u$ such that $d(u) \geq 3$, and let $S=\{u \mid d(u) \geq 3\}$. Choose $u \in S$ such that $d(v, u)<d\left(v, u^{\prime}\right)$ for all $u^{\prime} \in S-u$, and let $P$ be the $v u$-path in $T$. Additionally, let $y^{\prime} \in N(u) \cap V(P)$ and $x \in S_{1}(T)-v$. Clearly $x \notin V(P)$. Thus $r=\{v, u, x\} \in R_{p}(T)$ where $p$ is the length of $P$, and $g_{p}(r)=d(x)-d(u) \leq 1-3=-2$. Thus $T(r)$ is a $p$-ancestor of $T$.

If $\left|S_{1}(T(r))\right| \geq 3$, then $T(r) \not \not P_{n}$, and so $T(r)$ has a $p_{1}$-ancestor $T^{\prime}(r)$ with fewer leaves than $T(r)$. Hence $T^{\prime}(r)$ is a $p p_{1}$-ancestor of $T$. A reiteration of this process yields a tree with two leaves, i.e. $P_{n}$, that is a $p_{1}^{i_{1}}, p_{2}^{i_{2}}, \ldots, p_{j}^{i_{j}}$-ancestor of $T$ for some $j \geq 1$.

By Theorem 7.7, we see that for a given tree $T$, there is a sequences of trees $\left\{T_{i}\right\}_{i=0}^{\beta_{3}}$ such that $T_{0} \cong T, T_{\beta_{3}} \cong P_{n}$, and $T_{i+1}$ is a $p_{i+1}$-ancestor of $T_{i}$ for $0 \leq i \leq \beta_{3}$. We may now construct a sequence of trees that we define as a genealogy.

Definition 7.8. Let $T$ be a tree on $n$ vertices. Then the sequence of trees on $n$ vertices $\left\{T_{i}\right\}_{i=0}^{\beta}$ satisfying
(i) $T_{0} \cong P_{n}$,
(ii) $T_{\beta_{2}} \cong T$, for some $\beta_{2}, 0 \leq \beta_{2} \leq \beta$,
(iii) $T_{i+1}$ is a $p_{i+1}$-descendant of $T_{i}$ for $0 \leq i \leq \beta-1$,
(iv) $T_{\beta} \cong S_{n}$,
is said to be a genealogy of $T$.

The definition of a genealogy of a tree says that for a given tree, $T$, there is a sequence of trees starting with $P_{n}$ and ending with $S_{n}$ such that $T$ is a member of this sequence. Additionally, given a tree $T_{i}$ in this sequence, $T_{i+1}$ is a $p_{i+1}$-descendant for $0 \leq i \leq \beta-1$. In the subsequent sections, we will show that a genealogy of a tree creates a partial ordering of trees with respect to $f_{s}$ and $M_{1}$.

### 7.1. Independent Sets of a Tree and Its Descendants.

By investigating the relationship between a tree and other trees in a genealogy of that tree, we may generalize Theorem 7.1.

First define the family $\mathcal{F}$ of trees as follows; let $T \in \mathcal{F}$. Then $V(T)=\left\{u_{1}, \ldots, u_{n_{1}}\right\}$ $\cup\left\{v_{1}, \ldots, v_{n_{2}}\right\}$ where $v_{i} \in S_{1}(T)$ for $1 \leq i \leq n_{2}, G\left[\left\{u_{1}, \ldots, u_{n_{1}}\right\}\right] \cong P_{n_{1}}, N\left(v_{i}\right) \subseteq\left\{u_{1}, \ldots, u_{n_{1}}\right\}$ for $1 \leq i \leq n_{2}$. Thus we may state the following lemma.

Lemma 7.9. Let $T$ be a tree in $\mathcal{F}$ such that $r=\left\{u_{1}, v_{1}, v_{n_{1}}\right\} \in R_{1}(T)$ and $g_{1}(r) \geq 0$. Then $f_{s}(T) \leq f_{s}(T(r))$ for $s \geq 0$.

Proof. For the vertex set $S$, let $f_{s, S}(T)$ denote the number of independent sets of cardinality $s$ in $T$ containing $S$, and let $f_{s, \bar{S}}(T)$ denote the number of independent sets of cardinality $s$ in $T$ not containing $S$. Then $f_{s}(T)=f_{s, S}(T)+f_{s, \bar{S}}(T)$.

Let $v_{1}=v, u_{1}=u$, and $u_{n_{1}}=x$. Then $r=\{v, u, x\}$, and let $I$ be an independent set of $T$. If $\{v, x\} \nsubseteq I$, then $I$ is an independent set of $T(r)$. Thus $f_{s, \overline{\{v, x\}}}(T) \leq f_{s, \overline{\{v, x\}}}(T(r))$ for $s \geq 0$.

Let $I$ be an independent set of $T$ such that $\{v, x\} \subseteq I$. Then $I$ is not independent in $T(r)$. Note that as $T(r)$ is a 1-descendant of $T, d(x) \geq d(u)$, and thus $|N(x)| \geq|N(u)|$. Also it is clear that for any subset $S^{\prime}$ of $N(x), S^{\prime} \nsubseteq I$. Let $S=(N(u)-v) \cap I$. Then as $|S| \leq|N(u)|-1$ there exists at least one set $S^{\prime} \subseteq N(x)$ such that $\left|S^{\prime}\right|=|S|$ and $x^{\prime} \in S_{1}(T)$ for all $x^{\prime} \in S^{\prime}$. Hence there exists at least one independent set $I^{\prime}$ of $T(r)$ such that $I^{\prime}=(I-(\{x\} \cup S)) \cup\left(\{u\} \cup S^{\prime}\right)$ for some $S^{\prime} \subseteq N(x)$ such that $|S|=\left|S^{\prime}\right|$ and $x^{\prime} \in S_{1}(T)$ for all $x^{\prime} \in S^{\prime}$. Note that $\left|I^{\prime}\right|=|I|$, and $I^{\prime}$ is not an independent set of $T$. Thus $f_{s,\{v, x\}}(T) \leq f_{s,\{v, x\}}(T(r))$ for $s \geq 0$.

Then $f_{s}(T)=f_{s,\{v, x\}}(T)+f_{s, \overline{\{v, x\}}}(T) \leq f_{s,\{v, x\}}(T(r))+f_{s, \overline{\{v, x\}}}(T(r))=f_{s}(T(r))$ for $s \geq$ 0.

Now Theorem 7.1 and Theorem 7.3 of Wingard may now be extended.

Theorem 7.10. Let $T$ be a tree, and $r \in R_{1}(T)$. If $T(r)$ is a 1-descendant of $T$, then $f_{s}(T) \leq f_{s}(T(r))$ for $s \geq 0$.

Proof. Proceed by induction on the number of vertices $n$. There is nothing to show for $1 \leq n \leq 3$. Suppose that $n=4$, then $T \in\left\{P_{4}, S_{4}\right\}$. Now $P_{4}$ is an ancestor of $S_{4}$, and $f_{s}\left(P_{4}\right) \leq f_{s}\left(S_{4}\right)$ for $s \geq 0$. Suppose that for trees $T$ and $T^{\prime}$ on $4 \leq n^{\prime}<n$ vertices, such that $T^{\prime}$ is a descendant of $T, f_{s}(T) \leq f_{s}\left(T^{\prime}\right)$, and consider $T$ a tree on $n$ vertices.

Let $r=\{v, u, x\} \in R$ such that $g_{1}(r) \geq 0$. Then $T(r)$ is a descendant of $T$, and let $P$ be the $u x$-path in $T$. Suppose that there for every $v^{\prime} \in S_{1}(T)-v$ with support vertex $u^{\prime}$, $u^{\prime} \in V(P)$. Then $T \in \mathcal{F}$, and by Lemma $7.9 f_{s}(T) \leq f_{s}(T(r))$. Thus we may assume that there exists $v^{\prime} \in S_{1}(T)-v$ with support vertex $u^{\prime}$ such that $u^{\prime} \notin V(P)$. By the vertex reduction identity,

$$
\begin{aligned}
f_{s}(T) & =f_{s}\left(T-v^{\prime}\right)+f_{s-1}\left(T-N\left[v^{\prime}\right]\right) \\
f_{s}(T(r)) & =f_{s}\left(T(r)-v^{\prime}\right)+f_{s-1}\left(T(r)-N\left[v^{\prime}\right]\right)
\end{aligned}
$$

As $v^{\prime} \notin\{v, u, x\}, T(r)-v^{\prime}$ is a descendant of $T-v^{\prime}$. Thus, by induction $f_{s}\left(T-v^{\prime}\right) \leq$ $f_{s}\left(T(r)-v^{\prime}\right)$. Now $T-N\left[v^{\prime}\right]$ and $T(r)-N\left[v^{\prime}\right]$ are forrests on $l$ connected components. Also there are $l-1$ connected components of $T-N\left[v^{\prime}\right], H_{i}$ for $1 \leq i \leq l-1$ and $l-1$ connected components of $T(r)-N\left[v^{\prime}\right], H(r)_{i}$ for $i \leq i \leq l-1$ such that $\cup_{i}^{l-1} H_{i} \cong \cup_{i}^{l-1} H(r)_{i}$. Then $\Pi_{i}^{l-1} f_{s}\left(H_{i}\right)=\Pi_{i}^{l-1} f_{s}\left(H(r)_{i}\right)$. Let $H \cong T-\cup_{i}^{l-1} H_{i}$ and $H(r) \cong T(r)-\cup_{i}^{l-1} H(r)_{i}$. Then $f_{s}\left(T-N\left[v^{\prime}\right]\right)=\left(\Pi_{i}^{l-1} f_{s}\left(H_{i}\right)\right) f_{s}(H)$, and $f_{s}\left(T(r)-N\left[v^{\prime}\right]\right)=\left(\Pi_{i}^{l-1} f_{s}\left(H(r)_{i}\right)\right) f_{s}(H(r))$ for $s \geq 0$.

Note that $\{v, u, x\} \subseteq V(H),\{v, u, x\} \subseteq V(H(r))$, and $H-v \cong H(r)-v$. That is, $H(r)$ is a 1-descendant of $H$. Then, by induction $f_{s}(H) \leq f_{s}(H(r))$ for $s \geq 0$. Hence $f_{s}(T)=f_{s}\left(T-v^{\prime}\right)+f_{s-1}\left(T-N\left[v^{\prime}\right]\right)=f_{s}\left(T-v^{\prime}\right)+\left(\Pi_{i}^{l-1} f_{s-1}\left(H_{i}\right)\right) f_{s-1}(H) \leq f_{s}(T(r)-$ $\left.v^{\prime}\right)+\left(\Pi_{i}^{l-1} f_{s-1}\left(H(r)_{i}\right)\right) f_{s-1}(H(r))=f_{s}\left(T(r)-v^{\prime}\right)+f_{s-1}\left(T(r)-N\left[v^{\prime}\right]\right)=f_{s}(T(r))$. Thus by induction, the theorem holds.

Theorem 7.11. Let $T$ be a tree and $T^{\prime}$ be a $1^{j}$-descendant of $T$ for some $j \geq 1$. Then $f_{s}(T) \leq f_{s}\left(T^{\prime}\right)$ for $s \geq 0$.


Figure 14. $T_{5}$

Theorem 7.12. Let $T$ and $T^{\prime}$ be trees on $n$ vertices, and let $T^{\prime}$ be a p-descendant of $T$ for some $p \geq 1$. Then $f_{s}(T) \leq f_{s}\left(T^{\prime}\right)$ for $s \geq 0$.

Proof. We will show that for any $p$-ancestor $T^{\prime}$ of a given tree $T, f_{s}\left(T^{\prime}\right) \leq f_{s}(T)$ for $s \geq 0$ by induction on $n$. There is only one tree on $1 \leq n \leq 3$ vertices. Consider $n=4$, then $T^{\prime} \cong P_{4}$ and $T \cong S_{4}$, and the theorem holds. Suppose $n=5$. Then either $T^{\prime} \cong P_{5}$ and $T \in\left\{T_{5}, S_{5}\right\}$, or $T^{\prime} \cong T_{5}$ and $T \cong S_{5}$ where $T_{5}$ is the tree pictured in Figure 14. In either case, the theorem clearly holds.

Suppose that if $T^{\prime}$ is a tree on $1 \leq n^{\prime}<n$ vertices and is a $p$-ancestor of another tree $T$ for some $p \geq 1$, then $f_{s}\left(T^{\prime}\right) \leq f_{s}(T)$ for $s \geq 0$. Let $T$ be a tree on $n$ vertices, and let $T^{\prime}$ be a $p$-ancestor of $T$. Then there exists $r=\{v, u, x\} \in R_{p}(T)$ such that $g_{p}(r) \leq-2$ and $T(r) \cong T^{\prime}$. Let $P$ be the $v$, u-path in $T$ of length $p$, and let $y \in N(u) \cap V(P)$. By Proposition 3.1,

$$
\begin{align*}
& f_{s}(T)=f_{s}(T-v)+f_{s-1}\left(T-N_{T}[v]\right)  \tag{13}\\
& f_{s}\left(T^{\prime}\right)=f_{s}\left(T^{\prime}-v\right)+f_{s-1}\left(T^{\prime}-N_{T^{\prime}}[v]\right)
\end{align*}
$$

Suppose that $p=1$, then by Theorem $7.10 f_{s}(T) \geq f_{s}\left(T^{\prime}\right)$ for $s \geq 0$. Suppose that $p=2$, then $T-v$ is a 1 -descendant of $T^{\prime}-v$, and $T-N_{T}[v] \cong T^{\prime}-N_{T^{\prime}}[v]$. Hence, by (13), for
$s \geq 0$,

$$
f_{s}(T)=f_{s}(T-v)+f_{s-1}\left(T-N_{T}[v]\right) \geq f_{s}\left(T^{\prime}-v\right)+f_{s-1}\left(T^{\prime}-N_{T^{\prime}}[v]=f_{s}\left(T^{\prime}\right) .\right.
$$

If $p \geq 3$, then $T^{\prime}-v$ is a $(p-1)$-ancestor of $T-v$, and so by induction $f_{s}(T-v) \geq f_{s}\left(T^{\prime}-v\right)$ for $s \geq 0$. Also, $f_{s}\left(T-N_{T}[v]\right) \geq f_{s}\left(T^{\prime}-N_{T^{\prime}}[v]\right)$ for $s \geq 0$ by induction as $T^{\prime}-N_{T^{\prime}}[v]$ is a ( $p-2$ )-ancestor of $T-N_{T}[v]$. Hence, by (13),

$$
f_{s}(T)=f_{s}(T-v)+f_{s-1}\left(T-N_{T}[v]\right) \geq f_{s}\left(T^{\prime}-v\right)+f_{s-1}\left(T^{\prime}-N_{T^{\prime}}[v]\right)=f_{s}\left(T^{\prime}\right),
$$

for $s \geq 0$, and so the theorem holds by the Principle of Mathematical Induction.

If $T$ is a $p_{1}$-descendant of $T^{\prime}$, and $T^{\prime}$ is a $p_{2}$-descendant of $T^{\prime \prime}$, then $f_{s}\left(T^{\prime \prime}\right) \leq f_{s}\left(T^{\prime}\right) \leq f_{s}(T)$ for $s \geq 0$. By the transitive property, the following theorem immediately follows.

Theorem 7.13. Let $T$ and $T^{\prime}$ be trees on $n$ vertices such that $T^{\prime}$ is a $p_{1}^{i_{1}}, p_{2}^{i_{2}}, \ldots, p_{j}^{i_{j}}$ descendant of $T$. Then $f_{s}(T) \leq f_{s}\left(T^{\prime}\right)$ for $s \geq 0$.

By Theorem 7.10 and Theorem 7.12, we may state the following theorem.

Theorem 7.14. Let $T$ be a tree on $n$ vertices and $\left\{T_{i}\right\}_{i=0}^{\beta}$ be a genealogy of $T$. Then $f_{s}\left(T_{i}\right) \leq f_{s}\left(T_{i+1}\right)$ for $0 \leq i \leq \beta-1$ and $s \geq 0$.

Thereom 7.14 is an extension of Theorem 7.1 as for any tree $T$ we may find a sequence of trees $\left\{T_{i}\right\}_{i=0}^{\beta}$ such that $f_{s}\left(T_{i}\right) \leq f_{s}\left(T_{i+1}\right)$ for $0 \leq i \leq \beta-1$ and $s \geq 0$. Thus, a genealogy of $T$ along with $f_{s}$ for $s \geq 0$ yields a partial ordering of a set of trees on $n$ vertices.

### 7.2. Comparing $p$-descendants of a Tree.

By Theorem 7.10 and Theorem 7.12, we are able to generate a partial ordering of trees such that $f_{s}$ of a tree in this ordering is at least as large as $f_{s}$ of the previous tree in the ordering for $s \geq 0$. It is not difficult to show that a genealogy of a tree is not unique. Now we will consider the set of $p^{1}$-ancestors and $p^{1}$-descendants of a given tree and investigate $f_{s}$ values of trees in this set for $s \geq 0$.

Theorem 7.15. Let $T$ be a tree with starring triples $r_{i}=\left\{v_{i}, u_{i}, x\right\} \in R_{p}(T)$ for $i \in\{1,2\}$ such that $d\left(u_{2}\right)<d\left(u_{1}\right)$. Then $f_{s}\left(T\left(r_{1}\right)\right) \leq f_{s}\left(T\left(r_{2}\right)\right)$ for $s \geq 0$.

Proof. Consider $T\left(r_{1}\right)$. Then $r=\left\{v_{2}, u_{2}, u_{1}\right\} \in R_{p}\left(T\left(r_{1}\right)\right)$, and $g_{p}(r)=d_{T\left(r_{1}\right)}\left(u_{1}\right)-$ $d_{T\left(r_{1}\right)}\left(u_{2}\right) \geq 0$. By Theorem 7.10 and Theorem 7.12, $f_{s}\left(T\left(r_{1}\right)\right) \leq f_{s}\left(\left(T\left(r_{1}\right)\right)(r)\right)$ for $s \geq 0$. We claim that $\left(T\left(r_{1}\right)(r)\right) \cong T\left(r_{2}\right)$. Let $P_{i}=v_{i} \ldots y_{i} u_{i}$ be the $v_{i} u_{i}$-path in $T$ for $i \in\{1,2\}$. Note that $T\left(r_{2}\right)-\left\{y_{1} u_{1}\right\} \cong\left(T\left(r_{1}\right)\right)(r)-\left\{y_{2} u_{1}\right\}$, and $P_{1}-u_{1} \cong P_{2}-u_{2}$. Hence $V\left(T\left(r_{2}\right)\right) \cong$ $V\left(\left(T\left(r_{1}\right)\right)(r)\right)$ and $E\left(T\left(r_{2}\right)\right) \cong E\left(\left(T\left(r_{1}\right)\right)(r)\right)$. Thus the claim is true, and so $T\left(r_{1}\right)$ is an $p$-ancestor of $T\left(r_{2}\right)$. Hence, $f_{s}\left(T\left(r_{1}\right)\right) \leq f_{s}\left(T\left(r_{2}\right)\right)$ for $s \geq 0$.


Figure 15. $T, T\left(r_{1}\right), T\left(r_{2}\right)$

Theorem 7.16. Let $T$ be a tree with starring triples $r_{i}=\left\{v, u, x_{i}\right\} \in R_{p}(T)$ for $i \in\{1,2\}$ such that $d\left(x_{1}\right)<d\left(x_{2}\right)$. Then $f_{s}\left(T\left(r_{1}\right)\right) \leq f_{s}\left(T\left(r_{2}\right)\right)$.

Proof. Consider $r=\left\{v, x_{1}, x_{2}\right\} \in R_{p}\left(T\left(r_{1}\right)\right)$. Then as $d_{T\left(r_{1}\right)}\left(x_{2}\right)-d_{T\left(r_{1}\right)}\left(x_{1}\right) \geq 0, T\left(r_{1}\right)(r)$ is a $p$-descendant of $T\left(r_{1}\right)$. Hence by Theorem 7.10 and Theorem 7.12, $f_{s}\left(T\left(r_{1}\right)\right) \leq f_{s}\left(T\left(r_{1}\right)(r)\right)$ for $s \geq 0$. Let $P=v \ldots y u$ be the $u v$-path in $T$. Note that $V\left(T\left(r_{2}\right)\right)=V\left(T\left(r_{1}\right)\right)$ and $E\left(T\left(r_{2}\right)\right)=\left(E\left(T\left(r_{1}\right)\right)-\left\{y x_{1}\right\}\right) \cup\left\{y x_{2}\right\}$. Hence $T\left(r_{2}\right)$ is a $p$-descendant of $T\left(r_{1}\right)$, namely $T\left(r_{1}\right)(r)$.


T

$T\left(r_{1}\right)$

$T\left(r_{2}\right)$

Figure 16. $T^{\prime}, T^{\prime}\left(r_{1}\right), T^{\prime}\left(r_{2}\right)$

It has been shown that for a tree $T$ with starring triple $r \in R_{p}(T)$ such that $g_{p}(r) \geq 0$, $f_{s}(T) \leq f_{s}(T(r))$ for $s \geq 0$. Additionally, if $g_{p}(r) \leq-2, f_{s}(T) \geq f_{s}(T(r))$ for $s \geq 0$. However, if $g_{p}(r)=-1$, the relationship between $f_{s}(T)$ and $f_{s}(T(r))$ for $s \geq 0$ is inconclusive. It would be interesting to investigate what parameters determine that $f_{s}(T) \leq f_{s}(T(r))$ for $g_{p}(r)=-1$ and $s \geq 0$.

### 7.3. The First Zagreb Index of a Tree and Its Descendants.

In the same way that Theorem 7.10 and Theorem 7.12 extend the works of Wingard and Theorem 7.1. A genealogy of a tree also extends the works of Das and Gutman and Theorem 7.4.

Theorem 7.17. Let $T$ be a tree and $r \in R_{p}(T)$. Then $M_{1}(T(r))=M_{1}(T)+2 g_{p}(r)+2$.

Proof. Suppose that $p=1$, and let $r=\{v, u, x\} \in R_{1}(T)$. Then

$$
\begin{aligned}
M_{1}(T) & =M_{1}(T-v)+d_{T}(v)^{2}+\left(d_{T}(u)^{2}-d_{T-v}(u)^{2}\right) \\
& =M_{1}(T-v)+1+d_{T}(u)^{2}-\left(d_{T}(u)-1\right)^{2} \\
& =M_{1}(T-v)+2 d_{T}(u)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
M_{1}(T(r)) & =M_{1}(T(r)-v)+d_{T(r)}(v)^{2}+\left(d_{T(r)}(x)^{2}-d_{T(r)-v}(x)^{2}\right) \\
& =M_{1}(T(r)-v)+1+d_{T(r)}(x)^{2}-\left(d_{T(r)}(x)-1\right)^{2} \\
& =M_{1}(T(r)-v)+2 d_{T(r)}(x) .
\end{aligned}
$$

Note that $T-v \cong T(r)-v$, and $d_{T}(x)=d_{T(r)}(x)-1$. Then

$$
\begin{aligned}
M_{1}(T(r)) & =M_{1}(T)+2 d_{T}(x)+2-2 d_{T}(u) \\
& =M_{1}(T)+2 g_{1}(r)+2 .
\end{aligned}
$$

Suppose that $p \geq 2$, and let $r=\{v, u, x\} \in R_{p}(T)$, and let $P$ be the $v, u$-path of $T$ of lenght $p$. Let $y \in N(u) \cap V(P)$ and $y^{\prime} \in N(y) \cap V(P)-y$. If $p=2$, then

$$
\begin{aligned}
M_{1}(T) & =M_{1}(T-y)+d_{T}(y)^{2}+\left(d_{T}(u)^{2}-d_{T-y}(u)^{2}\right)+\left(d_{T}\left(y^{\prime}\right)^{2}-d_{T-y}\left(y^{\prime}\right)^{2}\right) \\
& =M_{1}(T-y)+4+d_{T}(u)^{2}-\left(d_{T}(u)-1\right)^{2}+1 \\
& =M_{1}(T-y)+2 d_{T}(u)+4 .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
M_{1}(T(r)) & =M_{1}(T(r)-v)+d_{T(r)}(v)^{2}+\left(d_{T(r)}(x)^{2}-d_{T(r)-v}(x)^{2}\right)+\left(d_{T(r)}\left(y^{\prime}\right)^{2}-d_{T(r)-y}\left(y^{\prime}\right)^{2}\right) \\
& =M_{1}(T(r)-v)+4+d_{T(r)}(x)^{2}-\left(d_{T(r)}(x)-1\right)^{2}+1 \\
& =M_{1}(T(r)-v)+2 d_{T(r)}(x)+4
\end{aligned}
$$

Note that $T-v \cong T(r)-v$, and $d_{T}(x)=d_{T(r)}(x)-1$. Then

$$
\begin{aligned}
M_{1}(T(r)) & =M_{1}(T)+2 d_{T}(x)+2-2 d_{T}(u) \\
& =M_{1}(T)+2 g_{2}(r)+2 .
\end{aligned}
$$

Suppose that $l \geq 3$, then

$$
\begin{aligned}
M_{1}(T) & =M_{1}(T-y)+d_{T}(y)^{2}+\left(d_{T}(u)^{2}-d_{T-y}(u)^{2}\right)+\left(d_{T}\left(y^{\prime}\right)^{2}-d_{T-y}\left(y^{\prime}\right)^{2}\right) \\
& =M_{1}(T-y)+4+d_{T}(u)^{2}-\left(d_{T}(u)-1\right)^{2}+3 \\
& =M_{1}(T-y)+2 d_{T}(u)+6 .
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
M_{1}(T) & =M_{1}(T(r)-v)+d_{T(r)}(v)^{2}+\left(d_{T(r)}(x)^{2}-d_{T(r)-v}(x)^{2}\right)+\left(d_{T(r)}\left(y^{\prime}\right)^{2}-d_{T(r)-y}\left(y^{\prime}\right)^{2}\right) \\
& =M_{1}(T(r)-v)+4+d_{T(r)}(x)^{2}-\left(d_{T(r)}(x)-1\right)^{2}+3 \\
& =M_{1}(T(r)-v)+2 d_{T(r)}(x)+6
\end{aligned}
$$

Note that $T-v \cong T(r)-v$, and $d_{T}(x)=d_{T(r)}(x)-1$. Then

$$
\begin{aligned}
M_{1}(T(r)) & =M_{1}(T)+2 d_{T}(x)+2-2 d_{T}(u) \\
& =M_{1}(T)+2 g_{p}(r)+2 .
\end{aligned}
$$

Corollary 7.18. Let $T$ be a tree and $r \in R_{p}(T)$. Then
(i) $M_{1}(T)<M_{1}(T(r))$ if $g_{p}(r) \geq 0$,
(ii) $M_{1}(T)=M_{1}(T(r))$ if $g_{p}(r)=-1$,
(iii) $M_{1}(T)>M_{1}(T(r))$ if $g_{p}(r) \leq-2$.

Thus for a given tree $T$, a genealogy of $T$ gives a sequence of trees such that the $M_{1}$ value of a tree in the sequence is larger than the $M_{1}$ value of any previous tree in this sequence. Hence, a genealogy of $T$ along with $M_{1}$ provides a partial ordering of a set of trees on $n$ vertices.

It should be noted that for $p \geq 1, g_{p}$ is not sufficient to build a sequence of trees such that the $M_{2}$ value of a tree in this sequence is larger that the $M_{2}$ value of any previous tree in this sequence. However, we believe that a similar function may be defined to generate such a sequence. It would be interesting to determine such a function and consequently determine a partial ordering of a set of trees with respect to $M_{2}$.

## 8. Potential Research in the Future

In Chapter 2, it was shown that maximal outerplanar graphs are tree-like 2-trees, and chordal planar graphs with toughness exceeding 1 are tree-like 3-trees with toughness exceeding 1. It would be interesting to classify graphs that are tree-like $k$-trees for $k \geq 4$.

The shell of a $k$-tree was introduced in Chapter 2, and it allowed us to define classes of $k$-trees such as path-like and tree-like $k$-trees. We say a clique is "maximal" if it is not contained in a larger clique. Thus for a $k$-tree, a $(k+1)$-clique is maximal. The shell may be generalized for any graph $G$ as follows.


Figure 17. A graph $G$ and its shell

Definition 8.1. Let $G$ be graph. Then the shell of the graph $G, S h(G)$, is a graph such that
(i) if $X$ is a maximal clique, then $X \in V(\operatorname{Sh}(G))$,
(ii) if $X$ and $Y$ are maximal cliques of size $r_{1}$ and $r_{2}$ respectfully, and $|V(X) \cap V(Y)|=$ $\min \left(r_{1}, r_{2}\right)-1$, then $X Y \in E(S h(G))$.

With this modified definition of the shell, it would be interesting to investigate the shells of graphs other than $k$-trees.

In Chapter 3, we defined families of trees $\mathcal{A}_{c}$ such that the independence polynomial of any tree in such a family has $c$ as a rational root. The family $\mathcal{A}_{-1}$ was determined to be unique, and families $\mathcal{A}_{-1 / 2}, \mathcal{A}_{-1 / 3}$, and $\mathcal{A}_{-1 / 4}$ were characterized. However it would be interesting to verify that $T \in \mathcal{A}_{i}$ if and only if $I(T ; i)=0$ for $i \in\left\{-\frac{1}{2},-\frac{1}{3},-\frac{1}{4}\right\}$.

It was shown that if $I\left(P_{n} ; c\right)=0$, then $c \in\left\{-1,-\frac{1}{2},-\frac{1}{3}\right\}$. It would be interesting to determine the set of rational numbers $\mathcal{C}$ such that, for the tree $T, I(T ; c)=0$ if and only if $c \in \mathcal{C}$.

Wingard determined that, for the graph $G$, if $I(G ;-1)=0$, then $G$ has the same number of independent sets of even cardinality as independent sets of odd cardinality. However, we were unsuccessful to find in the literature any significance to other rational roots of the independence polynomial of a graph. It would be an interesting question to ask what a given root of a graph's independence polynomial implies about the graph itself.

It is a natural parallel to generalize results about trees to $k$-trees. With this in mind, it would be interesting to investigate rational roots of independence polynomials of $k$-trees. Can the results about rational roots of the independence polynomials of paths be extended to the $k$-path or to path-like $k$-trees? Can families of $k$-trees be defined similarly to the families of trees defined in Chapter 3? There are many intriguing questions of this nature about the class of $k$-trees.

In Chapter 4, Wingard's bound, $|I(T ;-1)| \leq 1$, was generalized to $k$-degenerate graphs, and thus $k$-trees. We determined that for the $k$-degenerate graph $G,\left|I\left(G ;-\frac{1}{k}\right)\right| \leq 1$. However, we may state the following conjecture.

Conjecture 8.2. Let $G$ be a maximally $k$-degenerate graph and $k \geq 2$. Then $\left|I\left(G ;-\frac{1}{k}\right)\right|>0$.

In Chapter 4, the works of Alameddine were extended by showing a strict upper bound of $f_{s}$ of tree-like 2-trees for $s \geq 0$ that is uniquely obtained by the 2 -spiral. Additionally, it was shown that for tree-like 3 -trees with toughness exceeding 1 , the strict upper bound of $f_{s}$ is uniquely obtained by the 3 -spiral for $s \geq 0$. It was also conjectured that for path-like $k$-trees, the strict upper bound of $f_{s}$ is uniquely obtained by the $k$-spiral. In addition to verifying this conjecture, three other questions naturally follow:
(i) What is the strict upper bound of $f_{s}$ for tree-like $k$-trees for $s \geq 0$ ?
(ii) What is the strict upper bound of $f_{s}$ for tree-like $k$-trees with toughness exceeding $1, k \geq 3$, and $s \geq 0$ ?
(iii) What tree-like $k$-trees obtain these upper bounds?

Lower and upper bounds of the Zagreb indices for $k$-trees were demonstrated in Chapter 5 along with the unique $k$-trees that obtain these bounds for both $M_{1}$ and $M_{2}$. Furthermore, a strict upper bound of $M_{1}$-values for $k$-degenerate graphs was determined along with the $k$-degenerate graph that obtains this bound. The lower bound of the Zagreb indices for $k$-degenerate graphs is trivially zero as the empty graph is $k$-degenerate for $k \geq 0$. However, it would be interesting to deduce a strict lower bound of the Zagreb indices for maximally $k$-degenerate graphs. It is reasonable to think that this lower bound is obtained by the $k$-path, though maybe not uniquely. Likewise, it would be interesting to determine a strict upper bound of $M_{2}$-values for $k$-degenerate graphs and characterize the $k$-degenerate graphs that obtain this upper bound. It is again reasonable to believe that this bound is obtained by the $k$-star.

In Chapter 6, an upper bound of the first Zagreb index for tree-like $k$-trees was demonstrated along with the unique tree-like $k$-trees that obtain this bound. A strict upper bound
of the second Zagreb index for tree-like $k$-trees was partially solved. Conjecture 6.12 was presented, and it would be interesting to verify this unverified statement that the $k$-spiral uniquely obtains a strict upper bound for $M_{2}$ values among tree-like $k$-trees.

Genealogies of trees were introduced in Chapter 7, and it was shown that a genealogy of a tree helps provide a sequence of trees $\left\{T_{i}\right\}_{i=0}^{\beta}$ such that $f_{s}$ and $M_{1}$ are increasing as $i$ increases. It would be interesting to find a similar construction for $M_{2}$ of trees.

There is some difficulty in generalizing starring triples of trees to $k$-trees. If, however, the starring triples of a tree can be generalized to starring triples of $k$-trees, then a genealogy of a $k$-tree may be defined. If a genealogy of a $k$-tree can be successfully defined, then questions stated about finding a strict upper bound of $f_{s}$ of tree-like $k$-trees may be found. Given a tree-like $k$-tree, can we find a sequence of tree-like $k$-trees such that $f_{s}$ is increasing according to this sequence? This seems to be a reasonable question, and the graph in this sequence with the greatest index might obtain an upper bound of $f_{s}$ for tree-like $k$-trees. Similarly, a genealogy of a $k$-tree may provide the correct structure to verify Conjecture 6.12.

It would also be interesting to determine what other topological indices, such as the toughness, behave in a way similar to $f_{s}$ and $M_{1}$ in a genealogy of a tree.

A graph is said to be "hamiltonian" if it contains a cycle that passes through all of its vertices. Hamiltonicity has been a major area of research, and a common approach to questions of hamiltonicity is to examine a graph through its toughness. In 1973, Chvátal conjectured that there exists a number $t$ such that all $t$-tough graphs are hamiltonian. From the definition of toughness, it is clear that a cycle of length at least four is exactly 1-tough. It is thus clearly necessary for a hamiltonian graph to be 1-tough. For many years, it was thought that all 2-tough graphs are hamiltonian. However, this has been found to be untrue.

Theorem 8.3. [2] For every $\epsilon>0$, there exists $a\left(\frac{9}{4}-\epsilon\right)$-tough graph containing no hamiltonian path.

Chen, Jacobson, Kézdy, and Lehel proved Chvátal's conjecture for chordal graphs, and Böhme, Harant, and Tkác̃ solved the conjecture for chordal planar graphs with toughness exceeding 1.

Theorem 8.4. [7] Every 18-tough chordal graph is hamiltonian.

Theorem 8.5. [5] Every chordal planar graph with toughness exceeding 1 is hamiltonian.

In 2003, Broersma, Xiong, and Yoshimoto addressed hamiltonicity of $k$-trees.

Theorem 8.6. [4] If $T_{n}^{k} \neq K_{2}$ is a $\left(\frac{k+1}{3}\right)$-tough $k$-tree $(k \geq 2)$, then $T_{n}^{k}$ is hamiltonian.

Shook and Wei studied the hamiltonicity of $k$-trees through a parameter called the branch number, $\beta\left(T_{n}^{k}\right)$. Let the edge $e$ be contractible in $T_{n}^{k}$ if the graph resulting in contracting $e$ is a $k$-tree. The branch number may be calculated by $\beta\left(T_{n}^{k}\right)=\left|S_{1}\left(T_{n}^{k}\right)\right|+\left|A\left(T_{n}^{k}\right)\right|+k-n$ where $A$ is the set of contractible edges in $T_{n}^{k}$. Their result is a direct generalization of the result of Broersma.

Theorem 8.7. [38] For $k>1$, if $T_{n}^{k}$ is a $k$-tree with $\beta\left(T_{n}^{k}\right) \leq k$, then $T_{n}^{k}$ is hamiltonian.

Theorem 8.8. [38] If $T_{n}^{k} \neq K_{2}$ is a $\left(\frac{k+1}{3}\right)$-tough $k$-tree $(k \geq 2)$, then $\beta\left(T_{n}^{k}\right) \leq 2$.

For the class of tree-like $k$-trees, Broersma's bound on the toughness may be tightened. By making the connection that chordal planar graphs with toughness exceeding 1 are treelike 3 -trees with toughness exceeding 1 , we state the following conjecture which is a direct generalization of Theorem 8.5.

Conjecture 8.9. Let $T_{n}^{k}$ be a tree-like $k$-tree with toughness exceeding 1 and $k \geq 3$. Then $T_{n}^{k}$ is hamiltonian.

Even for the case of $k=4$, Conjecture 8.9 is a difficult question.
The connection between trees and $k$-trees is very interesting, and there are plenty of questions surrounding trees, $k$-trees, and tree-like $k$-trees. There is plenty of opportunity to propose and ask questions in regards to these graphs.

## LIST OF REFERENCES

[1] A. F. Alameddine, Bounds on the Fibonacci Number of a Maximal Outerplanar Graph, Fibonacci Quarterly 36.3 (1998) 206-210.
[2] D. Bauer, H.J. Broersma, H.J. Veldman, Not every 2-tough graph is Hamiltonian, in: Proceedings of the 5th Twente Workshop on Graphs and Combinatorial Optimization (Enschede, 1997), Discrete Appl. Math. 99 (2000) 317-321.
[3] L. W. Beineke and R. E. Pippet, The number of labeled $k$-dimensional trees, J. Combin. Theory 6 (1969) 200-205.
[4] H. Broersma, L. Xiong, K. Yoshimoto, Toughness and hamiltonicity in $k$-trees Discrete Mathmematics 307 (2007) 832-838.
[5] T. Böhme, J. Harant, M. Tkác̃, More than 1-tough chordal planar graphs are Hamiltonian, J. Graph Theory 32 (1999) 405-410.
[6] B. Bollobás, P Erdös, Graphs of extremal weights. Ars Comb 50 (1998) 225-233.
[7] G. Chen, M.S. Jacobson, A.E. Kézdy, J. Lehel, Tough enough chordal graphs are Hamiltonian, Networks 31 (1998) 29-38.
[8] S. Chen, H. Deng, Extremal $(n, n+1)$-graphs with respected to zeroth-order general Randić index. J Math Chem 42 (2007) 555-564.
[9] L. Chism, Independence Polynomials and Independence Equivalence in Graphs. dissertation. Univeristy of Mississippi (2009).
[10] M. Chudnovsky, P. Seymour. The roots of the independence polynomial of a clawfree graph. Journal of Combinatorial Theory. Series B. 97 (2007) 350-357.
[11] K. Das, Maximizing the sum of the squares of degrees of a graph. Discrete Math 257 (2004) 57-66.
[12] K. Das, I. Gutman, Some properties of the second Zagreb index. MATCH Commun Math Comput Chem 52 (2004) 103-112.
[13] D. de Caen, An upper bound on the sum of squares of degrees in a graph. Discrete Math 185 (1998) 245-248.
[14] Reinhard Diestel, Graph Theory third edition, New York, Springer Berlin Heidelberg, 2006.
[15] R. Euler. The Fibonacci Number of Grid graph and a New Class of Integer Sequence. Journal of Integer Sequences. Vol. 8 (2005) 1-16.
[16] R. Fröberg $A$ characterization of $k$-trees. Discrete Mathematics 104 (1992) 307-309.
[17] R. Garcia-Domenech, J. Galvez, JV de Julian-Ortiz, L Pogliani, Some new trends in chemical graph theory Chem Rev 108 (2008) 1127-1169.
[18] I. Gutman, Topological properties of benzenoid systems. Merrifield-Simmons indices and independence polynomials of unbranched catafusenes, Rev. Roum. Chim. 36 (1991) 379-388.
[19] I. Gutman, Some relations for the independence and matching polynomials and their chemical applications, Bul. Acad. Serbe Sci. arts 105 (1992) 39-49.
[20] I. Gutman, K. Das, The first Zagreb index 30 years after. MATCH Commun Math Comput Chem 50 (2004) 83-92.
[21] I. Gutman, F. Harary, Generalizations of the Matching Polynomial, Utilítas Mathematica, 24 (1983) 97-106.
[22] I. Gutman and O. E. Polansky, Mathematical Concepts in Organic Chemistry, Springer, Berlin, 1986.
[23] I. Gutman, N. Trinajstić, Graph theory and molecular orbitals. Total $\pi$-electron energy of alternant hydrocarbons. Chem Phys Lett 17 (1972) 535-538.
[24] R. Haggkvist, D. Andren, P.H. Lundrow and K. Markstrom, Graph theory and statistical physics, Case study: Discrete mathematics, Department of Mathematics, UMEA University (1999-2004).
[25] F. Harary and E. M. Palmer, On acyclic simplicial compleses, Mathematika 15 (1968) 115-122.
[26] C. Hoede, X. Li Clique Polynomials and Independent Sets of Graphs, Discrete Mathematics 125 (1994) 219-228.
[27] G. Hopkins and W. Staton. An Identity Arising from Counting Independent Sets. Congressus Numerantium. 44 (1984) 5-10.
[28] G. Hopkins and W. Staton. Some Identities Arising from the Fibonacci Numbers of Certain Graphs. The Fibonacci Quarterly 22.3 (1984) 255-258.
[29] A. Hou, S. Li, L. Song, B. Wei Sharp bounds for Zagreb indices of maximal outerplanar graphs. J Comb Optim 22 (2010) 252-269.
[30] V. Levit, E. Mandrescu The independence polynomial of a graph- a survey. (2005) 231-252
[31] S. Li, H Zhou, On the maximum and minimum Zagreb indices of graphs with connectivity at most $k$. Appl Math Lett 23 (2010) 128-132.
[32] D. R. Lick and A. T. White, k-degenerate subgraphs, Canad.J. Math. 22 (1970) 1082-1096.
[33] L. Markenzon, C. M. Justel, N. Paciornik, Subclasses of $k$-trees: Characterization and recognition, Discrete Applied Mathematics 154 (2006) 818-825.
[34] M. Kwaśnik, I. Wloch, The total number of generalized stable sets and kernels of graphs, Ars combinatoria, 55 (2000) 139-146.
[35] H. Prodinger and R.F. Tichy. Fibonacci Numbers of Graphs. The Fibonacci Quarterly 20.1 (1982) 16-21.
[36] M. Randić, On characterization of molecular branching. J Am Chem 97 (1975) 6609-6615
[37] D. J. Rose. On simple characterizations of $k$-trees. Discrete Mathematics 7 (1974) 317-322.
[38] J. Shook, B. Wei, Some properties of $k$-tree. Discrete Mathematics 310 (2010) 2415-2425.
[39] M. Startek, A. Wlcoh, I. Wloch, Fibonacci numbers of graphs, Discrete Applied Mathematics. 157 (2009) 864-868.
[40] L. Song, On Independence polynomials of $k$-trees and well-covered graphs. dissertation. University of Mississippi (2009).
[41] L. Song, W. Staton, B. Wei, Independence polynomials of $k$-tree related graphs. Discrete Math. 158 (2010) 943-950
[42] X. L. Li, H. X. Zhoa and I. Gutman, On the Merrifield-Simmons index of tree, MATCH Commun. Math. Comput. Chem. 54 (2005) 389.
[43] K. Xu, The Zagreb indices of graphs with a given clique number. Applied Mathematics Letters. Vol 24 (2011) 1026-1030.
[44] K. Wingard. On independence polynomials. dissertation. University of Mississippi (1995).

## LIST OF APPENDICES

Appendix A: Trees on $1 \leq n \leq 10$ vertices




# Appendix B: Independence Polynomials of Trees on $1 \leq n \leq 10$ vertices 

$1 \leq n \leq 6$
$1 I\left(P_{1} ; x\right)=1+x$
$2 I\left(P_{2} ; x\right)=1+2 x$
$3 I\left(P_{3} ; x\right)=1+3 x+x^{2}$
$4 I\left(P_{4} ; x\right)=1+4 x+3 x^{2}$
$5 I\left(S_{4} ; x\right)=1+4 x+3 x^{2}+x^{3}$
$6 I\left(P_{5} ; x\right)=1+5 x+6 x^{2}+x^{3}$
$7 I\left(T_{5} ; x\right)=1+5 x+6 x^{2}+2 x^{3}$
$8 I\left(S_{5} ; x\right)=1+5 x+6 x^{2}+4 x^{3}+x^{4}$
$9 I\left(P_{6} ; x\right)=1+6 x+10 x^{2}+4 x^{3}$
$10 I\left(T_{6,1} ; x\right)=1+6 x+10 x^{2}+5 x^{3}+x^{4}$
$11 I\left(T_{6,2} ; x\right)=1+6 x+10 x^{2}+5 x^{3}$
$12 I\left(T_{6,3} ; x\right)=1+6 x+10 x^{2}+6 x^{3}+x^{4}$
$13 I\left(T_{6,4} ; x\right)=1+6 x+10 x^{2}+7 x^{3}+2 x^{4}$
$14 I\left(S_{6} ; x\right)=1+6 x+10 x^{2}+10 x^{3}+5 x^{4}+x^{5}$
$n=7$
$1 I\left(P_{7} ; x\right)=1+7 x+15 x^{2}+10 x^{3}+x^{4}$
$2 I\left(T_{7,1} ; x\right)=1+7 x+15 x^{2}+11 x^{3}+3 x^{4}$
$3 I\left(T_{7,2} ; x\right)=1+7 x+15 x^{2}+11 x^{3}+2 x^{4}$
$4 I\left(T_{7,3} ; x\right)=1+7 x+15 x^{2}+11 x^{3}+x^{4}$
$5 I\left(T_{7,4} ; x\right)=1+7 x+15 x^{2}+12 x^{3}+3 x^{4}$
$6 I\left(T_{7,5} ; x\right)=1+7 x+15 x^{2}+12 x^{3}+5 x^{4}+x^{5}$
$7 I\left(T_{7,6} ; x\right)=1+7 x+15 x^{2}+13 x^{3}+6 x^{4}+x^{5}$
$8 I\left(T_{7,7} ; x\right)=1+7 x+15 x^{2}+13 x^{3}+4 x^{4}$
$9 I\left(T_{7,8} ; x\right)=1+7 x+15 x^{2}+14 x^{3}+6 x^{4}+x^{5}$
$10 I\left(T_{7,9} ; x\right)=1+7 x+15 x^{2}+16 x^{3}+9 x^{4}+2 x^{5}$
$11 I\left(S_{7} ; x\right)=1+7 x+15 x^{2}+20 x^{3}+15 x^{4}+6 x^{5}+x^{6}$
$n=8$
$1 I\left(P_{8} ; x\right)=1+8 x+21 x^{2}+20 x^{3}+5 x^{4}$
$2 I\left(T_{8,1} ; x\right)=1+8 x+21 x^{2}+21 x^{3}+8 x^{4}+x^{5}$
$3 I\left(T_{8,2} ; x\right)=1+8 x+21 x^{2}+21 x^{3}+7 x^{4}$
$4 I\left(T_{8,3} ; x\right)=1+8 x+21 x^{2}+21 x^{3}+7 x^{4}+x^{5}$
$5 I\left(T_{8,4} ; x\right)=1+8 x+21 x^{2}+22 x^{3}+9 x^{4}+x^{5}$
$6 I\left(T_{8,5} ; x\right)=1+8 x+21 x^{2}+22 x^{3}+10 x^{4}+2 x^{5}$
$7 I\left(T_{8,6} ; x\right)=1+8 x+21 x^{2}+22 x^{3}+11 x^{4}+2 x^{5}$
$8 I\left(T_{8,7} ; x\right)=1+8 x+21 x^{2}+22 x^{3}+8 x^{4}$
$9 I\left(T_{8,8} ; x\right)=1+8 x+21 x^{2}+23 x^{3}+13 x^{4}+3 x^{5}$
$10 I\left(T_{8,9} ; x\right)=1+8 x+21 x^{2}+23 x^{3}+11 x^{4}+2 x^{5}$
$11 I\left(T_{8,10} ; x\right)=1+8 x+21 x^{2}+21 x^{3}+6 x^{4}$
$12 I\left(T_{8,11} ; x\right)=1+8 x+21 x^{2}+23 x^{3}+9 x^{4}$
$13 I\left(T_{8,12} ; x\right)=1+8 x+21 x^{2}+22 x^{3}+8 x^{4}+x^{5}$
$14 I\left(T_{8,13} ; x\right)=1+8 x+21 x^{2}+24 x^{3}+13 x^{4}+3 x^{5}$
$15 I\left(T_{8,14} ; x\right)=1+8 x+21 x^{2}+24 x^{3}+16 x^{4}+6 x^{5}+x^{6}$
$16 I\left(T_{8,15} ; x\right)=1+8 x+21 x^{2}+23 x^{3}+11 x^{4}+2 x^{5}$
$17 I\left(T_{8,16} ; x\right)=1+8 x+21 x^{2}+24 x^{3}+12 x^{4}+2 x^{5}$
$18 I\left(T_{8,17} ; x\right)=1+8 x+21 x^{2}+26 x^{3}+19 x^{4}+7 x^{5}+x^{6}$
$19 I\left(T_{8,18} ; x\right)=1+8 x+21 x^{2}+26 x^{3}+16 x^{4}+4 x^{5}$
$20 I\left(T_{8,19} ; x\right)=1+8 x+21 x^{2}+26 x^{3}+17 x^{4}+6 x^{5}+x^{6}$
$21 I\left(T_{8,20} ; x\right)=1+8 x+21 x^{2}+27 x^{3}+19 x^{4}+7 x^{5}+x^{6}$
$22 I\left(T_{8,21} ; x\right)=1+8 x+21 x^{2}+30 x^{3}+25 x^{4}+11 x^{5}+2 x^{6}$
$23 I\left(S_{8} ; x\right)=1+8 x+21 x^{2}+35 x^{3}+35 x^{4}+21 x^{5}+7 x^{6}+x^{7}$
$n=9$
$1 I\left(P_{9} ; x\right)=1+9 x+28 x^{2}+35 x^{3}+15 x^{4}+x^{5}$
$2 I\left(T_{9,1} ; x\right)=1+9 x+28 x^{2}+36 x^{3}+19 x^{4}+4 x^{5}$
$3 I\left(T_{9,2} ; x\right)=1+9 x+28 x^{2}+36 x^{3}+18 x^{4}+2 x^{5}$
$4 I\left(T_{9,3} ; x\right)=1+9 x+28 x^{2}+36 x^{3}+18 x^{4}+3 x^{5}$
$5 I\left(T_{9,4} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+21 x^{4}+4 x^{5}$
$6 I\left(T_{9,5} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+22 x^{4}+7 x^{5}+x^{6}$
$7 I\left(T_{9,6} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+22 x^{4}+5 x^{5}$
$8 I\left(T_{9,7} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+23 x^{4}+7 x^{5}+x^{6}$
$9 I\left(T_{9,8} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+21 x^{4}+4 x^{5}$
$10 I\left(T_{9,9} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+20 x^{4}+3 x^{5}$
$11 I\left(T_{9,10} ; x\right)=1+9 x+28 x^{2}+38 x^{3}+26 x^{4}+9 x^{5}+x^{6}$
$12 I\left(T_{9,11} ; x\right)=1+9 x+28 x^{2}+38 x^{3}+24 x^{4}+6 x^{5}$
$13 I\left(T_{9,12} ; x\right)=1+9 x+28 x^{2}+36 x^{3}+17 x^{4}+2 x^{5}$
$14 I\left(T_{9,13} ; x\right)=1+9 x+28 x^{2}+36 x^{3}+17 x^{4}+x^{5}$
$15 I\left(T_{9,14} ; x\right)=1+9 x+28 x^{2}+38 x^{3}+24 x^{4}+8 x^{5}+x^{6}$
$16 I\left(T_{9,15} ; x\right)=1+9 x+28 x^{2}+38 x^{3}+23 x^{4}+5 x^{5}$
$17 I\left(T_{9,16} ; x\right)=1+9 x+28 x^{2}+38 x^{3}+25 x^{4}+8 x^{5}+x^{6}$
$18 I\left(T_{9,17} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+27 x^{4}+9 x^{5}+x^{6}$
$19 I\left(T_{9,18} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+29 x^{4}+12 x^{5}+2 x^{6}$ $20 I\left(T_{9,19} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+30 x^{4}+12 x^{5}+2 x^{6}$
$21 I\left(T_{9,20} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+26 x^{4}+8 x^{5}+x^{6}$
$22 I\left(T_{9,21} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+25 x^{4}+6 x^{5}$
$23 I\left(T_{9,22} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+28 x^{4}+11 x^{5}+2 x^{6}$
$24 I\left(T_{9,23} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+20 x^{4}+4 x^{5}$
$25 I\left(T_{9,24} ; x\right)=1+9 x+28 x^{2}+38 x^{3}+22 x^{4}+4 x^{5}$
$26 I\left(T_{9,25} ; x\right)=1+9 x+28 x^{2}+37 x^{3}+19 x^{4}+2 x^{5}$
$27 I\left(T_{9,26} ; x\right)=1+9 x+28 x^{2}+41 x^{3}+35 x^{4}+16 x^{5}+3 x^{6}$
$28 I\left(T_{9,27} ; x\right)=1+9 x+28 x^{2}+41 x^{3}+32 x^{4}+13 x^{5}+2 x^{6}$
$29 I\left(T_{9,28} ; x\right)=1+9 x+28 x^{2}+40 x^{3}+30 x^{4}+12 x^{5}+2 x^{6}$
$30 I\left(T_{9,29} ; x\right)=1+9 x+28 x^{2}+40 x^{3}+28 x^{4}+9 x^{5}+x^{6}$
$31 I\left(T_{9,30} ; x\right)=1+9 x+28 x^{2}+38 x^{3}+23 x^{4}+7 x^{5}+x^{6}$
$32 I\left(T_{9,31} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+26 x^{4}+9 x^{5}+x^{6}$
$33 I\left(T_{9,32} ; x\right)=1+9 x+28 x^{2}+39 x^{3}+24 x^{4}+5 x^{5}$
$34 I\left(T_{9,33} ; x\right)=1+9 x+28 x^{2}+41 x^{3}+29 x^{4}+8 x^{5}$
$35 I\left(T_{9,34} ; x\right)=1+9 x+28 x^{2}+42 x^{3}+35 x^{4}+16 x^{5}+3 x^{6}$
$36 I\left(T_{9,35} ; x\right)=1+9 x+28 x^{2}+42 x^{3}+33 x^{4}+13 x^{5}+2 x^{6}$
$37 I\left(T_{9,36} ; x\right)=1+9 x+28 x^{2}+42 x^{3}+39 x^{4}+22 x^{5}+7 x^{6}+x^{7}$
$38 I\left(T_{9,37} ; x\right)=1+9 x+28 x^{2}+41 x^{3}+37 x^{4}+21 x^{5}+7 x^{6}+x^{7}$
$39 I\left(T_{9,38} ; x\right)=1+9 x+28 x^{2}+41 x^{3}+31 x^{4}+12 x^{5}+2 x^{6}$
$40 I\left(T_{9,39} ; x\right)=1+9 x+28 x^{2}+45 x^{3}+45 x^{4}+26 x^{5}+8 x^{6}+x^{7}$

$$
\begin{aligned}
& 41 I\left(T_{9,40} ; x\right)=1+9 x+28 x^{2}+45 x^{3}+41 x^{4}+20 x^{5}+4 x^{6} \\
& 42 I\left(T_{9,41} ; x\right)=1+9 x+28 x^{2}+44 x^{3}+40 x^{4}+22 x^{5}+7 x^{6}+x^{7} \\
& 43 I\left(T_{9,42} ; x\right)=1+9 x+28 x^{2}+46 x^{3}+45 x^{4}+26 x^{5}+8 x^{6}+x^{7} \\
& 44 I\left(T_{9,43} ; x\right)=1+9 x+28 x^{2}+50 x^{3}+55 x^{4}+36 x^{5}+13 x^{6}+2 x^{7} \\
& 45 I\left(S_{9} ; x\right)=1+9 x+28 x^{2}+56 x^{3}+70 x^{4}+56 x^{5}+28 x^{6}+8 x^{7}+x^{8} \\
& n=10
\end{aligned}
$$

$1 I\left(P_{10} ; x\right)=1+10 x+36 x^{2}+56 x^{3}+35 x^{4}+6 x^{5}$
$2 I\left(T_{10,1} ; x\right)=1+10 x+36 x^{2}+57 x^{3}+40 x^{4}+12 x^{5}+x^{6}$
$3 I\left(T_{10,2} ; x\right)=1+10 x+36 x^{2}+57 x^{3}+39 x^{4}+9 x^{5}$
$4 I\left(T_{10,3} ; x\right)=1+10 x+36 x^{2}+57 x^{3}+39 x^{4}+10 x^{5}+x^{6}$
$5 I\left(T_{10,4} ; x\right)=1+10 x+36 x^{2}+57 x^{3}+39 x^{4}+10 x^{5}$
$6 I\left(T_{10,5} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+43 x^{4}+13 x^{5}+x^{6}$
$7 I\left(T_{10,6} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+44 x^{4}+17 x^{5}+3 x^{6}$
$8 I\left(T_{10,7} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+44 x^{4}+16 x^{5}+2 x^{6}$
$9 I\left(T_{10,8} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+44 x^{4}+15 x^{5}+2 x^{6}$
$10 I\left(T_{10,9} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+45 x^{4}+18 x^{5}+3 x^{6}$
$11 I\left(T_{10,10} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+42 x^{4}+11 x^{5}$
$12 I\left(T_{10,11} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+43 x^{4}+14 x^{5}+2 x^{6}$
$13 I\left(T_{10,12} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+43 x^{4}+12 x^{5}$
$14 I\left(T_{10,13} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+42 x^{4}+12 x^{5}+x^{6}$
$15 I\left(T_{10,14} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+49 x^{4}+22 x^{5}+4 x^{6}$
$16 I\left(T_{10,15} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+47 x^{4}+17 x^{5}+2 x^{6}$
$17 I\left(T_{10,16} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+47 x^{4}+19 x^{5}+3 x^{6}$
$18 I\left(T_{10,17} ; x\right)=1+10 x+36 x^{2}+57 x^{3}+38 x^{4}+7 x^{5}$
$19 I\left(T_{10,18} ; x\right)=1+10 x+36 x^{2}+57 x^{3}+38 x^{4}+8 x^{5}$
$20 I\left(T_{10,19} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+46 x^{4}+16 x^{5}+2 x^{6}$
$21 I\left(T_{10,20} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+47 x^{4}+17 x^{5}+2 x^{6}$
$22 I\left(T_{10,21} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+58 x^{4}+19 x^{5}+3 x^{6}$
$23 I\left(T_{10,22} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+47 x^{4}+18 x^{5}+3 x^{6}$
$24 I\left(T_{10,23} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+49 x^{4}+24 x^{5}+7 x^{6}+x^{7}$
$25 I\left(T_{10,24} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+45 x^{4}+13 x^{5}$
$26 I\left(T_{10,25} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+51 x^{4}+22 x^{5}+4 x^{6}$
$27 I\left(T_{10,26} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+53 x^{4}+28 x^{5}+8 x^{6}+x^{7}$
$28 I\left(T_{10,27} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+53 x^{4}+25 x^{5}+5 x^{6}$
$29 I\left(T_{10,28} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+54 x^{4}+28 x^{5}+8 x^{6}+x^{7}$
$30 I\left(T_{10,29} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+50 x^{4}+20 x^{5}+3 x^{6}$
$31 I\left(T_{10,30} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+49 x^{4}+18 x^{5}+2 x^{6}$
$32 I\left(T_{10,31} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+51 x^{4}+22 x^{5}+4 x^{6}$
$33 I\left(T_{10,32} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+52 x^{4}+23 x^{5}+4 x^{6}$
$34 I\left(T_{10,33} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+52 x^{4}+27 x^{5}+8 x^{6}+x^{7}$
$35 I\left(T_{10,34} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+49 x^{4}+19 x^{5}+3 x^{6}$
$36 I\left(T_{10,35} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+42 x^{4}+12 x^{5}+x^{6}$
$37 I\left(T_{10,36} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+45 x^{4}+13 x^{5}$
$38 I\left(T_{10,37} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+41 x^{4}+10 x^{5}+x^{6}$
$39 I\left(T_{10,38} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+42 x^{4}+11 x^{5}$
$40 I\left(T_{10,39} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+43 x^{4}+13 x^{5}+x^{6}$
$41 I\left(T_{10,40} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+43 x^{4}+15 x^{5}+2 x^{6}$
$42 I\left(T_{10,41} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+41 x^{4}+10 x^{5}$
$43 I\left(T_{10,42} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+45 x^{4}+15 x^{5}+2 x^{6}$
$44 I\left(T_{10,43} ; x\right)=1+10 x+36 x^{2}+57 x^{3}+38 x^{4}+9 x^{5}+x^{6}$
$45 I\left(T_{10,44} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+61 x^{4}+35 x^{5}+10 x^{6}+x^{7}$
$46 I\left(T_{10,45} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+58 x^{4}+29 x^{5}+6 x^{6}$
$47 I\left(T_{10,46} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+58 x^{4}+32 x^{5}+9 x^{6}+x^{7}$
$48 I\left(T_{10,47} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+50 x^{4}+20 x^{5}+3 x^{6}$
$49 I\left(T_{10,48} ; x\right)=1+10 x+36 x^{2}+61 x^{3}+54 x^{4}+25 x^{5}+5 x^{6}$
$50 I\left(T_{10,49} ; x\right)=1+10 x+36 x^{2}++61 x^{3}+56 x^{4}+29 x^{5}+8 x^{6}+x^{7}$
$51 I\left(T_{10,50} ; x\right)=1+10 x+36 x^{2}+61 x^{3}+57 x^{4}+31 x^{5}+9 x^{6}+x^{7}$
$52 I\left(T_{10,51} ; x\right)=1+10 x+36 x^{2}+61 x^{3}+52 x^{4}+21 x^{5}+3 x^{6}$
$53 I\left(T_{10,52} ; x\right)=1+10 x+36 x^{2}+61 x^{3}+55 x^{4}+28 x^{5}+8 x^{6}+x^{7}$
$54 I\left(T_{10,53} ; x\right)=1+10 x+36 x^{2}+61 x^{3}+53 x^{4}+23 x^{5}+4 x^{6}$
$55 I\left(T_{10,54} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+57 x^{4}+29 x^{5}+8 x^{6}+x^{7}$
$56 I\left(T_{10,55} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+61 x^{4}+37 x^{5}+13 x^{6}+2 x^{7}$
$57 I\left(T_{10,56} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+63 x^{4}+38 x^{5}+13 x^{6}+2 x^{7}$
$58 I\left(T_{10,57} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+55 x^{4}+24 x^{5}+4 x^{6}$
$59 I\left(T_{10,58} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+48 x^{4}+17 x^{5}+2 x^{6}$
$60 I\left(T_{10,59} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+45 x^{4}+15 x^{5}+2 x^{6}$
$61 I\left(T_{10,60} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+47 x^{4}+19 x^{5}+3 x^{6}$
$62 I\left(T_{10,61} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+55 x^{4}+24 x^{5}+4 x^{6}$
$63 I\left(T_{10,62} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+47 x^{4}+14 x^{5}$
$64 I\left(T_{10,63} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+50 x^{4}+21 x^{5}+4 x^{6}$
$65 I\left(T_{10,64} ; x\right)=1+10 x+36 x^{2}+59 x^{3}+45 x^{4}+14 x^{5}+x^{6}$
$66 I\left(T_{10,65} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+50 x^{4}+22 x^{5}+4 x^{6}$
$67 I\left(T_{10,66} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+48 x^{4}+17 x^{5}+2 x^{6}$
$68 I\left(T_{10,67} ; x\right)=1+10 x+36 x^{2}+60 x^{3}+52 x^{4}+25 x^{5}+5 x^{6}$
$69 I\left(T_{10,68} ; x\right)=1+10 x+36 x^{2}+58 x^{3}+40 x^{4}+8 x^{5}$
$70 I\left(T_{10,69} ; x\right)=1+10 x+36 x^{2}+63^{3}+62 x^{4}+35 x^{5}+10 x^{6}+x^{7}$
$71 I\left(T_{10,70} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+65 x^{4}+41 x^{5}+14 x^{6}+2 x^{7}$
$72 I\left(T_{10,71} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+66 x^{4}+41 x^{5}+14 x^{6}+2 x^{7}$
$73 I\left(T_{10,72} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+60 x^{4}+32 x^{5}+9 x^{6}+x^{7}$
$74 I\left(T_{10,73} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+59 x^{4}+29 x^{5}+6 x^{6}$
$75 I\left(T_{10,74} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+63 x^{4}+38 x^{5}+13 x^{6}+2 x^{7}$
$76 I\left(T_{10,75} ; x\right)=1+10 x+36 x^{2}+66 x^{3}+75 x^{4}+51 x^{5}+19 x^{6}+3 x^{7}$
$77 I\left(T_{10,76} ; x\right)=1+10 x+36 x^{2}+66 x^{3}+71 x^{4}+45 x^{5}+15 x^{6}+2 x^{7}$
$78 I\left(T_{10,77} ; x\right)=1+10 x+36 x^{2}+64 x^{3}+66 x^{4}+41 x^{5}+14 x^{6}+2 x^{7}$
$79 I\left(T_{10,78} ; x\right)=1+10 x+36 x^{2}+64 x^{3}+62 x^{4}+33 x^{5}+9 x^{6}+x^{7}$
$80 I\left(T_{10,79} ; x\right)=1+10 x+36 x^{2}+65 x^{3}+66 x^{4}+39 x^{5}+13 x^{6}+2 x^{7}$
$81 I\left(T_{10,80} ; x\right)=1+10 x+36 x^{2}+65 x^{3}+75 x^{4}+57 x^{5}+28 x^{6}+8 x^{7}+x^{8}$
$82 I\left(T_{10,81} ; x\right)=1+10 x+36 x^{2}+65 x^{3}+67 x^{4}+41 x^{5}+14 x^{6}+2 x^{7}$
$83 I\left(T_{10,82} ; x\right)=1+10 x+36 x^{2}+67 x^{3}+72 x^{4}+45 x^{5}+15 x^{6}+2 x^{7}$
$84 I\left(T_{10,83} ; x\right)=1+10 x+36 x^{2}+67 x^{3}+80 x^{4}+61 x^{5}+29 x^{6}+8 x^{7}+x^{8}$
$85 I\left(T_{10,84} ; x\right)=1+10 x+36 x^{2}+67 x^{3}+75 x^{4}+51 x^{5}+19 x^{6}+3 x^{7}$
$86 I\left(T_{10,85} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+60 x^{4}+32 x^{5}+9 x^{6}+x^{7}$
$87 I\left(T_{10,86} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+63 x^{4}+38 x^{5}+13 x^{6}+2 x^{7}$
$88 I\left(T_{10,87} ; x\right)=1+10 x+36 x^{2}+61 x^{3}+51 x^{4}+20 x^{5}+3 x^{6}$
$89 I\left(T_{10,88} ; x\right)=1+10 x+36 x^{2}+62 x^{3}+55 x^{4}+25 x^{5}+5 x^{6}$
$90 I\left(T_{10,89} ; x\right)=1+10 x+36 x^{2}+61 x^{3}+54 x^{4}+28 x^{5}+8 x^{6}+x^{7}$
$91 I\left(T_{10,90} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+57 x^{4}+25 x^{5}+4 x^{6}$
$92 I\left(T_{10,91} ; x\right)=1+10 x+36 x^{2}+63 x^{3}+61 x^{4}+35 x^{5}+10 x^{6}+x^{7}$
$93 I\left(T_{10,92} ; x\right)=1+10 x+36 x^{2}+66 x^{3}+67 x^{4}+36 x^{5}+8 x^{6}$
$94 I\left(T_{10,93} ; x\right)=1+10 x+36 x^{2}+71 x^{3}+90 x^{4}+71 x^{5}+34 x^{6}+9 x^{7}+x^{8}$
$95 I\left(T_{10,94} ; x\right)=1+10 x+36 x^{2}+71 x^{3}+85 x^{4}+61 x^{5}+24 x^{6}+4 x^{7}$
$96 I\left(T_{10,95} ; x\right)=1+10 x+36 x^{2}+73 x^{3}+91 x^{4}+71 x^{5}+34 x^{6}+9 x^{7}+x^{8}$
$97 I\left(T_{10,96} ; x\right)=1+10 x+36 x^{2}+72 x^{3}+90 x^{4}+71 x^{5}+34 x^{6}+9 x^{7}+x^{8}$
$98 I\left(T_{10,97} ; x\right)=1+10 x+36 x^{2}+77 x^{3}+105 x^{4}+91 x^{5}+49 x^{6}+15 x^{7}+2 x^{8}$
$99 I\left(T_{10,98} ; x\right)=1+10 x+36 x^{2}+68 x^{3}+78 x^{4}+58 x^{5}+28 x^{6}+8 x^{7}+x^{8}$
$100 I\left(S_{10} ; x\right)=1+10 x+36 x^{2}+84 x^{3}+126 x^{4}+126 x^{5}+84 x^{6}+36 x^{7}+9 x^{8}+x^{9}$

Appendix C: The Zagreb Indices of Trees on $1 \leq n \leq 10$ vertices

| $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $P_{1}$ | 0 | 0 | $T_{7,7}$ | 28 | 28 | $T_{8,17}$ | 38 | 36 |
| $P_{2}$ | 2 | 1 | $T_{7,8}$ | 30 | 30 | $T_{8,18}$ | 38 | 39 |
| $P_{3}$ | 6 | 4 | $T_{7,9}$ | 34 | 32 | $T_{8,19}$ | 38 | 40 |
| $P_{4}$ | 10 | 8 | $S_{7}$ | 42 | 36 | $T_{8,20}$ | 40 | 41 |
| $S_{4}$ | 12 | 9 | $P_{8}$ | 26 | 24 | $T_{8,21}$ | 46 | 44 |
| $P_{5}$ | 14 | 12 | $T_{8,1}$ | 28 | 26 | $S_{8}$ | 56 | 49 |
| $T_{5}$ | 16 | 14 | $T_{8,2}$ | 28 | 27 | $P_{9}$ | 30 | 28 |
| $S_{5}$ | 20 | 16 | $T_{8,3}$ | 28 | 27 | $T_{9,1}$ | 32 | 30 |
| $P_{6}$ | 18 | 16 | $T_{8,4}$ | 30 | 30 | $T_{9,2}$ | 32 | 31 |
| $T_{6,1}$ | 20 | 18 | $T_{8,5}$ | 30 | 29 | $T_{9,3}$ | 32 | 31 |
| $T_{6,2}$ | 20 | 19 | $T_{8,6}$ | 30 | 28 | $T_{9,4}$ | 34 | 34 |
| $T_{6,3}$ | 22 | 21 | $T_{8,7}$ | 30 | 31 | $T_{9,5}$ | 34 | 33 |
| $T_{6,4}$ | 24 | 22 | $T_{8,8}$ | 32 | 30 | $T_{9,6}$ | 34 | 33 |
| $S_{6}$ | 30 | 25 | $T_{8,9}$ | 32 | 32 | $T_{9,7}$ | 34 | 32 |
| $P_{7}$ | 22 | 20 | $T_{8,10}$ | 28 | 28 | $T_{9,8}$ | 34 | 34 |
| $T_{7,1}$ | 24 | 22 | $T_{8,11}$ | 32 | 34 | $T_{9,9}$ | 34 | 35 |
| $T_{7,2}$ | 24 | 23 | $T_{8,12}$ | 30 | 31 | $T_{9,10}$ | 36 | 34 |
| $T_{7,3}$ | 24 | 24 | $T_{8,13}$ | 34 | 35 | $T_{9,11}$ | 36 | 36 |
| $T_{7,4}$ | 26 | 26 | $T_{8,14}$ | 34 | 32 | $T_{9,12}$ | 32 | 32 |
| $T_{7,5}$ | 26 | 24 | $T_{8,15}$ | 32 | 33 | $T_{9,13}$ | 32 | 32 |
| $T_{7,6}$ | 28 | 26 | $T_{8,16}$ | 34 | 36 | $T_{9,14}$ | 36 | 36 |


| $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $T_{9,15}$ | 36 | 38 | $T_{9,36}$ | 44 | 42 | $T_{10,12}$ | 38 | 38 |
| $T_{9,16}$ | 36 | 36 | $T_{9,37}$ | 42 | 40 | $T_{10,13}$ | 38 | 39 |
| $T_{9,17}$ | 38 | 39 | $T_{9,38}$ | 42 | 46 | $T_{10,14}$ | 40 | 38 |
| $T_{9,18}$ | 38 | 37 | $T_{9,39}$ | 50 | 48 | $T_{10,15}$ | 40 | 40 |
| $T_{9,19}$ | 38 | 36 | $T_{9,40}$ | 50 | 52 | $T_{10,16}$ | 40 | 40 |
| $T_{9,20}$ | 38 | 40 | $T_{9,41}$ | 48 | 52 | $T_{10,17}$ | 36 | 36 |
| $T_{9,21}$ | 38 | 41 | $T_{9,42}$ | 52 | 54 | $T_{10,18}$ | 36 | 36 |
| $T_{9,22}$ | 38 | 38 | $T_{9,43}$ | 60 | 58 | $T_{10,19}$ | 40 | 42 |
| $T_{9,23}$ | 34 | 35 | $S_{9}$ | 72 | 64 | $T_{10,20}$ | 40 | 41 |
| $T_{9,24}$ | 36 | 38 | $P_{10}$ | 34 | 32 | $T_{10,21}$ | 40 | 40 |
| $T_{9,25}$ | 34 | 36 | $T_{10,1}$ | 36 | 34 | $T_{10,22}$ | 40 | 41 |
| $T_{9,26}$ | 42 | 40 | $T_{10,2}$ | 36 | 35 | $T_{10,23}$ | 40 | 39 |
| $T_{9,27}$ | 42 | 43 | $T_{10,3}$ | 36 | 35 | $T_{10,24}$ | 40 | 43 |
| $T_{9,28}$ | 40 | 42 | $T_{10,4}$ | 36 | 35 | $T_{10,25}$ | 42 | 43 |
| $T_{9,29}$ | 40 | 44 | $T_{10,5}$ | 38 | 38 | $T_{10,26}$ | 42 | 41 |
| $T_{9,30}$ | 36 | 38 | $T_{10,6}$ | 38 | 37 | $T_{10,27}$ | 42 | 41 |
| $T_{9,31}$ | 38 | 40 | $T_{10,7}$ | 38 | 37 | $T_{10,28}$ | 42 | 40 |
| $T_{9,32}$ | 38 | 42 | $T_{10,8}$ | 38 | 37 | $T_{10,29}$ | 42 | 44 |
| $T_{9,33}$ | 42 | 46 | $T_{10,9}$ | 38 | 36 | $T_{10,30}$ | 42 | 45 |
| $T_{9,34}$ | 44 | 46 | $T_{10,10}$ | 38 | 39 | $T_{10,31}$ | 42 | 43 |
|  | 48 | $T_{10,11}$ | 38 | 38 | $T_{10,32}$ | 42 | 42 |  |


| $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| $T_{10,33}$ | 42 | 42 | $T_{10,54}$ | 46 | 50 | $T_{10,75}$ | 54 | 52 |
| $T_{10,34}$ | 42 | 45 | $T_{10,55}$ | 46 | 46 | $T_{10,76}$ | 54 | 56 |
| $T_{10,35}$ | 38 | 39 | $T_{10,56}$ | 46 | 44 | $T_{10,77}$ | 50 | 53 |
| $T_{10,36}$ | 40 | 42 | $T_{10,57}$ | 46 | 52 | $T_{10,78}$ | 50 | 57 |
| $T_{10,37}$ | 38 | 40 | $T_{10,58}$ | 42 | 46 | $T_{10,79}$ | 52 | 59 |
| $T_{10,38}$ | 38 | 39 | $T_{10,59}$ | 40 | 43 | $T_{10,80}$ | 52 | 50 |
| $T_{10,39}$ | 38 | 38 | $T_{10,60}$ | 40 | 41 | $T_{10,81}$ | 52 | 58 |
| $T_{10,40}$ | 38 | 38 | $T_{10,61}$ | 46 | 50 | $T_{10,82}$ | 56 | 62 |
| $T_{10,41}$ | 38 | 40 | $T_{10,62}$ | 42 | 47 | $T_{10,83}$ | 56 | 54 |
| $T_{10,42}$ | 40 | 42 | $T_{10,63}$ | 42 | 44 | $T_{10,84}$ | 56 | 59 |
| $T_{10,43}$ | 36 | 36 | $T_{10,64}$ | 40 | 43 | $T_{10,85}$ | 48 | 54 |
| $T_{10,44}$ | 46 | 44 | $T_{10,65}$ | 42 | 44 | $T_{10,86}$ | 48 | 51 |
| $T_{10,45}$ | 46 | 47 | $T_{10,66}$ | 42 | 46 | $T_{10,87}$ | 44 | 50 |
| $T_{10,46}$ | 46 | 47 | $T_{10,67}$ | 42 | 42 | $T_{10,88}$ | 46 | 52 |
| $T_{10,47}$ | 42 | 45 | $T_{10,68}$ | 38 | 41 | $T_{10,89}$ | 44 | 47 |
| $T_{10,48}$ | 44 | 47 | $T_{10,69}$ | 48 | 50 | $T_{10,90}$ | 48 | 55 |
| $T_{10,49}$ | 44 | 45 | $T_{10,70}$ | 48 | 47 | $T_{10,91}$ | 48 | 51 |
| $T_{10,50}$ | 44 | 44 | $T_{10,71}$ | 48 | 46 | $T_{10,92}$ | 54 | 60 |
| $T_{10,51}$ | 44 | 49 | $T_{10,72}$ | 48 | 52 | $T_{10,93}$ | 64 | 62 |
| $T_{10,52}$ | 44 | 46 | $T_{10,73}$ | 48 | 53 | $T_{10,94}$ | 64 | 67 |
| $T_{10,53}$ | 44 | 48 | $T_{10,74}$ | 48 | 49 | $T_{10,95}$ | 60 | 66 |


| $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ | $T$ | $M_{1}$ | $M_{2}$ |
| :---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: | ---: |
| $T_{10,96}$ | 66 | 69 |  |  |  |  |  |  |
| $T_{10,97}$ | 76 | 74 |  |  |  |  |  |  |
| $T_{10,98}$ | 58 | 65 |  |  |  |  |  |  |
| $S_{10}$ | 90 | 81 |  |  |  |  |  |  |

## Vita

John Wheless Estes was born in Morristown Tennessee on March 27th 1985, the son of Michael Earl Estes and Suellen Wheless Estes. His family moved three times before finally settling in Blue Mountain Mississippi for the purpose of starting a church, Life Connection. There he volunteered time with construction projects, musical endeavors, and the leading of teenage groups.

John graduated from Ripley High School in 2003, after which he attented Oral Roberts University where he served as a floor chaplain and as a community outreach team leader. He graduated in 2007 with a Bachelors of Science degree in Mathematics and a Spanish Major.

May of that year, John married his high school sweetheart, Stephany Rangel, and the two began their life together in Oxford, Mississippi.

In the fall of 2007, John started as a full time graduate student in the Department of Mathematics at the University of Mississippi, and during his time there taught a variety of undergraduate mathematics courses. In May 2009, John obtained his Masters of Science in Mathematics under the supervision of Dr. William Staton.

On October 21st 2011, Stephany gave birth to the Estes' first son, Ethan. He weighed 6 lbs 11 oz.

In the latter part of 2012, the Estes family will be moving to Jackson Mississippi where John will start his professional career as a professor of mathematics at Belhaven University.

