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Milisha Hart-Simmons
University of Mississippi

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THE ELEMENT SPECTRUM OF A GRAPH

DISSERTATION

A Dissertation
presented in partial fulfillment of requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
The University of Mississippi

by

MILISHA HART-SIMMONS

August 2017

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ABSTRACT

Characterizations of graphs and matroids that have cycles or circuits of specified cardinality have been given by authors including Edmonds, Junior, Lemos, Murty, Reid, Young, and Wu. In particular, a matroid with circuits of a single cardinality is called a Matroid Design. We consider a generalization of this problem by assigning a weight function to the edges of a graph. We characterize when it is possible to assign a positive integer value weight function to a simple 3-connected graph G such that the graph G contains an edge that is only in cycles of two different weights. For example, as part of the main theorem we show that if this assignment is possible, then the graph G is an extension of a three-wheel, a four-wheel, a five-wheel , $K_{3,n}$, a prism, a certain seven-vertex graph, or a certain eight-vertex graph, or G is obtained from the latter three graphs by attaching triads in a certain manner. The reason for assigning weights is that if each edge of such a graph is subdivided according to the weight function, then the resulting subdivided graph will contain cycles through a fixed edge of just a few different cardinalities. We consider the case where the graph has a pair of vertex-disjoint cycles and the case where the graph does not have a pair of vertex-disjoint cycles. Results from graph structure theory are used to give these characterizations.

TABLE OF CONTENTS

1 INTRODUCTION

This chapter introduces the basic concepts discussed in the dissertation. Results from the literature as well as new results are given in the subsequent chapters. In section 1 of this chapter, we introduce the spectrum, e-spectrum, and weighted e-spectrum of a graph or matroid. In section 2, we discuss the basic terminology of matroid theory used throughout the dissertation. In section 3, we give background results and useful theorems needed in this research.

1.1 The Spectrum of a Matroid

There are results in the graph theory and matroid theory literature that investigate a graph or a matroid with cycles or circuits of one of just of few different cardinalities. Here we provide new such results on such graphs and matroids. We first define the spectrum of a graph or of a matroid and give an overview of the results from the literature related to this topic.

Let G be a graph and M be a matroid in the remainder of this chapter. The spectrum of G is the set of its cycle lengths, while the spectrum of M is the set of its circuit lengths. Note that the spectrum of a graph is typically associated with its eigenvalues. Here the spectrum of a graph is associated with its cycle lengths in order to agree with the terminology used for matroids by Lemos and others in research of this type. We let $\mathcal{C}(G)$ be the set of cycles of G and $\mathcal{C}(M)$ be the set of circuits of M. We denote the spectrum of G by spec(G) and denote the spectrum of M by spec(M). Then spec(G) = $\{|\mathrm{C}| \ : \mathrm{C} \in \mathcal{C}(\mathrm{G})\}$ while spec(M) = $\{|\mathrm{C}| \ : \mathrm{C} \in \mathcal{C}(\mathcal{M})\}$. By the term "circuit" of a graph we mean a "cycle" of that graph. Authors including Cordovil, Junior, Lemos, and Maia Jr. have determined all matroids M with spec(M) $\subseteq \{1,2,3,4,5\}$ and all 3- connected binary matroids M with spec(M) $\subseteq \{3,4,5,6,7\}$ (see [**?**, **?**]). A different direction of research on the circuit size of a matroid is to consider the problem of characterizing the matroids with a circuit-spectrum containing few elements. In general, this is a difficult problem. For example, even to characterize the non-binary matroids M with |spec(M)| = 1 would require the solution of problem in design theory as shown by Edmonds, Murty, and Young in several papers [**?**, **?**, **?**]. However, this problem is more tractable for restricted classes of matroids. Murty [**?**] completely characterized all binary matroids M with |spec(M)| = 1 (see Theorem 1.3.1). Lemos, Reid, and Wu [**?**] characterized all connected binary matroids M with a spectrum of cardinality two and largest circuit size odd (see Theorem 1.3.2). They provided a sharper characterization of the 3- connected binary matroids of this type (see Theorem 1.3.3). It is still an open question to characterize the 2- or 3- connected binary matroids with a circuit spectrum of cardinality two with the largest circuit size even. We will provide some information on these matroids here, but not a complete characterization.

Suppose that e is either a fixed edge of the graph G or a fixed element of the matroid M. In the former case, the spectrum of e is the set of cycle lengths of the cycles of G that contain e. In the latter case, spectrum of e is the set of circuit lengths of the circuits of M that contain e. We denote the spectrum of e in G by $\mathrm{spec_e(G)}$. We denote the spectrum of e in M by $\mathrm{spec_e(M)}$. Then $\mathrm{spec_e(G)} = \{|\mathrm{C}| \ : \mathrm{e} \in \mathrm{C} \in \mathcal{C}(\mathrm{G})\}$ while $\mathrm{spec_e(M)} = \{|\mathrm{C}| \ : \mathrm{e} \in \mathrm{C} \in \mathcal{C}(\mathrm{M})\}$.

Consider a pair (G, ω) where $\omega : E(G) \to \mathbb{Z}^+$ is a weight function on the edges of G. The weight $\omega(C)$ of a cycle C in the graph is the sum of the weights of its edges, following common usage. We define the ω-spectrum of an edge e of G to be $\mathrm{spec_{\omega(e)}(G)} := \{\omega(\mathrm{C}) \ : \mathrm{e} \in \mathrm{C} \in \mathcal{C}(\mathrm{G})\}$. The edge e is said to be ω-balanced (or sometimes just balanced) if it has an ω-spectrum of cardinality two. The graph G is said to be ω-balanced (or sometimes just balanced) if it contains an ω-balanced edge. For example, consider the graph G in Figure 1.1 with weights ω as shown on each edge other than e. Then the edge e is ω-balanced regardless of the weight assigned to the edge e as each cycle of G that contains the edge e has weight either $8 + \omega(e)$ or $12 + \omega(e)$. Hence $\mathrm{spec_{\omega(e)}(G)} = \{8 + \omega(\mathrm{e}), 12 + \omega(\mathrm{e})\}$ and we will say that the edge e is balanced as $|\mathrm{spec_{\omega(e)}(G)}| = 2$.

We mentioned results on matroids with a small circuit spectrum in the last paragraph. In this dissertation, we investigate classes of graphs that contains an

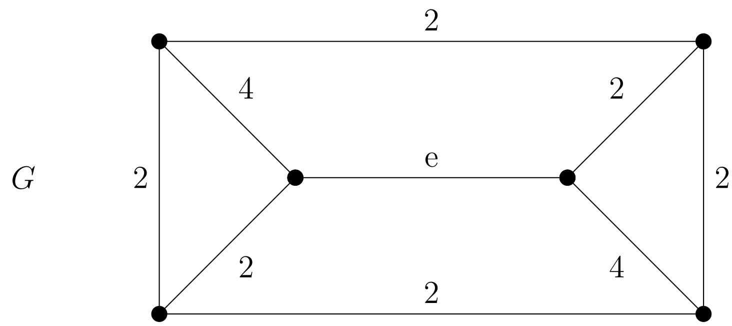


Figure 1.1: A balanced edge in a weighted graph

element e with a small edge-weighted e- spectrum set. We contain a complete characterization of a particular class of graphs with this property.

1.2 Concepts, Notation, and Terminology

In this section, we discuss the basic concepts of Matroid Theory needed in this dissertation. The matroid terminology used here mostly follows Oxley [**?**]. We begin with the definition of a matroid.

Definition 1.2.1. *A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three axioms:*

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.

(I3) If I_1 and I_2 are in $\mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.

The members of $\mathcal{I}$ are called the *independent sets* of M and E is called the *ground set* of M. It is also common to write $\mathcal{I}(M)$ for $\mathcal{I}$ and to write $E(M)$ for E. Any subset of E that is not *independent* is called *dependent*. A *minimal dependent set* is a dependent set with all proper subsets being independent. A matroid M can also be defined by its set of minimal dependent sets called *circuits*. The set of circuits of M is denoted by $\mathcal{C}$ or $\mathcal{C}(M)$.

Theorem 1.2.2. *A set of subsets $\mathcal{C}$ of a non-empty finite set E is the set of circuits of a matroid if and only if $\mathcal{C}$ satisfies the following three conditions.*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$. **(Circuit Elimination Axiom)**

Let $X \subset E(M)$. Then the *deletion* of X from M is denoted by $M \backslash X$. This is the matroid with groundset $E \backslash X$ and circuit set $\{C \in E(M) \backslash X : C \in \mathcal{C}(M)\}$. The *contraction* of X from M is denoted by M/X. This is the matroid with groundset $E \backslash X$ and circuit set consisting of the minimal non-empty members of $\{C - X : C \in \mathcal{C}(M)\}$.

Let $e \in E(M)$. Then e is said to be a loop of M when {e} is a circuit. If f and g be distinct elements of $E(M)$ such that $\{f, g\}$ is a circuit of M, then f and g are said to be in parallel. A *parallel class* of M is a maximal subset of X of $E(M)$

such that any two distinct members of X are in parallel. If elements f and g are distinct elements of $E(M)$ such that $\{f, g\}$ is a cocircuit of M, then f and g are said to be in series. A *series class* of M is a maximal subset X of $E(M)$ such that any two distinct members of X are in series. A matroid that is obtained from M by replacing each element by a series class of size k is called a *k-subdivision* of M. The simplification of the matroid M, denoted by $si(M)$, is obtained by deleting all loops and all but one element from each parallel class.

A maximum independent set of M is called a basis. The set of bases of M is denoted by $\mathcal{B}(M)$, or sometimes by $\mathcal{B}$. The members of $\mathcal{B}$ are equicardinal. In fact, if X is any subset of the ground set of a matroid M, then the maximal independent subsets of X are equicardinal. This common cardinality is called the *rank* of X. We denote this number by $r(X)$ and let $r(M) = r(E(M))$. The following theorem characterizes precisely when certain functions can be the rank function of a matroid.

Theorem 1.2.3. *Let E be a set. A function $r : 2^E \rightarrow Z^+ \cup \{0\}$ is the rank function of a matroid on E if and only if r satisfies the following conditions:*

(R1) If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.

(R2) If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.

(R3) If X and Y are subsets of E, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.

Let $X \subseteq E(M)$. Then $cl(X) = \{x \in E \ : \ r(X \cup x) = r(X)\}$. A set $X \subseteq E$ is a *flat* of M if $cl(X) = X$. A flat is sometimes called a *closed set*. A flat of M of rank $r(M) - 1$ is called a *hyperplane*.

Let G be a graph. Its vertex set is denoted by $V(G)$ while its edge set is denoted by $E(G)$. The cycle matroid of G is denoted by $M(G)$. This is the matroid with groundset E(G) and a circuit being the edge set of a cycle of G. The number of connected components of G is denoted by $\omega(G)$. Let r be the rank function of $M(G)$. Then, for $X \subseteq E(G)$, $r(X) = |V(G[X])| - \omega(X)$ where $G[X]$ is the subgraph of G induced by X. A matroid is said to be *graphic* if it is isomorphic to the cycle matroid of some graph.

For r and n integers with $0 \leq r \leq n$, the uniform matroid of rank r and order n is denoted by $U_{r,n}$. This is the matroid on a set E of n elements with $X \subseteq E$ independent if and only if $|X| \leq r$. The dual matroid of M is the matroid on $E(M)$ with $X \subseteq E(M)$ being a basis of the dual matroid if and only if $E(M)$-X is a basis of M. The dual matroid of M is denoted by M^*

Let A be a matrix with entries in a field $\mathbb{F}$. Let E be the set of column labels of A, and $\mathcal{I}$ be the collection of subsets I of E such that the columns labelled by I are linearly dependent over $\mathbb{F}$. Then $\mathcal{I}$ is the set of independent sets of a matroid on E. This matroid, denoted by $M[A]$, is called the vector matroid of A. A matroid M is representable over a given field if and only if $M \cong M[A]$ for some matrix A over $\mathbb{F}$.

A binary matroid M is a matroid that is representable over the field $GF(2)$. Each graphic matroid is also binary (see [**?**, Section 6.6]).

Several matroids and graphs mentioned in this research are next described. For an integer n exceeding two, the vector matroid of the matrix A_n consisting of all binary columns of length n with exactly 1, $n-1$, or n ones is called the binary spike of rank n. This matroid is denoted by S_n. The *tip* of S_n corresponds to the column of all ones. , Also, this element is called the (*cotip*) of (S_n^*) in Figure 1.2.

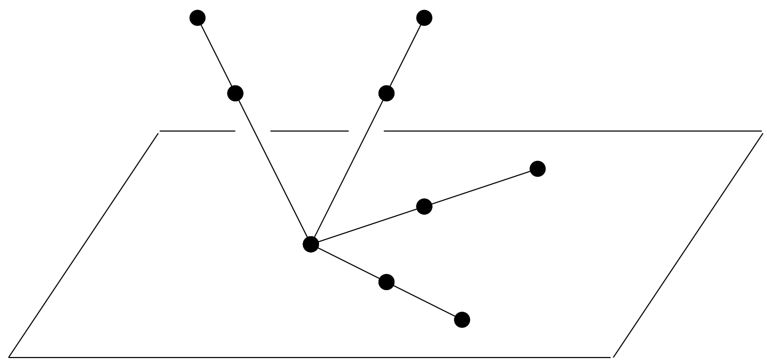


Figure 1.2: A partial geometric representation for S_4

Let n be a positive integer and q be an integer exceeding one. The finite field with q elements is denoted by $GF(q)$. The vector space of dimension n over the finite field with q elements is denoted by $V(n,q)$. The projective geometry of dimension n over $GF(q)$, denoted by $PG(n,q)$, is the matroid obtained from $V = V(n+1,q)$ by deleting the zero column and all but one element from each parallel class. The *affine geometry* $AG(n,q)$ is obtained from $PG(n,q)$ by deleting all the elements of a hyperplane. Matrix representation for the binary projective geometry and binary affine geometry of dimension two are given in Figure 1.3.

$$\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix} \end{array}$$

(a) $PG(2,2)$

$$\begin{array}{c} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array} \\ \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \end{pmatrix} \end{array}$$

(b) $AG(3,2)$

Figure 1.3: A projective geometry and an affine geometry

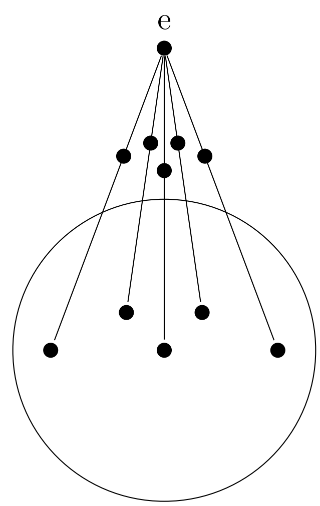


Figure 1.4: $B(r,2)$

Suppose $X \subseteq V(G)$. Then X is a vertex cut of G if $G-X$ has more connected components than G. The *connectivity* of G, denoted by $\kappa(G)$, is defined as follows. If G is connected and has a pair of non-adjacent vertices, then $\kappa(G)$ is the smallest j for which G has a j-element vertex cut. If G is connected but contains no pair of non-adjacent vertices, then $\kappa(G)$ is $|V(G)|-1$. If G is disconnected, then let $\kappa(G)=0$. If $k \in \mathbb{Z}^+$, then G is k-connected if $\kappa(G) \geq k$. For example, the graph H

in Figure 1.5 has a 2-vertex cut (for example $\{u, v\}$), but no 1-vertex cut so $\kappa(H) = 2$ and H is 2-connected.

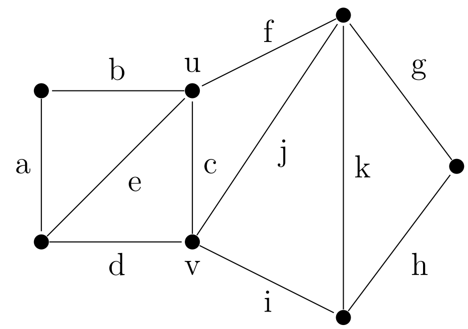


Figure 1.5: A 2-connected graph H

We next define what it means for a matroid to be n-connected for n a positive integer exceeding one. Let $X \subset E(M)$ in the remainder of the section. Then the *connectivity function* of M, denoted by λ_M or sometimes by λ, is defined as follows: $\lambda_M(X) = r(X)+r(E-X)-r(M)$. If k is a positive integer such that $\lambda(X) < k$, then both X and $(X, E-X)$ are said to be k-separating. If $(X, E-X)$ is a k-separating pair with $\min\{|X|, |E-X|\} \geq k$, then we say that $(X, E-X)$ is a k-separation of M. A matroid is n-connected if M has no k-separation with $k \in \{1, 2, \ldots, n-1\}$. For example, if M is the cycle matroid of the graph H of Figure 1.5, $X = \{a, b, c, d, e\}$, and $E - X = \{f, g, h, i, j, k\}$, then $\lambda(X) = r(X) + r(E-X) - r(M) = (4-1) + (5-1) - (7-1) = 1$, $|X| \geq 2$, and $|E-X| \geq 2$. Thus $(X, E-X)$ is a 2-separation. Hence M is not 3-connected. However, M has no 1-separations so that M is 2-connected.

The construction of the parallel connection or of the series connection of two matroids M_1 and M_2 is defined next. Suppose that these matroids are on disjoint sets and that $p_1 \in E(M_1)$ while $p_2 \in E(M_2)$. Let p be a new element that is not in $E(M_1) \cup E(M_2)$. Let $E = E(M_1 \backslash p_1) \cup E(M_2 \backslash p_2) \cup p$. Then we will define two matroids on E. Let $\mathcal{C}_P = \mathcal{C}(M_1 \backslash p_1) \cup \{(C_1 - p_1) \cup p : p_1 \in C_1 \in \mathcal{C}(M_1)\} \cup \mathcal{C}(M_2 \backslash p_2) \cup \{(C_2 - p_2) \cup p : p_2 \in C_2 \in \mathcal{C}(M_2)\} \cup \{(C_1 - p_1) \cup (C_2 - p_2) : p_1 \in \mathcal{C}(M_1) \text{ and } p_2 \in \mathcal{C}(M_2)\}$ and let $\mathcal{C}_S = \mathcal{C}(M_1 \backslash p_1) \cup \mathcal{C}(M_2 \backslash p_2) \cup \{(C_1 - p_1) \cup (C_2 - p_2) \cup p : p_1 \in \mathcal{C}(M_1) \text{ and } p_2 \in \mathcal{C}(M_2)\}$. Then $\mathcal{C}_P$ is the set of circuits of the parallel connection of M_1 and M_2, denoted by $P(M_1, M_2)$. Also, $\mathcal{C}_S$ is the set of circuits of the series connection of M_1 and M_2, denoted by $S(M_1, M_2)$.

1.3 Background Results

This section begins with the background results from the literature that motivate this research. This section concludes with some preliminary theorems that are used in this research. The first result given is by Murty [**?**]. This result completly determines the binary connected matroids with a circuit spectrum of cardinality one. This result motivates the subsequent research in this area.

Theorem 1.3.1. *Let M be a connected binary matroid. For $c \in \mathbb{Z}^+$,* spec(M) = {c} *if and only if M is isomorphic to one of the following matroids:*

(i) a c- subdivision of $U_{0,1}$,

(ii) a k- subdivision of $U_{1,n}$, where $c = 2k$ and $n \geq 3$

(iii) an l- subdivision of $\mathrm{PG(r,2)^*}$ *where* $c = 2^r l$ *and* $r \geq 2$

(iv) an l- subdivision of $\mathrm{AG(r+1,2)^*}$ *where* $c = 2^r l$ *and* $r \geq 2$

The next result is due to Lemos, Reid, and Wu [**?**]. This result determines the connected binary matroids with circuit spectrum of cardinality two in a special case. Note that the increasing complexity of the problem as the cardinality of the circuit spectrum set becomes larger.

Theorem 1.3.2. *Let $c,\ d \in \mathbb{Z}^+$ with $c < d$ and d odd. Let M be a connected binary matroid. Then* $\mathrm{spec(M)} = \{\mathrm{c}, \mathrm{d}\}$ *if and only if there are connected binary matroids $M_0, M_1, \ldots M_n$ for some $n \in \mathbb{Z}^+$ such that the following holds.*

(i) $E(M_i) \cap E(M_j) = \{e\}$ *, for distinct i and j in $\{0, 1, \ldots, n\}$.*

(ii) $E(M_0)$ *is the circuit of M_0.*

(iii) For $i \in \{0, 1, \ldots, n\}$, e is the series class of M_i, all other series classes of M_i have size l_i, and the cosimplification of M_i is isomorphic to one of the following matroids.

(a) $U_{1,n}$ *for some $n_i \geq 3$ where $c = 2l_i$.*

(b) $\mathrm{PG(r_i,2)^*}$*, for some $r_i \geq 2$ where $c = 2^{r_i} l_i$*

(c) $\mathrm{AG(r_i,2)^*}$ *for some $r_i \geq 3$ where $c = 2^{r_i - 1} l_i$*

(d) $S^*_{n_i}$ *for some $n_i \geq 4$ and e is the cotip, where $c = 4l_i$*

(e) $\mathrm{B(r_i,2)^*}$ *for some $r_i \geq 3$ and e is the cotip, where $c = 2^{r_i} l_i$*

(iv) $d = |E(M_0)| - 1 + d_1 + d_2 + \ldots + d_n > c$ *where* $d_i = \frac{c}{2}$ *where (iii) (a) holds,* $d_i = (2^{r_i} - 1)l_i$ *when (iii) (b) holds,* $d_i = (2^{r_i - 1} - 1)l_i$ *when (iii) (c) holds,* $d_i = n_i l_i$ *when (iii) (d) holds, and* $d_i = c$ *when (iii) (e) holds.*

(v) $M = S(M_0, M_1, \ldots M_n)/e$

Lemos, Reid, and Wu obtained the following consequences of the previous theorem by restricting their attention to the class of 3-connected matroids. An attractive result is obtained by restricting the largest cycle cardinality to an odd number.

Theorem 1.3.3. *Let M be a 3-connected binary matroid with largest circuit size odd. Then* $|\mathrm{spec(M)}| \leq 2$ *if and only if M is isomorphic to one of the following matroids.*

(i) $U_{0,1}$ *or* $U_{2,3}$

(ii) ${S_{2n}}^*$ *for some* $n \geq 2$

(iii) $\mathrm{B(r,2)}^*$ *for some* $r \geq 2$

We follow standard notation and denote the complete graph on n vertices by K_n. We denote the complete bipartite graph with vertex classes of size m and n by $K_{m,n}$. The graphs $K'_{3,n}$, $K''_{3,n}$, and $K'''_{3,n}$ are obtained by adding one, two, and three edges, respectively, to the partite class of size three in $K_{3,n}$. These graphs are given in Figure 1.6. The complete graph on five vertices is denoted by K_5 and the graph obtained from K_5 by deleting an edge is denoted by K_5^-. The wheel graph is given in Figure 1.7.

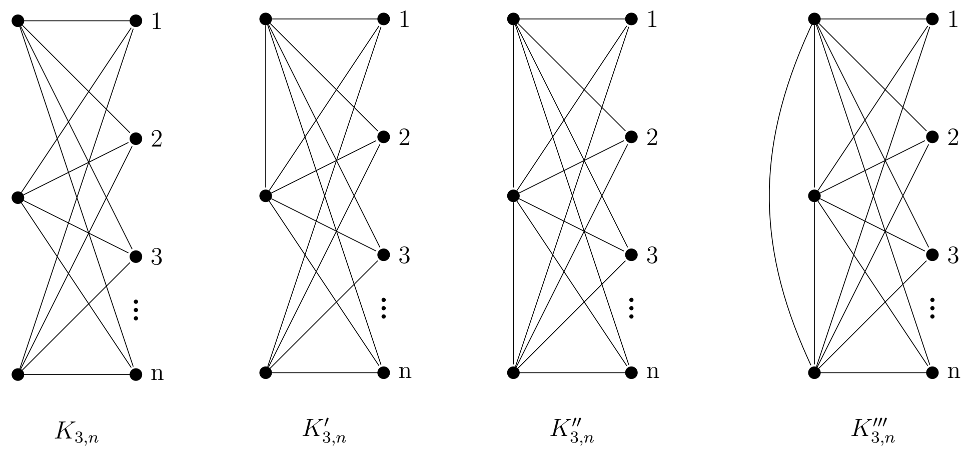


Figure 1.6: Some graphs without vertex-disjoint cycles

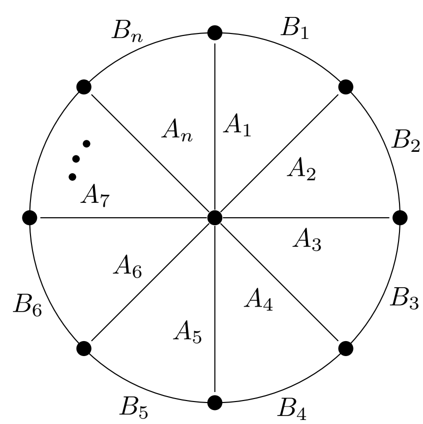


Figure 1.7: The wheel graph W_n

Dirac [?] characterized 3-connected graphs without vertex disjoint cycles in the following result. This is a useful result as two vertex disjoint cycles can be used to construct cycles of varying lengths in a highly connected graph. This result is crucial in the proof of the main result of the dissertation.

Theorem 1.3.4. *Let G be a simple 3-connected graph. Then G does not have two vertex disjoint cycles if and only if G is isomorphic to one of the following graphs; a wheel, K_5, K_5^-, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for some integer n exceeding two.*

Menger's Theorem (1927) is often used when studying n-connected graphs (see also, [?]). This is because it is useful for constructing cycles of varying lengths in a highly connected graph. In particular, we rely on this result in the dissertation.

Theorem 1.3.5. *A graph G is n-connected if and only if every pair of vertices of G are connected by at least n internally-disjoint paths.*

2 THE RESULTS

In the main result of the dissertation we give a characterization of all pairs (G, ω) where G is a simple 3-connected graph and ω is an positive integer valued weight function on the edges of G. The graph G further contains an edge such that there are only cycles of two different weights that contain this edge. In the first section of the chapter, we give an attractive statement of one direction of the main result. The statement of main result as an “if and only if” theorem waits until the third section of the chapter as the first two sections of the chapter gives such pairs (G, ω) for the graph G being in different classes of graphs. Section 1 of the chapter concentrates on graphs without vertex-disjoint cycles, while Section 2 of the chapter concentrates on graphs with a pair of vertex-disjoint cycles.

2.1 Graphs without disjoint cycles

We first give an attractive partial statement of one direction of the main result of the dissertation, Theorem 2.3.1. This next result indicates the graphs that are of interest in this dissertation. An *extension* of a graph is either the graph itself or a simple graph obtained by adding edges to the graph. The vertex labeling of the graphs Prism, Prism$^{\perp}$ and Prism$^{\perp\perp}$ are given in Figure 2.1, 2.2, and 2.3, respectively.

Theorem 2.1.1. *Let G be a simple 3-connected graph. If there exists a positive integer valued weight function on the edge set of G such that there exists an edge that is only in cycles of two different weights, then the graph is a three-wheel, a four-wheel, a five-wheel, $K_{3,n}$, the Prism, the Prism$^{\perp}$, or Prism$^{\perp\perp}$ graph or is isomorphic to one of the latter three graphs by attaching triads to either the vertex set $\{v_1, v_3, w_2\}$ or $\{v_3, w_1, w_2\}$.*

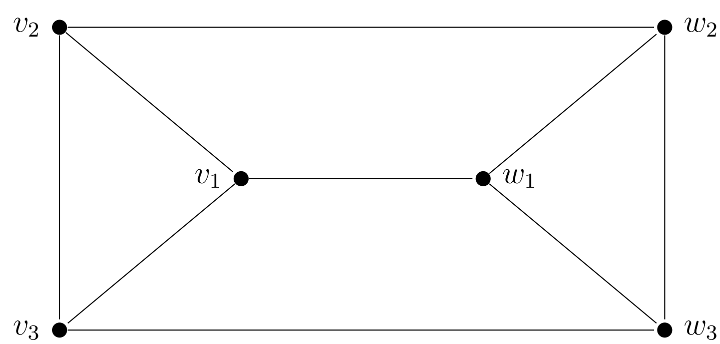


Figure 2.1: A vertex-labeled Prism Graph

The next result indicates why we consider graphs that contain an edge in cycles of two different weights. There are no simple 3-connected graphs that contain cycles of a single weight when the weights are assigned as positive integer values to the edges. We remind the reader, this theorem means that if a positive integer weight is assigned to each edge of such a graph, then there will be cycles of at least two different weights that contain each edge.

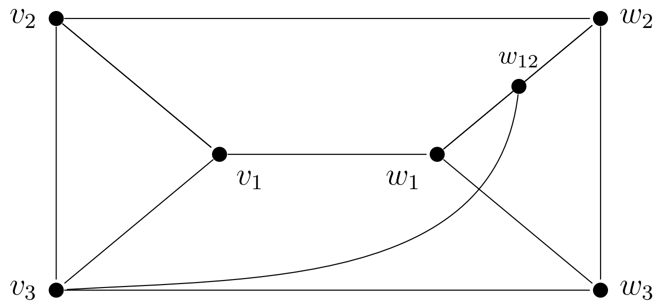


Figure 2.2: A vertex-labeled Prism$^{\perp}$ graph

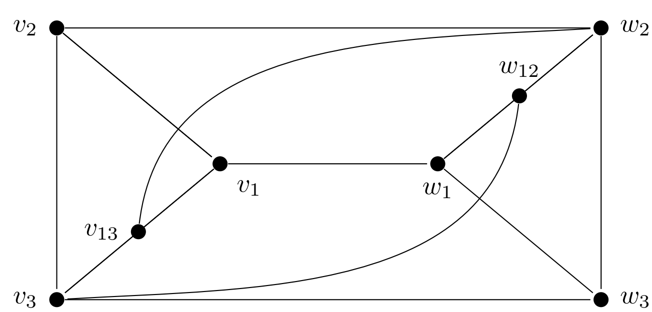


Figure 2.3: A vertex-labeled Prism$^{\perp\perp}$

Theorem 2.1.2. *Let G be a simple 3-connected graph and $\omega : E(G) \to \mathbb{Z}^+$. If $e \in E(G)$, then* $|\mathrm{spec}_{\omega(\mathrm{e})}(\mathrm{G})| \geq 2$.

In the remainder of the section we give results on simple 3-connected graphs that contain a balanced edge. The graphs considered in this section of the dissertation do not contain two vertex-disjoint cycles. An understanding of when edges are balanced in such graphs will be important to the characterization of the simple 3-connected graphs that contain a balanced edge that is given in Section 3 of this chapter.

An edge of G will always be assigned an uppercase letter as a label such as A_1, and the weight $\omega(A_1)$ of the edge A_1 will be represented by the corresponding lower case letter such as a_1. The edge labeling of the graph G in Theorems 2.1.3, 2.1.4, and 2.1.5 is as given in Figure 1.7 with n representing the appropriate number of spokes in the wheel graph under consideration. For convenience sake, we give a picture of the wheel with the appropriate number of spokes by the statement of each of these next four theorems.

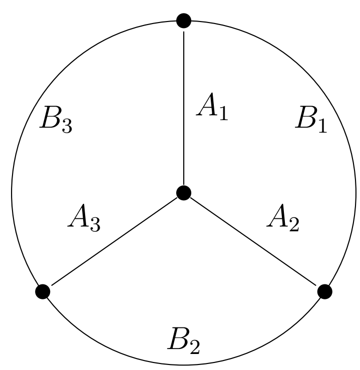


Figure 2.4: The wheel graph W_3

Theorem 2.1.3. *Let G be isomorphic to the wheel W_3 with the edge labels as pictured and with the associated weight function ω (indicated by lowercase letters). Then an edge E is balanced if and only if, up to relabeling, $E = A_1$ and one of the following conditions holds:*

(i) $a_i = a_j + b_2$ *and* $b_3 = b_1 + b_2$ *for* $i = 2$ *and* $j = 3$ *or for* $i = 3$ *and* $j = 2$, *or*

(ii) $a_i = a_j + b_2$ *and* $b_1 = b_2 + b_3$ *for* $i = 2$ *and* $j = 3$ *or for* $i = 3$ *and* $j = 2$, *or*

(iii) $a_2 = a_3$ *and* $b_1 = b_3$.

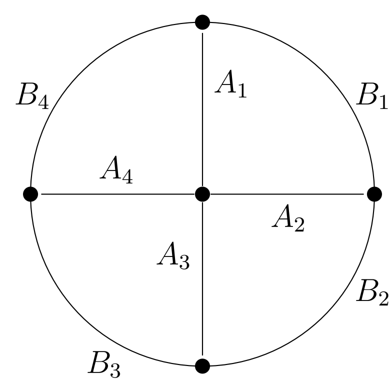


Figure 2.5: The wheel graph W_4

Theorem 2.1.4. *Let G be isomorphic to the wheel W_4 with the edge labels as pictured and with the associated weight function ω. Then the edge E is balanced if and only if, up to relabeling, either*

(i) $E = A_1$: $a_2 = a_4$, $b_1 = b_4$, $b_2 = b_3$, *and* $a_3 = a_2 \pm b_2$, *or*

(ii) $E = B_1$: *and condition (C1), (C2), or (C3) of Table 2.1 is satisfied.*

$(C1)\ a_1 = a_3 + b_4$	$(C2)\ a_1 = a_4 + b_4$	$(C3)\ a_2 = a_3 + b_2$
$a_2 = a_3 + b_2$	$a_2 = b_2$	$a_1 = b_4$
$a_3 = a_4$	$a_4 = b_3$	$a_3 = b_3$
$b_3 = 2a_3$	$a_3 = 2a_4$	$a_4 = 2a_3$

Table 2.1: Conditions that imply that the edge B_1 is balanced in a four-wheel

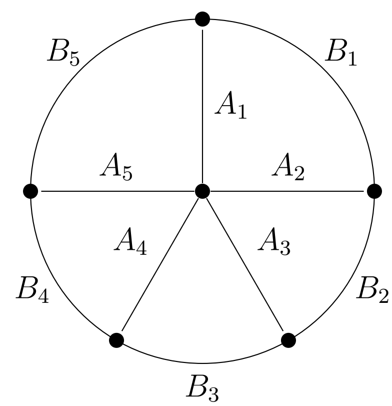


Figure 2.6: The wheel graph W_5

Theorem 2.1.5. *Let G be isomorphic to the wheel W_n for $n \geq 5$ with the edge labels as pictured and with the associated weight function ω. Then an edge E of G is balanced if and only if, up to relabeling, $E = B_1$:* $n = 5$, $a_1 = b_3 + b_5$, $a_2 = b_2 + b_3$, $a_3 = b_3$, $a_4 = 2b_3$, $a_5 = b_3$, *and* $b_3 = b_4$.

The edges of the graph G given in Theorem 2.1.6 are as labeled in Figure 2.7.

Theorem 2.1.6. *Let G be isomorphic to the graph K_5^- with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling, either*

(i) $E = A_1$: $a_2 = a_4$, $b_1 = b_4$, $b_2 = b_3$, $a_3 = a_2 - b_2$, *and* $c = 2b_2$ *or*

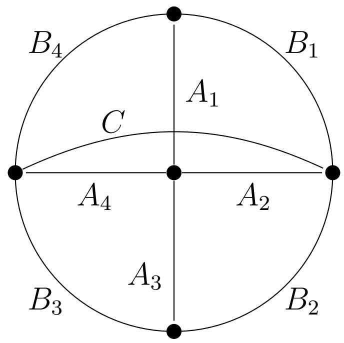


Figure 2.7: The graph K_5^-

(ii) $E = C$: $a_1 = a_3$, $b_1 = b_2$, $b_3 = b_4$, $a_2 = b_1 \pm a_1$, *and* $a_4 = b_3 \pm a_1$ *except that it is not true that both* $a_2 = b_1 - a_1$ *and* $a_4 = b_3 - a_1$.

Theorem 2.1.7. *Let G be isomorphic to the graph K_5 with the associated weight function $\omega : E(G) \to \mathbb{Z}^+$. Then no edge of G is balanced.*

The edges of the graph G in the following theorems are as labeled in Figure 2.8 with the appropriate subset of the set $\{U_1, U_2, U_3\}$ deleted from the edge set of the graph $K'''_{3,n}$.

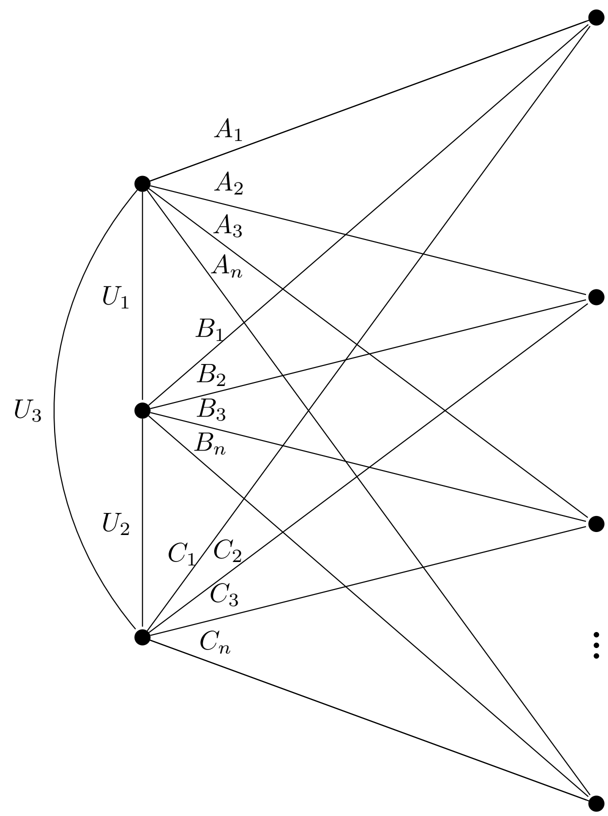


Figure 2.8: The graph $K'''_{3,n}$

Let $[n] = \{1, 2, \ldots, n\}$ for $n \in \mathbb{Z}^+$ throughout the dissertation. We refer in the following theorems to the edge labels given in the appropriate subgraph of Figure 2.8. In the next theorem the edges U_1, U_2, and U_3 are deleted. The graph $K_{3,3}$ is given in Figure 2.9 for the readers convenience in identifying the labeling mentioned in the theorem.

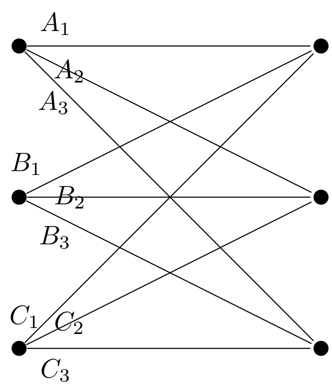


Figure 2.9: An edge labeled $K_{3,3}$ graph

Theorem 2.1.8. *Let G be isomorphic to the graph $K_{3,n}$, for $n \geq 3$ with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling, $E = A_1$: $b_1 = c_1$, and, for $i, j \in [n] \backslash 1$, $a_i = a_j$ and $b_i = c_j$.*

We refer in the following theorem to the edge labels given in Figure 2.8 where the edges U_2, and U_3 are deleted. We give a labeled $K'_{3,3}$ graph for the readers convenience.

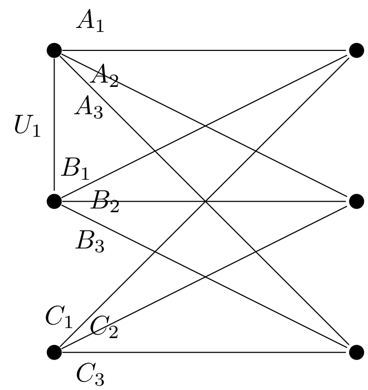


Figure 2.10: An edge labeled $K'_{3,3}$ graph

Theorem 2.1.9. *Let G be isomorphic to the graph $K'_{3,n}$, for $n \geq 3$ with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling,*

(i) $E = A_1$: $b_1 = c_1$ *and, for* $i, j \in [n]\backslash 1$, $a_i = a_j$, $b_i = c_j$, *and* $u_1 = a_2 + b_2$, *or*

(ii) $E = C_3$: $a_3 = b_3$, *and, for* $i, j \in [n]\backslash 3$, $a_i = b_j$, $c_i = c_j$, *and* $u_1 = 2a_1$, *or*

(iii) $E = U_1$: *for* $i, j \in [n]$, $a_i = a_j$, $b_i = b_j$, *and* $c_i = c_j$.

We refer in the following theorem to the edge labels given in Figure 2.8 where the edge U_3 is deleted. A labeled $K''_{3,3}$ graph is pictured as well for the readers convenience.

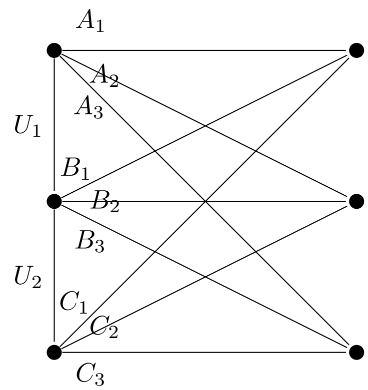


Figure 2.11: An edge labeled $K''_{3,3}$ graph

Theorem 2.1.10. *Let G be the graph $K''_{3,n}$, for $n \geq 3$ with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling,*

(i) $E = A_1$: $b_1 = c_1$, *for* $i, j \in [n]\backslash 1$, $a_i = a_j$, $b_i = c_j$, $u_1 = a_2 + b_2$, *and* $u_2 = 2b_2$, *or*

(ii) $E = B_1$: $a_1 = c_1$, *for* $i, j \in [n]\backslash 1$, $a_i = c_j$ *and* $b_i = b_j$, *and* $u_1 = u_2 = a_2 + b_2$, *or*

(iii) $E = U_1$: *for* $i, j \in [n]$, $a_i = a_j$, $b_i = b_j$, $c_i = c_j$, *and* $u_2 = b_1 \pm c_1$.

We refer in the following theorem to the edge labels given in Figure 2.8. We give a labeled $K'''_{3,3}$ graph in Figure 2.12 for the readers convenience.

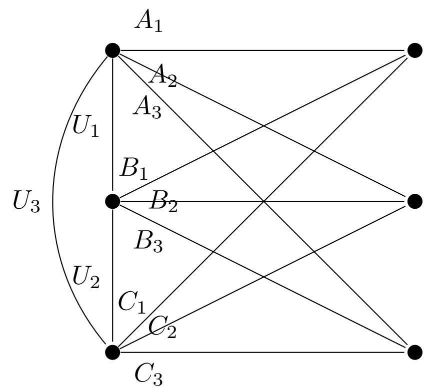


Figure 2.12: An edge labeled $K_{3,3}'''$ graph

Theorem 2.1.11. *Let G be the graph $K_{3,n}'''$, for $n \geq 3$ with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling,*

(i) $E = A_1$*:* $b_1 = c_1$, *for* $i, j \in [n]\backslash 1$, $a_i = a_j$, $b_i = c_j$, $u_1 = u_3 = a_2 + b_2$, *and* $u_2 = 2b_2$, *or*

(ii) $E = U_1$*:* *for* $i, j \in [n]$, $a_i = a_j$, $b_i = b_j$, $c_i = c_j$, $u_2 = b_1 \pm c_1$, *and* $u_3 = a_1 \pm c_1$.

2.2 Graphs with disjoint cycles

We give the simple 3-connected graphs with a balanced edge that contain a pair of disjoint cycles in this section of the dissertation. These graphs are extensions of either the Prism graph or of two other closely related graphs. We first give a useful lemma before giving the statements of these results. A graph H is called a topological minor of a graph G if a subdivision of H is isomorphic to a subgraph of G. So each edge of the topological minor H corresponds to a path of G. If ω_G is a positive integer valued weight function on the edge set of G, then the associated weight function on the edge set of H is obtained by letting the weight of each edge be the sum of the weights of the corresponding path in G. So an edge that can be balanced with respect to a weight function in a graph can be balanced with respect to another weight function in any topological minor that contains that edge.

Lemma 2.2.1. *Let G be a graph with weight function $\omega_G : E(G) \to \mathbb{Z}^+$ and H be a topological minor of G with associated weight function ω_H. If E is an edge of H, then* $|\mathrm{spec}_{\omega_{\mathrm{H}(\mathrm{E})}}(\mathrm{H})| \leq |\mathrm{spec}_{\omega_{\mathrm{G}(\mathrm{E})}}(\mathrm{G})|$.

The following theorem is a key part of the proof of Theorem 2.3.1. The graph G in Theorem 2.2.2 is as pictured in Figure 2.13.

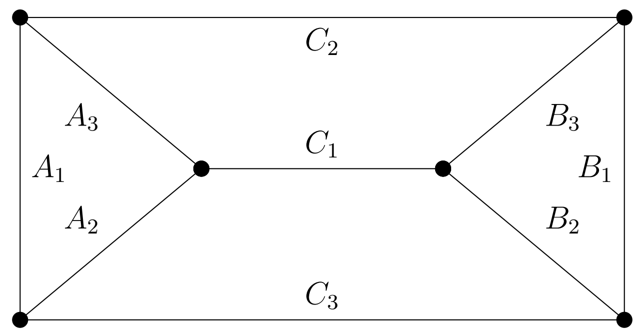


Figure 2.13: The labeled prism graph G

Theorem 2.2.2. *Let G be the Prism graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if either*

(i) $E = C_1$ *and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2, \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + 2\mathrm{a}_1\}$ *where* $a_1 = b_1$, $c_2 = c_3$, *and either* $a_3 = a_1 + a_2$ *and* $b_2 = b_1 + b_3$ *or* $a_2 = a_1 + a_3$ *and* $b_3 = b_1 + b_2$, *or*

(ii) $E = C_2$ *and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2, \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + 2\mathrm{a}_2\}$ *where* $a_2 = b_2$, $c_1 = c_3$, *and either* $a_1 = a_2 + a_3$ *and* $b_3 = b_1 + b_2$ *or* $a_3 = a_1 + a_2$ *and* $b_1 = b_2 + b_3$, *or*

(iii) $E = C_3$ *and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_2 + \mathrm{b}_2 + \mathrm{c}_1 + \mathrm{c}_3, \mathrm{a}_2 + \mathrm{b}_2 + \mathrm{c}_1 + \mathrm{c}_3 + 2\mathrm{a}_3\}$ *where* $a_3 = b_3$, $c_1 = c_2$, *and either* $a_2 = a_1 + a_3$ *and* $b_1 = b_2 + b_3$ *or* $a_1 = a_2 + a_3$ *and* $b_2 = b_1 + b_3$.

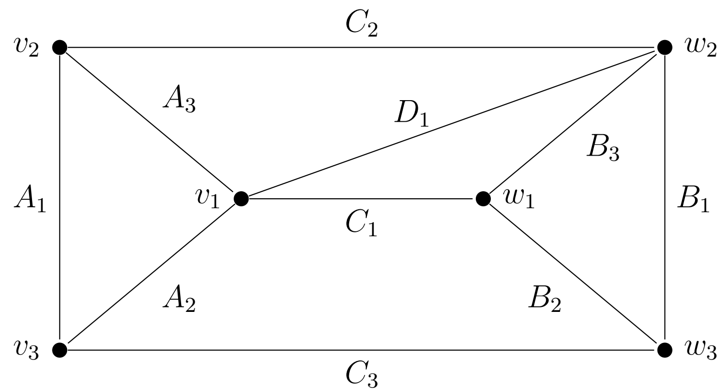


Figure 2.14: The labeled Prism+ Graph

Theorem 2.2.3. *Let G be the Prism+ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling,*

(i) $E = C_1$: $a_1 = b_1 = c_2 = c_3$, $a_2 = a_1 + a_3$, $b_3 = b_1 + b_2$, *and* $d_1 = a_3 + c_3$, *or*

(ii) $E = C_2$: $a_2 = b_2 = c_1 = c_3$, $a_3 = a_1 + a_2$, $b_1 = b_2 + b_3$, *and* $d_1 = b_3 + c_3$, *or*

(iii) $E = C_3$: $a_3 = b_3$, $c_1 = c_2$, $a_2 = a_1 + a_3$, $b_1 = b_2 + b_3$, *and either* $d_1 = a_3$ *and* $c_1 = 2a_3$ *or* $d_1 = 2a_3$ *and* $c_1 = a_3$.

Moreover, no other edge of G is balanced.

Let H be a subgraph of a simple 3-connected graph G. Then an H-bridge of G is a path between distinct vertices of H with the path being internally disjoint from the vertices of H. Now let H be a subdivision of a Prism such as the graph shown in Figure 2.13. For the purposes of the following definition, we consider the labels in that figure to correspond to paths so that C_1 is a path of H and not necessarily just an edge for example. Then an H-bridge of G that is a path from a vertex v_i to a vertex w_j for $i, j \in [3]$ with $i \neq j$ is called a *chord* of H (see the path D_1 in Figure 2.14). The following corollary of Theorem 2.2.3 will be useful in this research.

Corollary 2.2.4. *Let G be a simple 3-connected graph with a weight function $\omega : E(G) \to \mathbb{Z}^+$.*

(a) An edge of G that is contained in one of a pair of vertex-disjoint cycles of G is not balanced.

(b) Let H be a subgraph of G that is a subdivision of a Prism. If E is an edge of a chord of H, then E is not balanced.

The graph Prism1++ is as given in Figure 2.15.

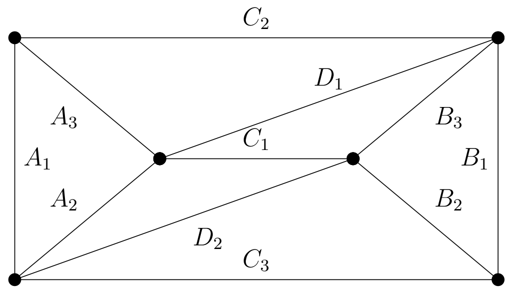


Figure 2.15: The labeled Prism1++ graph

Theorem 2.2.5. *Let G be the Prism1++ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if $E = C_1$: $a_1 = b_1 = c_2 = c_3$, $a_2 = a_1 + a_3$, $b_3 = b_1 + b_2$, $d_1 = a_3 + c_3$, and $d_2 = b_2 + c_2$.*

The edge labeled graph Prism2++ is given in Figure 2.16.

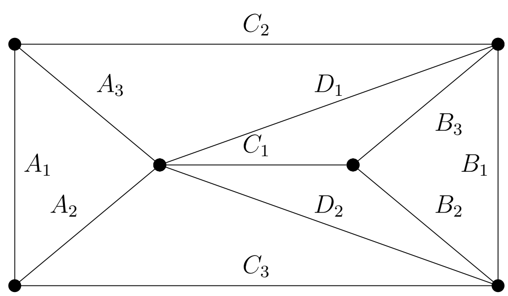


Figure 2.16: The labeled Prism2++ graph

Theorem 2.2.6. *Let G be the Prism2++ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling,*

(i) $E = C_2$: $a_2 = b_2 = c_1 = c_3,\ a_3 = a_1 + a_2,\ b_1 = b_2 + b_3,\ d_1 = b_3 + c_3,$ *and* $d_2 = 2a_2,$ *or*

(ii) $E = C_3$: $a_3 = b_3 = c_1 = c_2,\ a_2 = a_1 + a_3,\ b_1 = b_2 + b_3,\ d_1 = 2b_3,$ *and* $d_2 = b_2 + c_2.$

The edge labeled Prism3++ graph is given in Figure 2.17. This graph is mentioned in the next theorem.

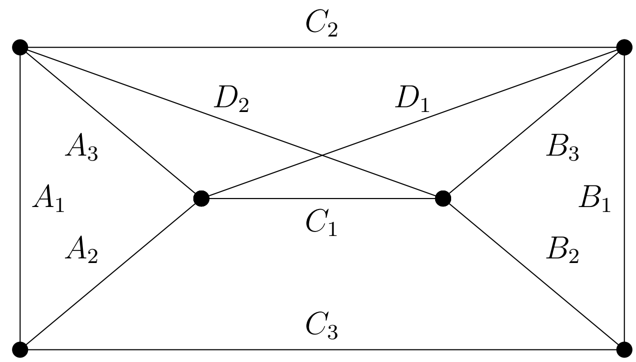


Figure 2.17: The labeled Prism3++ graph

Theorem 2.2.7. *Let G be the Prism3++ graph with the edge labels as pictured and with the associated weight function ω. Then G does not contain a balanced edge*

The graph Prism1+++ is as given in Figure 2.18. This graph is mentioned in Theorem 2.2.8.

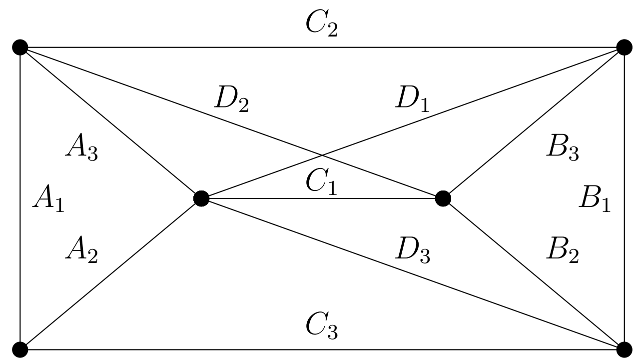


Figure 2.18: The labeled Prism1+++ graph

Theorem 2.2.8. *Let G be the Prism1+++ graph with the edge labels as pictured and with the associated weight function ω. Then G does not contain a balanced edge.*

The graph Prism2+++ is as given in Figure 2.19. This graph is mentioned in Theorem 2.2.9.

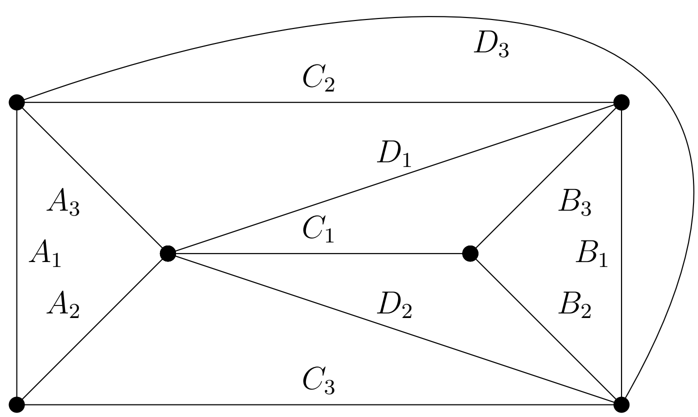


Figure 2.19: The labeled Prism2+++ graph

Theorem 2.2.9. *Let G be the Prism2+++ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if, up to relabeling, $E = C_2$: $a_2 = b_2 = c_1 = c_3$, $a_3 = a_1 + a_2$, $b_1 = b_2 + b_3$, $d_1 = b_3 + c_3$, $d_2 = 2b_2$, and $d_3 = a_1 + c_1$.*

The graph Prism3+++ is as given in Figure 2.20. This graph is mentioned in Theorem 2.2.10.

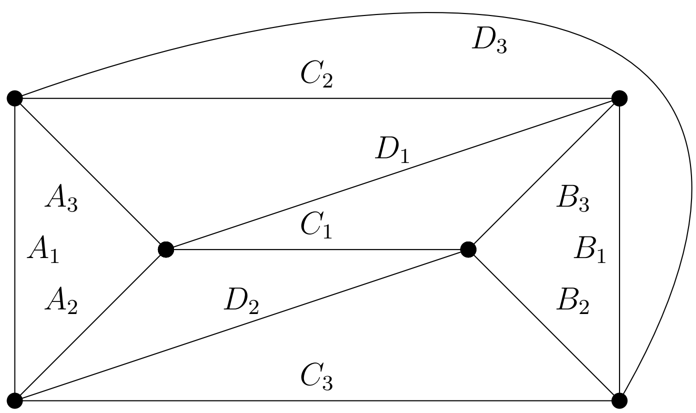


Figure 2.20: The labeled Prism3+++ graph

Theorem 2.2.10. *Let G be the Prism3+++ graph with the edge labels as pictured and with the associated weight function ω. Then G does not contain a balanced edge.*

Theorem 2.2.11. *Let G be an extension of the Prism graph obtained by adding four or more edges to that graph. Then G does not contain a balanced edge.*

Theorem 2.2.12. *Let G be the Prism$^{\perp}$ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if*

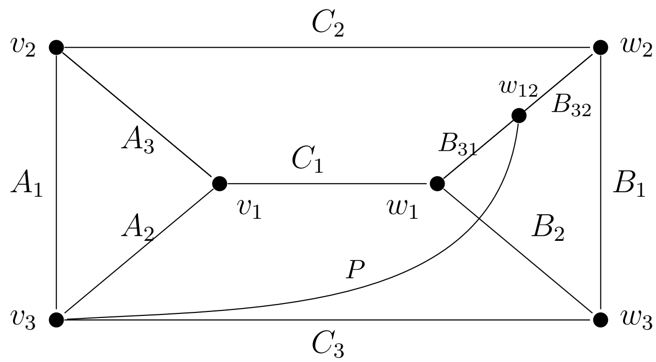


Figure 2.21: The graph Prism$^{\perp}$

(i) $E = C_1$: $p = a_1 = b_1 = c_2 = c_3,\ b_{31} = b_2,\ b_{32} = b_1,\ a_2 = a_1 + a_3,$ *and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2, \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + 2\mathrm{a}_1\}$ *or*

(ii) $E = C_2$: $p = a_2 = b_2 = c_1 = c_3,\ b_{31} = b_1,\ b_{32} = b_2,\ a_1 = a_2 + a_3,$ *and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2, \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + 2\mathrm{a}_2\}$.

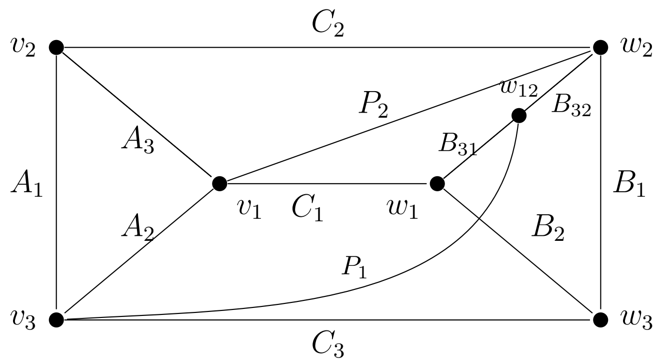


Figure 2.22: The graph Prism$^{\perp}$+a

Theorem 2.2.13. *Let G be the Prism$^{\perp}$ + a graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if $E = C_1$: $p_1 = a_1 = b_1 = c_2 = c_3$, $p_2 = a_3 + c_2$, $b_{31} = b_2$, $b_{32} = b_1$, and $a_2 = a_1 + a_3$.*

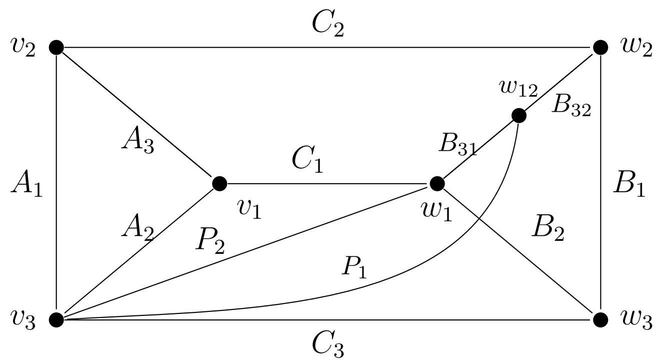


Figure 2.23: The graph Prism$^\perp$+b

Theorem 2.2.14. *Let G be the Prism$^\perp + b$ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if either*

(i) $E = C_1$: $p_1 = a_1 = b_1 = c_2 = c_3$, $p_2 = b_2 + c_3$, $b_{31} = b_2$, $b_{32} = b_1$, *and* $a_2 = a_1 + a_3$, *and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_1}\}$. *or*

(ii) $E = C_2$: $p_1 = a_2 = b_2 = c_1 = c_3$, $p_2 = 2a_2$, $b_{31} = b_1$, $b_{32} = b_2$, $a_1 = a_2 + a_3$, *and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_2}\}$.

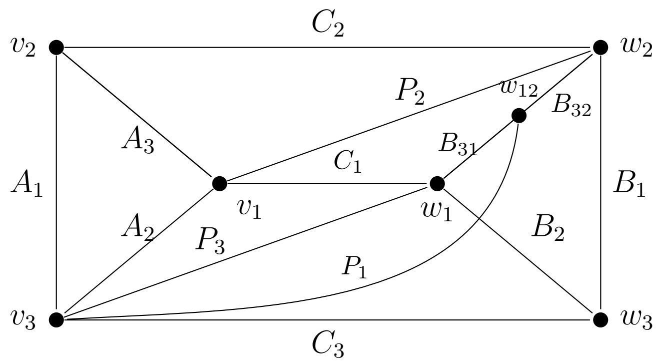


Figure 2.24: The graph Prism$^{\perp}$++

Theorem 2.2.15. *Let G be the Prism$^{\perp}$ $++$ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if $E = C_1$: $p_1 = a_1 = b_1 = c_2 = c_3$, $p_2 = b_2 + c_3$, $p_3 = a_3 + c_3$, $b_{31} = b_2$, $b_{32} = b_1$, and $a_2 = a_1 + a_3$, and* $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_1}\}$.

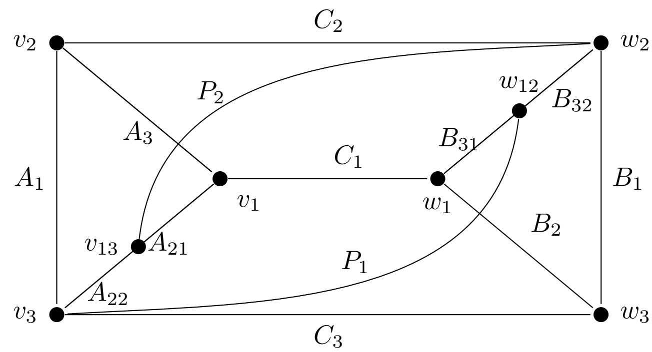


Figure 2.25: The graph Prism$^{\perp\perp}$

Theorem 2.2.16. *Let G be the Prism$^{\perp\perp}$ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if*

$E = C_1$: $p_1 = p_2 = a_1 = b_1 = c_2 = c_3$, $a_{21} = a_3$, $a_{22} = a_1$, $b_{31} = b_2$, $b_{32} = b_1$, $a_2 = a_1 + a_3$, *and* $b_3 = b_1 + b_2$.

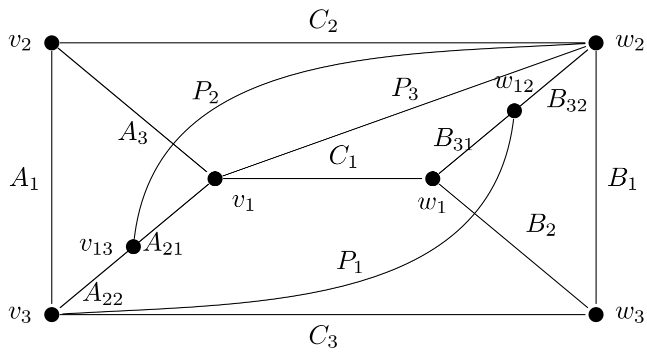


Figure 2.26: The graph Prism$^{\perp\perp}$+

Theorem 2.2.17. *Let G be the Prism$^{\perp\perp}$+ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if $E = C_1$:* $p_1 = p_2 = a_1 = b_1 = c_2 = c_3$, $p_3 = a_3 + c_2$, $a_{21} = a_3$, $a_{22} = a_1$, $b_{31} = b_2$, $b_{32} = b_1$, $a_2 = a_1 + a_3$, *and* $b_3 = b_1 + b_2$.

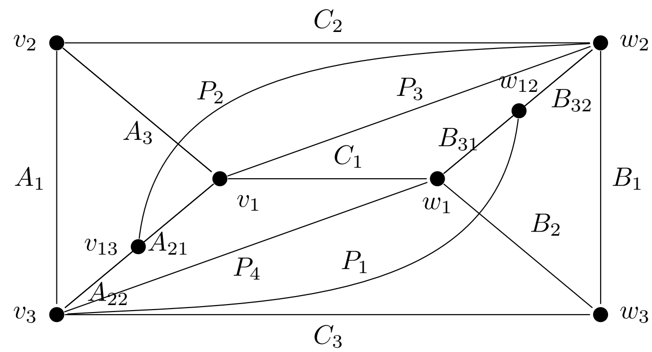


Figure 2.27: The graph Prism$^{\perp\perp}$++

Theorem 2.2.18. *Let G be the Prism$^{\perp\perp}$++ graph with the edge labels as pictured and with the associated weight function ω. Then an edge E is balanced if and only if $E = C_1$: $p_1 = p_2 = a_1 = b_1 = c_2 = c_3$, $p_3 = a_3 + c_2$, $p_4 = b_2 + c_2$, $a_{21} = a_3$, $a_{22} = a_1$, $b_{31} = b_2$, $b_{32} = b_1$, $a_2 = a_1 + a_3$, and $b_3 = b_1 + b_2$.*

2.3 The main result

We give the main result of the thesis in this section of the dissertation. A balanced three-wheel is a pair (G, ω) where G is a graph with $\omega : E(G) \to \mathbb{Z}^+$ where $G \cong W_3$ and, up to relabeling, ω is as given in the statement of Theorem 2.1.3. Likewise, we define a balanced four-wheel graph etc. to be a pair (G, ω) such as given in Theorems 2.1.4 through 2.2.18.

Suppose that the pair (G, ω) is balanced. Further suppose that there is a subgraph H of G that is either a Prism, Prism$^{\perp}$, or Prism$^{\perp\perp}$ graph as shown in

Figures 2.13, 2.21, and 2.25. By symmetry, we will assume that the edge C_1 is balanced and that the edge A_2 and B_3 are of largest weight in the cyles $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$, respectively. Here we let $A_2 = A_{21} \cup A_{22}$ and $B_3 = B_{31} \cup B_{32}$ when appropriate. Then we will refer to $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$ as the base triangles of H, A_2, A_3, B_2, and B_3 as diagonal edges, A_2 and B_3 as the long diagonal edges, and the path $A_2 \cup C_1 \cup B_3$ as the long path of H. Adding a *handle* to H is the operation of attaching a triad of G to H with the triad meeting the vertex set of H in both endvertices of the long path and in one interior vertex of the long path. There are two ways to add a handle to the subgraph H. An arbitrary number of handles may be added in the theorem that follows.

Theorem 2.3.1. *Let G be a simple 3-connected graph and $\omega : E(G) \to \mathbb{Z}^+$. Suppose that G contains an edge E such that $|\mathrm{spec}_\omega(\mathrm{E})(\mathrm{G})| = 2$. If G is isomorphic to a three-wheel, four-wheel, five-wheel, K_5^-, $K_{3,n}$, $K'_{3,n}$, $K''_{3,n}$, or $K'''_{3,n}$ for $n \geq 3$, Prism, Prism+, Prism1++, Prism2++, Prism2+++, Prism$^\perp$, Prism$^\perp$+a, Prism$^\perp$+b, Prism$^{\perp\perp}$, Prism$^{\perp\perp}$+, or the Prism$^{\perp\perp}$++ graph, then the edge E is balanced in G if and only if the Pair (G, ω) is as given in one of Theorems 2.1.3 through Theorem 2.2.18. The only remaining graphs that may be balanced are all obtained from extensions of the Prism, Prism$^\perp$, or Prism$^{\perp\perp}$ graph by attaching handles.*

So the previous theorem completely solves the problem of determining the balanced 3-connected graphs and their appropriate weight functions except that we do not explicitly determine the pairs (G, ω) in the latter three classes of graphs mentioned in the theorem statement.

3 THE PROOFS

We give the proofs of the results from Chapter 2 in this chapter of the dissertation. First we consider the graphs without two vertex-disjoint cycles. Then we consider graphs with disjoint cycles. Finally, we prove the main result of the dissertation.

3.1 Graphs without disjoint cycles

In this section we give the proofs of the results mentioned in Section 2.1 of the dissertation. Lemma 2.2.1 will be implicitly invoked throughout the dissertation so we prove it first.

Proof of Lemma 2.2.1. If C is a cycle of H of weight $\omega_H(C)$ that contains the edge E, then there exists a subdivision of C that is a cycle of G of weight $\omega_H(C)$ that contains the edge E. Hence the result follows from the observation $\mathrm{spec}_{\omega_{\mathrm{H}(\mathrm{E})}}(\mathrm{H}) \subseteq \mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G})$. $\square$

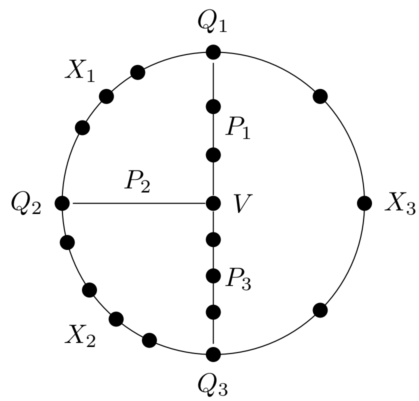


Figure 3.1: Cycles Containing e

Proof of Theorem 2.1.2. Suppose that $|\mathrm{spec}_{\omega(\mathrm{e})}(\mathrm{G})| = 1$. There exists a non-Hamiltonian cycle C that contains the edge e as G is 3-connected. Choose a vertex V that is not on C. By Menger's Theorem, there exist three internally disjoint paths from V to distinct vertices Q_1, Q_2, and Q_3 of C. Label these paths P_1, P_2, and P_3, respectively. Suppose that X_i is the arc of C from Q_i to Q_{i+1} for $i \in \{1, 2, 3\}$ modulo 3 (see Figure 3.1). Assume that e is an edge of the path X_3, without loss of generality. Then some of the cycles of G that contain the edge e are $X_1 \cup X_2 \cup X_3$, $P_1 \cup P_2 \cup X_2 \cup X_3$, $P_1 \cup P_3 \cup X_3$, and $P_2 \cup P_3 \cup X_1 \cup X_3$. Following standard usage throughout the dissertation, we denote the weight of a path or cycle by the lowercase letter of its uppercase label. Then $x_1 + x_2 + x_3 = p_1 + p_2 + x_2 + x_3 = p_1 + p_3 + x_3 = p_2 + p_3 + x_1 + x_3$. It follows from equating the first and second of these sums that $x_1 = p_1 + p_2$. It follows from equating the third and fourth of these sums that $p_1 = p_2 + x_1$. These two equations imply that $p_2 = 0$; a contradiction.Thus $|\mathrm{spec}_{\omega(\mathrm{e})}(\mathrm{G})| \geq 2$. $\square$

In the following proof we make a statement such as "By symmetry we may assume that $E = A_1$" to mean that the group of automorphisms of the given graph G is transitive on the edge set of G. We will make such statements in the subsequent proofs such as that we may assume that E is the edge A_1 or B_1 in a four-wheel graph. This statement means that every edge in a four-wheel graph can be mapped by an automorphism to either a spoke edge such as A_1 or a rim edge such as B_1.

Proof of Theorem 2.1.3. By symmetry we may assume that $E = A_1$. The cycles of G that contain the edge A_1 are $\{A_1, A_2, B_1\}$, $\{A_1, A_3, B_3\}$, $\{A_1, A_2, B_2, B_3\}$, and $\{A_1, A_3, B_1, B_2\}$. The weights of the cycles that contain the edge A_1 are listed in Table 3.1. Consider the edge isomorphism $(A_1)(A_2A_3)(B_2)(B_1B_3)$ of the graph G. The corresponding weight permutation

(†) $(a_1)(a_2a_3)(b_2)(b_1b_3)$

will feature prominently in this proof. Similar weight permutations will be used in subsequent proofs. If (†) is applied to the sums in (1), (2), (3), and (4) of Table 3.1, then the sums in (1) and (2) are interchanged, while the sums in (3) and (4) are also interchanged.

If case (i) of the theorem statement holds, then the cycles of G that contain A_1 have weight either $a_1 + a_2 + b_1$ or weight $a_1 + a_2 + b_1 + (2b_2)$. If case (ii) of the theorem statement holds, then the cycles of G that contain the edge A_1 have weight $a_1 + a_3 + b_3$ or $a_1 + a_3 + b_3 + (2b_2)$. If case (iii) of the theorem statement holds, then the cycles of G that contain the edge A_1 have weight $a_1 + a_2 + b_1$ or $a_1 + a_2 + b_1 + b_2$. Hence the edge A_1 is balanced if the weight function ω satisfies conditions (i), (ii), or (iii) of the theorem statement.

Conversely, suppose that the edge A_1 is balanced and that the weight function ω satisfies none of conditions (i), (ii), and (iii) of the theorem statement. The four sums in Table 3.1 have one of two different values.

(1) $a_1 + a_2 + b_1$	(2) $a_1 + a_3 + b_3$
(3) $a_1 + a_2 + b_2 + b_3$	(4) $a_1 + a_3 + b_1 + b_2$

Table 3.1: The cycles weights of the cycles containing the edge A_1 of the graph W_3

The following notation will be used for different tables of sum values throughout the remainder of the proof. There exist nonempty sets S and T that partition $\{1, 2, 3, 4\}$ such that all the sums in Table 3.1 corresponding to the elements of S have the same value, while all the sums in Table 3.1 corresponding to the elements of T have the other value. We will slightly corrupt notation and write, for example, $(S, T) = (123, 4)$ to mean that the sums in (1), (2), and (3) have the same value, while the sum in (4) has the other value. We summarize in each row of Table 3.2 the corresponding path length condition that occurs in the graph of Figure 1.7 when the corresponding sums in Table 3.1 have the same value.

Assume that $|S| \leq |T|$ without loss of generality. Then $|S| \in \{1, 2\}$ and $|T| = 4 - |S|$. Suppose $|S| = 1$. Suppose that $(S, T) = (1, 234)$ or $(S, T) = (3, 124)$. The former case implies that $b_3 = b_1 + b_2$ and $a_3 = a_2 + b_2$ and the latter case implies that $b_3 = b_1 + b_2$ and $a_2 = a_3 + b_2$. This a contradiction as then condition

$(1) = (2)$	$a_2 + b_1 = a_3 + b_3$
$(1) = (3)$	$b_1 = b_2 + b_3$
$(1) = (4)$	$a_2 = a_3 + b_2$
$(2) = (3)$	$a_3 = a_2 + b_2$
$(2) = (4)$	$b_3 = b_1 + b_2$
$(3) = (4)$	$a_2 + b_3 = a_3 + b_1$

Table 3.2: Consequences of Equal Sums in Table 3.1

(i) is satisfied. If $(S, T) = (2, 134)$ or $(S, T) = (4, 123)$, then condition (ii) of the theorem statement is satisfied by applying (†) to the cases $(S, T) = (1, 234)$ and $(S, T) = (3, 124)$, respectively. This is a contradiction.

Suppose that $|S| = 2$. Then $|T| = 2$. Assume that $1 \in S$ without loss of generality. Assume that $(S, T) = (12, 34)$. Then $a_2 = a_3$ and $b_1 = b_3$. This is a contradiction by (ii) in the theorem statement. If $(S, T) = (13, 24)$ or $(S, T) = (14, 23)$, then $b_2 = 0$; a contradiction. This completes the proof of Theorem 2.1.3. $\square$

(1) $a_1 + a_2 + b_1$	(4) $a_1 + a_3 + b_3 + b_4$
(2) $a_1 + a_4 + b_4$	(5) $a_1 + a_3 + b_1 + b_2$
(3) $a_1 + a_2 + b_2 + b_3 + b_4$	(6) $a_1 + a_4 + b_1 + b_2 + b_3$

Table 3.3: The weights of cycles containing the spoke edge A_1

Proof of Theorem 2.1.4. We may assume that E is either a spoke or a rim edge of G. Hence we may assume that $E = A_1$ or $E = B_1$. First suppose the former. The cycles of G that contain the edge A_1 are $\{A_1, A_2, B_1\}$, $\{A_1, A_4, B_4\}$, $\{A_1, A_2, B_2, B_3, B_4\}$, $\{A_1, A_3, B_3, B_4\}$, $\{A_1, A_3, B_1, B_2\}$, $\{A_1, A_4, B_1, B_2, B_3\}$. Hence the weights of the cycles of G that contain the edge A_1 are as given in Table 3.3. If the hypothesis condition (i) holds, then these weights are all either $a_1+a_2+b_1$ or $a_1+a_2+b_2+b_3+b_4$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G})| = 2$ and that (G, ω) is not as listed in the theorem statement. Then the sums in Table 3.3 are one of two different values. We summarize in each row of Table 3.4 the corresponding path length conditions that occur in the graph of Figure 1.7 when the corresponding sums in Table 3.3 have the same value. Some pairs of the equalities in Table 3.4 cannot occur simultaneously. From considering all pairs of possible equalities from Table 3.4 we obtain Table 3.5. For example, if (1)=(3) and (2)=(6), then one obtains the contradiction that $b_2 = b_3 = 0$. This information is summarized in the first row of Table 3.5. We use the same notation for nonempty sets S and T that partition $\{1, 2, 3, 4, 5, 6\}$ as before so that the sums in Table 3.3 corresponding to the elements of S have the same value, while all the sums in Table 3.3 corresponding to the elements of T have the other value.

Notice that $\min\{|S|, |T|\} \neq 1$ because in Table 3.5 each element of the set $\{1, 2, 3, 4, 5, 6\}$ is not mentioned in at least one row. For example, $(S, T) = (1, 23456)$ would contradict row (v) of the table. If $|S| = |T| = 3$, then, up to interchanging the sets S and T, we obtain ten possible combinations (S, T). We next list each of these ten pairs (S, T) followed in parentheses by the row number of Table 3.5 that implies that this pair cannot occur: $(123, 456)$ (ii), $(124, 356)$ (v), $(125, 346)$ (iii), $(126, 345)$ (vi), $(134, 256)$ (i), $(135, 246)$ (i), $(136, 245)$ (ii), $(145, 236)$ (vi), $(146, 235)$ (iv), $(156, 234)$ (iii). It follows that we may assume

$(1) = (2)$	$a_2 + b_1 = a_4 + b_4$
$(1) = (3)$	$b_1 = b_2 + b_3 + b_4$
$(1) = (4)$	$a_2 + b_1 = a_3 + b_3 + b_4$
$(1) = (5)$	$a_2 = a_3 + b_2$
$(1) = (6)$	$a_2 = a_4 + b_2 + b_3$
$(2) = (3)$	$a_4 = a_2 + b_2 + b_3$
$(2) = (4)$	$a_4 = a_3 + b_3$
$(2) = (5)$	$a_4 + b_4 = a_3 + b_1 + b_2$
$(2) = (6)$	$b_4 = b_1 + b_2 + b_3$
$(3) = (4)$	$a_2 + b_2 = a_3$
$(3) = (5)$	$a_2 + b_3 + b_4 = a_3 + b_1$
$(3) = (6)$	$a_2 + b_4 = a_4 + b_1$
$(4) = (5)$	$b_3 + b_4 = b_1 + b_2$
$(4) = (6)$	$a_3 + b_4 = a_4 + b_1 + b_2$
$(5) = (6)$	$a_3 = a_4 + b_3$

Table 3.4: Consequences of Equal Sums in Table 3.3

$|S| = 2$ and $|T| = 4$. There are fifteen pairs (S, T) to consider. Nine of these pairs cannot occur by Table 3.5. We next list each of these nine pairs (S, T) followed in parentheses by the row number of Table 3.5 that implies that this pair cannot occur: $(13, 2456)$ (i), $(15, 2346)$ (iii), $(16, 2345)$ (iv), $(23, 1456)$ (iv), $(24, 1356)$ (v), $(26, 1345)$ (i), $(34, 1256)$ (iii), $(45, 1236)$ (ii), $(56, 1234)$ (v). Thus we have six cases

(i)	$(1) = (3)$ and $(2) = (6)$
(ii)	$(1) = (3)$ and $(4) = (5)$
(iii)	$(1) = (5)$ and $(3) = (4)$
(iv)	$(1) = (6)$ and $(2) = (3)$
(v)	$(2) = (4)$ and $(5) = (6)$
(vi)	$(2) = (6)$ and $(4) = (5)$

Table 3.5: Pairs of equalities from Table 3.4 that do not simultaneously hold

remaining to consider for (S,T), namely $(12, 3456)$, $(14, 2356)$, $(25, 1346)$, $(35, 1246)$, $(36, 1245)$, and $(46, 1235)$.

Suppose $(S,T) = (14, 2356)$. We consider the information from Table 3.3 in the following sentences. It follows from (5)=(6) that $a_3 = a_4 + b_3$ and (2)=(6) $b_4 = b_1+b_2+b_3$. Substitute a_4+b_3 for a_3 and $b_1+b_2+b_3$ for b_4 into the two equations (1)=(4) $a_2 + b_1 = a_3 + b_3 + b_4$ and (3)=(5) $a_2 + b_3 + b_4 = a_3 + b_1$ to obtain that $a_2 = a_4 + b_2 + 3b_3$ and $a_4 = a_2 + b_2 + b_3$. Thus $b_2 = b_3 = 0$; a contradiction.

Suppose $(S,T) = (35, 1246)$. It follows from (3)=(5) that $a_2+b_3+b_4 = a_3+b_1$, (1)=(6) $a_2 = a_4 + b_2 + b_3$ and (2)=(6) $b_4 = b_1 + b_2 + b_3$. Substitute $a_4 + b_2 + b_3$ for a_2 and $b_1 + b_2 + b_3$ for b_4 into the equations (3)=(5) to obtain the equation $(a_4+b_2+b_3)+b_3+(b_1+b_2+b_3) = a_3+b_1$. Now use (2)=(4), where $a_4 = a_3+b_3$; we obtain the equation $[(a_3+b_3)+b_2+b_3]+b_3+b_1+b_2+b_3 = a_3+b_1$. Thus $b_2 = b_3 = 0$; a contradiction.

Assume that $(S,T) = (12, 3456)$. Then one obtains the following seven equations from Table 3.4: $a_2 + b_1 = a_4 + b_4$, $a_2 + b_2 = a_3$, $a_2 + b_3 + b_4 = a_3 + b_1$, $a_2 + b_4 = a_4 + b_1$, $b_1 + b_2 = b_3 + b_4$, $a_3 + b_4 = a_4 + b_1 + b_2$, and $a_3 = a_4 + b_3$. From combining the first and fourth of these equations we obtain that $b_1 = b_4$. One can show from these equations that $a_2 = a_4$, $b_1 = b_4$, $b_2 = b_3$, and $a_3 = a_2 + b_2$. Hence (G, ω) is as given in (i) of the theorem statement; a contradiction.

Assume that $(S,T) = (36, 1245)$. Then one obtains the following seven equations from Table 3.4: $a_2 + b_4 = a_4 + b_1$, $a_2 + b_1 = a_4 + b_4$, $a_2 + b_1 = a_3 + b_3 + b_4$,

$a_2 = a_3 + b_2$, $a_4 = a_3 + b_3$, $a_4 + b_4 = a_3 + b_1 + b_2$, and $b_1 + b_2 = b_3 + b_4$. It follows from the first two equations that $b_1 = b_4$. Then one can show that $a_2 = a_4$, $b_1 = b_4$, $b_2 = b_3$, and $a_2 = a_3 + b_2$. Hence (G, ω) is as given in (i) of the theorem statement; a contradiction.

Under the isomorphism $(A_1)(A_2, A_4)(A_3)(B_1, B_4)(B_2B_3)$ of the edge set of G we obtain the following permutation of the labels in Table 3.3, (12)(45)(36). Hence the remaining cases $(S, T) = (25, 1346)$ and $(46, 1235)$ become the cases $(14, 2356)$ and $(46, 1235)$, respectively, that we have shown cannot occur.

Now suppose that $E = B_1$. The cycles of G that contain the edge B_1 are $\{B_1, B_2, B_3, B_4\}$, $\{A_1, A_2, B_1\}$, $\{A_1, A_3, B_1, B_2\}$, $\{A_1, A_4, B_1, B_2, B_3\}$, $\{A_2, A_3, B_1, B_3, B_4\}$, $\{A_2, A_4, B_1, B_4\}$, and $\{A_3, A_4, B_1, B_2, B_4\}$. The weights of the cycles of G that contain the edge B_1 are labeled in Table 3.6.

Consider the edge isomorphism $(A_1, A_2)(A_3, A_4)(B_1)(B_2, B_4)(B_3)$ of G. Under the permutation (†) $(a_1a_2)(a_3a_4)(b_1)(b_2b_4)(b_3)$ of the corresponding weights in G, we obtain the following permutation (‡) (1)(2)(36)(45)(7) of the labels in Table 3.6.

If Condition (C1) of Table 2.1 holds, then $\mathrm{spec}_{\omega(\mathrm{B}_1)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1,\ \mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1 + \mathrm{b}_3\}$, If Condition (C2) of Table 2.1 holds, then $\mathrm{spec}_{\omega(\mathrm{B}_1)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1,\ \mathrm{a}_1 + \mathrm{a}_2 + \mathrm{a}_3 + \mathrm{b}_1\}$.

Conditions (C3) is obtained from condition $(C2)$ by applying the permutation (†). Hence if Condition (C3) of Table 2.1 holds, then $|\mathrm{spec}_{\omega(\mathrm{B}_1)}(\mathrm{G})| = 2$. This permutation (†) and its associated label permutation (‡) will be invoked again in this proof.

Conversely, suppose that the edge B_1 is balanced but that that (G, ω) is not as given in one of the three cases of Table 2.1.

(1) $b_1 + b_2 + b_3 + b_4$	(5) $a_2 + a_3 + b_1 + b_3 + b_4$
(2) $a_1 + a_2 + b_1$	(6) $a_2 + a_4 + b_1 + b_4$
(3) $a_1 + a_3 + b_1 + b_2$	(7) $a_3 + a_4 + b_1 + b_2 + b_4$
(4) $a_1 + a_4 + b_1 + b_2 + b_3$	

Table 3.6: B_1 is a rim edge of W_4

We summarize in each row of Table 3.7 the corresponding path length condition that occurs in the graph of Figure 1.7 when the corresponding sums in Table 3.6 have the same value.

Some pairs of the equalities in Table 3.7 cannot occur simultaneously. From considering all pairs of possible equalities from Table 3.7 we obtain Table 3.8. For example, if (1)=(3) and (2)=(5), then one obtains the contradiction that $a_3 = 0$. This information is summarized in the first row of Table 3.8. We use the same notation for nonempty sets S and T that partition $\{1, 2, 3, 4, 5, 6, 7\}$ as before so that the sums in Table 3.6 corresponding to the elements of S have the same value, while all the sums in Table 3.6 corresponding to the elements of T have the other value.

$(1) = (2)$	$b_2 + b_3 + b_4 = a_1 + a_2$
$(1) = (3)$	$b_3 + b_4 = a_1 + a_3$
$(1) = (4)$	$b_4 = a_1 + a_4$
$(1) = (5)$	$b_2 = a_2 + a_3$
$(1) = (6)$	$b_2 + b_3 = a_2 + a_4$
$(1) = (7)$	$b_3 = a_3 + a_4$
$(2) = (3)$	$a_2 = a_3 + b_2$
$(2) = (4)$	$a_2 = a_4 + b_2 + b_3$
$(2) = (5)$	$a_1 = a_3 + b_3 + b_4$
$(2) = (6)$	$a_1 = a_4 + b_4$
$(2) = (7)$	$a_1 + a_2 = a_3 + a_4 + b_2 + b_4$
$(3) = (4)$	$a_3 = a_4 + b_3$
$(3) = (5)$	$a_1 + b_2 = a_2 + b_3 + b_4$
$(3) = (6)$	$a_1 + a_3 + b_2 = a_2 + a_4 + b_4$
$(3) = (7)$	$a_1 = a_4 + b_4$
$(4) = (5)$	$a_1 + a_4 + b_2 = a_2 + a_3 + b_4$
$(4) = (6)$	$a_1 + b_2 + b_3 = a_2 + b_4$
$(4) = (7)$	$a_1 + b_3 = a_3 + b_4$
$(5) = (6)$	$a_3 + b_3 = a_4$
$(5) = (7)$	$a_2 + b_3 = a_4 + b_2$
$(6) = (7)$	$a_2 = a_3 + b_2$

Table 3.7: Consequences of Equal Sums in Table 3.6

(i)	(1) = (3) and (2) = (5)
(ii)	(1) = (4) and (2) = (5)
(iii)	(1) = (4) and (2) = (6)
(iv)	(1) = (4) and (3) = (7)
(v)	(1) = (5) and (2) = (3)
(vi)	(1) = (5) and (2) = (4)
(vii)	(1) = (5) and (6) = (7)
$(viii)$	(1) = (6) and (2) = (4)
(ix)	(1) = (7) and (3) = (4)
(x)	(1) = (7) and (5) = (6)
(xi)	(2) = (4) and (5) = (7)
(xii)	(2) = (5) and (4) = (7)
$(xiii)$	(3) = (4) and (5) = (6)
(xiv)	(3) = (5) and (4) = (6)

Table 3.8: Pairs of equalities from Table 3.7 that do not simultaneously hold

Notice that $\min\{|S|, |T|\} \neq 1$ as in Table 3.8 each element of the set $\{1, 2, 3, 4, 5, 6, 7\}$ is not mentioned in at least one row. For example, $(S, T) = (1, 234567)$ would contradict row (xi) of the Table 3.8.

Assume $|S| = 2$ and $|T| = 5$. There are twenty-one pairs (S, T) to consider. Only one of these pairs can occur by Table 3.8. We next list each of the twenty pairs (S, T) that cannot occur followed in parentheses by the row number of Table 3.8 that establishes that fact: $(12, 34567)$ $(xiii)$, $(13, 24567)$ (i), $(14, 23567)$ (ii), $(15, 23467)$ (v), $(16, 23457)$ $(viii)$, $(17, 23456)$ (ix), $(23, 14567)$ (v), $(24, 13567)$ (vi), $(25, 13467)$ (i), $(26, 13457)$ (iii), $(27, 13456)$ $(xiii)$, $(34, 12567)$ (vii), $(35, 12467)$ (iii), $(36, 12457)$ (ii), $(37, 12456)$ (ii), $(46, 12357)$ (i), $(47, 12356)$ (i), $(56, 12347)$ (iv), $(57, 12346)$ (iii), $(67, 12357)$ (i). Hence $(S, T) = (45, 12367)$. Notice that (4)=(5), (2)=(3), and (2)=(6) in Table 3.7 yield the equations $a_1 + a_4 + b_2 = a_2 + a_3 + b_4$, $a_2 = a_3 + b_2$, and $a_1 = a_4 + b_4$, respectively. Substituting $a_4 + b_4$ for a_1 and $a_3 + b_2$ for a_2 into $a_1 + a_4 + b_2 = a_2 + a_3 + b_4$, we obtain the equation $a_3 = a_4$. Since (1)=(7) in Table 3.7 yield the equations $b_3 = a_3 + a_4$ and $a_3 = a_4$, we obtain the equation $b_3 = 2a_3$. It follows from (2)=(3) in that table that $a_2 = a_3 + b_2$. It follows from (2)=(6) that $a_1 = a_4 + b_4 = a_3 + b_4$. In summary, $a_3 = a_4$, $b_3 = 2a_3$, $a_1 = a_3 + b_4$, and $a_2 = a_3 + b_2$. Thus (G, ω) is as given in (C1) of Table 2.1; a contradiction.

Assume $|S| = 3$ and $|T| = 4$. There are thirty-five pairs (S, T) to consider. Thirty-two of these pairs cannot occur by Table 3.8. We next list each of these thirty-two pairs (S, T) followed in parentheses by the row number of Table 3.8 that implies that this pair cannot occur: $(124, 3567)$ (iv), $(125, 3467)$ (vii)

$(127, 3456)$ (ix),$(134, 2567)$ (i), $(135, 2467)$ (vi), $(136, 2457)$ (i), $(137, 2456)$ (i), $(145, 2367)$ (iii), $(146, 2357)$ (ii), $(147, 2356)$ (ii), $(156, 2347)$ (v), $(157, 2346)$ (v), $(167, 2345)$ $(viii)$, $(234, 1567)$ (v), $(235, 1467)$ (ii), $(S, T) = (236, 1457)$ (v), $(237, 1456)$ (iv), $(245, 1367)$ (i), $(246, 1357)$ (vi), $(247, 1356)$ (vi), $(256, 1347)$ (i), $(257, 1346)$ (i), $(267, 1345)$ (iii), $(345, 1267)$ (ix), $(346, 1257)$ (ix), $(347, 1256)$ (xii), $(356, 1247)$ (x), $(357, 1246)$ (iii), $(367, 1245)$ (ii), $(456, 1237)$ (x), $(467, 1235)$ (i), and $(567, 1234)$ (xi).

We have three cases remaining to consider for (S, T), namely $(123, 4567)$, $(126, 3457)$, $(457, 1236)$. Assume that $(S, T) = (123, 4567)$. In Table 3.7, it follows from (1)=(3), (2)=(3), (4)=(7), and (5)=(6), respectively, that $b_3 + b_4 = a_1 + a_3$, $a_2 = a_3 + b_2$, $a_1 + b_3 = a_3 + b_4$, and $a_3 + b_3 = a_4$. One can show using these equations that $a_1 = b_4$, $a_3 = b_3$, $a_4 = 2a_3$, and $a_2 = a_3 + b_2$. Thus the pair (G, ω) is as given in condition (C3); a contradiction.

Assume that $(S, T) = (126, 3457)$. Under the permutation (‡), this condition becomes $(S, T) = (123, 4567)$ and under (†) condition (C3) becomes condition (C2); a contradiction.

Suppose $(S, T) = (457, 1236)$. Then, in Table 3.7, (4)=(7) and (1)=(3) imply that $a_1 + b_3 = a_3 + b_4$ and $b_3 + b_4 = a_1 + a_3$. Thus $a_1 = b_4$. However, (2)=(6) implies that $a_1 = a_4 + b_4$. Hence $a_4 = 0$; a contradiction. This completes the proof of Theorem 2.1.4. $\square$

Proof of Theorem 2.1.5. We may assume that $E \in \{A_1, B_1\}$. Suppose that $n = 5$. The cycles of G that contain the edge B_1 are $\{B_1, B_2, B_3, B_4, B_5\}$, $\{A_1, A_2, B_1\}$, $\{A_1, A_3, B_1, B_2\}$, $\{A_1, A_4, B_1, B_2, B_3\}$, $\{A_1, A_5, B_1, B_2, B_3, B_4\}$, $\{A_2, A_3, B_1, B_3, B_4, B_5\}$, $\{A_2, A_4, B_1, B_4, B_5\}$, $\{A_2, A_5, B_1, B_5\}$, $\{A_3, A_4, B_1, B_2, B_4, B_5\}$, $\{A_3, A_5, B_1, B_2, B_5\}$, and $\{A_4, A_5, B_1, B_2, B_3, B_5\}$. If the given conditions hold, then these cycles all have weight $b_1 + b_2 + b_3 + b_4 + b_5$ or $b_1 + b_2 + b_3 + b_4 + b_5 + 2b_3$.

Conversely, suppose that the edge $E \in \{A_1, B_1\}$ of G has $|\mathrm{spec}_\omega(\mathrm{G})(\mathrm{E})| = 2$ and it is not true that $n = 5$, $E = B_1$, and ω is as given in the theorem statement. Let H, K, and J be the graphs given in Figure 3.2 with weight functions ω_H, ω_K, and ω_J defined as in Table 3.9. It follows from the edge E being ω-balanced in G that the edge is also ω_H-balanced in H, is ω_K-balanced in K, and is ω_J-balanced in J.

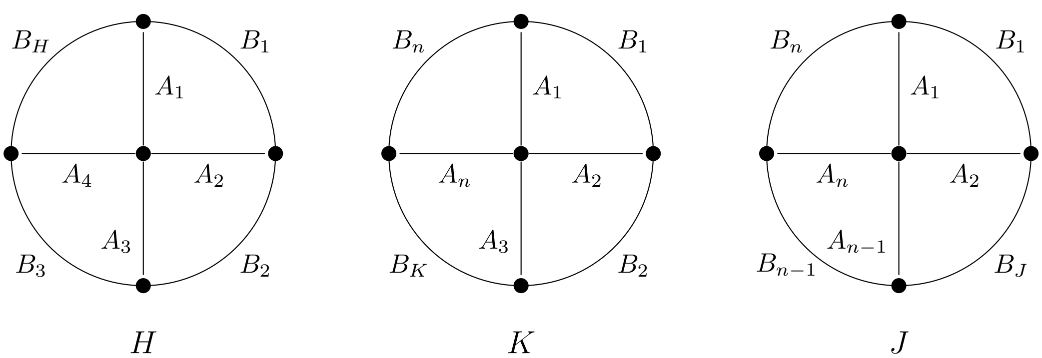


Figure 3.2: Three subgraphs of $G \cong W_5$

First assume that $E = A_1$. It follows from Theorem 2.1.4 that $\omega_H(B_1) = \omega_H(B_H)$ and that $\omega_J(B_1) = \omega_J(B_n)$. From Table 3.9 we obtain that $b_1 = \omega_H(B_1) =$

$\omega_H(A_1) = a_1$	$\omega_K(A_1) = a_1$	$\omega_J(A_1) = a_1$
$\omega_H(A_2) = a_2$	$\omega_K(A_2) = a_2$	$\omega_J(A_2) = a_2$
$\omega_H(A_3) = a_3$	$\omega_K(A_3) = a_3$	$\omega_J(A_{n-1}) = a_{n-1}$
$\omega_H(A_4) = a_4$	$\omega_K(A_n) = a_n$	$\omega_J(A_n) = a_n$
$\omega_H(B_1) = b_1$	$\omega_K(B_1) = b_1$	$\omega_J(B_1) = b_1$
$\omega_H(B_2) = b_2$	$\omega_K(B_2) = b_2$	$\omega_J(B_J) = b_2 + b_3 + \cdots + b_{n-2}$
$\omega_H(B_3) = b_3$	$\omega_K(B_K) = b_3 + b_4 + \cdots + b_{n-1}$	$\omega_J(B_{n-1}) = b_{n-1}$
$\omega_H(B_H) = b_4 + b_5 + \cdots + b_n$	$\omega_K(B_n) = b_n$	$\omega_J(B_n) = b_n$

Table 3.9: The pairs (H, ω_H), (K, ω_K), and (J, ω_J)

$\omega_H(B_H) = b_4 + b_5 + \cdots + b_n$ and $b_1 = \omega_J(B_1) = \omega_J(B_n) = b_n$. Hence $b_4 = 0$; a contradiction. Thus $E = B_1$. Then we have shown that (G, ω) is as given in one of the conditions (C1), (C2), and (C3). We obtain three analogous conditions, as given in Table 3.10, for each of the pairs (H, ω_H) and (K, ω_K).

We now substitute the values of Table 3.9 into Table 3.10 to obtain Table 3.11. Now exactly one of conditions (H1), (H2), and (H3) holds from that table. Also, exactly one of conditions (K1), (K2), or (K3) holds, while exactly one of conditions (J1), (J2), or (J3) holds. We first show that conditions (H1) and (H2) cannot occur.

$(H1)$ $\omega_H(A_1) = \omega_H(A_3) + \omega_H(B_H)$ $\omega_H(A_2) = \omega_H(A_3) + \omega_H(B_2)$ $\omega_H(A_3) = \omega_H(A_4)$ $\omega_H(B_3) = 2\omega_H(A_3)$	$(H2)$ $\omega_H(A_1) = \omega_H(A_4) + \omega_H(B_H)$ $\omega_H(A_2) = \omega_H(B_2)$ $\omega_H(A_4) = \omega_H(B_3)$ $\omega_H(A_3) = 2\omega_H(A_4)$
$(H3)$ $\omega_H(A_2) = \omega_H(A_3) + \omega_H(B_2)$ $\omega_H(A_1) = \omega_H(B_H)$ $\omega_H(A_3) = \omega_H(B_3)$ $\omega_H(A_4) = 2\omega_H(A_3)$	
$(K1)$ $\omega_K(A_1) = \omega_K(A_3) + \omega_K(B_n)$ $\omega_K(A_2) = \omega_K(A_3) + \omega_K(B_2)$ $\omega_K(A_3) = \omega_K(A_n)$ $\omega_K(B_K) = 2\omega_K(A_3)$	$(K2)$ $\omega_K(A_1) = \omega_K(A_n) + \omega_K(B_n)$ $\omega_K(A_2) = \omega_K(B_2)$ $\omega_K(A_n) = \omega_K(B_K)$ $\omega_K(A_3) = 2\omega_K(A_n)$
$(K3)$ $\omega_K(A_2) = \omega_K(A_3) + \omega_K(B_2)$ $\omega_K(A_1) = \omega_K(B_n)$ $\omega_K(A_3) = \omega_K(B_K)$ $\omega_K(A_n) = 2\omega_K(A_3)$	
$(J1)$ $\omega_J(A_1) = \omega_J(A_{n-1}) + \omega_J(B_n)$ $\omega_J(A_2) = \omega_J(A_{n-1}) + \omega_J(B_J)$ $\omega_J(A_{n-1}) = \omega_J(A_n)$ $\omega_J(B_{n-1}) = 2\omega_J(A_{n-1})$	$(J2)$ $\omega_J(A_1) = \omega_J(A_n) + \omega_J(B_n)$ $\omega_J(A_2) = \omega_J(B_J)$ $\omega_J(A_n) = \omega_J(B_{n-1})$ $\omega_J(A_{n-1}) = 2\omega_J(A_n)$
$(J3)$ $\omega_J(A_2) = \omega_J(A_{n-1}) + \omega_J(B_J)$ $\omega_J(A_1) = \omega_J(B_n)$ $\omega_J(A_{n-1}) = \omega_J(B_{n-1})$ $\omega_J(A_n) = 2\omega_J(A_{n-1})$	

Table 3.10: Conditions implied the weighted graphs H, K, and J

$(H1)$ $a_1 = a_3 + b_4 + b_5 + \cdots + b_n$ $a_2 = a_3 + b_2$ $a_3 = a_4$ $b_3 = 2a_3$	$(H2)$ $a_1 = a_4 + b_4 + b_5 + \cdots + b_n$ $a_2 = b_2$ $a_4 = b_3$ $a_3 = 2a_4$
$(H3)$ $a_2 = a_3 + b_2$ $a_1 = b_4 + b_5 + \cdots + b_n$ $a_3 = b_3$ $a_4 = 2a_3$	
$(K1)$ $a_1 = a_3 + b_n$ $a_2 = a_3 + b_2$ $a_3 = a_n$ $b_3 + b_4 + \cdots + b_{n-1} = 2a_3$	$(K2)$ $a_1 = a_n + b_n$ $a_2 = b_2$ $a_n = b_3 + b_4 + \cdots + b_{n-1}$ $a_3 = 2a_n$
$(K3)$ $a_2 = a_3 + b_2$ $a_1 = b_n$ $a_3 = b_3 + b_4 + \cdots + b_{n-1}$ $a_n = 2a_3$	
$(J1)$ $a_1 = a_{n-1} + b_n$ $a_2 = a_{n-1} + b_2 + b_3 + \cdots + b_{n-2}$ $a_{n-1} = a_n$ $b_{n-1} = a_{n-1}$	$(J2)$ $a_1 = a_n + b_n$ $a_2 = b_2 + b_3 + \cdots + b_{n-2}$ $a_n = b_{n-1}$ $a_{n-1} = 2a_n$
$(J3)$ $a_2 = a_{n-1} + b_2 + b_3 + \cdots + b_{n-2}$ $a_1 = b_n$ $a_{n-1} = b_{n-1}$ $a_n = 2a_{n-1}$	

Table 3.11: Exactly one pair of conditions (H,K) holds

Suppose that condition (H1) holds. If conditions (H1) and (K1) hold, then $a_3 + b_4 + b_5 + \cdots + b_n = a_1 = a_3 + b_n$ so that $b_4 = 0$; a contradiction. If conditions (H1) and (K2) hold, then $a_3 + b_2 = a_2 = b_2$ so that $a_3 = 0$; a contradiction. If conditions (H1) and (K3) hold, then $a_3 + b_4 + b_5 + \cdots + b_n = a_1 = b_n$ so that $a_3 = 0$; a contradiction. Thus condition (H1) doesn't hold.

Suppose that condition (H2) holds. If conditions (H2) and (K1) hold, then $b_2 = a_2 = a_3 + b_2$ so that $a_3 = 0$; a contradiction. If (H2) and (K2) hold, then $2a_4 = a_3 = 2a_n$ so that $a_4 = a_n$. Then $a_4 + b_n = a_n + b_n = a_1 = a_4 + b_4 + b_5 + \cdots + b_n$ so that $b_4 = 0$; a contradiction. If (H2) and (K3) hold, then $b_2 = a_2 = a_3 + b_2$ so that $a_3 = 0$; a contradiction. Thus condition (H2) does not hold.

We have shown that condition (H3) must occur. If (H3) and (J1) hold, then $b_3 + b_2 = a_3 + b_2 = a_2 = a_{n-1} + b_2 + b_3 + \cdots + b_{n-2}$ so that $a_{n-1} = 0$; a contradiction.

If (H3) and (J2) hold, then $b_{n-1} + b_n = a_n + b_n = a_1 = b_4 + b_5 + \cdots + b_n$. Thus $n = 5$. Hence $a_3 = b_3$ and $a_2 = b_2 + b_3$. Then $2a_3 = a_4 = 2a_5 = 2b_4$. Hence $a_3 = b_4$. Thus $b_3 = a_3 = b_4$. Also $a_4 = 2a_3 = 2b_3$ and $a_5 = b_4 = b_3$. Finally, $a_1 = b_4 + b_5 = b_3 + b_5$. Hence (G, ω) is as given in the theorem statement; a contradiction.

If (H3) and (J3) hold, then $b_n = a_1 = b_4 + b_5 + \cdots + b_n$ so that $b_4 = 0$; a contradiction. This completes the proof of Theorem 2.1.5. $\square$

The edge labeling of G in Theorem 2.1.6 was given in Figure 2.7.

Proof of Theorem 2.1.6. We may assume by symmetry that $E \in \{A_1, C_1\}$. First assume that $E = A_1$. The cycles of G that contain the edge A_1 are $\{A_1, A_2, B_1\}$, $\{A_1, A_4, B_4\}$, $\{A_1, A_2, B_2, B_3, B_4\}$, $\{A_1, A_3, B_3, B_4\}$, $\{A_1, A_3, B_1, B_2\}$, $\{A_1, A_4, B_1, B_2, B_3\}$, $\{C, A_1, A_2, B_4\}$, $\{C, A_1, A_4, B_1\}$, $\{C, A_1, A_3, B_3, B_4\}$, and $\{C, A_1, A_3, B_1, B_3\}$. If the conditions of the theorem statement are satisfied, then $\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1, \mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1 + 2\mathrm{b}_2\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement. It follows from applying Theorem 2.1.4 to the subgraph of G obtained by deleting the edge C that $a_2 = a_4$, $b_1 = b_4$, $b_2 = b_3$, and $a_3 = a_2 \pm b_2$. First suppose that $a_3 = a_2 + b_2$. Then the cycles of G that contain A_1 have length $a_1 + a_2 + b_1$ or $a_1 + a_2 + b_1 + 2b_2$. Hence the cycles of G that contain both of the edges A_1 and C have one of these two weights. The given conditions imply that these four cycles have weights $a_1 + a_2 + b_1 + c$ or $a_1 + a_3 + b_1 + b_2 + c$. Thus $\{a_1 + a_2 + b_1 + c, a_1 + a_3 + b_1 + b_2 + c\} = \{a_1 + a_2 + b_1, a_1 + a_2 + b_1 + 2b_2\}$. Then $c > 0$ implies that $a_1 + c + b_1 + a_2 = a_1 + a_2 + b_1 + 2b_2$ so that $c = 2b_2$.

Recall that $a_3 = a_2 \pm b_2$. Assume that $a_3 = a_2 + b_2$. Then $a_1 + a_3 + b_1 + b_2 + c = a_1 + a_2 + b_1 + 2b_2 + c$ has too large a weight. Hence $a_3 = a_2 - b_2$ and G is as given in the theorem statement; a contradiction.

Now suppose that $E = C$. The cycles of G that contain the edge C are $\{A_2, A_4, C\}$, $\{B_1, B_4, C\}$, $\{B_2, B_3, C\}$, $\{A_1, A_2, B_4, C\}$, $\{A_1, A_4, B_1, C\}$,

$\{A_2, A_3, B_3, C\}$, $\{A_3, A_4, B_2, C\}$, $\{A_1, A_3, B_1, B_3, C\}$, $\{A_1, A_3, B_2, B_4, C\}$. If the conditions of the theorem statement are satisfied, then $\mathrm{spec}_{\omega(\mathrm{C})}(\mathrm{G}) = \{\mathrm{b}_1 + \mathrm{b}_3 + \mathrm{c}, 2\mathrm{a}_1 + \mathrm{b}_1 + \mathrm{b}_3 + \mathrm{c}\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{C})}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement. Consider the two subgraphs G_1 and G_2 of G pictured in Figure 3.3. It follows from applying Theorem 2.1.4 to G_1 that $b_1 = b_2$, $b_3 = b_4$, $a_1 = a_3$, $a_2 = b_1 \pm a_1$. It follows from applying Theorem 2.1.4 to G_2 that $a_4 = b_3 \pm a_3$. Since $a_1 = a_3$, $a_4 = b_3 \pm a_1$. It follows from (G, ω) not being as given in the theorem statement that $a_2 = b_1 - a_1$ and $a_4 = b_3 - a_1$. Now the conditions obtained imply that $\mathrm{spec}_{\omega(\mathrm{C})}(\mathrm{G}) = \{\mathrm{b}_1 + \mathrm{b}_3 + \mathrm{c}, 2\mathrm{a}_1 + \mathrm{b}_1 + \mathrm{b}_3 + \mathrm{c}\}$. But then the cycle $\{A_2, A_4, C\}$ has weight $a_2 + a_4 + c = b_1 + b_3 + c - 2a_1$ which is of too small weight. This contradiction completes the proof of Theorem 2.1.6. $\square$

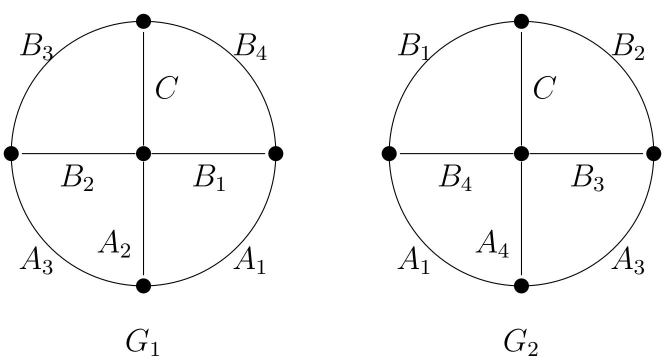


Figure 3.3: Two subgraphs of $G \cong K_5^-$ that use the edge e

Proof of Theorem 2.1.7. Suppose that $G \cong K_5$ with the edge labeling for G as given in Figure 3.4. Suppose that $|\mathrm{spec}_{\omega(\mathrm{D})}(\mathrm{G})| = 2$ without loss of generality. It follows

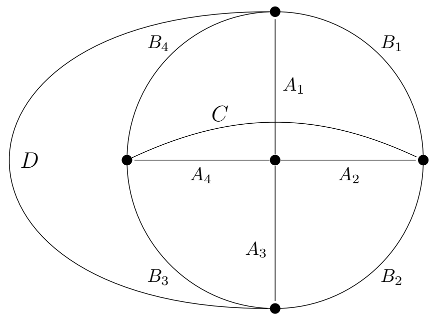


Figure 3.4: The graph K_5

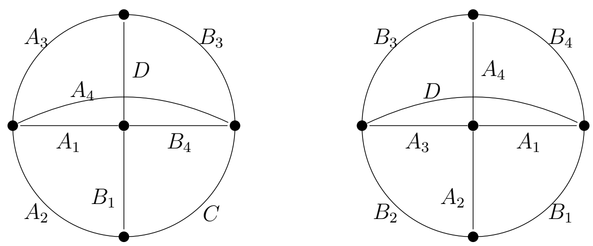


Figure 3.5: Two subgraphs of $G \cong K_5$

from applying Theorem 2.1.6 (i) to the left subgraph in Figure 3.5 that $b_1 = b_4 - c$. It follows from applying Theorem 2.1.6 (ii) to the right subgraph in that figure that $b_1 = b_4$. Thus $c = 0$; a contradiction. Hence the edge D is not balanced. By symmetry no edge of G is balanced. This completes the proof of Theorem 2.1.7. $\square$

Proof of Theorem 2.1.8. Suppose that $G \cong K_{3,n}$ with the edge labeling for G as given in Figure 2.8 with the edges U_1, U_2, and U_3 deleted. We may assume that $E = A_1$. cycles of G that contain the edge A_1 are, for $i \in [n]\backslash 1$, $\{A_1, B_1, A_i, B_i\}$ and $\{A_1, C_1, A_i, C_i\}$, and in addition for $j \notin \{1, i\}$, $\{A_1, B_1, B_i, C_i, C_j, A_j\}$ and

$\{A_1, C_1, B_i, B_j, C_i, A_j\}$. If the conditions in the theorem statement hold, then $\mathrm{Spec}_{\omega(\mathrm{A}_1)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{b}_1 + \mathrm{a}_2 + \mathrm{b}_2, \mathrm{a}_1 + \mathrm{b}_1 + \mathrm{a}_2 + 3\mathrm{b}_2\}$.

Conversely, suppose that $|\mathrm{Spec}_{\omega(\mathrm{A}_1)}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement. First suppose that $n = 3$. Then the eight sums in the Table 3.12 have one of two different values.

(1) $a_1 + a_2 + b_1 + b_2$	(5) $a_1 + a_2 + b_1 + b_3 + c_2 + c_3$
(2) $a_1 + a_2 + c_1 + c_2$	(6) $a_1 + a_2 + b_2 + b_3 + c_1 + c_3$
(3) $a_1 + a_3 + b_1 + b_3$	(7) $a_1 + a_3 + b_2 + b_3 + c_1 + c_2$
(4) $a_1 + a_3 + c_1 + c_3$	(8) $a_1 + a_3 + b_1 + b_2 + c_2 + c_3$

Table 3.12: Weights of cycles containing the edge A_1 in the graph $K_{3,3}$

There are three edge isomorphisms of $G \cong K_{3,3}$ that fix the edge A_1. These isomorphisms are $(A_1)(A_2)(A_3)(B_1C_1)(B_2C_2)(B_3C_3)$, $(A_1)(A_2A_3)(B_1)(B_2B_3)(C_1)(C_2C_3)$, and $(A_1)(A_2A_3)(B_1C_1)(B_2C_3)(B_3C_2)$. Thus in any equations involving the weights a_i, b_i, and c_i we may apply the permutations $(a_1)(a_2)(a_3)(b_1c_1)(b_2c_2)(b_3c_3)$, $(a_1)(a_2a_3)(b_1)(b_2b_3)(c_1)(c_2c_3)$, and $(a_1)(a_2a_3)(b_1c_1)(b_2c_3)(c_3c_2)$ to both sides of the equations. These three permutations of the edge weights of G yield the following three permutations of the equation labels of the equations in Table 3.12:

(A) (12)(34)(56)(78),

(B) (13)(24)(58)(67), and

(C) (14)(23)(57)(68).

$(1) = (2)$	$b_1 + b_2 = c_1 + c_2$
$(1) = (3)$	$a_2 + b_2 = a_3 + b_3$
$(1) = (4)$	$a_2 + b_1 + b_2 = a_3 + c_1 + c_3$
$(1) = (5)$	$b_2 = b_3 + c_2 + c_3$
$(1) = (6)$	$b_1 = b_3 + c_1 + c_3$
$(1) = (7)$	$a_2 + b_1 = a_3 + b_3 + c_1 + c_2$
$(1) = (8)$	$a_2 = a_3 + c_2 + c_3$
$(2) = (3)$	$a_2 + c_1 + c_2 = a_3 + b_1 + b_3$
$(2) = (4)$	$a_2 + c_2 = a_3 + c_3$
$(2) = (5)$	$c_1 = b_1 + b_3 + c_3$
$(2) = (6)$	$c_2 = b_2 + b_3 + c_3$
$(2) = (7)$	$a_2 = a_3 + b_2 + b_3$
$(2) = (8)$	$a_2 + c_1 = a_3 + b_1 + b_2 + c_3$
$(3) = (4)$	$b_1 + b_3 = c_1 + c_3$
$(3) = (5)$	$a_3 = a_2 + c_2 + c_3$
$(3) = (6)$	$a_3 + b_1 = a_2 + b_2 + c_1 + c_3$
$(3) = (7)$	$b_1 = b_2 + c_1 + c_2$
$(3) = (8)$	$b_1 = b_2 + c_1 + c_2$
$(4) = (5)$	$a_3 + c_1 = a_2 + b_1 + b_3 + c_2$
$(4) = (6)$	$a_3 = a_2 + b_2 + b_3$
$(4) = (7)$	$c_3 = b_2 + b_3 + c_2$
$(4) = (8)$	$c_1 = b_1 + b_2 + c_2$
$(5) = (6)$	$b_1 + c_2 = b_2 + c_1$
$(5) = (7)$	$a_2 + b_1 + c_3 = a_3 + b_2 + c_1$
$(5) = (8)$	$a_2 + b_3 = a_3 + b_2$
$(6) = (7)$	$a_2 + c_3 = a_3 + c_2$
$(6) = (8)$	$a_2 + b_3 + c_1 = a_3 + b_1 + c_2$
$(7) = (8)$	$b_3 + c_1 = b_1 + c_3$

Table 3.13: consequences of Equal Sums in Table 3.12

(i)	(1) = (2) and (3) = (7)
(ii)	(1) = (2) and (4) = (8)
(iii)	(1) = (3) and (2) = (7)
(iv)	(1) = (3) and (4) = (6)
(v)	(1) = (5) and (2) = (6)
(vi)	(1) = (5) and (3) = (8)
(vii)	(1) = (5) and (4) = (7)
$(viii)$	(1) = (6) and (2) = (5)
(ix)	(1) = (6) and (3) = (4)
(x)	(1) = (6) and (4) = (8)
(xi)	(1) = (6) and (7) = (8)
(xii)	(1) = (7) and (4) = (5)
$(xiii)$	(1) = (8) and (2) = (4)
(xiv)	(1) = (8) and (3) = (5)
(xv)	(1) = (8) and (4) = (6)
(xvi)	(1) = (8) and (6) = (7)
$(xvii)$	(2) = (4) and (3) = (5)
$(xviii)$	(2) = (5) and (3) = (4)
(xix)	(2) = (5) and (3) = (7)
(xx)	(2) = (5) and (7) = (8)
(xxi)	(2) = (6) and (3) = (8)
$(xxii)$	(2) = (6) and (4) = (7)
$(xxiii)$	(2) = (7) and (3) = (5)
$(xxiv)$	(2) = (7) and (4) = (6)
(xxv)	(2) = (7) and (5) = (8)
$(xxvi)$	(2) = (8) and (3) = (6)
$(xxvii)$	(3) = (5) and (6) = (7)
$(xxviii)$	(3) = (7) and (4) = (8)
$(xxix)$	(3) = (7) and (5) = (6)
(xxx)	(3) = (8) and (4) = (7)
$(xxxi)$	(4) = (6) and (5) = (8)
$(xxxii)$	(4) = (8) and (5) = (6)

Table 3.14: Pairs of equalities from Table 3.13 that do not simultaneously hold

We partition the equation labels in Table 3.12 in two classes of equal sums S and T as usual. Assume that $1 \in S$ without loss of generality. Suppose that $1 = 6$. Then Table 3.14 implies that $2 \neq 5$, $3 \neq 4$, $4 \neq 8$, and $7 \neq 8$. This implies that $3 = 8$ and $4 = 7$. This contradicts row (xxx) of Table 3.14. Hence $1 \in S$ and $6 \in T$. Hence from applying the fact that $1 \neq 6$ and the label permutations (A), (B), and (C) that $2 \neq 5$, $3 \neq 7$, and $4 \neq 8$.

Assume that $1 = 8$. Then Table 3.14 implies that $2 \neq 4$, $3 \neq 5$, $4 \neq 6$, and $6 \neq 7$. Then $\{1, 2, 3, 6, 8\} \subseteq S$. This contradicts row $(xxvi)$ of Table 3.14. Hence $1 \neq 8$ and $8 \in T$. It follows from $6 = 8$ and the label permutation (A) that $5 = 7$. First assume that $5, 7 \in S$. Then Table 3.14 implies that $2 \neq 6$, $3 \neq 8$, $4 \neq 7$, and $4 \neq 5$. So $\{1, 2, 3, 5, 7\} \subseteq S$ and $\{4, 6, 8\} \subseteq T/$ This contradicts row (i) of Table 3.14. Hence we have established that $1 \neq 8$ so that $1 \in S$ and $\{5, 6, 7, 8\} \subseteq T$. Then row $(xxxii)$ of Table 3.14 implies that $4 \in S$. Row (xxv) of Table 3.14 implies that $2 \in S$. Row $(xxvii)$ of Table 3.14 implies that $3 \in S$. Hence $S = \{1, 2, 3, 4\}$ and $T = \{5, 6, 7, 8\}$. So $a_2 = a_3$, $b_1 = c_1$, $b_2 = b_3$, and $c_2 = c_3$. Hence the pair (G, ω) is as given in Theorem 2.1.8 with $n = 3$; a contradiction. By symmetry, if $G \cong K_{3,n}$, then (G, ω) is as given in the theorem statement; a contradiction. This completes the proof of Theorem 2.1.8. $\square$

Proof of Theorem 2.1.9. Suppose that $G \cong K'_{3,n}$ as pictured in Figure 2.8 with the edges U_2 and U_3 deleted. By symmetry we may assume that $E = A_1$, $E = C_3$, or $E = U_1$. First assume that $E = A_1$. The cycles of G that contain the edge A_1 are as given in the previous proof together with cycles $\{A_1, B_1, U_1\}$ and $\{A_1, C_1, U_1, B_i, C_i\}$ for $i \in$

$[n]\backslash 1$. If (G,ω) satisfies conditions of the theorem statement (i), then $\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G}) = \{\mathrm{a}_1+\mathrm{b}_1+\mathrm{a}_2+\mathrm{b}_2, \mathrm{a}_1+\mathrm{b}_1+\mathrm{a}_2+3\mathrm{b}_2\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G})| = 2$ and that the pair (G,ω) is not as given in the theorem statement. Then Theorem 2.1.8 implies that $b_1 = c_1$, and, for $i,j \in [n]\backslash 1$, $a_i = a_j$ and $b_i = c_j$. These conditions imply that $\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G}) = \{\mathrm{a}_1+\mathrm{b}_1+\mathrm{a}_2+\mathrm{b}_2, \mathrm{a}_1+\mathrm{b}_1+\mathrm{a}_2+3\mathrm{b}_2\}$. Then there are cycles of length $a_1+b_1+u_1$ and $a_1+b_1+2b_2+u_1$ containing the edges A_1 and U_1. So $u_1 = a_2+b_2$ and (G,ω) is as given in Theorem 2.1.9 (i); a contradiction.

Now suppose that $E = C_3$. If (G,ω) satisfies the conditions in the theorem statement (ii), then $\mathrm{spec}_{\omega(\mathrm{C}_3)}(\mathrm{G}) = \{\mathrm{a}_1+\mathrm{a}_3+\mathrm{c}_1+\mathrm{c}_3, 3\mathrm{a}_1+\mathrm{a}_3+\mathrm{c}_1+\mathrm{c}_3\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{C}_3)}(\mathrm{G})| = 2$ and that (G,ω) is not as given in the theorem statement (ii). Then from interchanging the roles of the edges A_1 and C_3 in the proof of Theorem 2.1.8 we obtain that $a_3 = b_3$, $c_i = c_j$ for $i,j \in \{1,2,\cdots,n\}\backslash 3$ and $a_i = b_j$ for $i,j \in \{1,2,\cdots,n\}\backslash 3$. Thus $\mathrm{spec}_{\omega(\mathrm{C}_3)}(\mathrm{G}) = \{\mathrm{a}_1+\mathrm{a}_3+\mathrm{c}_1+\mathrm{c}_3, 3\mathrm{a}_1+\mathrm{a}_3+\mathrm{c}_1+\mathrm{c}_3\}$. The cycles of G that contain both the edges C_3 and U_1 all have weight $a_1+a_3+c_1+c_3+u_1$. Hence $u_1 = 2a_1$. Thus (G,ω) is as given in part (ii) of the theorem statement; a contradiction.

Now suppose that $E = U_1$. If (G,ω) satisfies the conditions in the theorem statement (iii), then $\mathrm{spec}_{\omega(\mathrm{U}_1)}(\mathrm{G}) = \{\mathrm{a}_1+\mathrm{b}_1+\mathrm{u}_1, \mathrm{a}_1+\mathrm{b}_1+\mathrm{u}_1+2\mathrm{c}_1\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{U}_1)}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement (iii). Suppose that $n = 3$ and then we generalize the result when $n > 3$. The cycles of G that contains the edge U_1 are $\{A_1, B_1, U_1\}$, $\{A_2, B_2, U_1\}$, $\{A_3, B_3, U_1\}$, $\{A_3, B_3, U_1\}$, $\{A_1, B_2, C_1, C_2, U_1\}$, $\{A_1, B_3, C_1, C_3, U_1\}$, $\{A_2, B_1, C_1, C_2, U_1\}$, $\{A_2, B_3, C_2, C_3, U_1\}$, $\{A_3, B_2, C_2, C_3, U_1\}$, and $\{A_3, B_1, C_1, C_3, U_1\}$. Then the sums in Table 3.15 are of two values. Define the sets S and T as before and assume that $1 \in S$ without loss of generality. Table 3.16 yields the consequences of different sums being equal in the Table 3.15 and Table 3.17 shows which pairs of equal sums cannot occur. Under the edge isomorphism $(A_1, B_1)(A_2, B_2)(A_3, B_3)(C_1)(C_2)(C_3)$ of G we obtain the following permutation of the label set of Table 3.15:

(†) (1)(2)(3)(46)(59)(78).

(1) $a_1 + b_1 + u_1$	(6) $a_2 + b_1 + c_1 + c_2 + u_1$
(2) $a_2 + b_2 + u_1$	(7) $a_2 + b_3 + c_2 + c_3 + u_1$
(3) $a_3 + b_3 + u_1$	(8) $a_3 + b_2 + c_2 + c_3 + u_1$
(4) $a_1 + b_2 + c_1 + c_2 + u_1$	(9) $a_3 + b_1 + c_1 + c_3 + u_1$
(5) $a_1 + b_3 + c_1 + c_3 + u_1$	

Table 3.15: Weights of cycles containing the edge U_1 in the graph $K_{3,3} + U_1$

(1) = (2)	$a_1 + b_1 = a_2 + b_2$
(1) = (3)	$a_1 + b_1 = a_3 + b_3$
(1) = (4)	$b_1 = b_2 + c_1 + c_2$
(1) = (5)	$b_1 = b_3 + c_1 + c_3$
(1) = (6)	$a_1 = a_2 + c_1 + c_2$
(1) = (7)	$a_1 + b_1 = a_2 + b_3 + c_2 + c_3$
(1) = (8)	$a_1 + b_1 = a_3 + b_2 + c_2 + c_3$
(1) = (9)	$a_1 = a_3 + c_1 + c_3$
(2) = (3)	$a_2 + b_2 = a_3 + b_3$
(2) = (4)	$a_2 = a_1 + c + 1 + c_2$
(2) = (5)	$a_2 + b_2 = a_1 + b_3 + c_1 + c_3$
(2) = (6)	$b_2 = b_1 + c_1 + c_2$
(2) = (7)	$b_2 = b_3 + c_2 + c_3$
(2) = (8)	$a_2 = a_3 + c_2 + c_3$
(2) = (9)	$a_2 + b_2 = a_3 + b_1 + c_1 + c_3$
(3) = (4)	$a_3 + b_3 = a_1 + b_2 + c_1 + c_2$
(3) = (5)	$a_3 = a_1 + c_1 + c_3$
(3) = (6)	$a_3 + b_3 = a_2 + b_1 + c_1 + c_2$
(3) = (7)	$a_3 = a_2 + c_2 + c_3$
(3) = (8)	$b_3 = b_2 + c_2 + c_3$
(3) = (9)	$b_3 = b_1 + c_1 + c_3$
(4) = (5)	$b_2 + c_2 = b_3 + c_3$
(4) = (6)	$a_1 + b_2 = a_2 + b_1$
(4) = (7)	$a_1 + b_2 + c_1 = a_2 + b_3 + c_3$
(4) = (8)	$a_1 + c_1 = a_3 + c_3$
(4) = (9)	$a_1 + b_2 + c_2 = a_3 + b_1 + c_3$
(5) = (6)	$a_1 + b_3 + c_3 = a_2 + b_1 + c_2$
(5) = (7)	$a_1 + c_1 = a_2 + c_2$
(5) = (8)	$a_1 + b_3 + c_1 = a_3 + b_2 + c_2$
(5) = (9)	$a_1 + b_3 = a_3 + b_1$
(6) = (7)	$b_1 + c_1 = b_3 + c_3$
(6) = (8)	$a_2 + b_1 + c_1 = a_3 + b_2 + c_3$
(6) = (9)	$a_2 + c_2 = a_3 + c_3$
(7) = (8)	$a_2 + b_3 = a_3 + b_2$
(7) = (9)	$a_2 + b_3 + c_2 = a_2 + b_1 + c_1$
(8) = (9)	$b_2 + c_2 = b_1 + c_1$

Table 3.16: consequences of Equal Sums in Table 3.15

(i)	$(1) = (4)$ and $(2) = (6)$
(ii)	$(1) = (4)$ and $(8) = (9)$
(iii)	$(1) = (5)$ and $(3) = (9)$
(iv)	$(1) = (5)$ and $(6) = (7)$
(v)	$(1) = (6)$ and $(2) = (4)$
(vi)	$(1) = (6)$ and $(5) = (7)$
(vii)	$(1) = (9)$ and $(3) = (5)$
$(viii)$	$(1) = (9)$ and $(4) = (8)$
(ix)	$(2) = (4)$ and $(5) = (7)$
(x)	$(2) = (6)$ and $(8) = (9)$
(xi)	$(2) = (7)$ and $(3) = (8)$
(xii)	$(2) = (7)$ and $(4) = (5)$
$(xiii)$	$(2) = (8)$ and $(3) = (7)$
(xiv)	$(2) = (8)$ and $(6) = (9)$
(xv)	$(3) = (5)$ and $(4) = (8)$
(xvi)	$(3) = (7)$ and $(6) = (9)$
$(xvii)$	$(3) = (8)$ and $(4) = (5)$
$(xviii)$	$(3) = (9)$ and $(6) = (7)$

Table 3.17: Pairs of equalities from Table 3.16 that do not simultaneously hold

Suppose that $4 \in S$. Then, by rows (i)and (ii) of Table 3.17, one of four cases occurs, $(1248, 69)$, $(1249, 68)$, $(1468, 29)$, or $(1469, 28)$ is a subset of (S, T). The first and last of these cases do not occur by row (xiv) of Table 3.17.

Suppose that $(1249, 68) \subset (S, T)$. It follows from row (viii) of Table 3.17 that $(3) \neq (5)$ so that $(12349, 568)$ or $(12459, 368)$ is a subset of (S, T). Suppose the former holds. Then $(1) = (4)$, $(2) = (7)$, and $(3) = (9)$ imply that $b_1 = b_2 + c_1 + c_2$, $b_2 = b_3 + c_2 + c_3$, and $b_3 = b_1 + c_1 + c_3$. Thus $b_1 = b_2 + c_1 + c_2 = (b_3 + c_2 + c_3) + c_1 + c_2 = b_3 + c_1 + 2c_2 + c_3 = (b_1 + c_1 + c_3) + c_1 + 2c_2 + c_3$. Hence $b_1 < b_1$; a contradiction. Suppose $(12459, 368)$ is a subset of (S, T). This contradicts row (xvii) of Table 3.17. Hence $(1468, 29)$ is a subset of (S, T).

It follows from row (xii) of Table 3.17 that $(14568, 279)$ or $(14678, 259)$ is a subset of (S, T). The former cannot occur by row (xii) of Table 3.17. If the latter occurs, then the equations for $(2) = (5)$, $(1) = (6)$, and $(3) = (8)$ imply that $a_2 + b_2 = a_1 + b_3 + c_1 + c_3$, $a_1 = a_2 + c_1 + c_2$, and $b_3 = b_2 + c_2 + c_3$. It follows from substituting the last two equations into the first we obtain that $a_2 + b_2 = a_1 + b_3 + c_1 + c_3 = (a_2 + c_1 + c_2) + (b_2 + c_2 + c_3) + c_1 + c_3 = a_2 + b_2 + 2(c_1 + c_2 + c_3)$ so that $a_2 + b_2 < a_2 + b_2$; a contradiction. Hence $(1) \neq (4)$. We may apply (†) to obtain that $(1) \neq (6)$.

Assume $(1) = (5)$. Then by rows *(iii)* and *(iv)* of Table 3.17, $(3) \neq (9)$ and $(6) \neq (7)$. Hence $7 \in S$ and $(157, 46)$ is a subset of (S, T), and either $(1357, 469)$ or $(1579, 346)$ is a subset of (S, T). Then former case cannot occur by row (xvi) of

Table 3.17. If the latter case occurs, then $8 \in S$ by row (viii). Then $2 \in S$ by row (ix) of Table 3.17. Then (3) = (4), (4) = (6), (2) = (8), and (1) = (5) imply that $a_3 + b_3 = a_1 + b_2 + c_1 + c_2$, $a_1 + b_2 = a_2 + b_1$, $a_2 = a_3 + c_2 + c_3$, and $b_1 = b_3 + c_1 + c_3$. Thus $a_3 + b_3 = (a_1 + b_2) + c_1 + c_2 = (a_2 + b_1) + c_1 + c_2 = (a_3 + c_2 + c_3) + b_1 + c_1 + c_2 = a_3 + c_2 + c_3 + (b_3 + c_1 + c_3) + c_1 + c_2 = a_3 + b_3 + 2(c_1 + c_2 + c_3)$ so that $a_3 + b_3 < a_3 + b_3$; a contradiction. So $(1) \neq (5)$, and by (†), $(1) \neq (9)$. Hence $(1, 4569)$ is a subset of (S, T).

Suppose (1) = (7) so that $(17, 4569)$ is a subset of (S, T). By row (xii) of Table 3.17, $2 \in T$. It follows from rows (xiv) and (xvi) of Table 3.17 that $(2) \neq (8)$ and $(3) \neq (7)$, respectively. Thus $(S, T) = (178, 234569)$. It follows from (1) = (7), (2) = (4), and (3) = (9) that $a_1 + b_1 = a_2 + b_3 + c_2 + c_3$, $a_2 = a_1 + c_1 + c_2$, and $b_3 = b_1 + c_1 + c_3$. It follows from combining these three equations that $a_1 + b_1 < a_1 + b_1$; a contradiction. Hence $(1) \neq (7)$, and by (†), $(1) \neq (8)$. Hence $(1, 456789)$ is a subset of (S, T). It follows from rows (ix) and (xvi) of Table 3.17 that $2 \in S$ and $3 \in S$, respectively. Hence $(S, T) = (123, 456789)$. Using the equalities obtained in Table 3.16 from (1)=(2)=(3) and (4)=(5)=(6)=(7)=(8)=(9) we obtain that, for $i, j \in \{1, 2, 3\}$, $a_i = a_j$, $b_i = b_j$, and $c_i = c_j$. Now if $n \geq 3$, by symmetry, for $i, j \in [n]$, $a_i = a_j$, $b_i = b_j$, and $c_i = c_j$. Hence (G, ω) is as given in the theorem statement (iii); a contradiction. This completes the proof of Theorem 2.1.9. $\square$

Proof of Theorem 2.1.10. Suppose that $G \cong K''_{3,n}$ as pictured in Figure 2.8 with the edge U_3 deleted. By symmetry, we may assume that $E = A_1$, $E = B_1$, or $E = U_1$.

First assume that $E = A_1$. If (G, ω) satisfies the conditions in the theorem statement (i), then $\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1 + \mathrm{b}_2, \mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1 + 3\mathrm{b}_2\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{A}_1)}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement (i). It follows from applying Theorem 2.1.9 (i) that $b_1 = c_1$, and, for $i, j \in [n]\backslash 1$, $a_i = a_j$, $b_i = c_j$, and $u_1 = a_2 + b_2$. The cycles of G that contain A_1 then must have weight $a_1 + a_2 + b_1 + b_2$ or $a_1 + a_2 + b_1 + 3b_2$. In addition, the cycle of G $\{A_1, C_1, U_1, U_2\}$ has weight $a_1 + b_1 + a_2 + b_2 + u_2$ so that $u_2 = 2b_2$. Hence (G, ω) is as given in the statement of the theorem (i); a contradiction.

Now suppose that $E = B_1$. If (G, ω) satisfies the conditions in the theorem statement (ii), then $\mathrm{spec}_{\omega(\mathrm{B}_1)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{a}_2 + \mathrm{b}_1 + \mathrm{b}_2, \mathrm{a}_1 + 3\mathrm{a}_2 + \mathrm{b}_1 + \mathrm{b}_2\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{B}_1)}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement (ii). It follows from Theorem 2.1.9 to the subgraph of G obtained by deleting the edge U_2 and symmetry that $a_1 = c_1$, for $i, j \in [n]\backslash 1$, $a_i = c_j$, $b_i = b_j$, and $u_1 = a_2 + b_2$. It follows from applying Theorem 2.1.9 to the subgraph of G obtained by deleting the edge U_1 that $u_2 = a_2 + b_2$ so that (G, ω) is as given in the theorem statement (ii); a contradiction.

Finally, assume that $E = U_1$. If (G, ω) satisfies the conditions in the theorem statement (iii), then $\mathrm{spec}_{\omega(\mathrm{U}_1)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{b}_1 + \mathrm{u}_1, \mathrm{a}_1 + \mathrm{b}_1 + 2\mathrm{c}_1 + \mathrm{u}_1\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(\mathrm{U}_1)}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement (iii). It follows from Theorem 2.1.9 (iii) that , for $i, j \in [n]$, $a_i = a_j$, $b_i = b_j$, and $c_i = c_j$ and hence the weights of the cycles that contain the edge U_1 are $a_1 + b_1 + u_1$ or $a_1 + b_1 + 2c_1 + u_1$.

The cycles of G that contain the edge U_1 but not U_2 have weight $a_1 + b_1 + u_1$ or $a_1 + b_1 + 2c_1 + u_1$.The cycles of G that contain the edges U_1 and U_2 have weight $a_1 + c_1 + u_1 + u_2$. Note that $a_1 + c_1 + u_1 + u_2 = a_1 + c_1 + u_1 + (b_1 \pm c_1)$ so that all cycles of G that contain the edge U_1 have weight $a_1 + b_1 + u_1$ or $a_1 + b_1 + 2c_1 + u_1$. There are cycles of G that contain both the edges U_1 and U_2 of weight $a_1 + c_1 + u_1 + u_2$. Thus $a_1 + c_1 + u_1 + u_2 \in \{a_1 + b_1 + u_1, a_1 + b_1 + 2c_1 + u_1\}$ and $u_2 = b_1 \pm c_1$. Hence (G, ω) is as given in the theorem statement (iii); a contradiction. This completes the proof of Theorem 2.1.10. $\square$

Proof of Theorem 2.1.11. Suppose that $G \cong K'''_{3,n}$ as pictured in Figure 2.8. We may assume by symmetry that $E = A_1$ or $E = U_1$. First assume that $E = A_1$. If (G, ω) satisfies the conditions in the theorem statement (i), then $\mathrm{spec}_{\omega(A_1)}(G) = \{a_1 + b_1 + a_2 + b_2, a_1 + b_1 + a_2 + 3b_2\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(A_1)}(G)| = 2$ and that the pair (G, ω) is not as given in the theorem statement (i). It follows from Theorem 2.1.10 (i) that $b_1 = c_1$, for $i, j \in [n]\backslash 1$, $a_i = a_j$, $b_i = c_j$, $u_1 = a_2 + b_2$, and $u_2 = 2b_2$. Thus $\mathrm{spec}_{\omega(A_1)}(G) = \{a_1 + b_1 + a_2 + b_2, a_1 + b_1 + a_2 + 3b_2\}$. Then the cycle $\{A_1, B_1, U_2, U_3\}$ has weight in $\mathrm{spec}_{\omega(A_1)}(G)$ so that $a_1 + b_1 + 2b_2 + u_3 = a_1 + b_1 + u_2 + u_3 \in \{a_1 + b_1 + a_2 + b_2, a_1 + b_1 + a_2 + 3b_2\}$. If $a_1 + b_1 + 2b_2 + u_3 = a_1 + b_1 + a_2 + 3b_2$, then $u_3 = a_2 + b_2$ and the pair (G, ω) is as given in the theorem statement (i); a contradiction. Hence

$a_1 + b_1 + 2b_2 + u_3 = a_1 + b_1 + a_2 + b_2$ and $u_3 = a_2 - b_2$. Then the cycle $\{A_1, C_1, U_3\}$ has weight $a_1 + c_1 + u_3 = a_1 + b_1 + (a_2 - b_2)$; a contradiction.

Suppose that $E = U_1$. If (G, ω) satisfies the conditions in the theorem statement, then $\mathrm{spec}_{\omega(U_1)}(\mathrm{G}) = \{\mathrm{a_1 + b_1 + u_1, a_1 + b_1 + 2c_1 + u_1}\}$.

Conversely, suppose that $|\mathrm{spec}_{\omega(U_1)}(\mathrm{G})| = 2$ and that the pair (G, ω) is not as given in the theorem statement (ii). It follows from Theorem 2.1.10 (iii) that , for $i, j \in [n]$, $a_i = a_j$, $b_i = b_j$, $c_i = c_j$, and $u_2 = b_1 \pm c_1$. Thus $\mathrm{spec}_{\omega(U_1)}(\mathrm{G}) = \{\mathrm{a_1 + b_1 + u_1, a_1 + b_1 + 2c_1 + u_1}\}$.

The cycle $\{U_1, U_2, U_3\}$ has weight $\{u_1 + u_2 + u_3\} \in \{a_1 + b_1 + u_1, a_1 + b_1 + 2c_1 + u_1\}$. The cycle $\{U_1, U_2, U_3\}$ has weight $u_1 + u_2 + u_3 = u_1 + (b_1 \pm c_1) + u_3$. Thus $u_1 + (b_1 \pm c_1) + u_3 = a_1 + b_1 + u_1$ or $u_1 + (b_1 \pm c_1) + u_3 = a_1 + b_1 + u_1 + 2c_1$. So $u_3 = a_1 \pm c_1$ or $u_3 = a_1 + 2c_1 \pm c_1$. Hence $u_3 \in \{a_1 \pm c_1, a_1 + 3c_1\}$. Since $u_3 = a_1 \pm c_1$ would imply that (G, ω) is as given in the theorem statement (iii), $u_3 = a_1 + 3c_1$. The cycle $\{B_1, C_1, U_1, U_3\}$ has weight $b_1 + c_1 + u_1 + u_3 = a_1 + b_1 + 4c_1 + u_1$ which is of too large a weight. This contradiction completes the proof of Theorem 2.1.11. $\square$

3.2 Graphs with disjoint cycles

In this section we give the proofs of the results mentioned in Section 2.2 of the dissertation.

Proof of Theorem 2.2.2. First suppose that the edge $E = A_1$ is balanced. The cycles of G that contain the edge A_1 are $\{A_1, A_2, A_3\}$, $\{A_1, B_1, C_2, C_3\}$, $\{A_1, A_2, B_3, C_1, C_2\}$, $\{A_1, B_2, B_3, C_2, C_3\}$, $\{A_1, A_3, B_2, C_1, C_3\}$, $\{A_1, A_3, B_1, B_3, C_1, C_3\}$, and $\{A_1, A_2, B_1, B_2, C_1, C_2\}$. Hence the seven sums in Table 3.18 are of two different values.

(1) $a_1 + a_2 + a_3$	(2) $a_1 + b_1 + c_2 + c_3$
(3) $a_1 + a_2 + b_3 + c_1 + c_2$	(4) $a_1 + b_2 + b_3 + c_2 + c_3$
(5) $a_1 + a_3 + b_2 + c_1 + c_3$	(6) $a_1 + a_3 + b_1 + b_3 + c_1 + c_3$
(7) $a_1 + a_2 + b_1 + b_2 + c_1 + c_2$	

Table 3.18: Weights of cycles containing the edge A_1 in the prism graph

Let S and T be nonempty sets that partition $\{1, 2, 3, 4, 5, 6, 7\}$ so that the elements of S have the same value in this table, while the elements of T have the other value in this table. Note that under the edge isomorphism of G $(A_1)(B_1)(C_1)(A_2A_3)(B_2B_3)(C_2C_3)$ we obtain the following permutation of the sum labels in Table 3.18:

(†) (1)(2)(35)(4)(67).

We summarize in each row of Table 3.19 the corresponding weight condition that occurs in the graph of Figure 2.13 when the corresponding sums in Table 3.18 have the same value.

Some pairs of the equalities in Table 3.19 cannot occur simultaneously. From considering all pairs of possible equalities from Table 3.19 we obtain Table 3.20. For

$(1) = (2)$	$a_2 + a_3 = b_1 + c_2 + c_3$
$(1) = (3)$	$a_3 = b_3 + c_1 + c_2$
$(1) = (4)$	$a_2 + a_3 = b_2 + b_3 + c_2 + c_3$
$(1) = (5)$	$a_2 = b_2 + c_1 + c_3$
$(1) = (6)$	$a_2 = b_1 + b_3 + c_1 + c_3$
$(1) = (7)$	$a_3 = b_1 + b_2 + c_1 + c_2$
$(2) = (3)$	$b_1 + c_3 = a_2 + b_3 + c_1$
$(2) = (4)$	$b_1 = b_2 + b_3$
$(2) = (5)$	$b_1 + c_2 = a_3 + b_2 + c_1$
$(2) = (6)$	$c_2 = a_3 + b_3 + c_1$
$(2) = (7)$	$c_3 = a_2 + b_2 + c_1$
$(3) = (4)$	$a_2 + c_1 = b_2 + c_3$
$(3) = (5)$	$a_2 + b_3 + c_2 = a_3 + b_2 + c_3$
$(3) = (6)$	$a_2 + c_2 = a_3 + b_1 + c_3$
$(3) = (7)$	$b_3 = b_1 + b_2$
$(4) = (5)$	$b_3 + c_2 = a_3 + c_1$
$(4) = (6)$	$b_2 + c_2 = a_3 + b_1 + c_1$
$(4) = (7)$	$b_3 + c_3 = a_2 + b_1 + c_1$
$(5) = (6)$	$b_2 = b_1 + b_3$
$(5) = (7)$	$a_3 + c_3 = a_2 + b_1 + c_2$
$(6) = (7)$	$a_3 + b_3 + c_3 = a_2 + b_2 + c_2$

Table 3.19: Consequences of Equal Sums in Table 3.18

example, if (1)=(3) and (2)=(6), then one obtains the contradiction that $b_3 = 0$. This information is summarized in the first row of Table 3.20.

Suppose $1 \in S$ without loss of generality. Assume $6 \in S$. It follows from Table 3.20 rows (v), (vi), and (vii) that exactly one of 2 and 7, exactly one of 2 and 3, and exactly one of 4 and 7, respectively, are in S. The first and last of these assertions imply that (2)=(4). The first two of these assertions imply that (3)=(7).

This contradicts Table 3.20 row (xi). So $6 \in T$. It follows from (†) that $7 \in T$ as well.

Assume $3 \in T$. Then, Table 3.20 row (xiii) implies that $5 \in S$. Then, by Table 3.20 row (iv) implies that $4 \in S$. Hence either $2 \in S$ which contradicts Table 3.20 row (xi) or $2 \in T$ which contradicts Table 3.20 row (iii). So $3 \in S$. We have shown to this point of the proof that $\{1, 3\} \subseteq S$ and $\{6, 7\} \subseteq T$.

It follows from Table 3.20 row (i) that $2 \in S$. It follows from Table 3.20 row (ii) that exactly one of 4 and 5 is in S. So either $4 \in S$ and $5 \in T$ or $5 \in S$ and $4 \in T$. The former contradicts Table 3.20 row (xii). Thus the latter holds so that $5 \in S$ while $4 \in T$. Hence $(S, T) = \{1235, 467\}$. It follows from (1)=(2), (1)=(3), and (1)=(5) in Table 3.18 that $b_1 = b_2 + b_3 + c_1 + c_1$. It follows from (4)=(7) and (1)=(5) in Table 3.18 that $b_3 + c_3 = (b_2 + c_1 + c_3) + (b_2 + b_3 + c_1 + c_1) + c_1$. Thus $0 = b_2 + b_2 + c_1 + c_1 + c_1 + c_1$; a contradiction. Hence the edge A_1 is not balanced. By symmetry, the edge E is not balanced if $E \in \{A_1, A_2, A_3, B_1, B_2, B_3\}$.

Let the pair (G, ω) be as given in the statement of the theorem for $i \in [3]$. Then $\mathrm{spec}_{\omega(C_i)} = \{\mathrm{a_i + a_j + b_k + c_i + c_j}, \mathrm{3a_i + a_j + b_k + c_i + c_j}\}$ so that the edge C_i is balanced.

Conversely, suppose that the edge $E = C_1$ is balanced and that (G, ω) is not as given in the theorem statement for i=1. We will obtain a contradiction. The result will then follow by symmetry when $E = C_2$ or $E = C_3$. There are eight cycles of G that contain the edge e. They are $A_2 \cup B_2 \cup C_1 \cup C_3$, $A_3 \cup B_3 \cup C_1 \cup C_2$,

$A_1 \cup A_2 \cup B_3 \cup C_1 \cup C_2$, $A_1 \cup A_3 \cup B_2 \cup C_1 \cup C_3$, $A_2 \cup B_1 \cup B_3 \cup C_1 \cup C_3$, $A_3 \cup B_1 \cup B_2 \cup C_1 \cup C_2$, $A_1 \cup A_2 \cup B_1 \cup B_2 \cup C_1 \cup C_2$, and $A_1 \cup A_3 \cup B_1 \cup B_3 \cup C_1 \cup C_3$. Hence the eight sums in Table 3.21 are of two values. Thus, again, there exist nonempty sets S and T that partition $\{1, 2, 3, 4, 5, 6, 7, 8\}$ such that all the sums in Table 3.21 corresponding to the elements of S have the same value, while all the sums in Table 3.21 corresponding to the elements of T have the other value.

(i)	(1) = (3) and (2) = (6)
(ii)	(1) = (3) and (4) = (5)
(iii)	(1) = (5) and (2) = (7)
(iv)	(1) = (5) and (3) = (4)
(v)	(1) = (6) and (2) = (3)
(vi)	(1) = (6) and (2) = (7)
(vii)	(1) = (6) and (4) = (7)
$(viii)$	(1) = (7) and (2) = (5)
(ix)	(1) = (7) and (2) = (6)
(x)	(1) = (7) and (4) = (6)
(xi)	(2) = (4) and (3) = (7)
(xii)	(2) = (4) and (5) = (6)
$(xiii)$	(3) = (6) and (5) = (7)
(xiv)	(3) = (7) and (5) = (6)

Table 3.20: Pairs of equalities from Table 3.18 that do not simultaneously hold

(1) $a_2 + b_2 + c_1 + c_3$	(2) $a_3 + b_3 + c_1 + c_2$	(3) $a_1 + a_2 + b_3 + c_1 + c_2$
(4) $a_1 + a_3 + b_2 + c_1 + c_3$	(5) $a_2 + b_1 + b_3 + c_1 + c_3$	(6) $a_3 + b_1 + b_2 + c_1 + c_2$
(7) $a_1 + a_2 + b_1 + b_2 + c_1 + c_2$	(8) $a_1 + a_3 + b_1 + b_3 + c_1 + c_3$	

Table 3.21: $E = C_1$ in the prism graph

(1) = (2)	$a_2 + b_2 + c_3 = a_3 + b_3 + c_2$
(1) = (3)	$b_2 + c_3 = a_1 + b_3 + c_2$
(1) = (4)	$a_2 = a_1 + a_3$
(1) = (5)	$b_2 = b_1 + b_3$
(1) = (6)	$a_2 + c_3 = a_3 + b_1 + c_2$
(1) = (7)	$c_3 = a_3 + b_3 + c_2$
(1) = (8)	$a_2 + b_2 = a_1 + a_3 + b_1 + b_3$
(2) = (3)	$a_3 = a_1 + a_2$
(2) = (4)	$b_3 + c_2 = a_1 + b_2 + c_3$
(2) = (5)	$a_3 + c_2 = a_2 + b_1 + c_3$
(2) = (6)	$b_3 = b_1 + b_2$
(2) = (7)	$a_3 + b_3 = a_1 + a_2 + b_1 + b_2$
(2) = (8)	$c_2 = a_1 + b_1 + c_3$
(3) = (4)	$a_2 + b_3 + c_2 = a_3 + b_2 + c_3$
(3) = (5)	$a_1 + c_2 = b_1 + c_3$
(3) = (6)	$a_1 + a_2 + b_3 = a_3 + b_1 + b_2$
(3) = (7)	$b_3 = b_1 + b_2$
(3) = (8)	$a_2 + c_2 = a_3 + b_1 + c_3$
(4) = (5)	$a_1 + a_3 + b_2 = a_2 + b_1 + b_3$
(4) = (6)	$a_1 + c_3 = b_1 + c_2$
(4) = (7)	$a_3 + c_3 = a_2 + b_1 + c_2$
(4) = (8)	$b_2 = b_1 + b_3$
(5) = (6)	$a_2 + b_3 + c_3 = a_3 + b_2 + c_2$
(5) = (7)	$b_3 + c_3 = a_1 + b_2 + c_2$
(5) = (8)	$a_2 = a_1 + a_3$
(6) = (7)	$a_3 = a_1 + a_2$
(6) = (8)	$b_2 + c_2 = a_1 + b_3 + c_3$
(7) = (8)	$a_2 + b_2 + c_2 = a_3 + b_3 + c_3$

Table 3.22: Consequences of equal sums in Table 3.21

(i)	(1) = (3) and (2) = (4)
(ii)	(1) = (4) and (2) = (3)
(iii)	(1) = (4) and (6) = (7)
(iv)	(1) = (5) and (2) = (6)
(v)	(1) = (5) and (3) = (7)
(vi)	(1) = (6) and (2) = (5)
(vii)	(1) = (7) and (2) = (8)
$(viii)$	(1) = (7) and (3) = (5)
(ix)	(1) = (7) and (4) = (6)
(x)	(1) = (8) and (2) = (7)
(xi)	(2) = (3) and (5) = (8)
(xii)	(2) = (6) and (4) = (8)
$(xiii)$	(2) = (8) and (3) = (5)
(xiv)	(2) = (8) and (4) = (6)
(xv)	(3) = (7) and (4) = (8)
(xvi)	(3) = (8) and (4) = (7)
$(xvii)$	(5) = (7) and (6) = (8)
$(xviii)$	(5) = (8) and (6) = (7)

Table 3.23: Pairs of equalities from Table 3.21 that do not simultaneously hold

We summarize in Table 3.22 the corresponding path length condition that occur in the graph of Figure 2.13 when the corresponding sums in Table 3.21 have the same values. Some of the pairs of equalities in Table 3.22 cannot occur simultaneously. This information is summarized in Table 3.23. For example, if (1) = (3) and (2) = (4) in Table 3.22, then $a_1 = 0$; a contradiction.

Note that under the weight permutation associated with the edge isomorphism $(A_1)(A_2, A_3)(B_1)(B_2, B_3)(C_1)(C_2, C_3)$ (‡) the sums (1) and (2) are interchanged, the sums (3) and (4) are interchanged, the sums (5) and (6) are interchanged; and the sums (7) and (8) are interchanged.

Suppose $1 \in S$ without loss of generality. assume $7 \in S$. It follows from Table 3.23 rows (vii), (viii), and (ix) that exactly one of 2 and 8, exactly one of 3 and 5, and exactly one of 4 and 6, respectively, are in S. This yields eight cases. We next list each of these eight cases and the row label of Table 3.23 that the case contradicts. $(S, T) = \{12457, 368\}$ row(xvi), $(S, T) = \{13478, 256\}$ row(xii), $(S, T) = \{12367, 458\}$ row(xi), $(S, T) = \{13678, 245\}$ row(i), $(S, T) = \{12347, 568\}$ row(i), $(S, T) = \{14578, 236\}$ row(ii), $(S, T) = \{12567, 348\}$ row(iv), and $(S, T) = \{15678, 234\}$ row(xi).

Hence we have shown that $7 \in T$. It follows from the symmetry of (‡) that exactly one of 2 and 8 is in S. However, $\{1, 8\} \subseteq S$ and $\{2, 7\} \subseteq T$ contradicts Table 3.23 row (x). Thus $\{1, 2\} \subseteq S$ and $\{7, 8\} \subseteq T$.

We will show that, up to symmetry, $S = \{1,2,3,5\}$ and $T = \{4,6,7,8\}$. Recall that up to this point in the proof, $\{1,2\} \subseteq S$ and $\{7,8\} \subseteq T$. First assume that $3 \in S$. If $5 \in T$, then (2)=(3) and (5)=(8) contradicting Table 3.23 row (xi). Hence $\{1,2,3,5\} \subseteq S$ and $\{7,8\} \subseteq T$. If $4 \in S$, then (1)=(3) and (2)=(4) contradicting Table 3.23 row (i). Hence $4 \in T$ and $\{1,2,3,5\} \subseteq S$ while $\{7,8\} \subseteq T$. If $6 \in S$, then (1)=(6) and (2)=(5) contradicting Table 3.23 row (vi). Hence $6 \in T$ We have shown that if $3 \in S$, then $(S,T) = \{1235, 4678\}$.

Now assume that $3 \in T$ so that $\{1,2\} \subseteq S$ and $\{3,7,8\} \subseteq T$. If $4 \in T$, then (3)=(7) and (4)=(8) contradicting Table 3.23 row (xv). Hence $4 \in S$ so that $\{1,2,4\} \subseteq S$ and $\{3,7,8\} \subseteq T$. If $5 \in S$, then (1)=(5) and (3)=(7) contradicting Table 3.23 row (v). Hence $5 \in T$ so that $\{1,2,4\} \subseteq S$ and $\{3,5,7,8\} \subseteq T$. If $6 \in T$, then (1)=(4) and (6)=(7) contradicting Table 3.23 row (iii). Hence $6 \in S$. We have shown that if $3 \in T$, then $(S,T) = \{1246, 3578\}$. We have established that $(S,T) = \{1235, 4678\}$ or $(S,T) = \{1246, 3578\}$. However, by symmetry induced in (‡), we may assume that $(S,T) = \{1235, 4678\}$. It follows from (1)=(2) and (7)=(8) that $c_2 = c_3$. It follows from that $c_2 = c_3$ and (3)=(5) that $a_1 = b_1$. It follows from (2)=(3) that $a_3 = a_1 + a_2$, while (1)=(5) implies that $b_2 = b_1 + b_3$. Thus the pair (G, ω) is as given in the theorem statement; a contradiction. This completes the proof of Theorem 2.2.2. $\square$

(1) $a_3 + c_2 + d_1$	(6) $a_1 + a_3 + b_1 + c_3 + d_1$
(2) $b_3 + c_1 + d_1$	(7) $a_2 + b_2 + b_3 + c_3 + d_1$
(3) $a_1 + a_2 + c_2 + d_1$	(8) $a_1 + a_3 + b_2 + b_3 + c_3 + d_1$
(4) $a_2 + b_1 + c_3 + d_1$	(9) $a_1 + b_2 + c_1 + c_2 + c_3 + d_1$
(5) $b_1 + b_2 + c_1 + d_1$	

Table 3.24: The weights of cycles containing the edge D_1 in the Prism+ graph

$(1) = (2)$	$a_3 + c_2 = b_3 + c_1$
$(1) = (3)$	$a_3 = a_1 + a_2$
$(1) = (4)$	$a_3 + c_2 = a_2 + b_1 + c_3$
$(1) = (5)$	$a_3 + c_2 = b_1 + b_2 + c_1$
$(1) = (6)$	$c_2 = a_1 + b_1 + c_3$
$(1) = (7)$	$a_3 + c_2 = a_2 + b_2 + b_3 + c_3$
$(1) = (8)$	$c_2 = a_1 + b_2 + b_3 + c_3$
$(1) = (9)$	$a_3 = a_1 + b_2 + c_1 + c_3$
$(2) = (3)$	$b_3 + c_1 = a_1 + a_2 + c_2$
$(2) = (4)$	$b_3 + c_1 = a_2 + b_1 + c_3$
$(2) = (5)$	$b_3 = b_1 + b_2$
$(2) = (6)$	$b_3 + c_1 = a_1 + a_3 + b_1 + c_3$
$(2) = (7)$	$c_1 = a_2 + b_2 + c_3$
$(2) = (8)$	$c_1 = a_1 + a_3 + b_2 + c_3$
$(2) = (9)$	$b_3 = a_1 + b_2 + c_2 + c_3$
$(3) = (4)$	$a_1 + c_2 = b_1 + c_3$
$(3) = (5)$	$a_1 + a_2 + c_2 = b_1 + b_2 + c_1$
$(3) = (6)$	$a_2 + c_2 = a_3 + b_1 + c_3$
$(3) = (7)$	$a_1 + c_2 = b_2 + b_3 + c_3$
$(3) = (8)$	$a_2 + c_2 = a_3 + b_2 + b_3 + c_3$
$(3) = (9)$	$a_2 = b_2 + c_1 + c_3$
$(4) = (5)$	$a_2 + c_3 = b_2 + c_1$
$(4) = (6)$	$a_2 = a_1 + a_3$
$(4) = (7)$	$b_1 = b_2 + b_3$
$(4) = (8)$	$a_2 + b_1 = a_1 + a_3 + b_2 + b_3$
$(4) = (9)$	$a_2 + b_1 = a_1 + b_2 + c_1 + c_2$
$(5) = (6)$	$b_2 + c_1 = a_1 + a_3 + c_3$
$(5) = (7)$	$b_1 + c_1 = a_2 + b_3 + c_3$
$(5) = (8)$	$b_1 + c_1 = a_1 + a_3 + b_3 + c_3$
$(5) = (9)$	$b_1 = a_1 + c_2 + c_3$
$(6) = (7)$	$a_1 + a_3 + b_1 = a_2 + b_2 + b_3$
$(6) = (8)$	$b_1 = b_2 + b_3$
$(6) = (9)$	$a_3 + b_1 = b_2 + c_1 + c_2$
$(7) = (8)$	$a_2 = a_1 + a_3$
$(7) = (9)$	$a_2 + b_3 = a_1 + c_1 + c_2$
$(8) = (9)$	$a_3 + b_3 = c_1 + c_2$

Table 3.25: Consequences of Equal Sums in Table 3.24

(i)	(1) = (3) and (4) = (6)
(ii)	(1) = (3) and (7) = (8)
(iii)	(1) = (6) and (3) = (4)
(iv)	(1) = (6) and (5) = (9)
(v)	(1) = (8) and (2) = (9)
(vi)	(1) = (8) and (3) = (7)
(vii)	(1) = (9) and (2) = (8)
$(viii)$	(1) = (9) and (5) = (6)
(ix)	(2) = (5) and (4) = (7)
(x)	(2) = (5) and (6) = (8)
(xi)	(2) = (7) and (3) = (9)
(xii)	(2) = (7) and (4) = (5)
$(xiii)$	(2) = (8) and (5) = (6)
(xiv)	(2) = (9) and (3) = (7)
(xv)	(3) = (4) and (5) = (9)
(xvi)	(3) = (9) and (4) = (5)

Table 3.26: Pairs of equalities from Table 3.25 that do not simultaneously hold

In the proofs that follow, when considering a pair (G, ω), where G is a graph and ω is a weight function on the edges we will slightly corrupt notation and refer to the pair (H, ω) when H is a subgraph of G with the edge labels of H being a subset of the edge labels of G.

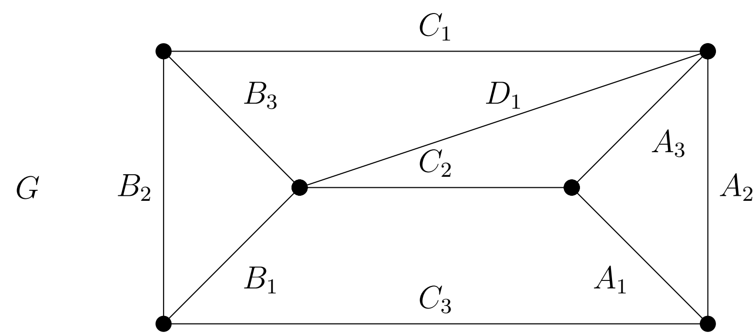


Figure 3.6: Another labeled prism graph when $E = C_2$

Proof of Theorem 2.2.3. Theorem 2.2.2 implies that the only edges of G that may be balanced are C_1, C_2, C_3, and D_1. If (G, ω) is as given in the theorem statement (i), then $\mathrm{spec}_{\omega(C_1)}(\mathrm{G}_1) = \{2\mathrm{a}_1 + \mathrm{a}_3 + \mathrm{b}_2 + \mathrm{c}_1, 4\mathrm{a}_1 + \mathrm{a}_3 + \mathrm{b}_2 + \mathrm{c}_1\}$ so that the edge C_1 is balanced. If (G, ω) is as given in the theorem statement (ii), then the edge C_2 is balanced by applying the weight permutation $(a_1, b_2)(a_2, b_1)(a_3, b_3)(c_1, c_2)(c_3)(d_1)$ associated with the edge isomorphism of G suggested by Figures 2.14 and 3.6. If the pair (G, ω) is as given in the statement of the theorem (iii), then $\mathrm{spec}_{\omega(C_3)}(\mathrm{G}_1) = \{\mathrm{a}_1 + \mathrm{a}_3 + \mathrm{b}_2 + \mathrm{c}_1 + \mathrm{c}_3, \mathrm{a}_1 + 3\mathrm{a}_3 + \mathrm{b}_2 + \mathrm{c}_1 + \mathrm{c}_3\}$ so that the edge C_3 is balanced.

We now consider the three cases where the edge E is C_1, C_2, or C_3. First suppose that $E = C_1$. Suppose that the edge C_1 is balanced but that the pair (G, ω) is not as given in the theorem statement (i). The cycles of G that contain the edges

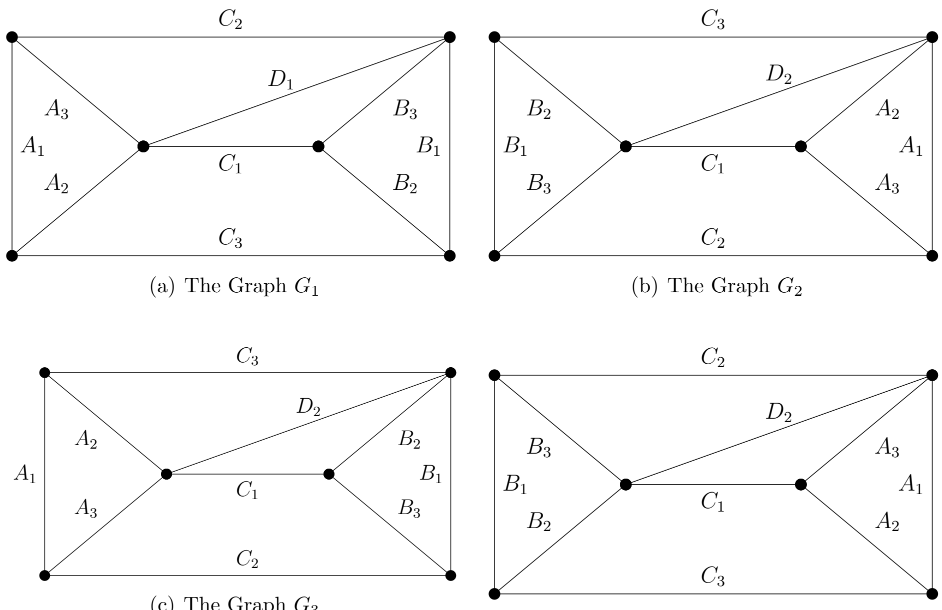
C_1 and D_1 are $\{B_3, C_1, D_1\}$, $\{B_1, B_2, C_1, D_1\}$, and $\{A_1, B_2, C_1, C_2, C_3, D_1\}$. Hence $\{b_3 + c_1 + d_1, b_1 + b_2 + c_1 + d_1, a_1 + b_2 + c_1 + c_2 + c_3 + d_1\} \subseteq \mathrm{spec}_{\omega(\mathrm{C}_1)}(\mathrm{G})$. So

(†) two of the three sums $b_3+c_1+d_1$, $b_1+b_2+c_1+d_1$, and $a_1+b_2+c_1+c_2+c_3+d_1$ must be the same.

It follows from Theorem 2.2.2 that $a_1 = b_1$ and $c_2 = c_3$ and either $a_2 = a_1+a_3$ and $b_3 = b_1 + b_2$ or $a_3 = a_1 + a_2$ and $b_2 = b_1 + b_3$. Suppose the latter holds. Then $b_3 + c_1 + d_1 < 2b_1 + b_3 + c_1 + d_1 = b_1 + b_2 + c_1 + d_1 = a_1 + b_2 + c_1 + d_1 < a_1 + b_2 + c_1 + c_2 + c_3 + d_1$. This contradicts (†).

We have established that the former holds and $a_2 = a_1+a_3$ and $b_3 = b_1+b_2$. So $\mathrm{spec}_{\omega(\mathrm{C}_1)}(\mathrm{G}) = \{\mathrm{a}_1+\mathrm{a}_3+\mathrm{b}_2+\mathrm{c}_1+\mathrm{c}_2, 3\mathrm{a}_1+\mathrm{a}_3+\mathrm{b}_2+\mathrm{c}_1+\mathrm{c}_2\}$. Now $\{b_1+b_2+c_1+d_1, a_1+b_2+c_1+c_2+c_3+d_1\} = \{a_1+a_3+b_2+c_1+c_2, 3a_1+a_3+b_2+c_1+c_2\}$ and the first listed element in each of these sets is smaller than the second listed element by the given conditions. So $b_1+b_2+c_1+d_1 = a_1+a_3+b_2+c_1+c_2$ implies that $d_3 = a_3+c_3$. Also, use the given conditions, $d_3 = a_3+c_3$, and $a_1+b_2+c_1+c_2+c_3+d_1 = 3a_1+a_3+b_2+c_1+c_2$ to obtain that $a_1 = c_3$. Thus the pair (G, ω) is as given in the theorem statement (i); a contradiction.

Next suppose that $E = C_2$. Then again the weight permutation $(a_1, b_2)(a_2, b_1)(a_3, b_3)(c_1, c_2)(c_3)(d_1)$ associated with the edge isomorphism of G suggested by Figures 2.14 and 3.6 implies that the pair (G, ω) is as given in the theorem statement (ii); a contradiction.



(a) The Graph G_1

(b) The Graph G_2

(c) The Graph G_3

(d) The Graph G_4

Figure 3.7: Several Labeled Prism+ Graphs

Finally suppose that $E = C_3$ and that the pair (G, ω) is not as given in the theorem statement (iii). The cycles of G that contain the edges C_3 and D_1 are $\{A_2, B_1, C_3, D_1\}$, $\{A_2, B_2, B_3, C_3, D_1\}$, $\{A_1, A_3, B_1, C_3, D_1\}$, $\{A_1, A_3, B_2, B_3, C_3, D_1\}$, and $\{A_1, B_2, C_1, C_2, C_3, D_1\}$. Then

(†) $\{a_2 + b_1 + c_3 + d_1, a_2 + b_2 + b_3 + c_3 + d_1, a_1 + a_3 + b_1 + c_3 + d_1, a_1 + a_3 + b_2 + b_3 + c_3 + d_1, a_1 + b_2 + c_1 + c_2 + c_3 + d_1\} \subseteq \mathrm{spec}_{\omega(\mathrm{C}_3)}(\mathrm{G})$.

Then Theorem 2.2.2 implies that $a_3 = b_3$ and $c_1 = c_2$ and either $a_2 = a_1 + a_3$ and $b_1 = b_2 + b_3$ or $a_1 = a_2 + a_3$ and $b_2 = b_1 + b_3$.

Suppose that the former holds. Then the given conditions and (†) imply that $\{a_1 + 2a_3 + b_2 + c_3 + d_1, a_1 + b_2 + 2c_1 + c_3 + d_1\} = \{a_2 + b_1 + c_3 + d_1, a_2 + b_2 + b_3 + c_3 + d_1, a_1 + a_3 + b_1 + c_3 + d_1, a_1 + a_3 + b_2 + b_3 + c_3 + d_1, a_1 + b_2 + c_1 + c_2 + c_3 + d_1\} \subseteq \mathrm{spec}_{\omega(\mathrm{C}_3)}(\mathrm{G}) = \{\mathrm{a}_1 + \mathrm{a}_3 + \mathrm{b}_2 + \mathrm{c}_1 + \mathrm{c}_3, \mathrm{a}_1 + 3\mathrm{a}_3 + \mathrm{b}_2 + \mathrm{c}_1 + \mathrm{c}_3\}$.

Thus $\{a_1 + 2a_3 + b_2 + c_3 + d_1, a_1 + b_2 + 2c_1 + c_3 + d_1\} \subseteq \{a_1 + a_3 + b_2 + c_1 + c_3, a_1 + 3a_3 + b_2 + c_1 + c_3\}$. The elements in the right set are distinct, while the elements in the left set may or may not be distinct. First assume that the elements in the left set are distinct. If the first two elements in each set are equal and the second two elements in each set are equal, then $d_1 = a_3$ and $c_1 = 2a_3$ so that the pair (G, ω) is as given in the theorem; a contradiction. If the first element in the first set is equal to the second element in the second set and the second element in the first set is equal to the first element in the second set, then the inconsistent pair of equations $d_1 = a_3 + c_1$ and $c_1 + d_1 = a_3$ hold; a contradiction. Second assume that the elements in the left set are equal. Then $a_3 = c_1$ Then $a_1 + a_3 + b_2 + c_1 + c_3 + d_1 = a_1 + 2a_3 + b_2 + c_3 + d_1 \in \{a_1 + a_3 + b_2 + c_1 + c_3, a_1 + 3a_3 + b_2 + c_1 + c_3\}$. This implies that $d_1 = 2a_3$ and so the pair (G, ω) is as given in the theorem statement; a contradiction.

We have shown that $a_1 = a_2 + a_3$ and $b_2 = b_1 + b_3$ together with $a_3 = b_3$ and $c_1 = c_2$. So $\{a_2 + b_1 + c_3 + d_1, a_2 + 2a_3 + b_1 + c_3 + d_1, a_2 + 2a_3 + b_1 + 2b_3 + c_3 + d_1\} \subseteq \{a_2 + b_1 + c_3 + d_1, a_2 + b_2 + b_3 + c_3 + d_1, a_1 + a_3 + b_1 + c_3 + d_1, a_1 + a_3 + b_2 + b_3 + c_3 + d_1, a_1 + b_2 + c_1 + c_2 + c_3 + d_1\} \subseteq \mathrm{spec}_{\omega(\mathrm{C}_3)}(\mathrm{G})$. However, the left set contains three distinct elements, while the right set only contains two; a contradiction.

Suppose $|\mathrm{spec}_{\omega(\mathrm{D}_1)}(\mathrm{G})| = 2$. Then the nine sums in Table 3.24 are of two different values. Let S and T be nonempty sets that partition $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ so that the elements of S have the same value in this table, while the elements of T have the other value in this table. Consequences of equal sums in this table are shown in Table 3.25 while pairs of sums that cannot be equal are given in Table 3.26.

Consider the edge isomorphism $(A_1, B_2)(A_2, B_1)(A_3B_3)(C_1, C_2)(C_3)(D_1)$. If we apply the associated weight permutation to the label set of Table 3.25 we obtain the following permutation that may be applied to these labels throughout the proof:

(†) (12)(35)(4)(67)(8)(9).

Suppose that $1 \in S$ without loss of generality. Assume that $3 \in S$. It follows from Table 3.26 that $4 \neq 6$ and $7 \neq 8$. Thus we obtain four cases to consider:

(α1) $(1347, 68) \subseteq (S, T)$

(α2) $(1348, 67) \subseteq (S, T)$

(α3) $(1367, 48) \subseteq (S, T)$

(α4) $(1368, 47) \subseteq (S, T)$

Assume that case (α1) holds. It follows from Table 3.26 row (xv) that $(13457, 689) \subseteq (S, T)$ or $(13479, 568) \subseteq (S, T)$. Suppose the former holds. It follows from row (ix) of that table that $(13457, 2689) \subseteq (S, T)$. This contradicts row (xiv). Hence the latter case holds. We obtain a contradiction from row (viii) of the table.

Assume that case (α2) holds. It follows from Table 3.26 row (xv) that $(13458, 679) \subseteq (S, T)$ or $(13489, 567) \subseteq (S, T)$ Suppose the former holds. It follows from row (xii) of the table that $(123458, 679) \subseteq (S, T)$. Then (6)=(9) in Table 3.25 implies that $a_3 + b_1 = b_2 + c_1 + c_2$. It follows from (2)=(8) that $c_1 = a_1 + a_3 + b_2 + c_3$. These two equations together yield $a_3 + b_1 = a_1 + a_3 + 2b_2 + c_2 + c_3 = (a_1 + c_2) + a_3 + 2b_2 + c_3$. It follows from (3)=(4) in the table that $a_1 + c_2 = b_1 + c_3$. Combine these two equations to obtain that $a_3 + b_1 = (b_1 + c_3) + a_3 + 2b_2 + c_3$. Hence $b_2 = 0$; a contradiction. Hence the latter case holds and $(13489, 567) \subseteq (S, T)$. This contradicts row (viii) of the table.

Assume that case (α3) holds. It follows from Table 3.26 row (iv) that (α3) $(13567, 489) \subseteq (S, T)$ or (α3) $(13679, 458) \subseteq (S, T)$. Suppose the former holds.

Then by row (xiii) of the table, $(123567, 489) \subseteq (S, T)$. It follows from (4)=(9) and (1)=(6), respectively, in Table 3.25 that $a_2+b_1 = a_1+b_2+c_1+c_2$ and $c_2 = a_1+b_1+c_3$. These two equations together $a_2 = 2a_1 + b_2 + c_1 + 2c_3$. Hence $a_2 > c_1$. However, (2)=(7) from Table 3.25 implies that $c_1 > a_2$; a contradiction. Hence the latter holds and $(13679, 458) \subseteq (S, T)$. This contradicts row (xvi) of Table 3.26.

We have shown that case ($\alpha 4$) $(1368, 47) \subseteq (S, T)$ holds. It follows from row (iv) of Table 3.26 that $(13568, 479) \subseteq (S, T)$ or $(13689, 457) \subseteq (S, T)$. Suppose the former case holds. It follows from row (v) of the table that $(123568, 479) \subseteq (S, T)$. This contradicts row (ix) of the table. Thus the latter case holds. This contradicts row (xvi) of the table.

We have shown that $1 \neq 3$. Hence (†) implies that $2 \neq 5$. Hence $(12, 35) \subseteq (S, T)$ or $(15, 23) \subseteq (S, T)$. Suppose the former holds. Assume 1=8. It follows from the table that $2 \neq 9$ and $3 \neq 7$. So $(1278, 359) \subseteq (S, T)$. This contradicts row (xi) of Table 3.26. Hence $(12, 358) \subseteq (S, T)$. Assume 1=9. Then $(129, 358) \subseteq (S, T)$. Hence row (viii) of Table 3.26 implies that $(1269, 358) \subseteq (S, T)$. It follows from row (xiv) of the table that $(12679, 358) \subseteq (S, T)$. Hence row (xii) of the table implies that $(124679, 358) \subseteq (S, T)$. But we established earlier in the proof that $4 = 9$, $2 = 7$, and $1 = 6$ cannot simultaneously occur. So $1 \neq 9$. Hence $(12, 3589) \subseteq (S, T)$. It follows from row (xvi) of Table 3.26 that $(124, 3589) \subseteq (S, T)$. Then row (iv) of the table implies that $(124, 35689) \subseteq (S, T)$. Then 2=4 and 3=9 in Table 3.25 imply that $b_3 = b_1 + b_2 + 2c_3$ so that $b_3 > b_1$. However, 6=8 in that table implies that $b_1 > b_3$; a contradiction.

We have established that $(15, 23) \subseteq (S, T)$. Assume 1=8. Then rows (v) and (vi) Table 3.26 imply that $(15789, 23) \subseteq (S, T)$. Then 5=9 and row (xv) of the table imply that $(145789, 23) \subseteq (S, T)$. So row (iv) of the table implies that $(145789, 236) \subseteq (S, T)$. Then 2=3 and 1=8 imply that $c_1 = 2a_1 + a_2 + b_2 + c_3 = 2a_1+b_2+(a_2+c_3)$. But 4=5 implies that $a_2+c_3 = b_2+c_1$. Hence $c_1 = 2a_1+b_2+(b_2+c_1)$ so that $b_2 = 0$; a contradiction. Thus $1 \neq 8$ and $(15, 238) \subseteq (S, T)$.

Row (vii) of Table 3.26 implies that $(15, 2389) \subseteq (S, T)$. Row (xiii) of Table 3.26 implies that $(15, 23689) \subseteq (S, T)$. Then row (xiii) of Table 3.26 implies that $(15, 234689) \subseteq (S, T)$. Then we obtain the triple 6=9, 2=8, and 3=4 that we have previously shown cannot occur. This completes the proof of Theorem 2.2.3. $\square$

Proof of Corollary 2.2.4. Recall that the weight of a path in a graph is the sum of the weights of its edges. Part (a) of the corollary follows from Theorem 2.2.3 where no edge of A_i or B_i is balanced in that theorem. Part (b) of the corollary follows by symmetry from Theorem 2.2.3 where the edge D_1 there is not balanced. $\square$

Proof of Theorem 2.2.5. It follows from Theorem 2.2.3 and symmetry that the only edges of G that may be balanced are C_1, C_2, and C_3. If the given conditions hold, then $\mathrm{Spec}_{\omega(C_1)}(G) = \{2a_1 + a_3 + b_2 + c_1, 4a_1 + a_3 + b_2 + c_1\}$.

Suppose that the edge C_1 is balanced and that the pair (G, ω) are not as given in the theorem statement (i). Then the graph G_1 in Figure 3.7 (a) is $\omega(C_1)$-balanced. Apply Theorem 2.2.3 (i) to the pair (G_1, ω) to obtain that that $a_1 = b_1 = c_2 = c_3$,

$a_2 = a_1 + a_3$ $b_3 = b_1 + b_2$, and $d_1 = a_3 + c_3$. The graph G_2 in Figure 3.7 (b) is $\omega(C_1)$-balanced. Apply Theorem 2.2.3 to the pair (G_2, ω) to obtain that $d_2 = b_2 + c_2$. Hence (G, ω) is as given in the theorem statement (i); a contradiction.

Suppose that the edge C_2 is balanced. Then the graph G_1 in Figure 3.7 (a) is $\omega(C_2)$-balanced. Apply Theorem 2.2.3 (ii) to the pair (G_1, ω) to obtain that $b_1 = b_2 + b_3$ among other weight conditions. The graph G_2 of Figure 3.7 (b) is $\omega(C_2)$-balanced. Apply Theorem 2.2.3 (iii) and the weight permutation of G_2 associated with the edge isomorphism between G_1 and G_2 given by $(A_1, B_1)(A_3, B_2)(A_2, B_3)(C_1)(C_2, C_3)(D_1, D_2)$ to transform the equation $a_2 = a_1 + a_3$ in that theorem statement to $b_3 = b_1 + b_2$. Hence $b_1 > b_3$ and $b_3 > b_1$; a contradiction. Hence the edge C_2 is not balanced. By symmetry, the edge C_3 is not balanced. This completes the proof of Theorem 2.2.5. □

Proof of Theorem 2.2.6. It follows from Theorem 2.2.3 that the only edges of G that may be balanced are C_1, C_2, and C_3. Let G_1 and G_3 be the subgraphs of G given in Figure 3.7 (a) and (c). Suppose that the edge C_1 is balanced. Then G_1 is $\omega(C_1)$-balanced. Apply Theorem 2.2.3 (i) to the pair (G_1, ω) to obtain that $a_2 = a_1 + a_3$. The graph G_3 is $\omega(C_1)$-balanced Apply Theorem 2.2.3 (i) to the pair (G_3, ω) and fix a_1 and interchange a_2 and a_3 in the equation $a_2 = a_1 + a_3$ to obtain that $a_3 = a_1 + a_2$. Then the two equations $a_2 = a_1 + a_3$ and $a_3 = a_1 + a_2$ are inconsistent so that the edge C_1 is not balanced.

Suppose that the edge C_2 is balanced. It was observed in the proof of Theorem 2.2.3 that $\mathrm{spec}_{\omega(\mathrm{C_2})}(\mathrm{G}) = \{\mathrm{2b_2 + a_1 + b_3 + c_2, 4b_2 + a_1 + b_3 + c_2}\}$. Apply Theorem 2.2.3 (ii) to the pair (G_1, ω) to obtain that $a_2 = b_2 = c_1 = c_3$, $a_3 = a_1 + a_2$, $b_1 = b_2 + b_3$, and $d_1 = b_3 + c_3$. It follows from G being $\omega(C_2)$-balanced that G_3 is $\omega(C_2)$-balanced. Then we obtain the following system of equations by applying the permutation $(a_1)(a_2, a_3)(b_1)(b_2, b_3)(c_1)(c_2, c_3)(d_1, d_2)$ to the conditions given in the statement of Theorem 2.2.3 (iii):

$a_2 = b_2$, $c_1 = c_3$, $a_3 = a_1 + a_2$, $b_1 = b_2 + b_3$, and either $d_2 = a_2$ and $c_1 = 2a_2$ or $d_2 = 2a_2$ and $c_1 = a_2$.

If $d_2 = a_2$ and $c_1 = 2a_2$, then the cycle $\{A_1, A_2, B_2, B_3, C_2, D_2\}$ of G has weight $a_1 + a_2 + b_2 + b_3 + c_2 + d_2 = 3b_2 + a_1 + b_3 + c_2 \notin \{2b_2 + a_1 + b_3 + c_2, 4b_2 + a_1 + b_3 + c_2\} = \mathrm{spec}_{\omega(\mathrm{C_2})}(\mathrm{G})$; a contradiction. Hence $d_2 = 2a_2$ and $c_1 = a_2$. Thus (G, ω) is as given in the theorem statement (i). In this case the edge C_2 is balanced.

If $E = C_3$, then the pair is as given in the theorem statement (iii) by applying the weight permutation $(a_1)(a_2, a_3)(b_1)(b_2, b_3)(c_1)(c_2, c_3)(d_1, d_2)$ to the statement of part (ii) of the theorem. □

Proof of Theorem 2.2.7. Only the edges C_1, C_2, and C_3 may be balanced by Theorem 2.2.3. Let the graphs G_1 and G_4 be as given in Figure 3.7 (a) and (d), respectively. Suppose the graph G is $\omega(C_1)$-balanced. Then the graph G_1 is $\omega(C_1)$-balanced. It follows from Theorem 2.2.3 that $a_2 = a_1 + a_3$. The graph G_4 is $\omega(C_1)$-balanced. It follows from Theorem 2.2.3 that $a_3 = a_1 + a_2$. These two equations are inconsistent. Hence the edge C_1 is not balanced. By symmetry, the edge C_2 is not balanced.

Suppose the edge C_3 is balanced. Then G_1 is $\omega(C_3)$-balanced. It follows from Theorem 2.2.3 (iii) that $a_2 = a_1 + a_3$. The graph G_4 is $\omega(C_3)$-balanced. Hence Theorem 2.2.3 (iii) implies that $a_1 = a_2 + a_3$. These two equations are inconsistent. Thus the edge C_3 is not balanced. □

Proof of Theorem 2.2.8. It follows from Theorems 2.2.3 and 2.2.5 that no edges of G are balanced. □

Proof of Theorem 2.2.9. It follows from Theorems 2.2.3 and 2.2.6 that the only edges of G that may be balanced are C_2 and C_3. First suppose that the edge C_2 is balanced. It follows from applying Theorem 2.2.6(i) that $a_2 = b_2 = c_1 = c_3$, $a_3 = a_1 + a_2$, $b_1 = b_2 + b_3$, $d_1 = b_3 + c_3$, and $d_2 = 2b_2$ Apply Theorem 2.2.5 and the weight permutation $(a_1, b_2)(a_2, b_1)(a_3, b_3)(c_1, c_2)(c_3)(d_1)(d_2, d_3)$ suggested by consideration of Figures 2.15 and 3.8 to obtain that $d_3 = a_1 + c_1$. Thus the pair (G, ω) is as given in the theorem statement. If the pair (G, ω) is as given in the theorem statement, then $\mathrm{spec}_{\omega(\mathrm{C_2})}(\mathrm{G}) = \{2\mathrm{b_2} + \mathrm{a_1} + \mathrm{b_3} + \mathrm{c_2}, 4\mathrm{b_2} + \mathrm{a_1} + \mathrm{b_3} + \mathrm{c_2}\}$.

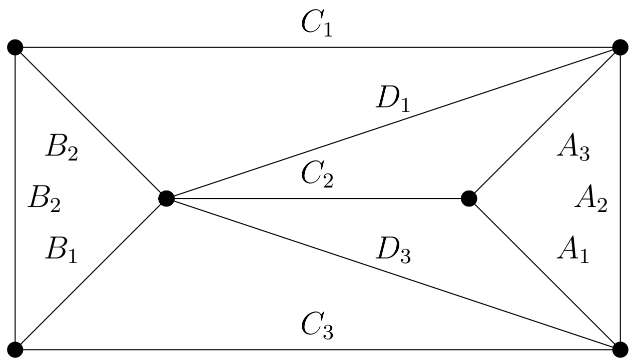


Figure 3.8: A relabeled extensions of the Prism graph

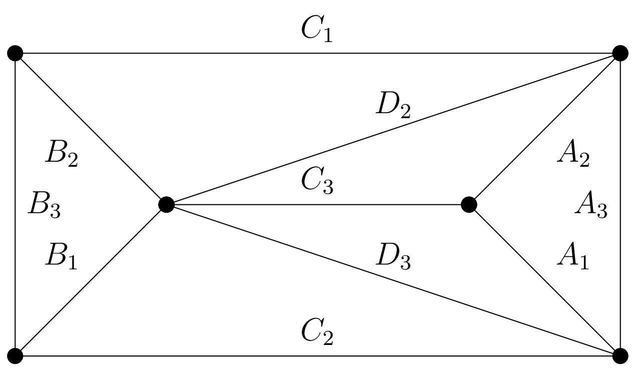


Figure 3.9: Another relabeled extensions of the Prism graph

Suppose that the edge C_3 is balanced. Then the edge C_3 is balanced in the subgraph of G given in Figure 3.9. This contradicts Theorem 2.2.6. □

Proof of Theorem 2.2.10. It follows from Theorem 2.2.5 that only the edge C_1 may be balanced. Suppose that the edge C_1 is balanced. Then the edge C_1 is balanced in the subgraph of G shown in Figure 3.11. This contradicts Theorem 2.2.5. Hence the edge C_1 is not balanced. □

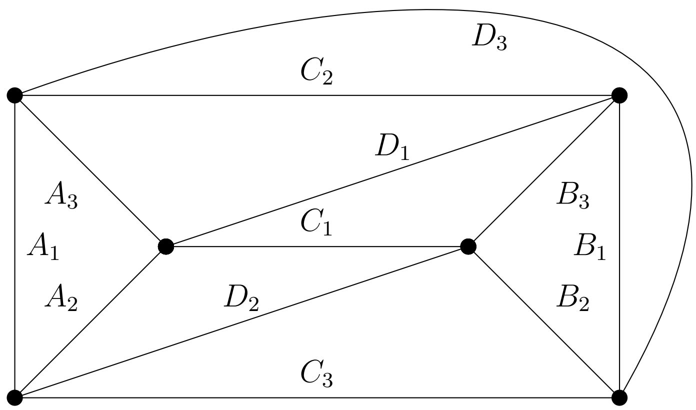


Figure 3.10: A three-edge extension of the Prism graph

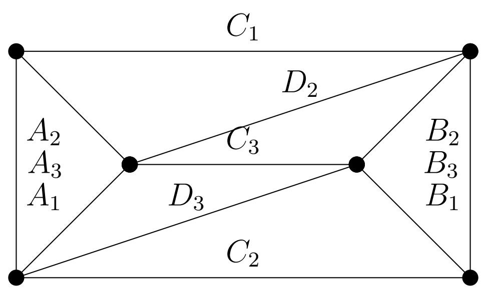


Figure 3.11: A relabled Prism1++ graph

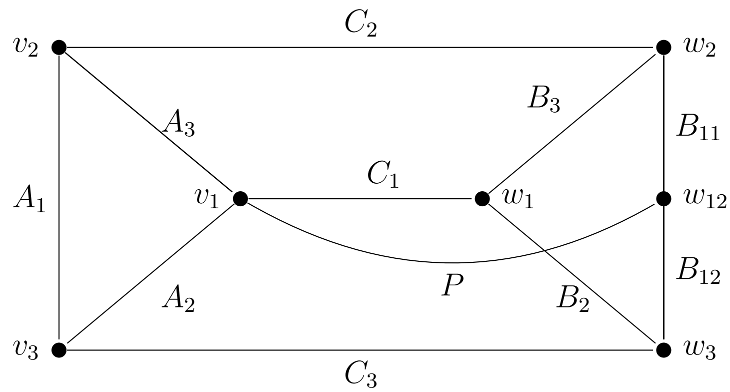


Figure 3.12: The edge C_1 is not balanced

Proof of Theorem 2.2.11. Suppose that E is a balanced edge of such a graph. A thirteen edge extension of a Prism graph is isomorphic to the Prism1+++, Prism2+++, or Prism3+++ graph. Hence G is obtained by adding edges to one of these three graphs. Thus there exists a fourteen edge subgraph H of G that uses the edge E and that is a single-edge extension of one of these three graphs. Each edge of a single-edge extension of a Prism1+++, Prism2+++, or Prism3+++ is in a subgraph of that graph that is isomorphic to the Prism1+++ graph. Hence the edge E is in a subgraph of G that is isomorphic to the Prism1+++ graph. Hence the edge E is not balanced in this subgraph. Thus the edge E is not balanced in G; a contradiction. $\square$

We next give a structural lemma that will be useful in many of the remaining proofs.

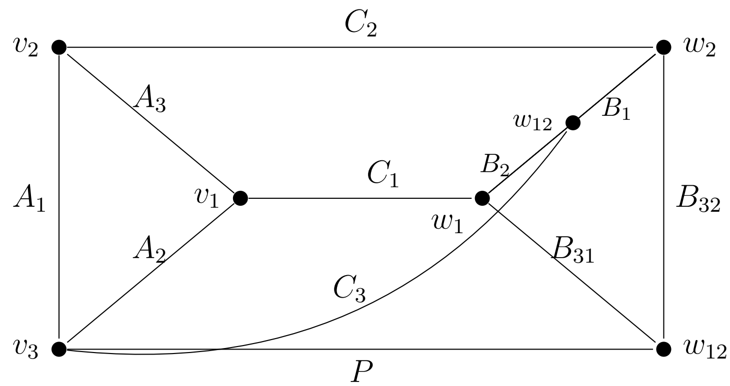


Figure 3.13: A redrawing of the graph Prism$^{\perp}$

Lemma 3.2.1. *Let G be the graph given in Figure 3.12 with associated weight function ω. Then the edge C_1 is not balanced.*

Proof of Lemma 3.2.1. Suppose that $E = C_1$ is balanced. Then Theorem 2.2.2 (iii) implies that $\mathrm{spec}_{\omega(E)}(\mathrm{G}) = \{\mathrm{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_1}\}$ where $a_1 = b_1$, $c_2 = c_3$, and either $a_3 = a_1 + a_2$ and $b_2 = b_1 + b_3$ or $a_2 = a_1 + a_3$ and $b_3 = b_1 + b_2$.

Then the three cycles $P \cup C_1 \cup B_3 \cup B_{11}$, $P \cup C_1 \cup B_2 \cup B_{12}$, and $P \cup C_1 \cup B_{12} \cup C_3 \cup A_1 \cup C_2 \cup B_3$ have weight in the set $S = \{p + c_1 + b_3 + b_{11}, p + c_1 + b_2 + b_{12}, p + c_1 + b_{12} + c_3 + a_1 + c_2 + b_3\}$. In either of the cases $a_3 = a_1 + a_2$ and $b_2 = b_1 + b_3$ or $a_2 = a_1 + a_3$ and $b_3 = b_1 + b_2$ the three elements of S are distinct as they can be listed in increasing order. Thus the edge C_1 is in cycles of at least three different weights; a contradiction. Hence the edge C_1 is not balanced. $\square$

Proof of Theorem 2.2.12. Suppose that the edge E is balanced. Then Theorem 2.2.2 implies that $E = C_i$ for $i \in [3]$. First suppose that $E = C_1$. It follows from Theorem 2.2.2 that $a_1 = b_1$, $c_2 = c_3$, $\mathrm{spec}_{\omega(E)}(G) = \{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_1\}$ and either $a_{31} + a_{32} = a_3 = a_1 + a_2$ and $b_2 = b_1 + b_3$ or $a_2 = a_1 + a_3$ and $b_3 = b_1 + b_2$. Consider the redrawing of the graph $\mathrm{Prism}^{\perp}$ of Figure 2.21 that is given in Figure 3.13. It follows from applying Theorem 2.2.2 to the redrawing that $b_{32} = a_1$, $p = c_2$, $b_1 = a_1 = b_{32}$, and $b_3 = b_{31} + b_{32} = b_1 + b_2$ so that $a_2 = a_1 + a_3$. Hence $b_{31} = b_2$. The cycle $P \cup B_{31} \cup C_1 \cup A_3 \cup C_2 \cup B_1 \cup C_3$ has weight in the set $\{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_1\}$ so that $a_1 = c_2$. Thus the pair (G, ω) is as given in (i) of the theorem statement. Likewise, by symmetry, if $E = C_2$, then the pair (G, ω) is as given in (ii) of the theorem statement. One can check that the conditions given in (i) or (ii) of the theorem statement hold, then $\mathrm{spec}_{\omega(E)}(G)$ is $\{a_3+b_3+c_1+c_2, a_3+b_3+c_1+c_2+2a_1\}$ or $\{a_3+b_3+c_1+c_2, a_3+b_3+c_1+c_2+2a_2\}$, respectively. The graph G can be redrawn as in Figure 3.12 with the edge C_3 in place of the edge C_1. Thus the edge C_3 is not balanced. $\square$

Proof of Theorem 2.2.13. It is straightforward to check that if the pair (G, ω) is as given in the theorem statement, then $\mathrm{spec}_{\omega(E)}(G) = \{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_1\}$ so that the edge E is balanced. Conversely, suppose that the edge E is balanced. Then Theorem 2.2.12 implies that $E = C_1$ or $E = C_2$. If the former holds, then all the conditions in the theorem statement not involving the quantity p_2 holds by applying Theorem 2.2.12 to the graph obtained by deleting the edge P_1.

The condition $p_2 = a_3 + c_2$ holds by Theorem 2.2.3. Hence the pair (G, ω) is as given in the theorem statement.

Suppose that $E = C_2$ is balanced. Then Theorem 2.2.12 implies that $a_1 > a_3$. Apply Theorem 2.2.3 (ii) to the graph G to obtain that $a_3 > a_1$; a contradiction

$\square$

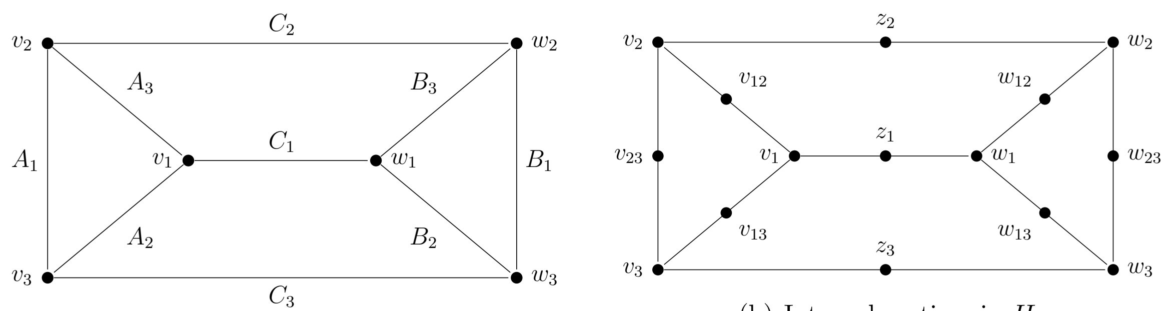
Proof of Theorem 3.2. It is straightforward to check that if the pair (G, ω) is as given in the theorem statement, then $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2, \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + 2\mathrm{a}_1\}$ so that the edge E is balanced. Conversely, suppose that the edge E is balanced. Then Theorem 2.2.12 implies that $E = C_1$ or $E = C_2$. If $E = C_1$, then the conditions in the Theorem statement hold by Theorems 2.2.12 and 2.2.3. One can check that the cycles that contain the edge P_2 all have weight either $a_3 + b_3 + c_1 + c_2$ or $a_3 + b_3 + c_1 + c_2 + 2a_1$ when these conditions hold.

Suppose that the edge C_2 is balanced . Then Theorem 2.2.12 (ii) implies that $p_1 = a_2 = b_2 = c_1 = c_3$, $b_{31} = b_1$, $b_{32} = b_2$, $a_1 = a_2 + a_3$, and $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2, \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + 2\mathrm{a}_2\}$. Apply the following permutation from left to right to the weights given in Theorem 2.2.3 (iii) $(c_3, c_2, c_1)(d_1, p_2)(a_1, a_3, a_2)(b_1, b_3, b_2)$ to obtain that either $p_2 = a_2$ and $c_3 = 2a_2$ or $p_2 = 2a_2$ and $c_3 = a_2$. This weight permutation is induced by the associated edge isomorphism of G that redraws the graph of Figure 2.23 as the graph in Figure 2.14. The former does not occur as $c_3 = a_2$. Hence the latter occurs and the pair (G, ω) is as given in (ii) of the theorem statement. Conversely, if condition (ii) of the theorem statement holds, then $\mathrm{spec}_{\omega(\mathrm{E})}(\mathrm{G}) = \{\mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2, \mathrm{a}_3 + \mathrm{b}_3 + \mathrm{c}_1 + \mathrm{c}_2 + 2\mathrm{a}_2\}$.

□

Proof of Theorem 2.2.16. It is straightforward to check that if the graph G satisfies the given conditions, then G has the stated spectrum. Now suppose that the edge E is balanced. Let G_i be the subgraph of G obtained by deleting the edge P_i for $i \in [2]$. It follows from applying Theorem 2.2.12 to G_1 that $E \in \{C_1, C_3\}$ and from applying Theorem 2.2.12 to G_2 that $E \in \{C_1, C_2\}$. Hence $E = C_1$, and moreover, the conditions of the theorem statement hold. □

Proof of Theorem 2.2.17 and 2.2.18. That a balanced edge in either of these graphs must satisfiy the theorem statements follows from Theorems 2.14 and 2.2.17. One can check that the edge C_1 is balanced if the weight function satisfies either of these conditions. □



(a) The Subgraph H

(b) Internal vertices in H

Figure 3.14: Labeled paths and vertices in the subgraph H

3.3 The main result

We prove the main result of the dissertation here. This proof relies on knowing the structure of balanced simple 3-connected graphs in both of the cases where the graph contains no two vertex-disjoint cycles and where the graph does not contain two vertex-disjoint cycles.

Proof of Theorem 2.3.1. Suppose that the graph G contains vertex-disjoint cycles and that (G, ω) is not as given in Theorems 2.1.3 through Theorem 2.2.18 or may be obtained from extensions of the Prism, Prism$^{\perp}$, or Prism$^{\perp\perp}$ graph by attaching handles.

If the graph G has six vertices, then G is obtained by adding edges to the Prism graph. Hence the pair (G, ω) is a balanced Prism, Prism+, Prism1++, Prism2++, or Prism2+++ graph by Theorems 2.2.2 through 2.2.11; a contradiction. Hence the graph G has seven or more vertices as well as a pair of disjoint cycles.

Let H be a subgraph of G that is a subdivision of a Prism. The existence of H is guaranteed by Menger's Theorem. Suppose that the subgraph H is as pictured in Figure 3.14 (a). There the uppercase letters label paths of G so that A_1 labels a path between the vertices v_2 and v_3, for example. Hence the subgraph H consists of two cycles $A_1 \cup A_2 \cup A_3$ and $B_1 \cup B_2 \cup B_3$ joined by three disjoint paths C_1, C_2, and C_3. Now each of the nine paths listed above may contain more than one edge. In this case the path contains at least one internal vertex. Even though there may be several internal vertices on these nine paths, we use a placeholder label v_{ij}, w_{ij}, or z_i to denote a canonical internal vertex of the path in question (see Figure 3.14 (b)). We use v_{ij} to denote an internal vertex on the path from v_i to v_j. We use w_{ij} to denote an internal vertex on the path from w_i to w_j. We use z_i to denote an internal vertex of the path C_i from vertex v_i to w_i . So the vertex v_{12} lies on the path A_3 of H where the path A_3 has endvertices v_1 and v_2. The vertex z_2 lies on the path C_2 from vertex v_2 to vertex w_2, for example.

Figure 3.14 (a) and (b) illustrate the placement of these nine internal vertices on the nine paths that comprise the edge set of H. If an internal vertex v_{ij}, w_{ij}, or z_i exists, then the two induced subpaths of A_i, B_i, or C_i, respectively are labeled by by double scripts as in Figure 3.15. For example, the vertex v_{23} will partition the path A_1 into subpaths A_{11} and A_{12} and the vertex z_2 will partition the path C_2 into subpaths C_{21} and C_{22} as shown in Figure 3.15. So $A_i = A_{i1} \cup A_{i2}$, $B_i = B_{i1} \cup B_{i2}$,

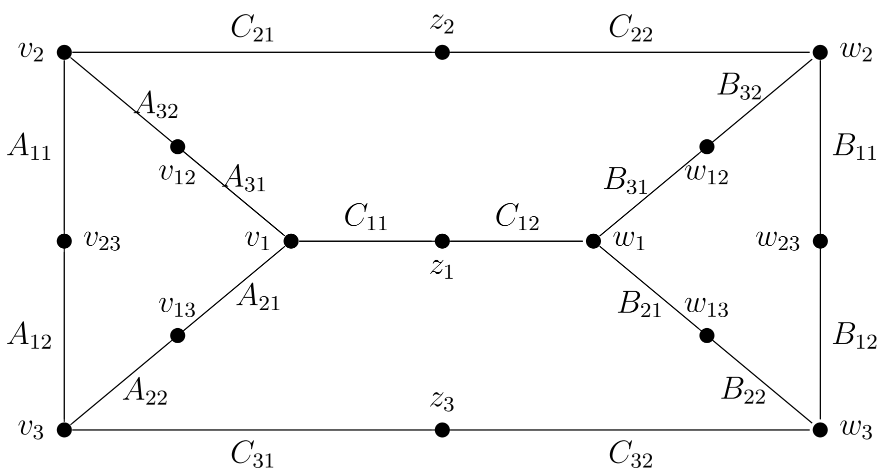


Figure 3.15: Subpaths of the paths of H

and $C_i = C_{i1} \cup C_{i2}$ for $i \in [3]$. Although the labeling in Figure 3.15 appears quite dense, we rarely consider more than two of these internal vertices in each case. We follow our usual convention of using a lower case letter to represent the weight of a path that is labeled by the corresponding upper case letter. So the path A_1 will have weight a_1 while the subpaths A_{11} and A_{12} will have weight a_{11} and a_{12}, respectively, for example.

We first discuss some terminology used to refer to an H-bridge of G. Recall that a path P in G is called an H-bridge when the distinct endvertices of P, say v and w, lie in the vertex set of H and the internal vertices of P are disjoint from the vertex set of H. Then we will write $P = P[v, w]$ even though there may many such paths between the vertex v and the vertex w. Throughout the proof, $P = P[v, w]$ will denote an H-bridge of G with v and w being distinct members of $\{v_1, v_2, v_3, w_1, w_2, w_3, z_1, z_2, z_3, v_{12}, v_{13}, v_{23}, w_{12}, w_{13}, w_{23}\}$where the vertices can be

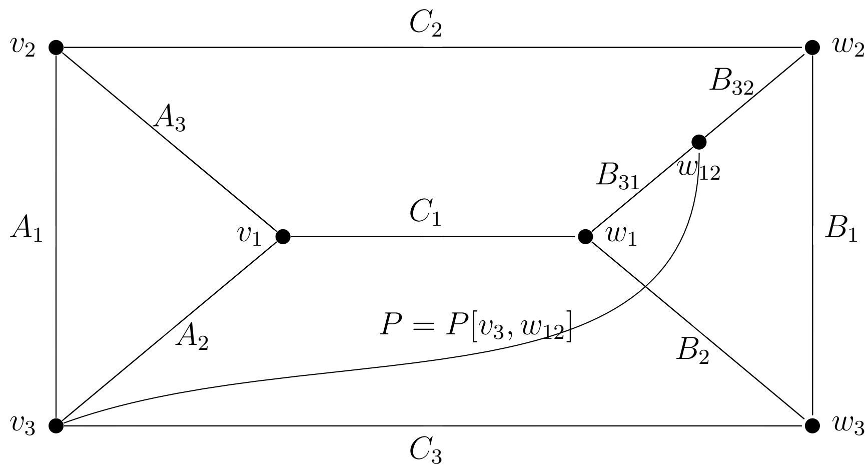


Figure 3.16: The graph $H \cup P[v_3, w_{12}]$

found in Figure 3.14 (b) and 3.15. The first claim will be a key part of the proof of the main theorem.

Claim 3.3.1. *No edge of the H-bridge $P = P[v, w]$ is balanced.*

Proof of Claim 3.3.1. Suppose that the balanced edge E in on the path $P = P[v, w]$ First assume that $v = v_3$. If w is in $\{v_1, v_2, w_3, z_1, z_2, z_3, v_{12}, v_{13}, v_{23}, w_{13}, w_{23}\}$, then there exist disjoint cycles of G with one of the cycles containing the edge E. This contradicts Corollary 2.2.4. If w is in $\{w_1, w_2\}$, then the edge E lies on a chord the subgraph H. This also contradicts Corollary 2.2.4. Hence we may assume that $w = w_{12}$. Then the graph H together with the H-bridge P is as given in Figure 3.16. It follows from Theorem 2.2.12 and 2.2.1 that no edge of P is balanced; a contradiction. Hence $w \neq w_{12}$. Hence $v \neq v_3$. Moreover, by symmetry, neither v nor w is in $\{v_1, v_2, v_3, w_1, w_2, w_3\}$.

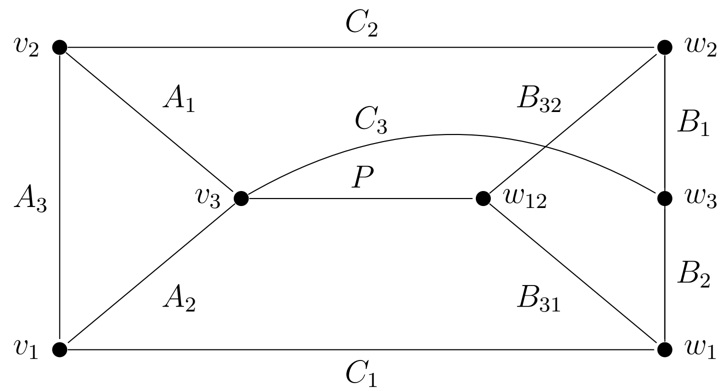


Figure 3.17: A redrawing of the graph $H \cup P[v_3, w_{12}]$

We may assume that $v \in \{v_{23}, z_3\}$ by symmetry. If $v = v_{23}$, then Corollary 2.2.4 implies that w cannot be any of the vertices labeled in Figure 3.14 other than w_1 as otherwise the edge E of the path P is in a cycle that is disjoint from another cycle of G. However, we have shown that neither v nor w is a vertex w_i; a contradiction. Suppose that $v = z_3$. Then Corollary 2.2.4 (a) implies that w cannot be any of the other vertices labeled in Figure 3.14 (b); a contradiction. Hence no edge of an H-bridge P is balanced. This completes the proof of Claim 3.3.1. $\square$

Suppose that the edge E is not in the subgraph H. Then Menger's Theorem implies that the edge E is in an H-bridge of G. This contradicts Claim 3.3.1. Hence E is an edge of H. It follows from Theorem 2.2.2 that we may assume that the edge E lies on the path C_1 subgraph H given in Figure 3.14. Hence the following claim holds by Theorem 2.2.2 (i).

Claim 3.3.2. *The following weight equations hold for the subgraph H:*
$a_1 = b_1$, $c_2 = c_3$, $\mathrm{spec}_\omega(\mathrm{E})(\mathrm{G}) = \{\mathrm{a_3 + b_3 + c_1 + c_2, a_3 + b_3 + c_1 + c_2 + 2a_1}\}$, *and either*

(i) $a_3 = a_1 + a_2$ *and* $b_2 = b_1 + b_3$, *or*

(ii) $a_2 = a_1 + a_3$ *and* $b_3 = b_1 + b_2$.

The next claim will be useful in the remainder of the proof.

Claim 3.3.3. *Let Z be a subgraph of G such that Z is a subdivision of a Prism. Suppose that P_1, P_2, and P_3 are the three pairwise disjoint paths that join the two vertex-disjoint cycles of Z. Then the edge E lies on one of these paths, say P_1. Moreover, the paths P_2 and P_3 have the same weight.*

Proof of Claim 3.3.2. It follows from Claim 3.3.1 and Menger's Theorem that E is an edge of Z. The result follows from Theorem 2.2.2. □

We next show that each of the nine paths of H listed in Figure 3.14 (a) consist of a single edge except in the case that exactly one of the paths A_2, A_3, B_2, and B_3 consists of two edges. Thus either none of the internal vertices v_{ij}, w_{ij}, and z_i listed in Figure 3.14 (b) will exist or exactly one vertex will exist in the set $\{v_{12}, w_{12}, v_{13}, w_{13}\}$. It follows from the assumptions, the 3-connectivity of G, and Menger's Theorem that the H-bridge P exists. Suppose that an internal vertex on C_2 exists so that $v = z_2$. Then we may assume that $w \in \{v_1, v_3, w_1, w_3, v_{12}, w_{12}, v_{23}, w_{23}, v_{13}, w_{13}, z_1, z_3\}$.

If $w \in \{v_{12}, w_{12}, v_{23}, w_{23}\}$, then there are two disjoint cycles in G with one of the cycles containing the edge E. This contradicts Corollary 2.2.4 (a). Recall that the edge E lies on the path C_1. If $w \in \{v_1, w_1, v_3, w_3\}$, then there is a pair of vertex-disjoint cycles with three disjoint paths between the cycles where one of the paths contains the edge E, another path is the path C_3, and a third path is a proper subpath of C_2. This contradicts Claim 3.3.3. Suppose that $w \in \{v_{13}, w_{13}\}$. By symmetry, suppose that $w = v_{13}$. Then again, there is a pair of vertex-disjoint cycles, namely $P \cup C_{21} \cup A_1 \cup A_{22}$ and $B_1 \cup B_2 \cup B_3$, joined by three disjoint paths $C_1 \cup A_{21}$, C_{22} and C_3. This contradicts Claim 3.3.3 as $c_{22} < c_3$. Thus $w \notin \{v_{13}, w_{13}\}$.

Suppose $w = z_1$. Then in either of the cases where the edge E lies between the vertices v_1 and z_1 or lies between the vertices z_1 and w_1 on the path C_1 we obtain a contradiction to Corollary 2.2.4 (a). Hence we may assume that $w = z_3$ and that $H \cup P$ is as shown in Figure 3.18 where the path P of weight p is the curved path from the vertex z_2 to the vertex z_3. It follows from applying Claim 3.3.3 to the pair of cycles $P \cup A_2 \cup A_3 \cup C_{21} \cup C_{31}$ and $B_1 \cup B_2 \cup B_3$ that $c_{22} = c_{32}$. By symmetry, $c_{21} = c_{31}$. Hence $c_2 = c_3$ implies that $c_{21} = c_{22} = c_{31} = c_{32} = \frac{1}{2}c_2$. It follows from applying Theorem 2.2.2 to these pair of cycles that $p = a_1$ as the edges P and A_1 are opposite from the path joining the two cycles that contains the edge E. We may assume by symmetry that case (i) of Claim 3.3.2 holds. Hence $a_3 = a_1 + a_2$ and $b_2 = b_1 + b_3$. Then the cycle $C_1 \cup A_3 \cup A_1 \cup C_{31} \cup P \cup C_{22} \cup B_1 \cup B_2$ has weight at most $a_3 + b_3 + c_1 + c_2 + 2a_1$ by Claim 3.3.2. Hence $c_1 + a_3 + a_1 + \frac{1}{2}c_2 + a_1 + \frac{1}{2}c_2 + b_1 + b_2 = c_1 + a_3 + a_1 + c_{31} + p + c_{22} + b_1 + b_2 \leq a_3 + b_3 + c_1 + c_2 + 2a_1$ so that $b_1 + b_2 \leq b_3$. This contradicts that $b_2 = b_1 + b_3$. Hence $w \neq z_3$ and the interior vertex z_2 does not exist. Thus the path C_2, and by symmetry the path C_3, consists of a single edge (see Figure 3.19).

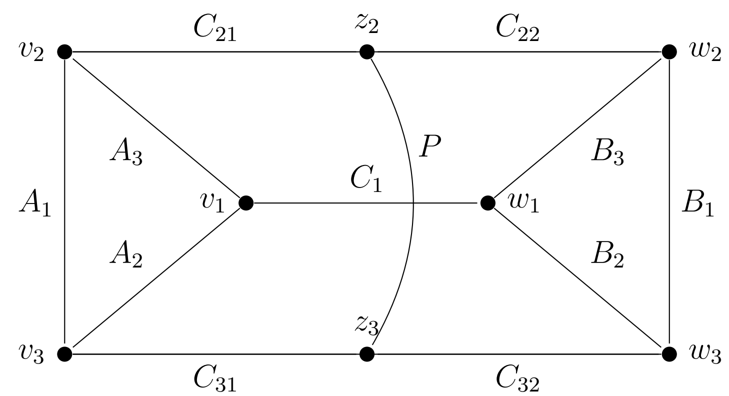


Figure 3.18: Adding a path to the subgraph H

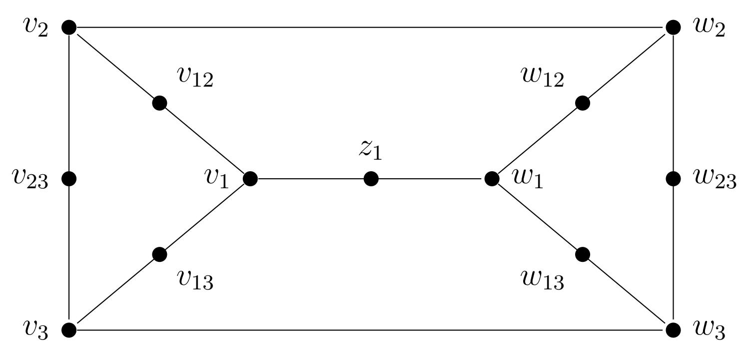
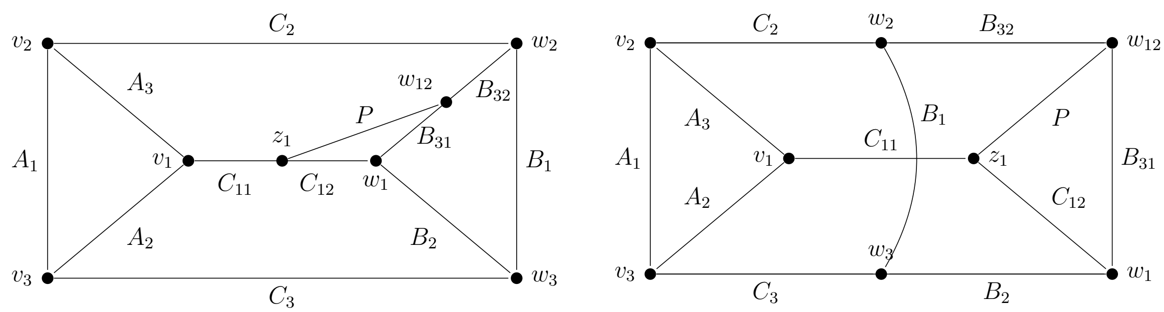


Figure 3.19: The subgraph H again



(a) There exists a path from z_1 to w_{12}

(b) A redrawing of the graph in (a)

Figure 3.20: The path C_{11} contains a balanced edge

Now suppose that the path C_1 contains at least two edges so that we may assume that an internal vertex z_1 exists. Let C_{11} and C_{12} be the subpaths of C_1 from v_1 to z_1 and from z_1 to w_1, respectively. Suppose that the edge E is on the subpath of C_1 from v_1 to z_1, i.e. E is an edge of the path C_{11}. Suppose that $v = z_1$. Then we may assume that $w \in \{v_2, v_3, w_2, w_3, v_{12}, v_{13}, v_{23}, w_{12}, w_{13}, w_{23}\}$. If $w \in \{v_2, v_3, v_{12}, v_{13}, v_{23}, \}$. Then there exists a cycle that contains the edge E and a disjoint cycle that does not meet this cycle contradicting Corollary 2.2.4.

Thus $w \in \{w_2, w_3, w_{12}, w_{13}, w_{23}\}$. Suppose that $w = w_{12}$ (see Figure 3.20) (a). Then redraw the graph in Figure 3.20 (a) as in Figure 3.20 (b). We have already shown that an edge of C_{11} in the right graph of Figure 3.20 cannot be balanced (see Figure 3.18). Hence $w \neq w_{12}$. By symmetry, $w \neq w_{13}$. Assume that $w = w_2$.

Now consider the graph $H \cup P$ shown in Figure 3.21. The cycles $A_1 \cup A_2 \cup A_3$ and $C_{12} \cup B_3 \cup P$ are joined by the disjoint paths C_{11} (that contains E), C_2 and $B_2 \cup C_3$. Hence $c_2 = b_2 + c_3$. Then $c_2 = c_3$ implies that $b_2 = 0$; a contradiction. Hence $x \neq w_2$.

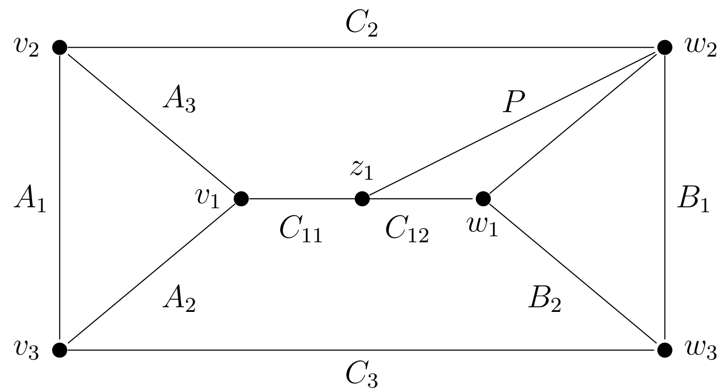


Figure 3.21: There is a path from z_1 to w_2

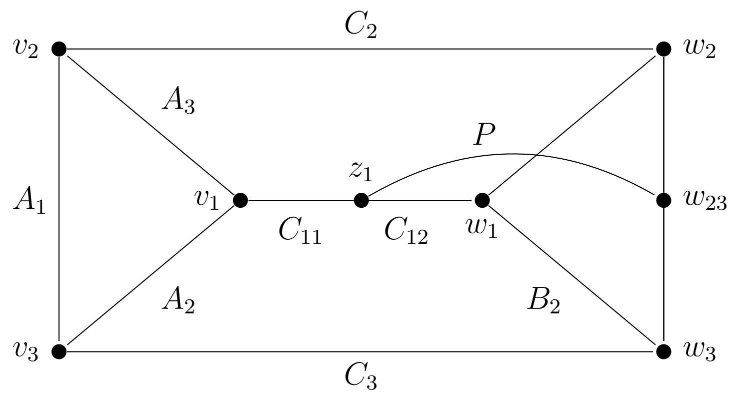


Figure 3.22: There is a path from z_1 to w_{23}

By symmetry, $w \neq w_3$. Hence $w = w_{23}$ (see Figure 3.22). Then redraw the graph in Figure 3.22 minus the subpath C_{12} to obtain the graph in Figure 3.23. We obtain that $b_1 = a_1 = b_2 + b_3$ by considering the left vertical path and the right vertical path of the graph. This contradicts that either $b_2 = b_1 + b_3$ or $b_3 = b_1 + b_2$ of Claim 3.3.2. Hence $v \neq w_{23}$ and the interior vertex z_1 does not exist.

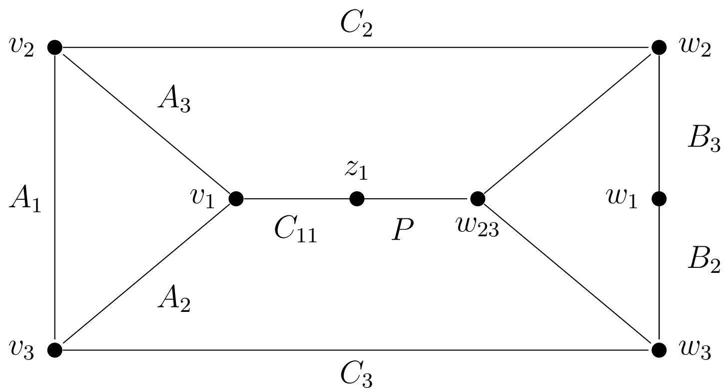


Figure 3.23: The left hand right paths have the same weight

It follows from Claim 3.3.1 and that the interior vertices z_1, z_2, and z_3 do not exist that we have established the following powerful observation.

Claim 3.3.4. *If U_1 and U_2 are disjoint cycles of G, then the edge E joins a vertex of U_1 to U_2. Any other pair of disjoint paths from U_1 to U_2 that are also vertex-disjoint from E have length one.*

Now suppose that an interior vertex $v = v_{23}$ exists. Then, by symmetry, we may assume that $w \in \{v_1, v_{12}, w_1, w_2, w_{12}, w_{23}\}$. It follows from Corollary 2.2.4 (a) that $w \notin \{v_{12}, w_2, w_{12}, w_{23}\}$. Suppose that $w = v_1$. Consider the cycle $B_1 \cup B_2 \cup B_3$ and the cycle $A_2 \cup P \cup A_{12}$. Then Claim 3.3.3 implies that the path $A_{11} \cup C_2$ consists of a single edge; a contradiction. Hence $w \neq v_1$. So $w = w_1$. But we obtain a graph that is isomorphic to the graph given in Figure 3.17. Then no edge of P is balanced by Lemma 3.2.1. So $w \neq w_1$. Thus $v \neq v_{23}$. By symmetry, $v \neq w_{23}$.

Suppose that an interior vertex $v = v_{12}$ exists. Then we may assume that $w \in \{v_3, v_{13}, w_1, w_{12}, w_2, w_{13}, w_3\}$. It follows from Corollary 2.2.4 (a) that $w \notin \{w1, w_{12}, w_2, w_{13}\}$. Suppose that $w = v_{13}$. Then consider the cycle $B_1 \cup B_2 \cup B_3$ together with the vertex-disjoint cycle formed from P together with the subpath of A_3 from v_{12} to v_1 and the subpath of A_2 from v_1 to v_{13} (see Figure 3.14). Then there are three vertex-disjoint paths between these two cycles with one of the paths being C_1. Another one of the paths is obtained by traversing A_3 from v_{12} to v_2 and then traversing C_2. The latter path must consist of a single edge by Claim 3.3.3; a contradiction. Hence $w \neq v_{13}$. It follows that $w = v_3$.

Claim 3.3.5. *The only internal vertices of the paths A_i, B_i, and C_i for $i \in [3]$ that may exist are in the set $\{v_{12}, v_{13}, w_{12}, w_{13}\}$. If such an internal vertex exists, then it is the only internal vertex on that path.*

(i) If v_{12} exists, then $a_3 > a_2$, $b_2 > b_3$, and there is an edge of G from v_{12} to w_3

(ii) If v_{13} exists, then $a_2 > a_3$, $b_3 > b_2$, and there is an edge of G from v_{13} to w_2.

(iii) If w_{12} exists, then $a_2 > a_3$, $b_3 > b_2$, and there is an edge of G from w_{12} to v_3.

(iv) If w_{13} exists, then $a_3 > a_2$, $b_2 > b_3$, and there is an edge of G from w_{13} to v_2.

(v) If a pair of these internal vertices exists, then the pair is either $\{v_{12}, w_{13}\}$ or $\{v_{13}, w_{12}\}$.

Proof of Claim 3.3.5. We have shown that the vertices listed are the only possible internal vertices of the paths of H in the previous arguments. Suppose that the

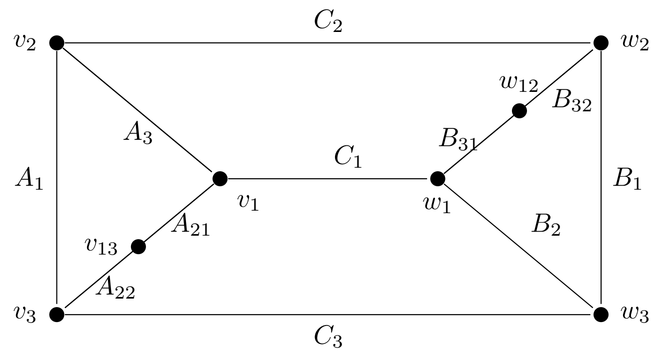


Figure 3.24: Possible vertices of the subgraph H

vertex v_{12} exits. Then we have shown that $P = P[v, w] = P[v_{12}, w_3]$. It follows from Theorem 2.2.12 that $a_3 = a_1 + a_2$ and $b_2 = b_1 + b_3$. Moreover, $a_{32} = a_1$. If another internal vertex v'_{12} existed on the path A_3, then the resulting subpath A'_{32} of A_3 would also have weight a_1 by Theorem 2.2.12. Then the weight of the subpath of A_3 from v_{12} to v'_{12} would be zero; a contradiction. Hence v_{12} is the only internal vertex on A_3 if it exists. By symmetry, each of the vertices v_{12}, v_{13}, w_{12}, w_{13} is unique if it exists. Conditions (i) through (iv) all follow from Theorem 2.2.12. The truth of conditions (i) through (iv) imply the truth of condition (v). $\square$

It follows from the above claim that we may assume that the vertex set of H consists of at most eight vertices, namely v_1, v_2, v_3, w_1, w_2, w_3 and possibly the vertices v_{13} and w_{12}, see Figure 3.25. Now let v be a vertex of G that is not in H. By Menger's Theorem there exists three internally disjoint paths Q_i for $i \in [3]$ from v to distinct vertices x_i of H.

Suppose that $\{x_1, x_2\} = \{w_2, w_3\}$. If $x_3 = v_1$, then we obtain a contradiction by Lemma 3.2.1. If $x_3 = w_1$, then we obtain a contradiction by Claim 3.3.4. If $x_3 \in \{v_2, v_3\}$, then we obtain a contradiction by Corollary refnobalance. Hence at most one vertex x_i is in the set $\{w_2, w_3\}$. By symmetry, at most one vertex x_i is in the set $\{v_{2,3}\}$. Likewise, Corollary 2.2.4 implies that at most one vertex is in each of the sets $\{v_i, w_i\}$ for $i \in [3]$. Hence we obtain four cases for the paths Q_i by these conditions. We may suppose that condition (i) holds by symmetry. Then conditions (ii) and (iii) cannot hold by Claim 3.3.4. So for each vertex v outside of H, either condition (i) or (iv) holds so that a triad is attached to the given vertices. In summary, we have shown that G is an extension of the Prism graph, the Prism$^{\perp}$ graph, or the Prism$^{\perp\perp}$ graph, or, up to relabeling, is obtained from an extension of these graphs by adding triads as in (i) or (iv). Hence the pair (G, ω) is as given in the theorem statement; a contradiction. This completes the proof of the main result.

(i) $\{x_1, x_2, x_3\} = \{v_1, v_3, w_2\}$

(ii) $\{x_1, x_2, x_3\} = \{v_1, v_2, w_3\}$

(iii) $\{x_1, x_2, x_3\} = \{v_2, w_1, w_3\}$

(iv) $\{x_1, x_2, x_3\} = \{v_3, w_1, w_2\}$

$\square$

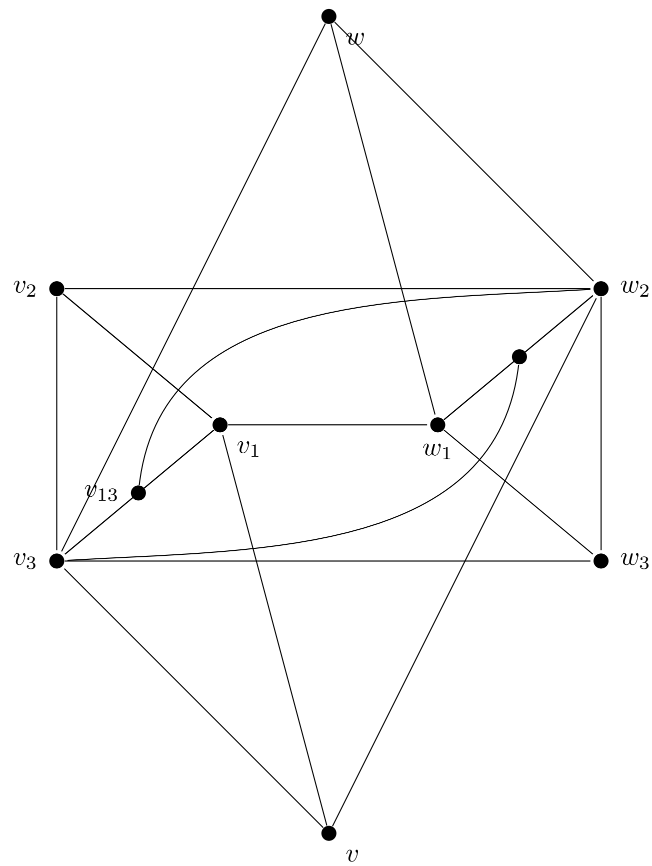


Figure 3.25: Adding triads to $H \cong Prism^{\perp\perp}$

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VITA

MILISHA YVONNE HART-SIMMONS

- 630 Edward Cove
- Jackson, Mississippi 39212
- (601) 573-1155
- Milisha.Hart@gmail.com

EXPERIENCE:

- 2014 2016 Graduate Assistant, University of Mississippi
- 2013 2016 Adjunct, Hinds Community College (Jackson Campus), Jackson, MS
- 2004 2010 Instructor , Jackson State University
- 2003 Graduate Assistant, Jackson State University

- 2001 2004 Instructor, Hardy Middle School, Jackson, MS

EDUCATION:

- 2014 - Present, Doctoral Student, Department of Mathematics, The University of Mississippi
- 2011 - 2014, Masters Student, Department of Mathematics, The University of Mississippi
- 2010 - 2011, Western Michigan University, Graduate Student
-

HONORS & ACTIVITIES

- Alpha Lambda Delta Chi Alpha Epsilon
- Habitat for Humanity
- National Deans List Scholar Whos Who Among Colleges and Universities