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Orthogonal Polynomials On An Arc Of The Unit Circle With Respect To A Generalized Jacobi Weight: A Riemann-Hilbert Method Approach

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ORTHOGONAL POLYNOMIALS ON AN ARC OF THE UNIT CIRCLE WITH RESPECT TO A GENERALIZED JACOBI WEIGHT: A RIEMANN-HILBERT METHOD APPROACH

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ABSTRACT

We investigate the asymptotic behavior of polynomials orthogonal over an arc of the unit circle $\gamma := \{z = e^{i\theta} : \theta_0 \le \theta \le 2\pi - \theta_0,\ 0 < \theta_0 < \pi\}$, with respect to a generalized Jacobi-type weight $|z - e^{i\theta_0}|^{2\alpha}|z - e^{-i\theta_0}|^{2\beta}h(z)$, where $\alpha, \beta > -\frac{1}{2}$, and h is a positive analytic weight on the arc. Full asymptotic expansions for the orthogonal polynomials are obtained at every point of the complex plane, extending previous results by Krasovsky [18] for the case $\alpha = \beta = 0$. Our results also extend those of Kuijlaars, McLaughlin, Van Assche, and Vanlessen [20] for polynomials orthogonal with respect to Jacobi weights on the real segment $[-1, 1]$. Our method of proof is based on a characterization, due to Baik, Deift, and Johansson [3], of the orthogonal polynomials as solutions of a 2×2 matrix Riemann-Hilbert problem, which extends to the unit circle the original Riemann-Hilbert characterization for orthogonal polynomials on the real line, first discovered by Fokas, Its, and Kitaev in [11]. In order to extricate the behavior of the polynomials from its Riemann-Hilbert matrix representation, we follow the steepest descent method of matrix transformations developed by Deift and Zhou in [9].

DEDICATION

For my grandmother, Betty Brooks

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Chapter 1 INTRODUCTION AND STATEMENTS OF THE MAIN RESULTS

1.1 Orthogonal polynomials

For a given positive, finite Borel measure μ in the complex plane $\mathbb{C}$ whose support is a compact set containing infinitely many points, there exists a unique sequence $\{\varphi_n\}_{n=0}^{\infty}$ of polynomials of a complex variable z with the property that

$$\varphi_n(z) = z^n + \cdots$$

is a polynomial of degree n and leading coefficient 1, and

$$\int \varphi_n \overline{\varphi_m} d\mu = 0, \quad n \neq m.$$

These polynomials φ_n are called the *monic orthogonal polynomials* associated with μ.

The polynomial φ_n has an important extremal property. Among all monic polynomials of degree n, φ_n is the only one with smallest L^2_μ-norm, that is, φ_n is characterized by the property that

$$\int |\varphi_n|^2 d\mu = \min_{P(z)=z^n+\cdots} \int |P|^2 d\mu.$$

Because the support of the measure μ contains infinitely many points, it is easily verified that the above minimum is strictly positive. If we now define

$$\kappa_n := \left(\int |\varphi_n|^2 d\mu \right)^{-1/2}, \quad \Phi_n(z) := \kappa_n \varphi_n(z), \quad n \geq 0,$$

then the polynomials Φ_n are *orthonormal* with respect to μ, that is,

$$\int \Phi_n \overline{\Phi_m} d\mu = \delta_{nm}, \quad n,\ m \geq 0.$$

The study of orthogonal polynomials is a vast and beautiful theory with important applications to many other branches of mathematics such as rational approximation, harmonic analysis, number theory, numerical analysis, and random point processes, to name a few. They have demonstrated utility outside the realm of pure mathematics. For example, they appear in classical mechanics, optics, and electrical engineering. The origin of this subject goes back to the eighteenth century, when Legendre was analyzing the motion of heavenly bodies. More recently, orthogonal polynomials have been found extremely useful in the study of random matrices [10, 21, 22]. The survey by Totik [31], meant for non-experts, offers a good review of different aspects of the theory of orthogonal polynomials.

The great applicability of orthogonal polynomials resides in the freedom of choosing the orthogonality measure μ. The theory is particularly rich when the support of μ is either a subset of the real line (this includes the classical families of Jacobi, Legendre, Laguerre, and Hermite), or when the support of the measure is the whole unit circle (see, for instance [14, 17, 28–30]). The richness of the theory emanates from the symmetry of the real line and of the unit circle, and in the case of a measure whose support is the unit circle, having at our disposal the whole machinery of complex function theory in the unit disk (e.g., Hardy spaces, Poisson representation of harmonic functions, etc).

A major breakthrough in the theory of orthogonal polynomials came in the papers [9,11]. It was proven in [11] that orthogonal polynomials on the real line can be characterized via the solution of a 2×2 matrix Riemann-Hilbert problem. Using this characterization, a steepest descent method was developed in [9] to obtain full asymptotic expansions for the orthogonal polynomials in every region of the complex plane. Based on this method, many new results have been obtained for both orthogonal polynomials on the real line (e.g., [8, 12, 20]) and on the unit circle [18, 24–26].

Orthogonal polynomials over more general curves of the complex plane have been studied to a certain extent. Here, the focus is mainly on their asymptotic behavior, since the great majority of algebraic properties that orthogonal polynomials on the real line or on the unit circle enjoy are no longer valid for a general arc, due to the lost of symmetry. This loss of algebraic properties means that the techniques available for the analysis of orthogonal polynomials over an arbitrary curve are much more limited; they essentially amount to finding an ingenious argument with tools from general complex function theory.

In this dissertation we will consider polynomials that are orthogonal with respect to a measure supported on an arc of the unit circle. Since circular arcs are second in simplicity only to straight line segments, it is natural to seek to extend results for orthogonal polynomials on a real segment to orthogonal polynomials on a circular arc. A number of works have been succesful in carrying out these extensions using different techniques, see for instance, [2, 4–6, 15, 16, 18, 19, 23].

In [20], the asymptotic behavior of orthogonal polynomials over the segment $[-1, 1]$ with respect to a modified Jacobi weight was investigated using the Riemann-Hilbert approach. In this dissertation, we carry out a similar investigation on an arc of the unit circle. Without loss of generality, we take a symmetric arc

$$\gamma := \{e^{i\theta} : \theta_0 \leq \theta \leq 2\pi - \theta_0\}, \quad \theta_0 \in (0, \pi), \tag{1.1.1}$$

and consider polynomials φ_n orthogonal over γ with respect to a modified Jacobi-type weight

$$w(z) = |z - e^{i\theta_0}|^{2\alpha}|z - e^{-i\theta_0}|^{2\beta}h(z), \tag{1.1.2}$$

where $\alpha, \beta > -1/2$ and $h(z)$ is positive on the arc and analytic in some neigborhood $U \supset \gamma$. Thus, we have that

$$\varphi_n(z) = z^n + \cdots, \quad n \geq 0$$

is a monic polynomial of degree n, and

$$\int_\gamma \varphi_n(z)\overline{\varphi_m(z)}|z - e^{i\theta_0}|^{2\alpha}|z - e^{-i\theta_0}|^{2\beta}h(z)|dz| = 0, \quad n \neq m.$$

The main goal of the dissertation is to establish full asymptotic expansions for φ_n, describing in great detail the behavior of φ_n as $n \to \infty$ in every region of the complex plane. Our results extend those of [18], where a similar investigation was carried out but without singularities at the end points of the arc, that is, for $\alpha = \beta = 0$.

1.2 Riemann-Hilbert characterization of the orthogonal polynomials

Every point on a smooth oriented curve that is neither a point of self-intersection nor an endpoint will be called an interior point. The collection of interior points of a curve ℓ will be denoted by ℓ^o.

The orientation of the curve defines a positive +side and a negative -side locally about every interior point. The positive/negative side lies to the left/right of a particle traveling the curve along the given orientation.

Given a function f defined on the complement of a curve ℓ, the positive and negative boundary values $f_\pm$ of f at $z \in \ell^o$ are defined to be

$$f_\pm(z) := \lim_{z \to t^\pm} f(z),$$

where $z \to t^\pm$ indicates that z approaches t within the positive/negative side. We say that f has continuous boundary values on ℓ^o if both f_+ and f_- are continuous functions on ℓ^o. For matrices whose entries are functions defined on the complement of a curve, the boundary values are defined entrywise.

Hereafter the arc γ will be oriented in clockwise motion along the unit circle. It is easy to see that the orthonormal polynomials are characterized by the non-hermitian orthogonality condition

$$\int_\gamma \Phi_n^*(z) z^m \frac{w(z)}{z^{n+1}} dz = \begin{cases} 0, & m = -1, 0, 1, ..., n-1, \\ i/\kappa_n, & m = n, \end{cases}$$

where $\Phi_n^*(z) = z^n \overline{\Phi_n(1/\overline{z})}$ denotes the reverse polynomial of Φ_n (for a full justification, see Section 5.1).

This characterization allows us to formulate a matrix Riemann-Hilbert problem (RHP) whose solution involves the orthogonal polynomials and their Cauchy transforms. Namely, we consider the problem of finding a 2×2 matrix Y satisfying the following conditions:

Y1 $Y(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$ with continuous boundary values on γ^o.

Y2 For all $t \in \gamma^o$,

$$Y_+(t) = Y_-(t) \begin{pmatrix} 1 & t^{-n} w(t) \\ 0 & 1 \end{pmatrix}.$$

Y3 As $z \to \infty$,

$$Y(z) = \left(I + O\left(\frac{1}{z}\right)\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Y4 As $z \to e^{i\theta_0}$,

$$Y(z) = \begin{cases} O\begin{pmatrix} 1 & |z-e^{i\theta_0}|^{2\alpha} \\ 1 & |z-e^{i\theta_0}|^{2\alpha} \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} 1 & \log|z-e^{i\theta_0}| \\ 1 & \log|z-e^{i\theta_0}| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \alpha > 0, \end{cases} \tag{1.2.1}$$

and as $z \to e^{-i\theta_0}$, Y behaves as in (1.2.1) with $e^{i\theta_0}$ and α replaced by $e^{-i\theta_0}$ and β, respectively.

Above in condition Y3 and in what follows, I denotes the identity matrix.

Theorem 1.2.1. *The Riemann-Hilbert problem Y1-Y4 has a unique solution given by*

$$Y(z) = \begin{pmatrix} \varphi_n(z) & \dfrac{1}{2\pi i}\displaystyle\int_\gamma \frac{\varphi_n(t)w(t)dt}{t^n(t-z)} \\ -2\pi\kappa_{n-1}\Phi^*_{n-1}(z) & -\dfrac{\kappa_{n-1}}{i}\displaystyle\int_\gamma \frac{\Phi^*_{n-1}(t)w(t)dt}{t^n(t-z)} \end{pmatrix}. \tag{1.2.2}$$

The proof of this theorem is standard and follows the same arguments given in [20], see also [3] and [11]. It is presented in Section 5.2.

To distill the asymptotics of the orthogonal polynomials from Theorem 1.2.1, we follow the steepest descent method as it was done in [20]. The objective of the steepest descent process for Riemann-Hilbert problems is to alter the RHP for Y through a series of transformations $Y \mapsto T \mapsto S \mapsto R$, where the latter matrix R satisfies, generally speaking, the following RHP on a system of contours $\hat{\Gamma}$:

R1 R is analytic on $\mathbb{C} \setminus \hat{\Gamma}$ with continuous boundary values on $\hat{\Gamma}^o$.

R2 $R_+(t) = R_-(t)V(t)$ for $t \in \hat{\Gamma}^o$ where $V(t)$ is a jump matrix that is uniformly close to the identity as $n \to \infty$.

R3 $R(z) \mapsto I$ as $z \mapsto \infty$.

Under a few more specific assumptions, one can prove that there is a unique solution R to the problem R1-R3, and that $R(z) \to I$ uniformly on compact subsets of $\mathbb{C} \setminus \hat{\Gamma}$ as $n \to \infty$. Since each of the transformations has an inverse, reversing the steepest descent process $R \mapsto S \mapsto T \mapsto Y$ yields the asymptotics of the matrix Y, and thus, of the orthogonal polynomials themselves.

1.3 Asymptotics for the orthogonal polynomials

In this section, we present our new results establishing the asymptotic behavior of the orthogonal polynomials on every region of the complex plane. The great strength of the Riemann-Hilbert approach resides in its ability to generate full asymptotic expansions for the orthogonal polynomials.

1.3.1 Asymptotics for the monic polynomials outside the arc

Our first theorem gives an asymptotic expansion for the nth monic orthogonal polynomial $\varphi_n(z)$ in powers of $1/n$ and for points z away from the arc. The expansions involve a few functions and quantities that we need to introduce first.

The function

$$\psi(z) := \frac{z + 1 + \sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}}{2c},$$

where

$$c := \cos(\theta_0/2),$$

is the unique conformal map of $\overline{\mathbb{C}} \setminus \gamma$ onto the exterior of the unit circle $\{w : |w| > 1\}$ such that $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$.

The Szegő function for the weight

$$w(z) = |z - e^{i\theta_0}|^{2\alpha} |z - e^{-i\theta_0}|^{2\beta} h(z)$$

is defined as

$$D(z) := \exp\left(\frac{g(z)}{2\pi i}\int_\gamma \frac{\log w(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta - z}\right), \quad z \in \overline{\mathbb{C}} \setminus \gamma,$$

where

$$g(z) := \sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}$$

is the square root function occurring in the definition of the conformal map ψ.

A thorough discussion on the conformal map ψ and the Szegő functions is presented in Section 2.2. It will be shown in Proposition 2.2.5 that

$$D(z) = \left[\frac{ie^{-i\theta_0/2}(z - e^{i\theta_0})}{\psi(z)}\right]^\alpha \left[\frac{-ie^{i\theta_0/2}(z - e^{-i\theta_0})}{\psi(z)}\right]^\beta \exp\left(\frac{g(z)}{2\pi i}\int_\gamma \frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta - z}\right).$$

From this expression, we can compute

$$|D(\infty)| = c^{\alpha+\beta}\exp\left(-\frac{1}{4\pi}\int_{\theta_0}^{2\pi-\theta_0}\frac{\sin(\theta/2)\log\, h(e^{i\theta})\, d\theta}{\sqrt{c^2 - \cos^2(\theta/2)}}\right), \tag{1.3.1}$$

$$D(\infty) = |D(\infty)|e^{i(\alpha-\beta)(\pi-\theta_0)/2}\exp\left(\frac{i}{4\pi}\int_{\theta_0}^{2\pi-\theta_0}\frac{\cos(\theta/2)\log\, h(e^{i\theta})\, d\theta}{\sqrt{c^2 - \cos^2(\theta/2)}}\right).$$

Let σ be a closed contour in U (the neighborhood where h is analytic), going around the arc γ once in the positive direction. Related to the Szegő function, we introduce numbers c_n and d_n defined as

$$c_n = \frac{1}{2\pi i}\int_\sigma \frac{\log h(\zeta)}{g(\zeta)}\frac{d\zeta}{(\zeta - e^{i\theta_0})^{n+1}}, \quad n \geq 0, \tag{1.3.2}$$

$$d_n = \frac{1}{2\pi i}\int_\sigma \frac{\log h(\zeta)}{g(\zeta)}\frac{d\zeta}{(\zeta - e^{-i\theta_0})^{n+1}}, \quad n \geq 0. \tag{1.3.3}$$

Theorem 1.3.1. *For $z \in \overline{\mathbb{C}} \setminus \gamma$, we have an asymptotic expansion of the form*

$$\frac{\varphi_n(z)}{c^{n+1}\psi^n(z)} \sim \frac{D(\infty)}{D(z)}\sqrt{\left(\frac{\psi(z) - e^{i\theta_0/2}}{z - e^{i\theta_0/2}}\right)\left(\frac{\psi(z) - e^{-i\theta_0/2}}{z - e^{-i\theta_0/2}}\right)}\left[1 + \sum_{k=1}^{\infty}\frac{\Pi_k(z)}{n^k}\right]$$

holding true as $n \to \infty$ uniformly for z on compact subsets of $\overline{\mathbb{C}} \setminus \gamma$. The functions Π_k are analytic in $\overline{\mathbb{C}} \setminus \gamma$ and can be computed explicitly. The first two are given by

$$\Pi_1(z) = -\frac{e^{i\theta_0/2}(16\alpha^2 - 1)}{4(\psi(z) - e^{i\theta_0/2})} - \frac{e^{-i\theta_0/2}(16\beta^2 - 1)}{4(\psi(z) - e^{-i\theta_0/2})},$$

$$\begin{aligned}\Pi_2(z) = &- \frac{i\cot(\theta_0/2)(16\alpha^2-1)(8-128\beta^2+(\alpha+\beta)10i\sin(\theta_0/2)+20ie^{i\theta_0}c_0\sin\theta_0)}{256(\psi(z)-e^{i\theta_0})} \\ &+ \frac{ie^{i\theta_0}\cot(\theta_0/2)(16\alpha^2-1)(21+112\alpha^2)}{256(z-e^{i\theta_0})} + \frac{e^{i\theta_0}(16\alpha^2-1)(12+128\alpha^2)}{256(\psi(z)-e^{i\theta_0/2})^2} \\ &+ \frac{i\cot(\theta_0/2)(16\beta^2-1)(8-128\alpha^2-(\alpha+\beta)10i\sin(\theta_0/2)-20ie^{-i\theta_0}d_0\sin\theta_0)}{256(\psi(z)-e^{-i\theta_0})} \\ &- \frac{ie^{-i\theta_0}\cot(\theta_0/2)(16\beta^2-1)(21+112\beta^2)}{256(z-e^{-i\theta_0})} + \frac{e^{-i\theta_0}(16\beta^2-1)(12+128\beta^2)}{256(\psi(z)-e^{-i\theta_0/2})^2}.\end{aligned}$$

1.3.2 Determination of the functions Π_k

The determination of Π_k rapidly becomes a computationally demanding task for $k \geq 3$. We briefly explain the steps needed in deriving these functions. We will see in the proof of Theorem 1.3.1 that

$$\Pi_k(z) = (R_k)_{11}(z) + \frac{1}{D(\infty)^2}\frac{\sin(\theta_0/2)}{\psi(z) - c}(R_k)_{12}(z),$$

where $(R_k)_{11}$ and $(R_k)_{12}$ are the entries in the first row of a matrix

$$R_k(z) = \begin{pmatrix} (R_k)_{11}(z) & (R_k)_{12}(z) \\ (R_k)_{21}(z) & (R_k)_{22}(z) \end{pmatrix}, \qquad k \geq 1.$$

For all sufficiently small $\delta > 0$, the open disks

$$U_\delta := \{z : |z - e^{i\theta_0}| < \delta\} \quad \text{and} \quad \tilde{U}_\delta := \{z : |z - e^{-i\theta_0}| < \delta\}$$

have disjoint closures, and for each integrable function f, the Cauchy transform

$$C(f)(z) := \frac{1}{2\pi i}\int_{\partial U_\delta \cup \partial \tilde{U}_\delta} \frac{f(\zeta)d\zeta}{\zeta - z}$$

defines an analytic function in the three components of $\mathbb{C} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$. Here, the circles ∂U_δ and $\partial \tilde{U}_\delta$ are clockwise oriented. The matrix functions R_k are then computed by using the residue theorem via the recursive formula

$$R_k = C(\Delta_k) + \sum_{j=1}^{k-1} C\left(\Delta_j (R_{k-j})_-\right), \qquad k \geq 1,$$

where the functions $\{\Delta_j\}_{j=1}^{\infty}$ are explicitly given in (3.5.9) and (3.5.10).

1.3.3 Asymptotics for the leading coefficient

Theorem 1.3.2. *The leading coefficient κ_n admits an expansion of the form*

$$\kappa_n^2 \sim \frac{\sin(\theta_0/2)}{2\pi c^{2(n+1)}|D(\infty)|^2}\left(1 + \sum_{k=1}^{\infty}\frac{\Gamma_k}{n^k}\right),$$

where the numbers Γ_k are expressed in terms of $\theta_0, \alpha, \beta, c_n,$ and d_n, and are explicitly computable. The first two are

$$\Gamma_1 = -\frac{i\cot(\theta_0/2)(16\alpha^2 - 1)}{4} + \frac{i\cot(\theta_0/2)(16\beta^2 - 1)}{4},$$

$$
\begin{aligned}
\Gamma_2 =& \frac{\cot(\theta_0/2)(16\alpha^2-1)(-8+128\beta^2-10ie^{-i\theta_0/2}(\alpha+\beta)\sin(\theta_0/2)-20ie^{-i\theta_0/2}c_0\sin\theta_0)}{256\sin(\theta_0/2)} \\
&+\frac{i\cot(\theta_0/2)(16\alpha^2-1)(21+112\alpha^2)}{256}-\frac{\cot^2(\theta_0/2)(16\alpha^2-1)(12+128\alpha^2)}{256} \\
&+\frac{\cot(\theta_0/2)(16\beta^2-1)(-8+128\alpha^2+10ie^{i\theta_0/2}(\alpha+\beta)\sin(\theta_0/2)+20ie^{i\theta_0/2}d_0\sin\theta_0)}{256\sin(\theta_0/2)} \\
&-\frac{i\cot(\theta_0/2)(16\beta^2-1)(21+112\beta^2)}{256}-\frac{\cot^2(\theta_0/2)(16\beta^2-1)(12+128\beta^2)}{256} \\
&+\frac{i\cot(\theta_0/2)(16\alpha^2-1)}{4}-\frac{i\cot(\theta_0/2)(16\beta^2-1)}{4}.
\end{aligned}
$$

The computation of further terms Γ_k can be achieved, once we have determined the matrix functions R_k, via the formula

$$
\Gamma_k = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left(\overline{(R_{k-j})_{22}(0)}-\cot(\theta_0/2)\overline{D^2(\infty)(R_{k-j})_{21}(0)}\right), \quad k\geq 1.
$$

1.3.4 Asymptotics for the monic polynomials on the arc

For $\theta_0 \leq \theta \leq 2\pi-\theta_0$, let us define

$$
\lambda(\theta) := \arccos\left(\frac{\cos(\theta/2)}{c}\right), \quad \chi(\theta) := -(\alpha+\beta)\lambda(\theta)+\alpha\pi+\frac{g_+(e^{i\theta})}{2\pi}\int_\gamma \frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta-e^{i\theta}},
$$

and

$$
\Lambda(\theta) := \sqrt{\frac{\cos(\theta/2)}{\sin(\theta_0/2)}}\sqrt{\tan(\theta/2)+\tan\lambda(\theta)}e^{i\frac{\lambda(\theta)}{2}}.
$$

The behavior of $\varphi_n(z)$ as $n\to\infty$ for points $z\in\gamma$ and away from the end points $e^{\pm i\theta_0}$ is given next.

Theorem 1.3.3. *For $\theta \in (\theta_0, 2\pi - \theta_0)$ we have an asymptotic expansion of the form*

$$\begin{aligned}\varphi_n(e^{i\theta}) \sim \frac{D(\infty) c^n e^{i\frac{(2n-1)\theta}{4}}}{\sqrt{w(e^{i\theta})}} \sqrt{\frac{\sin(\theta_0/2)}{2\sin\lambda(\theta)}} \\ \times \Bigg[\left(\Lambda(\theta) e^{i(n\lambda(\theta)-\chi(\theta))} + i\Lambda^{-1}(\theta) e^{-i(n\lambda(\theta)-\chi(\theta))} \right) \left(1 + \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^1(\theta)}{n^k} \right) \\ - \left(i\Lambda(\theta) e^{-i(n\lambda(\theta)-\chi(\theta))} + \Lambda^{-1}(\theta) e^{i(n\lambda(\theta)-\chi(\theta))} \right) \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^2(\theta)}{n^k} \Bigg]\end{aligned} \tag{1.3.4}$$

holding true uniformly on compact subsets of $(\theta_0, 2\pi - \theta_0)$. In particular,

$$\frac{\varphi_n(e^{i\theta})}{c^n e^{i\frac{(2n-1)\theta}{4}}} = \frac{D(\infty)}{\sqrt{w(e^{i\theta})}} \sqrt{\frac{\sin(\theta_0/2)}{2\sin\lambda(\theta)}} \left(\Lambda(\theta) e^{i(n\lambda(\theta)-\chi(\theta)} + i\Lambda^{-1}(\theta) e^{-i(n\lambda(\theta)-\chi(\theta))} \right) (1 + O(1/n)) \tag{1.3.5}$$

uniformly as $n \to \infty$ on compact subsets of $(\theta_0, 2\pi - \theta_0)$.

Remark 1.3.4. The functions $\mathcal{P}_k^1$ and $\mathcal{P}_k^2$ in the expansion of Theorem 1.3.3 are in fact given by

$$\mathcal{P}_k^1(\theta) = (R_k)_{11}(e^{i\theta}), \qquad \mathcal{P}_k^2(\theta) = \frac{i(R_k)_{12}(e^{i\theta})}{D(\infty)^2}, \quad k \geq 1. \tag{1.3.6}$$

The asymptotic behavior of $\varphi_n(z)$ for $z \in \gamma$ and near the endpoints $e^{\pm i\theta_0}$ involves the Bessel function of the first kind J_ν. For an arbitrary complex number ν, the Bessel function

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(k+\nu+1)} \left(\frac{z}{2}\right)^{2k} \tag{1.3.7}$$

is analytic in $\mathbb{C}$ with a branch cut along $(-\infty, 0]$ corresponding to the principal branch of $(z/2)^\nu$. It is a solution to the linear differential equation

$$z^2 \frac{d^2 w}{dz^2} + z\frac{dw}{dz} + (z^2 - \nu^2) w = 0.$$

With the help of the Bessel functions, it is now possible to express the behavior of φ_n near the endpoints. We first give the formulae for z near $e^{i\theta}$.

Theorem 1.3.5. *Let*

$$
\begin{aligned}
M_\alpha^1(\theta) &:= \Lambda(\theta)e^{i(\alpha\pi-\chi(\theta))} + \Lambda^{-1}(\theta)e^{-i(\alpha\pi-\chi(\theta))}, \\
M_\alpha^2(\theta) &:= \Lambda(\theta)e^{i(\alpha\pi-\chi(\theta))} - \Lambda^{-1}(\theta)e^{-i(\alpha\pi-\chi(\theta))}, \\
M_\alpha^3(\theta) &:= \Lambda(\theta)e^{-i(\alpha\pi-\chi(\theta))} + \Lambda^{-1}(\theta)e^{i(\alpha\pi-\chi(\theta))}, \\
M_\alpha^4(\theta) &:= \Lambda(\theta)e^{-i(\alpha\pi-\chi(\theta))} - \Lambda^{-1}(\theta)e^{i(\alpha\pi-\chi(\theta))}.
\end{aligned}
$$

There exists $\delta > 0$ such that the asymptotic expansion

$$
\begin{aligned}
\varphi_n(e^{i\theta}) \sim{} & \frac{e^{i\frac{(2n-1)\theta}{4}} c^n D(\infty)\sqrt{\sin(\theta_0/2)}}{2\sqrt{w(e^{i\theta})}} \sqrt{\frac{\pi n\lambda(\theta)}{\sin\lambda(\theta)}} \\
& \times \Bigg[\left(e^{i\pi/4} J_{2\alpha}(n\lambda(\theta)) M_\alpha^1(\theta) + e^{-i\pi/4} J'_{2\alpha}(n\lambda(\theta)) M_\alpha^2(\theta)\right) \left(1 + \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^1(\theta)}{n^k}\right) \\
& \quad - \left(e^{i\pi/4} J_{2\alpha}(n\lambda(\theta)) M_\alpha^3(\theta) - e^{-i\pi/4} J'_{2\alpha}(n\lambda(\theta)) M_\alpha^4(\theta)\right) \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^2(\theta)}{n^k} \Bigg]
\end{aligned}
$$

holds true uniformly for $\theta \in [\theta_0, \theta_0 + \delta)$ as $n \to \infty$. The functions $\mathcal{P}_k^1$ and $\mathcal{P}_k^2$ are given by (1.3.6).

The behavior of $\varphi_n(z)$ for z near $e^{-i\theta}$ is very similar and is given next.

Theorem 1.3.6. *Let*

$$
\begin{aligned}
M_\beta^1(\theta) &:= \Lambda(\theta)e^{-i(\beta\pi+\chi(\theta))} - \Lambda^{-1}(\theta)e^{i(\beta\pi+\chi(\theta))}, \\
M_\beta^2(\theta) &:= \Lambda(\theta)e^{-i(\beta\pi+\chi(\theta))} + \Lambda^{-1}(\theta)e^{i(\beta\pi+\chi(\theta))}, \\
M_\beta^3(\theta) &:= \Lambda(\theta)e^{i(\beta\pi+\chi(\theta))} - \Lambda^{-1}(\theta)e^{-i(\beta\pi+\chi(\theta))}, \\
M_\beta^4(\theta) &:= \Lambda(\theta)e^{i(\beta\pi+\chi(\theta))} + \Lambda^{-1}(\theta)e^{-i(\beta\pi+\chi(\theta))},
\end{aligned}
$$

and set $\lambda^*(\theta) := \pi - \lambda(\theta)$*. There exists* $\delta > 0$ *such that the asymptotic expansion*

$$\varphi_n(e^{i\theta}) \sim \frac{e^{i\frac{(2n-1)\theta}{4}} c^n D(\infty)\sqrt{\sin(\theta_0/2)}}{2\sqrt{w(e^{i\theta})}} \sqrt{\frac{\pi n \lambda^*(\theta)}{\sin \lambda^*(\theta)}}$$
$$\times \left[\left(e^{-i\pi/4} J_{2\beta}\left(n\lambda^*(\theta)\right) M_\beta^1(\theta) + e^{i\pi/4} J'_{2\beta}\left(n\lambda^*(\theta)\right) M_\beta^2(\theta) \right) \left(1 + \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^1(\theta)}{n^k} \right) \right.$$
$$\left. + \left(e^{-i\pi/4} J_{2\beta}\left(n\lambda^*(\theta)\right) M_\beta^3(\theta) - e^{i\pi/4} J'_{2\alpha}\left(n\lambda^*(\theta)\right) M_\beta^4(\theta) \right) \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^2(\theta)}{n^k} \right]$$

holds true uniformly for $2\pi - \theta_0 - \delta \leq \theta < 2\pi - \theta_0$ *as* $n \to \infty$*, where the functions* $\mathcal{P}_k^1$ *and* $\mathcal{P}_k^2$ *are given by* (1.3.6).

We remark that in Theorems 1.3.5 and 1.3.6, the asymptotic formulae at $z = e^{\pm i\theta_0}$ are to be understood in a limiting sense, that is, the behavior of $\varphi_n(e^{\pm i\theta_0})$ is obtained by finding the limiting values of the formulae as $z \to e^{\pm i\theta_0}$. We finish this section by stating such endpoint behavior.

Corollary 1.3.7. *At the endpoints of the arc* γ*, we have that as* $n \to \infty$*,*

$$\varphi_n(e^{i\theta_0}) \sim e^{i\frac{(2n-1)\theta_0+\pi}{4}} c^n D(\infty) \left(\frac{n}{2}\right)^{2\alpha} \sqrt{\frac{\pi n \sin(\theta_0/2)}{h(e^{i\theta_0})}} \frac{\tan^\alpha(\theta_0/2)}{(2\sin\theta_0)^\beta}$$
$$\times \left[\left(1 - \frac{\alpha\cot(\theta_0/2)}{4i}\right)\left(1 + \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^1(\theta_0)}{n^k}\right) - \left(1 + \frac{\alpha\cot(\theta_0/2)}{4i}\right) \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^2(\theta_0)}{n^k} \right],$$

and

$$\varphi_n(e^{-i\theta_0}) \sim e^{i\frac{(2n-1)(2\pi-\theta_0)+\pi}{4}} c^n D(\infty) \left(\frac{n}{2}\right)^{2\beta} \sqrt{\frac{\pi n \sin(\theta_0/2)}{h(e^{-i\theta_0})}} \frac{\tan^\beta(\theta_0/2)}{(2\sin\theta_0)^\alpha}$$
$$\times \left[\left(1 + \frac{\beta\cot(\theta_0/2)}{4i}\right)\left(1 + \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^1(-\theta_0)}{n^k}\right) + \left(1 - \frac{\beta\cot(\theta_0/2)}{4i}\right) \sum_{k=1}^{\infty} \frac{\mathcal{P}_k^2(-\theta_0)}{n^k} \right].$$

1.4 The case of a varying arc

We shall now briefly discuss the process in which the arc γ is allowed to vary. To be more precise, we fix numbers $\alpha, \beta > -1/2$ and $\epsilon \in (0, \pi)$, and for each $\theta_0 \in (0, \pi - \epsilon]$ and integer $n \geq 0$, we let φ_n be the nth monic orthogonal polynomial over the arc γ defined by (1.1.1) with respect to the weight given by

$$w(z) = |z - e^{i\theta_0}|^{2\alpha}|z - e^{-i\theta_0}|^{2\beta}h(z),$$

where this time h is analytic in a neighborhood U of, and positive on, the unit circle. Thus, φ_n actually depends on the two parameters θ_0 and n. It turns out that the asymptotic expansions of the previous sections remain valid with very little variation if the product $n\theta_0 \to \infty$ in such a way that

$$\lim \left(\frac{\ln \theta_0}{n\theta_0} \right) = 0. \tag{1.4.1}$$

The numbers c_n and d_n are similarly defined by

$$c_n = \frac{1}{2\pi i} \int_\sigma \frac{\log h(\zeta)}{g(\zeta)} \frac{d\zeta}{(\zeta - e^{i\theta_0})^{n+1}}, \quad n \geq 0,$$

$$d_n = \frac{1}{2\pi i} \int_\sigma \frac{\log h(\zeta)}{g(\zeta)} \frac{d\zeta}{(\zeta - e^{-i\theta_0})^{n+1}}, \quad n \geq 0,$$

with the difference that this time σ is chosen to be a cycle consisting of two circles centered at the origin, one negatively oriented and contained in $U \cap \{z : |z| < 1\}$, the other positively oriented and contained in $U \cap \{z : |z| > 1\}$.

For $\varepsilon > 0$, let Ω_ε be the set

$$\Omega_\varepsilon := \{z : \mathrm{d}(z, \gamma) \geq \varepsilon\},$$

where $\mathrm{d}(\cdot,\cdot)$ denotes the euclidean distance. The set Ω_ε varies if so does γ.

In this new setting, Theorem 1.3.1 takes the following form.

Theorem 1.4.1. *For every $\tau > 0$, we have*

$$\frac{\varphi_n(z)}{c^{n+1}\psi^n(z)} \sim \frac{D(\infty)}{D(z)}\sqrt{\left(\frac{\psi(z)-e^{i\theta_0/2}}{z-e^{i\theta_0/2}}\right)\left(\frac{\psi(z)-e^{-i\theta_0/2}}{z-e^{-i\theta_0/2}}\right)}\left[1+\sum_{k=1}^{\infty}\frac{\theta_0^k\Pi_k(z)}{(n\theta_0)^k}\right]$$

uniformly for $z \in \Omega_{\tau\theta_0}$ as $n\theta_0 \to \infty$ satisfying (1.4.1)*. The functions Π_k are the same functions occurring in the expansion of Theorem 1.3.1.*

We notice that, if in Theorem 1.4.1, we let $\theta_0 \to 0$, we are allowing z to get close to γ at a speed of order θ_0. It is possible to prove the same result under the sole assumption that $n\theta_0 \to \infty$, provided z remains within a fixed distance from γ when $\theta_0 \to 0$.

Similarly, there exist $\tau > 1$ such that as $n\theta_0 \to \infty$ satisfying (1.4.1), the asymptotic formulae of Theorems 1.3.3 and 1.3.5 remain valid uniformly for $\theta \in [\tau\theta_0, 2\pi - \tau\theta_0]$ and $\theta \in [\theta_0, \tau\theta_0]$, respectively, provided that the sums $\sum_{k=1}^{\infty}\frac{\mathcal{P}_k^1(\theta)}{n^k}$ and $\sum_{k=1}^{\infty}\frac{\mathcal{P}_k^2(\theta)}{n^k}$ get replaced by $\sum_{k=1}^{\infty}\frac{\theta_0^k\mathcal{P}_k^1(\theta)}{(n\theta_0)^k}$ and $\sum_{k=1}^{\infty}\frac{\theta_0^k\mathcal{P}_k^2(\theta)}{(n\theta_0)^k}$, respectively.

We finish this section with two corollaries illustrating the behavior of φ_n when the arc γ converges to the unit circle.

Corollary 1.4.2. *As $(\theta_0, n\theta_0) \to (0,\infty)$ satisfying* (1.4.1)*, we have that*

$$\frac{\varphi_n(z)}{c^n\psi^n(z)} \to \left(\frac{z}{z-1}\right)^{\alpha+\beta}\exp\left(\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{\zeta-z}\right), \qquad |z|>1,$$

and

$$\frac{\varphi_n(z)}{\sin(\theta_0/2)c^n\psi^n(z)} \to \frac{e^{i(\alpha-\beta)\pi}}{(1-z)^{\alpha+\beta+1}} \times \exp\left(-\frac{1}{\pi i}\fint_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta} - \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{\zeta-z}\right), \qquad |z|<1,$$

uniformly on compact subsets of the specified regions.

Corollary 1.4.3. *As* $(\theta_0, n\theta_0) \to (0, \infty)$ *satisfying* (1.4.1), *we have that*

$$\frac{\varphi_n(e^{i\theta_0})}{e^{in\theta_0/2}c^n n^{2\alpha+1/2}\sin(\theta_0/2)^{\alpha-\beta-1/2}} \to \frac{i\alpha e^{i(\alpha-\beta)\pi/2}}{4^{\alpha+\beta+1}}\sqrt{\frac{\pi}{h(1)}}\exp\left(-\frac{1}{2\pi i}\fint_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta}\right),$$

and if $\alpha = 0$, *then*

$$\frac{\varphi_n(e^{i\theta_0})}{e^{in\theta_0/2}c^n n^{1/2}\sin(\theta_0/2)^{1/2-\beta}} \to \frac{ie^{-i\beta\pi/2}}{4^{\beta+1}}\sqrt{\frac{\pi}{h(1)}}\exp\left(-\frac{1}{2\pi i}\fint_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta}\right).$$

In Corollaries 1.4.2 and 1.4.3, the symbol $\fint$ is used to denote the integral in principal value sense, that is, if $\mathbb{T}_\varepsilon$ is the positively oriented arc $\{\zeta : |\zeta| = 1,\ |\zeta - 1| \geq \varepsilon\}$, then

$$\fint_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta} := \lim_{\varepsilon\to 0}\int_{\mathbb{T}_\varepsilon}\frac{\log h(\zeta)d\zeta}{1-\zeta}.$$

1.5 Dissertation structure

The content of the dissertation has been linearly structured so as to have any result needed for a given topic well stated and developed beforehand. In Chapter 2, we state many auxiliary results needed for the development of the steepest descent method, such as boundary value properties of Cauchy transforms and properties of certain conformal maps and Szegő functions. For the sake of clarity, most of the proofs for Chapter 2 are relegated to Chapter 5. The steepest descent method is carried out in Chapter 3. This is a long and intricate chapter, from which all of our results will be derived. Chapter 4, where the proofs of our theorems are given, is essentially a continuation of Chapter 3.

Chapter 2 AUXILIARY RESULTS

2.1 The Sokhotskii-Plemelj formula

The proof of Theorem 1.2.1 and other facts we will encounter soon rely on the following fundamental results regarding the boundary values of Cauchy transforms, which can be found in the book [13].

Theorem 2.1.1 (Sokhotskii-Plemelj formula [27]). *Let Σ be a simple smooth path, and let $v(t)$ be a weight (non-negative integrable function) that is Hölder continuous on Σ^o, that is, there exist constants $A > 0$ and $0 < \lambda \leq 1$ such that*

$$|v(t) - v(t_1)| \leq A|t - t_1|^\lambda, \quad t,\, t_1 \in \Sigma^o.$$

Then, the Cauchy transform

$$C(z) = \frac{1}{2\pi i}\int_\Sigma \frac{v(t)}{t-z}dt$$

has continuous boundary values $C_+(z)$ and $C_-(z)$ at every interior point $z \in \Sigma^o$ given by

$$C_+(z) = \frac{v(z)}{2} + \frac{1}{2\pi i}\fint_\Sigma \frac{v(t)}{t-z}dt, \quad z \in \Sigma^o,$$

$$C_-(z) = -\frac{v(z)}{2} + \frac{1}{2\pi i}\fint_\Sigma \frac{v(t)}{t-z}dt, \quad z \in \Sigma^o,$$

where the symbol $\fint$ is understood as the integral in the sense of principal value. In particular,

$$C_+(z) = C_-(z) + v(z), \quad z \in \Sigma^o. \tag{2.1.1}$$

It is important to realize that the formula (2.1.1) is, indeed, of a local nature, as evidenced by the following corollary. We will use the notation

$$D(z_0, r) := \{z : |z - z_0| < r\}.$$

Corollary 2.1.2. *Let $v(t)$ be a weight defined on a system of contours Σ, and suppose $z_0 \in \Sigma$ is such that for some $r > 0$, $\Sigma_1 = D(z_0, r) \cap \Sigma$ is a simple smooth arc on which $v(t)$ is Hölder continuous. Then*

$$C_+(z) - C_-(z) = v(z), \quad z \in \Sigma_1.$$

Proof. Let us define

$$C^1(z) := \frac{1}{2\pi i} \int_{\Sigma_1} \frac{v(t)}{t - z} dt.$$

By Theorem 2.1.1, we have

$$C_+^1(z) - C_-^1(z) = v(z), \quad z \in \Sigma_1,$$

and since the function

$$z \mapsto \frac{1}{2\pi i} \int_{\Sigma \setminus \Sigma_1} \frac{v(t)}{t - z} dt$$

is continuous on the complement of $\Sigma \setminus \Sigma_1$, we see that

$$C_+(z) - C_-(z) = C_+^1(z) - C_-^1(z) = v(z), \quad z \in \Sigma_1.$$

$\square$

Corollary 2.1.3. *Let Σ be a smooth simple path in $\mathbb{C}$, and let $v(s)$ be Hölder continuous on Σ. The Cauchy transform*

$$f(z) = \frac{1}{2\pi i}\int_{\Sigma}\frac{v(s)}{s-z}ds \tag{2.1.2}$$

is a solution to the additive Riemann-Hilbert problem consisting of finding a function f such that

a1 *$f(z)$ is analytic in $\mathbb{C}\setminus\Sigma$;*

a2 *$f_+(s) = f_-(s) + v(s)$ for all $t \in \Sigma^o$;*

a3 *$f(z) \to 0$ as $z \to \infty$.*

Moreover, this problem has at most one solution satisfying that

a4 *if a and b are the endpoints of Σ, then $\lim_{z\to a}(z-a)f(z) = 0$ and $\lim_{z\to b}(z-b)f(z) = 0$.*

Proof. That $f(z)$ as defined by (2.1.2) is a solution follows from Theorem 2.1.1. Suppose f and g are two solutions to the Riemann-Hilbert problem solving a1-a3. For $t \in \Sigma^o$

$$(f-g)_+(t) = (f-g)_-(t),$$

so that by Morera's theorem, $(f-g)$ is analytic in $\mathbb{C}\setminus\{a,b\}$. Moreover,

$$\lim_{z\to a}(z-a)(f-g)(z) = 0 \quad \text{and} \quad \lim_{z\to b}(z-b)(f-g)(z) = 0,$$

so that the Laurent expansion of $(z-a)(f-g)(z)$ (resp., of $(z-b)(f-g)(z)$) about a (resp., about b) is a Taylor series with constant coefficient 0, and so $(f-g)(z)$ is entire. By Liouville's theorem and a3, $(f-g)(z) \equiv 0$, so $f(z) = g(z)$. □

2.2 Conformal maps and Szegő functions

2.2.1 The exterior conformal map of the arc

Let ℓ denote the segment $\left[e^{-i\theta_0}, e^{i\theta_0}\right]$, and let $G(z)$ be the branch of the square root of $(z-e^{i\theta_0})(z-e^{-i\theta_0})$ on $\mathbb{C}\setminus\ell$ that is positive for $x>\cos\theta_0$, namely, $G(x)=|x-e^{i\theta_0}|$. The properties of this function can be easily obtained from the well-known branch of $\sqrt{x^2-1}$ on $\mathbb{C}\setminus[-1,1]$. For instance, given that the transformation $z\mapsto 2\cos\theta_0 - z = -(z-\cos\theta_0)+\cos\theta_0$ is the reflection about the midpoint of ℓ, we have

$$G_\pm(t)=G_\pm(\bar{t}), \quad G_\pm(t)=-G_\mp(\bar{t}), \quad t\in(e^{-i\theta_0}, e^{i\theta_0}).$$

Notice also that $G(2\cos\theta_0 - z)$ is also a branch of the square root of $(z-e^{i\theta_0})(z-e^{-i\theta_0})$ on $\mathbb{C}\setminus\ell$. Hence

$$G(z)=-G(2\cos\theta_0 - z), \quad z\in\mathbb{C}\setminus\ell.$$

It follows that

$$g(z):=\begin{cases} G(z), & z\in \operatorname{ext}(\gamma\cup\ell)\cup\ell^o, \\ G(2\cos\theta_0 - z), & z\in\operatorname{int}(\gamma\cup\ell), \end{cases}$$

is the branch of the square root of $(z-e^{i\theta_0})(z-e^{-i\theta_0})$ on $\mathbb{C}\setminus\gamma$ that is positive for $z>-1$. Moreover,

$$g_+(t)=-g_-(t), \quad g_+(t)=G(t), \quad t\in\gamma^o. \tag{2.2.1}$$

We also record that if γ' is the reflection of the arc γ about the segment ℓ, then $g(z)=g(z_1)$ with $z\neq z_1$ only when $z_1=2\cos\theta_0 - z$ and $z,\, z_1\in\operatorname{int}(\gamma\cup\gamma')$.

In what follows we shall often denote $g(z)$ by $\sqrt{(z-e^{i\theta_0})(z-e^{-i\theta_0})}$.

Let us define

$$\psi(z) := \frac{z+1+g(z)}{2c}, \quad c := \cos(\theta_0/2), \quad z \in \overline{\mathbb{C}} \setminus \gamma. \tag{2.2.2}$$

Proposition 2.2.1. *The function ψ maps $\overline{\mathbb{C}} \setminus \gamma$ conformally onto $|w| > 1$, and satisfies $\psi(\infty) = \infty$, $\psi'(\infty) = c^{-1} > 0$. It has the following properties:*

i. The inverse ψ^{-1} of ψ is

$$\psi^{-1}(\omega) = \frac{\omega(c\omega - 1)}{\omega - c}, \quad |\omega| > 1. \tag{2.2.3}$$

ii.

$$\left|\frac{z}{\psi(z)}\right| < 1, \quad \textit{and therefore} \quad \left|\frac{z}{\psi(z)^2}\right| < 1, \quad z \in \overline{\mathbb{C}} \setminus \gamma. \tag{2.2.4}$$

iii. We have the boundary value properties

$$\psi(e^{\pm i\theta_0}) = e^{\pm i\theta_0/2},$$

$$\psi_+(t)\psi_-(t) = t, \quad t \in \gamma^o, \tag{2.2.5}$$

$$\psi_-(t) + \psi_+(t) = \frac{t+1}{c}, \quad t \in \gamma^o,$$

$$\psi_-(t) = \frac{c\psi_+(t) - 1}{\psi_+(t) - c}, \quad \psi_+(t) = \frac{c\psi_-(t) - 1}{\psi_-(t) - c}, \quad t \in \gamma^o. \tag{2.2.6}$$

iv. The geometric relation between $\psi_+(t)$ and $\psi_-(t)$ is given by

$$\begin{aligned}\psi_+(t) &= c - \frac{\sin^2(\theta_0/2)}{\psi_-(t) - c}\\ &= c - \frac{\sin^2(\theta_0/2)}{|\psi_-(t) - c|^2}\left(\overline{\psi_-(t)} - c\right),\end{aligned} \tag{2.2.7}$$

that is, $\psi_+(t)$ and $\overline{\psi_-(t)}$ are endpoints of a cord passing through $c = \cos(\theta_0/2)$.

Notice that from Proposition 2.2.1(iv.), it follows that

$$0 \le \arg\psi_-(e^{i\theta}) < \frac{\theta_0}{2} < \arg\psi_+(e^{i\theta}) \le \pi, \quad \theta \in (\theta_0, \pi],$$

$$-\pi \le \arg\psi_+(e^{i\theta}) < -\frac{\theta_0}{2} < \arg\psi_-(e^{i\theta}) \le 0, \quad \theta \in [-\pi, -\theta_0),$$

and since $\psi_+(e^{i\theta})\psi_-(e^{i\theta}) = e^{i\theta}$, then

$$\frac{\theta_0}{2} < \frac{\theta}{2} < \arg\psi_+(e^{i\theta}), \quad (\theta_0, \pi),$$

$$\arg\psi_+(e^{i\theta}) < \frac{\theta}{2} < -\frac{\theta_0}{2}, \quad (-\pi, -\theta_0).$$

2.2.2 Szegő functions

Definition 2.2.2. *Let $w(z)$ be an analytic weight on γ^o and let $\log w(z)$ be a branch of the logarithm of w on γ^o. Assuming that $\int_\gamma |\log w(z)||dz| < \infty$, we define the Szegő function for w (corresponding to the chosen branch of logarithm) as*

$$D(z; w) := \exp\left(\frac{g(z)}{2\pi i}\int_\gamma \frac{\log w(\zeta)}{G(\zeta)}\frac{d\zeta}{\zeta - z}\right), \quad z \in \overline{\mathbb{C}} \setminus \gamma.$$

Proposition 2.2.3. *The Szegő function has the following properties:*

i. $D(z;w)$ *is analytic and never zero in* $\overline{\mathbb{C}}\setminus\gamma$*, and*

$$D(\infty)=\exp\left(-\frac{1}{2\pi i}\int_\gamma \frac{\log w(\zeta)}{G(\zeta)}d\zeta\right).$$

ii. For two different branches of $\log w(z)$*, the corresponding Szegő functions differ at most by a factor of* -1*.*

iii. $D_+(t;w)D_-(t;w)=w(t)$ *for all* $t\in\gamma^o$*.*

iv. If $w(t)\geq 0$ *for* $t\in\gamma$*, then* $D(0)=\overline{D(\infty)}$*.*

Property i. is trivial from the definition. Property ii. follows from the fact that

$$\int_\gamma \frac{1}{G(\zeta)}\frac{d\zeta}{\zeta-z}=\frac{\pi i}{g(z)},\quad z\in\mathbb{C}\setminus\gamma. \tag{2.2.8}$$

We shall now compute the Szegő function for some relevant weights, starting with the orthogonality weight w. For this, we first need the following proposition.

Proposition 2.2.4. *The function*

$$F(z):=\frac{ie^{-i\theta_0/2}(z-e^{i\theta_0})}{\psi(z)}$$

is a conformal map of $\overline{\mathbb{C}}\setminus\gamma$ *onto the interior* $\mathcal{G}$ *of a cardioid shaped Jordan curve, which is symmetric with respect to the x-axis, intersecting it at* 0 *and* $2\sin\theta_0$*. The positive boundary values function* F_+ *takes* γ *to the lower half of the cardioid, and*

$$F_+(x)=\overline{F_-(x)},\qquad x\in\gamma.$$

Similarly, the function

$$\widetilde{F}(z):=\overline{F(\overline{z})}=\frac{-ie^{i\theta_0/2}(z-e^{-i\theta_0})}{\psi(z)}$$

is a conformal map of $\overline{\mathbb{C}} \setminus \gamma$ *onto the interior* $\mathcal{G}$*, with* $\widetilde{F}_+$ *taking* γ *to the upper half of the cardioid, and*

$$\widetilde{F}_+(x) = \overline{\widetilde{F}_-(x)}, \qquad x \in \gamma.$$

We will later need the following observation. Using (2.2.2), it is easy to see that

$$\psi(1) = \sec(\theta_0/2) + \tan(\theta_0/2),$$

and so

$$F(1) = \frac{ie^{-i\theta_0/2}(1 - e^{i\theta_0})}{\psi(1)} = 2\tan(\theta_0/2)(1 + \sin(\theta_0/2)).$$

Hence ψ maps $\mathbb{C} \setminus (\gamma \cup (-\infty, -1))$ onto $\{z : |z| > 1\} \setminus (-\infty, -1)$, and choosing the branch of $\log(z - e^{i\theta_0})$ in $\mathbb{C} \setminus (\gamma \cup (-\infty, -1))$ corresponding to $\arg(1 - e^{i\theta_0}) = (\theta_0 - \pi)/2$, we have

$$\mathrm{Log} F(z) = \log(z - e^{i\theta_0}) - \mathrm{Log}\,\psi(z) + i\frac{\pi - \theta_0}{2}, \quad z \in \mathbb{C} \setminus (\gamma \cup (-\infty, -1)) . \tag{2.2.9}$$

Similarly, if we choose the branch of $\log(z - e^{-i\theta_0})$ in $\mathbb{C} \setminus (\gamma \cup (-\infty, -1))$ corresponding to $\arg(1 - e^{-i\theta_0}) = (\pi - \theta_0)/2$, then

$$\mathrm{Log} \widetilde{F}(z) = \log(z - e^{-i\theta_0}) - \mathrm{Log}\,\psi(z) + i\frac{\theta_0 - \pi}{2}, \quad z \in \mathbb{C} \setminus (\gamma \cup (-\infty, -1)) . \tag{2.2.10}$$

Proposition 2.2.5. *The Szegő function for the weight*

$$w(z) = |z - e^{i\theta_0}|^{2\alpha} |z - e^{-i\theta_0}|^{2\beta}$$

and corresponding to the principal branch of the logarithm $\mathrm{Log}\, w(z)$ *is*

$$\begin{aligned} D(z; w) &= \exp\left\{\alpha\, \mathrm{Log} F(z) + \beta\, \mathrm{Log} \widetilde{F}(z)\right\} \\ &= \left[\frac{ie^{-i\theta_0/2}(z - e^{i\theta_0})}{\psi(z)}\right]^{\alpha} \left[\frac{-ie^{i\theta_0/2}(z - e^{-i\theta_0})}{\psi(z)}\right]^{\beta} . \end{aligned}$$

Proposition 2.2.6. *The function*

$$\mathcal{F}(z, \pm\theta_0) := \frac{i\sin(\pm\theta_0/2)(z - e^{\pm i\theta_0})}{\psi(z) - \cos(\theta_0/2)}, \quad z \in \mathbb{C} \setminus \gamma,$$

maps $\mathbb{C} \setminus \gamma$ *onto the interior of a cardioid symmetric about the line* $\{re^{\pm i(\theta_0 - \pi/2)} : r \in \mathbb{R}\}$*, with a cusp at the origin, and lying in* $\mathbb{C} \setminus \{-re^{\pm i(\theta_0 - \pi/2)} : r > 0\}$*, so that a branch of* $\log \mathcal{F}(z, \pm\theta_0)$ *exists for* $z \in \mathbb{C} \setminus \gamma$ *with*

$$\theta_0 - \frac{3\pi}{2} < \arg \mathcal{F}(z, \theta_0) < \theta_0 + \frac{\pi}{2},$$

$$-\theta_0 - \frac{\pi}{2} < \arg \mathcal{F}(z, -\theta_0) < -\theta_0 + \frac{3\pi}{2}.$$

Hereafter, when we write $\log \mathcal{F}(z, \pm\theta_0)$ and $\mathcal{F}(z, \pm\theta_0)^\alpha$, we mean the branches of these functions corresponding to the range of the $\arg \mathcal{F}(z, \pm\theta_0)$ specified in Proposition 2.2.6.

Proposition 2.2.7. *If the branches of the logarithmic functions* $\log(t - e^{i\theta_0})$ *and* $\log(t - e^{-i\theta_0})$ *on* γ^o *are chosen so that*

$$\pi/2 < \arg(t - e^{\pm i\theta_0}) < 3\pi/2, \quad t \in \gamma^o, \tag{2.2.11}$$

then

$$\int_\gamma \frac{\log(t - e^{\pm i\theta_0})}{G(t)} \frac{dt}{t - z} = \frac{\pi i \log \mathcal{F}(z, \pm\theta_0)}{g(z)} + \frac{(\pi i)^2}{g(z)},$$

so that for every $\alpha \in \mathbb{R}$*,*

$$D(z; (t - e^{\pm i\theta_0})^\alpha) = e^{i\pi\alpha/2} \left[\mathcal{F}(z, \pm\theta_0)\right]^{\alpha/2}.$$

The function $z \mapsto \frac{z-e^{i\theta_0}}{z-e^{-i\theta_0}}$ takes $\overline{\mathbb{C}} \setminus \gamma$ conformally onto $\mathbb{C} \setminus \{re^{i\theta_0} : r \geq 0\}$. We define

$$a(z) := \left(\frac{z - e^{i\theta_0}}{z - e^{-i\theta_0}}\right)^{\frac{1}{4}}, \quad z \in \overline{\mathbb{C}} \setminus \gamma,$$

with $a(\infty) = 1$, that is, the branch resulting from choosing

$$\theta_0 - 2\pi < \arg\left(\frac{z - e^{i\theta_0}}{z - e^{-i\theta_0}}\right) < \theta_0, \; z \in \overline{\mathbb{C}} \setminus \gamma \tag{2.2.12}$$

Proposition 2.2.8. *The function $a(z)$ is the Szegő function corresponding to the weight*

$$\nu(t) = \left(-\frac{t - e^{i\theta_0}}{t - e^{-i\theta_0}}\right)^{\frac{1}{2}}, \quad t \in \gamma^o,$$

where the branch of $\nu(t)$ and that of $\log \nu(t)$ are those corresponding to the choice $\arg \nu(t) = (\theta_0 - \pi)/2$.

Proposition 2.2.9. *If the branch of $\log \frac{1}{G(t)}$ is chosen according to*

$$\pi/2 < \arg\left(\frac{1}{G(t)}\right) < 3\pi/2, \quad t \in \gamma^o,$$

then the Szegő function for the weight $1/G(t)$ is given by

$$D\left(z; 1/G(t)\right) = i\left[\mathcal{F}(z, \theta_0)\right]^{-1/4}\left[\mathcal{F}(z, -\theta_0)\right]^{-1/4} = i\frac{a(z) + a^{-1}(z)}{\sqrt{2\sin\theta_0}}. \tag{2.2.13}$$

Proposition 2.2.10. *For $z \in \overline{\mathbb{C}} \setminus \gamma$, we have*

$$\frac{a(z) + a(z)^{-1}}{2} = \sqrt{\frac{c(\psi(z) - c)}{g(z)}} = \frac{c\sqrt{(\psi(z) - e^{i\theta_0/2})(\psi(z) - e^{-i\theta_0/2})}}{g(z)}, \tag{2.2.14}$$

$$\frac{a(z) - a(z)^{-1}}{2i} = \frac{-\sin\theta_0}{g(z)(a(z) + a(z)^{-1})}, \tag{2.2.15}$$

and

$$\begin{aligned}\frac{a(z)-a(z)^{-1}}{a(z)+a(z)^{-1}} &= \frac{-i\sin(\theta_0/2)}{\psi(z)-c} \\ &= \frac{-i\sin(\theta_0/2)(z-e^{i\theta_0})}{c(\psi(z)-e^{i\theta_0/2})^2} \\ &= \frac{-i\sin(\theta_0/2)(z-e^{-i\theta_0})}{c(\psi(z)-e^{-i\theta_0/2})^2}.\end{aligned} \tag{2.2.16}$$

We finish this section with some auxiliary results on boundary value limits that will be needed later.

Proposition 2.2.11. *With*

$$\lambda(\theta) := \arccos\left(c^{-1}\cos(\theta/2)\right), \qquad \theta_0 < \theta < 2\pi - \theta_0,$$

we have the following equalities holding true for all $\theta_0 < \theta < 2\pi - \theta_0$:

i.

$$\psi_+(e^{i\theta}) = e^{i(\lambda(\theta)+\theta/2)}.$$

$$|\psi_+(e^{i\theta}) - c| = \cos(\theta/2)[\tan(\theta/2) + \tan\lambda(\theta)].$$

ii.

$$\left(\frac{a+a^{-1}}{2}\right)_+(e^{i\theta}) = \sqrt{\frac{\cos(\theta/2)}{2\sin\lambda(\theta)}}\sqrt{\tan(\theta/2)+\tan\lambda(\theta)}e^{i\left(\frac{\lambda(\theta)}{2}-\frac{\theta}{4}\right)}.$$

iii.

$$\left(\frac{a-a^{-1}}{2i}\right)_+(e^{i\theta}) = \frac{i\sin(\theta_0/2)}{\sqrt{2\sin\lambda(\theta)}}\frac{e^{-i\left(\frac{\lambda(\theta)}{2}+\frac{\theta}{4}\right)}}{\sqrt{\cos(\theta/2)}\sqrt{\tan(\theta/2)+\tan\lambda(\theta)}}.$$

iv. $D_+(e^{i\theta}; w) = \sqrt{w(t)}\exp\{i\chi(\theta)\}$, *with*

$$\chi(\theta) = -(\alpha+\beta)\lambda(\theta) + \alpha\pi + \frac{g_+(e^{i\theta})}{2\pi}\int_\gamma \frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta - e^{i\theta}}.$$

Chapter 3 THE STEEPEST DESCENT METHOD

We have now established all the facts necessary for the application of the steepest descent method. We closely follow the ideas in [8, 20], and will be treating the fixed arc and the varying arc cases simultaneously. We depart from the Riemann-Hilbert problem Y1-Y4 stated in Chapter 1.2, whose solution is given by Theorem 1.2.1.

3.1 Transformation $Y \mapsto T$

This first transformation has for objective to normalize the behavior of Y at infinity. Let

$$T := \begin{pmatrix} c^{-n} & 0 \\ 0 & c^{n} \end{pmatrix} Y \begin{pmatrix} \psi(z)^{-n} & 0 \\ 0 & \psi(z)^{n} \end{pmatrix},$$

where

$$c = \cos(\theta_0/2).$$

Direct calculations show that T satisfies the following RHP:

T1 $T(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$ with continuous boundary values on γ^o.

T2 For all $t \in \gamma^o$,

$$T_+(t) = T_-(t) \begin{pmatrix} t^n \psi_+(t)^{-2n} & w(t) \\ 0 & t^n \psi_-(t)^{-2n} \end{pmatrix}.$$

T3 As $z \to \infty$,

$$T(z) = I + O\left(1/z\right).$$

T4 T has the same behavior as Y as $z \to e^{\pm i\theta_0}$, that is,

$$T(z) = \begin{cases} O\begin{pmatrix} 1 & |z-e^{i\theta_0}|^{2\alpha} \\ 1 & |z-e^{i\theta_0}|^{2\alpha} \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} 1 & \log|z-e^{i\theta_0}| \\ 1 & \log|z-e^{i\theta_0}| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \alpha > 0, \end{cases} \tag{3.1.1}$$

and as $z \to e^{-i\theta_0}$, T behaves as in (3.1.1) with $e^{i\theta_0}$ and α replaced by $e^{-i\theta_0}$ and β, respectively.

3.2 Transformation $T \mapsto S$

The weight w is given by

$$w(z) = |z-e^{i\theta_0}|^{2\alpha}|z-e^{-i\theta_0}|^{2\beta}h(z),$$

with $h(z)$ analytic in some neighborhood $U \supset \gamma$. In the case of varying θ_0, by assumption, h is positive and analytic in the whole unit circle $|z| = 1$, and we choose U to be a thin open annulus centered at the origin and containing the unit circle.

In either case, we can choose this neighborhood U in such a way that

$$0 < \inf_{z\in U} |h(z)| \leq \sup_{z\in U} |h(z)| < \infty. \tag{3.2.1}$$

We extend $w(z)$ analytically to $\Omega \cap U$,

$$\Omega := \mathbb{C} \setminus \{[0,\infty) \cup \{e^{i\theta} : -\theta_0 \leq \theta \leq \theta_0\}\},$$

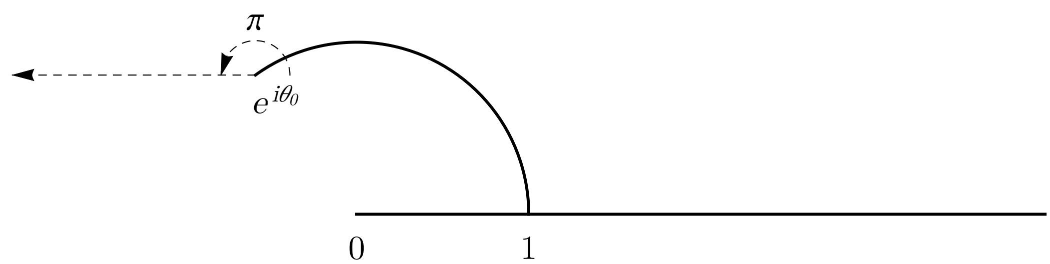

Figure 3.1: Domain of $w_\alpha(z)$

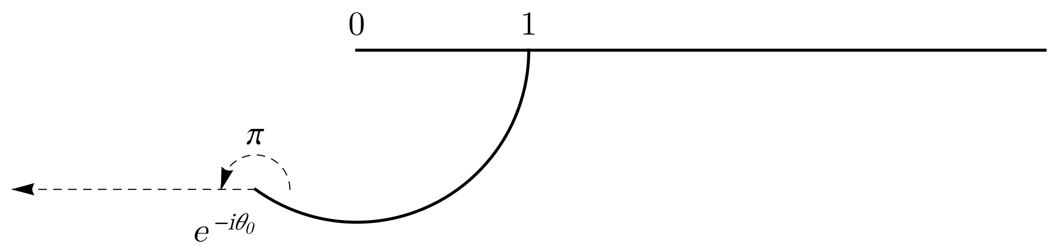

Figure 3.2: Domain of $w_\beta(z)$

via the equality

$$w(z) = w_\alpha(z) w_\beta(z) h(z),$$

with

$$\begin{aligned} w_\alpha(z) &:= e^{\alpha\left[2\log(z-e^{i\theta_0}) - \log z - (\pi+\theta_0)i\right]}, \quad z \in \mathbb{C} \setminus \{[0,\infty) \cup \{e^{i\theta} : 0 \le \theta \le \theta_0\}\}, \\ w_\beta(z) &:= e^{\beta\left[2\log(z-e^{-i\theta_0}) - \log z - (\pi-\theta_0)i\right]}, \quad z \in \mathbb{C} \setminus \{[0,\infty) \cup \{e^{i\theta} : -\theta_0 \le \theta \le 0\}\}. \end{aligned} \tag{3.2.2}$$

Here w_α and w_β are analytic in the specified domains with the branches of $\log(z - e^{\pm i\theta_0})$ and $\log z$ chosen so as to have $\arg(-1 + i\sin(\pm\theta_0) - e^{\pm i\theta_0}) = \pi$ and $\arg(-1) = \pi$, respectively.

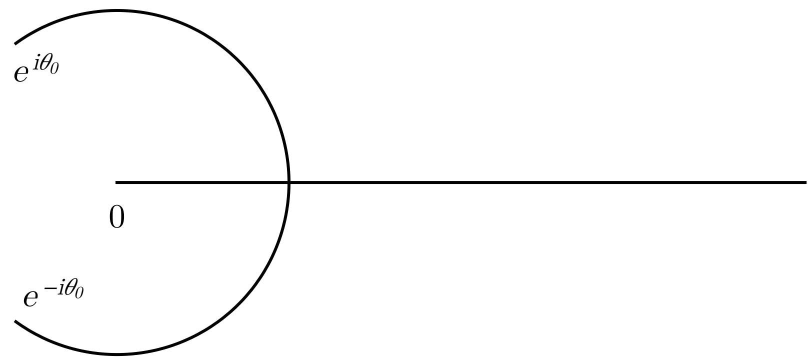

Figure 3.3: Domain of analytic continuation of $w_\alpha(z)w_\beta(z)$

From now on when we write $w(z)$, we will mean the analytic continuation as defined above (see Subsection 5.13 for details).

The purpose of the next transformation $T \mapsto S$ is to correct the oscillatory behavior of the jump in T2. We do this based on the factorization of the jump matrix

$$\begin{pmatrix} t^n\psi_+^{-2n} & w \\ 0 & t^n\psi_-^{-2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ w^{-1}t^n\psi_-^{-2n} & 1 \end{pmatrix} \begin{pmatrix} 0 & w \\ -w^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ w^{-1}t^n\psi_+^{-2n} & 1 \end{pmatrix}.$$

The analytic continuation of $w(z)$ allows us to extend the matrices

$$\begin{pmatrix} 1 & 0 \\ w^{-1}t^n\psi_-^{-2n} & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 \\ w^{-1}t^n\psi_+^{-2n} & 1 \end{pmatrix}$$

to the domains $\Omega \cap U \cap \{z : |z| < 1\}$ and $\Omega \cap U \cap \{z : |z| > 1\}$, respectively, and we denote both extensions by

$$\begin{pmatrix} 1 & 0 \\ w(z)^{-1}z^n\psi(z)^{-2n} & 1 \end{pmatrix}.$$

Since $|z\psi(z)^{-2}| < 1$ and $w(z)$ is non-zero for all $z \in \mathbb{C} \setminus \gamma$, see Propostion 2.2.1, we have

$$\frac{1}{w(z)}\left(\frac{z}{\psi(z)^2}\right)^n \to 0 \quad \text{as} \quad n \to 0$$

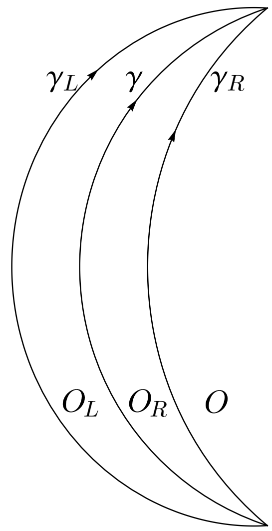

Figure 3.4: The lens Γ

for z away from γ. That is, these matrices tend to the identity (although not uniformly) as n tends to infinity. Using this information, we transform the RHP for T to a new RHP in an opened lens about γ, denoted by

$$\Gamma = \gamma_L \cup \gamma \cup \gamma_R$$

and shown in Figure 3.4, with the lens contained in $\Omega \cap U$.

Let O be the unbounded component of $\mathbb{C} \setminus \Gamma$, and let O_L and O_R be the interiors of $\gamma \cup \gamma_L$ and $\gamma \cup \gamma_R$, respectively.

Let us define

$$S = \begin{cases} T, & z \in O, \\ T \begin{pmatrix} 1 & 0 \\ -w(z)^{-1} z^n \psi(z)^{-2n} & 1 \end{pmatrix}, & z \in O_L, \\ T \begin{pmatrix} 1 & 0 \\ w(z)^{-1} z^n \psi(z)^{-2n} & 1 \end{pmatrix}, & z \in O_R. \end{cases}$$

Then S solves the following Riemann-Hilbert problem:

S1 $S(z)$ is analytic for $z \in \mathbb{C}\backslash\Gamma$ with continuous boundary values on Γ^o.

S2

$$S_+(t) = S_-(t) \begin{pmatrix} 0 & w(t) \\ -w(t)^{-1} & 0 \end{pmatrix}, \quad t \in \gamma^o,$$

$$S_+(t) = S_-(t) \begin{pmatrix} 1 & 0 \\ w(t)^{-1} t^n \psi(t)^{-2n} & 1 \end{pmatrix}, \quad t \in \gamma_L^o \cup \gamma_R^o.$$

S3 As $z \to \infty$,

$$S(z) = I + O\left(\frac{1}{z}\right).$$

S4 As $z \to e^{i\theta_0}$,

$$S(z) = \begin{cases} O\begin{pmatrix} 1 & |z-e^{i\theta_0}|^{2\alpha} \\ 1 & |z-e^{i\theta_0}|^{2\alpha} \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \\ \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \alpha > 0,\ z \in O, \\ O\begin{pmatrix} |z-e^{i\theta_0}|^{-2\alpha} & 1 \\ |z-e^{i\theta_0}|^{-2\alpha} & 1 \end{pmatrix}, & \alpha > 0,\ z \in O_L \cup O_R, \end{cases} \tag{3.2.3}$$

and as $z \to e^{-i\theta_0}$, S behaves as in (3.2.3) with $e^{i\theta_0}$ and α replaced by $e^{-i\theta_0}$ and β, respectively.

3.3 Outer parametrix

Here we seek to find a 2×2 matrix $N(z)$ satisfying the following Riemann-Hilbert problem:

N1 $N(z)$ is analytic for $z \in \mathbb{C} \setminus \gamma$.

N2 For all $t \in \gamma^o$,

$$N_+(t) = N_-(t) \begin{pmatrix} 0 & w(t) \\ -w(t)^{-1} & 0 \end{pmatrix}.$$

N3 As $z \to \infty$,

$$N(z) = I + O(1/z).$$

We now take advantage of the property (iii.) of the Szegő function for the weight w,

$$D(z) := D(z; w),$$

and write

$$N(z) = \begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1} \end{pmatrix} A(z) \begin{pmatrix} D(z)^{-1} & 0 \\ 0 & D(z) \end{pmatrix}. \tag{3.3.1}$$

Then, the 2×2 matrix $A(z)$ satisfies the following RHP:

A1 A is analytic in $\mathbb{C} \backslash \gamma$.

A2 For $t \in \gamma^o$,

$$A_+(t) = A_-(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

A3 $A \to I$ as $z \to \infty$.

If we, in addition, impose that

A4 $A(z) = O(|z - e^{\pm i\theta_0}|^{-1/4})$ as $z \to e^{\pm i\theta_0}$,

then as we show in Section 5.14, there is just one solution A given by

$$A(z) = \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ -\frac{a(z)-a^{-1}(z)}{2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix},$$

and we take

$$N(z) = \begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ -\frac{a(z)-a^{-1}(z)}{2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \begin{pmatrix} D(z)^{-1} & 0 \\ 0 & D(z) \end{pmatrix} \tag{3.3.2}$$

for solution to the RHP N1-N3.

We will later use that N is invertible, given that

$$\det N = \det A = \left(\frac{a(z) + a^{-1}(z)}{2}\right)^2 + \left(\frac{a(z) - a^{-1}(z)}{2i}\right)^2 = 1.$$

3.4 Local parametrices

Since the jump matrices across γ_L and γ_R are not uniformly close to the identity near the endpoints of γ, we must construct local parametrices in

$$U_\delta := \{z : |z - e^{i\theta_0}| < \delta\} \quad \text{and} \quad \tilde{U}_\delta := \{z : |z - e^{-i\theta_0}| < \delta\}$$

for some $\delta > 0$ small enough as to have U_δ and $\tilde{U}_\delta$ contained in the open upper half plane and lower half plane, respectively, with $(\overline{U}_\delta \cup \overline{\tilde{U}}_\delta) \subset U$.

In the case of a varying arc, because we allow θ_0 to vary, possibly approaching 0, we will specifically choose δ of the form

$$\delta = \rho\theta_0, \quad 0 < \rho < \rho_1,$$

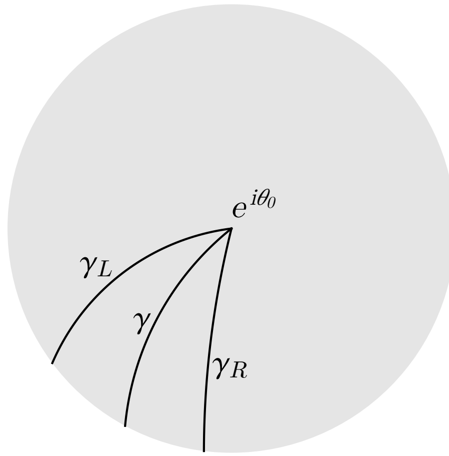

Figure 3.5: Neighborhood U_δ

where $\rho_1 > 0$ is a universal constant satisfying that

$$\rho_1\theta_0 < \sin(\theta_0), \quad 0 < \theta_0 \leq \pi - \epsilon.$$

We will better specify ρ_1 later in Section 3.4.3.

3.4.1 Riemann-Hilbert problem for P

We wish to find a P that satisfies the same jump relations as S, that matches N on the boundary ∂U_δ of U_δ, and that has the same behavior as S as $z \to e^{i\theta}$. More precisely, we seek to find P solving the following Riemann-Hilbert problem:

P1 $P(z)$ is analytic $U_\delta \setminus \Gamma$ and continuous in $\{z : |z - e^{i\theta_0}| \leq \delta\} \setminus \Gamma$.

P2 P has the same jumps as S on $\Gamma^o \cap U_\delta$:

$$P_+(t) = P_-(t) \begin{pmatrix} 1 & 0 \\ t^n \psi(t)^{-2n} w(t)^{-1} & 1 \end{pmatrix}, \qquad t \in U_\delta \cap (\gamma_L^o \cup \gamma_R^o),$$

$$P_+(t) = P_-(t)\begin{pmatrix} 0 & w(t) \\ -w(t)^{-1} & 0 \end{pmatrix}, \qquad t \in U_\delta \cap \gamma^o.$$

P3 (Fixed arc case) On the boundary of U_δ, we have as $n \to \infty$

$$P(z)N^{-1}(z) = I + O\left(\frac{1}{n}\right), \quad \text{uniformly for } z \in \partial U_\delta \setminus \Gamma.$$

P3 (Varying arc case) On the boundary of U_δ, we have as $n\theta_0 \to \infty$

$$P(z)N^{-1}(z) = I + O\left(\frac{1}{n\theta_0}\right), \quad \text{uniformly for } z \in \partial U_\delta \setminus \Gamma.$$

P4 As $z \to e^{i\theta_0}$,

$$P(z) = \begin{cases} O\begin{pmatrix} 1 & |z-e^{i\theta_0}|^{2\alpha} \\ 1 & |z-e^{i\theta_0}|^{2\alpha} \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \\ \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \alpha > 0,\ z \in O, \\ O\begin{pmatrix} |z-e^{i\theta_0}|^{-2\alpha} & 1 \\ |z-e^{i\theta_0}|^{-2\alpha} & 1 \end{pmatrix}, & \alpha > 0,\ z \in O_L \cup O_R. \end{cases} \tag{3.4.1}$$

We now have to apply a series of transformations to P in order to arrive to a Riemann-Hilbert problem with constant jumps on a systems of rays departing from the origin that can be explicitly solved in terms of special functions, namely, modified Bessel and Hankel functions.

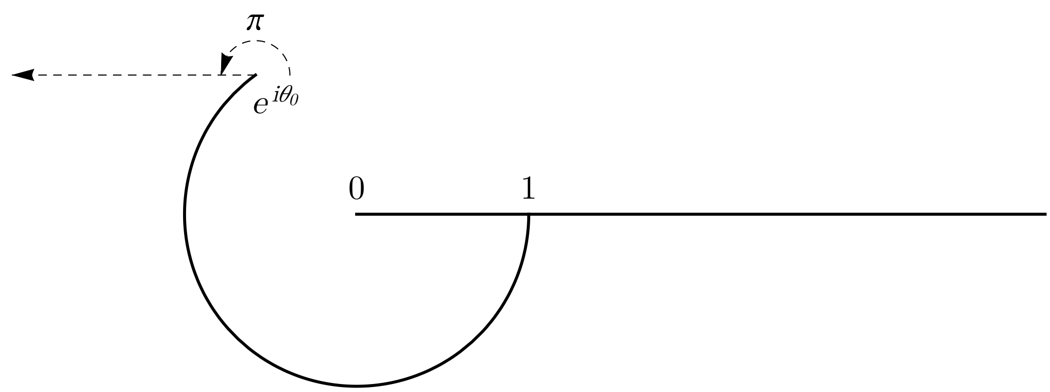

Figure 3.6: Domain of $W_\alpha(z)$

3.4.2 Reduction to constant jumps

Define

$$W_\alpha(z) := \exp\left\{\frac{\alpha}{2}\left(2\log(z - e^{i\theta_0}) - \log z + (\pi - \theta_0)i\right)\right\} \tag{3.4.2}$$

for $z \in \mathbb{C} \setminus \left(\{e^{i\theta} : \theta_0 \leq \theta \leq 2\pi\} \cup [0,\infty)\right)$, with $\log(z - e^{i\theta_0})$ and $\log z$ the branches of the logarithm in

$$\mathbb{C} \setminus \left(\{e^{i\theta} : \theta_0 \leq \theta \leq 2\pi\} \cup [0,\infty)\right)$$

corresponding to the values

$$\arg((-1 + i\sin\theta_0) - e^{i\theta_0}) = \pi \quad \text{and} \quad \arg(-1) = \pi,$$

respectively.

Define also

$$W(z) := W_\alpha(z) w_\beta(z)^{1/2} h(z)^{1/2}. \tag{3.4.3}$$

We have (see Section 5.15)

$$W(z)^2 = \begin{cases} w(z)e^{-2\pi\alpha i}, & |z|<1,\ z \notin [0,\infty), \\ w(z)e^{2\pi\alpha i}, & |z|>1,\ z \notin [0,\infty), \end{cases}$$

and

$$W_+(t)W_-(t) = w(t), \quad t \in U_\delta \cap \gamma^o. \tag{3.4.4}$$

We will later make use of the following lemma. Its proof is given in Section 5.15.

Lemma 3.4.1. *For a fixed arc γ, let σ be a closed contour in U going around the arc γ in the positive direction and leaving every point $z \in \overline{U}_\delta$ inside. For the case of a varying arc, let σ be a cycle consisting of two circles centered at the origin, one negatively oriented and contained in $U \cap \{z : |z| < 1\}$, the other positively oriented and contained in $U \cap \{z : |z| > 1\}$, and such that $\overline{U}_\delta$ is contained in the annulus bounded by these two circles. Then, for $z \in U_\delta \setminus \gamma$, we have*

$$\begin{aligned} \frac{W(z)^2}{D(z)^2} &= \left(\frac{\psi(z)}{\sqrt{z}}\right)^{2(\alpha+\beta)} \exp\left(\frac{g(z)}{2\pi i}\int_\sigma \frac{\log h(\zeta)}{g(\zeta)}\frac{d\zeta}{\zeta - z}\right) \\ &= \left(\frac{\psi(z)}{\sqrt{z}}\right)^{2(\alpha+\beta)} \exp\left(g(z)\sum_{n=0}^{\infty} c_n(z-e^{i\theta_0})^n\right), \end{aligned}$$

with c_n given by (1.3.2). *In particular,*

$$\frac{W(z)}{D(z)} = 1 + O(|z-e^{i\theta_0}|^{1/2}), \quad \textit{as } z \to e^{i\theta_0}.$$

Let us now define a matrix $P^{(1)}$ via the equality

$$P(z) = E_n(z)P^{(1)}(z)\begin{pmatrix} W(z)^{-1}\psi(z)^{-n}\sqrt{z}^n & 0 \\ 0 & W(z)\psi^n(z)\sqrt{z}^{-n} \end{pmatrix},$$

where $\sqrt{z}$ denotes the principal branch of the square root, and $E_n(z)$ is, at this moment, an arbitrary, invertible analytic matrix in U_δ, which will be more specifically chosen later. Then, $P^{(1)}$ satisfies the following RHP:

$\mathbf{P^{(1)}1}$ $P^{(1)}(z)$ is analytic $U_\delta \setminus \Gamma$ and continuous in $\{z : |z - e^{i\theta_0}| \leq \delta\} \setminus \Gamma$.

$\mathbf{P^{(1)}2}$ $P^{(1)}$ satisfies the jump relations

$$P^{(1)}_+(t) = P^{(1)}_-(t) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}, \qquad t \in U_\delta \cap \gamma_L^o,$$

$$P^{(1)}_+(t) = P^{(1)}_-(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad t \in U_\delta \cap \gamma^o,$$

$$P^{(1)}_+(t) = P^{(1)}_-(t) \begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i} & 1 \end{pmatrix}, \qquad t \in U_\delta \cap \gamma_R^o.$$

$\mathbf{P^{(1)}3}$ As $z \to e^{i\theta_0}$, $P^{(1)}(z)$ has the following behavior for $z \in U_\delta \setminus \Gamma$:

$$P^{(1)}(z) = \begin{cases} O\begin{pmatrix} |z-e^{i\theta_0}|^{\alpha} & |z-e^{i\theta_0}|^{\alpha} \\ |z-e^{i\theta_0}|^{\alpha} & |z-e^{i\theta_0}|^{\alpha} \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \\ \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} |z-e^{i\theta_0}|^{\alpha} & |z-e^{i\theta_0}|^{-\alpha} \\ |z-e^{i\theta_0}|^{\alpha} & |z-e^{i\theta_0}|^{-\alpha} \end{pmatrix}, & \alpha > 0, \text{ as } z \to e^{i\theta_0} \text{ with } z \in O, \\ O\begin{pmatrix} |z-e^{i\theta_0}|^{-\alpha} & |z-e^{i\theta_0}|^{-\alpha} \\ |z-e^{i\theta_0}|^{-\alpha} & |z-e^{i\theta_0}|^{-\alpha} \end{pmatrix}, & \alpha > 0, \text{ as } z \to e^{i\theta_0} \text{ with } z \in O_L \cup O_R. \end{cases}$$

3.4.3 Mapping U_δ onto a neighborhood of 0

Define

$$f(z) := \frac{1}{4}\left[\text{Log}\frac{\psi(z)}{\sqrt{z}}\right]^2, \quad z \in \mathbb{C} \setminus (\gamma \cup (-\infty, 0]) .$$

From the discussion following Proposition 2.2.1, we see that $f_+(z)/\sqrt{z}$ takes $\gamma^+ := \gamma \cap \{\Im z > 0\}$ onto the first quarter arc of the unit circle $\{e^{i\theta} : 0 \leq \theta \leq \pi/2\}$, while $f_-(z)/\sqrt{z}$ takes γ^+ onto the fourth quarter arc $\{e^{i\theta} : -\pi/2 \leq \theta \leq 0\}$. Moreover, $f_+(z)/\sqrt{z} = \sqrt{z}/f_-(z)$, so that $f_+(t) = f_-(t)$ for $t \in U_\delta \cap \gamma^o$, so $f(z)$ is analytic in U_δ. Also, for $|z - e^{i\theta_0}| < \sin\theta_0$, we have (see Section 3.6)

$$f(z) = \frac{ie^{-i\theta_0}\tan(\theta_0/2)}{4}(z - e^{i\theta_0}) - \frac{\tan^2(\theta_0/2)}{12e^{2i\theta_0}}(z - e^{i\theta_0})^2 + O\left((z - e^{i\theta_0})^3\right) .$$

Hence, for a sufficiently small δ, f is a conformal mapping of U_δ onto a neighborhood of 0. Moreover, $f(z)$ maps $\gamma \cap U_\delta$ into $(-\infty, 0]$, $\{z : |z| > 1\} \cap U_\delta$ to the upper half plane $\mathbb{C}_+$, and $\{z : |z| < 1\} \cap U_\delta$ to the lower half plane $\mathbb{C}_-$. Consequently,

$$f_+^{1/2}(t) = -f_-^{1/2}(t), \quad f_+^{1/4}(t) = if_-^{1/4}(t), \quad t \in \gamma^o \cap U_\delta. \tag{3.4.5}$$

To manage the case of a varying arc, we need in addition, some uniform estimates. From Rouche's theorem and the estimate (3.7.1) in Section 3.7 below, it is not difficult to see that there exists $0 < \rho_1 < 1$ such that for every $0 < \theta_0 \leq \pi - \epsilon$, the map f is conformal on

$$U_\delta = \{z : |z - e^{i\theta_0}| < \delta, \quad \delta = \rho\theta_0\}$$

for all $0 < \rho < \rho_1$. Moreover, there are positive constants C_{ρ_1} and c_{ρ_1} such that uniformly in $0 < \theta_0 \leq \pi - \epsilon$, $0 < \rho \leq \rho_1$, and $t \in [0, 2\pi]$,

$$c_{\rho_1}\rho\theta_0^2 \leq |f(z)| \leq C_{\rho_1}\rho\theta_0^2, \quad z = e^{i\theta_0} + \rho\theta_0 e^{it}. \tag{3.4.6}$$

3.4.4 Model RHP

We now use the mapping $\zeta = n^2 f(z)$ to transfer the RHP for $P^{(1)}$ in U_δ onto a neighborhood of 0 in the ζ plane on a contour Σ_Θ defined for $\Theta \in (0,\pi)$ as the union of the three infinite rays oriented towards the origin

$$\Sigma_{\Theta,1} := \{re^{\Theta i} : r \geq 0\}, \quad \Sigma_{\Theta,2} := \{re^{\pi i} : r \geq 0\}, \quad \Sigma_{\Theta,3} := \{re^{-\Theta i} : r \geq 0\}. \tag{3.4.7}$$

This leads us to the following RHP. For a fixed $\alpha > -1/2$, we wish to find Ψ such that

Ψ1 Ψ is analytic in $\mathbb{C} \setminus \Sigma_\Theta$;

Ψ2 Ψ satisfies the jump relations

$$\begin{aligned}
\Psi_+(t) &= \Psi_-(t) \begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}, \qquad t \in \Sigma^o_{\Theta,1}, \\
\Psi_+(t) &= \Psi_-(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad t \in \Sigma^o_{\Theta,2}, \\
\Psi_+(t) &= \Psi_-(t) \begin{pmatrix} 1 & 0 \\ e^{-2\alpha\pi i} & 1 \end{pmatrix}, \qquad t \in \Sigma^o_{\Theta,3};
\end{aligned}$$

Ψ3 As $\zeta \to 0$, $\Psi(\zeta)$ has the behavior

$$\Psi(\zeta) = \begin{cases} O\begin{pmatrix} |\zeta|^{\alpha} & |\zeta|^{\alpha} \\ |\zeta|^{\alpha} & |\zeta|^{\alpha} \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} \log|\zeta| & \log|\zeta| \\ \log|\zeta| & \log|\zeta| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} |\zeta|^{\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \alpha > 0, \text{ as } \zeta \to 0 \text{ with } |\arg\zeta| < \Theta, \\ O\begin{pmatrix} |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \\ |\zeta|^{-\alpha} & |\zeta|^{-\alpha} \end{pmatrix}, & \alpha > 0, \text{ as } \zeta \to 0 \text{ with } \Theta < |\arg\zeta| < \pi. \end{cases}$$

As shown in [20] (see Section 5.16 for details), a solution for this RHP is provided by a matrix Ψ_α defined as follows. For $|\arg\zeta| < \Theta$,

$$\Psi_\alpha(\zeta) = \begin{pmatrix} I_{2\alpha}(2\zeta^{1/2}) & \frac{i}{\pi}K_{2\alpha}(2\zeta^{1/2}) \\ 2\pi i\zeta^{1/2}I'_{2\alpha}(2\zeta^{1/2}) & -2\zeta^{1/2}K'_{2\alpha}(2\zeta^{1/2}) \end{pmatrix};$$

for $\Theta < \arg\zeta < \pi$,

$$\Psi_\alpha(\zeta) = \begin{pmatrix} \frac{1}{2}H^{(1)}_{2\alpha}(2(-\zeta)^{1/2}) & \frac{1}{2}H^{(2)}_{2\alpha}(2(-\zeta)^{1/2}) \\ \pi\zeta^{1/2}(H^{(1)}_{2\alpha})'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}(H^{(2)}_{2\alpha})'(2(-\zeta)^{1/2}) \end{pmatrix} \begin{pmatrix} e^{\alpha\pi i} & 0 \\ 0 & e^{-\alpha\pi i} \end{pmatrix}, \tag{3.4.8}$$

and for $-\pi < \arg\zeta < -\Theta$,

$$\Psi_\alpha(\zeta) = \begin{pmatrix} \frac{1}{2}H^{(2)}_{2\alpha}(2(-\zeta)^{1/2}) & -\frac{1}{2}H^{(1)}_{2\alpha}(2(-\zeta)^{1/2}) \\ -\pi\zeta^{1/2}(H^{(2)}_{2\alpha})'(2(-\zeta)^{1/2}) & \pi\zeta^{1/2}(H^{(1)}_{2\alpha})'(2(-\zeta)^{1/2}) \end{pmatrix} \begin{pmatrix} e^{-\alpha\pi i} & 0 \\ 0 & e^{\alpha\pi i} \end{pmatrix}, \tag{3.4.9}$$

where $I_{2\alpha}$, $K_{2\alpha}$ are modified Bessel functions and $H^{(1)}_{2\alpha}$, $H^{(2)}_{2\alpha}$ are Hankel functions (Bessel functions of the third kind).

We use the freedom we had in deforming $\gamma_L \cap U$ and $\gamma_R \cap U$ and take them to be the preimages of the rays $\Sigma_{\Theta,1}$ and $\Sigma_{\Theta,3}$, respectively, with respect to the map $f(z)$.

Remark 3.4.2. Notice that the expressions that define the solution Ψ_α are independent of Θ. If we let $C_{\Theta,1}$, $C_{\Theta,2}$ and $C_{\Theta,3}$ denote the three components of $\mathbb{C} \setminus \Sigma_\Theta$ in such a way that

$$C_{\Theta,1} = \{\zeta \in \mathbb{C} \setminus \Sigma_\Theta : \Theta < \arg\zeta < \pi\},$$

$$C_{\Theta,2} = \{\zeta \in \mathbb{C} \setminus \Sigma_\Theta : -\pi < \arg\zeta < -\Theta\},$$

and

$$C_{\Theta,3} = \{\zeta \in \mathbb{C} \setminus \Sigma_\Theta : |\arg\zeta| < \Theta\},$$

and if $\Psi_{\alpha,\Theta}$ and $\Psi_{\alpha,\Theta'}$ denote the solutions Ψ_α corresponding to the angles $\Theta' > \Theta$, then

$$\Psi_{\alpha,\Theta}(\zeta) = \Psi_{\alpha,\Theta'}(\zeta), \quad \zeta \in C_{\Theta',1} \cup C_{\Theta',2} \cup C_{\Theta,3},$$

so that $\Psi_{\alpha,\Theta'}|_{C_{\Theta',3}}$ is the analytic continuation of $\Psi_{\alpha,\Theta}|_{C_{\Theta,3}}$, $\Psi_{\alpha,\Theta}|_{C_{\Theta,1}}$ is the analytic continuation of $\Psi_{\alpha,\Theta'}|_{C_{\Theta',1}}$, and $\Psi_{\alpha,\Theta}|_{C_{\Theta,2}}$ is the analytic continuation of $\Psi_{\alpha,\Theta'}|_{C_{\Theta',2}}$.

Defining

$$P^{(1)}(z) = \Psi_\alpha(n^2 f(z))$$

we see that $P^{(1)}$ complies with $\mathrm{P}^{(1)}1 - \mathrm{P}^{(1)}3$, and so

$$P(z) = E_n(z)\Psi_\alpha(n^2 f(z)) \begin{pmatrix} W(z)^{-1}\psi(z)^{-n}\sqrt{z}^n & 0 \\ 0 & W(z)\psi^n(z)\sqrt{z}^{-n} \end{pmatrix} \qquad (3.4.10)$$

satisfies P1, P2, and P4. We now find E_n in such a way that, in addition, P3 is also satisfied.

We first consider the case of a fixed arc, that is, θ_0 is fixed, and so is U_δ. Using the asymptotic expansion for large $\zeta \in \mathbb{C}\setminus\Sigma_\Theta$, see (9.7.1)-(9.7.4) in [1] and formula (6.28) of [20], one finds that uniformly as $\zeta \to \infty$,

$$\Psi_\alpha(\zeta) = \begin{pmatrix} \frac{1}{\sqrt{2\pi}\zeta^{1/4}} & 0 \\ 0 & \sqrt{2\pi}\zeta^{1/4} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + O(\zeta^{-1/2}) & i + O(\zeta^{-1/2}) \\ i + O(\zeta^{-1/2}) & 1 + O(\zeta^{-1/2}) \end{pmatrix} \begin{pmatrix} e^{2\zeta^{1/2}} & 0 \\ 0 & e^{-2\zeta^{1/2}} \end{pmatrix}. \tag{3.4.11}$$

Here we note that when

$$\zeta = n^2 f(z) = \frac{n^2}{4}\left[\mathrm{Log}\frac{\psi(z)}{\sqrt{z}}\right]^2,$$

$e^{2\zeta^{1/2}} = \psi(z)^n \sqrt{z}^{-n}$, and so making $\zeta = n^2 f(z)$ yields that for every $\varepsilon \in (0, \delta)$,

$$\begin{aligned} \Psi_\alpha(n^2 f(z)) = &\begin{pmatrix} \frac{1}{\sqrt{2\pi n}} & 0 \\ 0 & \sqrt{2\pi n} \end{pmatrix} \begin{pmatrix} f(z)^{-1/4} & 0 \\ 0 & f(z)^{1/4} \end{pmatrix} \\ &\times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + O\left(\frac{1}{n}\right) & i + O\left(\frac{1}{n}\right) \\ i + O\left(\frac{1}{n}\right) & 1 + O\left(\frac{1}{n}\right) \end{pmatrix} \begin{pmatrix} \frac{\psi(z)^n}{\sqrt{z}^n} & 0 \\ 0 & \frac{\sqrt{z}^n}{\psi(z)^n} \end{pmatrix} \end{aligned}$$

uniformly on $\{z : \varepsilon < |z - e^{i\theta_0}| \leq \delta\} \setminus \Gamma$ as $n \to \infty$.

In the case of a varying weight, by (3.4.6), we also have, after making $\zeta = n^2 f(z)$, that for every $\rho_0 \in (0, \rho_1)$,

$$\begin{aligned} \Psi_\alpha(n^2 f(z)) = &\begin{pmatrix} \frac{1}{\sqrt{2\pi n}} & 0 \\ 0 & \sqrt{2\pi n} \end{pmatrix} \begin{pmatrix} f(z)^{-1/4} & 0 \\ 0 & f(z)^{1/4} \end{pmatrix} \\ &\times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + O\left(\frac{1}{n\theta_0}\right) & i + O\left(\frac{1}{n\theta_0}\right) \\ i + O\left(\frac{1}{n\theta_0}\right) & 1 + O\left(\frac{1}{n\theta_0}\right) \end{pmatrix} \begin{pmatrix} \frac{\psi(z)^n}{\sqrt{z}^n} & 0 \\ 0 & \frac{\sqrt{z}^n}{\psi(z)^n} \end{pmatrix} \end{aligned} \tag{3.4.12}$$

uniformly, as $n\theta_0 \to \infty$, for $\theta_0 \in (0, \pi - \epsilon]$ and z in the set

$$\{z : \rho_0\theta_0 < |z - e^{i\theta_0}| < \rho_1\theta_0\} \setminus \Gamma.$$

Combining (3.4.13) and (3.4.12) with (3.4.10), we obtain that for a fixed arc,

$$\begin{aligned} P(z) = E_n(z) & \begin{pmatrix} \frac{1}{\sqrt{2\pi n}} & 0 \\ 0 & \sqrt{2\pi n} \end{pmatrix} \begin{pmatrix} f(z)^{-1/4} & 0 \\ 0 & f(z)^{1/4} \end{pmatrix} \\ & \times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + O\left(\frac{1}{n}\right) & i + O\left(\frac{1}{n}\right) \\ i + O\left(\frac{1}{n}\right) & 1 + O\left(\frac{1}{n}\right) \end{pmatrix} \begin{pmatrix} W(z)^{-1} & 0 \\ 0 & W(z) \end{pmatrix} \end{aligned} \tag{3.4.13}$$

uniformly on $\{z : \varepsilon < |z - e^{i\theta_0}| \leq \delta\} \setminus \Gamma$ as $n \to \infty$, while for a varying arc, the equality (3.4.13) is also true when replacing the $O(1/n)$ by $O\left(\frac{1}{n\theta_0}\right)$ in a uniform sense, as $n\theta_0 \to \infty$, for $\theta_0 \in (0, \pi - \epsilon]$ and z in the set

$$\{z : \rho_0\theta_0 < |z - e^{i\theta_0}| < \rho_1\theta_0\} \setminus \Gamma.$$

Since we want to have $P(z)N^{-1}(z) = I + O\left(1/n\right)$ in the fixed arc case, and $P(z)N^{-1}(z) = I + O\left(n\theta_0\right)^{-1}$ in the varying case, uniformly for $z \in \partial U_\delta \setminus \Gamma$, we now choose

$$E_n(z) := N(z) \begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} f(z)^{1/4} & 0 \\ 0 & f(z)^{-1/4} \end{pmatrix} \begin{pmatrix} \sqrt{2\pi n} & 0 \\ 0 & \frac{1}{\sqrt{2\pi n}} \end{pmatrix}. \tag{3.4.14}$$

With this matrix E_n we then have

$$P(z) = N(z) \begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1} \end{pmatrix} \left(I + O\left(\frac{1}{n}\right) \right) \begin{pmatrix} W(z)^{-1} & 0 \\ 0 & W(z) \end{pmatrix}, \tag{3.4.15}$$

with $O(1/n)$ replaced by $O\left(\frac{1}{n\theta_0}\right)$ in the case of a varying arc. It is clear that this P complies with the condition P3 in case of a fixed arc, and moreover, we see that indeed

$$P(z)N^{-1}(z) = I + O\left(\frac{1}{n}\right) \tag{3.4.16}$$

uniformly on $\{z : \varepsilon < |z - e^{i\theta_0}| \le \delta\} \setminus \Gamma$ as $n \to \infty$.

For a varying arc, it follows from Lemma 3.4.1 that $W(z)/D(z)$ and $D(z)/W(z)$ remain uniformly bounded for $\theta_0 \in (0, \pi - \epsilon]$ and z in the set $\{z : |z - e^{i\theta_0}| < \rho_1\theta_0\} \setminus \Gamma$. In view of (1.3.1), the same is true for $|D(\infty)|$ and $|D(\infty)|^{-1}$. With these observations and the estimate (3.7.2), we see from the definition of N in (3.3.2) that

$$N(z)\begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} W(z)^{-1} & 0 \\ 0 & W(z) \end{pmatrix} N^{-1}(z)$$

remain uniformly bounded for $0 < \theta_0 \le \pi - \epsilon$ and z in the set

$$\{z : \rho_0\theta_0 < |z - e^{i\theta_0}| < \rho_1\theta_0\} \setminus \Gamma.$$

Then, by (3.4.15), we conclude that the condition P3 also holds for the varying arc case, and moreover,

$$P(z)N^{-1}(z) = I + O\left(\frac{1}{n\theta_0}\right) \tag{3.4.17}$$

uniformly, as $n\theta_0 \to \infty$, for $0 < \theta_0 \le \pi - \epsilon$ and z in the set $\{z : \rho_0\theta_0 < |z - e^{i\theta_0}| < \rho_1\theta_0\} \setminus \Gamma$.

Now, the matrix E_n is in principle only analytic on $U_\delta \setminus \gamma$, but we now show that E_n is indeed analytic in U_δ.

Using the condition N2 satisfied by the matrix $N(z)$ and (3.4.4), we get

$$N_+(t)\begin{pmatrix} W_+(t) & 0 \\ 0 & W_+(t)^{-1} \end{pmatrix} = N_-(t)\begin{pmatrix} W_-(t) & 0 \\ 0 & W_-(t)^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad t \in \gamma^o \cap U_\delta. \tag{3.4.18}$$

Using (3.4.5) and (3.4.18), we get from (3.4.14) that

$$\begin{aligned}
E_{n+}(t) &= E_{n-}(t) \begin{pmatrix} \frac{1}{\sqrt{2\pi n}} & 0 \\ 0 & \sqrt{2\pi n} \end{pmatrix} \begin{pmatrix} f_-(t)^{-1/4} & 0 \\ 0 & f_-(t)^{1/4} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\
&\quad \times \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} f_+(t)^{1/4} & 0 \\ 0 & f_+(t)^{-1/4} \end{pmatrix} \begin{pmatrix} \sqrt{2\pi n} & 0 \\ 0 & \frac{1}{\sqrt{2\pi n}} \end{pmatrix} \\
&= E_{n-}(t).
\end{aligned}$$

This proves that E_n is analytic in $U_\delta \setminus \{e^{i\theta_0}\}$. We now prove E_n is analytic at $e^{i\theta_0}$.

Using Lemma 3.4.1, we deduce that as $z \to e^{i\theta_0}$,

$$\begin{aligned}
N(z) \begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1} \end{pmatrix} &= \begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ -\frac{a(z)-a^{-1}(z)}{2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix} \begin{pmatrix} \frac{W(z)}{D(z)} & 0 \\ 0 & \frac{D(z)}{W(z)} \end{pmatrix} \\
&= O\begin{pmatrix} |z-e^{i\theta_0}|^{-1/4} & |z-e^{i\theta_0}|^{-1/4} \\ |z-e^{i\theta_0}|^{-1/4} & |z-e^{i\theta_0}|^{-1/4} \end{pmatrix} O\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned} \tag{3.4.19}$$

Since the function $f(z)$ has a simple zero at $e^{i\theta_0}$, we have that as $z \to e^{i\theta_0}$,

$$\begin{pmatrix} f(z)^{1/4} & 0 \\ 0 & f(z)^{-1/4} \end{pmatrix} = O\begin{pmatrix} |z-e^{i\theta_0}|^{1/4} & 0 \\ 0 & |z-e^{i\theta_0}|^{-1/4} \end{pmatrix}. \tag{3.4.20}$$

Then, from (3.4.14), (3.4.19) and (3.4.20) we see that as $z \to e^{i\theta_0}$,

$$E_n(z) = O\begin{pmatrix} 1 & |z-e^{i\theta_0}|^{-1/2} \\ 1 & |z-e^{i\theta_0}|^{-1/2} \end{pmatrix},$$

which implies that E_n has a removable singularity at $e^{i\theta_0}$, and so E_n is analytic in U_δ.

We conclude this section by writing out the solution P, which is the same for both the fixed arc and the varying arc cases:

$$P(z) = N(z)\begin{pmatrix} W(z) & 0 \\ 0 & W(z)^{-1} \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} f(z)^{1/4} & 0 \\ 0 & f(z)^{-1/4} \end{pmatrix}$$
$$\times\begin{pmatrix} \sqrt{2\pi n} & 0 \\ 0 & \frac{1}{\sqrt{2\pi n}} \end{pmatrix}\Psi_\alpha(n^2 f(z))\begin{pmatrix} W(z)^{-1}\psi(z)^{-n}\sqrt{z}^n & 0 \\ 0 & W(z)\psi^n(z)\sqrt{z}^{-n} \end{pmatrix}. \tag{3.4.21}$$

3.4.5 Invertibility of P

It is clear from (3.4.21) that $\det P = \det \Psi_\alpha(n^2 f)$. To evaluate $\det \Psi_\alpha(n^2 f)$, we use Liouville's theorem. We have that $\det \Psi_\alpha(\zeta)$ is analytic on $\mathbb{C} \setminus \Sigma_\Theta$. Each jump matrix for Ψ_α has for determinant 1, implying that $(\det \Psi_\alpha(\zeta))_+ = (\det \Psi_\alpha(\zeta))_-$ across $\Sigma^o_{\Theta,1}$, $\Sigma^o_{\Theta,2}$, and $\Sigma^o_{\Theta,3}$. That is, $\det \Psi_\alpha(\zeta)$ is analytic in $\mathbb{C} \setminus \{0\}$. Moreover, as ζ tends to the origin,

$$\det \Psi_\alpha(\zeta) = \begin{cases} O\left(|\zeta|^{2\alpha}\right), & \alpha < 0, \\ O\left(\log|\zeta|\right), & \alpha = 0, \\ O(1), & \text{for } \alpha > 0, \text{ as } \zeta \to 0 \text{ with } |\arg \zeta| < \Theta, \\ O\left(|\zeta|^{-2\alpha}\right), & \text{for } \alpha > 0, \text{ as } \zeta \to 0 \text{ with } \Theta < |\arg \zeta| < \pi. \end{cases}$$

Hence $\det \Psi_\alpha(\zeta)$ has a removable singularity at 0 and it is therefore entire. Finally, using (3.4.11), we observe that $\det \Psi_\alpha(\infty) = 1$, so that $\det \Psi_\alpha(\zeta) \equiv 1$.

3.4.6 Riemann-Hilbert problem for $\tilde{P}$

This subsection will only treat the case of a fixed arc, since the treatment for a varying arc mirrors that just done above for P.

We now seek to find $\tilde{P}$ solving the following Riemann-Hilbert problem:

$\tilde{\mathbf{P}}\mathbf{1}$ $\tilde{P}(z)$ is analytic $\tilde{U}_\delta \setminus \Gamma$ and continuous in $\{z : |z - e^{-i\theta_0}| \le \delta\} \setminus \Gamma$.

$\tilde{\mathbf{P}}\mathbf{2}$ $\tilde{P}$ has the same jumps as S on $\Gamma^o \cap \tilde{U}_\delta$, that is,

$$\tilde{P}_+(t) = \tilde{P}_-(t) \begin{pmatrix} 1 & 0 \\ t^n \psi(t)^{-2n} w(t)^{-1} & 1 \end{pmatrix}, \qquad t \in \tilde{U}_\delta \cap (\gamma_L^o \cup \gamma_R^o),$$

$$\tilde{P}_+(t) = \tilde{P}_-(t) \begin{pmatrix} 0 & w(t) \\ -w(t)^{-1} & 0 \end{pmatrix}, \qquad t \in \tilde{U}_\delta \cap \gamma^o.$$

$\tilde{\mathbf{P}}\mathbf{3}$ On the boundary of $\tilde{U}_\delta$, we have that as $n \to \infty$,

$$\tilde{P}(z) N^{-1}(z) = I + O\left(\frac{1}{n}\right), \quad \text{uniformly for } z \in \partial\tilde{U}_\delta \setminus \Gamma.$$

$\tilde{\mathbf{P}}\mathbf{4}$ As $z \to e^{-i\theta_0}$,

$$\tilde{P}(z) = \begin{cases} O\begin{pmatrix} 1 & |z - e^{-i\theta_0}|^{2\beta} \\ 1 & |z - e^{-i\theta_0}|^{2\beta} \end{pmatrix}, & \beta < 0, \\ O\begin{pmatrix} \log|z - e^{-i\theta_0}| & \log|z - e^{-i\theta_0}| \\ \log|z - e^{-i\theta_0}| & \log|z - e^{-i\theta_0}| \end{pmatrix}, & \beta = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \beta > 0,\ z \in O, \\ O\begin{pmatrix} |z - e^{-i\theta_0}|^{-2\beta} & 1 \\ |z - e^{-i\theta_0}|^{-2\beta} & 1 \end{pmatrix}, & \beta > 0,\ z \in O_L \cup O_R. \end{cases}$$

Let us define

$$W_\beta(z) := \exp\left\{\beta/2 \left[2\log(z - e^{-i\theta_0}) - \log z + (\pi + \theta_0)i\right]\right\} \tag{3.4.22}$$

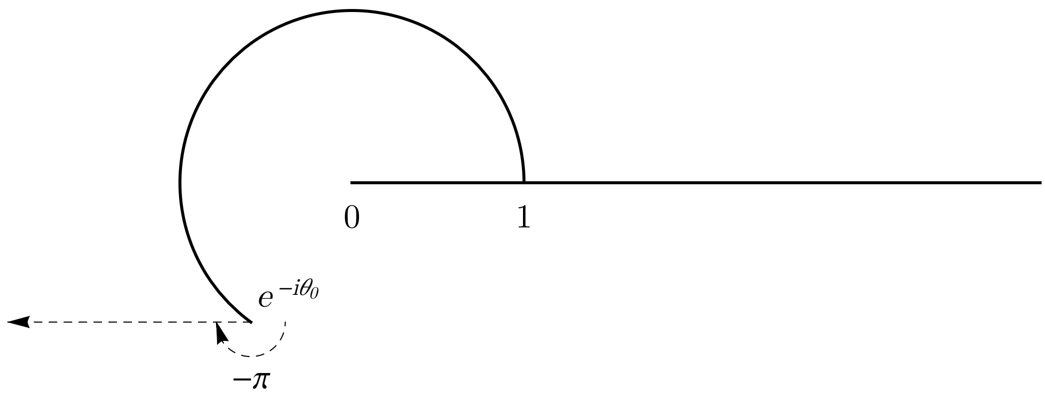

Figure 3.7: Domain of $W_\beta(z)$

for $z \in \mathbb{C} \setminus \big(\{e^{i\theta} : 0 \le \theta \le 2\pi - \theta_0\} \cup [0,\infty)\big)$, with $\log(z - e^{i\theta_0})$ and $\log z$ the branches of the logarithm in

$$\mathbb{C} \setminus \big(\{e^{i\theta} : 0 \le \theta \le 2\pi - \theta_0\} \cup [0,\infty)\big)$$

corresponding to $\arg(-1 + i\sin(-\theta_0) - e^{-i\theta_0}) = -\pi$ and $\arg(-1) = \pi$, respectively. Let us also set

$$\tilde{W}(z) = w_\alpha(z)^{1/2} W_\beta(z) h(z)^{1/2}.$$

As we verify in Section 5.15, we have

$$\tilde{W}(z)^2 = \begin{cases} w(z)e^{2\pi\beta i}, & |z| < 1,\ z \notin [0,\infty), \\ w(z)e^{-2\pi\beta i}, & |z| > 1,\ z \notin [0,\infty), \end{cases}$$

and

$$\tilde{W}_+(t)\tilde{W}_-(t) = w(t), \quad t \in \tilde{U}_\delta \cap \gamma^o \tag{3.4.23}$$

The analogue of Lemma 3.4.1 is then the following.

Lemma 3.4.3. *Let σ be a closed contour in U going around the arc γ in the positive direction and leaving every point $z \in \overline{\tilde{U}}_\delta \setminus \gamma$ inside. For the case of a varying arc, let σ be a cycle consisting of two circles centered at the origin, one negatively oriented and contained in $U \cap \{z : |z| < 1\}$, the other positively oriented and contained in $U \cap \{z : |z| > 1\}$, and such* *that $\overline{\tilde{U}_\delta}$ is contained in the annulus bounded by these two circles.For $z \in \tilde{U}_\delta \setminus \gamma$, we have*

$$\frac{\tilde{W}(z)^2}{D(z)^2} = \left(\frac{\psi(z)}{\sqrt{z}}\right)^{2(\alpha+\beta)} \exp\left(g(z)\sum_{n=0}^{\infty} d_n(z-e^{-i\theta_0})^n\right)$$

with d_n given by (1.3.3)*. In particular,*

$$\frac{\tilde{W}(z)}{D(z)} = 1 + O(|z-e^{-i\theta_0}|^{1/2}), \quad \text{as } z \to e^{-i\theta_0}.$$

Let us now define a matrix $\tilde{P}^{(1)}$ via the equality

$$\tilde{P}(z) = \tilde{E}_n(z)\tilde{P}^{(1)}(z)\begin{pmatrix} \tilde{W}(z)^{-1}\psi(z)^{-n}\sqrt{z}^n & 0 \\ 0 & \tilde{W}(z)\psi^n(z)\sqrt{z}^{-n} \end{pmatrix},$$

where $\tilde{E}_n(z)$ is an arbitrary, invertible analytic matrix in $\tilde{U}_\delta$. $\tilde{P}^{(1)}$ satisfies the following RHP:

$\tilde{\mathbf{P}}^{(\mathbf{1})}\mathbf{1}$ $\tilde{P}^{(1)}(z)$ is analytic $\tilde{U}_\delta \setminus \Gamma$ and continuous in $\{z : |z-e^{-i\theta_0}| \le \delta\} \setminus \Gamma$.

$\tilde{\mathbf{P}}^{(\mathbf{1})}\mathbf{2}$ $\tilde{P}^{(1)}$ satisfies the following jump relations:

$$\tilde{P}^{(1)}_+(t) = \tilde{P}^{(1)}_-(t)\begin{pmatrix} 1 & 0 \\ e^{-2\beta\pi i} & 1 \end{pmatrix}, \qquad t \in \tilde{U}_\delta \cap \gamma^o_L,$$

$$\tilde{P}^{(1)}_+(t) = \tilde{P}^{(1)}_-(t)\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad t \in \tilde{U}_\delta \cap \gamma^o,$$

$$\tilde{P}^{(1)}_+(t) = \tilde{P}^{(1)}_-(t)\begin{pmatrix} 1 & 0 \\ e^{2\beta\pi i} & 1 \end{pmatrix}, \qquad t \in \tilde{U}_\delta \cap \gamma^o_R.$$

$\tilde{\mathbf{P}}^{(\mathbf{1})}\mathbf{3}$ As $z \to e^{-i\theta_0}$, $\tilde{P}^{(1)}(z)$ has the following behavior for $z \in \tilde{U}_\delta \setminus \Gamma$:

$$\tilde{P}^{(1)}(z) = \begin{cases} O\begin{pmatrix} |z-e^{-i\theta_0}|^{\beta} & |z-e^{-i\theta_0}|^{\beta} \\ |z-e^{-i\theta_0}|^{\beta} & |z-e^{-i\theta_0}|^{\beta} \end{pmatrix}, & \beta < 0, \\ O\begin{pmatrix} \log|z-e^{-i\theta_0}| & \log|z-e^{-i\theta_0}| \\ \log|z-e^{-i\theta_0}| & \log|z-e^{-i\theta_0}| \end{pmatrix}, & \beta = 0, \\ O\begin{pmatrix} |z-e^{-i\theta_0}|^{\beta} & |z-e^{-i\theta_0}|^{-\beta} \\ |z-e^{-i\theta_0}|^{\beta} & |z-e^{-i\theta_0}|^{-\beta} \end{pmatrix}, & \beta > 0, \text{ as } z \to e^{-i\theta_0}, \ z \in O, \\ O\begin{pmatrix} |z-e^{-i\theta_0}|^{-\beta} & |z-e^{-i\theta_0}|^{-\beta} \\ |z-e^{-i\theta_0}|^{-\beta} & |z-e^{-i\theta_0}|^{-\beta} \end{pmatrix}, & \beta > 0, \text{ as } z \to e^{-i\theta_0}, \ z \in O_L \cup O_R. \end{cases}$$

The function $f(z)$ as defined in (3.4.3) satisfies that $f(z) = \overline{f(\overline{z})}$, so that $f(z)$ conformally maps $\gamma \cap \tilde{U}_\delta$ into $(-\infty, 0]$, $\{z : |z| > 1\} \cap \tilde{U}_\delta$ into $\mathbb{C}_-$, and $\{z : |z| < 1\} \cap \tilde{U}_\delta$ into $\mathbb{C}_+$. Consequently,

$$f_+^{1/2}(t) = -f_-^{1/2}(t), \quad f_+^{1/4}(t) = -i f_-^{1/4}(t), \quad t \in \gamma^o \cap \tilde{U}_\delta. \tag{3.4.24}$$

We now use the mapping $\zeta = n^2 f(z)$ to transfer the RHP for $\tilde{P}^{(1)}$ in $\tilde{U}_\delta$ onto a neighborhood of 0 in the ζ plane on the contour $\tilde{\Sigma}_\Theta$ that is the union of the three infinite rays $-\Sigma_{\Theta,1}$, $-\Sigma_{\Theta,2}$, and $-\Sigma_{\Theta,3}$. Here, $-\Sigma_{\Theta,k}$ denotes the ray $\Sigma_{\Theta,k}$ as defined in (3.4.7) but with reverse orientation, that is, departing from the origin. This leads us to the following RHP:

$\tilde{\mathbf{\Psi}}\mathbf{1}$ $\tilde{\Psi}$ is analytic in $\mathbb{C} \setminus \tilde{\Sigma}_\Theta$.

$\tilde{\Psi}$**2** $\tilde{\Psi}$ satisfies the jump relations

$$\tilde{\Psi}_+(t) = \tilde{\Psi}_-(t) \begin{pmatrix} 1 & 0 \\ e^{2\beta\pi i} & 1 \end{pmatrix}, \qquad t \in -\Sigma^o_{\Theta,1},$$

$$\tilde{\Psi}_+(t) = \tilde{\Psi}_-(t) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad t \in -\Sigma^o_{\Theta,2},$$

$$\tilde{\Psi}_+(t) = \tilde{\Psi}_-(t) \begin{pmatrix} 1 & 0 \\ e^{-2\beta\pi i} & 1 \end{pmatrix}, \qquad t \in -\Sigma^o_{\Theta,3}.$$

$\tilde{\Psi}$**3** As $\zeta \to 0$, $\tilde{\Psi}$ behaves exactly as Ψ with α replaced by β.

As shown in [20], see Section 5.16 for details, a solution to this RHP is given by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_\beta(\zeta) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where conjugation by the Pauli matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ corrects the reversal of the contours.

Then

$$\tilde{P}^{(1)}(z) := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_\beta(n^2 f(z)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

complies with $\tilde{\text{P}}^{(1)}1 - \tilde{\text{P}}^{(1)}3$, and thus

$$\tilde{P}(z) = \tilde{E}_n(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_\beta(n^2 f(z)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \tilde{W}(z)^{-1}\psi(z)^{-n}\sqrt{z}^n & 0 \\ 0 & \tilde{W}(z)\psi^n(z)\sqrt{z}^{-n} \end{pmatrix}$$

satisfies $\tilde{\mathrm{P}}1, \tilde{\mathrm{P}}2$, and $\tilde{\mathrm{P}}4$. We now find $\tilde{E}_n$ so that $\tilde{\mathrm{P}}3$ is also satisfied. As above, we use the asymptotic expansion of Ψ_β for large ζ to obtain

$$\tilde{P}(z) = \tilde{E}_n(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2\pi n}} & 0 \\ 0 & \sqrt{2\pi n} \end{pmatrix} \begin{pmatrix} f(z)^{-1/4} & 0 \\ 0 & f(z)^{1/4} \end{pmatrix}$$
$$\times \frac{1}{\sqrt{2}} \begin{pmatrix} 1 + O\left(\frac{1}{n}\right) & i + O\left(\frac{1}{n}\right) \\ i + O\left(\frac{1}{n}\right) & 1 + O\left(\frac{1}{n}\right) \end{pmatrix} \begin{pmatrix} \tilde{W}(z)^{-1} & 0 \\ 0 & \tilde{W}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

uniformly for $z \in \partial\tilde{U}_\delta \setminus \Gamma$ as $n \to \infty$. Since we want to have $\tilde{P}(z)N^{-1}(z) = I + O\left(\frac{1}{n}\right)$ uniformly for $z \in \partial\tilde{U}_\delta \setminus \Gamma$, we now choose

$$\begin{aligned} \tilde{E}_n(z) &:= N(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \tilde{W}(z) & 0 \\ 0 & \tilde{W}(z)^{-1} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \\ &\quad \times \begin{pmatrix} f(z)^{1/4} & 0 \\ 0 & f(z)^{-1/4} \end{pmatrix} \begin{pmatrix} \sqrt{2\pi n} & 0 \\ 0 & \frac{1}{\sqrt{2\pi n}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= N(z) \begin{pmatrix} \tilde{W}(z) & 0 \\ 0 & \tilde{W}(z)^{-1} \end{pmatrix} \begin{pmatrix} \sqrt{\pi n} f(z)^{1/4} & \frac{i}{2\sqrt{\pi n} f(z)^{1/4}} \\ i\sqrt{\pi n} f(z)^{1/4} & \frac{1}{2\sqrt{\pi n} f(z)^{1/4}} \end{pmatrix}. \end{aligned} \tag{3.4.25}$$

With this matrix $\tilde{E}_n$ we then have

$$\tilde{P}(z) = N(z) \begin{pmatrix} \tilde{W}(z) & 0 \\ 0 & \tilde{W}(z)^{-1} \end{pmatrix} (I + O(1/n)) \begin{pmatrix} \tilde{W}(z)^{-1} & 0 \\ 0 & \tilde{W}(z) \end{pmatrix},$$

and it is clear that this $\tilde{P}$ complies with the condition $\tilde{\text{P}}3$. We verify that the matrix $\tilde{E}_n$ is analytic in $\tilde{U}_\delta$. Using the condition N2 satisfied by the matrix $N(z)$ and (3.4.23), we get

$$N_+(t)\begin{pmatrix}\tilde{W}_+(t) & 0 \\ 0 & \tilde{W}_+(t)^{-1}\end{pmatrix} = N_-(t)\begin{pmatrix}\tilde{W}_-(t) & 0 \\ 0 & \tilde{W}_-(t)^{-1}\end{pmatrix}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}, \quad t \in \gamma^o \cap \tilde{U}_\delta. \tag{3.4.26}$$

Using (3.4.24) and (3.4.26), we get from (3.4.25) that for $t \in \gamma^o \cap \tilde{U}_\delta$,

$$\begin{aligned}\tilde{E}_{n+}(t) &= N_-(t)\begin{pmatrix}\tilde{W}_-(t) & 0 \\ 0 & \tilde{W}_-(t)^{-1}\end{pmatrix}\begin{pmatrix}0 & 1 \\ -1 & 0\end{pmatrix}\begin{pmatrix}\sqrt{\pi n}f_+(t)^{1/4} & \frac{i}{2\sqrt{\pi n}f_+(t)^{1/4}} \\ i\sqrt{\pi n}f_+(t)^{1/4} & \frac{1}{2\sqrt{\pi n}f_+(t)^{1/4}}\end{pmatrix} \\ &= \tilde{E}_{n-}(t).\end{aligned}$$

This proves that $\tilde{E}_n$ is analytic in $\tilde{U}_\delta \setminus \{e^{-i\theta_0}\}$. Using Lemma 3.4.3 we readily see that $\tilde{E}_n$ is analytic at $e^{-i\theta_0}$. Then, our solution $\tilde{P}$ is chosen to be

$$\begin{aligned}\tilde{P}(z) = N(z)&\begin{pmatrix}\tilde{W}(z) & 0 \\ 0 & \tilde{W}(z)^{-1}\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}\begin{pmatrix}f(z)^{1/4} & 0 \\ 0 & f(z)^{-1/4}\end{pmatrix}\begin{pmatrix}\sqrt{2\pi n} & 0 \\ 0 & \frac{1}{\sqrt{2\pi n}}\end{pmatrix} \\ &\times \begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\Psi_\beta(n^2 f(z))\begin{pmatrix}1 & 0 \\ 0 & -1\end{pmatrix}\begin{pmatrix}\tilde{W}(z)^{-1}\psi(z)^{-n}\sqrt{z}^n & 0 \\ 0 & \tilde{W}(z)\psi^n(z)\sqrt{z}^{-n}\end{pmatrix}.\end{aligned}$$

3.5 Transformation $S \mapsto R$

We now construct $\hat{\Gamma}$ in a convenient way. Given a sufficiently small δ and an angle $\Theta \in (\pi/2, \pi)$, let $z_{\delta,\Theta}$ and $\eta_{\delta,\Theta}$ be the unique points of ∂U_δ such that

$$f(z_{\delta,\Theta}) \in \Sigma_{\Theta,1}, \quad f(\eta_{\delta,\Theta}) \in \Sigma_{\Theta,3}.$$

Then, their conjugates $\overline{z}_{\delta,\Theta}$ and $\overline{\eta}_{\delta,\Theta}$ are the unique points of $\partial\tilde{U}_\delta$ such that

$$f(\overline{z}_{\delta,\Theta}) \in \Sigma_{\Theta,3}, \quad f(\overline{\eta}_{\delta,\Theta}) \in \Sigma_{\Theta,1}.$$

Let us define

$$\check{\gamma}_L := \{z \in U_\delta : f(z) \in \Sigma_{\Theta,1}\} \cup \{z \in \tilde{U}_\delta : f(z) \in \Sigma_{\Theta,3}\},$$
$$\check{\gamma}_R := \{z \in U_\delta : f(z) \in \Sigma_{\Theta,3}\} \cup \{z \in \tilde{U}_\delta : f(z) \in \Sigma_{\Theta,1}\},$$

and the clockwise oriented circular arcs

$$\hat{\gamma}_L := \{|z_{\delta,\Theta}|e^{i\theta} : \arg z_{\delta,\Theta} \leq \theta \leq 2\pi - \arg z_{\delta,\Theta}\},$$
$$\hat{\gamma}_R := \{|\eta_{\delta,\Theta}|e^{i\theta} : \arg \eta_{\delta,\Theta} \leq \theta \leq 2\pi - \arg \eta_{\delta,\Theta}\}.$$

We take

$$\gamma_L = \hat{\gamma}_L \cup \check{\gamma}_L, \quad \gamma_R = \hat{\gamma}_R \cup \check{\gamma}_R$$

and, as previously defined,

$$\Gamma = \Gamma_{\delta,\theta} = \gamma_L \cup \gamma \cup \gamma_R,$$

see Figure 3.8.

We now perform the final transformation, namely,

$$R(z) = R_{\delta,\Theta}(z) = \begin{cases} S(z)N^{-1}(z), & z \in \mathbb{C} \setminus (\overline{U}_\delta \cup \overline{\tilde{U}}_\delta \cup \Gamma), \\ S(z)P^{-1}(z), & z \in U_\delta \setminus \Gamma, \\ S(z)\tilde{P}^{-1}(z), & z \in \tilde{U}_\delta \setminus \Gamma. \end{cases} \tag{3.5.1}$$

By construction, the jumps of S and N across $\gamma \setminus (U_\delta \cup \tilde{U}_\delta)$ are the same, so R has an analytic continuation there. Since the jumps of S and P on $U_\delta \cap \Gamma^o$ are the same, R is analytic across $U_\delta \cap \Gamma^o$, and similarly, R is also analytic across $\tilde{U}_\delta \cap \Gamma^o$. This leaves R analytic on $\mathbb{C} \setminus \hat{\Gamma}$,

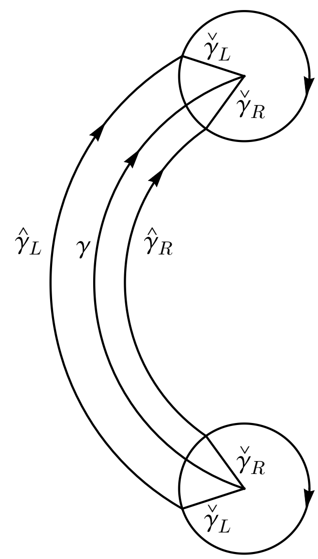

Figure 3.8: Contour $\Gamma = \Gamma_{\delta,\Theta}$

except for possible isolated singularities at $e^{\pm i\theta_0}$, where

$$\hat{\Gamma} = \hat{\Gamma}_{\delta,\Theta} := \partial U_\delta \cup \partial \tilde{U}_\delta \cup \hat{\gamma}_L \cup \hat{\gamma}_R,$$

see Figure 3.9.

We now check that R is analytic at the point $e^{i\theta_0}$, and by completely analogous arguments that we omit, it will follow that R is analytic at the point $e^{-i\theta_0}$ as well.

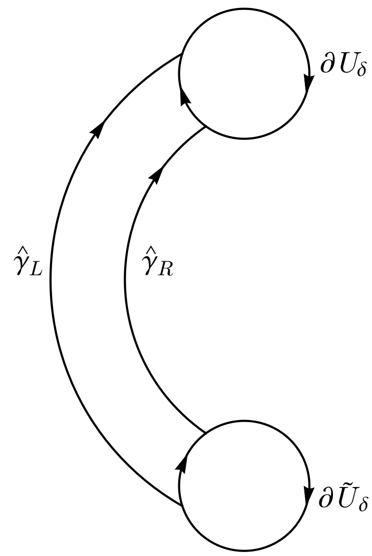

Figure 3.9: Contour $\hat{\Gamma} = \hat{\Gamma}_{\delta,\Theta}$

Since $\det(P) = 1$, we get from (3.4.1) that as $z \to e^{i\theta_0}$,

$$P^{-1}(z) = \begin{cases} O\begin{pmatrix} |z-e^{i\theta_0}|^{2\alpha} & |z-e^{i\theta_0}|^{2\alpha} \\ 1 & 1 \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \\ \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \alpha > 0,\ z \in O, \\ O\begin{pmatrix} 1 & 1 \\ |z-e^{i\theta_0}|^{-2\alpha} & |z-e^{i\theta_0}|^{-2\alpha} \end{pmatrix}, & \alpha > 0,\ z \in O_L \cup O_R. \end{cases} \tag{3.5.2}$$

Then, since $R = SP^{-1}$ in U_δ, it follows from (3.2.3) and (3.5.2) that as $z \to e^{i\theta_0}$,

$$R(z) = \begin{cases} O\begin{pmatrix} |z-e^{i\theta_0}|^{2\alpha} & |z-e^{i\theta_0}|^{2\alpha} \\ |z-e^{i\theta_0}|^{2\alpha} & |z-e^{i\theta_0}|^{2\alpha} \end{pmatrix}, & \alpha < 0, \\ O\begin{pmatrix} \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \\ \log|z-e^{i\theta_0}| & \log|z-e^{i\theta_0}| \end{pmatrix}, & \alpha = 0, \\ O\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, & \alpha > 0,\ z \in O, \\ O\begin{pmatrix} |z-e^{i\theta_0}|^{-2\alpha} & |z-e^{i\theta_0}|^{-2\alpha} \\ |z-e^{i\theta_0}|^{-2\alpha} & |z-e^{i\theta_0}|^{-2\alpha} \end{pmatrix}, & \alpha > 0,\ z \in O_L \cup O_R. \end{cases} \tag{3.5.3}$$

Since $2\alpha > -1$, it clearly follows from (3.5.3) that if $\alpha \le 0$, then the singularity that R has at $e^{i\theta_0}$ is removable. If $\alpha > 0$, and m is any integer with $m > 2\alpha$, then $(z - e^{i\theta_0})^m R(z)$ has a removable singularity at $e^{i\theta_0}$, and so R has no worse than a pole at that point, but since R remains bounded as $z \to e^{i\theta_0}$ with $z \in O$, we conclude that the singularity has to be removable.

It follows that $R = R_{\delta,\Theta}$ is a solution to the following Riemann-Hilbert problem:

R1 R is analytic for $z \in \mathbb{C}\backslash\hat{\Gamma}$ and has continuous boundary values on $\hat{\Gamma}$ from each component of $\mathbb{C} \setminus \hat{\Gamma}$.

R2 For $t \in \hat{\Gamma}^o$, $R_+(t) = R_-(t)V(t)$ with

$$V(t) = \begin{cases} N(t)\begin{pmatrix} 1 & 0 \\ w^{-1}(t)t^n\psi(t)^{-2n} & 1 \end{pmatrix} N^{-1}(t), & t \in \hat{\gamma}_L^o \cup \hat{\gamma}_R^o, \\ P(t)N^{-1}(t), & t \in \partial U_\delta, \\ \tilde{P}(t)N^{-1}(t), & t \in \partial\tilde{U}_\delta. \end{cases}$$

R3 As $z \to \infty$, $R(z) = I + O\left(\frac{1}{z}\right)$.

We now make an important observation, for which it is convenient to count with the following notation. We will write

$$A \subset B \quad \wedge \quad M << N$$

if A and B are domains in $\overline{\mathbb{C}}$ with $A \subset B$, and if M is an analytic matrix defined on A whose analytic continuation to B is given by N.

Remark 3.5.1. Since f is an angle preserving map, for any fixed $\Theta_0 \in (\pi/2, \pi)$ we can find a small enough $\delta_0 > 0$ such that for all Θ and δ with $\Theta_0 < \Theta < \pi$ and $0 < \delta < \delta_0$, the contour $\hat{\Gamma}_{\delta,\Theta}$ divides the complex plane in four open components U_δ, $\tilde{U}_\delta$, $\Omega_{\delta,\Theta}$, and $G_{\delta,\Theta}$, with $\Omega_{\delta,\Theta}$ denoting the unbounded component, see Figure 3.9. Let $R_{\delta,\Theta}$ be defined by (3.5.1), and let δ' and Θ' be such that $0 < \delta' < \delta_0$, and $\Theta_0 < \Theta' < \pi$. Then, by definition (3.5.1) (see also Remark 3.4.2), we have

$$\Omega_{\delta,\Theta} \subset \Omega_{\delta',\Theta'} \quad \wedge \quad R_{\delta,\Theta} << R_{\delta',\Theta'} \qquad \text{if } \delta' < \delta,\ \Theta < \Theta',$$

$$G_{\delta,\Theta} \subset G_{\delta',\Theta'} \quad \wedge \quad R_{\delta,\Theta} << R_{\delta',\Theta'} \qquad \text{if } \delta' < \delta,\ \Theta > \Theta',$$

$$U_{\delta'} \subset U_\delta \quad \wedge \quad R_{\delta',\Theta'} << R_{\delta,\Theta} \qquad \text{if } \delta' < \delta,$$

$$\tilde{U}_{\delta'} \subset \tilde{U}_\delta \quad \wedge \quad R_{\delta',\Theta'} << R_{\delta,\Theta} \qquad \text{if } \delta' < \delta.$$

This shows that R as defined in (3.5.1) admits an analytic continuation from and across the boundary of any of the components of $\mathbb{C} \setminus \hat{\Gamma}$.

The following corollary for a fixed arc is an immediate consequence of the discussion in Remark 3.5.1.

Corollary 3.5.2. *Let θ_0 be fixed. There exist numbers $\pi/2 < \Theta_0 < \Theta_1 < \pi$, $0 < \delta_0 < \delta_1$, and a constant $M > 0$, such that for the solution $R_{\delta,\Theta}(z)$ of the RHP* R1-R3 *we have the* *uniform bound*

$$|R_{\delta,\Theta}(z)| \le M, \quad z \in \overline{\mathbb{C}} \setminus \hat{\Gamma}_{\delta,\Theta}, \quad \delta_0 < \delta < \delta_1, \quad \Theta_0 < \Theta < \Theta_1.$$

We are now going to fix angles $\pi/2 < \Theta_0 < \Theta_1 < \pi$ and numbers $0 < \rho_0 < \rho_1 < 1$ such that, as we had in Remark 3.5.1, for all Θ and ρ with $\Theta_0 < \Theta < \Theta_1$ and $\rho_0 < \rho < \rho_1$, the contour $\hat{\Gamma}_{\delta,\Theta}$, with $\delta = \rho\theta_0$ divides the complex plane in four open components. We also write

$$\delta_0 = \rho_0\theta_0, \qquad \delta_1 = \rho_1\theta_0.$$

If θ_0 is fixed, by (2.2.4) and the maximum principle for analytic functions, there exists some $\lambda > 0$ such that for all $n \ge 0$,

$$\left|\frac{t}{\psi(t)^2}\right|^n \le e^{-\lambda n}, \; t \in \hat{\Gamma}_{\delta,\Theta} \setminus (\partial U_\delta \cup \partial\tilde{U}_\delta), \quad \delta_0 < \delta < \delta_1, \quad \Theta_0 < \Theta < \Theta_1.$$

Since $1/w$ is bounded in U, c.f. (3.2.1), we then have

$$\begin{aligned} N \begin{pmatrix} 1 & 0 \\ \frac{t^n}{w(t)\psi(t)^{2n}} & 1 \end{pmatrix} N^{-1} &= \begin{pmatrix} N_{11}N_{22} - N_{12}N_{21} + O(e^{-\lambda n}) & O(e^{-\lambda n}) \\ O(e^{-\lambda n}) & N_{11}N_{22} - N_{12}N_{21} + O(e^{-\lambda n}) \end{pmatrix} \\ &= \begin{pmatrix} \det N + O(e^{-\lambda n}) & O(e^{-\lambda n}) \\ O(e^{-\lambda n}) & \det N + O(e^{-\lambda n}) \end{pmatrix} \\ &= I + O(e^{-\lambda n}) \end{aligned} \tag{3.5.4}$$

uniformly, as $n \to \infty$, for $t \in \hat{\Gamma}_{\delta,\Theta} \setminus (\partial U_\delta \cup \partial\tilde{U}_\delta)$, $\delta_0 < \delta < \delta_1$, $\Theta_0 < \Theta < \Theta_1$.

This observation, together with (3.4.16), implies that as $n \to \infty$,

$$R_+(t) = R_-(t)\left(I + O\left(\frac{1}{n}\right)\right), \quad t \in \hat{\Gamma}_{\delta,\Theta},$$

uniformly for $\delta_0 < \delta < \delta_1$, $\Theta_0 < \Theta < \Theta_1$.

When θ_0 is allowed to vary in $(0, \pi - \epsilon]$, we have to be more careful. If $t \in \hat{\Gamma}_{\delta,\Theta} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$, then

$$|t| \geq 1 + \tau_1 \rho_0 \theta_0, \qquad |t| > 1,$$

and

$$|t| \leq 1 - \tau_2 \rho_1 \theta_0, \qquad |t| < 1,$$

for some positive constants τ_1, τ_2 independent of θ_0, ρ, and Θ. Since $|t/\psi(t)| < 1$ for $t \notin \gamma$, we have that for $|t| > 1$,

$$\left| \frac{t}{\psi(t)^2} \right|^n \leq \frac{1}{|\psi(t)|^n} \leq \frac{1}{|t|^n} \leq \frac{1}{|1 + \tau_1 \rho_0 \theta_0|^n},$$

while for $|t| < 1$,

$$\left| \frac{t}{\psi(t)^2} \right|^n \leq |t|^n \leq |1 - \tau_2 \rho_1 \theta_0|^n.$$

Combining these two last estimates with the series

$$\log(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots,$$

we find that if ρ_1 is chosen sufficiently small, then we can find some universal constant $\lambda > 0$ such that for all $\theta_0 \in (0, \pi - \epsilon]$, $\Theta_0 < \Theta < \Theta_1$, $\rho_0 < \rho < \rho_1$, and with $\delta = \rho\theta_0$,

$$\left| \frac{t}{\psi(t)^2} \right|^n \leq e^{-\lambda n \theta_0}, \quad t \in \hat{\Gamma}_{\delta,\Theta} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta).$$

Just as we argued for the fixed arc case, this implies that

$$N \begin{pmatrix} 1 & 0 \\ \frac{t^n}{w(t)\psi(t)^{2n}} & 1 \end{pmatrix} N^{-1} = I + O(e^{-\lambda n \theta_0}) \tag{3.5.5}$$

uniformly for $z \in \hat{\Gamma}_{\delta,\Theta} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$ as $n\theta_0 \to \infty$. This together with (3.4.17) implies that as $n\theta_0 \to \infty$, and with $\delta = \rho\theta_0$,

$$R_+(t) = R_-(t)\left(I + O\left(\frac{1}{n\theta_0}\right)\right), \quad t \in \hat{\Gamma}_{\delta,\Theta},$$

uniformly for $\theta_0 \in (0, \pi - \epsilon]$, $\Theta_0 < \Theta < \Theta_1$, $\rho_0 < \rho < \rho_1$.

3.5.1 Asymptotic expansion of R

Following [8], in order to obtain the asymptotic expansion of $R(z)$, we first give estimates for the jump matrix $V(t)$, for $t \in \hat{\Gamma}$. Define

$$\Delta(t) := V(t) - I.$$

By (3.5.4)-(3.5.5), we have

$$|\Delta(t)| \leq e^{-\lambda n}, \qquad |\Delta(t)| \leq e^{-\lambda n\theta_0}, \tag{3.5.6}$$

uniformly for $t \in \hat{\Gamma}_{\delta,\Theta} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$ as $n \to \infty$, $n\theta_0 \to \infty$, in the fixed and varying arc cases, respectively.

Proposition 3.5.3. *There exist $\pi/2 < \Theta_0 < \Theta_1 < \pi$ and $0 < \rho_0 < \rho_1 < 1$ such that with $\delta = \rho\theta_0$, the matrix Δ has an asymptotic expansion on $\hat{\Gamma}_{\delta,\Theta}$ of the following form:*

i. If θ_0 is fixed, then

$$\Delta(t) \sim \sum_{k=1}^{\infty} \frac{\Delta_k(t)}{n^k}, \quad t \in \hat{\Gamma}_{\delta,\Theta}, \tag{3.5.7}$$

uniformly in the parameters δ and Θ as $n \to \infty$. This means that for every integer $K \geq 0$, there is a constant C_K such that for all $n \geq 1$, $\rho \in (\rho_0, \rho_1)$ and $\Theta \in (\Theta_0, \Theta_1)$,

$$\left| \Delta(t) - \sum_{k=1}^{K} \frac{\Delta_k(t)}{n^k} \right| \leq \frac{C_K}{n^{K+1}}, \quad t \in \hat{\Gamma}_{\delta,\Theta}.$$

ii. If θ_0 is allowed to vary, then

$$\Delta(t) \sim \sum_{k=1}^{\infty} \frac{\theta_0^k \Delta_k(t)}{(n\theta_0)^k}, \quad t \in \hat{\Gamma}_{\delta,\Theta}, \tag{3.5.8}$$

uniformly, as $n\theta_0 \to \infty$, for $\theta_0 \in (0, \pi - \epsilon]$, $\rho \in (\rho_0, \rho_1)$ and $\Theta \in (\Theta_0, \Theta_1)$.

The functions Δ_k in (3.5.7) and (3.5.8) are one and the same.

Proof. We will prove only part i. corresponding to a fixed arc, since the proof for a varying arc follows with the help of (3.4.6) along the same lines. By (3.5.4), $\Delta_k(t) = 0$ for all t in the outer lips $\hat{\Gamma}_{\delta,\Theta} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$. For $t \in \partial U_\delta$, using again (9.7.1)-(9.7.4) in [1], we find that the asymptotic expansion

$$\begin{aligned}
\Delta(t) &= P(t)N(t)^{-1} - I \\
&\sim N(t) \begin{pmatrix} W(t) & 0 \\ 0 & W(t)^{-1} \end{pmatrix} \left[\sum_{k=1}^{\infty} \frac{(2\alpha, k-1)}{2^k (n^2 f(t))^{k/2}} \begin{pmatrix} \frac{(-1)^k}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) & -i(k - \frac{1}{2}) \\ (-1)^k (k - \frac{1}{2}) i & \frac{1}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) \end{pmatrix} \right] \\
&\quad \times \begin{pmatrix} W(t)^{-1} & 0 \\ 0 & W(t) \end{pmatrix} N(t)^{-1}
\end{aligned}$$

holds uniformly on ∂U_δ, $\delta = \rho\theta_0$, $\rho \in (\rho_0, \rho_1)$. Here $(\alpha, 0) = 1$ and

$$(\alpha, k) = \frac{(4\alpha^2 - 1)(4\alpha^2 - 9)\cdots(4\alpha^2 - (2k-1)^2)}{2^{2k} k!}, \quad k \geq 1.$$

That is, for $t \in \partial U_\delta$,

$$\Delta_k(t) = N(t) \begin{pmatrix} W(t) & 0 \\ 0 & W(t)^{-1} \end{pmatrix} \frac{(2\alpha, k-1)}{2^k (f(z))^{k/2}} \begin{pmatrix} \frac{(-1)^k}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) & -i(k - \frac{1}{2}) \\ (-1)^k (k - \frac{1}{2}) i & \frac{1}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) \end{pmatrix}$$
$$\times \begin{pmatrix} W(t)^{-1} & 0 \\ 0 & W(t) \end{pmatrix} N(t)^{-1}. \tag{3.5.9}$$

Similarly, we find that for $t \in \partial \tilde{U}_\delta$,

$$\Delta_k(t) = N(t) \begin{pmatrix} \tilde{W}(t) & 0 \\ 0 & \tilde{W}(t)^{-1} \end{pmatrix} \frac{(2\beta, k-1)}{2^k (f(z))^{k/2}} \begin{pmatrix} \frac{(-1)^k}{k}(4\beta^2 + \frac{k}{2} - \frac{1}{4}) & i(k - \frac{1}{2}) \\ (-1)^{k+1} (k - \frac{1}{2}) i & \frac{1}{k}(4\beta^2 + \frac{k}{2} - \frac{1}{4}) \end{pmatrix}$$
$$\times \begin{pmatrix} \tilde{W}(t)^{-1} & 0 \\ 0 & \tilde{W}(t) \end{pmatrix} N(t)^{-1}. \tag{3.5.10}$$

□

We now notice that the expressions defining Δ_k are analytic in $(U_\delta \cup \tilde{U}_\delta) \setminus \gamma$. We can actually say more.

Lemma 3.5.4. *For each $k \geq 1$ and $\delta \in (0, \delta_1)$, the restrictions $\Delta_k\big|_{U_\delta \setminus \gamma}$ and $\Delta_k\big|_{\tilde{U}_\delta \setminus \gamma}$ have meromorphic continuations to U_δ and $\tilde{U}_\delta$, respectively, whose only pole is of order at most $(k+1)/2$ and located at $e^{\pm i\theta_0}$.*

Proof. We consider the restriction $\Delta_k\big|_{U_\delta \setminus \gamma}$. By definition, Δ_k is analytic in $U_\delta \setminus \gamma$. By (3.4.5), $(f_+(t))^{k/2} = (-1)^k (f_-(t))^{k/2}$ for $t \in U_\delta \cap \gamma^o$. Using (3.4.18) and its inverse counterpart, we see that for $t \in \gamma^o$,

$$\begin{pmatrix} W_+(t)^{-1} & 0 \\ 0 & W_+(t) \end{pmatrix} N_+(t)^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_-(t)^{-1} & 0 \\ 0 & W_-(t) \end{pmatrix} N(t)_-^{-1},$$

and so

$$\begin{aligned}
\Delta_{k+}(t) &= N_-(t) \begin{pmatrix} W_-(t) & 0 \\ 0 & W_-(t)^{-1} \end{pmatrix} \frac{(2\alpha, k-1)}{2^k (f_-(t))^{k/2}} (-1)^k \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\
&\quad \times \begin{pmatrix} \frac{(-1)^k}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) & -i(k - \frac{1}{2}) \\ (-1)^k (k - \frac{1}{2}) i & \frac{1}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_-(t)^{-1} & 0 \\ 0 & W_-(t) \end{pmatrix} N_-(t)^{-1} \\
&= N_-(t) \begin{pmatrix} W_-(t) & 0 \\ 0 & W_-(t)^{-1} \end{pmatrix} \frac{(2\alpha, k-1)}{2^k (f_-(t))^{k/2}} (-1)^k \begin{pmatrix} \frac{1}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) & (-1)^{k+1}(k - \frac{1}{2}) i \\ i(k - \frac{1}{2}) & \frac{(-1)^k}{k}(4\alpha^2 + \frac{k}{2} - \frac{1}{4}) \end{pmatrix} \\
&\quad \times \begin{pmatrix} W_-(t)^{-1} & 0 \\ 0 & W_-(t) \end{pmatrix} N_-(t)^{-1} \\
&= \Delta_{k-}(t).
\end{aligned}$$

Hence Δ_k is analytic across $U_\delta \cap \gamma^o$. Near the pole $e^{i\theta_0}$, we recall (3.4.19) and that f has a simple zero at $e^{i\theta_0}$, so that as $z \to e^{i\theta_0}$,

$$\begin{aligned}
\Delta_k(z) &= O \begin{pmatrix} |z - e^{i\theta_0}|^{-1/4} & |z - e^{i\theta_0}|^{-1/4} \\ |z - e^{i\theta_0}|^{-1/4} & |z - e^{i\theta_0}|^{-1/4} \end{pmatrix} O \begin{pmatrix} |z - e^{i\theta_0}|^{-k/2} & |z - e^{i\theta_0}|^{-k/2} \\ |z - e^{i\theta_0}|^{-k/2} & |z - e^{i\theta_0}|^{-k/2} \end{pmatrix} \\
&\quad \times O \begin{pmatrix} |z - e^{i\theta_0}|^{-1/4} & |z - e^{i\theta_0}|^{-1/4} \\ |z - e^{i\theta_0}|^{-1/4} & |z - e^{i\theta_0}|^{-1/4} \end{pmatrix} \\
&= O \begin{pmatrix} |z - e^{i\theta_0}|^{-1/2-k/2} & |z - e^{i\theta_0}|^{-1/2-k/2} \\ |z - e^{i\theta_0}|^{-1/2-k/2} & |z - e^{i\theta_0}|^{-1/2-k/2} \end{pmatrix}.
\end{aligned}$$

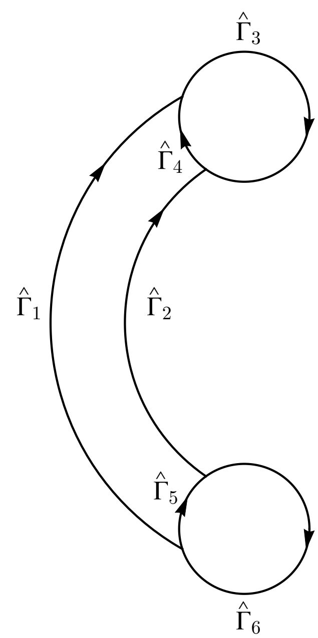

Figure 3.10: Splitting $\hat{\Gamma}$ into six contours

The statement for $\Delta_k\big|_{\tilde{U}_\delta \setminus \gamma}$ is proven analogously. $\square$

Lemma 3.5.5. *For the matrix R defined by* (3.5.1) *and satisfying the RHP* R1-R3*, we have the representation*

$$R(z) - I = \frac{1}{2\pi i} \int_{\hat{\Gamma}} \frac{R_-(t)\Delta(t)}{t-z} dt, \quad z \in \mathbb{C} \setminus \hat{\Gamma}. \tag{3.5.11}$$

Proof. The contour $\hat{\Gamma}$ divides $\overline{\mathbb{C}}$ in four components, U_δ, $\tilde{U}_\delta$, $\Omega = \Omega_{\delta,\Theta}$, and $G = \Omega_{\delta,\Theta}$, where Ω denotes the unbounded component. We consider the matrix $R(z) - I$ in each component and split $\hat{\Gamma}$ into the six paths $\hat{\Gamma}_1, \hat{\Gamma}_2, \ldots, \hat{\Gamma}_6$ defined via Figure 3.10.

For $z \in G$, we have by Cauchy's integral formula,

$$R(z) - I = \frac{1}{2\pi i} \left(\int_{\hat{\Gamma}_5} + \int_{\hat{\Gamma}_2} + \int_{\hat{\Gamma}_4} \right) \frac{R_+(t) - I}{t-z} dt - \frac{1}{2\pi i} \int_{\hat{\Gamma}_1} \frac{R_-(t) - I}{t-z} dt.$$

Moreover, by Cauchy's theorem, we have that for $z \in G$,

$$\left(\int_{\hat{\Gamma}_5} + \int_{\hat{\Gamma}_6} \right) \frac{R_-(t) - I}{t-z} dt = 0 = \left(\int_{\hat{\Gamma}_4} + \int_{\hat{\Gamma}_3} \right) \frac{R_-(t) - I}{t-z} dt,$$

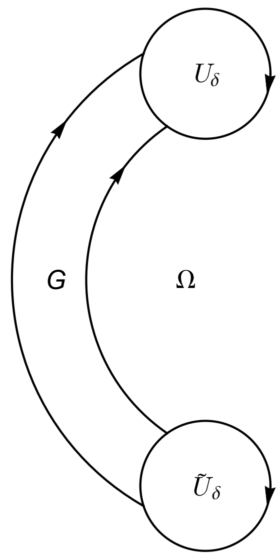

Figure 3.11: Components of $\overline{\mathbb{C}} \setminus \hat{\Gamma}$.

and after deforming to infinity and using that $R(z) - I = O\left(\frac{1}{z}\right)$, we also have that

$$\left(\int_{\hat{\Gamma}_1} + \int_{\hat{\Gamma}_3} + \int_{\hat{\Gamma}_6}\right) \frac{R_+(t) - I}{t - z} dt - \int_{\hat{\Gamma}_2} \frac{R_-(t) - I}{t - z} dt = 0.$$

Hence for $z \in G$,

$$\begin{aligned} R(z) - I &= \frac{1}{2\pi i} \left(\int\limits_{\hat{\Gamma}_1} + \int\limits_{\hat{\Gamma}_2} + \int\limits_{\hat{\Gamma}_3} + \int\limits_{\hat{\Gamma}_4} + \int\limits_{\hat{\Gamma}_5} + \int\limits_{\hat{\Gamma}_6} \right) \frac{(R_+(t) - I) - (R_-(t) - I)}{t - z} dt \\ &= \frac{1}{2\pi i} \int_{\hat{\Gamma}} \frac{R_-(t)\Delta(t)}{t - z} dt, \end{aligned}$$

where we have used that by definition $\Delta := V - I$, and so

$$R_+(t) - I - (R_-(t) - I) = R_-(t)V(t) - R_-(t) = R_-(t)\Delta(t).$$

Similar arguments hold for $z \in U_\delta$ and $z \in \tilde{U}_\delta$.

Since $R(z) - I = O(1/z)$ as $z \to \infty$, we get from Cauchy's integral formula for simply connected neighborhoods of infinity that for $z \in \Omega$,

$$R(z) - I = \frac{1}{2\pi i}\left(\int_{\hat{\Gamma}_6} + \int_{\hat{\Gamma}_1} + \int_{\hat{\Gamma}_3}\right)\frac{R_+(t) - I}{t - z}dt - \frac{1}{2\pi i}\int_{\hat{\Gamma}_2}\frac{R_-(t) - I}{t - z}dt.$$

We again note that, by Cauchy's theorem, the integrals over the remaining Jordan contours vanish, and we have

$$\begin{aligned} R(z) - I &= \frac{1}{2\pi i}\left(\int_{\hat{\Gamma}_6} + \int_{\hat{\Gamma}_1} + \int_{\hat{\Gamma}_3}\right)\frac{R_+(t) - I}{t - z}dt - \frac{1}{2\pi i}\int_{\hat{\Gamma}_2}\frac{R_-(t) - I}{t - z}dt \\ &\quad - \frac{1}{2\pi i}\left(\int_{\hat{\Gamma}_3} + \int_{\hat{\Gamma}_4} + \int_{\hat{\Gamma}_5} + \int_{\hat{\Gamma}_6}\right)\frac{R_-(t) - I}{t - z}dt \\ &\quad + \frac{1}{2\pi i}\left(\int_{\hat{\Gamma}_5} + \int_{\hat{\Gamma}_2} + \int_{\hat{\Gamma}_4}\right)\frac{R_+(t) - I}{t - z}dt - \frac{1}{2\pi i}\int_{\hat{\Gamma}_1}\frac{R_-(t) - I}{t - z}dt \\ &= \frac{1}{2\pi i}\int_{\hat{\Gamma}}\frac{(R_+(t) - I) - (R_-(t) - I)}{t - z}dt \\ &= \frac{1}{2\pi i}\int_{\hat{\Gamma}}\frac{R_-(t)\Delta(t)}{t - z}dt. \end{aligned}$$

□

By Proposition 3.5.3 and Lemma 3.5.4, and (3.5.6), we have for a fixed arc,

$$\|\Delta\|_{L^\infty(\partial U_\delta \cup \partial \tilde{U}_\delta)} = O(n^{-1}), \quad \|\Delta\|_{L^\infty(\hat{\Gamma} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta))} = O(e^{-\lambda n}), \tag{3.5.12}$$

while for a varying arc

$$\|\Delta\|_{L^\infty(\partial U_\delta \cup \partial \tilde{U}_\delta)} = O((n\theta_0)^{-1}), \quad \|\Delta\|_{L^\infty(\hat{\Gamma} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta))} = O(e^{-\lambda n\theta_0}). \tag{3.5.13}$$

We now consider the bounded linear operator

$$C_- : L_2(\hat{\Gamma}; \mathbb{C}^{2x2}) \to L_2(\hat{\Gamma}; \mathbb{C}^{2x2})$$

given by

$$C_-(f)(t) = \lim_{z \to t-} \frac{1}{2\pi i} \int_{\hat{\Gamma}} \frac{f(\zeta) d\zeta}{\zeta - z}, \quad t \in \hat{\Gamma}^o, \quad f \in L_2(\hat{\Gamma}; \mathbb{C}^{2x2}),$$

whose norm remains uniformly bounded as $\hat{\Gamma}$ varies, see e.g., Appendix A of [7]. We also consider the operator

$$C_\Delta : L_2(\hat{\Gamma}; \mathbb{C}^{2x2}) \to L_2(\hat{\Gamma}; \mathbb{C}^{2x2})$$

defined as

$$C_\Delta(f) := C_-(f\Delta), \qquad f \in L_2(\hat{\Gamma}; \mathbb{C}^{2x2}).$$

By (3.5.12)-(3.5.13), C_Δ is a bounded linear operator from $\mathbb{C}^{2x2}$ onto itself with operator norm

$$\|C_\Delta\| = \begin{cases} O\left(1/n\right), & \theta_0 \text{ fixed}, \\ O\left((n\theta_0)^{-1}\right), & \theta_0 \text{ varying}, \end{cases} \tag{3.5.14}$$

so that for n large enough, we can invert $\mathbb{1} - C_\Delta$ via the Neumann series

$$(\mathbb{1} - C_\Delta)^{-1} = \sum_{n=0}^{\infty} C_\Delta^n,$$

uniformly in the respective parameters. Here $\mathbb{1}$ denotes the identity operator.

We now use

$$\mu := (\mathbb{1} - C_\Delta)^{-1}(C_-(\Delta))$$

to construct an explicit expression for R. Note that μ remains uniformly bounded on $\hat{\Gamma}$ as $\hat{\Gamma}$ varies with θ_0.

Lemma 3.5.6. *If R is a solution of the Riemann-Hilbert problem* R1-R3*, then R is unique and given by*

$$R = I + C(\Delta + \mu\Delta).$$

Proof. Taking limiting values in (3.5.11) we get

$$R_- - I = C_\Delta(R_-),$$

or equivalently,

$$R_- - I - C_\Delta(R_- - I) = C_-(\Delta).$$

Acting on both sides by $(\mathbb{1} - C_\Delta)^{-1}$ and rearranging terms, we get

$$R_- = I + (\mathbb{1} - C_\Delta)^{-1}(C_-(\Delta)) = I + \mu. \tag{3.5.15}$$

Plugging (3.5.15) into (3.5.11) completes the proof. □

Corollary 3.5.7. *For the case of a varying arc, we have*

$$\theta_0 R(z) = O(1), \quad z \in \overline{\mathbb{C}} \setminus \hat{\Gamma}_{\delta,\Theta}, \quad \delta = \rho\theta_0,$$

uniformly for $\theta_0 \in (0, \pi - \epsilon]$, $\rho \in (\rho_0, \rho_1)$ and $\Theta \in (\Theta_0, \Theta_1)$.

Proof. By increasing ρ_0 and Θ_0, and decreasing ρ_1 and Θ_1, if necessary, we may assume that for some small $\varepsilon > 0$, the estimates (3.5.13) are true uniformly for $\theta_0 \in (0, \pi - \epsilon]$, $\rho \in (\rho_0 - \varepsilon, \rho_1 + \varepsilon)$ and $\Theta \in (\Theta_0 - \varepsilon, \Theta_1 + \epsilon)$. Hence, and by Remark 3.5.1, we can find a small $\tau > 0$ such that for every $\rho \in (\rho_0, \rho_1)$, $\Theta \in (\Theta_0, \Theta_1)$, and $z \in \mathbb{C} \setminus \hat{\Gamma}_{\delta,\Theta}$ with $\delta = \rho\theta_0$, there is $\rho' \in (\rho_0 - \varepsilon, \rho_1 + \varepsilon)$ and $\Theta' \in (\Theta_0 - \varepsilon, \Theta_1 + \varepsilon)$ with

$$\mathrm{d}(z, \hat{\Gamma}_{\delta',\Theta'}) > \tau\theta_0, \quad R_{\delta,\Theta}(z) = R_{\delta',\Theta'}(z),$$

where $\delta' = \rho'\theta_0$. By Lemma 3.5.6, we then have

$$|\theta_0 R(z)| \leq |\theta_0| + \frac{1}{2\pi\tau} \int_{\hat{\Gamma}_{\delta',\Theta'}} |(\Delta + \mu\Delta)(s)||ds|,$$

with the absolute value understood in an entrywise sense, and the corollary follows in view of (3.5.13). □

Proposition 3.5.8. *The matrix R has an asymptotic expansion on $\mathbb{C} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$ of the following form:*

i. If θ_0 is fixed, then

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{R_k(z)}{n^k} \quad \text{as} \quad n \to \infty, \tag{3.5.16}$$

with each $R_k(z)$ analytic for $z \in \mathbb{C} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$ and

$$R_k(z) = O\left(\frac{1}{z}\right) \quad \text{as} \quad z \to \infty.$$

This expansion is uniform for $z \in \mathbb{C} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$.

ii. When θ_0 is allowed to vary, there exist $0 < \rho_0 < \rho_1 < 1$ such that with $\delta = \rho\theta_0$,

$$R(z) \sim I + \sum_{k=1}^{\infty} \frac{\theta_0^k R_k(z)}{(n\theta_0)^k}, \quad z \in \mathbb{C} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta), \tag{3.5.17}$$

uniformly for $\theta_0 \in (0, \pi - \epsilon]$ and $\rho \in (\rho_0, \rho_1)$ as $n\theta_0 \to \infty$ with

$$\lim \frac{\ln \theta_0}{n\theta_0} = 0.$$

The functions R_k in (3.5.16) *and* (3.5.17) *are one and the same.*

Proof. We will only prove part ii. since part i. is, in fact, a corollary of part ii. It will be sufficient to show that there are ρ_0, ρ_1, and ε such that for all $\rho \in (\rho_0 - \varepsilon, \rho_1 + \varepsilon)$, $\Theta \in (\Theta_0 - \varepsilon, \Theta_1 + \varepsilon)$, and $\tau > 0$, the matrix $R = R_{\delta,\Theta}$ with $\delta = \rho\theta_0$, has an expansion of the form

$$R_{\delta,\Theta}(z) \sim I + \sum_{k=1}^{\infty} \frac{\theta_0^k R_{\delta,\Theta,k}(z)}{(n\theta_0)^k}$$

uniformly on $\{z \in \mathbb{C} : \mathrm{d}(z, \hat{\Gamma}_{\delta,\Theta}) > \tau\theta_0\}$. For suppose this is the case. Then, by Remark 3.5.1 (see also the proof of Corollary 3.5.7), there exists a small $\tau > 0$ with the property that for every $\rho \in (\rho_0, \rho_1)$ and $\mathcal{O}$ any of the four components of $\overline{\mathbb{C}} \setminus \hat{\Gamma}_{\delta,\Theta}$, we can find $\rho' \in (\rho_0 - \varepsilon, \rho_1 + \varepsilon)$ and $\Theta' \in (\Theta_0 - \varepsilon, \Theta_1 + \varepsilon)$ such that with $\delta' = \rho'\theta_0$, we have that $\overline{\mathcal{O}} \subset \overline{\mathbb{C}} \setminus \hat{\Gamma}_{\delta',\Theta'}$, $\mathrm{d}(\overline{\mathcal{O}}, \hat{\Gamma}_{\delta',\Theta'}) > \tau\theta_0$, and the solutions $R_{\delta,\Theta}$ and $R_{\delta',\Theta'}$ coincide on $\overline{\mathcal{O}}$. By assumption, we have an expansion

$$R_{\delta',\Theta'}(z) \sim I + \sum_{k=1}^{\infty} \frac{\theta_0^k R_{\delta',\Theta',k}(z)}{(n\theta_0)^k}$$

on $\overline{\mathcal{O}}$. Since the expansion is, if it exists, unique, it follows that $R_{\delta',\Theta',k}(z) = R_{\delta,\Theta,k}(z)$ for all $z \in \mathcal{O}$ and $k \geq 1$.

Then, for $l \in \mathbb{N}$, define $S_l(t)$ to be the partial sum of Δ containing the first l terms. That is,

$$S_l(t) := \sum_{k=1}^{l} \frac{\theta_0^k \Delta_k(t)}{(n\theta_0)^k}.$$

Let C denote the Cauchy transform taken over $\hat{\Gamma}$, and define

$$C_{S_l}(f) := C_-(fS_l), \quad f \in L^2(\hat{\Gamma}),$$
$$\mu_l := \sum_{j=0}^{l} (C_{S_l})^j (C_-(S_l)),$$
$$r_l := I + C(S_l + \mu_l S_l).$$

Notice that since $S_l \equiv 0$ on $\hat{\Gamma} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$, $r_l(z)$ is analytic for $z \in \mathbb{C} \setminus (\partial U_\delta \cup \partial \tilde{U}_\delta)$. Let $\tilde{C}$ denote the Cauchy transform taken over $\partial U_\delta \cup \partial \tilde{U}_\delta$. By Lemma 3.5.6, we have

$$\begin{aligned} \|R - r_l\| &= \|I + C(\Delta + \mu\Delta) - (I + C(S_l + \mu_l S_l))\| \\ &\leq \|C(\Delta + \mu\Delta) - \tilde{C}(\Delta + \mu\Delta)\| + \|\tilde{C}(\Delta + \mu\Delta) - C(S_l + \mu_l S_l)\| \\ &= \|C(\Delta + \mu\Delta) - \tilde{C}(\Delta + \mu\Delta)\| + \|\tilde{C}(\Delta + \mu\Delta - S_l - \mu_l S_l)\|. \end{aligned} \tag{3.5.18}$$

We examine the first term in the sum on the right hand side of (3.5.18). Using (3.5.15), we have

$$C(\Delta + \mu\Delta) - \tilde{C}(\Delta + \mu\Delta) = \frac{1}{2\pi i} \int_{\hat{\Gamma}_1 \cup \hat{\Gamma}_2} \frac{\Delta(s) + \mu(s)\Delta(s)}{s - z} ds.$$

By (3.5.6), it then follows that

$$\|C(\Delta + \mu\Delta) - \tilde{C}(\Delta + \mu\Delta)\| = O(\theta_0^{-1} e^{-\lambda n \theta_0})$$

uniformly for $\{z \in \mathbb{C} : \mathrm{d}(z, \hat{\Gamma}) > \tau\theta_0\}$ as $n\theta_0 \to \infty$.

If we now use that $n\theta_0 \to \infty$ in such a way that

$$\lim \frac{\ln \theta_0}{n\theta_0} = 0,$$

then for $n\theta_0$ large enough, we have $-\frac{\ln\theta_0}{n\theta_0} \le \frac{\lambda}{2}$, so that $\theta_0^{-1}e^{-\lambda n\theta_0} = e^{-\lambda n\theta_0 - \ln\theta_0} < e^{-\frac{\lambda}{2}n\theta_0}$. Therefore, for the first term we have

$$\|C(\Delta + \mu\Delta) - \tilde{C}(\Delta + \mu\Delta)\| = O(e^{-\frac{\lambda}{2}n\theta_0}).$$

Now we examine the second term in (3.5.18), which is the norm of the function matrix

$$\begin{aligned}\tilde{C}(\Delta + \mu\Delta - S_l - \mu_l S_l) = \frac{1}{2\pi i}\int_{\partial U_\delta \cup \partial\tilde{U}_\delta} &\frac{\Delta(s) - S_l(s)}{s - z}ds \\ + \frac{1}{2\pi i}\int_{\partial U_\delta \cup \partial\tilde{U}_\delta} &\frac{\mu(s)\Delta(s) - \mu_l(s)S_l(s)}{s - z}ds.\end{aligned} \tag{3.5.19}$$

By Proposition 3.5.3, we have

$$\|\Delta - S_l\|_{L^\infty(\hat{\Gamma})} = O\left(\frac{1}{(n\theta_0)^{l+1}}\right), \tag{3.5.20}$$

which yields

$$\|C_\Delta - C_{S_l}\| \le \|C_-\|\|\Delta - S_l\|_{L^2(\hat{\Gamma})} = O\left(\frac{1}{(n\theta_0)^{l+1}}\right),$$

and since $\|C_\Delta\| = O\left(1/(n\theta_0)\right)$, see (3.5.14), we also have

$$\|C_{S_l}\| \le \|C_\Delta\| + \|C_\Delta - C_{S_l}\| = O\left(\frac{1}{n\theta_0}\right).$$

In particular, $\mathbb{1} - C_{S_l}$ is invertible for all n sufficiently large, and

$$\|(\mathbb{1} - C_{S_l})^{-1}\| \le \sum_{n=0}^{\infty}\|C_{S_l}\|^n = O(1), \qquad \|(\mathbb{1} - C_\Delta)^{-1}\| = O(1)$$

as $n\theta_0 \to \infty$.

From the definition of μ_l, we also find

$$\|\mu_l\|_{L^\infty(\hat{\Gamma})} \leq \|C_-\|\|S_l\|_{L^2(\hat{\Gamma})} \sum_{j=0}^{l} \|C_{S_l}\|^j = O\left(\frac{1}{n\theta_0}\right). \tag{3.5.21}$$

Finally, we estimate the L^2-norm of $\mu - \mu_l$. Since

$$\begin{aligned}\mu - \mu_l &= (\mathbb{1} - C_\Delta)^{-1}(C_-(\Delta)) - \sum_{j=0}^{l}(C_{S_l})^j(C_-(S_l)) \\ &= \left((\mathbb{1} - C_\Delta)^{-1} - (\mathbb{1} - C_{S_l})^{-1}\right)(C_-\Delta) + \sum_{j=l+1}^{\infty}(C_{S_l})^j(C_-S_l)\end{aligned}$$

and

$$\begin{aligned}\|(\mathbb{1} - C_\Delta)^{-1} - (\mathbb{1} - C_{S_l})^{-1}\| &= \|(\mathbb{1} - C_\Delta)^{-1}(C_\Delta - C_{S_l})(\mathbb{1} - C_{S_l})^{-1}\| \\ &\leq \|(\mathbb{1} - C_\Delta)^{-1}\|\|C_\Delta - C_{S_l}\|\|(\mathbb{1} - C_{S_l})^{-1}\| \\ &= O\left(\frac{1}{(n\theta_0)^{l+1}}\right),\end{aligned}$$

it follows that

$$\|\mu - \mu_l\|_{L^2(\hat{\Gamma})} \leq O\left(\frac{1}{(n\theta_0)^{l+2}}\right) + \|C_-\|\|S_l\|_{L^2(\hat{\Gamma})} \sum_{j=l+1}^{\infty} \|C_{S_l}\|^j = O\left(\frac{1}{(n\theta_0)^{l+2}}\right).$$

Combining this with (3.5.21) and (3.5.20) we get

$$\begin{aligned}\|\mu\Delta - \mu_l S_l\|_{L^2(\hat{\Gamma})} &\leq \|\mu - \mu_l\|_{L^2(\hat{\Gamma})} \, \|\Delta\|_{L^\infty(\hat{\Gamma})} + \|\mu_l\|_{L^\infty(\hat{\Gamma})} \, \|\Delta - S_l\|_{L^2(\hat{\Gamma})} \\ &= O\left(\frac{1}{(n\theta_0)^{l+2}}\right).\end{aligned} \tag{3.5.22}$$

Then, from (3.5.19), (3.5.20), and (3.5.22), we deduce that

$$\|\tilde{C}(\Delta + \mu\Delta - S_l - \mu_l S_l)\| = O\left(\frac{1}{(n\theta_0)^{l+1}}\right)$$

uniformly for $\{z \in \mathbb{C} : \mathrm{d}(z, \hat{\Gamma}) > \tau\theta_0\}$ as $n \to \infty$. Thus, we have proven that

$$\|R - r_l\| = O\left(\frac{1}{(n\theta_0)^{l+1}}\right).$$

Now, writing

$$\tilde{\Delta}_k = \theta_0^k \Delta_k, \quad k \geq 1,$$

we have

$$(C_{S_l})^j = \left(\sum_{k=1}^{l} \frac{C_{\tilde{\Delta}_k}}{(n\theta_0)^k}\right)^j = \sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \cdots \sum_{k_j=1}^{l} \frac{C_{\tilde{\Delta}_{k_1}} C_{\tilde{\Delta}_{k_2}} \cdots C_{\tilde{\Delta}_{k_j}}}{(n\theta_0)^{k_1+k_2+\cdots+k_j}}.$$

By the definition of μ_l, we then have

$$\mu_l = \sum_{j=0}^{l} \left(\sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \cdots \sum_{k_{j+1}=1}^{l} \frac{(C_{\tilde{\Delta}_{k_1}} C_{\tilde{\Delta}_{k_2}} \cdots C_{\tilde{\Delta}_{k_{j+1}}})(1)}{(n\theta_0)^{k_1+k_2+\cdots+k_{j+1}}} \right),$$

and so

$$\begin{aligned} R = r_l + O\left(\frac{1}{(n\theta_0)^{l+1}}\right) &= I + C(S_l(1+\mu_l)) + \left(\frac{1}{(n\theta_0)^{l+1}}\right) \\ &= I + \sum_{k=1}^{l} \frac{C(\tilde{\Delta}_k)}{(n\theta_0)^k} + O\left(\frac{1}{(n\theta_0)^{l+1}}\right) \\ &\quad + \sum_{j=0}^{l} \left(\sum_{k_1=1}^{l} \sum_{k_2=1}^{l} \cdots \sum_{k_{j+2}=1}^{l} \frac{C(\tilde{\Delta}_{k_1}(C_{\tilde{\Delta}_{k_2}} C_{\tilde{\Delta}_{k_3}} \cdots C_{\tilde{\Delta}_{k_{j+2}}})(1))}{(n\theta_0)^{k_1+k_2+\cdots+k_{j+2}}} \right). \end{aligned}$$

Hence for every $l \geq 1$,

$$R_l = \sum_{j=1}^{l} \left(\sum_{\substack{1 \leq k_1, k_2, \ldots, k_j \leq l+1-j \\ k_1+k_2+\cdots+k_j = l}} \frac{C(\tilde{\Delta}_{k_1}(C_{\tilde{\Delta}_{k_2}} C_{\tilde{\Delta}_{k_3}} \cdots C_{\tilde{\Delta}_{k_j}})(1))}{(n\theta_0)^{k_1+k_2+\cdots+k_j}} \right),$$

completing the proof. Notice that

$$
\begin{aligned}
R_l &= (n\theta_0)^l \sum_{\substack{1 \le k_1, k_2, \ldots, k_j \\ k_1 + k_2 + \cdots + k_j = l}} \frac{C(\tilde{\Delta}_{k_1}(C_{\tilde{\Delta}_{k_2}} C_{\tilde{\Delta}_{k_3}} \cdots C_{\tilde{\Delta}_{k_j}})(1))}{(n\theta_0)^{k_1 + k_2 + \cdots + k_j}} \\
&= C(\tilde{\Delta}_l) + (n\theta_0)^l \sum_{k_1=1}^{l-1} \frac{1}{(n\theta_0)^{k_1}} C \left(\tilde{\Delta}_{k_1} \sum_{\substack{1 \le k_2, \ldots, k_j \\ k_2 + \cdots + k_j = l - k_1}} \frac{C_-(\tilde{\Delta}_{k_2}(C_{\tilde{\Delta}_{k_3}} \cdots C_{\tilde{\Delta}_{k_j}})(1))}{(n\theta_0)^{k_2 + \cdots + k_j}} \right) \\
&= C(\tilde{\Delta}_l) + \sum_{k=1}^{l-1} C\left(\tilde{\Delta}_k (R_{l-k})_- \right).
\end{aligned}
$$

Since $\tilde{\Delta}_k = \theta_0^k \Delta_k$, this formula for R_l remains valid in the case of a fixed arc, provided that we replace $\tilde{\Delta}_k$ by Δ_k, that is,

$$
R_l = C(\Delta_l) + \sum_{k=1}^{l-1} C\left(\Delta_k (R_{l-k})_- \right).
$$

□

3.6 Series expansions

In this section we derive several series expansions that will be needed for the determination of the functions R_k. The series hold for fixed θ_0, but not uniformly as θ_0 varies. The needed uniform estimates will be given in the next section.

If we choose the branch of $(z - e^{i\theta_0})^{\pm 1/4}$ in $\mathbb{C} \setminus \{ri : r \le \sin\theta_0\}$ corresponding to

$$
-\frac{\pi}{2} < \arg(z - e^{i\theta_0}) < \frac{3\pi}{2},
$$

then for $|z - e^{i\theta_0}| < 2\sin\theta_0$,

$$\begin{aligned} a(z) &= \frac{e^{-i\frac{\pi}{8}}}{\sqrt[4]{2\sin\theta_0}}(z - e^{i\theta_0})^{1/4} \sum_{k=0}^{\infty} \frac{\binom{-1/4}{k}}{(2i\sin\theta_0)^k}(z - e^{i\theta_0})^k, \\ a(z)^{-1} &= e^{i\frac{\pi}{8}}\sqrt[4]{2\sin\theta_0}(z - e^{i\theta_0})^{-1/4} \sum_{k=0}^{\infty} \frac{\binom{1/4}{k}}{(2i\sin\theta_0)^k}(z - e^{i\theta_0})^k, \end{aligned} \tag{3.6.1}$$

and if we choose the branch of $(z - e^{-i\theta_0})^{\pm 1/4}$ in $\mathbb{C} \setminus \{ri : -\sin\theta_0 \le r\}$ corresponding to

$$-\frac{3\pi}{2} < \arg(z - e^{-i\theta_0}) < \frac{\pi}{2},$$

then for $|z - e^{-i\theta_0}| < 2\sin\theta_0$,

$$\begin{aligned} a(z) &= e^{-i\frac{\pi}{8}}\sqrt[4]{2\sin\theta_0}(z - e^{-i\theta_0})^{-1/4} \sum_{k=0}^{\infty} \frac{\binom{1/4}{k}(-1)^k}{(2i\sin\theta_0)^k}(z - e^{-i\theta_0})^k, \\ a(z)^{-1} &= \frac{e^{i\frac{\pi}{8}}}{\sqrt[4]{2\sin\theta_0}}(z - e^{-i\theta_0})^{1/4} \sum_{k=0}^{\infty} \frac{\binom{-1/4}{k}(-1)^k}{(2i\sin\theta_0)^k}(z - e^{i\theta_0})^k. \end{aligned}$$

The expansion

$$\begin{aligned} \frac{\psi(z)}{\sqrt{z}} &= \frac{z - e^{i\theta_0} + 2ce^{i\theta_0/2} + \sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}}{2c}(z - e^{i\theta_0} + e^{i\theta_0})^{-1/2} \\ &= \left(1 + \frac{z - e^{i\theta_0}}{2ce^{i\theta_0/2}} + \frac{\sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}}{2ce^{i\theta_0/2}}\right) \sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} e^{-ik\theta_0}(z - e^{i\theta_0})^k \\ &= 1 + \sum_{k=1}^{\infty} \binom{\frac{-1}{2}}{k} e^{-ik\theta_0}(z - e^{i\theta_0})^k \\ &\quad + \frac{1}{2ce^{i\theta_0/2}} \left(\sum_{k=0}^{\infty} \binom{\frac{-1}{2}}{k} e^{-ik\theta_0}(z - e^{i\theta_0})^k\right) \\ &\quad \times \left(z - e^{i\theta_0} + e^{i\pi/4}\sqrt{2\sin\theta_0}(z - e^{i\theta_0})^{1/2} \sum_{k=0}^{\infty} \frac{\binom{1/2}{k}}{(2i\sin\theta_0)^k}(z - e^{i\theta_0})^k\right) \end{aligned} \tag{3.6.2}$$

is valid for $\{z : |z - e^{i\theta}| < 2\sin\theta_0\} \setminus \gamma$, provided that the branch of $(z - e^{i\theta_0})^{1/2}$ in the plane cut along $\gamma \cup \{ri : r < -\sin\theta_0\}$ is chosen so as to have

$$\arg(ri - e^{i\theta_0}) = \pi/2, \qquad r > \sin\theta_0.$$

In particular, we have

$$\begin{aligned}\frac{\psi(z)}{\sqrt{z}} &= 1 - \frac{e^{-i\theta_0}}{2}(z - e^{i\theta_0}) + \frac{3e^{-2i\theta_0}}{8}(z - e^{i\theta_0})^2 + O(z - e^{i\theta_0})^3 \\ &\quad + e^{-i\theta_0/2}\left(1 - \frac{e^{-i\theta_0}}{2}(z - e^{i\theta_0}) + \frac{3e^{-2i\theta_0}}{8}(z - e^{i\theta_0})^2\right) \\ &\quad \times \Bigg(e^{i\pi/4}\sqrt{\tan(\theta_0/2)}(z - e^{i\theta_0})^{1/2} + \frac{1}{2c}(z - e^{i\theta_0}) - \frac{ie^{i\pi/4}\sqrt{\tan(\theta_0/2)}}{4\sin\theta_0}(z - e^{i\theta_0})^{3/2} \\ &\quad + \frac{e^{i\pi/4}\sqrt{\tan(\theta_0/2)}}{32\sin^2\theta_0}(z - e^{i\theta_0})^{5/2}\Bigg).\end{aligned}$$

Collecting similar terms and simplifying the coefficients yields

$$\begin{aligned}\frac{\psi(z)}{\sqrt{z}} &= 1 + \frac{e^{i\pi/4}\sqrt{\tan(\theta_0/2)}}{e^{i\theta_0/2}}(z - e^{i\theta_0})^{1/2} + \frac{ie^{-i\theta_0}\tan(\theta_0/2)}{2}(z - e^{i\theta_0}) \\ &\quad + \frac{ie^{i\pi/4}e^{-5i\theta_0/2}\sqrt{\tan(\theta_0/2)}}{4\sin\theta_0}(z - e^{i\theta_0})^{3/2} + \frac{e^{-2i\theta_0}(1 - 2i\tan(\theta_0/2))}{8}(z - e^{i\theta_0})^2 \\ &\quad + \frac{e^{i\pi/4}e^{-5i\theta_0/2}\sqrt{\tan(\theta_0/2)}(4 - 3e^{-2i\theta_0})}{32\sin^2\theta_0}(z - e^{i\theta_0})^{5/2} + O(z - e^{i\theta_0})^3.\end{aligned} \tag{3.6.3}$$

For $|z| < 1$, we know that

$$\log z = (z-1) - \frac{(z-1)^2}{2} + \frac{(z-1)^3}{3} - \cdots.$$

So for $|\psi(z)/\sqrt{z} - 1| < 1$, we have

$$\log\left(\frac{\psi(z)}{\sqrt{z}}\right) = \left(\frac{\psi(z)}{\sqrt{z}} - 1\right) - \frac{1}{2}\left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^2 + \frac{1}{3}\left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^3 - \frac{1}{4}\left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^4 + \frac{1}{5}\left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^5 + O(z - e^{i\theta_0})^6$$

and

$$\left[\log\left(\frac{\psi(z)}{\sqrt{z}}\right)\right]^2 = \left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^2 - \left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^3 + \frac{11}{12}\left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^4 - \frac{5}{6}\left(\frac{\psi(z)}{\sqrt{z}} - 1\right)^5 + O(z - e^{i\theta_0})^6. \tag{3.6.4}$$

These last two expansions together with (3.6.3) give that as $z \to e^{i\theta_0}$,

$$\begin{aligned}
f(z) &= \frac{1}{4}\log^2\left(\frac{\psi(z)}{\sqrt{z}}\right) = \frac{i\tan(\theta_0/2)}{4e^{i\theta_0}}(z - e^{i\theta_0}) + \frac{\tan^2(\theta_0/2)}{12e^{2i\theta_0}}(z - e^{i\theta_0})^2 + O(z - e^{i\theta_0})^3, \\
f(z)^{1/2} &= \frac{1}{2}\log\left(\frac{\psi(z)}{\sqrt{z}}\right) = \frac{e^{i\pi/4}e^{-i\theta_0/2}\sqrt{\tan(\theta_0/2)}}{2}(z - e^{i\theta_0})^{1/2} + O(z - e^{i\theta_0})^{3/2}, \\
f(z)^{-1/2} &= \frac{2e^{-i\pi/4}e^{i\theta_0/2}}{\sqrt{\tan(\theta_0/2)}}(z - e^{i\theta_0})^{-1/2} + O(z - e^{i\theta_0})^{1/2}.
\end{aligned}$$

Since $\overline{\psi(\overline{z})} = f(z)$ and $\overline{f(\overline{z})} = f(z)$, we immediately get that as $z \to e^{-i\theta_0}$,

$$\begin{aligned}
f(z) &= \frac{-ie^{i\theta_0}\tan(\theta_0/2)}{4}(z - e^{-i\theta_0}) + \frac{e^{2i\theta_0}\tan^2(\theta_0/2)}{12}(z - e^{-i\theta_0})^2 + O(z - e^{-i\theta_0})^3, \\
f(z)^{1/2} &= \frac{1}{2}\log\left(\frac{\psi(z)}{\sqrt{z}}\right) = \frac{e^{-i\pi/4}e^{i\theta_0/2}\sqrt{\tan(\theta_0/2)}}{2}(z - e^{-i\theta_0})^{1/2} + O(z - e^{-i\theta_0})^{3/2}, \\
f(z)^{-1/2} &= \frac{2e^{i\pi/4}e^{-i\theta_0/2}}{\sqrt{\tan(\theta_0/2)}}(z - e^{-i\theta_0})^{-1/2} + O(z - e^{-i\theta_0})^{1/2}.
\end{aligned}$$

3.7 Uniform estimates

Writing $z = e^{i\theta_0} + \rho \sin(\theta_0/2)e^{it}$, $0 \le t \le 2\pi$, we get from (3.6.2) that uniformly in $0 < \theta_0 < \pi - \epsilon$ and $0 \le \rho < 1$

$$\frac{\psi(z)}{\sqrt{z}} = 1 + \frac{e^{i\pi/4}\sqrt{2\sin\theta_0}}{2ce^{i\theta_0/2}}(\rho\sin(\theta_0/2)e^{it})^{1/2} + O(\rho).$$

Combining this with (3.6.4), we see that there exists $0 < \rho_1 < 1$ such that uniformly in $0 < \theta_0 \le \pi - \epsilon$ and $0 \le \rho < \rho_1$

$$f(z) = \frac{ie^{i(t-\theta_0)}\sin^2(\theta_0/2)\rho}{4c}(1 + O(\rho)). \tag{3.7.1}$$

Similarly, it follows from (3.6.1) that

$$a(z) = O(\rho^{1/4}), \quad a(z)^{-1} = O(\rho^{-1/4}), \tag{3.7.2}$$

uniformly in $0 < \theta_0 \le \pi - \epsilon$, $0 \le \rho < \rho_1$, and z in the set

$$\{z : \rho_0 \sin(\theta_0/2) < |z - e^{i\theta_0}| < \rho_1 \sin(\theta_0/2)\} \setminus \Gamma.$$

3.7.1 Determination of R_1

It suffices to treat only the case of θ_0 fixed. At the end of the proof of Proposition 3.5.8 we found the recursive formula

$$R_k = C(\Delta_k) + \sum_{j=1}^{k-1} C\left(\Delta_j (R_{k-j})_-\right). \tag{3.7.3}$$

In particular, we have

$$R_1(z) = C(\Delta_1) = \frac{1}{2\pi i}\int_{\partial U_\delta} \frac{\Delta_1(t)dt}{t-z} + \frac{1}{2\pi i}\int_{\partial \tilde{U}_\delta} \frac{\Delta_1(t)dt}{t-z}. \tag{3.7.4}$$

By Lemma 3.5.4, we have

$$\Delta_1(z) = \frac{A^{(1)}}{z-e^{i\theta_0}} + O(1), \quad \text{as } z \to e^{i\theta_0}, \quad \Delta_1(z) = \frac{B^{(1)}}{z-e^{-i\theta_0}} + O(1), \quad \text{as } z \to e^{-i\theta_0},$$

where $A^{(1)}$ and $B^{(1)}$ are the constant matrices

$$A^{(1)} = \frac{ce^{i\theta_0/2}(16\alpha^2-1)}{4}\begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1}\end{pmatrix}\begin{pmatrix} -1 & i \\ i & 1\end{pmatrix}\begin{pmatrix} D(\infty)^{-1} & 0 \\ 0 & D(\infty)\end{pmatrix}$$

and

$$B^{(1)} = \frac{ce^{-i\theta_0/2}(16\beta^2-1)}{4}\begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1}\end{pmatrix}\begin{pmatrix} -1 & -i \\ -i & 1\end{pmatrix}\begin{pmatrix} D(\infty)^{-1} & 0 \\ 0 & D(\infty)\end{pmatrix}$$

that we compute using the expansions involving the functions $f(z)$ found in Section 3.6, the expansion for $W(z)/D(z)$ and its inverse (see (5.15.3)), together with the expansions for $g(z)$ and $a(z)$ for $z \in U_\delta$.

By the residue theorem, we immediately get

$$R_1(z) = \begin{cases} \dfrac{A^{(1)}}{z-e^{i\theta_0}} + \dfrac{B^{(1)}}{z-e^{-i\theta_0}}, & z \in \mathbb{C} \setminus \left(\overline{U}_\delta \cup \overline{\tilde{U}}_\delta\right), \\ \dfrac{A^{(1)}}{z-e^{i\theta_0}} + \dfrac{B^{(1)}}{z-e^{-i\theta_0}} - \Delta_1(z), & z \in U_\delta \cup \tilde{U}_\delta. \end{cases} \tag{3.7.5}$$

3.7.2 Determination of R_2

Using the recursive relation (3.7.3), we have

$$\begin{aligned} R_2(z) &= C(\Delta_2 + R_{1-}\Delta_{1-}) \\ &= \frac{1}{2\pi i}\int_{\partial U_\delta} \frac{\Delta_1(t) + R_{1-}(t)\Delta_{1-}(t)dt}{t-z} + \frac{1}{2\pi i}\int_{\partial \tilde{U}_\delta} \frac{\Delta_1(t) + R_{1-}(t)\Delta_{1-}(t)dt}{t-z}. \end{aligned}$$

We can find constant matrices $A^{(2)}$, $B^{(2)}$ such that

$$\begin{aligned} R_1(z)\Delta_1(z) + \Delta_2(z) &= \frac{A^{(2)}}{z - e^{i\theta_0}} + O(1), \qquad \text{as } z \to e^{i\theta_0}, \\ R_1(z)\Delta_1(z) + \Delta_2(z) &= \frac{B^{(2)}}{z - e^{-i\theta_0}} + O(1), \qquad \text{as } z \to e^{-i\theta_0}, \end{aligned}$$

so that once again from the residue theorem we get

$$R_2(z) = \begin{cases} \dfrac{A^{(2)}}{z - e^{i\theta_0}} + \dfrac{B^{(2)}}{z - e^{-i\theta_0}}, & z \in \mathbb{C} \setminus \left(\overline{U}_\delta \cup \overline{\tilde{U}}_\delta\right), \\ \dfrac{A^{(2)}}{z - e^{i\theta_0}} + \dfrac{B^{(2)}}{z - e^{-i\theta_0}} - R_1(z)\Delta_1(z) - \Delta_2(z), & z \in U_\delta \cup \tilde{U}_\delta. \end{cases} \tag{3.7.6}$$

Using two terms in each of the expansions used above and with the aid of Mathematica, we compute

$$\begin{aligned} A^{(2)} =& \frac{(16\alpha^2 - 1)\cot(\theta_0/2)}{256} \begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1} \end{pmatrix} \begin{pmatrix} iA_2(\alpha,\beta,c_0) & B_2(\alpha,\beta,c_0) \\ C_2(\alpha,\beta,c_0) & -iD_2(\alpha,\beta,c_0) \end{pmatrix} \\ &\times \begin{pmatrix} D(\infty)^{-1} & 0 \\ 0 & D(\infty) \end{pmatrix}, \end{aligned}$$

with

$$
\begin{aligned}
A_2(\alpha,\beta,c_0) =&8+21e^{i\theta_0}-128\beta^2+112\alpha^2e^{i\theta_0}+(\alpha+\beta)e^{i\theta_0/2}10i\sin(\theta_0/2)+20ic_0\sin(\theta_0)e^{i\theta_0},\\
B_2(\alpha,\beta,c_0) =&8+12e^{i\theta_0}-128\beta^2+128\alpha^2e^{i\theta_0}+(\alpha+\beta)e^{i\theta_0/2}10i\sin(\theta_0/2)+20ic_0\sin(\theta_0)e^{i\theta_0},\\
C_2(\alpha,\beta,c_0) =&-8-12e^{i\theta_0}+128\beta^2-128\alpha^2e^{i\theta_0}+(\alpha+\beta)e^{i\theta_0/2}10i\sin(\theta_0/2)+20ic_0\sin(\theta_0)e^{i\theta_0},\\
D_2(\alpha,\beta,c_0) =&-8-21e^{i\theta_0}+128\beta^2-112\alpha^2e^{i\theta_0}+(\alpha+\beta)e^{i\theta_0/2}10i\sin(\theta_0/2)+20ic_0\sin(\theta_0)e^{i\theta_0},
\end{aligned}
$$

and c_0 defined in (1.3.2). Similarly, we find

$$
\begin{aligned}
B^{(2)} =&\frac{(16\beta^2-1)\cot(\theta_0/2)}{256}\begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1}\end{pmatrix}\begin{pmatrix} -i\overline{A_2(\beta,\alpha,d_0)} & \overline{B_2(\beta,\alpha,d_0)} \\ \overline{C_2(\beta,\alpha,d_0)} & i\overline{D_2(\beta,\alpha,d_0)}\end{pmatrix}\\
&\times\begin{pmatrix} D(\infty)^{-1} & 0 \\ 0 & D(\infty)\end{pmatrix},
\end{aligned}
$$

with d_0 defined in (1.3.3).

Chapter 4 PROOFS OF THE ASYMPTOTIC RESULTS

4.1 Proof of Theorems 1.3.1 and 1.4.1

Given a compact set $K \subset \mathbb{C}\backslash\gamma$, we may open the lens about γ and pick neighborhoods U_δ and $\tilde{U}_\delta$ in such a way as to leave K in the exterior of $\hat{\Gamma}$. Then, reversing the steepest descent process we find that for $z \in K$,

$$Y(z) = \begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} R(z)N(z) \begin{pmatrix} \psi(z)^n & 0 \\ 0 & \psi(z)^{-n} \end{pmatrix},$$

where

$$N(z) = \begin{pmatrix} D(\infty) & 0 \\ 0 & D(\infty)^{-1} \end{pmatrix} \begin{pmatrix} A_{11}(z) & A_{12}(z) \\ -A_{12}(z) & A_{11}(z) \end{pmatrix} \begin{pmatrix} D(z)^{-1} & 0 \\ 0 & D(z) \end{pmatrix}$$

and

$$A_{11}(z) = \frac{a(z) + a^{-1}(z)}{2}, \qquad A_{12}(z) = \frac{a(z) - a^{-1}(z)}{2i}.$$

Hence,

$$\begin{pmatrix} Y_{11} \\ Y_{21} \end{pmatrix} = \begin{pmatrix} c^n\psi^n(z)[R_{11}(z)N_{11}(z) + R_{12}N_{21}(z)] \\ c^{-n}\psi^n(z)[R_{21}(z)N_{11}(z) + R_{22}N_{21}(z)] \end{pmatrix}.$$

Using that $Y_{11} = \varphi_n(z)$ and the expansion of Proposition 3.5.8

$$R(z) = I + \sum_{k=1} \frac{R_k(z)}{n^k},$$

we get

$$\frac{\varphi_n(z)}{c^n\psi^n(z)} \sim \frac{D(\infty)}{D(z)}\frac{a(z)+a(z)^{-1}}{2} \\ \times\left[1+\sum_{k=1}^{\infty}\frac{(R_k)_{11}(z)}{n^k}+\frac{i}{D(\infty)^2}\frac{a(z)-a(z)^{-1}}{a(z)+a(z)^{-1}}\sum_{k=1}^{\infty}\frac{(R_k)_{12}(z)}{n^k}\right].$$

By Proposition 2.2.10 we can the write

$$\frac{\varphi_n(z)}{c^{n+1}\psi^n(z)} \sim \frac{D(\infty)}{D(z)}\sqrt{\left(\frac{\psi(z)-e^{i\theta_0/2}}{z-e^{i\theta_0/2}}\right)\left(\frac{\psi(z)-e^{-i\theta_0/2}}{z-e^{-i\theta_0/2}}\right)}\left[1+\sum_{k=1}^{\infty}\frac{\Pi_k(z)}{n^k}\right],$$

where for each k,

$$\Pi_k(z)=(R_k)_{11}(z)+\frac{1}{D(\infty)^2}\frac{\sin(\theta_0/2)}{\psi(z)-c}(R_k)_{12}(z).$$

Using this formula together with (3.7.5) and (3.7.6), the expressions for Π_1 and Π_2 are obtained by direct computation.

The proof of Theorem 1.4.1 is just a repetition of the above arguments, since the expansion for (3.5.17) for R is valid as θ_0 varies, possibly approaching 0.

4.2 Proof of Theorem 1.3.2

For each n, let $Y^{(n)}$ denote the solution to the RHP Y1-Y4. From (1.2.2), we have

$$\overline{Y_{21}^{(n+1)}\left(\overline{z^{-1}}\right)}=-2\pi\kappa_n^2\frac{\varphi_n}{z^n}.$$

Since φ_n is a monic polynomial of degree n, we have that

$$\kappa_n^2=-\frac{1}{2\pi}\lim_{z\to\infty}\overline{Y_{21}^{(n+1)}\left(\overline{z^{-1}}\right)}=-\frac{1}{2\pi}\overline{Y_{21}^{(n+1)}(0)}.$$

As in the proof of Theorem 1.3.1, we have

$$Y_{21}^{(n+1)}(z) = c^{-(n+1)}\psi^{n+1}(z)[R_{21}^{(n+1)}(z)N_{11}(z) + R_{22}^{(n+1)}(z)N_{21}(z)],$$

where $R^{(n+1)}$ denotes the solution to R1-R3 corresponding to $n+1$. One can easily compute

$$\psi(0) = c^{-1}, \quad N_{11}(0) = \frac{cD(\infty)}{D(0)}, \quad N_{21}(0) = \frac{-\sin(\theta_0/2)}{D(\infty)D(0)},$$

so that

$$\begin{aligned} Y_{21}^{(n+1)}(0)c^{2(n+1)} &\sim \frac{cD(\infty)}{D(0)}\sum_{k=0}^{\infty}\frac{(R_k)_{21}(0)}{(n+1)^k} - \frac{\sin(\theta_0/2)}{D(\infty)D(0)}\left(1+\sum_{k=1}^{\infty}\frac{(R_k)_{22}(0)}{(n+1)^k}\right) \\ &\sim -\frac{\sin(\theta_0/2)}{D(\infty)D(0)}\left(1+\sum_{k=1}^{\infty}\frac{(R_k)_{22}(0)-\cot(\theta_0/2)D^2(\infty)(R_k)_{21}(0)}{(n+1)^k}\right). \end{aligned}$$

Since $D(0) = \overline{D(\infty)}$, see Proposition 2.2.3(iv), it follows that

$$\kappa_n^2 \sim \frac{\sin(\theta_0/2)}{c^{2(n+1)}2\pi|D(\infty)|^2}\left(1+\sum_{k=1}^{\infty}\frac{\Gamma_k}{n^k}\right),$$

where

$$\Gamma_k = \sum_{j=0}^{k-1}\binom{k-1}{j}(-1)^j\left(\overline{(R_{k-j})_{22}(0)} - \cot(\theta_0/2)\overline{D^2(\infty)(R_{k-j})_{21}(0)}\right).$$

Having already computed the first two functions R_1 and R_2, by evaluating them at zero we find Γ_1 and Γ_2, whose values are given in the statement of Theorem 1.3.2.

4.3 Proof of Theorem 1.3.3

Given a compact set $K \subset (\theta_0, 2\pi-\theta_0)$, we may pick neighborhoods U_δ and $\tilde{U}_\delta$ so small as to have the compact set $K^* = \{e^{i\theta} : \theta \in K\}$ lying exterior to $\overline{U_\delta \cup \tilde{U}_\delta}$. Then, reversing the steepest descent process, we find that for $z \in O_L \setminus \overline{U_\delta \cup \tilde{U}_\delta}$,

$$Y(z) = \begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} R(z)N(z) \begin{pmatrix} 1 & 0 \\ w(z)^{-1}z^n\psi(z)^{-2n} & 1 \end{pmatrix} \begin{pmatrix} \psi(z)^n & 0 \\ 0 & \psi(z)^{-n} \end{pmatrix}. \qquad (4.3.1)$$

Now, taking into account that R is continuous on the complement of $\hat{\Gamma}$, in particular, on the neighborhood $(O_L \cup O_R \cup \gamma) \setminus \overline{U_\delta \cup \tilde{U}_\delta}$ of K^*, we get after multiplying the matrices on the right of (4.3.1) and taking limits from O_L toward K^* that

$$\begin{aligned} \varphi_n(t) = D(\infty)c^n \left(\frac{\psi_+^n(t)A_{11+}(t)}{D_+(t)} + \frac{t^n D_+(t)A_{12+}(t)}{w(t)\psi_+^n(t)} \right) \left(1 + \sum_{k=1}^{\infty} \frac{(R_k)_{11}(t)}{n^k} \right) \\ + \frac{c^n}{D(\infty)} \left(\frac{\psi_+^n(t)A_{21+}(t)}{D_+(t)} + \frac{t^n D_+(t)A_{22+}(t)}{w(t)\psi_+^n(t)} \right) \sum_{k=1}^{\infty} \frac{(R_k)_{12}(t)}{n^k}. \end{aligned}$$

By plugging in the formulae obtained in Proposition 2.2.11, we find

$$\begin{aligned} \frac{\psi_+^n(e^{i\theta})A_{11+}(e^{i\theta})}{D_+(e^{i\theta})} + \frac{e^{in\theta}D_+(e^{i\theta})A_{12+}(e^{i\theta})}{w(e^{i\theta})\psi_+^n(e^{i\theta})} &= \frac{e^{i\frac{(2n-1)\theta}{4}}}{\sqrt{w(e^{i\theta})}} \sqrt{\frac{\sin(\theta_0/2)}{2\sin\lambda(\theta)}} \\ &\times \left(\Lambda(\theta)e^{i(n\lambda(\theta)-\chi(\theta))} + i\Lambda^{-1}(\theta)e^{-i(n\lambda(\theta)-\chi(\theta))} \right), \end{aligned}$$

and

$$\begin{aligned} \frac{\psi_+^n(e^{i\theta})A_{21+}(e^{i\theta})}{D_+(e^{i\theta})} + \frac{e^{in\theta}D_+(e^{i\theta})A_{22+}(e^{i\theta})}{w(e^{i\theta})\psi_+^n(e^{i\theta})} &= \frac{e^{i\frac{(2n-1)\theta}{4}}}{\sqrt{w(e^{i\theta})}} \sqrt{\frac{\sin(\theta_0/2)}{2\sin\lambda(\theta)}} \\ &\times \left(\Lambda(\theta)e^{-i(n\lambda(\theta)-\chi(\theta))} - i\Lambda^{-1}(\theta)e^{i(n\lambda(\theta)-\chi(\theta))} \right), \end{aligned}$$

where

$$\Lambda(\theta) = \sqrt{\frac{\cos(\theta/2)}{\sin(\theta_0/2)}} \sqrt{\tan(\theta/2) + \tan\lambda(\theta)} e^{i\frac{\lambda(\theta)}{2}}.$$

Theorem 1.3.3 then readily follows by combining the last three equalities. Notice that since

$$\sqrt{\cos(\theta/2)}\sqrt{\tan(\theta/2) + \tan\lambda(\theta)} = \sqrt{\sin(\theta/2) + \sqrt{c^2 - \cos^2(\theta/2)}},$$

we have $|\Lambda(\theta)| > 1$, and consequently

$$0 < |\Lambda(\theta)| - |\Lambda(\theta)|^{-1} < \left|\Lambda(\theta)e^{i(n\lambda(\theta)-\chi(\theta))} + i\Lambda^{-1}(\theta)e^{-i(n\lambda(\theta)-\chi(\theta))}\right| < |\Lambda(\theta)| + |\Lambda(\theta)|^{-1}.$$

Therefore, we are allowed to factor the term $\Lambda(\theta)e^{i(n\lambda(\theta)-\chi(\theta))} + i\Lambda^{-1}(\theta)e^{-i(n\lambda(\theta)-\chi(\theta))}$ out of (1.3.4) and get (1.3.5).

4.4 Proof of Theorems 1.3.5 and 1.3.6

For $\theta \in (\theta_0, 2\pi - \theta_0)$, we have

$$f(e^{i\theta}) = \frac{1}{4}\left[i\mathrm{Arg}\left(\frac{\psi_+(e^{i\theta})}{\sqrt{e^{i\theta}}}\right)\right]^2 = \begin{cases} -\frac{1}{4}\lambda(\theta)^2, & e^{i\theta} \in U_\delta \cap \gamma^o, \\ -\frac{1}{4}[\pi - \lambda(\theta)]^2, & e^{i\theta} \in \tilde{U}_\delta \cap \gamma^o. \end{cases}$$

From the definitions of W_+ and $\tilde{W}_+$, we see that

$$\begin{aligned} W_+(e^{i\theta}) &= \sqrt{w(e^{i\theta})}e^{\alpha\pi i}, \quad e^{i\theta} \in U_\delta \cap \gamma^o, \\ \tilde{W}_+(e^{i\theta}) &= \sqrt{w(e^{i\theta})}e^{-\beta\pi i}, \quad e^{i\theta} \in \tilde{U}_\delta \cap \gamma^o. \end{aligned} \tag{4.4.1}$$

We now concentrate on $t \in U_\delta \cap \gamma^o$. After reverting the steepest descent process, we arrive

$$Y(t) = \begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} R(t)E_n(t)[\Psi_\alpha(n^2 f(t))]_+ \begin{pmatrix} W_+(t)^{-1}\psi_+(t)^{-n}\sqrt{t}^n & 0 \\ 0 & W_+(t)\psi_+(t)^n\sqrt{t}^{-n} \end{pmatrix}$$
$$\times \begin{pmatrix} 1 & 0 \\ w(t)^{-1}\psi_+(t)^{-2n}t^n & 1 \end{pmatrix} \begin{pmatrix} \psi_+(t)^n & 0 \\ 0 & \psi_+(t)^{-n} \end{pmatrix}.$$

Multiplying the last three matrices and using (4.4.1) we get

$$\begin{pmatrix} Y_{11}(t) \\ Y_{21}(t) \end{pmatrix} = \begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} R(t)E_n(t)\frac{\sqrt{t}^n}{\sqrt{w(t)}}\Psi_{\alpha+}(n^2 f(t)) \begin{pmatrix} e^{-\alpha\pi i} \\ e^{\alpha\pi i} \end{pmatrix}.$$

Recall that $f(z)$ maps $U_\delta \cap O_L$ into $\Im z > 0$ with $2\pi/3 < \arg f(z) < \pi$, so that

$$[f(t)^{1/4}]_+ = \frac{e^{i\pi/4}\sqrt{\lambda(\theta)}}{\sqrt{2}}, \quad t = e^{i\theta},$$

and if we use the notation

$$\zeta = n^2 f(t) = -\frac{n^2}{4}\lambda^2(\theta), \quad t = e^{i\theta},$$

then

$$2(-\zeta)_+^{1/2} = n\lambda(\theta), \quad \zeta_+^{1/2} = \frac{in}{2}\lambda(\theta).$$

Then, using (3.4.8) and the fact that

$$H_{2\alpha}^{(1)} + H_{2\alpha}^{(2)} = 2J_{2\alpha},$$

see (9.1.3)-(9.1.4) in [1], we arrive at

$$\begin{aligned}\Psi_{\alpha+}(n^2 f(t))\begin{pmatrix} e^{-\alpha\pi i} \\ e^{\alpha\pi i}\end{pmatrix} &= \begin{pmatrix} \frac{1}{2}H_{2\alpha}^{(1)}(n\lambda(\theta)) & \frac{1}{2}H_{2\alpha}^{(2)}(n\lambda(\theta)) \\ \frac{i\pi n\lambda(\theta)}{2}(H_{2\alpha}^{(1)})'(n\lambda(\theta)) & \frac{i\pi n\lambda(\theta)}{2}(H_{2\alpha}^{(2)})'(n\lambda(\theta))\end{pmatrix}\begin{pmatrix} 1 \\ 1\end{pmatrix} \\ &= \begin{pmatrix} J_{2\alpha}\left(n\lambda(\theta)\right) \\ i\pi n\lambda(\theta) J_{2\alpha}'\left(n\lambda(\theta)\right)\end{pmatrix}.\end{aligned}$$

Across γ^o, $E_n = E_{n+}$, so that (c.f. (3.4.14))

$$E_n(t) = N_+(t) \begin{pmatrix} W_+(t) & 0 \\ 0 & W_+(t)^{-1} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2\pi n}[f(t)^{1/4}]_+ & 0 \\ 0 & \sqrt{2\pi n}^{-1}[f(t)^{-1/4}]_+ \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{A_{11+}(t)D(\infty)W_+(t)}{D_+(t)} & \frac{A_{12+}(t)D(\infty)D_+(t)}{W_+(t)} \\ \frac{-A_{12+}(t)W_+(t)}{D(\infty)D_+(t)} & \frac{A_{11+}(t)D_+(t)}{D(\infty)W_+(t)} \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} e^{i\pi/4}\sqrt{\pi n\lambda(\theta)} & 0 \\ 0 & e^{-i\pi/4}\sqrt{\pi n\lambda(\theta)}^{-1} \end{pmatrix}.$$

Therefore, for $t = e^{i\theta} \in U_\delta \cap \gamma^o$,

$$\begin{pmatrix} Y_{11}(t) \\ Y_{21}(t) \end{pmatrix} = \frac{D(\infty)\sqrt{t}^n\sqrt{\pi n\lambda(\theta)}}{\sqrt{2w(t)}} \begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} \left(I + \sum_{k=1}^{\infty} \frac{R_k(t)}{n^k} \right)$$

$$\times \begin{pmatrix} A_{11+}(t)e^{i(\alpha\pi - \chi(\theta))} & A_{12+}(t)e^{-i(\alpha\pi-\chi(\theta))} \\ -\frac{A_{12+}(t)e^{i(\alpha\pi-\chi(\theta))}}{D(\infty)^2} & \frac{A_{11+}(t)e^{-i(\alpha\pi-\chi(\theta))}}{D(\infty)^2} \end{pmatrix} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} e^{i\pi/4}J_{2\alpha}(n\lambda(\theta)) \\ ie^{-i\pi/4}J'_{2\alpha}(n\lambda(\theta)) \end{pmatrix}.$$

Using the formulas in Proposition 2.2.11 and the definition of $\Lambda(\theta)$ we can write

$$\begin{aligned} A_{11_+}(e^{i\theta}) &= e^{-i\theta/4}\sqrt{\frac{\sin(\theta_0/2)}{2\sin\lambda(\theta)}}\Lambda(\theta), \\ A_{12_+}(e^{i\theta}) &= ie^{-i\theta/4}\sqrt{\frac{\sin(\theta_0/2)}{2\sin\lambda(\theta)}}\Lambda(\theta)^{-1}. \end{aligned} \tag{4.4.2}$$

Thus,

$$\begin{pmatrix} Y_{11}(t) \\ Y_{21}(t) \end{pmatrix} = \frac{D(\infty)e^{i\frac{(2n-1)\theta}{4}}\sqrt{\pi n\lambda(\theta)}}{2\sqrt{w(t)}}\sqrt{\frac{\sin(\theta_0/2)}{\sin\lambda(\theta)}}\begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix}\left(I+\sum_{k=1}^{\infty}\frac{R_k(t)}{n^k}\right)$$
$$\times\begin{pmatrix} \Lambda(\theta)e^{i(\alpha\pi-\chi(\theta))}+\Lambda(\theta)^{-1}e^{-i(\alpha\pi-\chi(\theta))} & -i\left[\Lambda(\theta)e^{i(\alpha\pi-\chi(\theta))}-\Lambda(\theta)^{-1}e^{-i(\alpha\pi-\chi(\theta))}\right] \\ -i\frac{\Lambda(\theta)e^{-i(\alpha\pi-\chi(\theta))}+\Lambda(\theta)^{-1}e^{i(\alpha\pi-\chi(\theta))}}{D(\infty)^2} & \frac{\Lambda(\theta)e^{-i(\alpha\pi-\chi(\theta))}-\Lambda(\theta)^{-1}e^{i(\alpha\pi-\chi(\theta))}}{D(\infty)^2} \end{pmatrix}$$
$$\times\begin{pmatrix} e^{i\pi/4}J_{2\alpha}\left(n\lambda(\theta)\right) \\ ie^{-i\pi/4}J'_{2\alpha}\left(n\lambda(\theta)\right) \end{pmatrix},$$

and Theorem 1.3.5 follows after multiplying these matrices.

Next, we concentrate on $t = e^{i\theta} \in \tilde{U}_\delta \cap \gamma^o$. After reverting the steepest descent process, this time we arrive at

$$Y = \begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} R(t)\tilde{E}_n(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}[\Psi_\beta(n^2 f(t))]_+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$\times\begin{pmatrix} \frac{\sqrt{t}^n}{\tilde{W}_+(t)\psi_+^n(t)} & 0 \\ 0 & \frac{\tilde{W}_+(t)\psi_+^n(t)}{\sqrt{t}^n} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \frac{t^n}{w(t)\psi_+(t)^{2n}} & 1 \end{pmatrix}\begin{pmatrix} \psi_+(t)^n & 0 \\ 0 & \psi_+(t)^{-n} \end{pmatrix}.$$

Multiplying the last three matrices and using (4.4.1) we get

$$\begin{pmatrix} Y_{11} \\ Y_{21} \end{pmatrix} = \frac{\sqrt{t}^n}{\sqrt{w(t)}}\begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} R(t)\tilde{E}_n(t)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}[\Psi_\beta(n^2 f(t))]_+\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} e^{\beta\pi i} \\ e^{-\beta\pi i} \end{pmatrix}.$$

Recall that $f(z)$ maps $\tilde{U}_\delta \cap O_L$ into $\Im z < 0$ with $-\pi < \arg f(z) < -2\pi/3$, so that

$$[f(t)^{1/4}]_+ = \frac{e^{-i\pi/4}\sqrt{\pi-\lambda(\theta)}}{\sqrt{2}},$$

and if we again rename

$$\zeta = n^2 f(t) = -\frac{n^2}{4}[\pi-\lambda(\theta)]^2,$$

then

$$2(-\zeta)_+^{1/2} = n[\pi - \lambda(\theta)], \quad (\zeta^{1/2})_+ = -\frac{in}{2}[\pi - \lambda(\theta)],$$

and using (3.4.9), we arrive at

$$\begin{aligned}
&\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Psi_{\beta_+}(n^2 f(t)) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} e^{\beta\pi i} \\ e^{-\beta\pi i} \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{2} H_{2\beta}^{(1)}(n[\pi - \lambda(\theta)]) & \frac{1}{2} H_{2\beta}^{(2)}(n[\pi - \lambda(\theta)]) \\ -\frac{i\pi n[\pi-\lambda(\theta)]}{2} (H_{2\beta}^{(1)})'(n[\pi - \lambda(\theta)]) & -\frac{i\pi n[\pi-\lambda(\theta)]}{2} (H_{2\beta}^{(2)})'(n[\pi - \lambda(\theta)]) \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} J_{2\beta}\left(n[\pi - \lambda(\theta)]\right) \\ -i\pi n[\pi - \lambda(\theta)] J_{2\beta}'\left(n[\pi - \lambda(\theta)]\right) \end{pmatrix}.
\end{aligned}$$

We also have

$$\begin{aligned}
\tilde{E}_n(t) &= N_+(t) \begin{pmatrix} \tilde{W}_+(t) & 0 \\ 0 & \tilde{W}_+(t)^{-1} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2\pi n}[f(t)^{1/4}]_+ & 0 \\ 0 & \sqrt{2\pi n}^{-1}[f(t)^{-1/4}]_+ \end{pmatrix} \\
&= \begin{pmatrix} \frac{A_{11+}(t) D(\infty) \tilde{W}_+(t)}{D_+(t)} & \frac{A_{12+}(t) D(\infty) D_+(t)}{\tilde{W}_+(t)} \\ \frac{-A_{12+}(t) \tilde{W}_+(t)}{D(\infty) D_+(t)} & \frac{A_{11+}(t) D_+(t)}{D(\infty) \tilde{W}_+(t)} \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \\
&\quad \times \begin{pmatrix} e^{-i\pi/4} \sqrt{\pi n[\pi - \lambda(\theta)]} & 0 \\ 0 & e^{i\pi/4} \sqrt{\pi n[\pi - \lambda(\theta)]}^{-1} \end{pmatrix}.
\end{aligned}$$

Therefore, for $t = e^{i\theta} \in \tilde{U}_\delta \cap \gamma^o$,

$$\begin{aligned}
\begin{pmatrix} Y_{11}(t) \\ Y_{21}(t) \end{pmatrix} &= \frac{D(\infty)\sqrt{t}^n \sqrt{\pi n[\pi - \lambda(\theta)]}}{\sqrt{2w(t)}} \begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix} R(t) \\
&\quad \times \begin{pmatrix} A_{11+}(t) e^{-i(\beta\pi + \chi(\theta))} & A_{12+}(t) e^{i(\beta\pi + \chi(\theta))} \\ -\frac{A_{12+}(t)}{D(\infty)^2} e^{-i(\beta\pi + \chi(\theta))} & \frac{A_{11+}(t)}{D(\infty)^2} e^{i(\beta\pi + \chi(\theta))} \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} \begin{pmatrix} e^{-i\pi/4} J_{2\beta}\left(n[\pi - \lambda(\theta)]\right) \\ -ie^{i\pi/4} J_{2\beta}'\left(n[\pi - \lambda(\theta)]\right) \end{pmatrix}.
\end{aligned}$$

With the help of (4.4.2), we thus obtain

$$\begin{pmatrix} Y_{11}(t) \\ Y_{21}(t) \end{pmatrix} = \frac{D(\infty)e^{i\frac{(2n-1)\theta}{4}}\sqrt{\pi n[\pi-\lambda(\theta)]}}{2\sqrt{w(t)}}\sqrt{\frac{\sin(\theta_0/2)}{\sin\lambda(\theta)}}\begin{pmatrix} c^n & 0 \\ 0 & c^{-n} \end{pmatrix}\left(I+\sum_{k=1}^{\infty}\frac{R_k(t)}{n^k}\right)$$
$$\times \begin{pmatrix} \Lambda(\theta)e^{-i(\beta\pi+\chi(\theta))}-\Lambda(\theta)^{-1}e^{i(\beta\pi+\chi(\theta))} & i\left[\Lambda(\theta)e^{-i(\beta\pi+\chi(\theta))}+\Lambda(\theta)^{-1}e^{i(\beta\pi+\chi(\theta))}\right] \\ i\frac{\Lambda(\theta)e^{i(\beta\pi+\chi(\theta))}-\Lambda(\theta)^{-1}e^{-i(\beta\pi+\chi(\theta))}}{D(\infty)^2} & \frac{\Lambda(\theta)e^{i(\beta\pi+\chi(\theta))}+\Lambda(\theta)^{-1}e^{-i(\beta\pi+\chi(\theta))}}{D(\infty)^2} \end{pmatrix}$$
$$\times \begin{pmatrix} e^{-i\pi/4}J_{2\beta}\left(n[\pi-\lambda(\theta)]\right) \\ -ie^{i\pi/4}J'_{2\beta}\left(n[\pi-\lambda(\theta)]\right) \end{pmatrix},$$

and Theorem 1.3.6 follows after multiplying these matrices.

4.5 Proof of Corollary 1.3.7

As $\theta \to \theta_0+$, we have

$$e^{i\lambda(\theta)/2} = 1 + O(\theta-\theta_0)^{1/2}, \quad e^{i(\alpha\pi-\chi(\theta))} = 1 + O(\theta-\theta_0)^{1/2}$$

and

$$\begin{aligned} \Lambda(\theta)^2 e^{-i\lambda(\theta)} &= \frac{1}{\sin(\theta_0/2)}\left(\sin(\theta/2)+\sqrt{c^2-\cos^2(\theta/2)}\right) \\ &= \frac{1}{\sin(\theta_0/2)}\left(\sin(\theta/2)+\sqrt{\sin(\theta/2+\theta_0/2)\sin(\theta/2-\theta_0/2)}\right) \\ &= \frac{\sin(\theta/2)}{\sin(\theta_0/2)}\left(1+O(\theta-\theta_0)^{1/2}\right). \end{aligned}$$

Hence,

$$\begin{aligned} M_\alpha^{1,3}(\theta) &= 2 + O(\theta-\theta_0)^{1/2}, \\ M_\alpha^{2,4}(\theta) &= \left(\sqrt{\frac{\sin(\theta/2)}{\sin(\theta_0/2)}} - \sqrt{\frac{\sin(\theta_0/2)}{\sin(\theta/2)}}\right)\left(1+O(\theta-\theta_0)^{1/2}\right). \end{aligned}$$

Also,

$$\lim_{\theta\to\theta_0+}\frac{\lambda(\theta)^2}{e^{i\theta}-e^{i\theta_0}} = e^{-i(\pi/2+\theta_0)}\tan(\theta_0/2),$$
$$\lim_{\theta\to\theta_0+}\frac{\lambda(\theta)'}{(e^{i\theta}-e^{i\theta_0})^{-1/2}} = \frac{e^{i(\pi/4+\theta_0/2)}\sqrt{\tan(\theta_0/2)}}{2}.$$

Using these observations together with (1.3.7) and (1.1.2) (or (3.2.2)), we obtain

$$\lim_{\theta\to\theta_0}\frac{J_{2\alpha}(n\lambda\theta)}{\sqrt{w(e^{i\theta})}} = \left(\frac{n}{2}\right)^{2\alpha}\frac{\tan^\alpha(\theta_0/2)}{2^\beta\sin^\beta(\theta_0)\sqrt{h(e^{i\theta_0})}},$$
$$\lim_{\theta\to\theta_0}\frac{M_\alpha^{2,4}J'_{2\alpha}(n\lambda(\theta))}{\sqrt{w(e^{i\theta})}} = \left(\frac{n}{2}\right)^{2\alpha}\frac{\alpha\cot(\theta_0/2)\tan^\alpha(\theta_0/2)}{2(2\sin\theta_0)^\beta\sqrt{h(e^{i\theta_0})}},$$

and plugging these into the formula of Theorem 1.3.5 completes the proof. The behavior at $e^{-i\theta_0}$ is found analogously.

4.6 Proof of Corollaries 1.4.2 and 1.4.3

We first prove Corollary 1.4.2. If the arc γ approaches the unit circle, that is, if $\theta_0 \to 0$, then

$$\psi(z) \to \begin{cases} z, & |z| > 1, \\ 1, & |z| < 1. \end{cases}$$

Also, for $|z| < 1$, and as $\theta_0 \to 0$, we have

$$\psi(z) - c = \frac{z-\cos\theta_0+\sqrt{(z-e^{i\theta_0})(z-e^{-i\theta_0})}}{2c} = \frac{(z-e^{i\theta_0})(z-e^{-i\theta_0})-(z-\cos\theta_0)^2}{2c(\sqrt{(z-e^{i\theta_0})(z-e^{-i\theta_0})}-(z-\cos\theta_0))}$$
$$= \frac{c\sin^2(\theta_0/2)}{(1-z)}(1+o(1)).$$

Then, by (2.2.10) we have that for $|z|<1$,

$$\lim_{\theta_0\to 0}\frac{\sqrt{\left(\frac{\psi(z)-e^{i\theta_0/2}}{z-e^{i\theta_0/2}}\right)\left(\frac{\psi(z)-e^{-i\theta_0/2}}{z-e^{-i\theta_0/2}}\right)}}{\sin(\theta_0/2)}=\lim_{\theta_0\to 0}\sqrt{\frac{\psi(z)-c}{c\sin^2(\theta_0/2)g(z)}}=\frac{1}{1-z}.$$

Hence,

$$\sqrt{\left(\frac{\psi(z)-e^{i\theta_0/2}}{z-e^{i\theta_0/2}}\right)\left(\frac{\psi(z)-e^{-i\theta_0/2}}{z-e^{-i\theta_0/2}}\right)}=\begin{cases}1+o(1), & |z|>1,\\ \frac{\sin(\theta_0/2)}{1-z}(1+o(1)), & |z|<1.\end{cases}$$

We now show that as $\theta_0\to 0$,

$$D(z)\to\begin{cases}e^{i(\alpha-\beta)\pi/2}\left(\frac{z-1}{z}\right)^{\alpha+\beta}\exp\left(-\frac{1}{2\pi i}⨍_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta}-\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{\zeta-z}\right), & |z|>1,\\ e^{i(\beta-\alpha)\pi/2}(1-z)^{\alpha+\beta}\exp\left(\frac{1}{2\pi i}⨍_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta}+\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{\zeta-z}\right), & |z|<1,\end{cases}$$

and, consequently,

$$D(\infty)\to e^{i(\alpha-\beta)\pi/2}\exp\left(-\frac{1}{2\pi i}⨍_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta}\right).$$

This will follow from the expression

$$D(z)=\left[\frac{ie^{-i\theta_0/2}(z-e^{i\theta_0})}{\psi(z)}\right]^{\alpha}\left[\frac{-ie^{i\theta_0/2}(z-e^{-i\theta_0})}{\psi(z)}\right]^{\beta}\exp\left(\frac{g(z)}{2\pi i}\int_\gamma\frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta-z}\right)$$

once we understand what the exponential factor approaches to.

By Cauchy's theorem,

$$\begin{aligned}\frac{g(z)}{\pi i}\int_\gamma\frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta-z}&=\frac{g(z)}{2\pi i}\int_\gamma\frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta-z}+\frac{g(z)}{2\pi i}\int_{-\gamma}\frac{\log h(\zeta)}{g_-(\zeta)}\frac{d\zeta}{\zeta-z}\\&=\frac{g(z)}{2\pi i}\int_{\sigma_2}\frac{\log h(\zeta)}{g(\zeta)}\frac{d\zeta}{\zeta-z}+\frac{g(z)}{2\pi i}\int_{\sigma_1}\frac{\log h(\zeta)}{g(\zeta)}\frac{d\zeta}{\zeta-z},\end{aligned}$$

where $\sigma=\sigma_1\cup\sigma_2\subset U$, with σ_1 and σ_2 circles centered at the origin and of radii less than 1 and greater than 1, respectively. Here σ_1 is positively oriented and σ_2 is negatively oriented, and they are chosen so close to the unit circle as to not have z in between them. Then, for $|z| > 1$,

$$\begin{aligned}\lim_{\theta_0\to 0}\frac{g(z)}{\pi i}\int_\gamma \frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta - z} &= \lim_{\theta_0\to 0}\frac{z-1}{2\pi i}\int_{\sigma_1}\frac{\log h(\zeta)}{1-\zeta}\frac{d\zeta}{\zeta - z} + \lim_{\theta_0\to 0}\frac{z-1}{2\pi i}\int_{\sigma_2}\frac{\log h(\zeta)}{\zeta - 1}\frac{d\zeta}{\zeta - z}\\ &= -\lim_{\theta_0\to 0}\frac{1}{2\pi i}\int_{\sigma_1}\frac{\log h(\zeta)d\zeta}{1-\zeta} - \lim_{\theta_0\to 0}\frac{1}{2\pi i}\int_{\sigma_1}\frac{\log h(\zeta)d\zeta}{\zeta - z}\\ &\quad + \lim_{\theta_0\to 0}\frac{1}{2\pi i}\int_{\sigma_2}\frac{\log h(\zeta)d\zeta}{1-\zeta} + \lim_{\theta_0\to 0}\frac{1}{2\pi i}\int_{\sigma_2}\frac{\log h(\zeta)d\zeta}{\zeta - z}\\ &= -\frac{1}{\pi i}\int_{\mathbb{T}_\epsilon}\frac{\log h(\zeta)d\zeta}{1-\zeta} - \frac{1}{2\pi i}\int_{C_\epsilon}\frac{\log h(\zeta)d\zeta}{1-\zeta} - \frac{1}{\pi i}\int_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{\zeta - z},\end{aligned}$$

where $\mathbb{T}_\epsilon = \{z : |z| = 1,\ |z-1| \geq \epsilon\}$ and C_ϵ consists of two circular arcs

$$C_\epsilon := \{z : |z| \geq 1,\ |z-1| = \epsilon\} \cup \{z : |z| \leq 1,\ |z-1| = \epsilon\},$$

the first arc being oriented in counter clockwise direction, the second one in clockwise direction. Letting $\epsilon \to \infty$, we get that for $|z| > 1$,

$$\lim_{\theta_0\to 0}\frac{g(z)}{\pi i}\int_\gamma \frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta - z} = -\frac{1}{\pi i}⨍_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta} - \frac{1}{\pi i}\int_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{\zeta - z}$$

For $|z| < 1$, we can follow the same argument, the change in sign of $g(z)$ being the only difference, and so we have that for $|z| < 1$,

$$\lim_{\theta_0\to 0}\frac{g(z)}{\pi i}\int_\gamma \frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta - z} = \frac{1}{\pi i}⨍_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{1-\zeta} + \frac{1}{\pi i}\int_{|\zeta|=1}\frac{\log h(\zeta)d\zeta}{\zeta - z}.$$

Having found all the necessary limits, we can now use them in the formula of Theorem 1.4.1 to get Corollary 1.4.2.

Corollary 1.4.3 follows from Corollary 1.3.7 by a straight forward limiting computation, since in the sense explained in Section 1.4, the asymptotic formula of Corollary 1.3.7 remains valid as the angle $\theta_0 \to 0$.

Chapter 5 PROOFS OF THE AUXILIARY RESULTS

5.1 Non-hermitian representation of the orthogonal polynomials

For $z = e^{i\theta}$, we have

$$dz = ie^{i\theta}d\theta, \qquad |dz| = |ie^{i\theta}|d\theta = d\theta = \frac{dz}{iz}.$$

Since $\overline{z} = 1/z$ for $|z| = 1$, we see that for all $0 \leq k \leq n-1$,

$$\begin{aligned}\frac{1}{\kappa_n}\delta_{nk} &= \int_\gamma \Phi_n(z)\overline{z^k}w(z)|dz| == \int_\gamma \Phi_n(z)z^{-k}w(z)\frac{dz}{iz} \\ &= -i\int_\gamma \Phi_n(z)z^{-k-1}w(z)dz = -i\int_\gamma \Phi_n(z)z^{n-k-1}\frac{w(z)}{z^n}dz.\end{aligned}$$

In particular, the last integral is 0 whenever $0 \leq k \leq n-1$. That is,

$$\int_\gamma \Phi_n(z)z^m\frac{w(z)}{z^n} = 0, \quad 0 \leq m \leq n-1.$$

Also, we note that

$$\int_\gamma \overline{\Phi_n(z)}z^k w(z)|dz| = -i\int_\gamma z^n\overline{\Phi_n(1/\overline{z})}z^{k-1}\frac{w(z)}{z^n}dz = -i\int_\gamma \Phi_n^*(z)z^{k-1}\frac{w(z)}{z^n}dz.$$

Thus,

$$\int_\gamma \Phi_{n-1}^*(z)z^m\frac{w(z)}{z^n}dz = \begin{cases} 0, & -1 \leq m \leq n-2, \\ \frac{i}{\kappa_{n-1}}, & m = n-1. \end{cases}$$

5.2 Proof of Theorem 1.2.1

Note that the jump condition Y2 implies that

$$Y_{11_+}(t) = Y_{11_-}(t)$$

for all $t \in \gamma^o$. Since the singularities of Y are removable, we have that Y_{11} is entire. The asymptotic condition Y3 yields $Y_{11} = z^n + O(z^{n-1})$ as $z \to \infty$, so by Liouville's theorem, Y_{11} is a monic polynomial of degree n.

The condition Y2 implies that for all $z \in \gamma^o$,

$$Y_{12_+}(t) = Y_{12_-}(t) + Y_{11_-}(t) z^{-n} w(t).$$

That is, Y_{12} can be solved as an additive Riemann-Hilbert problem. By Corollary 2.1.3, we know this problem has the solution

$$Y_{12}(z) = \frac{1}{2\pi i} \int_\gamma \frac{Y_{11}(t) w(t) dt}{t^n (t-z)}.$$

We also note the relation

$$\frac{1}{t-z} = -\sum_{k=0}^{n-1} \frac{t^k}{z^{k+1}} + \frac{t^n}{z^n (t-z)},$$

so that we may rewrite

$$Y_{12}(z) = O(z^{-n-1}) - \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_\gamma Y_{11}(t) t^k \frac{w(t)}{t^n} dt \left(\frac{1}{z^{k+1}} \right).$$

Y-RH3 demands that as $z \to \infty$, $Y_{12} = O(z^{-n-1})$. In order for this to be true, it must be the case that each coefficient of z^{k+1} is 0 for $0 \leq k \leq n-1$. That is to say, Y_{11} must satisfy the non-hermitian orthogonal requirements. Since we already deduced that Y_{11} is monic, it must be the case that $Y_{11}(z) = \varphi_n(z)$ for all $z \in \mathbb{C}$.

Similarly, Y-RH2 implies $Y_{21_+} = Y_{21_-}$ on γ^o, and Y-RH4/5 imply Y_{21} is bounded at the endpoints of γ, so Y_{21} is entire. Meanwhile, Y-RH3 demands that as $z \to \infty$, $Y_{21} = O(z^{n-1})$. Y-RH2 also implies that for all $z \in \gamma^o$,

$$Y_{22_+}(t) = Y_{22_-}(t) + Y_{21_-}(t) t^{-n} w(t).$$

So that, as $z \to \infty$,

$$Y_{22}(z) = \frac{1}{2\pi i} \int_\gamma \frac{Y_{21}(t) w(t) dt}{t^n (t-z)} = O(z^{-n-1}) - \sum_{k=0}^{n-1} \frac{1}{2\pi i} \int_\gamma Y_{21}(t) t^k \frac{w(t)}{t^n} dt \left(\frac{1}{z^{k+1}} \right).$$

To satisfy $Y_{22} = z^{-n} + O(z^{-n-1})$ near ∞, we must have

$$\frac{1}{2\pi i} \int_\gamma Y_{21}(t) t^k \frac{w(t)}{t^n} dt = \begin{cases} -1, & k = n-1, \\ 0, & 0 \le k \le n-2. \end{cases}$$

After inspecting the non-Hermitian orthogonality condition, we may conclude that

$$Y_{21}(z) = -2\pi \kappa_{n-1} \Phi^*_{n-1}(z),$$

and the construction of Y_{22} follows.

5.3 Proof of Proposition 2.2.1

By standard arguments, the only conformal map $\psi(z)$ of the exterior of the arc γ onto the exterior of the unit circle satisfying $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$ is

$$\psi(z) := i \left(\frac{\varphi\left(\frac{i}{\cot(\theta_0/2)} \frac{z+1}{z-1}\right) \overline{\varphi\left(\frac{i}{\cot(\theta_0/2)}\right)} - 1}{\varphi\left(\frac{i}{\cot(\theta_0/2)} \frac{z+1}{z-1}\right) - \varphi\left(\frac{i}{\cot(\theta_0/2)}\right)} \right),$$

where $\varphi(z) = z + \sqrt{z^2 - 1}$ is the conformal map of the exterior of the segment $[-1, 1]$ onto the exterior of the unit disk, with the branch of the square root chosen to be positive for $z > 1$. Using $\varphi(z) + 1/\varphi(z) = 2z$ and the understanding that $\sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})} = g(z)$, exhaustive computation shows that $\psi(z)$ can be written as

$$\psi(z) = \frac{z + 1 + \sqrt{(z - e^{i\theta_0})(z - e^{-i\theta_0})}}{2\cos(\theta_0/2)}.$$

We now prove the listed properties.

i. To find the inverse of $\psi(z)$, we first notice that $(z - e^{i\theta_0})(z - e^{-i\theta_0}) = (z+1)^2 - 4c^2 z$. Hence

$$\frac{1}{\psi(z)} = \frac{2c}{z + 1 + g(z)} = \frac{z + 1 - g(z)}{2c},$$

so that

$$2c\psi(z) + \frac{2c}{\psi(z)} = 2z + 2$$

and

$$z = \frac{\psi(z)[c\psi(z) - 1]}{\psi(z) - c}. \tag{5.3.1}$$

ii. Using that

$$\left|\frac{c\psi(z) - 1}{\psi(z) - c}\right| < 1 \iff |\psi(z)| > 1$$

and relation (5.3.1), we find

$$\left|\frac{z}{\psi(z)}\right| = \left|\frac{c\psi(z) - 1}{\psi(z) - c}\right| < 1, \quad z \in \overline{\mathbb{C}} \setminus \gamma.$$

iii. Orienting γ clockwise from $e^{-\theta_0}$ to e^{θ_0}, the positive side of γ lies outside the unit circle and the negative side lies inside. From (2.2.2) we have

$$\psi(e^{\pm i\theta_0}) = e^{\pm i\theta_0/2},$$

and so the function given by the positive boundary values ψ_+ takes γ to the clockwise oriented arc of the unit circle going from $e^{-i\theta_0/2}$ to $e^{i\theta_0/2}$, while the negative boundary values ψ_- take γ to the clockwise oriented arc of the unit circle going from $e^{i\theta_0/2}$ to $e^{-i\theta_0/2}$.

We now want to understand how to obtain $\psi_+(t)$ from $\psi_-(t)$ for any given value $t \in \gamma^o$. Using (2.2.1), we have

$$\begin{aligned}\psi_+(t)\psi_-(t) &= \frac{1}{4c^2}\left(t+1+g_+(t)\right)\left(t+1+g_-(t)\right)\\ &= \frac{1}{4c^2}\Big((t+1)^2 + (t+1)\left(g_+(t)+g_-(t)\right) - G^2(t)\Big) = t\end{aligned}$$

and

$$\psi_+(t)+\psi_-(t) = \frac{1}{2c}\left(2t+2+g_+(t)+g_-(t)\right) = \frac{t+1}{c}.$$

Taking limits as $z \to t$ from each side in (5.3.1), we get that

$$t = \frac{\psi_+(t)[c\psi_+(t)-1]}{\psi_+(t)-c} = \frac{\psi_-(t)[c\psi_-(t)-1]}{\psi_-(t)-c}, \quad t \in \gamma^o. \tag{5.3.2}$$

Since we know that $\psi_+(t)\psi_-(t) = t$, this yields

$$\psi_-(t) = \frac{c\psi_+(t)-1}{\psi_+(t)-c}, \qquad \psi_+(t) = \frac{c\psi_-(t)-1}{\psi_-(t)-c}, \quad t \in \gamma^o.$$

Alternatively, one can also argue by cross multiplying the two fractions in (5.3.2) and rearranging terms, which gives

$$[\psi_+(t) - \psi_-(t)][\psi_-(t) - c]\left[\psi_+(t) - \frac{c\psi_-(t) - 1}{\psi_-(t) - c}\right] = 0.$$

This proves (2.2.5), and together with (5.3.2) gives another proof that $\psi_+(t)\psi_-(t) = t$.

We can now also deduce from (2.2.5) that

$$\begin{aligned}\psi_-(t) + \psi_+(t) &= \frac{c\psi_+(t) - 1}{\psi_+(t) - c} + \psi_+(t) = \frac{\psi_+^2(t) - 1}{\psi_+(t) - c}\\ &= \frac{c^{-1}\psi_+(t)[c\psi_+(t) - 1 + 1] - 1}{\psi_+(t) - c}\\ &= \frac{\psi_+(t)\psi_-(t)}{c} + \frac{1}{c} = \frac{t+1}{c},\end{aligned}$$

which allows one to prove the property that $g_+(t) = -g_-(t)$. In effect, since $g(z) = 2c\psi(z) - z - 1$, we have

$$g_+(t) + g_-(t) = 2c[\psi_+(t) + \psi_-(t)] - 2t - 2 = 0.$$

iv. From (2.2.5) we get

$$\begin{aligned}\psi_+(t) &= \frac{c\psi_-(t) - 1}{\psi_-(t) - c} = \frac{c[\psi_-(t) - c] + c^2 - 1}{\psi_-(t) - c} = c - \frac{\sin^2(\theta_0/2)}{\psi_-(t) - c}\\ &= \cos(\theta_0/2) - \frac{\sin^2(\theta_0/2)}{|\psi_-(t) - \cos(\theta_0/2)|^2}\left(\overline{\psi_-(t)} - \cos(\theta_0/2)\right).\end{aligned}$$

5.4 Proof of Proposition 2.2.3

ii. Let Σ be a Jordan contour oriented clockwise and passing through $e^{\pm i\theta_0}$ (such as, i.e. the circle $-1 + |e^{i\theta_0} + 1|e^{-it}$, $0 \le t \le 2\pi$). Let Σ_1 and Σ_2 be, respectively, the arcs of Σ from $e^{-i\theta_0}$ to $e^{i\theta_0}$ and from $e^{i\theta_0}$ to $e^{-i\theta_0}$, both respecting the clockwise orientation. Then, we see that for every z outside Σ,

$$\int_\gamma \frac{1}{G(\zeta)}\frac{d\zeta}{\zeta-z} = \frac{1}{2}\left\{\int_\gamma \frac{1}{g_+(\zeta)}\frac{d\zeta}{\zeta-z} + \int_{-\gamma}\frac{1}{g_-(\zeta)}\frac{d\zeta}{\zeta-z}\right\}$$
$$= \frac{1}{2}\left\{\int_{\Sigma_1}\frac{1}{g(\zeta)}\frac{d\zeta}{\zeta-z} + \int_{\Sigma_2}\frac{1}{g(\zeta)}\frac{d\zeta}{\zeta-z}\right\} = \frac{1}{2}\oint_\Sigma \frac{1}{g(\zeta)}\frac{d\zeta}{\zeta-z}.$$

If now $C_R = \left\{Re^{i\theta} : 0 \le \theta \le 2\pi\right\}$, then by the residue theorem we find that for all $R > |z|$,

$$\frac{\pi i}{g(z)} = \frac{1}{2}\oint_{C_R}\frac{1}{g(\zeta)}\frac{d\zeta}{\zeta-z} + \frac{1}{2}\oint_\Sigma \frac{1}{g(\zeta)}\frac{d\zeta}{\zeta-z},$$

and the result follows from the uniqueness principle for analytic functions, since

$$\lim_{R\to\infty}\oint_{C_R}\frac{1}{g(\zeta)}\frac{d\zeta}{\zeta-z} = 0.$$

iii. Define

$$C(z) := \frac{1}{2\pi i}\int_\gamma \frac{\log w(\zeta)}{G(\zeta)}\frac{d\zeta}{\zeta-z},$$

so that $C_+(t) - C_-(t) = G^{-1}(t)\log w(t)$, for every $t \in \gamma^o$. Since $g_+ = -g_-$ across γ^o, we have

$$D_+(t;w)D_-(t;w) = \exp\left[g_+(t)C_+(t) + g_-(t)C_-(t)\right] = \exp\left[g_+(t)C_+(t) - g_+(t)C_-(t)\right] = w(t).$$

iv. By definition, and since $w(t) \ge 0$,

$$\overline{D(0)} = \exp\left(-\frac{1}{2\pi i}\int_\gamma \frac{\log w(\zeta)}{\overline{g_+(\zeta)}}\frac{\overline{d\zeta}}{\overline{\zeta}}\right) = \exp\left(-\frac{1}{2\pi i}\int_\gamma \frac{\log w(\zeta)d\zeta}{g_+(\zeta)}\right) = D(\infty),$$

where we have used that $\overline{g_+(\zeta)} = -g_+(\zeta)/\zeta$, see (5.12.1) below.

5.5 Proof of Proposition 2.2.4

Writing $\omega = \psi(z)$, we obtain from (2.2.5) that

$$-ie^{i\theta_0/2}F(z) = \; - c\tan^2(\theta_0/2)\rho^{-2}\left[1 - c\omega^{-1} + \frac{\rho^2}{1 - c\omega^{-1}} - 2\rho\right],$$

where

$$\rho := \frac{i\tan(\theta_0/2)}{1 + i\tan(\theta_0/2)} = \frac{1}{1 - i\cot(\theta_0/2)}.$$

Thus, making the change of variables

$$v = \rho^{-1}(1 - c\omega^{-1}),$$

we get

$$-ie^{i\theta_0/2}F(z) = \; - c\tan^2(\theta_0/2)\rho^{-1}\left[v + \frac{1}{v} - 2\right].$$

That is,

$$F(z) = \; - \tan(\theta_0/2)\left[v + \frac{1}{v} - 2\right].$$

We know that $\omega = \psi(z)$ takes $\overline{\mathbb{C}} \setminus \gamma$ onto the exterior of the unit circle, and so $\rho^{-1}(1 - c\omega^{-1})$ takes $\overline{\mathbb{C}} \setminus \gamma$ onto the interior of the circle

$$C := \{v : |v - \rho^{-1}| = c/|\rho|\} = \{v : |v - (1 - i\cot(\theta_0/2)| = \cot(\theta_0/2)\}.$$

The function $v + 1/v$ is conformal on $\Im z < 0$ and doubles angles at 1, so that it maps the circle C onto a cardioid shaped curve. The circle C and the unit circle $\mathbb{T}_1$ intersect at 1 and

$-e^{i\theta}$, being orthogonal to each other, and thus C is the reflection of itself about $\mathbb{T}_1$. Thus the portion of C interior (resp. exterior) to $\mathbb{T}_1$ goes onto the upper (resp. lower) half plane, and the cardioid is symmetric about the x-axis, intersecting this axis at 2 and $-2\cos\theta_0$. Therefore, the proposition will follow once we prove that if

$$v_\pm := \rho^{-1}(1 - c/\omega_\pm), \quad \omega_\pm = \psi_\pm(x),$$

then $v_- = 1/\overline{v}_+$. Using (2.2.5), that $|\omega_\pm| = 1$, and that $|\rho| = \sin(\theta_0/2)$, we compute

$$v_- = \rho^{-1}\left(1 - c/\omega_-\right) = \rho^{-1}\left(1 - \frac{c}{\frac{c\omega_+ - 1}{\omega_+ - c}}\right) = -\rho^{-1}\frac{\sin^2(\theta_0/2)}{c\omega_+ - 1} = \rho(1 - c/\overline{\omega_+})^{-1} = 1/\overline{v}_+.$$

5.6 Proof of Proposition 2.2.5

By Proposition 2.2.4, the principal logarithm of $F(z)$ exists in $\mathbb{C} \setminus \gamma$, and

$$[\mathrm{Log}F(z)]_+ + [\mathrm{Log}F(z)]_- = 2\ln|z - e^{i\theta_0}|.$$

Therefore, for $z \in \mathbb{C} \setminus \gamma$,

$$\begin{aligned}
\int_\gamma \frac{\log|t - e^{i\theta_0}|^{2\alpha}}{g_+(t)}\frac{dt}{t - z} &= \alpha\int_\gamma \frac{2\ln|t - e^{i\theta_0}|}{g_+(t)}\frac{dt}{t - z} \\
&= \alpha\left[\int_\gamma \frac{[\mathrm{Log}F(t)]_+}{g_+(t)}\frac{dt}{t - z} + \int_\gamma \frac{[\mathrm{Log}F(t)]_-}{g_+(t)}\frac{dt}{t - z}\right] \\
&= \alpha\left[\int_\gamma \frac{[\mathrm{Log}F(t)]_+}{g_+(t)}\frac{dt}{t - z} - \int_\gamma \frac{[\mathrm{Log}F(t)]_-}{g_-(t)}\frac{dt}{t - z}\right] \\
&= \alpha\oint_\Sigma \frac{\mathrm{Log}F(t)}{g(t)}\frac{dt}{t - z},
\end{aligned}$$

where Σ is the closed contour (oriented clockwise and passing through the points $e^{\pm i\theta_0}$) sufficiently close to γ so that z is in its exterior.

For $R > |z|$, let $C_R = \left\{Re^{i\theta} : 0 \le \theta \le 2\pi\right\}$. By the residue theorem,

$$\oint_{\Sigma} \frac{\mathrm{Log}F(t)}{g(t)} \frac{dt}{t-z} = \frac{2\pi i \mathrm{Log}F(z)}{g(z)} - \oint_{C_R} \frac{\mathrm{Log}F(t)}{g(t)} \frac{dt}{t-z}.$$

Letting R tend to infinity, the integral vanishes, and we find

$$\frac{g(z)}{2\pi i} \int_{\gamma} \frac{\log|z-e^{i\theta_0}|^{2\alpha}}{g_+(t)} \frac{dt}{t-z} = \alpha \mathrm{Log}F(z).$$

By similar arguments,

$$\frac{g(z)}{2\pi i} \int_{\gamma} \frac{\log|z-e^{-i\theta_0}|^{2\beta}}{g_+(t)} \frac{dt}{t-z} = \beta \mathrm{Log}\widetilde{F}(z),$$

and so

$$D(z;w) = \exp\left\{\alpha \mathrm{Log}F(z) + \beta \mathrm{Log}\widetilde{F}(z)\right\} = \left[\frac{ie^{-i\theta_0/2}(z-e^{i\theta_0})}{\psi(z)}\right]^{\alpha} \left[\frac{-ie^{i\theta_0/2}(z-e^{-i\theta_0/2})}{\psi(z)}\right]^{\beta}.$$

5.7 Proof of Proposition 2.2.6

Again, we let $\omega = \psi(z)$. From (2.2.3), we have

$$\begin{aligned} \frac{\mathcal{F}(z,\pm\theta_0)}{i\sin(\pm\theta_0/2)} &= \frac{\frac{\omega(c\omega-1)}{\omega-c} - e^{\pm i\theta_0}}{\omega - c} = \frac{c\omega^2 - \omega - \omega e^{\pm i\theta_0} + ce^{\pm i\theta_0}}{(\omega-c)^2} \\ &= \frac{c\omega^2 - 2ce^{\pm i\theta_0/2}\omega + ce^{\pm i\theta_0}}{(\omega-c)^2} = c\left(\frac{\omega - e^{\pm i\theta_0/2}}{\omega - c}\right)^2 \\ &= c\left(1 - \frac{i\sin(\pm\theta_0/2)}{\omega-c}\right)^2, \end{aligned}$$

or

$$\begin{aligned}\mathcal{F}(z, \pm\theta_0) &= i\sin(\pm\theta_0/2)\cos(\theta_0/2)\left(1 - \frac{i\sin(\pm\theta_0/2)}{\omega - c}\right)^2 \\ &= \frac{i\sin(\pm\theta_0)}{2}\left(1 - \frac{i\sin(\pm\theta_0/2)}{\omega - c}\right)^2.\end{aligned}$$

We know that $\omega - c$ maps $\mathbb{C}\backslash\gamma$ onto the exterior of the circle of radius 1 centered at $-c$. Hence $\frac{\omega-c}{\sin(\theta_0/2)}$ maps $\mathbb{C}\backslash\gamma$ onto the exterior of the circle centered at $-\cot(\theta_0/2)$ with radius $\frac{1}{\sin(\theta_0/2)}$, which crosses the x-axis at $-\cot(\theta_0/4)$ and $\tan(\theta_0/4)$, so that $\frac{\sin(\theta_0/2)}{\omega-c}$ maps to the interior of the same circle. Multiplying by $\pm i$ shifts the circle to one centered at $\mp i\cot(\theta_0/2)$ and passing through the points $\mp i\cot(\theta_0/4)$ and $\pm i\tan(\theta_0/4)$. By the Pythagorean Theorem, this circle intersects the x-axis at both 1 and -1, so that $1 \mp \frac{i\sin(\theta_0/2)}{\omega-c}$ maps $\mathbb{C}\backslash\gamma$ onto the interior of the circle centered at $1 \mp i\cot(\theta_0/2)$ and passing through the origin and 2. Moreover, the line passing through the origin and the center of this circle is of the form $\ell_1 = \{re^{\pm i(\theta_0-\pi)/2} : r \in \mathbb{R}\}$. That is, when we "square this circle", we find that $\left(1 - \frac{i\sin(\pm\theta_0/2)}{\omega-c}\right)^2$ maps to a cardioid symmetric about the line $\ell_2 = \{re^{\pm i(\theta_0-\pi)} : r \in \mathbb{R}\}$, and lastly, multiplying by $\pm i$ and rescaling by $\frac{\sin(\theta_0)}{2}$, we find that the co-domain of $\mathcal{F}(z, \pm\theta_0)$ is the interior of the cardioid symmetric about the line $\ell = \{re^{\pm i(\theta_0-\pi/2)} : r \in \mathbb{R}\}$, passing through the origin, and lying in $\mathbb{C} \setminus \{-re^{\pm i(\theta_0-\pi/2)} : r > 0\}$.

Hence a branch of $\log\mathcal{F}(z, \pm\theta_0)$ exists with

$$\theta_0 - \frac{3\pi}{2} < \arg\mathcal{F}(z, \theta_0) < \theta_0 + \frac{\pi}{2}$$

and

$$-\theta_0 - \frac{\pi}{2} < \arg\mathcal{F}(z, -\theta_0) < -\theta_0 + \frac{3\pi}{2}.$$

5.8 Proof of Proposition 2.2.7

Recall from (2.2.7) that

$$\psi_+(t) - \cos(\theta_0/2) = \frac{-\sin^2(\theta_0/2)}{\psi_-(t) - \cos(\theta_0/2)}.$$

By uniqueness of branches of the logarithm, there exists $m_\pm \in \mathbb{Z}$ such that

$$\log \mathcal{F}(t, \pm\theta_0)_+ + \log \mathcal{F}(t, \pm\theta_0)_- + 2m_\pm \pi i = 2\log(t - e^{\pm i\theta_0}). \tag{5.8.1}$$

Note that

$$\mathcal{F}(e^{\mp i\theta_0}, \pm\theta_0) = \frac{i\sin(\pm\theta_0/2)(e^{\mp i\theta_0} - e^{\pm i\theta_0})}{e^{\mp i\theta_0/2} - \cos(\theta_0/2)} = 2i\sin(\pm\theta_0).$$

Taking limits as $t \to e^{\mp i\theta_0}$ in (5.8.1), we deduce

$$\ln|2i\sin(\theta_0)|^2 + i\pi + 2m_+\pi i = \ln|-4\sin^2(\theta_0)| + 3\pi i,$$

$$\ln|2i\sin(\theta_0)|^2 - \pi i + 2m_-\pi i = \ln|-4\sin^2(\theta_0)| + \pi i.$$

Hence, $m_\pm = 1$ and

$$\begin{aligned}
&\int_\gamma \frac{\log(t - e^{\pm i\theta_0})}{g_+(t)} \frac{dt}{t-z} = \frac{1}{2}\left[\int_\gamma \frac{2\log(t - e^{\pm i\theta_0})}{g_+(t)} \frac{dt}{t-z}\right] \\
&= \frac{1}{2}\left[\int_\gamma \frac{\log \mathcal{F}(t, \pm\theta_0)_+}{g_+(t)} \frac{dt}{t-z} - \int_\gamma \frac{\log \mathcal{F}(t, \pm\theta_0)_-}{g_-(t)} \frac{dt}{t-z} + \int_\gamma \frac{2\pi i}{g_+(t)} \frac{dt}{t-z}\right] \\
&= \frac{1}{2}\left[\oint \frac{\log \mathcal{F}(t, \pm\theta_0)}{g(t)} \frac{dt}{t-z} + \int_\gamma \frac{\pi i}{g_+(t)} \frac{dt}{t-z} - \int_\gamma \frac{\pi i}{g_-(t)} \frac{dt}{t-z}\right] \\
&= \frac{1}{2}\left(\frac{2\pi i \log \mathcal{F}(t, \pm\theta_0)}{g(z)} + \frac{2(\pi i)^2}{g(z)}\right) = \frac{\pi i \log \mathcal{F}(z, \pm\theta_0)}{g(z)} + \frac{(\pi i)^2}{g(z)}.
\end{aligned}$$

5.9 Proof of Proposition 2.2.8

Choosing the branches of $\log(t-e^{\pm i\theta_0})$ on γ^o according to (2.2.11), and since $\arg(-1-e^{i\theta_0}) = \pi + \theta_0/2$ and $\arg(-1-e^{-i\theta_0}) = \pi - \theta_0/2$, we see that

$$2\log\, w(t) = \log(t - e^{i\theta_0}) - \log(t - e^{-i\theta_0}) - i\pi.$$

Applying Proposition 2.2.7 and (2.2.8), we find

$$D(z;\omega(t)) = \exp\left(\frac{1}{4}\log \mathcal{F}(z,\theta_0) - \frac{1}{4}\log \mathcal{F}(z,-\theta_0) - \frac{i\pi}{4}\right) = \exp\left(\frac{1}{4}\log\left[\frac{-\mathcal{F}(z,\theta_0)}{\mathcal{F}(z,-\theta_0)}\right]\right),$$

the last equality being true in view of (2.2.12) and the fact that $\arg \mathcal{F}(0,\pm\theta_0) = \pm(\theta_0 - \pi/2)$.

5.10 Proof of Proposition 2.2.9

Clearly, there exists some $m \in \mathbb{Z}$ such that

$$2\log\frac{1}{G(t)} = -\log(t - e^{i\theta_0}) - \log(t - e^{-i\theta_0}) + 2m\pi i.$$

Since the branches of $\log(t - e^{\pm i\theta_0})$ on γ^o are chosen according to (2.2.11), we have $\arg(-1-e^{i\theta_0}) = \pi + \theta_0/2$, $\arg(-1-e^{-i\theta_0}) = \pi - \theta_0/2$, so that being $1/G(-1) = -|e^{i\theta_0} - 1|^{-1}$ we must have $m = 2$. Therefore,

$$\log\frac{1}{G(t)} = -\frac{1}{2}\log(t - e^{i\theta_0}) - \frac{1}{2}\log(t - e^{-i\theta_0}) + 2\pi i,$$

and (2.2.13) follows from Proposition 2.2.7 and (2.2.8).

Let us now define the function

$$b(z) := \left(\frac{z - e^{i\theta_0}}{z - e^{-i\theta_0}}\right)^{\frac{1}{2}}, \quad z \in \overline{\mathbb{C}} \setminus \gamma,$$

with $b(\infty) = 1$. Then, clearly $b(z) = a(z)^2$.

Now,

$$\begin{aligned}
\left[b(z) + b(z)^{-1}\right]^2 &= \frac{(z - e^{i\theta_0})^2 + (z - e^{-i\theta_0})^2 + 2(z - e^{i\theta_0})(z - e^{-i\theta_0})}{(z - e^{i\theta_0})(z - e^{-i\theta_0})} \\
&= \frac{4z^2 - 4ze^{i\theta_0} - 4ze^{-i\theta_0} + e^{2i\theta_0} + e^{-2i\theta_0} + 2}{(z - e^{i\theta_0})(z - e^{-i\theta_0})} \\
&= \frac{4z^2 - 8z\cos\theta_0 + 2(\cos(2\theta_0) + 1)}{(z - e^{i\theta_0})(z - e^{-i\theta_0})} \\
&= \frac{4z^2 - 8z\cos(\theta_0) + 4\cos^2\theta_0}{(z - e^{i\theta_0})(z - e^{-i\theta_0})} \\
&= \frac{4(z - \cos\theta_0)^2}{(z - e^{i\theta_0})(z - e^{-i\theta_0})} \\
&= \left[\frac{2(z - \cos\theta_0)}{g(z)}\right]^2,
\end{aligned}$$

while

$$\lim_{z\to\infty} b(z) + b(z)^{-1} = 2 = \lim_{z\to\infty} \frac{2(z - \cos\theta_0)}{g(z)}.$$

Thus,

$$b(z) + b(z)^{-1} = \frac{2(z - \cos\theta_0)}{g(z)}.$$

Also,

$$[b(z) - b(z)^{-1}][b(z) + b(z)^{-1}] = \frac{z - e^{i\theta_0}}{z - e^{-i\theta_0}} - \frac{z - e^{-i\theta_0}}{z - e^{i\theta_0}} = \frac{-4i(z - \cos\theta_0)\sin\theta_0}{g^2(z)},$$

so that

$$[a(z) - a(z)^{-1}][a(z) + a(z)^{-1}] = b(z) - b(z)^{-1} = \frac{-2i\sin\theta_0}{g(z)}. \tag{5.10.1}$$

Hence, we have

$$\begin{aligned}\left[a(z)+a(z)^{-1}\right]^4 = \left[b(z)+b(z)^{-1}+2\right]^2 &= \left[\frac{2(z-\cos\theta_0)}{g(z)}+2\right]^2 \\ &= 4\left[\frac{z+1+g(z)-(1+\cos\theta_0)}{g(z)}\right]^2 \\ &= 16\cos^2(\theta_0/2)\left(\frac{\psi(z)-\cos(\theta_0/2)}{g(z)}\right)^2 = L(z)^4,\end{aligned} \tag{5.10.2}$$

where

$$L(z) = \sqrt{2\sin\theta_0}\,[\mathcal{F}(z,\theta_0)]^{-1/4}\,[\mathcal{F}(z,-\theta_0)]^{-1/4}.$$

So $a+a^{-1}$ and F are both branches of the fourth root of a same function, and since they have the same limit at infinity, they must be equal.

5.11 Proof of Proposition 2.2.10

The equality (2.2.15) is exactly what we got in (5.10.1). Taking square roots in (5.10.2) we get

$$a(z)+a(z)^{-1} = 2\sqrt{\frac{c(\psi(z)-c)}{g(z)}}. \tag{5.11.1}$$

Because of (2.2.3), we see that

$$(\psi(z)-c)(z-e^{i\theta_0}) = (\psi(z)-c)\left(\frac{c\psi(z)-1}{\psi(z)-c}-e^{i\theta_0}\right) = c(\psi(z)-e^{i\theta_0/2})^2,$$

or what is the same,

$$\psi(z)-c = \frac{c(\psi(z)-e^{i\theta_0/2})^2}{z-e^{i\theta_0}} = \frac{c(\psi(z)-e^{-i\theta_0/2})^2}{z-e^{-i\theta_0}}, \tag{5.11.2}$$

where the last equality is true since $\overline{\psi(\overline{z})} = \psi(z)$.

Then, (2.2.14) follows from (5.11.1) and (5.11.2), while (2.2.16) follows from (2.2.14), (2.2.15) and (5.11.2).

5.12 Proof of Proposition 2.2.11

To deduce (i), we combine the second and third relations in Proposition 2.2.1(iii) to get that

$$\psi_-(e^{i\theta}) = e^{i\theta}/\psi_+(e^{i\theta}),$$

$$\frac{1}{2}\left(\frac{\psi_+(e^{i\theta})}{e^{i\theta/2}} + \frac{e^{i\theta/2}}{\psi_+(e^{i\theta})}\right) = \frac{\cos(\theta/2)}{c}.$$

Writing $\psi_+(e^{i\theta}) = e^{i\arg\psi_+(e^{i\theta})}$ with $\theta_0/2 \leq \arg\psi_+(e^{i\theta}) \leq 2\pi - \theta_0/2$, we find

$$\cos(\arg\psi_+(e^{i\theta}) - \theta/2) = \frac{\cos(\theta/2)}{c},$$

and so either

$$\arg\psi_+(e^{i\theta}) - \theta/2 = \arccos\left(\frac{\cos(\theta/2)}{c}\right) + 2k\pi \quad (k \in \mathbb{Z})$$

or

$$\arg\psi_+(e^{i\theta}) - \theta/2 = -\arccos\left(\frac{\cos(\theta/2)}{c}\right) + 2k\pi \quad (k \in \mathbb{Z}).$$

Since $\psi_+(-1) = -1$, it follows that $\arg\psi_+(t) = \theta/2 + \arccos\left(c^{-1}\cos(\theta/2)\right)$. Hence,

$$\psi_+(e^{i\theta}) = \exp\left\{i(\lambda(\theta) + \theta/2)\right\},$$

with

$$\lambda(\theta) = \arccos\left(c^{-1}\cos(\theta/2)\right).$$

Then,

$$\begin{aligned}|\psi_+(e^{i\theta})-c| &= \sqrt{[\cos(\lambda(\theta)+\theta/2)-c]^2+[\sin(\lambda(\theta)+\theta/2)]^2}\\ &= \sqrt{c^2-2c\cos(\lambda(\theta)+\theta/2)+1}\\ &= \sqrt{c^2\sin^2\lambda(\theta)+2c\sin\lambda(\theta)\sin(\theta/2)+\sin^2(\theta/2)}\\ &= \sin(\theta/2)+c\sin\lambda(\theta)=\cos(\theta/2)[\tan(\theta/2)+\tan\lambda(\theta)].\end{aligned}$$

We now prove (ii). First, we note that

$$\begin{aligned}g_+(e^{i\theta}) &= 2c\psi_+(e^{i\theta})-e^{i\theta}-1=2ce^{i(\lambda(\theta)+\theta/2)}-2\cos(\theta/2)e^{i\theta/2}\\ &= e^{i\theta/2}(2c\cos\lambda(\theta)+2ci\sqrt{1-\cos^2\lambda(\theta)}-\cos(\theta/2))\\ &= 2ie^{i\theta/2}\sqrt{c^2-\cos^2(\theta/2)}=2ie^{i\theta/2}c\sin\lambda(\theta).\end{aligned} \tag{5.12.1}$$

Second, using the second and fourth relations of Proposition 2.2.1(iii), we get

$$-\frac{\psi_+^2(t)}{t}=\frac{\psi_+(t)(\psi_+(t)-c)}{1-c\psi_+(t)}=\frac{\psi_+(t)-c}{\overline{\psi_+(t)}-c}\cdot\frac{\psi_+(t)-c}{\psi_+(t)-c}=\frac{(\psi_+(t)-c)^2}{|\psi_+(t)-c|^2}.$$

That is,

$$(\psi_+(t)-c)^2=|\psi_+(t)-c|^2\exp\left\{i\left[\pi+2\ \arccos\left(c^{-1}\cos(\theta/2)\right)\right]\right\}.$$

Using (2.2.14), we then find that for some integer k,

$$\left(\frac{a+a^{-1}}{2}\right)_+(e^{i\theta})=\frac{\sqrt{\cos(\theta/2)[\tan(\theta/2)+\tan\lambda(\theta)]}}{\sqrt{2\sin\lambda(\theta)}}\exp\left\{\frac{i}{2}\lambda(\theta)-\frac{i\theta}{4}+\frac{k\pi i}{2}\right\}.$$

Knowing that $(a+a^{-1})_+(-1)=2\cos(\theta_0/4)$, we conclude that (ii.) holds true. To derive (iii) we simply use (2.2.15).

Finally, (iv) is a direct consequence of Theorem 2.1.1 and the definition of the Szegő function for w.

5.13 Analytic Continuation of $w(z)$

We will continue $|z - e^{i\theta_0}|^{2\alpha}$ analytically from γ, to the domain $\mathbb{C} \setminus ([0,\infty) \cup \{e^{i\theta} : 0 \le \theta \le \theta_0\})$. If $|z| = 1$, then

$$\begin{aligned}
|z - e^{i\theta_0}|^{2\alpha} &= \left[(z - e^{i\theta_0})(\overline{z} - e^{-i\theta_0})\right]^{\alpha} = \left[(z - e^{i\theta_0})(\frac{1}{z} - e^{-i\theta_0})\right]^{\alpha} \\
&= \left[\frac{(z - e^{i\theta_0})(1 - ze^{i\theta_0})}{z}\right]^{\alpha} = \left[\frac{(z - e^{i\theta_0})e^{-i\theta_0}(e^{i\theta_0} - z)}{z}\right]^{\alpha} \\
&= \left[\frac{(z - e^{i\theta_0})^2 e^{-i(\pi+\theta_0)}}{z}\right]^{\alpha}.
\end{aligned}$$

If we wish for the function

$$w_\alpha(z) = \left[\frac{(z - e^{i\theta_0})^2 e^{-i(\pi+\theta_0)}}{z}\right]^{\alpha} = \exp\left\{\alpha\left[2\log(z - e^{i\theta_0}) - \log z - (\pi + \theta_0)i\right]\right\},$$

defined for $z \in \mathbb{C} \setminus ([0,\infty) \cup \{e^{i\theta} : 0 \le \theta \le \theta_0\})$, to be the analytic continuation of $|z - e^{i\theta_0}|^{2\alpha}$, it must be the case that $\arg w_\alpha(-1) = 0$. Since

$$-1 - e^{i\theta_0} = -e^{i\theta_0/2} 2\cos(\theta_0/2),$$

the choice of the branches of $\log z$ and $\log(z - e^{i\theta_0})$ corresponding to $\arg(-1) = \pi$ and $\arg(-1 - e^{i\theta_0}) = \pi + \theta_0/2$, respectively, yields

$$\arg w_\alpha(-1) = \alpha\left(2\pi + \theta_0 - \pi - (\pi + \theta_0)\right) = 0,$$

as desired. A similar analysis justifies the choice of arguments for w_β in $\mathbb{C} \setminus ([0,\infty) \cup \{e^{i\theta} : -\theta_0 \le \theta \le 0\})$.

5.14 Solution of the Riemann-Hilbert problem for A

Condition A2 implies that for $t \in \gamma^o$

$$A_{11+}(t) = -A_{12-}(t), \qquad A_{12+}(t) = A_{11-}(t).$$

Define $k(z) = g(z)A_{11}(z)A_{12}(z)$. Across γ^o, $k_+(t) = k_-(t)$, and k is bounded near $e^{\pm i\theta_0}$, so $k(z)$ is entire. Since $A_{12} \to 0$ as $z \to \infty$, A_{12} has an expansion of the form

$$A_{12}(z) = \frac{\kappa}{z} + \frac{\kappa_2}{z^2} + \frac{\kappa_3}{z^3} + \cdots .$$

Hence

$$\lim_{z\to\infty} k(z) = \kappa,$$

and by Liouville's theorem, $k(z) \equiv \kappa$.

Note that if $\kappa = 0$, then $g(z)A_{11}(z)A_{12}(z) \equiv 0$. For in such a case, since $A_{11} \to 1$ as $z \to \infty$, we also have $A_{12} \equiv 0$. But then $A_{11+} = A_{11-} = 0$ implies that A_{11} is entire, so $A_{11} \equiv 0$, contradicting A3. So $\kappa \not\equiv 0$. Therefore, $A_{11}(z) \neq 0$ and $A_{12}(z) \neq 0$ for all $z \in \mathbb{C}\backslash\gamma^o$.

Now we may write

$$A_{12}(z) = \frac{\kappa}{g(z)A_{11}(z)}$$

and use $A_{11+} = -A_{12-}$ to deduce

$$A_{11+}(t)A_{11-}(t) = \frac{-\kappa}{g_-(t)} = \frac{\kappa}{G(t)}.$$

Since $A_{11} \to 1$ as $z \to \infty$, A_{11} can be solved as a multiplicative Riemann-Hilbert problem with a Szegő function solution. In order to compute $D\left(z; \kappa/G\right)$, we rely on the multiplicative property of the Szegő function and Proposition 2.2.9 to get

$$A_{11} = D\left(z\cdot; \kappa/G\right) = D\left(z;\kappa\right) D\left(z; 1/G\right) = \pm\kappa i \frac{a(z) + a^{-1}(z)}{\sqrt{2\sin\theta_0}}.$$

Notice that

$$
\begin{aligned}
D(\kappa, z) &= \exp\left\{\frac{g(z)}{2\pi i}\int_\gamma \frac{\log\kappa}{G(t)}\frac{dt}{t-z}\right\} \\
&= \exp\left\{\log\kappa\left[\frac{g(z)}{2\pi i}\left(\frac{1}{2}\int_\gamma \frac{1}{g_+(t)}\frac{dt}{t-z} + \frac{1}{2}\int_\gamma \frac{1}{g_+(t)}\frac{dt}{t-z}\right)\right]\right\} \\
&= \exp\left\{\log\kappa\cdot\frac{1}{2}\right\} = \sqrt{|\kappa|}e^{\frac{i\arg\kappa}{2}}.
\end{aligned}
$$

From the asymptotic condition on A_{11} and the fact that

$$
\lim_{z\to\infty}\mathcal{F}(z,\pm\theta_0) = \lim_{z\to\infty}\frac{i\sin(\pm\theta_0/2)z}{\frac{2z}{2\cos(\theta_0/2)}} = \pm i\sin(\theta_0/2)\cos(\theta_0/2),
$$

which implies that

$$
\lim_{z\to\infty} D\left(\frac{1}{G(z)}, z\right) = \lim_{z\to\infty} i\left[\mathcal{F}(z,\theta_0)\right]^{-1/4}\left[\mathcal{F}(z,-\theta_0)\right]^{-1/4} = \frac{i}{\sqrt{\sin(\theta_0/2)\cos(\theta_0/2)}},
$$

we find $|\kappa| = \sin(\theta_0/2)\cos(\theta_0/2)$ and $\dfrac{\arg\kappa}{2} = \dfrac{-\pi}{2}$. Hence

$$
\kappa = -\sin(\theta_0/2)\cos(\theta_0/2) \quad \text{and} \quad D(\kappa, z) = -i\sqrt{\sin(\theta_0/2)\cos(\theta_0/2)}.
$$

That is,

$$
A_{11} = \sqrt{\sin(\theta_0/2)\cos(\theta_0/2)}\left[\mathcal{F}(z,\theta_0)\right]^{-1/4}\left[\mathcal{F}(z,-\theta_0)\right]^{-1/4} = \frac{a(z)+a^{-1}(z)}{2}
$$

and

$$
\begin{aligned}
A_{12} &= \frac{\kappa}{g(z)A_{11}(z)} = -\frac{\sin(\theta_0/2)\cos(\theta_0/2)}{g(z)(a+a^{-1})/2} \\
&= -\frac{2\sin(\theta_0/2)\cos(\theta_0/2)}{g(z)}\left(\frac{1}{a+a^{-1}}\right)\left(\frac{a-a^{-1}}{a-a^{-1}}\right) \\
&= -\frac{2\sin(\theta_0/2)\cos(\theta_0/2)}{g(z)}\left(\frac{a-a^{-1}}{a^2-(a^{-1})^2}\right) \\
&= -(a-a^{-1})\left(\frac{2\sin(\theta_0/2)\cos(\theta_0/2)}{-2i\sin\theta_0}\right) = \frac{a-a^{-1}}{2i}.
\end{aligned}
\tag{5.14.1}
$$

Proceeding in the same manner as above, and writing

$$
A_{21}(z) = \frac{\mu}{z} + \frac{\mu_2}{z^2} + \frac{\mu_3}{z^3} + \cdots ,
$$

we obtain

$$
A_{21}(z) = \frac{\mu}{g(z)A_{22}(z)}
$$

and

$$
A_{22+}(t)A_{22-}(t) = \frac{-\mu}{g_+(t)}, \quad t \in \gamma^o.
$$

We now see that in order to satisfy the asymptotic condition for A_{22} we must have $D\left(-\mu, z\right) = D\left(\kappa, z\right)$. Hence, $|\mu| = \sin(\theta_0/2)\cos(\theta_0/2)$ and $\frac{\arg(-\mu)}{2} = \frac{-\pi}{2}$ (i.e. $\mu = -\kappa = \sin(\theta_0/2)\cos(\theta_0/2)$), and

$$
\begin{aligned}
A_{22} &= \sqrt{\sin(\theta_0/2)\cos(\theta_0/2)}\,[\mathcal{F}(z,\theta_0)]^{-1/4}\,[\mathcal{F}(z,-\theta_0)]^{-1/4}, \\
A_{21} &= \frac{\sqrt{\sin(\theta_0/2)\cos(\theta_0/2)}}{g(z)}\,[\mathcal{F}(z,\theta_0)]^{1/4}\,[\mathcal{F}(z,-\theta_0)]^{1/4}.
\end{aligned}
$$

Finally, we observe that $A_{21} = -A_{12} = \frac{a(z)-a^{-1}(z)}{-2i}$. That is,

$$
A(z) = \begin{pmatrix} \frac{a(z)+a^{-1}(z)}{2} & \frac{a(z)-a^{-1}(z)}{2i} \\ \frac{a(z)-a^{-1}(z)}{-2i} & \frac{a(z)+a^{-1}(z)}{2} \end{pmatrix}.
$$

5.15 Relations involving W and $\tilde{W}$

We aim to verify that

$$W_\alpha^2(z) = \begin{cases} w_\alpha(z)e^{-2\pi i\alpha}, & |z|<1,\ z \notin [0,\infty), \\ w_\alpha(z)e^{2\pi i\alpha}, & |z|>1,\ z \notin [0,\infty). \end{cases} \tag{5.15.1}$$

Let us denote by $w_\alpha^{1/2}$ the branch of the square root of w_α given by

$$w_\alpha^{1/2}(z) := \exp\left\{\frac{\alpha}{2}\left(2\log(z-e^{i\theta_0}) - \log z - (\pi+\theta_0)i\right)\right\},$$

and recall that

$$W_\alpha(z) = \exp\left\{\frac{\alpha}{2}\left(2\log(z-e^{i\theta_0}) - \log z + (\pi-\theta_0)i\right)\right\}.$$

From the way the branches of $\log(z-e^{i\theta_0})$ and $\log z$ are chosen in (3.2.2) and (3.4.2), we see that

$$W_\alpha(z) = w_\alpha^{1/2}(z)e^{i\alpha\pi}, \quad |z|>1,\ z\notin[0,\infty),$$

and

$$W_\alpha(z) = w_\alpha^{1/2}(z)e^{-i\alpha\pi}, \quad |z|<1,\ z\notin[0,\infty),$$

which, after taking squares, yields (5.15.1), and moreover, it also yields that for $t \in U_\delta \cap \gamma^o$,

$$W_{\alpha+}(t)W_{\alpha-}(t) = w_\alpha(t).$$

Since w_β and h are analytic in U_δ, we have

$$W_+(t)W_-(t) = w(t), \quad t\in U_\delta\cap\gamma^o.$$

Similarly, we want to show

$$W_\beta^2(z) = \begin{cases} w_\beta(z)e^{2\pi i\beta}, & |z|<1,\ z\notin[0,\infty), \\ w_\beta(z)e^{-2\pi i\beta}, & |z|>1,\ z\notin[0,\infty). \end{cases} \tag{5.15.2}$$

Let us denote by $w_\beta^{1/2}$ the branch of the square root of w_β given by

$$w_\beta^{1/2}(z) := \exp\left\{\frac{\beta}{2}\left(2\log(z-e^{-i\theta_0}) - \log z - (\pi-\theta_0)i\right)\right\},$$

and recall that

$$W_\beta(z) = \exp\left\{\frac{\beta}{2}\left(2\log(z-e^{-i\theta_0}) - \log z + (\pi+\theta_0)i\right)\right\}.$$

From the way the branches of $\log(z-e^{-i\theta_0})$ and $\log z$ are chosen in (3.2.2) and (3.4.22), we see that

$$W_\beta(z) = w_\beta^{1/2}(z)e^{-i\beta\pi}, \quad |z|>1,\ z\notin[0,\infty),$$

and

$$W_\beta(z) = w_\beta^{1/2}(z)e^{i\beta\pi}, \quad |z|<1,\ z\notin[0,\infty),$$

which, after taking squares, yields (5.15.2), and moreover, it also yields that for $t\in\tilde{U}_\delta\cap\gamma^o$,

$$W_{\beta_+}(t)W_{\beta_-}(t) = w_\beta(t),$$

and

$$\tilde{W}_+(t)\tilde{W}_-(t) = w(t), \quad t\in\tilde{U}_\delta\cap\gamma^o.$$

We now prove Lemmas 3.4.1 and 3.4.3. From Proposition 2.2.5, we see that the Szegő function $D(z)$ for the weight w is given, for all $z \in \overline{\mathbb{C}} \setminus \gamma$, by

$$D(z) = \left[\frac{ie^{-i\theta_0/2}(z - e^{i\theta_0})}{\psi(z)}\right]^\alpha \left[\frac{-ie^{i\theta_0/2}(z - e^{-i\theta_0})}{\psi(z)}\right]^\beta \exp\left(\frac{g(z)}{2\pi i}\int_\gamma \frac{\log h(\zeta)}{G(\zeta)}\frac{d\zeta}{\zeta - z}\right).$$

Combining (2.2.9) and (2.2.10) with the definitions (3.4.2) and (3.2.2), we find that for $z \in U_\delta \setminus \gamma$,

$$\begin{aligned} D(z)^2 &= \frac{e^{2\alpha \log(z - e^{i\theta_0}) + i\alpha(\pi - \theta_0)} e^{2\beta \log(z - e^{-i\theta_0}) + i\beta(\theta_0 - \pi)}}{\psi(z)^{2(\alpha+\beta)}} \exp\left(\frac{g(z)}{\pi i}\int_\gamma \frac{\log h(\zeta)}{G(\zeta)}\frac{d\zeta}{\zeta - z}\right). \\ &= \frac{W_\alpha(z)^2 w_\beta(z) z^\alpha z^\beta}{\psi(z)^{2(\alpha+\beta)}} \exp\left(\frac{g(z)}{\pi i}\int_\gamma \frac{\log h(\zeta)}{G(\zeta)}\frac{d\zeta}{\zeta - z}\right). \end{aligned}$$

Let σ be a closed contour in U going around γ is the positive direction and leaving every point $z \in U_\delta \setminus \gamma$ inside. Using (2.2.1) and the residue theorem we find

$$\frac{g(z)}{\pi i}\int_\gamma \frac{\log h(\zeta)}{g_+(\zeta)}\frac{d\zeta}{\zeta - z} = \log h(z) - \frac{g(z)}{2\pi i}\int_\sigma \frac{\log h(\zeta)}{g(\zeta)}\frac{d\zeta}{\zeta - z}.$$

Hence (see (3.4.3))

$$\begin{aligned} \frac{W(z)^2}{D(z)^2} &= \left(\frac{\psi(z)}{\sqrt{z}}\right)^{2(\alpha+\beta)} \exp\left(\frac{g(z)}{2\pi i}\int_\sigma \frac{\log h(\zeta)}{g(\zeta)}\frac{d\zeta}{\zeta - z}\right) \\ &= \left(\frac{\psi(z)}{\sqrt{z}}\right)^{2(\alpha+\beta)} \exp\left(g(z)\sum_{n=0}^\infty c_n(z - e^{i\theta_0})^n\right), \end{aligned}$$

with c_n given by (1.3.2).

Now, from (2.2.2) we see that for some $b_0 \neq 0$,

$$\frac{\psi(z)}{\sqrt{z}} - 1 = b_0\sqrt{z - e^{i\theta_0}} + O(|z - e^{i\theta_0}|^{3/2}), \quad z \in U_\delta \setminus \gamma.$$

Using the Maclaurin expansions of $\text{Log}(1+z)$ and of the exponential function, we get that for z near $e^{i\theta_0}$,

$$\left(\frac{\psi(z)}{\sqrt{z}}\right)^{\alpha+\beta} = 1 + (\alpha+\beta)e^{-i\theta_0/2}e^{i\pi/4}\sqrt{\tan(\theta_0/2)}(z-e^{i\theta_0})^{1/2} + \frac{(\alpha+\beta)^2 ie^{-i\theta_0}\tan(\theta_0/2)}{2}(z-e^{i\theta_0}) + O(z-e^{i\theta_0})^{3/2}.$$

Then

$$\frac{W(z)^2}{D(z)^2} = \left(1+2(\alpha+\beta)b_0\sqrt{z-e^{i\theta_0}} + O(|z-e^{i\theta_0}|)\right)\left(1+c_0 g(z) + O(|z-e^{i\theta_0}|^{3/2})\right),$$

and moreover,

$$\begin{aligned}\frac{W(z)}{D(z)} &= \left(\frac{\psi(z)}{\sqrt{z}}\right)^{\alpha+\beta}\exp\left(\frac{g(z)}{2}\sum_{n=0}^{\infty}c_n(z-e^{i\theta_0})^n\right)\\ &= \left(\frac{\psi(z)}{\sqrt{z}}\right)^{\alpha+\beta}\exp\left(\frac{c_0 e^{i\pi/4}\sqrt{2\sin\theta_0}}{2}(z-e^{i\theta_0})^{1/2} + O(z-e^{i\theta_0})^{3/2}\right) \qquad (5.15.3)\\ &= 1 + e^{i\pi/4}\sqrt{\tan(\theta_0/2)}\left((\alpha+\beta)e^{-i\theta_0/2} + c_0 c\right)(z-e^{i\theta_0})^{1/2} + O(z-e^{i\theta_0})\end{aligned}$$

and

$$\frac{D(z)}{W(z)} = 1 - e^{i\pi/4}\sqrt{\tan(\theta_0/2)}\left((\alpha+\beta)e^{-i\theta_0/2} + c_0 c\right)(z-e^{i\theta_0})^{1/2} + O(z-e^{i\theta_0}).$$

Similarly, from (2.2.9), (2.2.10) and the definitions (3.4.2) and (3.2.2), we find that for $z \in \tilde{U}_\delta \setminus \gamma$,

$$D(z)^2 = \frac{W_\beta(z)^2 w_\alpha(z) z^\alpha z^\beta}{\psi(z)^{2(\alpha+\beta)}}\exp\left(\frac{g(z)}{\pi i}\int_\gamma \frac{\log h(\zeta)}{G(\zeta)}\frac{d\zeta}{\zeta - z}\right),$$

and the remainder of the proof follows from here along the same lines.

5.16 The solution Ψ_α

To observe that Ψ_α is a solution, we utilize the following properties (see [1]), where $J_{2\alpha}$, $Y_{2\alpha}$ are ordinary Bessel functions of the first and second kind:

(i) $H^{(1)}_{2\alpha}(z)$, $H^{(2)}_{2\alpha}(z)$, $I_{2\alpha}(z)$, and $K_{2\alpha}(z)$ (and their derivatives) are analytic in $\mathbb{C}\setminus(-\infty, 0]$ (cf. (9.1), (9.6) in [1]).

(ii) $H^{(1)}_{2\alpha}(z) = J_{2\alpha}(z) + iY_{2\alpha}(z)$ (cf. (9.1.3) in [1]).
$H^{(2)}_{2\alpha}(z) = J_{2\alpha}(z) - iY_{2\alpha}(z)$ (cf. (9.1.4) in [1]).

(iii) $e^{-2\alpha\pi i}J_{2\alpha}(ze^{\pi i}) = J_{2\alpha}(z)$ (cf. (9.1.35) in [1]).

(iv) $I_{2\alpha}(z) = e^{-\alpha\pi i}J_{2\alpha}(ze^{\pi i/2})$ for $-\pi < \arg z \leq \pi/2$ (cf. (9.6.3) in [1]).

(v) $K_{2\alpha}(z) = \frac{\pi i}{2}e^{\alpha\pi i}H^{(2)}_{2\alpha}(ze^{-\pi i/2})$ for $-\pi/2 < \arg z \leq \pi$ (cf. (9.6.4) in [1]).

That $\Psi_\alpha(\zeta)$ satisfies $\Psi 1$ is a direct application of (i).

We now consider the jump condition $\Psi 2$. Across Σ_2^o, a straightforward calculation along with (i) proves the jump condition for the (11) and (12) entries. For the (21) and (22) entries, we use (i) together with the fact that $(\zeta^{1/2})_+ = -(\zeta^{1/2})_-$.

Across Σ_1^o, we verify the jump condition

$$\Psi_{\alpha+}(\zeta) = \Psi_{\alpha-}(\zeta)\begin{pmatrix} 1 & 0 \\ e^{2\alpha\pi i} & 1 \end{pmatrix}$$

in each entry:

(11) $I_{2\alpha}(2\zeta^{1/2}) = \frac{e^{\alpha\pi i}}{2}\left(H^{(1)}_{2\alpha}(2(-\zeta)^{1/2}) + H^{(2)}_{2\alpha}(2(-\zeta)^{1/2})\right)$.

(21) $2\pi i\zeta^{1/2}I'_{2\alpha}(2\zeta^{1/2}) = \pi e^{\alpha\pi i}\zeta^{1/2}\left((H^{(1)}_{2\alpha})'(2(-\zeta)^{1/2}) + (H^{(1)}_{2\alpha})'(2(-\zeta)^{1/2})\right)$.

(12) $\frac{i}{\pi}K_{2\alpha}(2\zeta^{1/2}) = \frac{e^{-\alpha\pi i}}{2}H^{(2)}_{2\alpha}(2(-\zeta)^{1/2})$.

(22) $-2\zeta^{1/2}K'_{2\alpha}(2\zeta^{1/2}) = \pi e^{-\alpha\pi i}\zeta^{1/2}(H^{(2)}_{2\alpha})'(2(-\zeta)^{1/2})$.

To show equality of the (11) entries, we consider the right hand side and apply (ii), (iii) and (iv), together with the fact that for $\zeta \in \Sigma_1^o$, $(-\zeta)^{1/2} = \zeta^{1/2}e^{-\pi i/2}$:

$$\begin{aligned}\frac{e^{\alpha\pi i}}{2}\left(H^{(1)}_{2\alpha}(2(-\zeta)^{1/2}) + H^{(2)}_{2\alpha}(2(-\zeta)^{1/2})\right) &= e^{\alpha\pi i}J_{2\alpha}(2\zeta^{1/2}e^{-\pi i/2}) = e^{-\alpha\pi i}J_{2\alpha}(2\zeta^{1/2}e^{i\pi/2}) \\ &= I_{2\alpha}(2\zeta^{1/2}).\end{aligned}$$

Equality for (21) follows by taking the derivatives of the (11) entries and multiplying by a factor of $2\pi\zeta e^{\pi i/2}$.

Proving equality forn (12) is a direct application of (v), and for (22) we take the derivatives and again multiply by $2\pi\zeta e^{\pi i/2}$.

With the understanding that for $\zeta \in \Sigma_3^o$, $(-\zeta)^{1/2} = \zeta^{1/2}e^{i\pi/2}$, similar calculations show that Ψ_α satisfies the jump condition across Σ_3^o

We deduce the limiting behavior near zero for these functions by examining the following formulae in [1]:

(vi) as $z \to 0$, $I_{2\alpha}(z) \sim (\frac{1}{2}z)^{2\alpha}/\Gamma(2\alpha+1)$ (9.6.7).

(vii) For $\mathcal{L} = I_{2\alpha}$ or $\mathcal{L} = e^{2\alpha\pi i}K_{2\alpha}$, $\mathcal{L}'_{2\alpha}(z) = \mathcal{L}_{2\alpha+1}(z) + \frac{2\alpha}{z}\mathcal{L}_{2\alpha}(z)$ (9.6.26).

(viii) as $z \to 0$, if $2\alpha > 0$, $K_{2\alpha}(z) \sim \frac{1}{2}\Gamma(2\alpha)(\frac{1}{2}z)^{-2\alpha}$ (9.6.9).

(ix) $I_{-2\alpha}(z) = I_{2\alpha}(z)$, $K_{-2\alpha}(z) = K_{2\alpha}(z)$ (9.6.6).

(x) as $z \to 0$, $K_0(z) \sim -\ln z$ (9.6.8).

(xi) $K_{2\alpha}(z) = \frac{1}{2}\pi i e^{\alpha\pi i}H^{(1)}_{2\alpha}(ze^{\pi i/2})$ for $-\pi < \arg z \leq \pi/2$ (9.6.4).

In the domain $\{\zeta : |\arg z| < 2\pi/3\}$, for $\alpha > 0$, applying (vi), (vii), and (viii.) shows that $\Psi_\alpha(\zeta)$ satisfies $\Psi3$. For $\alpha < 0$, the same arguments together with (x) yield the desired results. Finally, for $\alpha = 0$, (vi) together with (x) shows that $\Psi_\alpha(z)$ satisfies $\Psi3$ in the given region.

In the two remaining regions, we apply (v) and (xi) together with the previous arguments for $K_{2\alpha}$.

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