2012

Characterizations Of Zero Divisor Graphs Determined By Equivalence Classes Of Zero Divisors

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CHARACTERIZATIONS OF ZERO DIVISOR GRAPHS DETERMINED BY EQUIVALENCE CLASSES OF ZERO DIVISORS

A Thesis
presented in partial fulfillment of requirements
for the degree of Master of Science
in the Department of Mathematics
The University of Mississippi

by
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May 2012

ABSTRACT

We study zero divisor graphs of commutative rings determined by equivalence classes of zero divisors, specifically for a Noetherian ring R. We study the classification of these graphs. Specifically, we add more criteria to the list of characterizations that disqualify a graph as the zero divisor graph of a ring. We also briefly discuss Sage, a mathematical software, which was an aid in providing visual pictures for the graphs under study.

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1 INTRODUCTION

The overall goal of this thesis is to continue the classification of zero divisor graphs of commutative rings determined by equivalence classes of zero divisors. While the classification is a difficult problem and remains incomplete, we add more criteria to the list of graph properties. To further the classification, we study the six vertices zero divisor graphs determined by equivalence classes of zero divisors, with an eye towards determining which of these graphs is realizable.

In section 2, we provide the necessary background of Algebra terms and results. In section 3, we provide the necessary background of Graph Theory terms. Section 4 contains history and examples regarding zero divisor graphs of commutative rings determined by equivalence classes of zero divisors. In section 5, we provide the working list of graph properties which disqualify a graph G as the zero divisor graph of a commutative ring, R, determined by equivalence classes. Section 6 contains the new results obtained in the research for this paper. In section 7, we provide further examples of the new results from section 6. The graphs for these examples contain 7 vertices. Section 8 provides explanation of the mathematical software, Sage. Lastly, the Appendices classify the six vertices graphs under study by method of disqualification as the zero divisor graph of a commutative ring determined by equivalence classes of zero divisors.

In this paper, R will represent a commutative Noetherian ring with unity

2 ALGEBRA

In this section, we give the necessary Algebra background by collecting definitions and results. Some good general references for the material in this section are [4], [7], and [11].

Definition 2.1 *A* **ring** *R is a set with two binary operations, addition (denoted $a + b$) and multiplication (denoted by ab) , such that for all a, b, c in R:*

1. $a + b = b + a$.

2. $(a + b) + c = a + (b + c)$.

3. There is an additive identity 0. That is, there is an element 0 in R such that $a + 0 = a$ for all a in R.

4. There is an element $-a$ in R such that $a + (-a) = 0$.

5. $a(bc) = (ab)c$.

6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$.

Definition 2.2 *When multiplication in a ring, R, is commutative, we say the ring is* **commutative**.

Definition 2.3 *A* **unity** *(or identity) in a ring, R, is a nonzero element, denoted 1_R, that is an identity under multiplication.*

Definition 2.4 *A nonzero element, a, of a commutative ring, R, is a* **unit** *if a^{-1} exists such that $aa^{-1} = 1_R$.*

Definition 2.5 *A non-empty subset S of a ring R is a* **subring** *of R if S is itself a ring with the operations of R.*

Definition 2.6 *A subring A of a ring R is called a (two-sided)* **ideal** *of R if for every r in R and every a in A both ra and ar are in A.*

Definition 2.7 *A* **prime ideal**, *A, of a commutative ring, R, is a proper ideal of R such that a, $b \in R$ and $ab \in A$ imply $a \in A$ or $b \in A$.*

Definition 2.8 *Let a be any element of a commutative ring R. The* **annihilator ideal** *of a in R is denoted $Ann(a) = \{ r \in R \mid ra = 0 \}$.*

Definition 2.9 *A* **zero-divisor** *is a nonzero element, a, of a commutative ring, R, such that* $\mathrm{Ann}(a) \neq (0)$*.* $\mathbf{Z}^*(\mathbf{R})$ *denotes the zero divisors of R and* $\mathbf{Z}(\mathbf{R}) = Z^*(R) \cup \{0\}$.

Definition 2.10 *If $\mathfrak{p}$ is a prime ideal of a commutative ring R such that $\mathfrak{p} = \mathrm{ann}(y)$ for some $0 \neq y \in R$, then $\mathfrak{p}$ is an* **associated prime**. *$Ass(R)$ denotes the set of associated primes.*

Definition 2.11 *A* **maximal associated prime** *is an associated prime, $\mathfrak{p}$ of a commutative ring R, such that $\mathfrak{p}$ is not properly contained in any other associated prime.*

Definition 2.12 *A commutative ring R is said to be a* **Noetherian ring** *if any ascending chain of ideals, $I_1 \subseteq I_2 \subseteq \ldots I_{k-1} \subseteq I_k \subseteq I_{k+1}$, contains only a finite number of distinct ideals; i.e., there exists $n \in \mathbb{N}$ such that $I_n = I_{n+k}$ for all $k \in \mathbb{N}$.*

The following result is utilized in proving that a complete graph G cannot be realized as the zero divisor graph of a commutative ring R determined by equivalence classes. See Proposition 4.14.

Proposition 2.13 *Let R be a commutative ring, I, $P_1, .., P_r$ ideals of R, and suppose that $P_3, ..., P_r$ are prime, and that I is not contained in any of the P_i; then there exists an element $x \in I$ not contained in any P_i.*

Proof We want to show that there exists an $x \in I$ not contained in any P_i. Assume no inclusion among the $P_i's$ because if $P_1 \subseteq P_2$, then any $x \in I \backslash P_2$ also satisfies $x \notin P_1$. Note that for $r = 1$, $I \not\subseteq P_1$ implies there exists $x \in I \backslash P_1$. Consider $r = 2$. There exists $x \in I \backslash P_1$ and there exists $y \in I \backslash P_2$. Hence, we have $x + y \in I$. We will now show that one of x, y or $x + y$ is the element we seek. If $x \in I \backslash (P_1 \cup P_2)$, we are done. Similarly, if $y \in I \backslash (P_1 \cup P_2)$, we are done. Now, suppose $x \in P_2$ and $y \in P_1$. If $x + y \in P_1$, then this implies a contradiction since $x = x + y - y$ and $y \in P_1$ implies $x \in P_1$. If $x + y \in P_2$, then this implies a contradiction since $y = x + y - x$ and $x \in P_2$ implies $y \in P_2$. Thus, $x + y$ is in neither P_1 nor P_2. We now proceed by induction on $r \in \mathbb{N}$. Note that we have already shown above the base case for $r = 1$ and $r = 2$. For the inductive hypothesis, assume the result holds for all integers k with $2 \leq k \leq r - 1$. We want to show the result holds for r. Since P_r is prime, then $IP_1P_2 \cdots P_{r-1} \not\subseteq P_r$ as none of $I, P_1,\ P_2, ..., P_{r-1}$ is contained in P_r. Thus, $P_r \not\supseteq IP_1P_2 \cdots P_{r-1}$. Choose $x \in IP_1P_2 \cdots P_{r-1} \backslash P_r$. Note that we have $a_r \in I \cap P_1 \cap P_2 \cap \cdots \cap P_{r-1}$ since $a_r \in IP_1P_2 \cdots P_{r-1}$. By the inductive hypothesis, choose $y \in I \backslash (P_1 \cup P_2 \cup \cdots \cup P_{r-1})$. Consider y and $x + y$. If $y \notin P_r$, then we have found the element which we seek. Suppose $y \in P_r$. Consider $x + y$. If $x + y \in P_r$, then $x = x + y - y \in P_r$ which is a contradiction. If $x + y \in P_i$, $i < r$, then $y = y + x - x \in P_r$ which implies $y \in P_1 \cup P_2 \cup \cdots \cup P_{r-1}$, a contradiction to the inductive hypothesis.

3 GRAPH THEORY

In this section, we give the necessary Graph Theory background by collecting definitions and results. The definitions for the material in this section were taken verbatim from the following general references [8] and [12].

Definition 3.1 *A* **relation** *on a set S is a collection of ordered pairs from S. We say S is an* **equivalence relation** *if S is reflexive, symmetric, and transitive.*

Definition 3.2 *A* **graph** *G consists of a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates with each edge two vertices (not necessarily distinct) called its* **endpoints**.

Definition 3.3 *Given an equivalence relation $\sim$ on a set S, the* **equivalence class** *of $s \in S$ is $\{t \in S \mid s \sim t\}$.*

Definition 3.4 *A* **loop** *is an edge whose endpoints are equal.*

Definition 3.5 **Multiple edges** *are edges having the same pair of endpoints.*

Definition 3.6 *Let u and v be vertices. We say u and v are* **adjacent** *and are* **neighbors** *when u and v are the endpoints of an edge.*

Definition 3.7 *A graph G is* **finite** *if its vertex set and edge set are finite.*

Definition 3.8 *A* **simple graph** *is a graph G having no loops or multiple edges.*

Definition 3.9 *G is a* **cycle graph** *if G is an n-gon for some integer $n \geq 3$.*

Definition 3.10 *A simple graph G is a* **path** *if all vertices in G can be ordered such that two vertices are adjacent if and only if they are consecutive in the list. A* **path** *in a graph is a sequence of vertices such that from each of its vertices there is an edge to the next vertex in the sequence.*

Definition 3.11 *A* **subgraph** *of a graph G is a graph H such that V(H)* $\subseteq$ *V(G) and E(H)* $\subseteq$ *E(G) and the assignment of endpoints to edges in H is the same as in G.*

Definition 3.12 *A graph G is* **connected** *if each pair of vertices in G belongs to a path.*

Definition 3.13 *A graph G is* **disconnected** *if G is not connected.*

Definition 3.14 *Let v be a vertex and e be an edge. We say that v and e are* **incident** *if v is an endpoint of e.*

Definition 3.15 *The* **degree** *of vertex v in a simple graph is the number of incident edges.*

Definition 3.16 *An* **end** *is a vertex of degree one.*

Definition 3.17 *A graph G is* **regular** *if every vertex has the same degree.*

Definition 3.18 *A simple graph G is* **complete** *if all vertices in G are pairwise adjacent.*

Definition 3.19 *A graph G is* **complete bipartite** *if there is a partition of all vertices into two subsets* $\{u_i\}$ *and* $\{v_j\}$ *such that* u_i *is adjacent to* v_j *for all pairs i,j, but no two elements of the same subset are adjacent.*

Definition 3.20 *A graph G is* **complete k-partite graph** *if the vertices can be partitioned into k distinct subsets such that each element of a subset is adjacent to every element not in the same subset, but no two elements of the same subset are adjacent.*

Definition 3.21 *The* **girth** *of a graph G with a cycle is the length of a shortest cycle. If G contains no cycles, then G has infinite girth.*

Definition 3.22 *The* **length** *of a path is its number of edges.*

Definition 3.23 *Let G be a connected graph. The* **distance** *from u to v, denoted* $d(u, v)$, *is the least length of any path from u to v.*

Definition 3.24 *Let G be a connected graph. The* **diameter** *of G, denoted diam G, is the greatest distance between any two vertices of G.*

Definition 3.25 *The* **neighborhood** *of a vertex v in a graph G, denoted N(v), is the* *set of all vertices adjacent to v.*

Definition 3.26 *The* **closure of a neighborhood** *of a vertex v in a graph G, denoted $\overline{N(v)}$, is the neighborhood of v along with v itself; i.e., $N(v) \cup \{v\}$.*

Definition 3.27 *Two cycles joined by a path consisting of at least one edge is called a* **bicycle**.

4 HISTORY, EXAMPLES, AND PRIOR RESULTS

All rings R are assumed to be commutative, Noetherian and contain a unity.

4.1 History and Examples

In 1988, Istvan Beck first introduced the idea of associating to a ring a graphic representation. Subsequently, other researchers altered Beck's original definition. In this section we detail the three main definitions of graphs associated to a ring R.

Definition 4.1 *[3] Given a finite ring R, let $G(R)$ denote the graph whose vertices are the elements of R, and with pair of distinct vertices r and s joined by an edge if and only if $rs = 0$.*

Remark 4.2 *The graph $G(R)$ is connected and simple with diameter at most two since 0 $\in R$ and every element in R is adjacent to 0.*

Example 4.3 *Consider $R = \mathbb{Z}/8\mathbb{Z} = \{\overline{0}, \overline{1}, \overline{2}, \overline{3}, \overline{4}, \overline{5}, \overline{6}, \overline{7}\}$. The graph $G(\mathbb{Z}/8\mathbb{Z})$ is given below.*

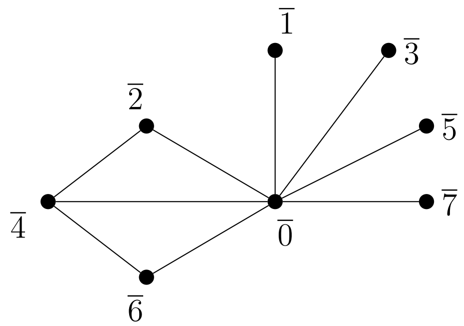

Figure 1: The Beck Graph $G(\mathbb{Z}/8\mathbb{Z})$

In 1999, D. Anderson and P. Livingston introduced a simplified version of Beck's zero divisor graph. While the edges of the graph are defined as in Definition 4.1, the vertex set is smaller.

Definition 4.4 *[1] Given a ring R, let $Z^*(R)$ denote the set of (non-zero) zero divisors of R. Let $\Gamma(R)$ denote the graph whose vertices are the elements of $Z^*(R)$, and with each pair of distinct vertices r and s joined by an edge if and only if $rs = 0$.*

Remark 4.5 *The graph $\Gamma(R)$ is connected and simple with diameter at most three.*

Example 4.6 *Consider $R = \mathbb{Z}/8\mathbb{Z}$ where $\{\overline{2}, \overline{4}, \overline{6}\}$ is the set of zero divisors of R. The graph $\Gamma(\mathbb{Z}/8\mathbb{Z})$ is given below.*

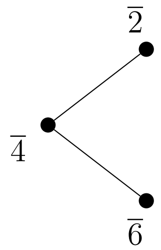

Figure 2: The Zero Divisor Graph $\Gamma(\mathbb{Z}/8\mathbb{Z})$

The following theorem provides motivation for studying zero divisor graphs. Note that a *reduced ring* is a ring that does not contain any nonzero nilpotent elements; i.e., if $a^n = 0$ for some $a \in R$ and some $n \in \mathbb{N}$, then $a = 0$.

Theorem 4.7 *[2, Theorem 4.1] Let R and S be finite reduced commutative rings which are not fields. Then $\Gamma(R) \cong \Gamma(S)$ as graphs if and only if $R \cong S$ as rings.*

In 2002, S.B. Mulay introduced the concept of constructing a graph from equivalence classes of zero divisors, further simplifying the zero divisor graph. Roughly speaking, this graph is a condensed version of the Anderson and Livingston graph.

Definition 4.8 *[10] Given a ring R, two zero divisors r, $s \in Z^*(R)$ are equivalent if $Ann(r) = Ann(s)$. This is an equivalence relation. The equivalence class of r is denoted $[r]$.*

Definition 4.9 *[11, Definition 1.1] The graph of equivalence classes of zero divisors of* *a ring R, denoted $\Gamma_E(R)$, is the graph associated to R whose vertices are the classes of elements in $Z^*(R)$, and with each pair of distinct classes $[x], [y]$ joined by an edge if and only if $xy = 0$ where classes are multiplied in the obvious way.*

Remark 4.10 *The graph $\Gamma_E(R)$ is connected and simple with diameter at most three.*

Example 4.11 *Consider $R = \mathbb{Z}/8\mathbb{Z}$ where $\{[\overline{2}], [\overline{4}]\}$ is the set of equivalence classes of zero divisors of R. Note that $\mathrm{ann}(\overline{2}) = \mathrm{ann}(\overline{6})$, hence $[\overline{2}] = [\overline{6}]$. The graph $\Gamma_E(\mathbb{Z}/8\mathbb{Z})$ is given below.*

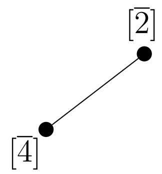

Figure 3: The (Condensed) Zero Divisor Graph $\Gamma_E(\mathbb{Z}/8\mathbb{Z})$

Remark 4.12 *The graph $\Gamma_E(R)$ can be finite even if R is infinite.*

Example 4.13 *Consider the ring $R = (\mathbb{Z}/8\mathbb{Z})[X]$. R consists of polynomials in the variable X where the coefficients belong to the ring $\mathbb{Z}/8\mathbb{Z}$. Note that R has infinitely many elements, but R contains only two equivalence classes of zero divisors. The graph $\Gamma_E(\mathbb{Z}/8\mathbb{Z})[X]$ takes the same form as the graph $\Gamma_E(\mathbb{Z}/8\mathbb{Z})$ given in Example 4.11.*

4.2 Prior Results

We now collect past research on zero divisor graphs determined by equivalence classes of zero divisors of a commutative ring R. These results are utilized in proving new results in section 6.

Proposition 4.14 *[11, Proposition 1.5] If G is complete with at least 3 vertices, then $G \neq \Gamma_E(R)$ for any R.*

Proof Take any three vertices $[u], [v]$, and $[z]$. Without loss of generality, $\mathrm{ann}(z) \not\subset \mathrm{ann}(v)$ and $\mathrm{ann}(z) \not\subset \mathrm{ann}(u)$; otherwise, $\mathrm{ann}(z) = \mathrm{ann}(v) = \mathrm{ann}(u)$. Thus, there exists $x \in \mathrm{ann}(z) \backslash \mathrm{ann}(v)$, and there exists $y \in \mathrm{ann}(z) \backslash \mathrm{ann}(u)$. Note that $x \notin \mathrm{ann}(v)$ and $y \notin$

ann(u) implies we must have $[x] = [v]$ and $[y] = [u]$. Thus, $x \in \text{ann}(u)$ and $y \in \text{ann}(v)$. By Proposition 2.13, $x + y \in \text{ann}(z)\backslash(\text{ann}\,(v) \cup \text{ann}\,(u))$. Contradiction, there is no such element in the graph.

Remark 4.15 *Since every associated prime is of the form* ann(v), *we will often abuse terminology and refer to* $[v]$ *as an associated prime.*

Lemma 4.16 *[11, Lemma 1.2] Any two distinct elements of* Ass(R) *are adjacent. Furthermore, every vertex* $[v]$ *of* $\Gamma_E(R)$ *is either an associated prime or adjacent to a maximal associated prime.*

Proposition 4.17 *[11, Proposition 1.7] Let* R *be a ring such that* $\Gamma_E(R)$ *is complete* r*-partite. Then* $r = 2$ *and* $\Gamma_E(R) = K_{n,1}$ *for some* $n \geq 1$.

Proposition 4.18 *[11, Proposition 3.2] If* R *is a ring such that* $|\Gamma_E(R)| > 3$, *then no associated prime of* R *is an end.*

Theorem 4.19 *[11, Theorem 3.6] Let* R *be such that* $2 < |\Gamma_E(R)| < \infty$. *Then any vertex of maximal degree is a maximal associated prime.*

Proposition 4.20 *[6, Proposition 5.2] Associated primes are "edge dominating"; i.e., every associated prime contains an end of every edge in the graph* $\Gamma_E(R)$.

Corollary 4.21 *[6, Corollary 5.3] If* $|\Gamma_E(R)| > 3$ *and the graph has at least one vertex [x] with two or more ends, then*

(a) Ass$\,(R) = \{\text{Ann}\,(x)\}$;

(b) every vertex [y] $\neq$ *[x] must be adjacent to [x]; in particular,*

(1) deg([x]) $= |\Gamma_E(R)|$ *- 1 if* $|\Gamma_E(R)|$ *is finite, and*

(2) no vertex other than [x] can have an end.

5 CHARACTERIZATIONS OF ZERO DIVISOR GRAPHS DETERMINED BY EQUIVALENCE CLASSES

Below is the current known list of characterizations concerning whether or not a graph G is realizable as a zero divisor graph. Characterizations 9 - 20 represent the new results, and form the basis for the work in this thesis.

A graph G cannot be the (condensed) zero divisor graph of a Noetherian, commutative ring R with unity 1 if:

1. (Connected) G is not connected.

2. (Diameter) The diameter of G is greater than 3.

3. (Girth) The girth of G is greater than 3 and finite.

4. (Regular) For $2 < |G| < \infty$, G is a regular graph.

5. (Complete) For $|G| > 2$, G is complete. See Propostition 4.14.

6. (Complete r-Partite) For $|G| > 2$ and $r \neq 2$, G is complete r-partite. See Proposition 4.17.

7. (Cycle) G is a cycle graph.

8. (Associated Primes and Ends Theorem) For $|G| > 3$, G has a vertex $[x]$ with two or more ends, and

(a) there exists a vertex [y] $\neq [x]$ not adjacent to $[x]$, or

(b) there exists another vertex other than $[x]$ that has an end. See Corollary 4.21.

9. (Hinge Theorem) G is connected with at least five vertices and contains a hinged end. See Theorem 6.6.

10. (Maximum Degree Theorem) G is finite and connected with two non-adjacent vertices of maximal degree. See Theorem 6.7.

11. ("Looped End" Theorem) G is a connected graph with a vertex d that has an end represented by a self-annihilating element and another vertex $b \notin \overline{N([d])}$. See Theorem 6.2.

12. ("Looped End" Corollary 1) G is connected with two distinct vertices each having an end and one of the ends is represented by a self-annihilating element. See Corollary 6.3.

13. ("Looped End" Corollary 2) G is a connected graph with a vertex d having two ends, and there exists $[b] \notin \overline{N([d])}$. See Corollary 6.4.

14. (Bridge Theorem) G is a connected graph such that $G = \overline{N([a])} \cup \overline{N([c])}$, where $|\mathrm{N}([a])| \geq 1, |N([c])| \geq 2$, and where a and c are two distinct vertices such that $ac = 0$, and no vertex in $\mathrm{N}([a])$ is adjacent to any vertex in $\mathrm{N}([c])$. See Theorem 6.9.

15. (Bicycle Corollary) G is a bicycle. See Corollary 6.11.

16. (The Book Theorem) G is a book graph. See Theorem 6.13.

17. (The Modified Book Theorem) G is a modified book graph. See Theorem 6.15.

18. (Missing Chord Theorem) G is a connected graph with two distinct non-adjacent vertices a, b such that for every $v \in G$, $N([a]) \cup N([b]) \not\subset \overline{N([v])}$. See Theorem 6.18.

19. (Missing Sum Theorem 1) G is a connected graph with two distinct adjacent vertices a, b such that

(a) there exists a vertex [v] $\in N([a]) \backslash \overline{N([b])}$;

(b) there exists a vertex [w] $\in N([b]) \backslash \overline{N([a])}$;

(c) $\overline{N([a])} \cap \overline{N([b])} \neq (0)$,

and $u \neq [a+b]$ for any $u \in G$. See Theorem 6.19.

20. (Missing Sum Theorem 2) G is a connected graph with two distinct non-adjacent vertices a, b such that

(a) there exists a vertex [v] $\in \overline{N([a])} \backslash \overline{N([b])}$;

(b) there exists a vertex [w] $\in \overline{N([b])} \backslash \overline{N([a])}$;

(c) $N([a]) \cap N([b]) \neq (0)$,

and $u \neq [a+b]$ for any $u \in G$. See Theorem 6.20.

6 NEW RESULTS

All rings R are assumed to be commutative, Noetherian, with unity. The following remark describes two strategies which are utilized throughout this paper to disprove a graph as $\Gamma_E(R)$.

Remark 6.1 *We note the following two strategies given in [9].*

> *I. If two vertices in the zero divisor graph are adjacent to the same set of vertices, but are not adjacent to one another, then at least one is represented by a self-annihilating element; otherwise, the two vertices would represent the same class.*
>
> *II. If two vertices in the zero divisor graph are adjacent to the same set of vertices and are also adjacent to one another, then at least one of the vertices must not be represented by a self-annihilating element; otherwise, the two vertices would represent the same class.*

The above strategies are utilized throughout Section 6 and Appendix II.

6.1 Negative Results

In this section, we collect the new characterizations that disqualify a graph from being realized as $\Gamma_E(R)$, developed during the research for this paper.

A graph G cannot be the (condensed) zero divisor graph of a Noetherian, commutative ring R with unity 1 if:

9. (Hinge Theorem) G is connected with at least five vertices and contains a hinged end. See Theorem 6.6.

10. (Maximum Degree Theorem) G is finite and connected with two non-adjacent vertices of maximal degree. See Theorem 6.7.

11. ("Looped End" Theorem) G is a connected graph with a vertex d that has an end represented by a self-annihilating element and another vertex $b \notin \overline{N([d])}$. See Theorem 6.2.

12. ("Looped End" Corollary 1) G is connected with two distinct vertices each having an end and one of the ends is represented by a self-annihilating element. See Corollary 6.3.

13. ("Looped End" Corollary 2) G is a connected graph with a vertex d having two ends, and there exists $[b] \notin \overline{N([d])}$. See Corollary 6.4.

14. (Bridge Theorem) G is a connected graph such that $G = \overline{N([a])} \cup \overline{N([c])}$, where $|\mathrm{N}([a])| \geq 1, |N([c])| \geq 2$, and where a and c are two distinct vertices such that $ac = 0$, and no vertex in $\mathrm{N}([a])$ is adjacent to any vertex in $\mathrm{N}([c])$. See Theorem 6.9.

15. (Bicycle Corollary) G is a bicycle. See Corollary 6.11.

16. (The Book Theorem) G is a book graph. See Theorem 6.13.

17. (The Modified Book Theorem) G is a modified book graph. See Theorem 6.15.

18. (Missing Chord Theorem) G is a connected graph with two distinct non-adjacent vertices a, b such that for every $v \in G$, $N([a]) \cup N([b]) \not\subset \overline{N([v])}$. See Theorem 6.18.

19. (Missing Sum Theorem 1) G is a connected graph with two distinct adjacent vertices a, b such that

(a) there exists a vertex [v] $\in N([a])\backslash\overline{N([b])}$;

(b) there exists a vertex [w] $\in N([b])\backslash\overline{N([a])}$;

(c) $\overline{N([a])} \cap \overline{N([b])} \neq (0)$,

and $u \neq [a+b]$ for any $u \in G$. See Theorem 6.19.

20. (Missing Sum Theorem 2) G is a connected graph with two distinct non-adjacent vertices a, b such that

(a) there exists a vertex [v] $\in \overline{N([a])}\backslash\overline{N([b])}$;

(b) there exists a vertex [w] $\in \overline{N([b])}\backslash\overline{N([a])}$;

(c) $N([a]) \cap N([b]) \neq (0)$,

and $u \neq [a+b]$ for any $u \in G$. See Theorem 6.20.

Theorem 6.2 ("Looped End" Theorem) *If G is a connected graph with a vertex d that has an end represented by a self-annihilating element and another vertex $b \notin \overline{N([d])}$, then G is not the zero divisor graph of a ring.*

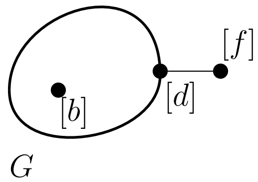

Figure 6.2a

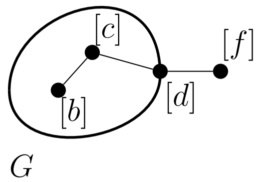

Figure 6.2b

Figure 4: "Looped End" Graphs

Proof As G is a connected graph, there exists a path between b and d, see Figure 6.2a. But, $b \notin \operatorname{ann}(d)$ and $\operatorname{diam}(G) \leq 3$ which implies there exists a distinct vertex c creating a path between b and d, see Figure 6.2b. Note that we have $\operatorname{ann}(c) \cup \operatorname{ann}(f) \subseteq \operatorname{ann}(cf)$. Thus, $f \in \operatorname{ann}(cf)$ implies $[cf] = [d]$ or $[f]$, but $b \in \operatorname{ann}(cf)$, contradiction.

Corollary 6.3 *If G is a connected graph with two distinct vertices each having an end, and one of the ends is represented by a self-annihilating element, then G is not the zero divisor graph of a ring.*

Proof This follows immediately from the Theorem.

Corollary 6.4 *If G is a connected graph with a vertex d having two ends, and there exists $[b] \notin \overline{N([d])}$, then G is not the zero divisor graph of a ring.*

Proof By Strategy I from Remark 6.1 one of the ends of d is represented by a self-annihilating element, say f. The proof now follows immediately from the Theorem.

Definition 6.5 *A* **hinged end** *in a graph G (containing at least three vertices) is a pair* *of adjacent vertices e and f such that* $\deg f = 1$ *and* $\deg e = 2$.

Theorem 6.6 (Hinge Theorem) *If G is a connected graph with at least five vertices and contains a hinged end, then G is not the zero divisor graph of a ring.*

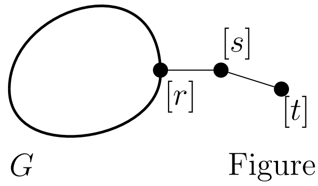

Figure 5: Hinge Graph

Proof Consider cs for some vertex $[c]$ which lies in the remaining graph other than $[r]$. It is not 0 since no edge exists between c and s, but it is a zero divisor since $r(cs) = (rc)s = (0)s = 0$. Note that cs is annihilated by r and t. Thus, we have $[cs] = [s]$. We now consider the vertex $[b]$ which lies in the remaining graph other than $[c]$ and $[r]$. Note that we must have $br = 0$, else the diameter of G is greater than 3. Consider bs. It is not 0 since no edge is drawn, but it is a zero divisor since $r(bs) = (rb)s = (0)s = 0$. Note that bs is annihilated by r and t. Thus, we have $[bs] = [s]$. We now consider the following two cases; otherwise, $[b] = [c]$.

<u>Case I</u> deg $b =$ deg $c = 1$ implies $b^2 = 0$ or $c^2 = 0$.
Without loss of generality, $b^2 = 0$. Thus, we have that b also annihilates bs; i.e., $b(bs) = (bb)s = 0(s) = 0$. Contradiction, since $[bs] = [s]$, but $bs \neq 0$.

<u>Case II</u> at least one of b or c has deg > 1
Without loss of generality, deg $b \geq 2$. Thus, there exists $[d]$ other than $[b]$ and $[c]$ in the remaining graph such that $bd = 0$. Hence, we have that d also annihilates bs; i.e., $d(bs) = (db)s = 0(s) = 0$. Contradiction, since $[bs] = [s]$, but $ds \neq 0$.

We now provide an alternate proof to the Hinge Theorem using an associated prime argument.

Proof No end, if $|G| > 3$, can be an associated prime by Proposition 4.18. (See Remark 4.15.) Thus, by Lemma 4.16, $\mathfrak{q} = \text{ann}(s)$ is an associated prime. Consider some $[x]$ and $[y]$ in the remaining graph other than $[b]$, $[c]$ or $[r]$. The fact above implies x or y must be contained in $\mathfrak{q}$, a contradiction as deg $(s) = 2$ by definition of a hinged end.

Theorem 6.7 (Maximum Degree Theorem) *If G is a finite, connected graph with two non-adjacent vertices of maximal degree, then G is not the zero divisor graph of a ring.*

Proof Each vertex of maximal degree, by Theorem 4.19, must be an associated prime. By Lemma 4.16, all of these vertices must be adjacent, which is a contradiction if a pair of these vertices is not adjacent.

Example 6.8 *By Theorem 6.7, the following graph G cannot be realized as $\Gamma_E(R)$ for any ring R since $[a]$ and $[c]$ have maximal degree 3, but are not adjacent.*

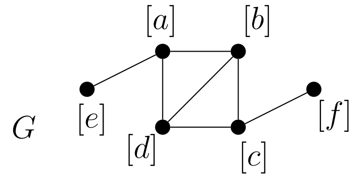

Figure 6: Maximum Degree Theorem Example

Theorem 6.9 (Bridge Theorem) *If G is a connected graph such that $G = \overline{N([a])} \cup \overline{N([c])}$, where $|N([a])| \geq 1, |N([c])| \geq 2$, and where a and c are two distinct vertices such that $ac = 0$, and no vertex in $N([a])$ is adjacent to any vertex in $N([c])$, then G is not the zero divisor graph of a ring.*

Proof Without loss of generality, $N([v])$ has two vertices, x and y, and $N([a])$ has one vertex, b. For diameter reasons, both x and y must be adjacent to v. Similarly, b must be adjacent to a. By Proposition 4.18 for $|\Gamma_E(R)| > 3$ as no ends are associated primes, if x and y were associated primes, then by edge domination b or $a \in \text{ann}\,(x)$, contradiction. By a similar argument, b is not an associated prime. Hence, a and v are the only associated primes. Note that if any vertices, $x_i \in N([v])$ are adjacent to one another, then implies a contradiction as $x_i, x, y \notin \text{ann}\,(a)$. Thus, we have x and y are non-adjacent which implies x or y is self-annihilating. Without loss of generality, say x is self-annihilating. Thus, by edge domination, we must have $x \in \text{ann}_R\,(a)$, contradiction. A similar argument holds for y.

Example 6.10 *By Theorem 6.9, the following graph G cannot be realized as $\Gamma_E(R)$ for any ring R.*

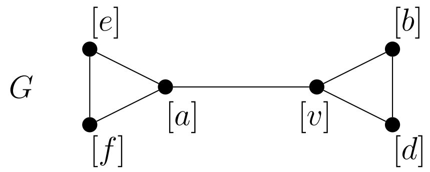

Figure 7: Bridge Theorem Example

Corollary 6.11 *If G is a bicycle, see Definition 3.27, then $G \neq \Gamma_E(R)$.*

Proof The proof follows from the Bridge Theorem. See Theorem 6.9.

Definition 6.12 *A (possibly infinite) graph G is called a* **book** *if G takes the form of the graph given below.*

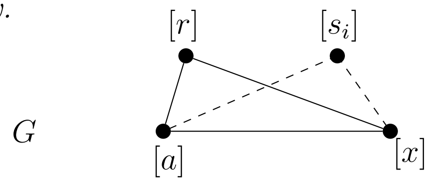

Figure 8: Book Graph

We note that the dashed edges of the graph in Definition 6.12 represent a (possibly infinite) number of pages, or vertices, in the book.

Theorem 6.13 (The Book Theorem) *If a graph G is a book, then $G \neq \Gamma_E(R)$.*

Proof Suppose that R is a ring such that $\Gamma_E(R)$ takes the form of a book graph, G. If $i = 0$ (i.e., $\{s_i\} = \varnothing$), then G is a 3-cycle. Thus by [11, Proposition 1.8], $G \neq \Gamma_E(R)$. Suppose $i \geq 1$ (i.e., $\{s_i\} \neq \varnothing$). Then, G is complete tri-partite with partitioning sets $\{a\}, \{x\}, \{r, s_i\}_{i \in I}$, and thus by [11, Proposition 1.7], $G \neq \Gamma_E(R)$.

Definition 6.14 *A (possibly infinite) graph G is called a* **modified book** *if G takes the form of the graph given below where there must exist vertices b, y such that $ab = 0$ and $xy = 0$.*

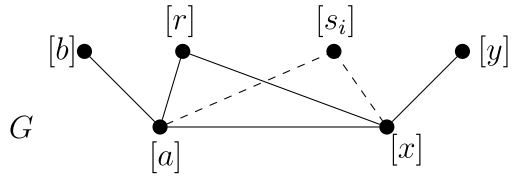

Figure 9: Modified Book Graph

We note that the dashed edges of the graph in Definition 6.14 represent a (possibly infinite) number of vertices in the modified book.

Theorem 6.15 (The Modified Book Theorem) *If G is a modified book graph, then $G \neq \Gamma_E(R)$.*

Proof Suppose that R is a ring such that $\Gamma_E(R)$ takes the form of a modified book graph, G. If $i = 0$ (i.e., $\{s_i\} = \varnothing$), then, by [9, Proposition 3.1], the result holds. Suppose $i \geq 1$ (i.e., $\{s_i\} \neq \varnothing$). Consider $a + x$. It is not 0 since a and x represent distinct classes, but it is a zero divisor since it is annihilated by r and s_i, but not b or y. Thus, the only candidates for $[a + x]$ are $[r]$ and $[s_1]$. Without loss of generality, say $[a + x] = [s_1]$. But, $r(a + x) = ra + rx = 0$ which implies $r(s_1) = 0$. This implies $[r] = [s_1]$ which is a contradiction.

Remark 6.16 *It is natural to consider graphs which can be classified as intermediates between a book graph and a modified book graph, namely G which takes the form of the following graph given below.*

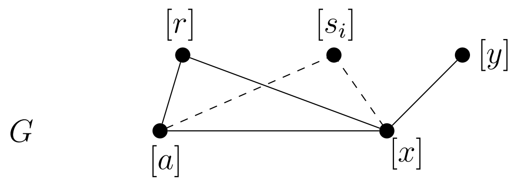

Figure 10: Intermediate Book Graph

The following example provides two graphs from this class. At this time, it is not clear *whether all graphs taking this general form are realizable.*

Example 6.17

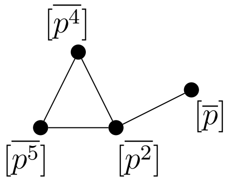

Figure 11: The Zero Divisor Graph $\Gamma_E(\mathbb{Z}/p^5\mathbb{Z})$

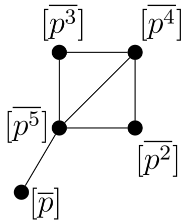

Figure 12: The Zero Divisor Graph $\Gamma_E(\mathbb{Z}/p^6\mathbb{Z})$

Theorem 6.18 (Missing Chord Theorem) *Let G be a connected graph with two distinct non-adjacent vertices a, b. If for every $v \in G$ $N([a]) \cup N([b]) \not\subset \overline{N([v])}$, then $G \neq \Gamma_E(R)$.*

Proof Suppose G is a connected graph with two distinct non-adjacent vertices a, b. Note that $[ab] \in G$ and $N([a]) \cup N([b]) \subseteq \overline{N[ab]}$ by [6, Proposition 5.1 (a)] or [5, Theorem 1 (4)]. By hypothesis, there is no $v = [ab]$. Hence, $G \neq \Gamma_E(R)$.

Theorem 6.19 (Missing Sum Theorem 1) *Let G be a connected graph with two distinct adjacent vertices a, b. If*

(a) there exists a vertex $[v] \in N([a]) \backslash \overline{N([b])}$;

(b) there exists a vertex $[w] \in N([b]) \backslash \overline{N([a])}$;

(c) $\overline{N([a])} \cap \overline{N([b])} \neq (0)$,

and $u \neq [a+b]$ for any $u \in G$, then $G \neq \Gamma_E(R)$.

Proof Note that $[a+b]$ is a zero divisor since $N([a]) \cap N([b]) \neq (0)$. But, $[a+b]$ represents a new class. Thus, $G \neq \Gamma_E(R)$.

Theorem 6.20 (Missing Sum Theorem 2) *Let G be a connected graph with two dis-*

tinct non-adjacent vertices a, b. If

(a) there exists a vertex $[v] \in \overline{N([a])} \backslash \overline{N([b])}$;

(b) there exists a vertex $[w] \in \overline{N([b])} \backslash \overline{N([a])}$;

(c) $N([a]) \cap N([b]) \neq (0)$,

and $u \neq [a+b]$ for any $u \in G$, then $G \neq \Gamma_E(R)$.

Proof Note that $[a+b]$ is a zero divisor since $N([a]) \cap N([b]) \neq (0)$. But, $[a+b]$ represents a new class. Thus, $G \neq \Gamma_E(R)$.

6.2 Positive Results

The graphs in the following examples can be realized as the zero divisor graph determined by equivalence classes of a ring R.

Example 6.21 *[6, Example 2.14] If $R = \frac{(\mathbb{Z}/2\mathbb{Z})[X_1, X_2, Y]}{(X_1^2, X_2^2, X_1X_2, Y^2, X_1X_2Y)}$, then $\Gamma_E(R)$ takes the form of the graph below. Note that lower case letters represent the cosets in R of the upper case letters. Let $\mathfrak{m}$ be the ideal (x_1, x_2, y) in R.*

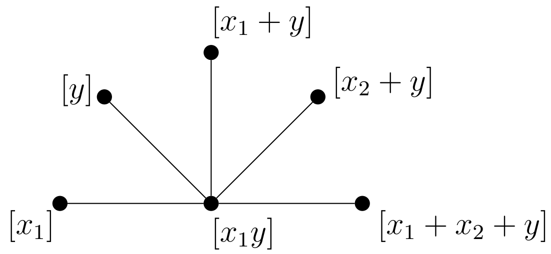

Figure 13: The Zero Divisor Graph $\Gamma_E(\frac{(\mathbb{Z}/2\mathbb{Z})[X_1, X_2, Y]}{(X_1^2, X_2^2, X_1X_2, Y^2, X_1X_2Y)})$

<u>First Class</u> $\text{Ann}(x_1y) = (x_1, x_2, y) = \text{Ann}(x_2y)$

<u>Second Class</u> $\text{Ann}(x_2) = (x_1, x_2, \mathfrak{m}^2) = \text{Ann}(x_2y) = \text{Ann}(x_1) = \text{Ann}(x_1 + x_2)$

<u>Third Class</u> $\text{Ann}(y) = (y, \mathfrak{m}^2)$

<u>Fourth Class</u> $\text{Ann}(x_1 + y) = (x_1 + y, \mathfrak{m}^2)$

<u>Fifth Class</u> $\text{Ann}(x_2 + y) = (x_2 + y, \mathfrak{m}^2)$

Sixth Class $\mathrm{Ann}(x_1 + x_2 + y) = (x_1 + x_2 + y, \mathfrak{m}^2)$

Example 6.22 *If $R = \mathbb{Z}/p^7\mathbb{Z}$, then $\Gamma_E(R)$ takes the form of the graph below.*

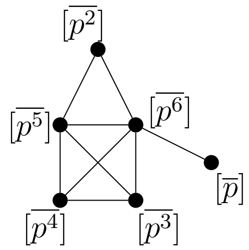

Figure 14: The Zero Divisor Graph $\Gamma_E(\mathbb{Z}/p^7\mathbb{Z})$

Example 6.23 *If $R = \mathbb{Z}/pqr\mathbb{Z}$, then $\Gamma_E(R)$ takes the form of the graph below.*

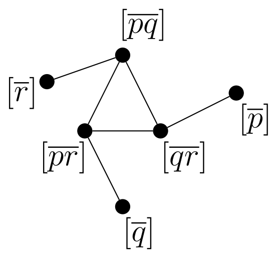

Figure 15: The Zero Divisor Graph $\Gamma_E(\mathbb{Z}/pqr\mathbb{Z})$

Example 6.24 *If $R = \mathbb{F}_2 \times \mathbb{Z}/8\mathbb{Z}$, then $\Gamma_E(R)$ takes the form of the graph below. See [6, Proposition 4.8].*

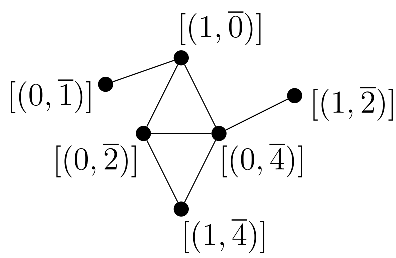

Figure 16: The Zero Divisor Graph $\Gamma_E(\mathbb{F}_2 \times \mathbb{Z}/8\mathbb{Z})$

7 GRAPHS WITH 7 VERTICES

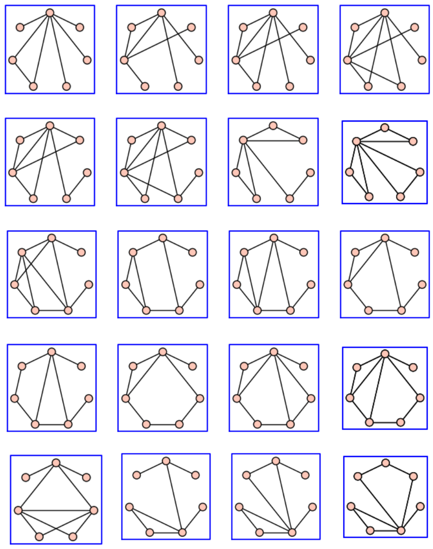

Figure 17: Some Examples of Graphs with 7 Vertices

The above graphs were drawn in Sage. The following section provides information about Sage. We will refer to the graphs using matrix notation. Note that a_{ij} denotes the entry of the above matrix, A, which is in the ith row and the jth column. We will refer to this entry as the (i, j) entry of A. A is a 5 x 4 matrix.

We now prove that several of the entries in A cannot be realized as $\Gamma_E(R)$ for any ring R using the list of characterizations in Section 5 that disprove a graph as the (condensed) zero divisor graph.

Entries (2, 1) and (2, 3) of A are not realizable by the Hinge Theorem. See Thm. 6.6.

The (5, 2) entry is not realizable by the Bridge Theorem. See Thm. 6.9.

The (3, 2) entry is not realizable by the Maximum Degree Theorem. See Thm. 6.7.

The (1, 2), (1, 3) and (5, 1) entries are not realizable by Theorem 4.21.

The (3, 4) and (4, 1) entries are not realizable by the Missing Chord Theorem. See Thm. 6.18.

8 SAGE

Sage is a free mathematics open-source software which can be used alternatively to Mathematica or Matlab. The code referred to in this section is indented and italicized.
The following code, which is a list comprehension, defines a function which lists all the graphs with 6 vertices.

$$graphs6 = list(graphs(6))$$

The next line of code determines the total number of graphs with 6 vertices.

$$len(graphs6)$$

There are a total of 156 graphs with 6 vertices. The next line of code will produce a visual picture for each graph.

$$show(graphs6)$$

As there are so many of these graphs we will not provide the pictures of the 156 graphs here. The following line of code defines a function to determine and filter all connected graphs in the set of graphs with 6 vertices.

$$connectedgraphs6{=}[g\ for\ g\ in\ graphs6\ if\ g.is\_connected()]$$

By using the following line of code, we are able to determine from the original 156 graphs with 6 vertices only 112 are connected.

$$len(connectedgraphs6)$$

The next line of code defines a filter function to determine and filter all graphs with diameter less than 4 in the set of 112 connected graphs.

$$connectedgraphs6diameter4 = filter(lambdax : x.diameter() < 4, connectedgraphs6)$$

By using the following line of code, we are able to determine from the original 112 connected graphs with 6 vertices only 103 are connected.

$$len(connectedgraphs6diameter4)$$

The next line of code will produce a visual picture for each graph of these 103 graphs. We provide a visual picture for these graphs in Appendix I.

$$show(connectedgraphs6diameter4)$$

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LIST OF APPENDICES

APPENDIX I

A APPENDIX I

We will refer to the graphs in this appendix using matrix notation. Note that a_{jk} denotes the entry of the matrices, on pages 1 - 5 of this appendix, which is in the jth row and the kth column. We let i denote the page number of this appendix containing the matrix. We will refer to this entry as the (i, j, k) entry of the matrix. The matrices on pages 1 - 4 are 5 x 4 matrices. On page 5, the matrix is a 6 x 4 matrix with a null value in entry (6, 4). The vertices are labeled, starting at the twelve o'clock position and going clockwise, as r, s, t, u, v, and w.

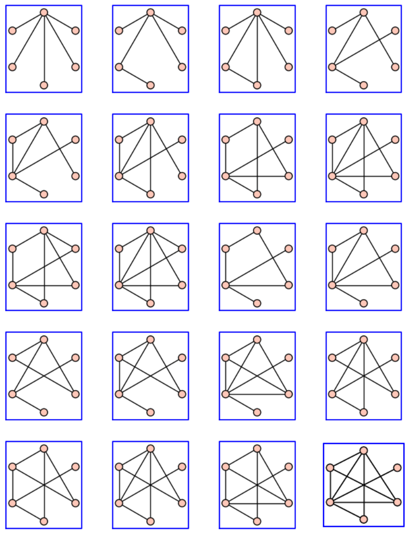

Figure 18: 6 Vertices Graphs page 1

The entries of the above matrix are referred to in the following notation: (1, j, k).

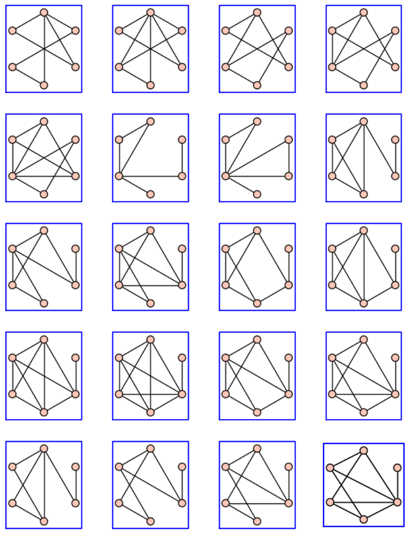

Figure 19: 6 Vertices Graphs page 2

The entries of the above matrix are referred to in the following notation: $(2,\ j,\ k)$.

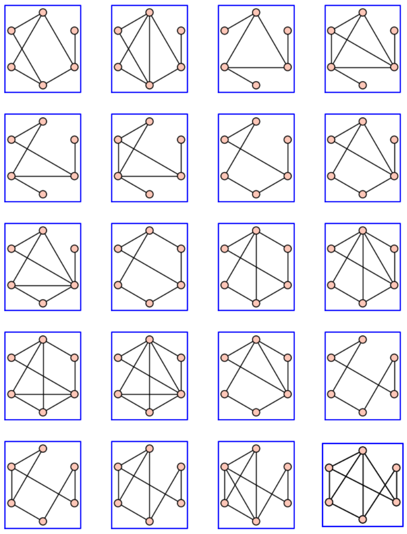

Figure 20: 6 Vertices Graphs page 3

The entries of the above matrix are referred to in the following notation: (3, j, k).

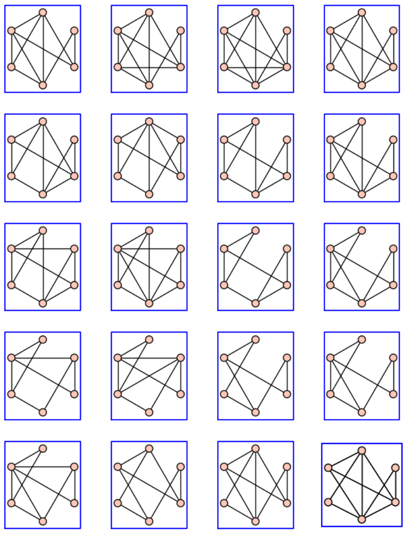

Figure 21: 6 Vertices Graphs page 4

The entries of the above matrix are referred to in the following notation: $(4,\, j,\, k)$.

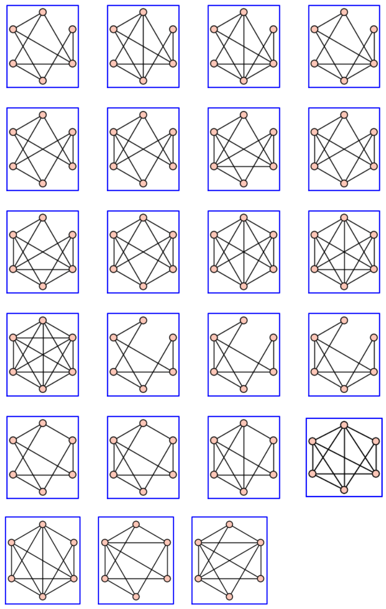

Figure 22: 6 Vertices Graphs page 5

The entries of the above matrix are referred to in the following notation: $(5,\ j,\ k)$.

APPENDIX II

A APPENDIX II

The following appendix categorizes the graphs of Appendix I which are not realizable, detailing the method of disproof, as well as the graphs which are realizable, and the remaining 6 graphs which have not yet been determined. Strategy I and II are stated in Remark 6.1.

Remark A.1 *Tables 10 - 13 use a version of the Missing Sum Theorems. Many of the graphs in these tables use a version of the theorems, in conjunction with Strategies I and II.*

Table 1: Bridge

(1,1,2)	(1,1,4)	(1,4,1)	(2,1,3)

Table 2: Maximum Degree

Missing Edge		
ru	(3,1,1)	(4,5,2)
rt	(3,3,2)	(3,4,1)
tu	(4,1,2)	(5,1,2)
uw	(3,5,2)	(4,3,1)
rv	(1,5,1)	
sv	(4,3,3)	
tv	(5,2,4)	
vw	(5,4,2)	

Table 3: Chordal

Missing Edge				
tv	(1,4,4)	(2,3,3)	(2,3,4)	(3,2,3)
tu	(2,3,1)	(2,5,2)	(3,5,4)	
rv	(1,3,3)			
st	(2,1,1)			
sv	(3,5,1)			
sw	(2,1,4)			
tw	(3,1,2)			
uw	(3,2,1)			

Table 4: Hinged End

(1,1,2)	(2,2,2)	(2,2,4)	(2,5,1)

Table 5: Book

(1,3,1)	(1,3,2)

Table 6: Modified Book

(1,2,2)

Table 7: Regular

(5,2,1)	(5,3,2)	(5,5,1)

Table 8: Miscellaneous (Violations of Cycle, Girth, Complete, respectively)

(3,4,4)	(4,4,3)	(5,4,1)

Table 9: Realizable

(1,1,1)	(1,5,4)	(3,1,3)	(3,2,2)

Table 10: Neither Strategy I nor II

Missing Element						
$s+v$	(3,5,3)	(4,4,1)	(4,4,2)	(4,4,4)	(4,5,4)	(3,3,4)
$u+w$	(4,3,4)	(5,2,3)				
$t+v$	(4,2,4)					
$r+t$	(5,3,1)					

Table 11: Strategy I

Missing Element	**Self-Ann.**		
$r+w$	w	(2,5,3)	(5,1,1)
$u+w$	w	(3,4,3)	(5,1,4)
$v+w$	r, t, v, w	(3, 3, 3)	
$r+t$	r	(2, 4, 3)	
$r+w$	r, w	(2,5,4)	
$v+w$	w	(3,2,4)	
$s+w$	s	(4,2,2)	
$s+v$	v	(4,1,1)	
$s+t$	s, t	(5,3,3)	
$r+v$	r, s, t	(5, 6, 2)	

Table 12: Strategy II

Missing Element	**Not Self-Ann.**					
$r+w$	r, w	(1,4,3)	(2,2,1)	(2,4,1)	(2,4,2)	(4,1,3)
$t+w$	w	(2,3,2)	(4,1,4)			
$t+w$	t, w	(5, 4, 4)				
$u+v$	u, v	(5, 5, 3)				
$u+v$	t, u, v	(5, 3, 4)				
$r+v$	v	(5, 5, 2)				
$t+u$	u	(4, 3, 2)				
$t+u$	t, u	(5, 6, 1)				

Table 13: Combination of Strategy I and II

Missing Element	**Not Self-Ann.**	**Self-Ann.**	
$r + w$	w	r	(2,4,4)
$r + s$	s	r	(5,6,3)
$u + v$	u	v	(5,5,4)
$s + v$	r, t	s, v	(4,5,1)
$t + v$	t	w	(5,1,3)
$t + w$	w	t	(5,4,3)

Table 14: Unclassified

(1,1,3)	(2,2,3)	(1,2,4)	(3,1,4)	(1,3,4)	(3,4,2)	(2,2,1)	(3,3 1)

B VITA

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