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Generalized Characteristics Of A Generic Polytope

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GENERALIZED CHARACTERISTICS OF A GENERIC POLYTOPE

DISSERTATION

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ABSTRACT

For a smooth hypersurface $S \subset \mathbb{R}^{2n}$ given by the level set of a Hamiltonian function H, a symplectic form ω on $\mathbb{R}^{2n}$ induces a vector field X_H which flows tangent to S. By the nondegeneracy of ω, there exists a distinguished line bundle $\mathcal{L}_S$ whose characteristics are the integral curves of X_H. When S is the boundary of a smooth convex domain $K \subseteq \mathbb{R}^{2n}$, then the least action among closed characteristics of $\mathcal{L}_S$ is equal to the Ekeland-Hofer-Zehnder capacity, a symplectic invariant. From a result due to Artstein-Avidan and Ostrover, there exists a continuous extension of this capacity to nonsmooth convex domains $\tilde{K} \subseteq \mathbb{R}^{2n}$, and from the work of Künzle, there is a generalization of the notion of characteristics of $\tilde{K}$. The existence of corners in $\partial\tilde{K}$, however, prevents the analogous uniqueness/existence result found in the smooth case, coming from the characteristic initial value problem. First, we will define a generic class of polyhedral sets, called "symplectic-faced", which avoid certain obstructions to uniqueness. We will show that, for symplectic-faced 4-polytopes Σ, we have the existence and local uniqueness of generalized characteristics of Σ. Then, we will show that symplectic-faced polytopes $\Sigma \subset \mathbb{R}^{2n}$ admit only characteristics with piecewise-linear trajectories. Finally, we will extend our existence/uniqueness result from 4-polytopes to the relative interior of low-codimension faces of symplectic-faced $2n$-polytopes.

DEDICATION

I dedicate this thesis to the state of Mississippi and its people for providing me with the most wonderful opportunities to work, love, and grow.

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1 INTRODUCTION

Motivated by classical mechanics, symplectic geometry formalizes the geometry of the phase space of a Hamiltonian system. Although symplectic manifolds do not, in general, represent the phase space of a physical system, the existence of a symplectic form ω on a manifold M can induce similar dynamical objects on M. For smooth functions $H : M \to \mathbb{R}$, we get the existence of a volume-preserving vector field X_H on M whose flow is analogous to the evolution of a conservative classical mechanical system.

The flow of X_H is not only volume-preserving, but it also preserves ω. In 1985, Gromov [8] proved that maps which preserve ω must satisfy much more rigid conditions than preserving volume. His non-squeezing theorem shows that, for the unit ball $B \subset \mathbb{R}^{2n}$ and the cylinder of radius r

$$Z(r) = \{(x_1, \dots x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n} \ : \ \sum_{i=1}^{n} x_i^2 + y_i^2 \leq r^2\}$$

if $r < 1$, then there does not exist a symplectomorphism ϕ such that $\phi(B) \subset Z(r)$.

Another important distinguishing feature of symplectic manifolds is that, by Darboux's theorem, there do not exist any local invariants. Global invariants such as the Ekeland-Hofer capacity c_{EH} and the Hofer-Zehnder capacity c_{HZ} are the main invariants considered in modern symplectic geometry [10], although they are difficult to compute. One alternative quantity, the Ekeland-Hofer-Zehnder capacity c_{EHZ}, considers characteristics γ on M, which are given by the image of the flow of X_H on ∂M. This quantity is attained by taking the symplectic action of closed characteristics γ, $\mathcal{A}(\gamma)$ and minimizing it. Ostrover [1] introduced this quantity and showed that, on bounded convex domains $K \subset \mathbb{R}^{2n}$, we have $c_{HZ}(K) = c_{EH}(K) = c_{EHZ}(K)$.

Additionally, Ostrover showed that c_{EHZ} admits a continuous extension to non-smooth convex domains. In order to compute such capacities, one must understand the nature of characteristics on non-smooth convex domains. We will outline the notion of a characteristic in the non-smooth case, using the technology introduced by Künzle ([12],[13]). We will then show that, for a generic polytope $\Sigma \subset \mathbb{R}^4$, there exists a unique characteristic passing through each point in $\partial\Sigma$.

1.1 Linear Symplectic Geometry

Many objects and concepts in Euclidean geometry can be expressed by using the Euclidean inner product on $\mathbb{R}^n$. We will see that a symplectic structure on $\mathbb{R}^{2n}$ exists if a certain anti-symmetric bilinear form on $\mathbb{R}^{2n}$ exists.

Proposition 1.1.1. *[10, p. 1] A finite dimensional real vector space V is symplectic if there exists a bilinear form $\omega : V \times V \to \mathbb{R}$ such that*

1. $\omega(u, v) = -\omega(v, u), \; u, v \in V$ *(antisymmetry)*

2. *The map $\phi : V \to V^*$ given by*

$$v \mapsto \omega(v, \cdot) \qquad \textit{(nondegeneracy)}$$

   is a linear isomorpism.

We call such a bilinear form ω on $V \times V$ a symplectic form, and note that ω exists on V only if V is even-dimensional. Now we will show that the typical symplectic structure on $\mathbb{R}^{2n}$ is given by a matrix deformation of the Euclidean inner product.

Proposition 1.1.2. *[10, p. 1] If*

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

where I is the real, $n \times n$ identity matrix, then the bilinear form ω_0 on $\mathbb{R}^{2n}$ given by

$$\omega_0(u, v) := \langle Ju, v \rangle$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^{2n}$, is a symplectic form on $\mathbb{R}^{2n}$.

We call the symplectic form above generated by J the standard symplectic form on $\mathbb{R}^{2n}$. Since J is skew-symmetric, observe that for every vector $v \in \mathbb{R}^{2n}$, we have $\omega_0(v, v) = 0$.

Proposition 1.1.3. *[10, p.8] Let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be a smooth function. Then,*

1. *There exists a unique vector field X_H on $\mathbb{R}^{2n}$ such that*

$$\omega_0\Big(X_H(x), v\Big) = -dH(x)v$$

for all $v \in \mathbb{R}^{2n}$ and $x \in \mathbb{R}^{2n}$.

2. *X_H is given by*

$$X_H(x) = J\nabla H(x), \;\; x \in \mathbb{R}^{2n}$$

By the nondegeneracy of ω_0, we get the uniqueness of $X_H(x)$, and by the definition of the gradient we get the explicit form $X_H(x) = J\nabla H(x)$. We call H a **Hamiltonian** on $\mathbb{R}^{2n}$, and X_H its **Hamiltonian vector field**.

1.2 Smooth Energy Surfaces in $(\mathbb{R}^{2n}, \omega_0)$

We will now consider hypersurfaces in $\mathbb{R}^{2n}$ which are analogous to the fixed energy levels of the phase space of a physical system.

Definition 1.2.1. *[10, p. 19] Let H be a Hamiltonian on $(\mathbb{R}^{2n}, \omega_0)$. We call $S \subseteq \mathbb{R}^{2n}$ a **regular energy surface** if $H^{-1}(c) = S$ and $dH(x) \neq 0$, for all $x \in S$.*

Definition 1.2.2. *[10, p. 19] The* ***tangent space*** *of a regular energy surface $S \subset \mathbb{R}^{2n}$ at x is the set*

$$T_xS = \{v \in \mathbb{R}^{2n} \ : \ dH(x)v = 0\}$$

For an energy surface $S \subset \mathbb{R}^{2n}$, we have that for every $x \in S$, T_xS is $2n-1$ dimensional. By the nondegeneracy of ω_0 and the rank-nullity theorem, when we restrict ω_0 to T_xS, we will get a one-dimensional kernel.

Definition 1.2.3. *[10, p. 22] The* ***characteristic line bundle*** *of a regular energy surface S is the set*

$$\mathcal{L}_S = \{(x,\xi) \in TS \ : \omega_0(\xi, v) = 0 \ \forall v \in T_xS\}$$

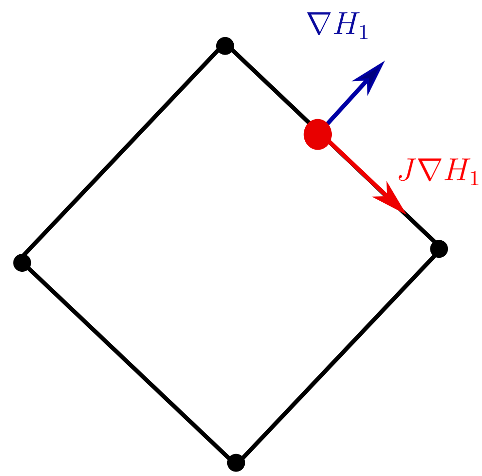
Since X_H flows tangent to level sets of H, then for all $x \in S$ we have that $X_H(x) \in T_xS$. By the definition of X_H, we see that

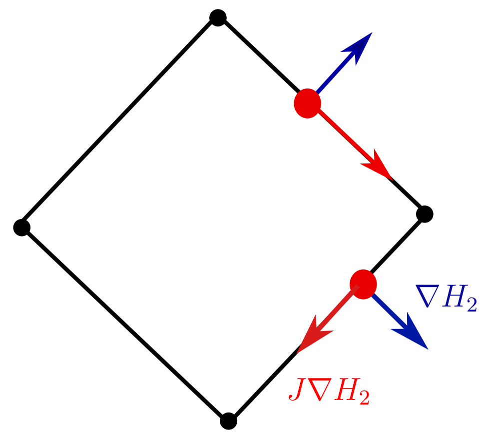
$$\omega_0(X_H(x), v) = -dH(x)v$$

and by definition of T_xS, we have $-dH(x)v = 0$ for all $v \in T_x$.

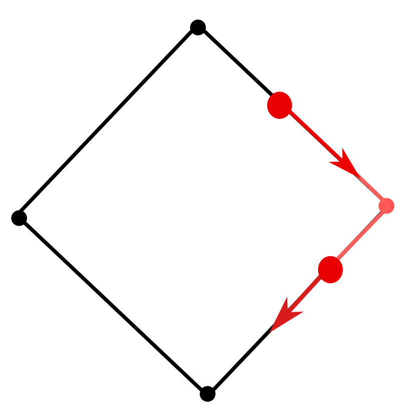
1.3 Non-smooth Energy Surfaces in $(\mathbb{R}^{2n}, \omega_0)$

We will now illustrate some difficulties in considering a concept analogous to characteristics of non-smooth convex domains. The ℓ_2 unit ball $\Sigma_0 \subset \mathbb{R}^2$ is given by a level set of the Hamiltonian $H(x,y) = |x| + |y|$. On the interior of a one-dimensional face $F_1 \subset \partial\Sigma_0$, points are given locally by the smooth function $H_1(x,y) = x + y$ and so the tangent space at each relative interior point is well defined. Then, characteristics passing through relative interior points $x \in F_1$ are given by integral curves of $X_{H_1(x)} = J\nabla H_1(x)$.
Similarly, we can derive the characteristics passing through points of an adjacent face F_2.





However, at a vertex $x_0 \in \partial\Sigma_0$, the tangent space at x_0 is undefined and the outward normal direction is not given by a single vector. Fortunately though, in this one-dimensional example, the adjacent faces fill in the unique characteristic passing through x_0.



Now we consider a pathological non-smooth energy surface in dimension four, which demonstrates a problem with uniqueness of characteristics at corners. From an example given in

[12], we consider the hypercube

$$\mathcal{C} = \left\{ (p_1, p_2, q_1, q_2) \in \mathbb{R}^4 \ : \ p_i, q_i \in \left[\frac{-1}{2}, \frac{1}{2} \right] \right\}$$

We can generate a two-dimensional corner $F = F_1 \cap F_2$ by considering the intersection of two top-dimensional faces F_1, F_2 given by the normal vectors $s_1 = (1,0,0,0)$ and $s_2 = (0,1,0,0)$ and their corresponding affine hyperplanes $\mathrm{Aff}\,(F_1) = \{s_1 \cdot x = 1\}$, $\mathrm{Aff}\,(F_2) = \{s_2 \cdot x = 1\}$. Note that

$$\omega_0(s_1, s_2) = \omega_0(s_1, s_1) = \omega_0(s_2, s_2) = 0$$

Note that ω_0 vanishes on $\mathrm{Span}^+\{s_1, s_2\}$, which is a cone analogous to the outward normal direction. Then, we have that $\mathrm{Span}^+\{Js_1, Js_2\}$ not only generates the characteristic direction (a rotation of the normal direction $\mathrm{Span}^+\{s_1, s_2\}$), but is also in the space tangent to F (orthogonal to normal direction).

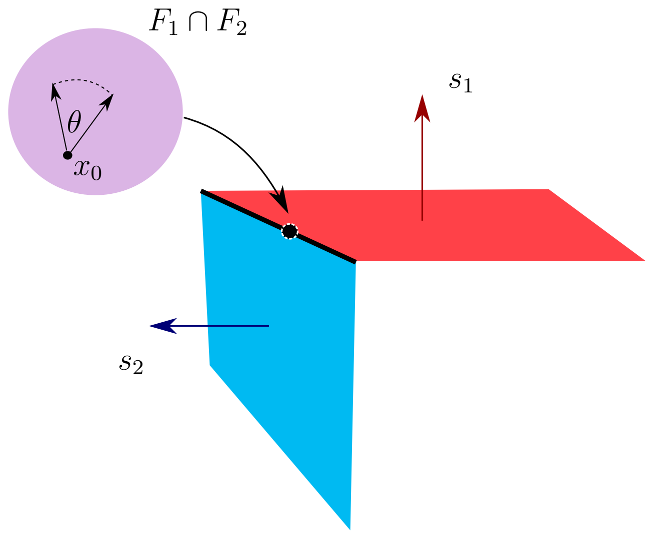


Figure 1.1: At $x_0 \in \mathrm{Int}\,(F_1 \cap F_2)$, there exists a one-dimensional family of characteristics passing through x_0, parameterized by θ. Although in this example we have a two-dimensional family of characteristic directions given by $\mathrm{Span}^+\{Js_1, Js_2\}$, this figure reflects the actual definition of a generalized characteristic which has a normalization constraint.

Therefore, we have constructed a pathological convex domain $\mathcal{C}$ on which there exists faces where uniqueness of characteristics is false. In the main result, we will restrict our attention to a generic class of polyhedral sets with the property of being "symplectic-faced" which do not suffer from such obstructions to uniqueness. Before we show our main result, we will put a topology on the space of polyhedral sets in $\mathbb{R}^4$ generated by m normal vectors, denoted $\tilde{P}^4_m$, and then construct an open, dense set of symplectic-faced polyhedral sets.

Theorem 1.3.1. *There exists an open and dense set $\mathcal{F} \subset \tilde{P}^4_m$ such that for every $S_\Sigma \in \mathcal{F}$ we have that Σ is symplectic-faced.*

For symplectic-faced 4-polytopes Σ, in addition to uniqueness of characteristics, for any $x \in \partial\Sigma$ we will demonstrate the existence of a characteristic which passes through x.

Theorem 1.3.2. *Let Σ be a symplectic-faced 4-polytope. Then, for $x_0 \in \partial\Sigma$, there exists a generalized characteristic X of Σ and $\exists\epsilon > 0$ such that*

i $x_0 \in X$

ii for any other generalized characteristic X_0 of Σ with $x_0 \in X_0$, we have $B_\epsilon(x_0) \cap X_0 = B_\epsilon(x_0) \cap X$.

For each face F in a polytope, we can consider the space of possible characteristic directions, denoted $\mathrm{Char}\,(F)$. We will show that if the dimension of $\mathrm{Char}\,(F)$ is strictly less than two, then we have uniqueness of characteristics within the relative interior of F. For symplectic-faced $2n$-polytopes, we will show that every face has this uniqueness property within its relative interior.

Theorem 1.3.3. *Let F be a face of a symplectic-faced $2n$-polytope. Then* $\dim(\mathrm{Char}\,(F)) = 0$ *or* $\dim(\mathrm{Char}\,(F)) = 1$.

We will then generalize our result of existence and uniqueness of characteristics to the relative interior $\mathrm{ri}\,(F)$ of low-codimension faces F of symplectic-faced $2n$-polytopes.

Theorem 1.3.4. *Let Σ be a symplectic-faced $2n$-polytope and let $x_0 \in \mathrm{ri}\,(F)$ for F a face of Σ with* $\mathrm{codim}\,(F) \leq 4$. *Then, there exists a generalized characteristic X of Σ and $\exists\epsilon > 0$ such that* *i* $x_0 \in X$

ii *for any other generalized characteristic* X_0 *of* Σ *with* $x_0 \in X_0$, *we have* $B_\epsilon(x_0) \cap X_0 = B_\epsilon(x_0) \cap X$.

2 POLYTOPES

A (convex) polytope can be seen either as the convex hull of a finite set or as a bounded intersection of half-spaces. Since piecewise-linear energy surfaces are the boundaries of polytopes, in Section 2.1 we will introduce some fundamental facts about polytopes and their representation as an intersection of half-spaces. In Section 2.2, we construct an object dual to a polytope which, in Section 3.1, will be helpful in determining the direction of a characteristic. In Section 2.3, we will introduce the first fundamental concept needed to adapt our notion of a characteristic to the nonsmooth structure of a polytope. The "corners" which exist on the boundary of a polytope Σ force any characteristic of Σ to be nonsmooth, which turns our characteristic's differential equation of $\dot{x} = J\nabla H(x)$ into a differential *inclusion*. Before we can present the differential inclusion, in this section we must first formalize a notion of being normal to Σ (like how ∇H is normal to an energy surface) at "corners" by defining an *outward normal cone*.

2.1 Polyhedral Sets in $\mathbb{R}^n$

Definition 2.1.1. *A* ***polyhedral set*** Σ *is a finite intersection of closed half-spaces in* $\mathbb{R}^n$ *($n \geq 1$), i.e. there exists a finite set* $S_\Sigma \subseteq \mathbb{R}^n$ *such that*

$$\Sigma = \bigcap_{s \in S_\Sigma} K(s, \alpha_s)$$

where

$$K(s, \alpha_s) = \{x \in \mathbb{R}^n \ : \ x \cdot s \leq \alpha_s\}$$

We say that Σ *is* ***generated*** *by* S_Σ *and* $\{\alpha_s\}_{S_\Sigma}$*. Additionally, we define the* ***supporting hyperplane*** $H(s, \alpha_s)$ *of the halfspace to be the set*

$$H(s, \alpha_s) = \{x \in \mathbb{R}^n \ : \ x \cdot s = \alpha_s\}$$

Definition 2.1.2. *[6, p. 52] If, for a polyhedral set*

$$\Sigma = \bigcap_{s \in S_\Sigma} K(s, \alpha_s) \tag{2.1.1}$$

we have $|S_\Sigma| = 1$*, or if for every* $\tilde{s} \in S_\Sigma$ *we have*

$$\Sigma \subsetneq \bigcap_{s \in S_\Sigma \setminus \{\tilde{s}\}} K(s, \alpha_s)$$

then we call Equation 2.1.1 an ***irreducible representation*** *of* Σ*.*

Definition 2.1.3. *For a polyhedral set* $\Sigma \subseteq \mathbb{R}^n$ *with an irreducible representation generated by* S_Σ *and* $\{\alpha_s\}_{S_\Sigma}$*, we define a* ***facet*** *generated by* $s \in S_\Sigma$ *to be the set*

$$F_s = H(s, \alpha_s) \cap \Sigma$$

Definition 2.1.4. *Two distinct facets* $F, \tilde{F}$ *are* ***adjacent*** *if* $F \cap \tilde{F} \neq \emptyset$

Definition 2.1.5. *An* ***affine subspace*** *of* $\mathbb{R}^n$ *is a set of the form* $A = x + L$*, where* $x \in \mathbb{R}^n$ *and* L *is a linear subspace of* $\mathbb{R}^n$*.*

Definition 2.1.6. *For an affine subspace* $A = x + L \subset \mathbb{R}^n$ *we define* $\dim(A) := \dim(L)$*.*

Definition 2.1.7. *Let* $S \subseteq \mathbb{R}^n$ *be a set and let* $\mathcal{A}$ *be the collection of affine subspaces of* $\mathbb{R}^n$ *containing* S*. Then, we define the* ***affine hull*** *of* S *to be*

$$\mathrm{Aff}(S) = \bigcap_{\mathcal{A}} A$$

Definition 2.1.8. *Let $S \subseteq \mathbb{R}^n$ be a convex set. Then, we define the* ***dimension*** *of S to be the quantity*

$$\dim(S) = \dim(\mathrm{Aff}(S))$$

Definition 2.1.9. *For $\Sigma \subseteq \mathbb{R}^n$ a polyhedral set, a* ***face*** *F of Σ is a set*

$$F = \bigcap_{S_F} F_s$$

*such that $S_F \subseteq S_\Sigma$ is nonempty. If $\dim(F) = d$, we may refer to F as a d-**face**.*

Proposition 2.1.10. *Let Σ be a polyhedral set and let F be a face of Σ. Then, $\Sigma \backslash F$ is convex.*

Proof. For $x, y \in \Sigma$ distinct, denote

$$[x, y] = \{\alpha x + (1-\alpha)y \ : \ \alpha \in [0,1]\}$$

$$]x, y[= \{\alpha x + (1-\alpha)y \ : \ \alpha \in (0,1)\}$$

By [6, p. 30], F is a face of Σ if and only if for two distinct points $y, x \in \Sigma$, $]x, y[\cap F \neq \emptyset$ implies that $[x, y] \subset F$. Let $x, y \in \Sigma \backslash F$ be distinct. By Σ convex, we know that $[x, y] \subset \Sigma$. Since $x, y \notin F$, then we cannot have $[x, y] \subset F$ and therefore we have $]x, y[\cap F = \emptyset$, as desired. □

Since each face of a polyhedral set in $\mathbb{R}^n$ is of dimension at most $n-1$, then every face has empty interior under the standard topology of $\mathbb{R}^n$. Additionally, every point in a face is a limit point of that face, so the boundary of a face must be the entire face. We will now introduce the notion of *relative* interior of a face and *relative* boundary of a face by considering the ambient affine space in which a face lies.

Definition 2.1.11. *The* ***relative interior*** $\mathrm{ri}(F)$ *of a face F is the interior of $F \subseteq \mathrm{Aff}(F)$ under the subspace topology of $\mathrm{Aff}(F) \subseteq \mathbb{R}^n$.*

Definition 2.1.12. *The **relative boundary*** $\mathrm{rb}\,(F)$ *of a face* F *is the boundary of* $F \subseteq \mathrm{Aff}\,(F)$ *under the subspace topology of* $\mathrm{Aff}\,(F) \subseteq \mathbb{R}^n$*.*

Proposition 2.1.13. *[6, Corollary 5.7] Let* F, G *be two distinct faces. Then,* $\mathrm{ri}\,(F) \cap \mathrm{ri}\,(G) = \emptyset$*.*

$\square$

It is convenient to prove the existence of an irreducible, "normalized" form of a polytope's half-space representation, since such a form is unique and will provide computational simplicity.

Proposition 2.1.14. *If* Σ *is a polyhedral set with* $0 \in \mathrm{Int}\,(\Sigma)$ *and* Σ *is generated by* S_Σ*,* $\{\alpha_s\}_{S_\Sigma}$*, then there must exist a set* $\tilde{S}_\Sigma$ *such that*

$$\Sigma = \bigcap_{s \in \tilde{S}_\Sigma} K(s, 1)$$

Proof. Want to first show that all $\{\alpha_s\}_{S_\Sigma}$ are nonzero. Suppose there exists at least one $\alpha_{s_0} = 0$. By S_Σ, $\{\alpha_s\}_{S_\Sigma}$ irreducible, $H(s, \alpha_{s_0}) \cap \Sigma = H(s, 0) \cap \Sigma$ is a facet of Σ. Since $0 \in H(s, 0)$, this means $0 \in \partial\Sigma$, but we assumed $0 \in \mathrm{Int}\,(\Sigma)$, so this is impossible. Suppose we have at least one $\alpha_{s_0} < 0$. However, then $0 \notin \Sigma$, so this is also impossible.
So, we can construct a set

$$\tilde{S}_\Sigma = \left\{ \frac{1}{\alpha_s} s \right\}_{S_\Sigma}$$

and since $K(s, \alpha_s) = K(\frac{1}{\alpha_s} s, 1)$, we still have

$$\Sigma = \bigcap_{s \in S_\Sigma} K(\frac{1}{\alpha_s} s, 1)$$

$\square$

Proposition 2.1.15. *Let* $\Sigma \subseteq \mathbb{R}^n$ *be a polytope. If* S *and* $\tilde{S}$ *are both irreducible unit generating sets of* Σ*, we must have* $S = \tilde{S}$*.*

Proof. Note that the collection of facets $\mathcal{F}$ of a polytope Σ depends only on the set Σ and not on its half-space representation. By [6, Theorem 8.2], since both S and $\tilde{S}$ are irreducible, we must have

$$\mathcal{F} = \left\{H(s,1) \cap \Sigma\right\}_{s \in S} = \left\{H(\tilde{s},1) \cap \Sigma\right\}_{\tilde{s} \in \tilde{S}}$$

Let $m := |S| = |\tilde{S}|$. Using $\mathcal{I} = \{1, \dots, m\}$, we can make a choice of indexing for all $s \in S$ and all $\tilde{s} \in \tilde{S}$ so that

$$\Sigma \cap H(s_i, 1) = \Sigma \cap H(\tilde{s}_i, 1)$$

for all $i \in \mathcal{I}$. Then, we must have

$$\Sigma \cap H(s_i, 1) = \Sigma \cap H(s_i, 1) \cap H(\tilde{s}_i, 1) \subseteq H(s_i, 1) \cap H(\tilde{s}_i, 1)$$

and since each $\Sigma \cap H(s_i, 1)$ is a facet of Σ, it must have dimension $n - 1$, which means $n > \dim(H(s_i, 1) \cap H(\tilde{s}_i, 1)) \geq n - 1$. Then we must have $H(s_i, 1) = H(\tilde{s}_i, 1)$ for all $i \in \mathcal{I}$, which implies that $S = \tilde{S}$. $\square$

Definition 2.1.16. *For a polyhedral set Σ with $0 \in \operatorname{Int}(\Sigma)$, we define the* ***unit generating set*** *of Σ to be the unique set S^1_Σ that generates the irreducible representation of Σ:*

$$\Sigma = \bigcap_{s \in S^1_\Sigma} K(s, 1)$$

Definition 2.1.17. *A* ***polytope*** *is a bounded polyhedral set. If for a polytope Σ we have $\dim(\Sigma) = d$, then we call Σ a d-**polytope**.*

In order to easily discriminate between characteristics contained in the relative interior or relative boundary of a given face, we observe that the relative interior of a face is given geometrically by an intersection of hyperplanes which generate the face, and an intersection of open half-spaces from the remaining hyperplanes. We can then concisely express this as a system of equalities and strict inequalities, using the normal vectors in our polytope's half-space representation.

Proposition 2.1.18. *Let Σ be a polyhedral set with unit generating set S_Σ and let F be a face of Σ. Then, $x \in \mathrm{ri}\,(F)$ if and only if $x \cdot s = 1$ for all $s \in S_F$ and $x \cdot \tilde{s} < 1$ for all $\tilde{s} \in S_\Sigma \backslash S_F$.*

Proof. Let $F = \cap_{S_F} F_s$ be a face of Σ. Then, by Definition 2.1.9 and Definition 2.1.16 we know that

$$F = \left(\bigcap_{S_F} H(s,1)\right) \bigcap \left(\bigcap_{S_\Sigma \backslash S_F} K(s,1)\right) \tag{2.1.2}$$

$$\mathrm{Aff}\,(F) = \bigcap_{S_F} H(s,1) \tag{2.1.3}$$

It suffices to show that

$$x \in F, \text{ and } \exists \epsilon > 0 \text{ such that } B_\epsilon(x) \bigcap \mathrm{Aff}\,(F) \subseteq F$$

if and only if

$$x \in \left(\bigcap_{S_F} H(s,1)\right) \bigcap \left(\bigcap_{S_\Sigma \backslash S_F} \mathrm{Int}\,\Big(K(s,1)\Big)\right)$$

First we show " $\Longleftarrow$ ". Since $S_\Sigma \backslash S_F$ is finite, then $\cap_{S_\Sigma \backslash S_F} \mathrm{Int}\,\Big(K(s,1)\Big) = \mathrm{Int}\,\Big(\cap_{S_\Sigma \backslash S_F} K(s,1)\Big)$. Since $x \in \mathrm{Int}\,(\cap_{S_\Sigma \backslash S_F} K(s,1))$, we can take $\epsilon > 0$ such that $B_\epsilon(x) \subseteq \cap_{S_\Sigma \backslash S_F} K(s,1)$. By Eqns 2.1.2 and 2.1.3, we then have that $B_\epsilon(x) \bigcap \mathrm{Aff}\,(F) \subseteq F$.

Now we show " $\Longrightarrow$ ". Let $x \in F$. By Eqn 2.1.2 and Eqn 2.1.3, we know $x \in \mathrm{Aff}\,(F) \cap \Big(\bigcap_{S_\Sigma \backslash S_F} K(s,1)\Big)$. It remains to be shown that x is an interior point of $\Big(\bigcap_{S_\Sigma \backslash S_F} K(s,1)\Big)$. Suppose not. Then, x is a boundary point of $\Big(\bigcap_{S_\Sigma \backslash S_F} K(s,1)\Big)$, i.e. $x \in H(\hat{s},1)$ for all $\hat{s}$ in a nonempty set $X \subseteq S_\Sigma \backslash S_F$. Then, we have that

$$x \in \left(\bigcap_{s \in S_F \cup X} H(s,1) \right) \cap \left(\bigcap_{S_\Sigma \backslash (S_F \cup X)} \mathrm{Int}\left(K(s,1) \right) \right)$$

Which, by the previous direction, shows that for the face

$$\tilde{F} = \left(\bigcap_{S_F \cup X} H(s,1) \right) \cap \left(\bigcap_{S_\Sigma \backslash (S_F \cup X)} K(s,1) \right)$$

we have $x \in \mathrm{ri}\,(\tilde{F})$. This implies that $\tilde{F} \neq F$, and by Prop 2.1.13, we have $\mathrm{ri}\,(\tilde{F}) \cap \mathrm{ri}\,(F) = \emptyset$. However, by hypothesis we had that $x \in \mathrm{ri}\,(F)$, a contradiction. Therefore, x must be an interior point of $\left(\bigcap_{S_\Sigma \backslash S_F} K(s,1) \right)$. □

2.2 Polar of a Polytope

Since certain "corners" of the boundary of a polytope Σ can be generated by many adjacent facets (e.g. 0-faces, 1-faces in a 4-polytope), a characteristic which begins in such a corner has many possible faces to move into in forward time. Theorem 3.1.23 establishes the criteria for a characteristic to move into a certain face, from a corner, using dot-product inequalities of nearby normal vectors. These inequalities can be realized geometrically as a particular arrangement of hyperplanes which intersect a new polytope, Σ°, which is dual to the original polytope Σ. We will show that we can transform Σ into its dual Σ° by taking the convex hull of S_Σ (normal vectors become 0-faces).

Definition 2.2.1. *Let $K \subseteq \mathbb{R}^n$. We define the **polar** of K to be the set*

$$K^\circ = \{ y \in \mathbb{R}^n \ : \ x \cdot y \leq 1, \forall x \in K \}$$

Definition 2.2.2. *Let $K \subseteq \mathbb{R}^n$. We say that K and K° are **mutually polar** if $(K^\circ)^\circ = K$.*

Theorem 2.2.3. *[6, Theorem 6.1] For any subset $M \subseteq \mathbb{R}^n$, we have:*

i If M is bounded, then $0 \in \operatorname{Int}(M^\circ)$.

ii If $0 \in \operatorname{Int}(M)$, then M° is bounded.

$\square$

Theorem 2.2.4. *Let Σ be a polytope with $0 \in \operatorname{Int}\Sigma$. Then, Σ and $\operatorname{Conv}(S_\Sigma)$ are mutually polar polytopes.*

Proof. By [6, Theorem 9.2], a bounded intersection of half-spaces and a convex hull of a finite set are both polytopes. Applying [6, Theorem 6.1], we get that $0 \in \operatorname{Int}(M^\circ)$ and applying [6, Theorem 9.1] we get that Σ and $\operatorname{Conv}(S_\Sigma)$ are mutually polar. $\square$

We have shown that the dual of a polytope Σ can be obtained by taking the convex hull of S_Σ. Similarly, we can focus our attention on a face F of Σ and find a "conjugate" face $F^\Delta \subseteq \Sigma^\circ$.

Definition 2.2.5. *Let F be a face of a polytope Σ. Then, we define the* ***conjugate*** *of F to be the set*

$$F^\Delta = \{y \in \Sigma^\circ \mid \forall x \in F \ : \ x \cdot y = 1\}$$

Proposition 2.2.6. *[6, Theorem 6.6] If Σ and Σ° are mutually polar polytopes, and F is a face of Σ, then F^Δ is a face of Σ°.*

Theorem 2.2.7. *[6, Theorem 9.8] Let F be a face of a d-polytope Σ. Then, F^Δ is a face of Σ° with dimension*

$$\dim(F^\Delta) = d - 1 - \dim(F)$$

$\square$

Proposition 2.2.8. *Let $F = \cap_{S_F} F_s$ be a face of a polytope Σ. Then,*

$$F^\Delta = \operatorname{Conv}(S_F)$$

Proof. First we show "$\subseteq$". By the definition of F^{Δ}, we have $F^{\Delta} \subseteq \Sigma^{\circ}$. By Theorem 2.2.4, we then have "$\subseteq$".

Let $y \in \operatorname{Conv}(S_F)$. Then, by Theorem 2.2.4 we have that $y \in \Sigma^{\circ}$. Since any element of a convex hull of finitely many points can be expressed as a convex combination of those points, we can write $y = \sum_{S_F} \alpha_s s$ with $\sum_{S_F} \alpha_s = 1$, $\alpha_s \geq 0$. Then, for any $x \in F = \bigcap F_s$ we must have $x \cdot s = 1$, and so

$$x \cdot y = \sum_{S_F} \alpha_s (x \cdot s) = 1$$

□

2.3 The Normal Cone of a Polyhedral Set

In the case of an energy surface $\Sigma_0 \subseteq \mathbb{R}^{2n}$ and a point $x \in \Sigma_0$, we know that $\nabla H(x)$ is normal to Σ_0 at x. Additionally, if we take $J\nabla H(x)$, we get a vector pointing in the direction of the Hamiltonian vector field constrained to flow in Σ_0. In the case of a piecewise-linear energy surface, we will similarly be interested in the direction normal to the surface in order to determine the direction of a characteristic. With the challenge of having "corners" present in a polytope, we encounter places on the boundary of a polytope for which the normal direction is a space spanned by several vectors.

For a facet F_s in a polytope Σ, the outward normal direction to F_s is given by the normal vector s. For a lower dimensional face, a notion of "outward normal" cannot be expressed by a single vector. We will define the outward normal cone of a set, and then show that for $\partial\Sigma$, the outward normal cone is generated by the normal vectors of neighboring facets.

Definition 2.3.1. *The* ***outward normal cone*** *of a convex set $K \subseteq \mathbb{R}^n$ at x is*

$$N_K(x) = \left\{ v \in \mathbb{R}^n \ : \ (x - y) \cdot v \geq 0, \ \forall y \in K \right\}$$

Definition 2.3.2. *The* ***unit normal cone*** *of a convex set* $K \subseteq \mathbb{R}^n$ *at* x *is*

$$n_K(x) = \{v \in N_K(x) \ : \ |v| = 1\}$$

Theorem 2.3.3. *If* $\Sigma \subset \mathbb{R}^n$ *is a polyhedral set with irreducible unit generating set* S^1_Σ*, then for* $x \in \partial\Sigma$*, we have*

$$N_\Sigma(x) = \text{Span}^+(S_x)$$

where

$$S_x = \{s \in S^1_\Sigma \ : \ s \cdot x = 1\}$$

Proof. We will first show that $N_\Sigma(x) \subseteq \text{Span}(S_x)$, followed by $\text{Span}^+(S_x) \subseteq N_\Sigma(x)$. Then, we will show that, for $\xi \in \text{Span}(S_x)$ with $\xi \cdot t \geq 0$ for all $t \in S_x$, we must have $n \cdot \xi \leq 0$ for all $n \in N_\Sigma(x)$. Finally, we will show that $N_\Sigma(x) \subseteq \text{Span}^+(S_x)$.

We show our first claim by showing that, for any $n \in N_\Sigma(x)$, we must have $n \in (\langle S_x \rangle^\perp)^\perp$. Let $\sigma \in \langle S_x \rangle^\perp$. We claim that there exists a real number $t > 0$ such that $y := x + t\sigma \in \Sigma$ and $\tilde{y} := x - t\sigma \in \Sigma$. We show this by finding a real number t such that:

$$y \cdot s \leq 1 \tag{2.3.1}$$

$$\tilde{y} \cdot s \leq 1 \tag{2.3.2}$$

for all $s \in S_x$ and

$$y \cdot s \leq 1 \tag{2.3.3}$$

$$\tilde{y} \cdot s \leq 1 \tag{2.3.4}$$

for all $\tilde{s} \in S_\Sigma \backslash S_x$. Expanding Eqn 2.3.1, we have:

$$(x + t\sigma) \cdot s = x \cdot s + t(\sigma \cdot s) = 1$$

by $\sigma \in \langle S_x \rangle^{\perp}$. So, Eqn 2.3.1 is satisfied for any $t > 0$. Similarly, Eqn 2.3.2 is satisfied for any $t > 0$. In Eqn 2.3.3, Since $x \cdot s < 1$ for all $s \in S_\Sigma \backslash S_x$ and since $S_\Sigma \backslash S_x$ is finite, there must exist $\delta > 0$ such that $x \cdot \tilde{s} = 1 - \delta > 0$ for all $\tilde{s} \in S_\Sigma \backslash S_x$. Then, expanding Eqn 2.3.3 we have:

$$x \cdot \tilde{s} + t(\sigma \cdot \tilde{s}) = 1 - \delta + t(\sigma \cdot \tilde{s})$$

If we take t such that $t(\sigma \cdot \tilde{s}) < \delta$, then Eqn 2.3.3 and Eqn 2.3.4 are both satisfied and we must have that $y, \tilde{y} \in \Sigma$. If we let $n \in N_\Sigma(x)$, then by definition of $N_\Sigma(x)$, by $y, \tilde{y} \in \Sigma$, and by $t > 0$ we have:

$$\begin{aligned} n \cdot (x - y) = n \cdot (-t\sigma) = -t(n \cdot \sigma) &\geq 0 \\ (n \cdot \sigma) &\leq 0 \end{aligned}$$

Similarly, we have

$$\begin{aligned} n \cdot (x - \tilde{y}) = t(n \cdot \sigma) &\geq 0 \\ n \cdot \sigma &\geq 0 \end{aligned}$$

Since we have both $n \cdot \sigma \geq 0$ and $n \cdot \sigma \leq 0$, we must have $n \cdot \sigma = 0$. This implies that $n \in (\langle S_x \rangle^{\perp})^{\perp}$, which also means that $n \in \mathrm{Span}(S_x)$, as desired.

Now we show our second claim. Consider $v = \sum_{s \in S_x} k_s s \in \mathrm{Span}^+ (S_x)$. Let $y \in \Sigma$. Then, we have

$$v \cdot (x - y) = (\sum_{s \in S_x} k_s s) \cdot x - (\sum_{s \in S_x} k_s s) \cdot y = \sum_{s \in S_x} k_s (s \cdot x - s \cdot y)$$

We know that $s \cdot x = 1$ for all $s \in S_x$ and since $y \in \Sigma$, we know $s \cdot y \leq 1$ for all $s \in S_x$. Since $k_s \geq 0$ for all $s \in S_x$, we have that $v \cdot (x - y) \geq 0$, and therefore $\mathrm{Span}^+ (S_x) \subseteq N_\Sigma(x)$.

To begin showing our third claim, let $\xi \in \mathrm{Span}(S_x)$ be such that $\xi \cdot s \leq 0$ for all $s \in S_x$ and let $n \in N_\Sigma(x)$. We proceed by first showing the existence of an $\epsilon > 0$ such that $x + \epsilon\xi \in \Sigma$.

It suffices to show that there exists an $\epsilon > 0$ such that

$$s \cdot (x + \epsilon\xi) \leq 1 \tag{2.3.5}$$

for all $s \in S_x$ and that

$$\tilde{s} \cdot (x + \epsilon\xi) \leq 1 \tag{2.3.6}$$

for all $\tilde{s} \in S_\Sigma \backslash S_x$.

Expanding the left hand side of 2.3.5, we have

$$s \cdot x + \epsilon(s \cdot \xi) = 1 + \epsilon(s \cdot \xi)$$

and since, by hypothesis $s \cdot \xi \leq 0$, then Eqn 2.3.5 holds for all $\epsilon > 0$. Expanding the left hand side of 2.3.6, we get

$$= \tilde{s} \cdot x \ + \epsilon(\tilde{s} \cdot \xi)$$

We know that $\tilde{s} \cdot x < 1$, so there exists $\delta > 0$ such that

$$= (1 - \delta) + \epsilon(\tilde{s} \cdot \xi)$$

If $\tilde{s} \cdot \xi \leq 0$, then we are done since $(1 - \delta) + \epsilon(\tilde{s} \cdot \xi) \leq 1$. If not, then we take $\epsilon > 0$ such that $\epsilon(\tilde{s} \cdot \xi) < \delta$, and Eqn 2.3.6 is satisfied. By Eqns 2.3.5 and 2.3.6, we have
$x + \epsilon\xi \in \Sigma$. Then, by the definition of $N_\Sigma(x)$, we must have

$$n \cdot (x - (x + \epsilon\xi)) = -\epsilon(n, \xi) \geq 0$$

and so $\epsilon(n \cdot \xi) \leq 0$. Since $\epsilon > 0$, then we have $n \cdot \xi \leq 0$ as desired.

To show our final claim, we define the following sets:

$$S := \text{Span}^+(S_x)$$
$$\mathcal{S} := \{K(u,0) : S \subseteq K(u,0)\}$$
$$\Xi := \{\xi \in \text{Span}\,(S_x) \ : \ \xi \cdot s \geq 0 \ \forall s \in S_x \text{ and } \xi \cdot n \leq 0 \ \forall n \in N_\Sigma(x)\}$$
$$\mathcal{K} := \{K_\xi(u,0) \ : \xi \in K(u,0) \text{ for at least one } \xi \in \Xi\}$$

We make the following observation: since S is a closed convex set, then $\overline{\text{Conv}\,(S)} = S$. Then, by [4, Theorem 14], we have that $S = \bigcap_{K \in \mathcal{S}} K$. Therefore, we want to show that $\bigcap_{\Xi,\mathcal{K}} K_\xi^c \subseteq \bigcap_{K \in \mathcal{S}} K$. Let $v \in \bigcap_{\Xi,\mathcal{K}} K_\xi^c$. Then, $v \in K_\xi^c$, for all $\xi \in \Xi$ and all $K_\xi \in \mathcal{K}$. Suppose there exists a half space $\tilde{K}_S \in \mathcal{S}$ for which $v \notin \tilde{K}_S$. Since $S \subseteq N_\Sigma(x)$, then there does not exist any $K_\xi \in \mathcal{K}$ with $S \subseteq K_{xi}$, there must exist some $\tilde{\xi} \in \Xi$ such that $\tilde{\xi} \in \tilde{K}_S^c$. However, then both $v, \xi \in \tilde{K}_S^c$, but by hypothesis, we assumed that $v \in \bigcap_{\Xi,\mathcal{K}} K_\xi^c$. So we must have that $v \in \bigcap_{K \in \mathcal{S}} K$, as desired.
Finally, in since in our third claim we showed that $N_\Sigma(x) \subseteq \bigcap_{\Xi,\mathcal{K}} K_\xi^c$, and since $\bigcap_{\Xi,\mathcal{K}} K_\xi^c \subseteq S$, we must have that $N_\Sigma(x) \subseteq S$. Combined with our second claim, we have that $N_\Sigma(x) = S$. $\square$

The following corollary expresses that, for a face of a polyhedral set, the normal cone at every point in F is identical and equals the positive span the generating normal vectors.

Corollary 2.3.4. *Let Σ be a polyhedral set and let F be a face of Σ. Then, for $x, \tilde{x} \in \text{ri}\,(F)$, we have*

$$N_\Sigma(x) = N_\Sigma(\tilde{x}) = \text{Span}^+\,(S_F)$$

Proof. Let $x \in F$. By the definition of a face, $x \in F \iff x \cdot s = 1$ for all $s \in S_F$. Then, $S_x = S_F \ \forall x \in F$, and by Theorem 2.3.3, we have $N_\Sigma(F) = \text{Span}^+\,(S_F)$. Similarly, by $\tilde{x} \in F$, we have $S_{\tilde{x}} = S_F$ and so $N_\Sigma(F) = \text{Span}^+\,(S_F)$. $\square$

By Corollary 2.3.4, we can now identify each face F of a polyhedral set Σ with the normal cone associated to any point in F.

Definition 2.3.5. *Let Σ be a polyhedral set and F a face of Σ. The* ***normal cone of*** *F is the set*

$$N_\Sigma(F) = \mathrm{Span}^+(S_F)$$

In order to easily compute a normal cone's dimension for a particular face, we can express face membership as a matrix equation and appeal to the rank-nullity theorem.

Definition 2.3.6. *Let $\Sigma \subseteq \mathbb{R}^n$ be a polyhedral set and let F be a face of Σ. Then, the* ***normal matrix*** *$\mathcal{F}$ of F is the $|S_F| \times n$ matrix with elements of S_F as row vectors.*

Proposition 2.3.7. *Let $\Sigma \subseteq \mathbb{R}^n$ be a polyhedral set and let F a face of Σ. Then,*

i $\dim(\mathrm{null}(\mathcal{F})) = \dim(F)$

ii $\dim(\mathrm{rank}(\mathcal{F})) = \dim(N_\Sigma(F))$

Proof. First we show (i). For some $w \in \mathbb{R}^n$, we have

$$\begin{aligned} \mathrm{Aff}(F) &= \{v \in \Sigma \ : \ \mathcal{F}v = 1\} \\ &= \mathrm{null}(\mathcal{F}) + w \end{aligned} \tag{2.3.7}$$

then

$$\begin{aligned} \dim(\mathrm{Aff}(F)) &= \dim(\mathrm{null}(\mathcal{F}) + w) \\ &= \dim(\mathrm{null}(\mathcal{F})) \end{aligned}$$

and by Definition 2.1.8, $\dim(F) = \dim(\mathrm{Aff}(F))$.

Now we show (ii). By (i) and applying the rank-nullity theorem to the linear map $\mathcal{F}$, it suffices to show that $\mathrm{Aff}(N_\Sigma(F)) = \langle N_\Sigma(F) \rangle$. Note that by Theorem 2.3.3, we have $0 \in N_\Sigma(F)$. Let

A be an affine space containing $N_\Sigma(F)$. By definition of an affine space, $\exists x \in \mathbb{R}^n$ and there exists a subspace $L \subseteq \mathbb{R}^n$ such that $A = x + L$. Since $0 \in N_\Sigma(F)$, then $0 \in A$ and $\exists \ell \in L$ such that $x + \ell = 0$, which implise $\ell = -x$. Since L is a vector space we must have $x \in L$, which implies that $x + L = L$ and therefore $A = L$.

Since A was arbitrary, then

$$\operatorname{Aff}\left(N_\Sigma(F)\right) = \bigcap_{N_\Sigma(F) \subseteq L} L$$

and an intersection of subspaces must also be a subspace, therefore we must have $\operatorname{Aff}\left(N_\Sigma(F)\right) = \langle N_\Sigma(F) \rangle$. $\square$

3 GENERALIZED CHARACTERISTICS OF A POLYTOPE in $\mathbb{R}^4$

Defining the characteristic line bundle of a smooth energy surface $\Sigma_0 \subset \mathbb{R}^{2n}$ required the existence of a symplectic structure on $\mathbb{R}^{2n}$. Since a nonsmooth energy surface $\Sigma \subset \mathbb{R}^{2n}$ has points at which a tangent space cannot be defined, the characteristic line bundle of Σ cannot be defined. However, at each point $x \in \Sigma$ we can make use of the existence of a symplectic matrix J on $\mathbb{R}^{2n}$ to rotate the outward normal cone at x, giving a family of possible directions analogous to the direction of a characteristic flow.

With the prevalence of "corners" in a polytope Σ, a characteristic path beginning in at x in a corner F has a variety of adjacent faces in which to move, including a direction tangent to F (provided that F is not a 0-face). In Section 3.1, we define the characteristic of a nonsmooth energy surface, and formalize the distinct trajectories of a characteristic path "gliding" along F (flowing tangent to F), and "bouncing" off of F (flowing transverse to F).

In Section 3.2, we define a generic class of polytopes Σ which has the property that for $x \in \Sigma$, locally there exists a unique generalized characteristic X such that $x \in \operatorname{Int}(X)$. This is accomplished by determining the trajectory of a generalized characteristic path γ, given that $\gamma(0) \in \operatorname{ri}(F)$ for F a 3-face, followed by determining the trajectory of γ for F a 2-face, and so on.

3.1 Generalized Characteristics

Definition 3.1.1. *The **characteristic cone** of a convex set $K \subseteq \mathbb{R}^{2n}$ at $x \in \partial K$ is*

$$JN_K(x) = \{Jv \ : \ v \in N_K(x)\}$$

Definition 3.1.2. *The* ***unit characteristic cone*** *of a convex set* $K \subseteq \mathbb{R}^{2n}$ *at* $x \in \partial K$ *is*

$$Jn_K(x) = \{Jv \ : \ v \in n_K(x)\}$$

Definition 3.1.3. *A* ***convex body*** $K \subseteq \mathbb{R}^n$ *is a convex set with* $0 \in \mathrm{Int}\,(K)$ *and for which* $\dim\,(K) = n$.

Proposition 3.1.4. *Let* $K \subseteq \mathbb{R}^n$ *be a convex body with* $x \in K$. *If* $v \in N_K(x)$ *and* $-v \in N_K(x)$, *then we must have* $v = 0$.

Proof. Let $v \in N_K(x)$ and $-v \in N_K(x)$. Then, by definition of $N_K(x)$, we must have

$$(x-y) \cdot v \geq 0 \quad \forall y \in K$$
$$(x-y) \cdot v \leq 0 \quad \forall y \in K$$

which implies that $(x-y) \cdot v = 0$ for all $y \in K$. Since x is fixed and $x \cdot v = y \cdot v$, then the linear functional $\xi(y)$ given by $\xi(y) = v \cdot y$ must be constant on K. By K a convex body, we have $0 \in \mathrm{Int}\,(K)$ and since ξ is constant on K, we must have $\xi(y) = \xi(0) = 0$ for all $y \in K$. By K a convex body, there exists an open ball $B_r(0) \subseteq K$. For any basis $\{w_i\} \subseteq \mathbb{R}^n$, we can choose $k > 0$ such that $\{kw_i\} \subseteq B(0, r)$. Then, by ξ identically zero on K and $\{kw_i\} \subseteq B(0, r) \subseteq K$, we have that ξ is identically zero on $\mathbb{R}^n$, and so $v \in (\mathbb{R}^n)^\perp$, which implies that $v = 0$. □

With the characteristic direction encoded in the cone $JN_K(x)$ and then normalized in the cone $Jn_K(x)$, we can now define a generalized characteristic. Since the corners of $\partial\Sigma$ make up a set of measure zero in $\partial\Sigma$, and since any characteristic path γ which intersects a corner of $\partial\Sigma$ cannot be differentiable, then we must require γ to be differentiable almost everywhere. In contrast to the characteristic equation for a smooth energy surface in the introduction, γ will need to satisfy a characteristic differential inclusion. This definition of a generalized characteristic will resemble that of K'́unzle [12, p. 176].

Definition 3.1.5. *A* ***generalized characteristic path*** *of a convex body $K \subseteq \mathbb{R}^{2n}$ is a Lipschitz continuous function $\gamma : \mathcal{I} \to \partial K$ (for some interval $\mathcal{I} \subseteq \mathbb{R}$) for which we have:*

$$\dot{\gamma}(t) \in Jn_K(x) \textit{ for almost all } t \in \mathcal{I}$$

We call $\operatorname{Im}(\gamma)$ *a* ***generalized characteristic*** *of K.*

The definition of a generalized characteristic on a polytope Σ shows us that the direction $\dot{\gamma}(t)$ of a generalized characteristic path γ must lie in the unit characteristic cone at $\gamma(t)$. However, at corners of $\partial\Sigma$, γ may not remain tangent to $\partial\Sigma$ in forward time, since $Jn_\Sigma(x)$ will not be orthogonal to the normal cone at $\gamma(t)$. In order to determine $\dot{\gamma}$ for γ lying in a face $F \subset \partial\Sigma$, we will also need for $\dot{\gamma}$ to be orthogonal to $N_\Sigma(F)$.

Definition 3.1.6. *For a $2n$-polytope Σ, the* ***characteristic cone restricted to a face*** *$F \subseteq \Sigma$ is*

$$\operatorname{Char}(F) = JN_\Sigma(F) \cap N_\Sigma(F)^\perp$$

Definition 3.1.7. *Let γ be a generalized characteristic path of a polytope Σ with $\gamma(0) \in \operatorname{ri}(F)$ for some face F of Σ. We say that a face $F^+_{\gamma(0)}$ is a* ***forward face*** *of F if the quantity*

$$\bar{t} := \textit{inf}\,\{t > 0 : \gamma(t) \notin \operatorname{ri}(F)\}$$

exists and $\gamma(\bar{t}) \in F^+_{\gamma(0)}$. We say that a face $F^-_{\gamma(0)}$ is a ***backward face*** *of F if the quantity*

$$\underline{t} := \textit{sup}\,\{t < 0 : \gamma(t) \notin \operatorname{ri}(F)\}$$

exists and $\gamma(\underline{t}) \in F^-_{\gamma(0)}$.

Definition 3.1.8. *Let F be a face of Σ. If γ is a generalized characteristic for which $\gamma(0) \in \mathrm{ri}\,(F)$ and for which there exists a nonzero $v \in \mathbb{R}^{2n}$ such that*

$$\gamma(t) = \gamma(0) + tv, \;\; t \in (\underline{t}, \overline{t}); \;\; \underline{t} \neq \overline{t}$$

*then we say that γ **glides in** F **via** v.*

A generalized characteristic path γ gliding in F would represent the simplest possible trajectory in F, rather than any type of non-linear trajectories which may exist for $\dim\,(\mathrm{Char}\,(F)) > 1$. We will establish two types of generalized characteristic paths γ in a face F: bounce-only ($\mathrm{Im}\,(\gamma)$ intersects F transversely) and glide-only ($\mathrm{Im}\,(\gamma)$ intersects F tangentially).

Definition 3.1.9. *Let F be a face of Σ. If there exists $v \in \mathbb{R}^{2n}$ such that every generalized characteristic path γ with $\gamma(t_0) \in \mathrm{ri}\,(F)$ glides in F via v, then we say that F is **glide-only via** v.*

Definition 3.1.10. *Let F be a face of Σ. If, for every open interval $\mathcal{I}$, there does not exist a generalized characteristic path γ such that $\gamma|_{\mathcal{I}} \subseteq F$, then we say that F is **bounce-only**.*

Proposition 3.1.11. *If F is a 0-face of a $2n$-polytope Σ, then* $\mathrm{Char}\,(F) = \langle 0 \rangle$.

Proof. Since F is a 0-face, by Equation 2.3.7 of Proposition 2.3.7 and the rank-nullity theorem, we have that $\mathrm{rank}\,(\mathcal{F}) = 4$, which implies that $N_{\Sigma}(F)^{\perp}$ is zero dimensional. □

Using the concept of $\mathrm{Char}\,(F)$, we can now associate $\dim\Big(\mathrm{Char}\,(F)\Big)$ with the type of generalized characteristic paths allowed on F.

Proposition 3.1.12. *Let F be a face of a $2n$-polytope Σ. Then,* $\mathrm{Char}\,(F) = \langle 0 \rangle$ *if and only if F is bounce-only.*

Proof. " $\implies$ " Suppose there exists an interval $\mathcal{I}$ such that $\gamma|_{\mathcal{I}} \subseteq F$. Then, we know for all $s \in S_F$,

$$\begin{aligned} &\gamma(t) \cdot s = 1 \\ &\dot{\gamma}(t) \cdot s = 0 && \text{for almost all } t \in \mathcal{I} \end{aligned}$$

So, we must have $\dot{\gamma}(t) \in JN_\Sigma(F) \cap N_\Sigma(F)^\perp$ a.e., which by hypothesis implies that $\dot{\gamma}(t) = 0$ a.e. However, by definition of a generalized characteristic we must have $|\dot{\gamma}(t)| = 1$ a.e., which is impossible.

" $\impliedby$ " By Proposition 3.1.11, we only need to consider the case of $\mathrm{ri}\,(F) \neq \emptyset$, i.e. F not a 0-face. Suppose $\mathrm{Char}\,(F)$ not zero-dimensional and take $v \in \mathrm{Char}\,(F)$ such that $|v| = 1$. Let $x \in \mathrm{ri}\,(F)$. Then, by $v \in \mathrm{Char}\,(F)$ and by Σ bounded, we can take $\epsilon > 0$ such that for $t \in [0, \epsilon]$ we have

$$\begin{aligned} &(x \pm tv) \cdot s = 1 \quad \forall s \in S_F \\ &(x \pm tv) \cdot \tilde{s} < 1 \quad \forall \tilde{s} \in S_\Sigma \backslash S_F \end{aligned}$$

However, the function $\gamma(t) := x + tv$ for $t \in (-\epsilon, \epsilon)$ is a generalized characteristic path for which $\gamma(t) \in F$ on an interval. This contradicts that F is bounce-only, and therefore we must have that $\mathrm{Char}\,(F)$ is zero-dimensional. $\square$

Proposition 3.1.13. *A face F is glide-only via $kv \in \mathbb{R}^{2n}$ for some $k \in \mathbb{R}_{>0}$ if and only if*

$$\mathrm{Char}\,(F) = \mathrm{Span}^+\{v\}$$

Proof. Let γ be a generalized characteristic path of Σ for which $\gamma(0) \in \mathrm{ri}\,(F)$ and for which $\underline{t_0} \neq \overline{t_0}$.

" $\implies$ "

First we show "$\supseteq$". By F glide-only via kv, we have $\dot{\gamma}(t) = kv$ for $t \in (t_0 + \underline{t_0}, t_0 + \overline{t_0})$. By the definition of a generalized characteristic, $v \in Jn_\Sigma(F) \subseteq JN_\Sigma(F)$. By $\gamma(t) \in F$ for $t \in (\underline{t}, \overline{t})$, we know for all $s \in S_F$:

$$s \cdot \dot{\gamma}(t) = 0$$
$$s \cdot kv = 0$$

and so $v \in N_\Sigma(F)^\perp$. Now we show "$\subseteq$". Let $\underline{v} \in \mathrm{Char}\,(F)$. Then, for $t \in (\underline{t}, \overline{t})$ and for all $s \in S_F$:

$$\begin{aligned}\Big(\gamma(0) + t\underline{v}\Big) \cdot s &= \gamma(0) \cdot s + t\underline{v} \cdot s \\ &= 1\end{aligned}$$

Note that $\exists \delta > 0$ such that for all $\tilde{s} \in S_\Sigma \backslash S_F$, we have $\gamma(0) \cdot \tilde{s} < 1 - \delta < 1$. If we take $\epsilon > 0$ such that $\epsilon < \min\left\{\left|\frac{\delta}{t\underline{v} \cdot \tilde{s}}\right| : \tilde{s} \in S_\Sigma \backslash S_F\right\}$, then for $t \in (-\epsilon, \epsilon)$, we have that

$$\Big(\gamma(0) + t\underline{v}\Big) \cdot \tilde{s} < 1$$

Therefore, the function $\tilde{\gamma}$ given by

$$\tilde{\gamma}(t) := \gamma(0) + t\underline{v} \quad \text{for } t \in (-\epsilon, \epsilon)$$

is a generalized characteristic path and glides in F via $\underline{v}$, and by F glide-only via v, we must have $t\underline{v} = tv$. So, have $\underline{v} \in \langle v \rangle$.

"$\Longleftarrow$"

By definition of a generalized characteristic path, we have $\dot{\gamma}(t) \in JN_\Sigma(F)$ for almost all $t \in (\underline{t}, \overline{t})$. Additionally, we have $\dot{\gamma}(t) \in N_\Sigma(F)^\perp$ for almost all $t \in (\underline{t}, \overline{t})$. So, $\dot{\gamma} \in \mathrm{Char}\,(F)$

which means that for almost all $t \in (\underline{t}, \overline{t})$, we have $\dot{\gamma}(t) \in \langle v \rangle$. Additionally, by definition of a generalized characteristic path, we must have that $|\dot{\gamma}| = 1$. Since $\langle v \rangle$ is generated by a single vector, we know $\exists k \in \mathbb{R}$ such that $\dot{\gamma}(t) = \sigma(t)kv$ for $\sigma(t) = \pm 1$. Because $\dot{\gamma} \in JN(F)$ then by Proposition 3.1.4, $\sigma(t)$ must equal a constant $\sigma \in \{-1, 1\}$. By γ absolutely continuous, for $\epsilon \in (\underline{t}, \overline{t})$:

$$\gamma(\epsilon) - \gamma(0) = \int_0^\epsilon \sigma kv \; dt$$

$$\gamma(\epsilon) = \gamma(0) + \epsilon(\sigma kv)$$

so F must be glide-only via v. □

We will now see that the interior of top-dimensional faces (facets) possess the simplest generalized characteristic paths. This is visually clear, because $Jn_\Sigma(x)$ and $N_\Sigma(F)$ each consist of a single vector and they are mutually orthogonal.

Theorem 3.1.14. *Every facet F_s of a $2n$-polytope Σ is glide-only via kJs, for some $k \in \mathbb{R}_{>0}$.*

Proof. By Proposition 3.1.13, it suffices to show that $\mathrm{Char}\,(F_s) = \mathrm{Span}^+\{Js\}$. By Theorem 2.3.4, we know that $N_\Sigma(F_s) = \mathrm{Span}^+\{s\}$ and so $JN_\Sigma(F_s) = \mathrm{Span}^+\{Js\}$. Additionally, we have that $s \cdot Js = 0$ which implies that $\langle Js \rangle = N_\Sigma(F_s)^\perp$. Therefore, we must have that $\mathrm{Char}\,(F_s) = \mathrm{Span}^+\{Js\}$, as desired. □

Understanding a glide-only face F helps in understanding characteristics which intersect the lower-dimensional faces lying on its boundary. In particular, since we have shown that every facet F_s possesses a constant vector field in $\mathrm{ri}\,(F_s)$ which determines every generalized characteristic lying in $\mathrm{ri}\,(F_s)$, we can now work towards understanding generalized characteristics in the lower dimensional faces of Σ. This is because every lower dimensional face must lie in $\mathrm{rb}\,(F_s)$, for some facet F_s.

We will first define regions of $\mathrm{rb}\,(F)$, for a glide-only face F, based on where characteristics in F intersect $\mathrm{rb}\,(F)$ transversely, and whether they intersect in forward/backward time of a generalized characteristic path. We will then understand how these regions can be attached to one another at their respective boundaries.

Definition 3.1.15. *If F is a glide-only face, then the **forward fence** of F is the set*

$$F^+ = \{\cup_\gamma \operatorname{ri}(F^+_{\gamma(0)}) \ : \ \gamma \text{ a generalized characteristic path with } \gamma(0) \in \operatorname{ri}(F) \text{ and } \gamma(\bar{t}) \in \operatorname{ri}(F^+_{\gamma(0)})\}$$

*and the **backward fence** of F is the set*

$$F^- = \{\cup_\gamma \operatorname{ri}(F^-_{\gamma(0)}) \ : \ \gamma \text{ a generalized characteristic path with } \gamma(0) \in \operatorname{ri}(F) \text{ and } \gamma(\underline{t}) \in \operatorname{ri}(F^-_{\gamma(0)})\}$$

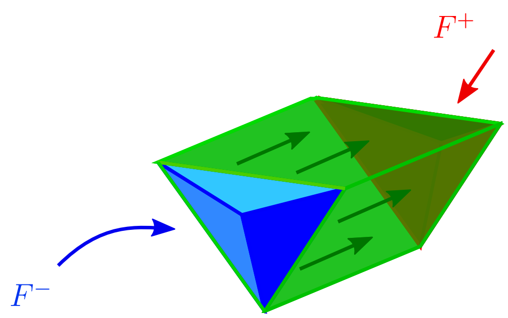


Figure 3.1: The backward and forward fence of a glide-only 3-face.

In Figure 3.1, the interior of this polyhedron represents the relative interior of a glide-only facet in a 4-polytope. The forward flow on the inside of the polyhedron is transverse to the red forward fence, and the backward flow is transverse to the blue backward fence. The polyhedron is extruded out by the green direction in order to see the flow lines, but in reality the edges of the red and blue fence must be glued together. Notice that the edges on the outside of the two fence portions are not included in either fence (colored in green). This is because the flow is not transverse to these outside edges, therefore the relative interior of these edges is not included in either fence. For the same reason, the vertices outside of each fence are not included in either fence.

Definition 3.1.16. *If F is a glide-only face, then the* ***boundary*** *of the forward fence F^+ is the set*

$$\mathrm{bd}\,(F^+) = \{\cup_\gamma \mathrm{rb}\,(F^+_{\gamma(0)})\ :\ \gamma \text{ a generalized characteristic path with } \gamma(0) \in \mathrm{ri}\,(F) \text{ and } \gamma(\bar{t}) \in \mathrm{ri}\,(F^+_{\gamma(0)})\}$$

and the ***boundary*** *of the backward fence F^- is the set*

$$\mathrm{bd}\,(F^-) = \{\cup_\gamma \mathrm{rb}\,(F^-_{\gamma(0)})\ :\ \gamma \text{ a generalized characteristic path with } \gamma(0) \in \mathrm{ri}\,(F) \text{ and } \gamma(\underline{t}) \in \mathrm{ri}\,(F^-_{\gamma(0)})\}$$

We will now define the closure of the forward and backward fence of a glide-only face F, in order to decompose $\mathrm{rb}\,(F)$ into two closed sets which correspond with either forward or backward flow.

Proposition 3.1.17. *The set*

$$\overline{F^+} = F^+ \cup \mathrm{bd}\,(F^+)$$

is the closure of F^+ in $\mathrm{Aff}\,(F)$ and the set

$$\overline{F^-} = F^- \cup \mathrm{bd}\,(F^-)$$

is the closure of F^- in $\mathrm{Aff}\,(F)$.

Proof. Let $F_i \subseteq F^+$ be a face of Σ. By definition of F^+, we have that $\mathrm{rb}\,(F_i) \subseteq \overline{F^+}$ and $\mathrm{ri}\,(F_i) \subseteq F^+$. Note that a polytope Σ has a finite number of faces and the closure of a finite union is the union of closures. Then, it suffices to show that for each $F_i \subseteq F^+$, we have $\mathrm{cl}\,(F_i) = \mathrm{rb}\,(F_i) \cup \mathrm{ri}\,(F_i)$. By definition of $\mathrm{rb}\,(F_i)$ and $\mathrm{ri}\,(F_i)$, we know that the closure of F_i in $\mathrm{Aff}(F_i)$ is equal to $\mathrm{rb}\,(F_i) \cup \mathrm{ri}\,(F_i)$. Under the subspace topology of $\mathrm{Aff}\,(F)$ inherited from $\mathbb{R}^{2n}$, we have that $\Big(\mathrm{rb}\,(F_i) \cup \mathrm{ri}\,(F_i)\Big) \cap \mathrm{Aff}\,(F)$ is closed in $\mathrm{Aff}\,(F)$. Since $F = \mathrm{rb}\,(F_i) \cup \mathrm{ri}\,(F_i)$, this is the smallest closed set containing F. Therefore, $\mathrm{cl}\,(F_i) = \mathrm{rb}\,(F_i) \cup \mathrm{ri}\,(F_i)$, as desired. $\square$

Proposition 3.1.18. *For a glide-only face F, we have $F^+ \cap F^- = \emptyset$.*

Proof. Suppose not, i.e. suppose that there exist faces $F_{\gamma(0)} \subset F^-$, $F_{\tilde{\gamma}(0)} \subset F_+$ such that $\mathrm{ri}\,(F_{\gamma(0)}) \cap \mathrm{ri}\,(F_{\tilde{\gamma}(0)}) \neq \emptyset$. These faces have corresponding generalized characteristic paths $\gamma, \tilde{\gamma}$ with initial conditions $\gamma(0), \tilde{\gamma}(0) \in \mathrm{ri}\,(F)$ and by the contrapositive of Proposition 2.1.13, we have that $\gamma(\bar{t}), \tilde{\gamma}(\underline{\tilde{t}}) \in \mathrm{ri}\,(F_{\gamma(0)}) = \mathrm{ri}\,(F_{\tilde{\gamma}(0)})$.

Since $\mathrm{ri}\,(F_{\gamma(0)}) = \mathrm{ri}\,(F_{\tilde{\gamma}(0)})$ is a convex set and F a glide-only face via the vector $v \in \mathbb{R}^{2n}$, then for any $\alpha \in [0,1]$, we must have

$$\alpha\Big(\gamma(0) + \bar{t}v\Big) + (1-\alpha)\Big(\tilde{\gamma}(\tilde{t}) + \underline{\tilde{t}}v\Big) \in \mathrm{ri}\,(F_{\gamma(0)}) = \mathrm{ri}\,(F_{\tilde{\gamma}(0)})$$
$$\Big(\alpha\gamma(0) + (1-\alpha)\tilde{\gamma}(0)\Big) + \Big(\alpha\bar{t} + (1-\alpha)\underline{\tilde{t}}\Big)v \in \mathrm{ri}\,(F_{\gamma(0)}) = \mathrm{ri}\,(F_{\tilde{t}})$$

If $\exists\alpha \in [0,1]$ such that

$$\alpha\bar{t} + (1-\alpha)\underline{\tilde{t}} = 0$$

then we would have $\alpha\gamma(0) + (1-\alpha)\tilde{\gamma}(0) \in \mathrm{ri}\,(F_{\gamma(0)})$ and $\alpha\gamma(0) + (1-\alpha)\tilde{\gamma}(0) \in \mathrm{ri}\,(F_{\tilde{\gamma}(0)})$, which is impossible since $F_{\gamma(0)}$,$F_{\tilde{\gamma}(0)}$ are distinct faces and by Proposition 2.1.13, we must have $\mathrm{ri}\,(F_{\gamma(0)}) \cap \mathrm{ri}\,(F_{\tilde{\gamma}(0)})$ disjoint. Consider $\alpha = \frac{-\underline{\tilde{t}}}{\bar{t}-\underline{\tilde{t}}}$. Since $-\underline{\tilde{t}} > 0$ and $\bar{t} - \underline{\tilde{t}} > 0$, we indeed have $\alpha \in [0,1]$. ☐

Lemma 3.1.19. *Let F be a glide-only face. Then we must have*

i $F^+ \cap \mathrm{bd}\,(F^-) = \emptyset$

ii $F^- \cap \mathrm{bd}\,(F^+) = \emptyset$

Proof. Suppose not, i.e. suppose that there exists $y \in F^+ \cap \mathrm{bd}\,(F^-)$. Then, there exists a generalized characteristic path γ with $\gamma(0) \in \mathrm{ri}\,(F)$ and there exists a face $F^+_{\gamma(0)}$ with $y \in F^+_{\gamma(0)}$. Additionally, there exists a a generalized characteristic path $\tilde{\gamma}$ with $\tilde{\gamma}(0) \in \mathrm{ri}\,(F)$ and a face $F^-_{\tilde{\gamma}(0)}$ with $y \in \mathrm{rb}\,(F^-_{\tilde{t}_0})$.

Take $\epsilon > 0$ so that $B_\epsilon(\gamma(0)) \cap \mathrm{Aff}\,(F) \subseteq F$. Then, by $y \in \mathrm{rb}\,(F^-_{\tilde{\gamma}(0)})$, we know that $\exists x \in \mathrm{ri}\,(F^-_{\tilde{\gamma}(0)})$ such that $x \in B_\epsilon(y) \cap \mathrm{rb}\,(F)$ (where $B_\epsilon(y)$ is an open ball in $\mathbb{R}^{2n}$). Consider the isometry ϕ on $\mathbb{R}^{2n}$ given by a translation:

$$\phi(w) = w - \bar{t}Jv$$

Since $y \in F^+_{\gamma(0)}$, we know $y = (\gamma\bar{t})$ and so

$$\phi(y) = \gamma(\bar{t}) - \bar{t}Jv = \gamma(0)$$

and

$$\phi(x) = \tilde{\gamma}(\underline{\tilde{t}}) - \bar{t}Jv$$

Since ϕ is an isometry and $d(x, y) < \epsilon$, we must have that $d\Big(\phi(y), \phi(x)\Big) = d\Big(\gamma(t_0), \phi(x)\Big) < \epsilon$, which implies that $\phi(x) \in \mathrm{ri}\,(F)$. Since translations of absolutely continuous functions by a vector are again absolutely continuous, the function $\tilde{\phi}(t) := \phi(x) + tJv$ is a generalized characteristic path with $\tilde{\phi}(0) = \phi(x) \in \mathrm{ri}\,(F)$. However, we have $\tilde{\phi}(\overline{t_0}) = x \in \mathrm{ri}\,(F^-_{\tilde{\gamma}(0)})$, which implies that $x \in F^+$ and $x \in F^-$, and by Proposition3.1.18, this is impossible.
By a symmetric argument, we get (ii). □

Lemma 3.1.20. *Let F be a glide-only face and let $d(p, q)$ be the Euclidean metric on $\mathbb{R}^{2n}$. If $x \in F^+$, then the quantity*

$$d_x := \inf\{d(x, y) \ : \ y \in \overline{F^-}\}$$

exists and is nonzero. Likewise, if $\tilde{x} \in F^-$, then the quantity

$$d_{\tilde{x}} := \inf\{d(\tilde{x}, y) \ : \ y \in \overline{F^+}\}$$

exists and is nonzero.

Proof. First we will show that $x \notin \overline{F^+}$. By Proposition 3.1.18, we know that $x \notin F^+$. By Lemma 3.1.19, we know that $x \notin \mathrm{bd}\,(F^+)$. Since $x \notin F^+$ and $x \notin \mathrm{bd}\,(F^+)$, then by

Proposition 3.1.17, we know that $x \notin \overline{F^+}$ and so $d_x \neq 0$. Since $\overline{F^+}$ is a compact set in $\mathbb{R}^{2n}$ and $d(x, \cdot)$ is a continuous function, then by the extreme value theorem the infimum must be attained. By a similar argument, for $\tilde{x} \in F^-$, we have that $d_{\tilde{x}}$ exists and is nonzero. $\square$

Now we prove a theorem that shows the impossibility of a pathological generalized characteristic which begins in F^+ for some glide-only face F, but which returns to $\mathrm{ri}\,(F)$ infinitely often and in arbitrarily small amounts of forward time.

Lemma 3.1.21. *Let γ be a generalized characteristic and let F be a glide-only face. If $\gamma(0) \in \mathrm{ri}\,(F)$ then $\exists \overline{\epsilon} > 0$ such that*

$$t_0 \in (\overline{t}, \overline{t} + \overline{\epsilon}) \implies \gamma(t_0) \notin \mathrm{ri}\,(F)$$

and $\exists \underline{\epsilon} > 0$ such that

$$t_0 \in (\underline{t} - \underline{\epsilon}, \underline{t}) \implies \gamma(t_0) \notin \mathrm{ri}\,(F)$$

Proof. Suppose that there does not exist such an $\overline{\epsilon} > 0$. Consider $\{\epsilon_n\}_{n=1}^{\infty} > 0$ with $\epsilon_n \downarrow 0$. Then, by assumption for each $n \in \mathbb{N}$, $\exists \delta_n > 0$ such that $\delta_n < \epsilon_n$ and $\gamma(\overline{a} + \delta_n) \in \mathrm{ri}\,(F)$. Consider $\underline{\overline{t} + \delta_n} < 0$ from Definition 3.1.7. We then have that either

$$|\underline{\overline{t} + \delta_n}| > \delta_n \text{ or } |\underline{\overline{t} + \delta_n}| \leq \delta_n$$

Suppose that ">" is true; we would then have that $-\delta_n > \underline{\overline{t} + \delta_n}$. However, $-\delta_n \in \{t < 0 \ : \ \gamma(\overline{t} + \delta_n + t) \notin \mathrm{ri}\,(F)\}$ and by Definition 3.1.7, we must have $-\delta_n \leq \underline{\overline{t} + \delta_n}$. Then, $|\underline{\overline{t} + \delta_n}| \leq \delta_n$ for all n, and so as $n \to \infty$, we have that $|\underline{\overline{t} + \delta_n}| \downarrow 0$. Then, by continuity of γ, we must have that

$$\gamma\Big(\overline{t} + \delta_n + \underline{\overline{t} + \delta_n}\Big) \to \gamma(\overline{t}) \in F^+_{\gamma(0)}$$

Consider

$$d_{\gamma(\overline{t})} := \inf\{d(\gamma(\overline{t}, y) \ : \ y \in \overline{F^-}\}$$

which exists and is nonzero by Lemma 3.1.20. If we take N large enough so that $\delta_N + \underline{\bar{t} + \delta_n} < d$, then by γ Lipschitz, we have

$$\begin{aligned} ||\gamma\left(\bar{t} + \delta_n + \underline{\bar{t} + \delta_n}\right) - \gamma(\bar{t})|| &< \delta_N + \underline{\bar{t} + \delta_n} \\ &< d \end{aligned} \tag{3.1.1}$$

However, $\gamma\left(\bar{t} + \delta_n + \underline{\bar{t} + \delta_n}\right) \in F^-$ and Eqn 3.1.1 violates that d is a minimum.
By a symmetric argument, we get the existence of $\underline{\epsilon} > 0$. $\square$

The previous lemma allows us to demonstrate that, from a facet F_1 into the interior of a $(2n-2)$-dimensional face F, characteristics will bounce into the "next" facet, F_2.

Theorem 3.1.22. *Let F be a $(2n-2)$-face which is bounce-only and such that $F = F_1 \cap F_2$, where F_1 and F_2 are distinct facets. If $F \subseteq F_1^+$ and γ is a generalized characteristic path with $\gamma(0) \in \mathrm{ri}\,(F_1)$ and $\gamma(\bar{t}) \in \mathrm{ri}\,(F)$, then, $\exists \delta > 0$ such that*

$$\mathrm{Im}\,(\gamma)|_{(\bar{t},\ \bar{t}+\delta)} \subseteq \mathrm{ri}\,(F_2)$$

Proof. Let $x \in \mathrm{ri}\,(F)$. Take $\epsilon = \min\{\bar{\epsilon}, \epsilon^*\}$, where $\bar{\epsilon}$ is the quantity from Lemma 3.1.21 and $\epsilon^* > 0$ is such that $B_{\epsilon^*}(x) \cap (F \cup F_2) \subseteq \mathrm{ri}\,(F) \cup \mathrm{ri}\,(F_2)$. Suppose first that

$$\mathrm{Im}\,(\gamma)|_{(\bar{t},\bar{t}+\epsilon)} \subseteq \mathrm{ri}\,(F)$$

However, since F is bounce-only, then by Proposition 3.1.12, this is impossible. Additionally, by Lemma 3.1.21, we have that $\mathrm{Im}\,(\gamma) \cap \mathrm{ri}\,(F_1) = \emptyset$, so there must exist $t_0 \in (\bar{t},\ \bar{t}+\epsilon)$ such that $\gamma(t_0) \in \mathrm{ri}\,(F_2)$. We can write t_0 as $t_0 = \bar{t} + \delta$ for some $0 < \delta < \epsilon$, and by F_2 glide-only, we know $\exists \underline{t_0} < 0$ such that $\gamma(t_0 + \underline{t_0}) \notin \mathrm{ri}\,(F_2)$. If $|\underline{t_0}| = \delta$, then $\gamma(\tilde{t}) \in \mathrm{ri}\,(F_2)$ for values of $\tilde{t}$:

$$\begin{aligned}\tilde{t} \in (t_0 + \underline{t_0}, t_0) &\iff \tilde{t} \in (\bar{t} + \delta - \delta, \bar{t} + \delta) \\ &\iff \tilde{t} \in (\bar{t}, \bar{t} + \delta)\end{aligned}$$

as desired. Therefore, we want to show that $|\underline{t_0}| = \delta$, i.e. that $|\underline{t_0}| \leq \delta$ and that $|\underline{t_0}| \geq \delta$. By $\underline{t_0}$ a least upper bound, we have that $\delta \geq |\underline{t_0}|$. It remains to show that $\delta \leq |\underline{t_0}|$.
Suppose not, i.e. that $\delta > |\underline{t_0}|$. Take $\eta = \min\{\delta - |\underline{t_0}|, \epsilon_{\underline{t_0}}\}$, where $\epsilon_{\underline{t_0}} > 0$ is the quantity from Lemma 3.1.21, given by the intial condition $\gamma(t_0) \in \mathrm{ri}\,(F_2)$. Then, by Lemma 3.1.21, we have that $\mathrm{Im}\,(\gamma)|_{(\bar{t},\ \bar{t}+\eta)} \cap \mathrm{ri}\,(F_2) = \emptyset$, and therefore we must have $\mathrm{Im}\,(\gamma)|_{(\bar{t},\ \bar{t}+\eta)} \subseteq \mathrm{ri}\,(F)$. However, by F bounce-only, γ cannot be contained in $\mathrm{ri}\,(F)$ for any interval of time.
Therefore, we must have that $\delta = |\underline{t_0}|$, and so $\mathrm{Im}\,(\gamma)|_{(\bar{t},\ \bar{t}+\delta)} \subseteq \mathrm{ri}\,(F_2)$. $\square$

The following theorem gives the necessary and sufficient conditions for a generalized characteristic to bounce into or out of a face F which lies on the relative boundary of a glide-only face F^g. The theorem admits a geometric realization involving certain supporting hyperplanes of the relevant faces' conjugate faces, and this interpretation will be used later in determining the bouncing into/out of edges in a generic 4-polytope (see Lemma 3.2.14).

Theorem 3.1.23. *Let $F = \bigcap_{g \in G} F^g$ be a face of a polytope Σ, where each F^g is a glide-only face of Σ via g. Then,*

i $F \subseteq (F^g)^+$ if and only if $v \cdot g > 0$ for all $v \in N_\Sigma(F) \backslash N_\Sigma(F^g)$.

ii $F \subseteq (F^g)^-$ if and only if $v \cdot g < 0$ for all $v \in N_\Sigma(F) \backslash N_\Sigma(F^g)$.

Proof. "$\implies$"
Suppose not, i.e. suppose that there exists a $v \in N_\Sigma(F) \backslash N_\Sigma(F^g)$ such that $v \cdot g \leq 0$. By Cor 2.3.4, $N_\Sigma(F) = \mathrm{Span}^+(S_F)$, so for the case of $v \cdot g = 0$, we must have that either $s \cdot g = 0$ for all $s \in S_F \backslash S_{F^g}$ or that $\exists \tilde{s} \in S_F \backslash S_{F^g}$ such that $\tilde{s} \cdot g < 0$. For the case of $v \cdot g < 0$, we would also have that $\exists \tilde{s} \in S_F \backslash S_{F^g}$ such that $\tilde{s} \cdot g < 0$. So, it suffices to show the impossibility of having

$s \cdot g = 0$ for all $s \in S_F \backslash S_{F^g}$ and the impossibility of there existing such an $\tilde{s} \in S_F \backslash S_{F^g}$.

Suppose that $s \cdot g = 0$ for all $s \in S_F \backslash S_{F^g}$. By definition of $(F^g)^+$, for any $y \in F$ there exists a generalized characteristic path γ such that $\gamma(0) \in \mathrm{ri}\,(F^g)$ and $\gamma(\bar{t}) = y$. By F^g glide-only via g, we have that, for $\epsilon \in (0, \bar{t})$:

$$\gamma(\epsilon) = \gamma(0) + \epsilon g$$

and $\gamma(\epsilon) \in \mathrm{ri}\,(F^g)$. However, $s \cdot g = 0$ implies that

$$\gamma(\epsilon) \cdot s = 1 \qquad \forall s \in S_F \backslash S_{F^g}$$

but by Proposition 2.1.18, this contradicts that $\gamma(\epsilon) \in \mathrm{ri}\,(F^g)$.

Now we show that the case in which $\exists \tilde{s} \in S_F \backslash S_{F^g}$ such that $\tilde{s} \cdot g < 0$ is impossible. Consider y, γ as in the previous case. By $\tilde{s} \cdot g < 0$, by Proposition 2.1.18 and since $\bar{t} > 0$, we must have that $y \cdot \tilde{s} < 1$. However, by Proposition 2.1.18 this contradicts that $y \in F$.

"$\Longleftarrow$"

Take $x \in \mathrm{ri}\,(F)$. We want to show that there exists an $\epsilon > 0$ such that for $t \in (0, \epsilon)$, we have $x - tg \in \mathrm{ri}\,(F^g)$. By Proposition 2.1.18 we know that $x \cdot s = 1$ $\forall s \in S_F$ and $x \cdot \tilde{s} < 1$ $\forall \tilde{s} \in S_\Sigma \backslash S_F$. First, we show that if a quantity $t > 0$ is such that

$$t < \max \left\{ \left| \frac{x \cdot \tilde{s} - 1}{g \cdot \tilde{s}} \right| \ : \ \tilde{s} \in S_\Sigma \backslash S_{F^g}, \ g \cdot s \neq 0 \right\}$$

then we have that $(x - tg) \cdot \tilde{s} < 1$ $\forall \tilde{s} \in S_\Sigma \backslash S_{F^g}$. We consider each possibility of the sign of $g \cdot \tilde{s}$, for $\tilde{s} \in S_\Sigma \backslash S_{F^g}$.

Let $\tilde{s} \in S_\Sigma \backslash S_{F^g}$ be such that $g \cdot \tilde{s} = 0$. By hypothesis, we have $\tilde{s} \in S_\Sigma \backslash S_F$ and by $x \in \mathrm{ri}\,(F)$ we have $x \cdot \tilde{s} < 1$. Then, for any value of $\epsilon > 0$ and any $t \in (0, \epsilon)$, we must have $(x - tg) \cdot \tilde{s} = x \cdot \tilde{s} < 1$.

Now, let $\tilde{s} \in S_\Sigma \backslash S_{F^g}$ be such that $g \cdot \tilde{s} < 0$. Then, we see that

$$\begin{aligned}(x - tg) \cdot \tilde{s} = x \cdot \tilde{s} - tg \cdot \tilde{s} < 1 \\ -tg \cdot \tilde{s} < 1 - x \cdot \tilde{s} \\ t < \frac{x \cdot \tilde{s} - 1}{g \cdot \tilde{s}}\end{aligned}$$

and since $x \cdot \tilde{s} - 1 < 0$, we see that $\frac{x \cdot \tilde{s} - 1}{g \cdot \tilde{s}} > 0$, so for any $t \in [0, \frac{x \cdot \tilde{s} - 1}{g \cdot \tilde{s}})$, we have that $(x - tg) \cdot \tilde{s} < 1$.

Finally, let $\tilde{s} \in S_\Sigma \backslash S_{F^g}$ be such that $g \cdot \tilde{s} > 0$. Then, we see that

$$\begin{aligned}x \cdot \tilde{s} - tg \cdot \tilde{s} < 1 \\ -tg \cdot \tilde{s} < 1 - x \cdot \tilde{s} \\ t > \frac{x \cdot \tilde{s} - 1}{g \cdot \tilde{s}}\end{aligned}$$

Since $\frac{x \cdot \tilde{s} - 1}{g \cdot \tilde{s}} > 0$, then for any $t > 0$, we have that $(x - tg) \cdot \tilde{s} < 1$.

So, if we let

$$\epsilon := \max \left\{ \left| \frac{x \cdot s - 1}{g \cdot s} \right| \ : \ s \in S_\Sigma \backslash S_{F^g}, \ g \cdot s \neq 0 \right\}$$

then, for $t \in (0, \epsilon)$, we have that

$$\begin{aligned}(x - tg) \cdot s < 1 \qquad \forall s \in S_\Sigma \backslash S_{F^g} \\ (x - tg) \cdot s = 1 \qquad \forall s \in S_{F^g}\end{aligned}$$

which implies that, for $t \in (0, \epsilon)$, we have $x - tg \in \mathrm{ri}\,(F^g)$. □

3.2 Flow on a Symplectic-Faced 4-Polytope

For any point x in a smooth energy surface, the IVP existence/uniqueness coming from the characteristic equation guarantees the existence and uniqueness of a characteristic curve which intersects x. For a polytope Σ, however, we may have even-dimensional faces, such as the 2-faces of the hypercube, for which there is not a unique characteristic direction. On such faces, we have a 1-dimensional family of characteristic directions which makes uniqueness of characteristics impossible.

However, we can restrict our attention to the family of 4-polytopes for which even-dimensional faces are bounce-only. Making this assumption, Theorem 3.1.14, Theorem 3.1.22 and Theorem 3.1.23 give us the existence and uniqueness of characteristics on facets and 2-faces. All that remains is to determine the existence and uniqueness of characteristics which intersect 1-faces (edges) and 0-faces (vertices).

3.2.1 Genericity of Symplectic-Faced Polyhedral Sets

We will now proceed to define a class of polyhedral sets in $\mathbb{R}^4$, for which we can prove the existence and uniqueness of generalized characteristics.

Definition 3.2.1. *A polyhedral set $\Sigma \subset \mathbb{R}^{2n}$ is **symplectic-faced** if for every $2k$-face F of Σ with $0 < k < n$, we have that F is bounce-only.*

Proposition 3.2.2. *Let $\Sigma \subset \mathbb{R}^4$ be a polyhedral set. Then, Σ is symplectic-faced if and only if for every 2-face $F_s \cap F_{\tilde{s}}$ of Σ, we have $Js \cdot \tilde{s} \neq 0$.*

Proof. For a polyhedral set $\Sigma \subseteq \mathbb{R}^4$, Σ is symplectic-faced if and only if every $2k$-face F of Σ is bounce-only, for $0 < k < 2$. Then, Σ is symplectic-faced if and only if every 2-face is bounce-only. Let $F = F_s \cap F_{\tilde{s}}$ be a 2-face of Σ. Note that, by Corollary 2.3.4, we have

$JN_\Sigma(F) = \mathrm{Span}^+ (\{Js, J\tilde{s}\})$ and so

$$\begin{aligned} F \text{ bounce-only } &\iff JN_\Sigma(F) \cap N_\Sigma^\perp(F) = \langle 0 \rangle \\ &\iff \text{there does not exist a nonzero } v \in JN_\Sigma(F) \cap N_\Sigma^\perp(F) \\ &\iff \forall k_1, k_2 \geq 0, \text{ we have } (k_1 Js + k_2 J\tilde{s}) \cdot s \neq 0, (k_1 Js + k_2 J\tilde{s}) \cdot \tilde{s} \neq 0 \\ &\iff Js \cdot \tilde{s} \neq 0 \text{ and } J\tilde{s} \cdot s \neq 0 \end{aligned}$$

as desired. □

Definition 3.2.3. *We denote the* ***collection of polyhedral generating sets with size*** *m to be the set*

$$\tilde{\mathcal{P}}_m^4 = \{S_\Sigma \subset \mathbb{R}^4 \ : \ \Sigma \text{ a polyhedral set }, \ |S_\Sigma| = m\}$$

and we denote the collection of S_Σ with $|S_\Sigma| = m$ and Σ a polytope by $\mathcal{P}_m^4 \subset \tilde{\mathcal{P}}_m^4$.

In order to prove our genericity result, we will be considering a particular function which computes the dot product of pairs of potential normal vectors of a polyhedral set. This function will be defined on pairs of factors in a product of Euclidean spaces. However, in order to establish a association between generating sets and a list of vectors, we must ensure that each list of m vectors is a set of m vectors. Therefore, from the product of Euclidean spaces, we must define and delete a closed subspace $\mathcal{D}_m^n$ which accounts for ordered lists of vectors with repeated entries.

Definition 3.2.4. *The* ***fat diagonal*** *$\mathcal{D}_m^n$ of $\prod^m \mathbb{R}^n$ is the following closed subspace:*

$$\mathcal{D}_m^n = \left\{(v_1, v_2, \ldots, v_m) \in \prod^m \mathbb{R}^n \ : \ v_i = v_j \text{ for at least one pair of } i, j \text{ with } i \neq j\right\}$$

Once we have defined the set of ordered lists of normal vectors, we can then take a quotient by permutations of these lists to obtain a collection of unordered normal vectors. Note that the following space $\tilde{\mathbb{R}}^4_m$ is an example of a symmetric product and is metrizable, by a paper from K. Borsuk and S. Ulam[5].

Definition 3.2.5. *Let S_m be the symmetric group on m letters. Then we define the topological space*

$$\tilde{\mathbb{R}}^4_m = \Big((\textstyle\prod_{i=1}^m \mathbb{R}^4_i) \backslash \mathcal{D}^4_m \Big) \big/ S_m \tag{3.2.1}$$

Where $\prod_{i=1}^m \mathbb{R}^4_i$ is equipped with the product topology and $\tilde{\mathbb{R}}^4_m$ is equipped with the quotient topology.

Proposition 3.2.6. *The quotient map*

$$q : (\prod_{i=1}^m \mathbb{R}^4_i) \backslash \mathcal{D}^4_m \to \tilde{\mathbb{R}}^4_m$$

is open.

Proof. Let $X := (\prod_{i=1}^m \mathbb{R}^4)$ and let $\tilde{q} : X \to X \big/ S_m$ be the induced quotient map. Note that for any $Y \subset \tilde{\mathbb{R}}^4_m$, we have $q^{-1}(Y) = \tilde{q}^{-1}(Y) \cap (\mathcal{D}^4_m)^c$. Since $(\mathcal{D}^4_m)^c$ is open, it suffices to show that $\tilde{q}$ is an open map. Let $U \subset X$ be open. Then, we know that $U = \prod_{i=1}^m U_i$ such that each U_i is open in $\mathbb{R}^4_i$. By definition of $\tilde{q}$, we know that

$$\tilde{q}(U) = \bigcup_{x \in U} \{[x]\}$$

We will first show that $\bigcup_{x \in U} \{[x]\}$ is open in $X \big/ S_m$ by showing that $\tilde{q}^{-1}\big(\bigcup_{x \in U} \{[x]\}\big)$ is open in X. By definition of the quotient map under the action of S_m, we know that

$$\tilde{q}^{-1}\big(\bigcup_{x \in U} \{[x]\}\big) = \bigcup_{\sigma \in S_m} \sigma(U)$$

We now show that $\sigma(U)$ is open in X for each $\sigma \in S_m$. Let $\sigma \in S_m$. Since each factor in the product space X is homemorphic to $\mathbb{R}^4$, and since each U_i is open in $\mathbb{R}^4_i$, we know that $U_{\sigma(i)}$ is also open in $\mathbb{R}^4_{\sigma(i)}$. Therefore, since $\sigma(U) = \prod_{i=1}^m U_{\sigma(i)}$, we know that $\sigma(U)$ is open in X.

Then we have that $\tilde{q}(U)$ is open in $X\big/_{S_m}$.

Let $U \subset \tilde{\mathbb{R}}^4_m$ be open. Since $q^{-1}(U) = \tilde{q}^{-1}(U) \cap (\mathcal{D}^4_m)^c$, then we have that $q^{-1}(U)$. Therefore, we have that q is an open map. $\square$

We will now formally establish our identification of polyhedral generating sets with elements of $\tilde{\mathbb{R}}^4_m$.

Proposition 3.2.7. *There exists a bijective map $\phi : \tilde{\mathcal{P}}^4_m \to \tilde{\mathbb{R}}^4_m$.*

Proof. Let $\Sigma \in \tilde{\mathcal{P}}^4_m$. Then, we can define the function $\phi : \tilde{\mathcal{P}}^4_m \to \tilde{\mathbb{R}}^4_m$ given by $S_\Sigma \mapsto [(s)]_{s \in S_\Sigma}$. First we show that ϕ is injective. Let $S_\Sigma, S_{\tilde{\Sigma}} \in \tilde{\mathcal{P}}^4_m$. Then, for some $(s_1, \dots, s_m) \in \big(\prod_{i=1}^m \mathbb{R}^4_i\big) \backslash \mathcal{D}^4_m$, we have

$$\phi(S_\Sigma) = \phi(\tilde{S}_\Sigma) = \{\sigma\big(s_1, \dots, s_m\big)\}_{\sigma \in S_m}$$

By definition of ϕ, we must have $S_\Sigma = S_{\tilde{\Sigma}} = \{s_1, \dots, s_m\}$, so we have injectivity. Now we show surjectivity. Let $[(s_1, \dots, s_m)] \in \tilde{\mathbb{R}}^4_m$, for some $(s_1, \dots, s_m) \in \big(\prod_{i=1}^m \mathbb{R}^4_i\big) \backslash \mathcal{D}^4_m$. Since the list $(s_1, \dots, s_m)$ contains distinct vectors from $\mathbb{R}^4$, then we have that $\{s_1, \dots, s_m\}$ is a set of m vectors from $\mathbb{R}^4$. Therefore, for the set S, the intersection $\cap_{s \in S} K(s, 1)$ is a polyhedral set in $\mathbb{R}^4$ with generating set S, and we have $\phi(S) = [(s_1, \dots, s_m)]$.

$\square$

Lemma 3.2.8. *Let $\tilde{P}^4_m := \phi(\tilde{\mathcal{P}}^4_m)$. Then, there exists an open set $\mathcal{F} \subseteq \tilde{P}^4_m$ such that for every $S_\Sigma \in \tilde{\mathcal{F}}$, we have that Σ is symplectic-faced.*

Proof. First we construct $\mathcal{F}$. Since being symplectic-faced checks a J-orthogonality condition on 2-faces of a polyhedral set Σ, and since 2-faces are generated by a pair $s_i, s_j \in S_\Sigma$, we will consider the collection of index-pairs in order to label pairs of normal vectors. Define

$$N^2 := \{(i,j) \in \{1, 2, \dots, m\}^2 \ : i \neq j\}$$

and let $(i,j) \in N^2$. To check for J-orthogonality, we define the continuous function $f_{(i,j)} : \mathbb{R}^4_i \times \mathbb{R}^4_j \to \mathbb{R}$ given by

$$f_{(i,j)}(s_i, s_j) = s_i \cdot Js_j$$

Since each $f_{(i,j)}$ is continuous, we know that for any $(i,j) \in N^2$, the set $f^{-1}_{(i,j)}(\{0\}^c) = U_i^{(i,j)} \times U_j^{(i,j)}$ is open in $(\mathbb{R}^4_i \times \mathbb{R}^4_j) \backslash \mathcal{D}^4_2$. Additionally, since N^2 is finite, we know that

$$\tilde{\mathcal{F}} := \prod_{k=1}^{m} \big(\cap_{(i,j)\in N^2} U_k^{(i,j)} \big)$$

is open in $(\prod_{i=1}^m \mathbb{R}^4) \backslash \mathcal{D}^4_m$. By Proposition 3.2.6, we know that $\mathcal{F} := q(\tilde{\mathcal{F}})$ is open in $\tilde{\mathbb{R}}^4_m$. Now, let $S_\Sigma \in \tilde{\mathcal{F}}$ and let $N \subset N^2$ be the set of (i,j) such that for the facets F_{s_i}, F_{s_j} of Σ, we have that $F_{s_i} \cap F_{s_j}$ is a 2-face. Then, for any $(i,j) \in N$, by definition of $\tilde{\mathcal{F}}$ we have $(s_i, s_j) \in f^{-1}(\{0\}^c)$. By Proposition 3.2.2, this implies that Σ is symplectic-faced, as desired. $\square$

Theorem 3.2.9. *There exists an open and dense set $\mathcal{F} \subset \tilde{P}^4_m$ such that for every $S_\Sigma \in \mathcal{F}$ we have that Σ is symplectic-faced.*

Proof. By Lemma 3.2.8, we know that the set $\mathcal{F}$ is open in $\tilde{P}^4_m$. Now it remains to show that $\mathcal{F}$ is dense in $\tilde{P}^4_m$. Consider the collection N^2 and the function $f_{(i,j)}$ from the proof of Lemma 3.2.8. We have that each $f^{-1}_{(i,j)}(\{0\}^c)$ is open and dense in $(\mathbb{R}^4_i \times \mathbb{R}^4_j) \backslash \mathcal{D}^4_2$ and since N^2 is a finite set, then we have that $\tilde{\mathcal{F}}$ is dense in $\prod_{i=1}^m \mathbb{R}^4 \backslash \mathcal{D}^m_4$.
By Proposition 3.2.6, we have that $q(\tilde{\mathcal{F}})$ is open in $\tilde{\mathbb{R}}^4_m$. Additionally, since q is surjective and $\tilde{\mathcal{F}}$ is dense in $\prod_{i=1}^m \mathbb{R}^4_i \backslash \mathcal{D}^4_m$, we have that

$$\begin{aligned} q(\overline{\tilde{\mathcal{F}}}) &= q\Big(\prod_{i=1}^m \mathbb{R}^4_i \backslash \mathcal{D}^4_m \Big) \\ &= \tilde{P}^4_m \end{aligned}$$

This implies that $\tilde{P}^4_m \subseteq \overline{q(\tilde{\mathcal{F}})} = \overline{\mathcal{F}}$, and so we have that $\mathcal{F}$ is dense in $\tilde{P}^4_m$. $\square$

3.2.2 Edges of a Symplectic-Faced 4-Polytope

Edges are the first type of face we've encountered with a characteristic dimension that allows either bounce-only or glide-only behavior. In fact, the cross-polytope in $\mathbb{R}^4$ is an example of a polytope which contains both bounce-only edges and glide-only edges. Due to edges being one-dimensional, a characteristic trajectory in a glide-only edge is constrained to simply flow from an initial vertex to a terminal vertex. For the case of a bounce-only edge however, we must show the less obvious fact that there exists a unique facet from which we bounce into the edge, followed by a bounce out to a unique facet.

Definition 3.2.10. *In a 4-polytope Σ, we call a 0-face of Σ a* ***vertex****, a 1-face of Σ an* ***edge****, a 2-face of Σ a* ***face*** *and a 3-face of Σ a* ***facet****.*

Proposition 3.2.11. *An edge E of a $2n$-polytope is either bounce-only or glide-only.*

Proof. By Proposition 2.3.7, we must have that $\dim\left(N_\Sigma(E)^\perp\right) = 1$ and so $\dim \operatorname{Char}(E) \leq 1$. By Proposition 3.1.12 and 3.1.13, we then have that E can only be either glide-only or bounce-only. $\square$

By Theorem 3.1.23, showing that a characteristic must flow from a particular facet into an edge requires a geometric argument involving affine spaces intersecting the conjugate face of an edge. The conjugate face of an edge will be a polygon in a three-dimensional space, and in the future we will consider the affine plane containing this polygon.

Proposition 3.2.12. *Let E be an edge of a 4-polytope Σ. Then, there exists an affine plane $P_E \subseteq \mathbb{R}^4$ such that* $\operatorname{Aff}(\operatorname{Conv}(S_E)) = P_E$*.*

Proof. By Theorem 2.2.8, we know that $E^\Delta = \operatorname{Conv}(S_E)$ and by Theorem 2.2.7, we then have that $\dim(\operatorname{Conv}(S_E)) = 2$. Since $\operatorname{Conv}(S_E)$ is 2-dimensional, its affine hull must be an affine plane in $\mathbb{R}^4$. $\square$

In our geometric representation of Theorem 3.1.23 as applied to bounce-only edges, we will investigate hyperplanes which intersect $N_\Sigma^\perp(E)$ and the plane P_E from Proposition 3.2.12.

For bounce-only edges, the following proposition will force the line $N_\Sigma^\perp(E)$ and the plane P_E to both lie in the ambient three-dimensional space $\langle JN_\Sigma(E)\rangle$, which we will later show to force intersection between $N_\Sigma^\perp(E)$ and P_E.

Proposition 3.2.13. *If E is an edge of a 4-polytope Σ, then we have $N_\Sigma^\perp(E) \subseteq \langle JN_\Sigma(E)\rangle$.*

Proof. Since $\dim\Big(N_\Sigma(E)^\perp\Big) = 1$ and $\dim\Big(JN_\Sigma(E)\Big) = 3$, we cannot have "$\supseteq$". Since $N_\Sigma(E)^\perp$ and $\langle JN_\Sigma(E)\rangle$ are both vector spaces, if there exists a nonzero $v \in N_\Sigma(E)^\perp \cap \langle JN_\Sigma(E)\rangle$, then $\langle v\rangle = N_\Sigma(E)^\perp$ and we have proven our claim.

By Proposition 2.3.7, we have that $\dim(JN_\Sigma(E)) = 3$ and so $m := |S_E| \geq 3$. Suppose that m is odd. Then, if we enumerate elements of $S_E = \{s_i\}_{1\leq i\leq m}$ we can construct an $m \times m$ skew-symmetric matrix $M = [s_i \cdot Js_j]$. An odd-dimensional skew-symmetric matrix must have a nontrivial kernel, i.e. there must exist a nonzero $m \times 1$ vector $[k_\ell]_{1\leq \ell\leq m}$ such that $Mk = 0$. Therefore, for every $i \in \{1, \ldots, m\}$, we have $s_i \cdot \sum_{j=1}^m k_j Js_j = 0$ which implies that $v := \sum_{j=1}^m k_j Js_j$ is our desired vector.

Now suppose that $m > 3$ and is even. Since E is an edge, we know that $\operatorname{rank}\mathcal{E} = 3$ and S_E is linearly dependent. Then, $\exists \check{s} \in S_E$ such that for $\check{S}_E := S_E \backslash \{\check{s}\}$ we have $\operatorname{Span}(S_E) = \operatorname{Span}(\check{S}_E)$. If we enumerate elements of $\check{S}_E = \{\check{s}_i\}_{1\leq i\leq m-1}$ we can construct an $(m-1) \times (m-1)$ skew-symmetric matrix $\check{M} = [\check{s}_i \cdot J\check{s}_j]$. Since $m - 1$ odd, then by our previous case there must exist a nonzero $(m-1) \times 1$ vector $[\check{k}_\ell]_{1\leq \ell\leq m-1}$ such that $\check{M}\check{k} = 0$. Since $\operatorname{Span}(S_E) = \operatorname{Span}(\check{S}_E)$, then $\check{v} := \sum_{j=1}^{m-1} \check{k}_j J\check{s}_j \in \langle JN_\Sigma(E)\rangle$ is our desired vector. $\square$

We will now show the existence of vectors $\overline{s}, \underline{s}$ for which $J\overline{s}, J\underline{s}$ are the glide vectors of two (and only two) facets generating an edge E, one of which posseses a characteristic bouncing into F, and one of which posseses a characteristic bouncing out of E.

Lemma 3.2.14. *If $E = \bigcap F_i$ is a bounce-only edge of a symplectic-faced 4-polytope Σ, then $\exists!\underline{s} \in S_E$ such that $\underline{s} \cdot Js < 0$ for all $s \in S_E\backslash\{\underline{s}\}$ and $\exists!\overline{s} \in S_E$ such that $\overline{s} \cdot Js > 0$ for all $s \in S_E\backslash\{\overline{s}\}$*

Proof. By Proposition 3.2.13, we know that $N_\Sigma(E)^\perp \subseteq \langle JN_\Sigma(E)\rangle$ and by Proposition 3.2.12, we know that there exists an affine plane JP_E with $JE^\Delta \subseteq JP_E$. Since $N_\Sigma(E)^\perp$ contains the origin and JP_E does not, then we have that $N_\Sigma(E)^\perp \cap JP_E$ contains at most one point. Suppose that this intersection is empty. Then, we have that $N_\Sigma(E)^\perp$ and JP_E are parallel, i.e. that there exists a vector $v \in \langle JN_\Sigma(E)\rangle$ such that

$$N_\Sigma(E)^\perp \subseteq \{x \in \langle JN_\Sigma(E)\rangle : x \cdot v = 0\} \tag{3.2.2}$$

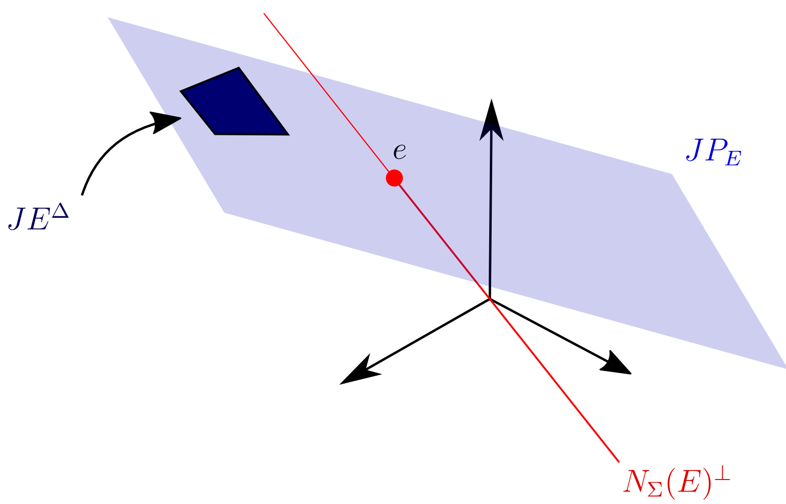
$$JP_E = \{x \in \langle JN_\Sigma(E)\rangle : x \cdot v = 1\} \tag{3.2.3}$$

Line 3.2.2 implies that $v \in N_\Sigma(E)$, i.e. $v = \sum_{S_E} k_s s$. We observe that $\frac{1}{\sum_{S_E} k_s} v \in \operatorname{Conv}(S_E)$, and by Proposition 2.2.8, we have $\frac{1}{\sum_{S_E} k_s} v \in E^\Delta$ and therefore $\frac{1}{\sum_{S_E} k_s} Jv \in JE^\Delta \subseteq JP_E$. However, by Line 3.2.3, we then have that

$$v \cdot \frac{1}{\sum_{S_E} k_s} Jv = 1$$

$$v \cdot Jv = \sum_{S_E} k_s$$

and this is impossible, since $v \cdot Jv = 0$ and $\sum_{S_E} k_s > 0$. Therefore, there must exist a vector $e \in N_\Sigma(E)^\perp \cap JP_E$.



Since E is bounce-only, we have that $JN_\Sigma(E) \cap N_\Sigma(E)^\perp = \langle 0 \rangle$. Since $JE^\Delta \subseteq JN_\Sigma(E)$, then we must have $e \in JP_E \backslash JE^\Delta$.

For $s \in S_E$, let P_s denote the plane

$$P_s := \{x \in \langle JN_\Sigma(E) \rangle \ : \ x \cdot s = 0\}$$

and since $\forall s \in S_E$, we have $Js \cdot s = 0$, then $Js \in P_s \cap JP_E$ and so the set

$$L_s := P_s \cap JP_E$$

is a line lying in the plane JP_E. Observe that, since $e \in N_\Sigma(E)^\perp$, and since for any $s \in S_E$ we have $s \cdot Js = 0$, then each L_s has $e, Js \in L_s$. If we let P_s^+, P_s^- denote the closed upper and lower halfspaces determined by P_s, respectively, then we can similarly define $L_s^+ = P_s^+ \cap JP_E$ and $L_s^- = P_s^- \cap JP_E$.

Consider the collection of half-planes of JP_E, determined by S_E:

$$\mathcal{V} := \{L_s^+, L_s^-\}_{s \in S_E}$$

We now show that there exist exactly two elements $\underline{s}, \overline{s} \in S_E$ such that for $V_{\underline{s}}, V_{\overline{s}} \in \mathcal{V}$, we have $JE^\Delta \subseteq V_{\underline{s}}$, $JE^\Delta \subseteq V_{\overline{s}}$.

First, we map JP_E to $\mathbb{R}^2$ by translating e to the origin:

$$f : JP_E \to \mathbb{R}^2$$

$$w \mapsto w - e$$

and we let $JE_e^\Delta := f(JE_e^\Delta)$. Since 0 and JE_e^Δ are disjoint, closed convex sets, then by the Hahn-Banach separation theorem, there exists $v \in \mathbb{R}^2$ such that for

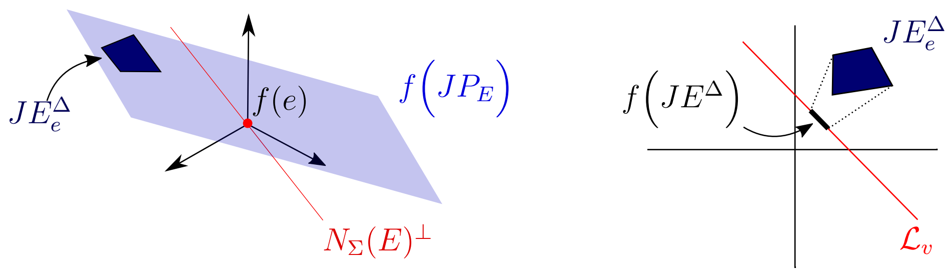
$$\mathcal{L}_v := \{w \in \mathbb{R}^2 : w \cdot v = 1\}$$

we have $0 \in \mathcal{L}_v^-$ and $JE_e^\Delta \subseteq \mathcal{L}_v^+$. Then, we can define the function

$$g : \mathbb{R}^2 \backslash \{0\} \to \mathcal{L}_v$$

given by

$$w \mapsto kw \;\; : \;\; kw \cdot v = 1$$



Observe that g is continuous, g preserves convex sets, and that the continuous image of a compact set is compact. Since $\mathcal{L}_v$ is one-dimensional, we then have that the compact, convex set $g(JE_e^\Delta)$ must be parameterized by an interval $\mathcal{I}$. Let $\beta : \mathbb{R} \to \mathcal{L}_v$ be a parameterization such that $\mathcal{I} = \beta\Big([a, b]\Big)$ for $a < b$.

Since $g(JE_e^\Delta) = \beta\Big([a, b]\Big)$, then there exists only one real number $c \in \beta(\mathcal{L}_v)$ with $g^{-1}(\beta(c)) \in JE_e^\Delta$ and such that for sufficiently large $M > 0$, $JE_e^\Delta \subseteq g^{-1}\Big(\beta\Big([c, c+N]\Big)\Big) \forall N > M$, in particular, $c = a$. Similarly, there exists only one $c \in \beta(\mathcal{L}_v)$ with $g^{-1}(\beta(c)) \in JE_e^\Delta$ and such that for sufficiently large $M > 0$, $JE^\Delta \subseteq g^{-1}\Big(\beta\Big([c-N, c]\Big)\Big) \forall N > M$, in particular, $c = b$.

Since $g^{-1}\Big(\beta(a)\Big), g^{-1}\Big(\beta(b)\Big)$ are limit points of $g^{-1}\Big(\beta\Big([a, b]\Big)\Big)$, then $g^{-1}\Big(\beta(a)\Big) \cap JE_e^\Delta \subseteq \partial JE_e^\Delta$ and $g^{-1}\Big(\beta(b)\Big) \cap JE_e^\Delta \subseteq \partial JE_e^\Delta$.

We claim that $g^{-1}\Big(\beta(a)\Big) \cap JE_e^\Delta, g^{-1}\Big(\beta(b)\Big) \cap JE_e^\Delta$ each consist of a single point, i.e. that each is a vertex of JE_e^Δ. Suppose not, i.e. that there exists $g^{-1}\Big(\beta(a)\Big) \cap JE_e^\Delta$ such that $x = \alpha f(Js_1) + (1-\alpha) f(Js_2)$ for some $\alpha \in (0, 1)$ and for some adjacent vertices $Js_1, Js_2 \in JE^\Delta$. Then, we must have that

$$f^{-1}(\langle x \rangle) \subseteq \{v \in \langle JN_\Sigma(E) \rangle \;\; : \;\; v \cdot s_1 = 0\}$$

However, this would imply that $Js_2 \cdot s_1 = 0$, which violates that Σ is symplectic-faced. Therefore, $g^{-1}\Big(\beta(a)\Big) \cap JE_e^{\Delta}$ must be a single point. By a similar argument, we get that $g^{-1}\Big(\beta(b)\Big)$ consists of one point. Therefore, we must have $f^{-1}\Big(g^{-1}\Big(\beta(a)\Big)\Big) = J\overline{s}$, $f^{-1}\Big(g^{-1}\big(\beta(b)\big)\Big) = J\underline{s}$ for some $\underline{s}, \overline{s} \in S_E$.

For the two elements $\underline{s}, \overline{s}$, we now show that either $JE^{\Delta} \subseteq L_{\underline{s}}^{-}$ and $JE^{\Delta} \subseteq L_{\overline{s}}^{+}$, or that $JE^{\Delta} \subseteq L_{\underline{s}}^{+}$ and $JE^{\Delta} \subseteq L_{\overline{s}}^{-}$. Assume not, i.e. WLOG that $JE^{\Delta} \subseteq L_{\underline{s}}^{+}$, $JE^{\Delta} \subseteq L_{\overline{s}}^{+}$. Then, we have

$$J\underline{s} \cdot \overline{s} \geq 0 \qquad\qquad J\overline{s} \cdot \underline{s} \geq 0$$

Which, by skew-symmetry of J, implies that $J\underline{s} \cdot \overline{s} = 0$. Then, for any x such that

$$x \in \{\alpha J\underline{s} + (1-\alpha)J\overline{s} : \alpha \in [0,1]\}$$

we have $x \cdot \underline{s} = 0$, which implies that $x \in L_{\underline{s}}$. By a previous argument, we know that

$$\{\alpha J\underline{s} + (1-\alpha)J\overline{s} : \alpha \in [0,1]\} \subseteq \partial JE^{\Delta}$$

and so this set is an edge of JE^{Δ}. However, this implies that the facets $F_{\underline{s}}$ and $F_{\overline{s}}$ are adjacent, and $J\underline{s} \cdot \overline{s} = 0$ violates that Σ is symplectic-faced.
So, WLOG, there must exist $\underline{s}, \overline{s} \in S_E$ such that $JE^{\Delta} \subseteq L_{\underline{s}}^{-}$ and such that $JE^{\Delta} \subseteq L_{\overline{s}}^{+}$, as desired. $\square$

Theorem 3.2.15. *Let E be a bounce-only edge of a symplectic-faced 4-polytope Σ. Then, there exists a unique $\underline{s} \in S_E$ such that $E \subset F_{\underline{s}}^{+}$ and there exists a unique $\overline{s} \in S_E$ such that $E \subset F_{\overline{s}}^{-}$.*

Proof. By Lemma 3.2.14 and Theorem 3.1.23, there exist two unique vectors $\underline{s}, \overline{s} \in S_E$, for which we have $E \subset F_{\underline{s}}^{+}$ and $E \subset F_{\overline{s}}^{-}$. $\square$

3.2.3 Vertices of a Symplectic-Faced 4-Polytope

Due to having the highest dimension conjugate face, demonstrating the existence of flow through a vertex proves to be the most challenging. In order to prove the existence of flow through a vertex, we will need to introduce some concepts from graph theory (Appendix A) and apply it to the conjugate face of a vertex (a polyhedron). Before introducing all of the technology needed in the existence argument, we will first show the uniqueness of a glide-only face which flows into a vertex. In particular, for a vertex V, we first show that if generalized characteristic paths from both a facet and an edge exist and intersect V transversely, then they cannot both intersect V in forward/backward time. This will then be generalized to give a unique glide trajectory (to/from either facet or edge) which begins/terminates in V.

Lemma 3.2.16. *Let F_s be a facet of Σ and let E^g be a glide-only edge of Σ, for which there is a vertex V of Σ such that $V \subseteq F_s \cap E^g$. Then at most one of the following is true:*

i $V \subseteq F_s^+$ (or $V \subseteq F_s^-$)

ii $V \subseteq (E^g)^+$ (resp. $V \subseteq (E^g)^-$)

Proof. Suppose both (i) and (ii) are true. Since E^g is one-dimensional, we must have either $V = F_s \cap E^g$ or $V \subseteq E^g \subseteq F_s$. First, we show that (i) and (ii) cannot both be true in the case that $V = F_s \cap E^g$.

By $V \subseteq (E^g)^+$ and by Theorem 3.1.23, we have that

$$n \cdot g > 0 \qquad \forall n \in N_\Sigma(V) \backslash N_\Sigma(E^g)$$

Since $E^g \not\subseteq F_s$ and $V \subseteq F_s$, then $s \in N_\Sigma(V) \backslash N_\Sigma(E)$. Additionally, we know that $g = J\hat{n}$ for some $\hat{n} \in N_\Sigma(E^g)$, and so

$$\begin{aligned} s \cdot J\hat{n} &> 0 \\ Js \cdot \hat{n} &< 0 \end{aligned} \tag{3.2.4}$$

By $V \subseteq F_s^+$ and by Theorem 3.1.23,

$$\tilde{n} \cdot Js > 0 \qquad \forall \tilde{n} \in N_\Sigma(V) \setminus \operatorname{Span}^+\{s\} \tag{3.2.5}$$

Since $E^g \not\subseteq F_s$, we have $N_\Sigma(E^g) \subseteq N_\Sigma(V) \setminus \operatorname{Span}^+\{s\}$ and so if we take $\tilde{n} := \hat{n}$, then by Equations 3.2.4 and 3.2.5, we have that $\tilde{n} \cdot Js > 0$ and $\tilde{n} \cdot Js < 0$, a contradiction.

Now we show that (i) and (ii) cannot both be true for the case that $E^g \subseteq F_s$. Take $x \in \operatorname{ri}(E^g), y \in \operatorname{ri}(F_s)$ such that for the generalized characteristic paths γ_x, γ_y with

$$\gamma_x(0) = x \qquad\qquad \gamma_y(0) = y$$

we have $\bar{t} := \overline{t_{\gamma_x}} = \overline{t_{\gamma_y}}$. Since V consists of a single point, $v \in V$, we have that $\gamma_x(\bar{t}) = v = \gamma_y(\bar{t})$ and so

$$\begin{aligned} x + \bar{t}g &= y + \bar{t}Js \\ x &= y + \bar{t}Js - \bar{t}g \end{aligned} \tag{3.2.6}$$

Note that the function

$$\gamma(t) := \frac{1}{2}\gamma_x(t) + \frac{1}{2}\gamma_y(t)$$

is a generalized characteristic path with $\gamma(0) \in \operatorname{ri}(F_s)$. Additionally, by Eqn 3.2.6 we have that

$$\begin{aligned}
\gamma(\bar{t}) &= y + \bar{t}Js - \frac{\bar{t}}{2}g \\
&= v - \frac{\bar{t}}{2}g \\
&= x + \bar{t}g - \frac{\bar{t}}{2}g \\
&= x + \frac{\bar{t}}{2}g
\end{aligned}$$

and we know that $x + \frac{\bar{t}}{2}g \in \operatorname{ri}(E^g)$. However, for $t \in (0, \bar{t})$, $\gamma(t)$ glides via Js, and this contradicts that γ is a generalized characteristic path with $\gamma(\bar{t}) \in \operatorname{ri}(E^g)$ and E^g glide-only via g, since $g \neq Js$. $\square$

Theorem 3.2.17. *Let V be a vertex of Σ and let $F^g, F^{\tilde{g}}$ be glide-only faces of Σ. If $V \subseteq (F^g)^+$ and $V \subseteq (F^{\tilde{g}})^+$, or if $V \subseteq (F^g)^-$ and $V \subseteq (F^{\tilde{g}})^-$, then $F^g = F^{\tilde{g}}$.*

Proof. Suppose $V \subseteq (F^g)^+$, $V \subseteq (F^{\tilde{g}})^+$ and $F^g \neq F^{\tilde{g}}$. First we consider the case that there exists at least one $s_0 \in S_{F^g} \backslash S_{F^{\tilde{g}}}$ and at least one $\tilde{s}_0 \in S_{F^{\tilde{g}}} \backslash S_{F^g}$ (i.e. $S_{F^g} \not\subset S_{F^{\tilde{g}}}$ and $S_{F^g} \not\subset S_{F^{\tilde{g}}}$). Take $x \in \operatorname{ri}(F^g)$, $y \in \operatorname{ri}(F^{\tilde{g}})$ such that for the generalized characteristic paths γ_x, γ_y with

$$\gamma_x(0) = x \qquad\qquad \gamma_y(0) = y$$

we have $\bar{t} := \overline{t_{\gamma_x}} = \overline{t_{\gamma_y}}$. Then,

$$\gamma_x(\bar{t}) = v = \gamma_y(\bar{t}) \qquad \text{where } V = \{v\}$$

Then

$$
\begin{aligned}
v \cdot \tilde{s} &= 1 \quad \forall \tilde{s} \in S_{F^{\tilde{g}}} \backslash S_{F^g} \\
(x + \bar{t}g - \bar{t}\tilde{g}) \cdot \tilde{s} &= 1
\end{aligned}
$$

since $x \cdot \tilde{s} < 1$, since $g = Jn$ for some $n \in N_\Sigma(F^g)$, and since $\tilde{g} = J\tilde{n}$ for some $\tilde{n} \in N_\Sigma(F^{\tilde{g}})$, we have

$$
\begin{aligned}
g \cdot \tilde{s} &> 0 \quad \forall \tilde{s} \in S_{F^{\tilde{g}}} \backslash S_{F^g} \\
n \cdot J\tilde{s} &< 0 \qquad (3.2.7)
\end{aligned}
$$

We know that $\tilde{n} = \sum_{S_{F^{\tilde{g}}}} k_s J\tilde{s}$ with $k_s \geq 0$ and for every $\tilde{s} \in S_{F^{\tilde{g}}} \cap S_{F^g}$, we have $n \cdot J\tilde{s} = 0$ by $J\tilde{s} \in \mathrm{Span}^+\{S_{F^{\tilde{g}}}\}^\perp \subseteq \mathrm{Char}\,(F^{\tilde{g}})$. So, by Inequality 3.2.7, we have

$$
n \cdot J\tilde{n} < 0 \qquad (3.2.8)
$$

Similarly, since $v = y + \bar{t}\tilde{g}$, we get

$$
\tilde{n} \cdot Jn > 0 \quad \forall s \in S_{F^g} \backslash S_{F^{\tilde{g}}}
$$

However, this contradicts Inequality 3.2.8. Therefore, we must have $F^g = F^{\tilde{g}}$.

Now, suppose WLOG that $S_{F^g} \subsetneq S_{F^{\tilde{g}}}$. If $F^g, F^{\tilde{g}}$ are both facets then our initial assumption of $F^g \neq F^{\tilde{g}}$ is impossible and we must have $F^g = F^{\tilde{g}}$. Otherwise, Lemma 3.2.16 implies that F^g and $F^{\tilde{g}}$ are both edges. However, this is impossible by Proposition 2.1.13. $\square$

Now we employ some directed graph technology (see Appenix A) to begin our argument of the existence of flow through a vertex. Polytopes naturally possess a graph structure, where we identify vertices with 0-faces and edges with 1-faces. We will make use of this structure, however, we will be doing this in the dual polytope, Σ°. Therefore, we will be associating graph vertices with facets of Σ and edges with $(2n-2)$-faces of Σ.

Definition 3.2.18. *Let Σ be a $2n$-polytope and let $F \subset \Sigma$ be a face. The* ***conjugate graph*** *generated by F, denoted G_F is the graph given by*

$$V(G_F) = \{v_s \in V(G_\Sigma) \ : \ F_s \text{ adjacent to } F\}$$

and

$$E(G_F) = \{v_s v_t \ : \ \dim(F_s \cap F_t) = 2n-2\}$$

By the symplectic-faced property of a polytope, we get a sign relation on codimension-2 faces, given by $Js \cdot \tilde{s}$, where F_s and $F_{\tilde{s}}$ are facets. This sign relation provides a natural orientation to the graph of a conjugate face, considering that the conjugate face of a codimension-2 face is an edge.

Definition 3.2.19. *Let Σ be a symplectic-faced $2n$-polytope. We define the* ***conjugate digraph*** *D_Σ generated by Σ to be the following orientation of G_Σ:*

$$A(D_\Sigma) = \{v_s v_t \ : \ Js \cdot t > 0\}$$

Similarly, we can define the conjugate digraph generated by a face $F \subset \Sigma$ to be the digraph D_F given by the same orientation on F^Δ.

Using this directed structure, we will show that the direction of arcs can imply the existence of a desired supporting hyperplane, as those implicitly seen in Theorem 3.1.23. From this point forward, unless otherwise noted, we will let V be a vertex of a symplectic-faced 4-polytope Σ and we will let D_V be the conjugate digraph generated by V. Before we proceed to associate graph-theoretic features of D_V with the flow of generalized characteristics, we will first prove a general fact about polytopes which shows that a particular hyperplane supporting adjacent vertices implies that the hyperplane will support the entire polytope.

Lemma 3.2.20. *Let v_0 be a vertex of a d-polytope Σ and let H be a hyperplane of $\mathbb{R}^d$ with $v_0 \in H$, given by*

$$H = \{x \in \mathbb{R}^d \ : \ x \cdot w = \alpha\}$$

for some $s \in \mathbb{R}^d$ and some $\alpha \in \mathbb{R}$. Let V be the set of all vertices v of Σ with the property that $\mathrm{Conv}\,(\{v, v_0\})$ *is a 1-face of Σ. If, for the open half-space*

$$H^+ = \{x \in \mathbb{R}^d \ : \ x \cdot w > \alpha\}$$

we have that $V \subset H^+$ (or H^-), then $\Sigma \backslash \{v_0\} \subset H^+$ (resp. H^-).

Proof. Translating by $-v_0$, WLOG $v_0 = 0$ and $\alpha = 0$.

First, for a vertex $v \in V$, define $R_v^+ := \{kv \ : \ k \geq 0\}$ to be the nonnegative ray generated by v. For a face F of Σ for which $0 \in F$, we can define a polyhedral set $\tilde{F}$ which contains F. Let K_F be the set of all $s' \in S_\Sigma \backslash S_F$ such that for the facet $F_{s'}$ of Σ, we have $0 \in F_{s'}$. Then, we define

$$\tilde{F} := \Big(\bigcap_{S_F} H(s, 0) \Big) \cap \Big(\bigcap_{K_F} K(s', 0) \Big) \tag{3.2.9}$$

where we recall from Definition 2.1.1 that $H(s, 0)$ is a hyperplane with normal vector s, and $K(s, 0)$ is a half-space with normal vector s. We will first show that for each k-face $F_k \subset \partial\Sigma$ ($1 \leq k \leq d-1$) adjacent to 0, we have

$$\tilde{F}_k \subset C_{F_k} := \mathrm{Conv}\Big(\bigcup_{v \in V \cap F_k} R_v^+ \Big)$$

We proceed by induction on k. For the base case of $k = 1$, we have that F_1 is an edge of Σ, and therefore $V \cap F_1 = \{v\}$ for a single nonzero vertex of Σ. Because $V \cap F_1 = \{v\}$ and since R_v^+ is a convex set, we must have $C_{F_1} = R_v^+$. Note that, since $\cap_{S_{F_1}} H(s, 0)$ is one-dimensional, for any nonzero $x \in \tilde{F}_1$ we have $x = cv$ for some $c \in \mathbb{R}$. For $c \geq 0$, we also have $cv \in \tilde{F}_1$. Additionally, since $v \in \cap_{K_{F_1}} K(s', 0)$, we cannot have $-cv \in \tilde{F}_1$ for any $c > 0$. Therefore, we must have $x \in R^+_{F_1}$.

Now, for our induction hypothesis assume that for each k-face $F_k \subset \partial\Sigma$ $(1 \leq k \leq d-1)$ adjacent to 0, we have $\tilde{F}_k \subset C_{F_k}$. Let $x \in \partial\tilde{F}_{k+1} \subseteq \cap_{S_{F_{k+1}}} H(s, 0)$. By Proposition 2.1.18, there must exist at least one $s' \in K_{F_{k+1}}$ such that

$$x \in \Big(\bigcap_{S_{F_{k+1}} \cup \{s'\}} H(s, 0) \Big) \cap \Big(\bigcap_{K_{F_{k+1}} \setminus \{s'\}} K(s, 0) \Big)$$

Since $\dim \left(\bigcap_{S_{F_{k+1}} \cup \{s'\}} H(s, 0) \right) = k$ and by Equation (3.2.9), this implies that $x \in \tilde{F}^i_k$ for some k-face F^i_k of Σ adjacent to 0.

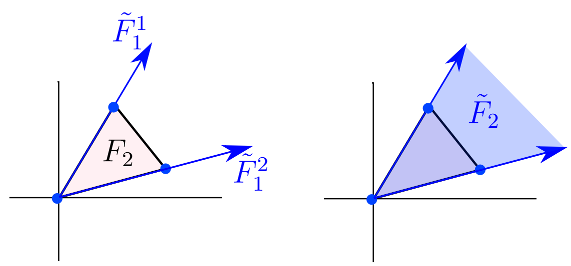


Figure 3.2: Edges F^1_1, F^2_1 adjacent to F_2 have polyhedral sets $\tilde{F}^1_1$, $\tilde{F}^2_1$ which generate the polyhedral set $\tilde{F}_2$.

Then, if $\{F^i_k\}_{\mathcal{I}}$ is the collection of k-faces adjacent to both F_{k+1} and 0, by our inductive hypothesis we have that $\partial\tilde{F}_{k+1} \subset \bigcup_{\mathcal{I}} C_{F^i_k}$. By [15, Lemma 1.4.1], we know that $\mathrm{Conv}\,(\partial\tilde{F}_{k+1}) = \tilde{F}_{k+1}$. Therefore, we will finish our induction argument by showing that $\mathrm{Conv}\left(\bigcup_{\mathcal{I}} C_{F^i_k}\right) \subset C_{F_{k+1}}$. Since $F^i_k \subset F_{k+1}$ for each $i \in \mathcal{I}$, we have that $V \cap F^i_k \subset V \cap F_{k+1}$ for each i. Therefore, for each i we must have

$$\mathrm{Conv}\Big(\bigcup_{V \cap F^i_k} R^+_v \Big) \subset \mathrm{Conv}\Big(\bigcup_{V \cap F_{k+1}} R_v \Big) = C_{F_{k+1}}$$

and so

$$\bigcup_{\mathcal{I}} C_{F_k^i} \subseteq C_{F_{k+1}}$$

which completes our induction argument.

We will now use a similar argument to show that $\Sigma \subseteq \cup_V R_v^+$. Let K_Σ be the set of $s' \in \Sigma$ for which $F_{s'}$ is adjacent to 0. Then we can define the following polyhedral set which contains Σ:

$$\tilde{\Sigma} := \bigcap_{K_\Sigma} K(s, 0)$$

Using an argument similar to one above, we will show that $\partial\Sigma \subset \cup_{K_\Sigma} \tilde{F}_s$. Let $x \in \partial\Sigma$. By Proposition 2.1.18, there exists at least one $s' \in K_\Sigma$ such that $x \in H(s', 0) \bigcap (\cap_{K_\Sigma \setminus \{s'\}})$. This implies that $x \in \tilde{F}_{s'}$ for some $s' \in K_\Sigma$, and so we have that $\partial\Sigma \subset \cup_{K_\Sigma} \tilde{F}_{s'}$. Since $\mathrm{Conv}\,(\partial\Sigma) = \Sigma$, and since $V \cap F_{s'} \subseteq V$ for each $s' \in K_F$, we have that $\tilde{\Sigma} \subseteq \mathrm{Conv}\,(\cup_V R_v^+)$, as desired.

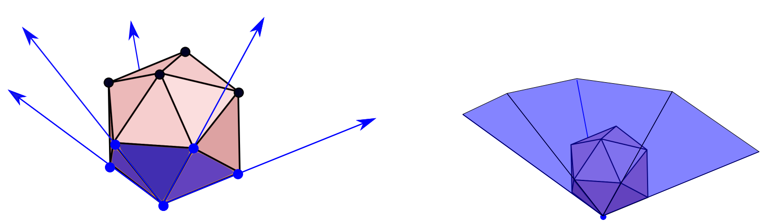


Figure 3.3: The convex hull of rays emanating from the origin contains the entire polyhedron.

Let $x \in \Sigma \backslash \{0\}$. Since $\Sigma \subseteq \mathrm{Conv}\,(\cup_V R_v^+)$, then by [6, Theorem 2.2], x can be written as a convex combination $x = \sum_{i=1}^n \alpha_i y_i$ for some $\{y_i\} \subseteq \cup_V R_v^+$, with at least one $y_{i'}$ nonzero and at least one $\alpha_{i'} > 0$. Since each $\alpha_i \geq 0$, since each $y_i = k_i v_i$ for some $k_i \geq 0$, $v_i \in V \backslash \{0\}$ and since $V \subset H^+$, we have that $x \cdot w > 0$. Therefore, we have that $\Sigma \backslash \{0\} \subset H^+$.

$\square$

Theorem 3.2.21. *If the vertex representation of a facet F_s in D_V is a source (sink) in D_V, then $V \subset F_s^+$ ($V \subset F_s^-$).*

Proof. Consider the affine hyperplane $H_V := \mathrm{Aff}\,(V^\Delta)$ in $\langle N_\Sigma(V)\rangle$ and the hyperplane H_{Js} given by

$$H_{Js} := \{x \in \langle N_\Sigma(V)\rangle \ : \ x \cdot Js = 0\}$$

Since $Js \cdot s = 0$ and by the symplectic-faced property of Σ, for any $\tilde{s} \in V$ with $\dim(F_s \cap F_{\tilde{s}}) = 2$ we have $J\tilde{s} \cdot s \neq 0$, then we know $H_{Js} \neq H_V$. Since $\dim(\langle N_\Sigma(V)\rangle) = 4$, we have that $P_{Js} := H_{Js} \cap H_V$ is two-dimensional. By Theorem 3.1.23, it suffices to show that $V^\Delta \subset P_{Js}^+$.

Suppose that the vertex representation $v_s \in V(D_V)$ of a facet F_s is a source in D_V. Let $\tilde{S}$ be all $\tilde{s} \in S_V$ for which $\dim(F_s \cap F_{\tilde{s}}) = 2$. By v_s a source and by Definition 3.2.19, we know that $Js \cdot \tilde{s} > 0$, for all $\tilde{s} \in \tilde{S}$ and therefore we have $\tilde{S} \subset P_{Js}^+$. Since V^Δ is a polytope and since P_{Js} is a hyperplane in $\mathrm{Aff}\,(V^\Delta)$, then by Lemma 3.2.20, we have $V^\Delta \subset P_{Js}^+$.

□

We have shown that sinks and/or sources of D_V represent facets whose glide-only trajectories flow into V, and now we will show that certain cycles in D_V represent glide-only edges whose trajctories flow into V.

Proposition 3.2.22. *Let $E \subset \Sigma$ be an edge adjacent to V. If the subdigraph $D_E \subset D_V$ is a cycle, then E is glide-only and either $V \subset E^+$ or $V \subset E^-$.*

Proof. By Proposition 3.2.11, E is either glide-only or bounce-only, so suppose E is bounce-only. By Definition 3.2.19, by D_E a cycle, and by Σ symplectic-faced, for each vertex $s \in S_E$, there must exist $\underline{s}, \overline{s} \in S_E$ such that $Js \cdot \underline{s} < 0$, $Js \cdot \overline{s} > 0$. However, by Theorem 3.1.23, this implies that there does not exist a facet F such that $E \subset F^+$, and this contradicts Theorem 3.2.15. Therefore, E cannot be bounce-only and so E is glide-only. Since E is one-dimensional and glide-only, then we must have either $V \subset E^+$ or $V \subset E^-$. □

Our main theorem will be an algorithm that searches the directed graph D_V for either sources, sinks, or cycles, in order to find glide-only faces which flow into V. The algorithm will first consider a cycle in D_V (if one exists), then it will consider the two hemispheres of D_V created by this cycle. Inside of each hemisphere, we will search for nested cycles until we find either a source/sink or we find that the cycle bounds a face of D_V. The following lemma corresponds with our main algorithm terminating: when we are unable to find smaller cycles.

Lemma 3.2.23. *Let C be a directed topological cycle of a topological planar digraph D for which $\pi(D)$ is homeomorphic to a disk. If $H_{\overline{\mathring{C}}}$ does not contain a cycle C_1 with $C_1 \neq C$. Then, either*

i $H_{\mathring{C}}$ contains a vertex which is a sink or a source in D

ii $H_{\mathring{C}}$ is empty

Proof. If $V(H_{\mathring{C}})$, $A(H_{\mathring{C}})$ are both empty then we are done. Therefore, we will first consider the case that $V(H_{\mathring{C}})$ is nonempty and consider the subdigraph $\tilde{D}$ of D with vertex set

$$V(\tilde{D}) := \Big\{ v \in V(D) \ : \ vu \in A(D) \ \text{ for some } u \in V(H_{\mathring{C}}) \Big\} \bigcup V(H_{\mathring{C}})$$

and arcs of $\tilde{D}$ are given by

$$A(\tilde{D}) := A(D) \backslash A(C)$$

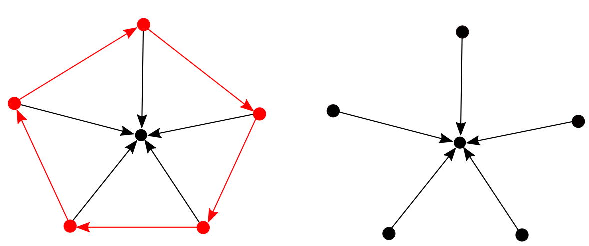


Figure 3.4: From D, we can construct a digraph $\tilde{D}$ which deletes the arcs of the cycle bounding D, but which keeps the arcs attaching to points on the interior of D.

Since $\tilde{D}$ is a subdigraph of $H_{\overline{\mathring{C}}}$ and since C is the only cycle in $H_{\overline{\mathring{C}}}$, then we have that $\tilde{D}$ is acyclic, and therefore $\tilde{D}$ must have a sink and a source. By definition of $A(\tilde{D})$, if any vertex in $V(H_{\mathring{C}})$ is a sink/source of $\tilde{D}$, then this vertex is also a sink/source in D. Now, suppose that no vertex $v \in V(H_{\mathring{C}})$ is a sink or a source of $\tilde{D}$, i.e. suppose that for any source/sink $\underline{v}, \overline{v}$ of $\tilde{D}$, we must have $\underline{v}, \overline{v} \in V(C)$. We will first show that for a source $\underline{v} \in V(C)$, there exists a path $\underline{v} \cdots v_i \cdots \overline{v}$ in $\tilde{D}$ for $i \geq 1$ and for some sink $\overline{v} \in V(\tilde{D})$. Since $\underline{v}$ is a source and by definition of $\tilde{D}$, there must exist $v_1 \in V(H_{\mathring{C}})$ such that $\underline{v}v_1 \in A(\tilde{D})$ and we take $P = \underline{v}v_1$. We now construct P recursively. Given v_i in P, if there exists an arc $v_i u \in A(\tilde{D})$ for some $u \in V(\tilde{D})$, take $v_{i+1} = u$ and let $P = \underline{v}v_1 \cdots v_i v_{i+1}$, otherwise let $P = \underline{v}v_1 \cdots v_i v_{i+1}$ with v_{i+1} as the terminal vertex of P. Since $\tilde{D}$ is finite and acyclic, we know that P is finite and we know that P consists of distinct vertices, i.e. P is indeed a path. By definition of P, its terminal vertex must be some sink $\overline{v}$ of $\tilde{D}$ and so by assumption $\overline{v} \in V(C)$. However, since C is a cycle with $\underline{v}, \overline{v} \in V(C)$, then there exist vertices $u_i \in V(C)$ such that $\underline{v}u_1u_2 \cdots \overline{v}$ is a path in D (or possibly $\underline{v}\overline{v} \in A(C)$). So, we can construct a path $C_1 = \underline{v}v_1 \cdots \overline{v}u_1 \cdots \underline{v}$.

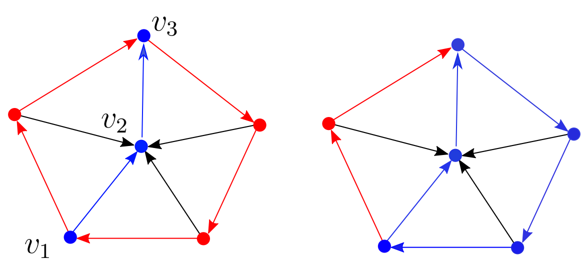


Figure 3.5: The existence of a path between cycle vertices v_1 and v_3 (left, in blue) implies the existence of a smaller cycle inside of $H_{\overline{\mathring{C}}}$ (right, in blue)

However, $C_1 \neq C$ and C_1 is a cycle, which is impossible. Therefore, there must exist a vertex in $V(H_{\mathring{C}})$ which is either a sink or source of D. $\square$

The following lemma corresponds with the iteration of our algorithm: finding a smaller cycle which bounds a strictly smaller number of faces of D_V.

Lemma 3.2.24. *Let C be a directed topological cycle of a connected topological planar digraph D for which $\pi(D)$ is homeomorphic to a disk. If $H_{\overline{\mathring{C}}}$ contains a cycle C_1 such that $C_1 \neq C$, then $f_{H_{\overline{\mathring{C}_1}}} < f_{H_{\overline{\mathring{C}}}}$.*

Proof. By Proposition A.2.13, we know that $H_{\overline{\mathring{C}}}$ is connected. Since $H_{\overline{\mathring{C}}}$ is planar and connected, then by [11, p.65-66] the Euler characteristic χ of $H_{\overline{\mathring{C}}}$ is given by

$$\chi = v - e + f = 1$$

Let v_1, e_1, f_1 be the number of vertices, edges and faces of $H_{\overline{\mathring{C}_1}}$ (respectively). Note that since $C_1 \neq C$, then $v_1 \leq v$, $e_1 < e$. By Proposition A.2.13 and $H_{\overline{\mathring{C}_1}}$ planar, for the Euler characteristic of $H_{\overline{\mathring{C}_1}}$ given by χ_1, we have $\chi_1 = 1$.
Let the vertices, edges and faces of $H_{\overline{\mathring{C}_1}}$ be given (respectively) by $v_1 = v - k$, $e_1 = e - m$ and $f_1 = f - n$ for some $0 \leq k \leq v$, $1 < m \leq e$, $0 \leq n \leq f$. Then, we must have

$$\begin{aligned}\chi_1 = & v_1 - e_1 + f_1 = 1 \\ & v - k - e + m + f - n = 1 \\ & 1 + m - k - n = 1 \\ & m - k = n\end{aligned}$$

Since deleting a vertex deletes at least two edges, we must have $m \geq 2k$ and so we are done. □

Proposition 3.2.25. *Let V be a vertex of a symplectic-faced 4-polytope Σ. Then, D_V is planar.*

Proof. By Theorem 2.2.7, D_V is the graph of a 3-polytope. By Steinitz' Theorem [16, Theorem 4.1], we then have that D_V is planar. □

Theorem 3.2.26. *For a vertex V of a symplectic-faced 4-polytope Σ, there exist unique glide-only faces $F_{\underline{s}}$, $F_{\overline{s}}$ of Σ such that $V \subset F^+_{\underline{s}}$ and $V \subset F^-_{\overline{s}}$.*

Proof. If D_V is acyclic, then by Proposition A.1.15, D_V has a source and a sink. By Theorem 3.2.21 and Theorem 3.2.17, there is only a single source in D_V and it is the vertex representation of a facet $F_{\underline{s}}$, with the property that $V \subset F_{\underline{s}}^+$. Similarly, there is only a single sink in D_V which is the vertex representation of a facet $F_{\overline{s}}$ with the property that $V \subset F_{\overline{s}}^-$. Suppose now that D_V has at least one cycle, C. Let π be an embedding of D_V into S^2 and let $\infty \in S^2$ for which $\infty \notin \pi(G)$. Since $\pi(C)$ is homeomorphic to S^1, then by the Jordan Curve Theorem [11, p.75], $S^2 = \pi(C)^+ \cup \pi(C)^-$ where $\pi(C)^+, \pi(C)^-$ are each homeomorphic to a closed disk with boundary $\pi(C)$ and such that $\infty \in \pi(C)^+$. Consider the underlying graph of $(\pi(C)^+)^\circ$, $H_{\mathring{C}_+}$. By Lemmas 3.2.23 and 3.2.24, we will have one of three possible cases.

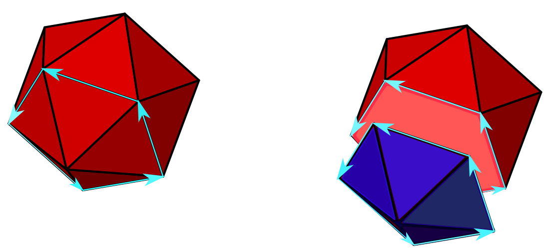


Figure 3.6: The cycle C (light blue) cuts the polyhedron V^Δ into two hemispheres $\pi(C)^+, \pi(C)^-$, each of which is homeomorphic to a closed disk with boundary $\pi(C)$.

Case 1: $H_{\mathring{C}_+}$ is empty, which by definition implies that C bounds a face of D_V and by Proposition 3.2.22, $H_{\overline{\mathring{C}_+}}$ is the representation of a glide-only edge F_0 of Σ, and so we have either $V \subset F_0^+$ or $V \subset F_0^-$.

Case 2: $H_{\mathring{C}_+}$ contains a sink or a source v. We will now show that v is also a sink/source in D_V, and WLOG we will consider the case that v is a sink in D_V. Suppose not, i.e. suppose that there exists a vertex $v_0 \in V(D_V)\backslash V(H_{\overline{\mathring{C}_+}})$ such that $vv_0 \in A(D_V)$. However, by continuity $\exists y \in \pi\Big(f_{vv_0}((0,1))\Big)$ such that for some edge $E \subset C$ and some $x \in E$ we have $\pi(x) = y$ and this contradicts that π is the embedding of a planar digraph D_V.

By Theorem 3.2.21, v is the vertex representation of a facet F_0 for which either $V \subset F_0^+$ or $V \subset F_0^-$

Case 3: $H_{\mathring{C}_+}$ contains a cycle $C_1 \neq C$, and by Lemma 3.2.24 we have $f_{H_{\overline{\mathring{C}_1}}} < f_{H_{\overline{\mathring{C}}}}$. We then must have that $H_{\mathring{C}_{1+}}$ meets the criteria of Case 1, 2, or 3, and we can iterate the procedure of examining $H_{\mathring{C}_{i+}}$ and checking for a sink/source in $H_{\mathring{C}_{i+}}$ or a smaller cycle C_{i+1}. Since a cycle must bound at least one face and C_{i+1} must bound strictly fewer faces than C_i , we know that this procedure will eventually find either a sink/source (representing a facet F_0) or find a face which represents a glide-only edge F_0 of Σ.

We can proceed similarly for $H_{\mathring{C}_-}$ and we will find a facet or glide-only edge F_1 for which either $V \subset F_1^+$ or $V \subset F_1^-$. By Theorem 3.2.17, we know F_1, F_0 are the only such glide-only faces. $\square$

3.2.4 Existence/Uniqueness of Generalized Characteristics in a Symplectic-Faced 4-Polytope

Before we demonstrate the existence/uniqueness of generalized characteristics passing through a point $x \in \partial\Sigma$, we will identify two natural "small" generalized characteristics which begin and end in face(s) containing x.

Definition 3.2.27. *Let F^g be a glide-only face of a symplectic-faced $2n$-polytope. A generalized characteristic $X = \mathrm{Im}\,(\gamma)$ of Σ is a* ***short GC*** *of F^g if $\gamma(t_0) \in \mathrm{ri}\,(F^g)$ for some $t_0 \in \mathrm{Int}\,(\mathcal{I}_\gamma)$ and $\mathcal{I}_\gamma = (\underline{t_0}, \overline{t_0})$.*

Proposition 3.2.28. *Let $x_0 \in \mathrm{ri}\,(F^g)$ for some glide-only face F^g of a symplectic-faced $2n$-polytope. Then, there exists a unique short GC X of F^g for which $x_0 \in \mathrm{ri}\,(F^g)$.*

Proof. By F^g a glide-only face, we can construct a GCP $\gamma : (\underline{t}, \overline{t}) \to \partial\Sigma$ with $\gamma(0) \in \mathrm{ri}\,(F^g)$ given by

$$\gamma(t) = \gamma(0) + tg$$

for any $t \in [\underline{t}, \overline{t}]$ and we then have that X_γ is a short GC with $x_0 \in \mathrm{Int}\,(X_\gamma)$. Let X_{γ_0} be a short GC with $x_0 \in \mathrm{Int}\,(X_{\gamma_0})$. By definition of a short GC containing x_0, there must exist $t_0 \in \mathcal{I}_{\gamma_0}$ such that $\gamma_0(t_0) = x_0 \in \mathrm{ri}\,(F^g)$. By definition of glide-only via g, we also have that

$$\gamma_0(t) = \gamma_0(t_0) + tg$$

for $t \in [\underline{t_0}, \overline{t_0}]$. Since $\gamma(0) = x_0 = \gamma_0(t_0)$ and g is a nonzero vector, we must have that $t_0 = 0$, and therefore $X_\gamma = X_{\gamma_0}$. $\square$

Proposition 3.2.29. *Let $x_0 \in \mathrm{ri}\,(F)$ for some bounce-only face F of a symplectic-faced 4-polytope Σ. Then, there exists a unique pair of short GCs X^-, X^+ such that $x_0 \in X^- \cup X^+$.*

Proof. By definition of a symplectic-faced polyhedral set, by Theorem 3.1.14, and by Propositions 3.1.11 and 3.2.11, this claim needs to be shown for the case of F a face, F an edge, or F a vertex. By Theorems 3.1.22, 3.2.15, 3.2.26, there exists a unique pair of glide-only faces $F^g, F^{\tilde{g}}$, there exists points $y_0, \tilde{y}_0 \in F^g, F^{\tilde{g}}$ and there exists $t > 0$ such that

$$y_0 + tg = x_0 = \tilde{y}_0 - t\tilde{g}$$

By Proposition 3.2.28, there exists a unique pair of short GCs X^-, X^+ for which $y_0 \in X^-$, $\tilde{y}_0 \in X^+$. Since $y_0, \tilde{y}_0$ were arbitrary and X^-, X^+ are unique for any $y_0 \in \mathrm{ri}\,(F^g)$, $\tilde{y}_0 \in \mathrm{ri}\,(F^{\tilde{g}})$, then our claim is shown. $\square$

Theorem 3.2.30. *Let Σ be a symplectic-faced 4-polytope. Then, for $x_0 \in \partial\Sigma$, there exists a generalized characteristic X of Σ and $\exists \epsilon > 0$ such that*

i $x_0 \in X$

ii for any other generalized characteristic X_0 of Σ with $x_0 \in X_0$, we have $B_\epsilon(x_0) \cap X_0 = B_\epsilon(x_0) \cap X$.

Proof. Let Σ be a symplectic-faced 4-polytope. Then, we know that $x_0 \in \mathrm{ri}\,(F)$ for a glide-only or bounce-only face F of Σ. In the case that F is a glide-only face, by Proposition 3.2.28, there exists a unique short GC X such that $x_0 \in X$. Let X_0 be a generalized characteristic of Σ with $x_0 \in X_0$. Since $x \in \mathrm{ri}\,(F)$, there exists an $\epsilon > 0$ such that $B_\epsilon(x_0) \subseteq \mathrm{ri}\,(F)$. Then, by definition of a glide-only face, we have $B_\epsilon(x_0) \cap X_0 = B_\epsilon(x_0) \cap X$.

In the case that F is a bounce-only face, by Proposition 3.2.29, there exists a unique pair of short GCs X^-, X^+ of $F^g, F^{\tilde{g}}$ (respectively) such that $x_0 \in X^- \cup X^+$. By $x_0 \in \mathrm{ri}\,(F)$, $\exists \epsilon > 0$ such that $B_\epsilon(x_0) \cap (F^g \cup F \cup F^{\tilde{g}}) \subseteq \mathrm{ri}\,(F^g) \cup F \cup \mathrm{ri}\,(F^{\tilde{g}})$. Then, by definition of a glide-only face, we have $B_\epsilon(x_0) \cap (X^- \cup X^+) = B_\epsilon(x_0) \cap X_0$. $\square$

4 GENERALIZED CHARACTERISTICS OF SYMPLECTIC-FACED POLYTOPES IN $\mathbb{R}^{2n}$

In order to generalize the previous uniqueness/existence results of generalized charachteristics of symplectic-faced polytopes to higher dimensions, one must first establish the same bounce-only/glide-only dichotomy present in $\mathbb{R}^4$. In 6-polytopes, one can not immediately rule out the possibility of a 3-face admitting a 2-dimensional family of characteristic directions. In this chapter, we will show that the bounce-only/glide-only dichotomy which was immediate in symplectic-faced 4-polytopes also extends to faces of a symplectic-faced $2n$-polytope. Once the bounce-only/glide-only dichotomy is established for symplectic-faced $2n$-polytopes, we will then take the existence/uniqueness arguments used for edges E of symplectic-faced 4-polytopes and generalize them to codimension-three faces of a symplectic-faced $2n$-polytope.

4.1 Bounce-only / Glide-only Dichotomy in $\mathbb{R}^{2n}$

For our general argument, we will return to the setting of conjugate faces. The following proposition shows us that a characteristic vector of a face F can be scaled in a way that places it inside the conjugate face F^{Δ}.

Proposition 4.1.1. *Let F be a face of a polytope Σ for which $\dim(\mathrm{Char}(F)) \geq 1$. Then, there exists an affine hyperplane H_F in $\langle JN_{\Sigma}(F)\rangle$ such that for $v \in \mathrm{Char}(F)$, there exists $k > 0$ such that $kv \in JF^{\Delta} \subset H_F$.*

Proof. By Theorem 2.2.7, we know that $H_F := \mathrm{Aff}(JF^{\Delta})$ is an affine hyperplane in $\langle JN_{\Sigma}(F)\rangle$. Let $v \in \mathrm{Char}(F)$ be nonzero. Then, $v = \sum_{S_F} k_s Js$ for some $k_s \geq 0$. Take $k := \frac{1}{\sum_S k_s}$. Since $kv \in \mathrm{Conv}(JS_F)$, then by Proposition 2.2.8, we have $kv \in JF^{\Delta} \subset H_F$. $\square$

When working in the conjugate face of an odd-dimensional face F of Σ, the following lemma gives us a particular correspondence between boundary faces of JF^{Δ} and even-dimensional faces of Σ.

Lemma 4.1.2. *Let F be a face of a $2n$-polytope Σ with $\dim(F)$ odd and $\dim(F) < 2n-2$. Then, for any $y \in \mathrm{rb}(JF^{\Delta})$, there exists an even dimensional face F_0 of Σ such that $JF_0^{\Delta} \subseteq \mathrm{rb}(JF^{\Delta})$ and $y \in JF_0^{\Delta}$.*

Proof. By Proposition 2.2.8 and Theorem 2.2.7, we know that $\mathrm{rb}(JF^{\Delta})$ is the boundary of a $(\mathrm{codim}(F)-1)$-polytope. Then, $\dim(\mathrm{rb}(JF^{\Delta})) = \mathrm{codim}(F) - 2$ must be odd, and so facets of JF^{Δ} are of odd-dimension. Let $y \in \mathrm{rb}(JF^{\Delta})$. Then, by Corollary 6.5 ([6], p. 41), there exists a face F_0 of Σ such that F_0^{Δ} is a facet of $\mathrm{rb}(JF^{\Delta})$ and $y \in F_0^{\Delta}$.
By Theorem 2.2.7 and Corollary 6.5 ([6], p. 41), we get that

$$\dim(F_0) = \dim((F_0^{\Delta})^{\Delta}) = 2n - 1 - \dim(F_0^{\Delta})$$

and since $\dim(F_0^{\Delta})$ is odd, $F_0 \subset \Sigma$ is even-dimensional. $\square$

The following theorem establishes the bounce-only/glide-only dichotomy of symplectic-faced $2n$-polytopes.

Theorem 4.1.3. *Let F be a face of a symplectic-faced $2n$-polytope. Then $\dim(\mathrm{Char}(F)) = 0$ or $\dim(\mathrm{Char}(F)) = 1$.*

Proof. Let $c_F := \mathrm{codim}(F)$, $d_F := \dim(F)$ and $ch_F := \dim(\mathrm{Char}(F))$. By Proposition 3.1.11, by Theorem 3.1.14, and by definition of a symplectic-faced polytope, it remains to show our claim for the case of $3 \leq d_F \leq 2n-3$ and d_F odd, which implies that $3 \leq c_F \leq 2n-3$ and c_F is odd.
Consider the case that $ch_F = c_F = d_F$. Then, we have that $\langle JN_{\Sigma}(F)\rangle = N_{\Sigma}(F)^{\perp}$. However, there must exist $s, \tilde{s} \in S_F$ such that $\dim(F_s \cap F_{\tilde{s}}) = 2n-2$, and $Js \in N_{\Sigma}(F)^{\perp}$ implies that $Js \cdot \tilde{s} = 0$, which violates that Σ is symplectic-faced.

Now consider the case that $ch_F \geq 2$ and let $v \in \mathrm{Char}(F)$. We will show that $\exists w \in N_\Sigma(F)^\perp \cap \mathrm{rb}(kJF^\Delta)$ for some $k > 0$. By Proposition 4.1.1, $\exists k > 0$ such that $kv \in JF^\Delta$ and for the hyperplane H_F from Proposition 4.1.1 with $kv \in H_F$, we have $H_F \cap N_\Sigma(F)^\perp \neq \emptyset$. Let $A_F := H_F \cap N_\Sigma(F)^\perp$. Since $kv \in A_F$, we know A_F isn't empty. Note that H_F is given by exactly one linear constraint in $\langle JN_\Sigma(F)\rangle$ and since $c_F \geq 3$ and $ch_F \geq 2$, we know $N_\Sigma(F)^\perp \cap \langle JN_\Sigma(F)\rangle$ is given by at most $ch_F - 2$ constraints. Therefore, A_F is given by at most $ch_F - 1$ constraints, which means that $\dim(A_F) \geq 1$.

We will now show that A_F must intersect $\mathrm{rb}(JF^\Delta)$. If $kv \in \mathrm{rb}(JF^\Delta)$, we are done. Suppose not, i.e. suppose that $kv \in \mathrm{ri}(JF^\Delta)$. Since JF^Δ is compact and A_F is unbounded, then there must exist $z \in A_F \cap (JF^\Delta)^c$. By A_F convex, we have that $\alpha kv + (1-\alpha)z \in A_F$ for all $\alpha \in [0,1]$. Then, there must exist $\tilde{\alpha} \in (0,1)$ such that $w := \tilde{\alpha}kv + (1-\tilde{\alpha})z \in \mathrm{rb}(JF^\Delta)$. Since $w \in A_F$, we know that $w \in N_\Sigma(F)^\perp$, and so $w \in N_\Sigma(F)^\perp \cap \mathrm{rb}(JF^\Delta)$.

Now we will show that the existence of such a w violates that Σ is symplectic-faced. By Lemma 4.1.2, there exists a $(c_F - 2)$-dimensional face G of JF^Δ such that $G = JF_0^\Delta$ for some even-dimensional face F_0 of Σ. By Proposition 2.2.8, we know $G = \mathrm{Conv}(JS_{F_0})$, $JF^\Delta = \mathrm{Conv}(JS_F)$ and since $G \subset JF^\Delta$, we have $S_{F_0} \subset S_F$. Then, we must have $w \in N_\Sigma(F)^\perp \subset N_\Sigma(F_0)^\perp$. Additionally, we know $w \in \mathrm{Conv}(JS_{F_0})$ and so by Proposition 2.2.8, we have $w \in \mathrm{Char}(F_0)$. However, F_0 is even-dimensional and having $\dim(\mathrm{Char}(F_0)) > 0$ violates that Σ is symplectic-faced. $\square$

4.2 Flow in a Symplectic-Faced Polytope in $\mathbb{R}^{2n}$

In Chapter 3, the flow occuring transverse to a codimension-2 face in a symplectic-faced $2n$-polytope Σ was described by Theorem 3.1.14, Lemma 3.1.21 and Theorem 3.1.22. By Σ symplectic-faced, any flow transverse to a codimension-3 face will originate from the relative interior of a facet. We will now generalize the previous result of existence/uniqueness of flow from a facet to an edge in a symplectic-faced 4-polytope to the case of a facet flowing to a codimension-3 face in a symplectic-faced $2n$-polytope.

Theorem 4.2.1. *Let F be a face of a symplectic-faced $2n$-polytope Σ with* $\operatorname{codim}(F) > 1$. *If F_s, $F_{\tilde{s}}$ are facets of Σ for which $F \subset F_s^+$ ($F \subset F_s^-$) and $F \subset F_{\tilde{s}}^+$ (resp. $F \subset F_{\tilde{s}}^-$), then we must have $F_s = F_{\tilde{s}}$.*

Proof. WLOG, suppose that $F \subset F_s^+$ and $F \subset F_{\tilde{s}}^+$ for $s \neq \tilde{s}$. Then, by Theorem 3.1.23, we have that

$$Js \cdot \tilde{s} > 0$$

$$s \cdot J\tilde{s} > 0$$

However, these inequalities contradict the skew-symmetry of J. Therefore, we must have $s = \tilde{s}$. □

Theorem 4.2.2. *Let D_F be the conjugate digraph of a face F of a symplectic-faced $2n$-polytope Σ, and let F_s be a facet of Σ with $s \in S_F$. Then, v_s is source (sink) of D_F if and only if $F \subset F_s^+$ ($F \subset F_s^-$).*

Proof. First we show " $\implies$ ". By Lemma 3.2.20, we have that $Js \cdot \tilde{s} > 0$ for all facets $F_{\tilde{s}}$ adjacent to F. Then, by Theorem 3.1.23, we have $F \subset F_s^+$. Now we show " $\impliedby$ ". By Theorem 3.1.23, we have that $Js \cdot \tilde{s} > 0$ for all vertices $v_{\tilde{s}} \in D_F$. In particular, we have that $Js \cdot \tilde{s} > 0$ for all vertices $v_{\tilde{s}}$ which are adjacent to v_s, and so by definition of D_F, we have that v_s is a source of D_F. □

4.2.1 Codimension 3 Faces

By Theorem 4.1.3, we know that a codimension-three face F is either glide-only or bounce-only. Since the behavior of generalized characteristics in the relative interior of glide-only faces is already well understood in arbitrary dimension, we will generalize the existence result of Theorem 3.2.15 to bounce-only faces of codimension-three. We do this by demonstrating that the original three-dimensional picture of $\langle JN_\Sigma(F)\rangle$ (first seen in Lemma 3.2.14) is still relevant in the setting of a $2n$-polytope.

Proposition 4.2.3. *Let F be a bounce-only face of a symplectic-faced $2n$-polytope Σ for which* $\operatorname{codim}(F) = 3$. *Then, we have that* $\dim\left(N_\Sigma^\perp(F) \cap \langle JN_\Sigma(F)\rangle\right) \geq 1$.

Proof. The proof of Proposition 3.2.13 proves exactly this claim by showing the existence of a nonzero vector $v \in N_\Sigma^\perp(F) \cap \langle JN_\Sigma(F)\rangle$ given that, for the matrix $(\mathcal{F})$ from Proposition 2.3.7, we have $\operatorname{rank}(\mathcal{F}) = 3$. □

Theorem 4.2.4. *Let F be a bounce-only face of a symplectic-faced $2n$-polytope Σ for which* $\operatorname{codim}(F) = 3$. *Then, there exists a unique $\underline{s} \in S_F$ such that $F \subset F_{\underline{s}}^+$ and there exists a unique $\overline{s} \in S_F$ such that $F \subset F_{\overline{s}}^-$.*

Proof. By Proposition 4.2.3 and since Σ is symplectic-faced, we must have $1 \leq \dim\left(N_\Sigma^\perp(F) \cap \langle JN_\Sigma(F)\rangle\right) < 3$. By selecting a nonzero vector $v \in N_\Sigma^\perp(F)$, the line $\langle v\rangle$ has properties identical to the line $N_\Sigma^\perp(F)$ in the proof of Lemma 3.2.14, and we can guarantee the existence of our desired $\underline{s}, \overline{s} \in S_F$. By Theorem 4.2.1, we know that these $\underline{s}, \overline{s}$ are unique. □

4.2.2 Codimension 4 Faces

For a codimension-4 face F of a $2n$-polytope Σ, the same proof of Theorem 3.2.26 can still find a unique pair of glide-only faces adjacent to F, given by either a source/sink of D_F or by 2-faces of D_F which are bounded by a directed cycle. However, since 2-faces of D_F (which are the conjugate faces of codimension-3 faces of Σ) no longer represent one-dimensional faces (edges) of Σ, the existence of a glide-only codimension-three face E does not immediately guarantee that $F \subset E^+$ or $F \subset E^-$. We will now show that glide-only codimension-three faces do indeed have flow transverse to F.

Proposition 4.2.5. *Let Σ be a symplectic-faced $2n$-polytope and let F be a face of Σ with* $\operatorname{codim}(F) = 4$. *Suppose E, $\tilde{E}$ are glide-only faces of Σ with* $\operatorname{codim}(E) = 3$ *and such that $F \subset E^+$ and $F \subset \tilde{E}^+$ (or $F \subset E^-$ and $F \subset \tilde{E}^-$). Then, we must have $E = \tilde{E}$.*

Proof. Suppose that E is glide-only via g, $\tilde{E}$ is glide-only via $\tilde{g}$ and $E \neq \tilde{E}$. WLOG, assume that $F \subset E^+$ and $F \subset \tilde{E}^+$. First we will show that $g \in \operatorname{ri}(JE^\Delta)$ and $\tilde{g} \in \operatorname{ri}(J\tilde{E}^\Delta)$. WLOG, we consider g first. Suppose $g \in \mathrm{rb}\,(JE^{\Delta})$. By Proposition 2.2.8, we know that JE^{Δ} is a polygon, and so $g = \alpha J s_1 + (1-\alpha) J s_2$ for some $s_1, s_2 \in S_E$. However, by E glide-only via g, we would have that $g \cdot s_1 = 0$, which violates that Σ is symplectic-faced. Therefore, we must have $g \in \mathrm{ri}\,(JE^{\Delta})$ and $\tilde{g} \in \mathrm{ri}\,(J\tilde{E}^{\Delta})$. Additionally, by Proposition 2.1.13, we know that $g \notin J\tilde{E}^{\Delta}$.

We know that $g = \sum_{s \in S_E} k_s J s$ and $\tilde{g} = \sum_{\tilde{s} \in S_{\tilde{E}}} k_{\tilde{s}} J \tilde{s}$. By Theorem 3.1.23 and skew symmetry, we have

$$g \cdot \Big(\sum_{\tilde{s} \in S_{\tilde{E}}} k_{\tilde{s}} \tilde{s} \Big) > 0$$

$$\Big(\sum_{s \in S_E} k_s J s \Big) \cdot \tilde{g} < 0 \tag{4.2.1}$$

$$\tag{4.2.2}$$

However, by $F \subset \tilde{E}^{+}$ and Theorem 3.1.23, we have

$$\Big(\sum_{s \in S_E} k_s J s \Big) \cdot \tilde{g} > 0$$

which contradicts Inequality 4.2.1. Therefore, we must have $E = \tilde{E}$. □

Lemma 4.2.6. *Let Σ be a symplectic-faced $2n$-polytope and let F be a face of Σ with* $\mathrm{codim}\,(F) = 4$. *Let E be a face adjacent to F with* $\mathrm{codim}\,(E) = 3$. *If the subdigraph $D_E \subset D_F$ is a directed cycle, then E is glide-only and either $F \subset E^{+}$ or $F \subset E^{-}$.*

Proof. If D_E is a directed cycle, then for every vertex representation of a facet $v_s \in V(D_E)$, we have $d^{-}(v_s) \geq 1$ and $d^{+}(v_s) \geq 1$, and so by Theorem 4.2.2, we know that each v_s is neither a source nor a sink. Therefore, E is not in the forward or backward fence of any facet F_s. By Theorem 4.2.4, we must then have that E is not bounce-only, and so by Theorem 4.1.3, we must have that E is glide-only.

If E is glide-only, then there exists $g \in JN_\Sigma(E)$ such that $g \cdot s = 0$ for all $s \in S_E$. By Proposition 2.2.8, we then have that $g \cdot x = 0$ for all $x \in E^\Delta$. Consider the plane

$$H_g = \{x \in \text{Aff}\,(F^\Delta) \ : \ x \cdot g = 0\}$$

Since $H_g \cap \text{Aff}\,(E^\Delta)$ is two-dimensional with both H_g and $\text{Aff}\,(E^\Delta)$ two-dimensional, then we must have that $H_g = \text{Aff}\,(E^\Delta)$. Additionally, since E^Δ is a facet of F^Δ, we have that $H_g \cap F^\Delta = E^\Delta$.

By Theorem 3.1.23, we must show that $F^\Delta \backslash E^\Delta \subset H_g^+$ or $F^\Delta \backslash E^\Delta \subset H_g^-$. Suppose not, i.e. suppose that there exists $y, \tilde{y} \in F^\Delta \backslash E^\Delta$ such that $g \cdot y > 0$ and $g \cdot \tilde{y} < 0$. Note that, since E^Δ is a face of F^Δ, by Proposition 2.1.10 we have that $F^\Delta \backslash E^\Delta$ is convex. Then, by convexity of $F^\Delta \backslash E^\Delta$ and the intermediate value theorem, there exists $\tilde{\alpha} > 0$ such that $\tilde{\alpha} y + (1 - \tilde{\alpha})\tilde{y} \in F^\Delta \backslash E^\Delta$ and $\big(\tilde{\alpha} y + (1 - \tilde{\alpha})\tilde{y}\big) \cdot g = 0$. However, this contradicts that $\tilde{\alpha} y + (1 - \tilde{\alpha})\tilde{y} \notin E^\Delta$. Therefore, for all $x \in F^\Delta \backslash E^\Delta$, we must have either $x \cdot g > 0$ or $x \cdot g < 0$, as desired. □

Theorem 4.2.7. *Let Σ be a symplectic-faced $2n$-polytope and let F be a face of Σ with* $\text{codim}\,(F) = 4$. *Then, there exist unique glide-only faces $F_{\underline{s}}$, $F_{\overline{s}}$ of Σ such that $F \subset F_{\underline{s}}^+$ and $F \subset F_{\overline{s}}^-$.*

Proof. By Proposition 2.2.8, we know that F^Δ is a 3-polytope and so by Steinitz' Theorem [16, Theorem 4.1], D_F is a planar digraph. By Theorem 4.2.2 and Lemma 4.2.6, the proof of Theorem 3.2.26, along with Theorem 4.2.1 and Proposition 4.2.5 guarantees the existence and uniqueness of the desired glide-only faces $F_{\underline{s}}, F_{\overline{s}}$. □

Proposition 4.2.8. *Let Σ be a symplectic-faced $2n$-polytope and let $x_0 \in$* $\text{ri}\,(F)$ *for F a bounce-only face of Σ with* $\text{codim}\,(F) \leq 4$. *Then, there exists a unique pair of short GCs X^-, X^+ such that $x_0 \in X^- \cup X^+$.*

Proof. By definition of a symplectic-faced polyhedral set, by Theorem 3.1.14, this claim needs to be shown for the case of $\text{codim}\,(F) = 2$ and $\text{codim}\,(F) = 3$. By Theorems 3.1.22, 4.2.4 and

4.2.7 there exists a unique pair of glide-only faces $F^g, F^{\tilde{g}}$, there exists points $y_0, \tilde{y}_0 \in F^g, F^{\tilde{g}}$ and there exists $t > 0$ such that

$$y_0 + tg = x_0 = \tilde{y}_0 - t\tilde{g}$$

By Proposition 3.2.28, there exists a unique pair of short GCs X^-, X^+ for which $y_0 \in X^-$, $\tilde{y}_0 \in X^+$. Since $y_0, \tilde{y}_0$ were arbitrary and X^-, X^+ are unique for any $y_0 \in \mathrm{ri}\,(F^g)$, $\tilde{y}_0 \in \mathrm{ri}\,(F^{\tilde{g}})$, then our claim is shown. $\square$

Theorem 4.2.9. *Let Σ be a symplectic-faced $2n$-polytope and let $x_0 \in \mathrm{ri}\,(F)$ for F a face of Σ with $\mathrm{codim}\,(F) \leq 4$. Then, there exists a generalized characteristic X of Σ and $\exists \epsilon > 0$ such that*

i $x_0 \in X$

ii for any other generalized characteristic X_0 of Σ with $x_0 \in X_0$, we have $B_\epsilon(x_0) \cap X_0 = B_\epsilon(x_0) \cap X$.

Proof. Let Σ be a symplectic-faced $2n$-polytope and let $x_0 \in \mathrm{ri}\,(F)$ for F a face of Σ with $\mathrm{codim}\,(F) \leq 3$. Then, we know that $x_0 \in \mathrm{ri}\,(F)$ for a glide-only or bounce-only face F of Σ. In the case that F is a glide-only face, by Proposition 3.2.28, there exists a unique short GC X such that $x_0 \in X$. Let X_0 be a generalized characteristic of Σ with $x_0 \in X_0$. Since $x \in \mathrm{ri}\,(F)$, there exists an $\epsilon > 0$ such that $B_\epsilon(x_0) \subseteq \mathrm{ri}\,(F)$. Then, by definition of a glide-only face, we have $B_\epsilon(x_0) \cap X_0 = B_\epsilon(x_0) \cap X$.

In the case that F is a bounce-only face, by Proposition 4.2.8, there exists a unique pair of short GCs X^-, X^+ of $F^g, F^{\tilde{g}}$ (respectively) such that $x_0 \in X^- \cup X^+$. By $x_0 \in \mathrm{ri}\,(F)$, $\exists \epsilon > 0$ such that $B_\epsilon(x_0) \cap (F^g \cup F \cup F^{\tilde{g}}) \subseteq \mathrm{ri}\,(F^g) \cup F \cup \mathrm{ri}\,(F^{\tilde{g}})$. Then, by definition of a glide-only face, we have $B_\epsilon(x_0) \cap (X^- \cup X^+) = B_\epsilon(x_0) \cap X_0$. $\square$

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APPENDIX

Appendix A Graph Theory

A.1 Graphs and Digraphs

The following facts are taken from *Digraphs: Theory, Algorithms and Applications*[3], pages 2-19.

Definition A.1.1. *A **graph** G is a non-empty finite set $V(G)$ of elements called vertices and a finite set of $E(G)$ of unordered pairs of distinct vertices, called edges.*

Definition A.1.2. *A **digraph** D is a non-empty finite set of vertices $V(D)$ and a finite set of ordered pairs of distinct vertices $A(D)$, whose elements are called arcs.*

Definition A.1.3. *For an arc $(u, v) \in A(D)$, we say that the vertices $u, v \in V(D)$ are the **end-vertices** of (u, v), and we say that u is the **tail** of (u, v), v is the **head** of (u, v).*

Definition A.1.4. *A digraph H is a **subdigraph** of D if $V(H) \subseteq V(D)$, $A(H) \subseteq A(D)$ and every arc in $A(H)$ has both end-vertices in $V(H)$.*

Definition A.1.5. *Let $v \in V(D)$. The **in-degree** $d^-(v)$ of v is the number of arcs of D whose head is v. The **out-degree** $d^+(v)$ of v is the number of arcs of D whose tail is v.*

Definition A.1.6. *We say $v \in V(D)$ is a **source** in D if $d^-(v) = 0$ and we say v is a **sink** in D if $d^+(v) = 0$.*

Definition A.1.7. *A **walk** in D is a sequence*

$$W = x_1 x_2 x_3 \ldots x_{k-1} x_k$$

of $x_i \in V(D)$ such that $x_i x_{i+1} \in A(D)$ for every $1 \leq i \leq k-1$. We say that W is a (x_1, x_k)-walk. Similarly, we can define a walk in a graph G for $x_i \in V(G)$ and $x_i x_{i+1} \in E(G)$.

Definition A.1.8. *A **path** is a walk W with distinct vertices.*

Definition A.1.9. *A graph G is **connected** if for every vertex $u, v \in V(G)$ there exists a (u, v)-walk in G.*

Definition A.1.10. *A **cycle** of a digraph D is a (x_1, x_k) walk for which $x_1 = x_k$ and $\{x_i\}$ are distinct for $1 \leq i \leq k-1$*

Definition A.1.11. *A digraph D is **acyclic** if it contains no cycles.*

Definition A.1.12. *A digraph D is an **orientation** of a graph G if D is obtained from G by replacing each edge of G by the ordered pair (x, y) or (y, x). If D is an orientation of a graph G, we say such a graph G is **oriented**.*

Definition A.1.13. *The **underlying graph** $UG(D)$ of a digraph D is the unique graph G such that D is an orientation of G.*

Definition A.1.14. *A digraph D is said to be **connected** if $UG(D)$ is connected.*

Proposition A.1.15. *([3], p. 13) Every acyclic digraph has a source and a sink.*

A.2 Topological Graphs

Definition A.2.1. *([7]) A **topological graph** G is a topological space $G = V \cup E$, where*

1. *V is a finite discrete set*
2. *E is a finite disjoint union of intervals called edges*
3. *For each edge $e \subset E$, there is a continuous map $f_e : [0,1] \to G$ mapping $(0,1)$ homeomorphically onto e and sending $\{0,1\}$ to V.*

Definition A.2.2. *A **topological subgraph** $H = V_H \cup E_H$ of a topological graph $G = V \cup E$ is a topological graph with*

1. $V_H \subset V$

2. $E_H = \{e \subset E \ : \ f_e(\{0,1\}) \subset V_H\}$

Definition A.2.3. *Given a topological graph G, we define the **underlying graph** of G to be the graph $\mathcal{UG}(G)$ given by*

1. $V(\mathcal{UG}(G)) = V$

2. $E(\mathcal{UG}(G)) = \{v_1v_2 \ : \ v_1, v_2 \in V$ *and* $\exists$ *edge* $e \subset E$ *such that* $v_1, v_2 \in f_e(\{0,1\})\}$

Note that we will refer to both a topological graph and its underlying graph as G, and we will refer to the vertices and edges of $\mathcal{UG}(G)$ as $V(G)$, $E(G)$ respectively.

Definition A.2.4. *A topological graph is said to be **connected** if its underlying graph is connected.*

Definition A.2.5. *A **topological cycle** is a topological graph which is homeomorphic to S^1.*

Definition A.2.6. *A topological graph G is **planar** if there exists a map $\pi : G \to S^2$ such that $\pi(G)$ is homeomorphic to G (i.e. π is an embedding).*

Definition A.2.7. *Let (G, π) be a planar topological graph. We call each region in $\pi(G)^c$ a **face** of G.*

Definition A.2.8. *Let (G, π) be a planar topological graph. We use f_G to mean the number of faces of G.*

By the Jordan Curve Theorem ([11], p. 249), we can define a particular region of the plane cut out by the embedding of a cycle.

Definition A.2.9. *Let (G, π) be a planar topological graph and C a cycle of G. We define $\mathring{C}$ to be the bounded open domain with $\partial\mathring{C} = \pi(C)$.*

Definition A.2.10. *Let (G, π) be a planar topological graph and X a subset of S^2. We define the **underlying topological graph** H_X of X to be the topological subgraph $H_X \subset G$ such that $\pi(H_X) = X \cap \pi(G)$.*

Definition A.2.11. *A **topological digraph** is a topological graph whose underlying graph is oriented.*

Definition A.2.12. *A **directed topological cycle** is a topological cycle whose underlying digraph is a cycle.*

Proposition A.2.13. *Let (D, π) be a planar topological digraph whose underlying digraph is connected. If C is a directed topological cycle of D, then $H_{\overline{\mathring{C}}}$ is connected.*

Proof. Let $u, \tilde{u} \in V(H_{\overline{\mathring{C}}})$. Since D is connected, there exists a path $P = uu_1u_2 \ldots \tilde{u}$ for $u_i \in V(D)$. If all $u_i \in V(H_{\overline{\mathring{C}}})$, then we are done. Consider each sub-path $P_i = v_1^i v_2^i \ldots v_{n_i}^i$ of P such that for all vertices $v_j^i$of P_i, we have $v_j^i \notin V(H_{\overline{\mathring{C}}})$. Let N be the number of such sub-paths P_i. Since D is planar and since $\overline{\mathring{C}}$ is homemorphic to a closed disk, then for each i there must exist $\{u^i, \tilde{u}^i\}_{i=1}^N \subset C$ and paths $\tilde{P}_i$ in $H_{\overline{\mathring{C}}}$ such that P can be written as

$$P = u\tilde{P}_0 \cdots u^1 P_1 \tilde{u}^1 \tilde{P}_1 u^2 P_2 \tilde{u}^2 \cdots u^n P_n \tilde{u}_n \tilde{P}_n \tilde{u}$$

However, all of u_i are in a cycle C, and so we can construct a path $\tilde{P}$ from P by replacing each P_i with the unique path in C which has initial vertex u^i and terminal vertex $\tilde{u}^i$. Then, $\tilde{P}$ is a path from u to $\tilde{u}$ with vertices contained in $H_{\overline{\mathring{C}}}$. $\square$

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