View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by eGrove (Univ. of Mississippi)

University of Mississippi

eGrove

Electronic Theses and Dissertations | Graduate School

2013

On Binary And Regular Matroids Without Small Minors

Kayla Davis Harville
University of Mississippi

Follow this and additional works at: https://egrove.olemiss.edu/etd

Part of the Mathematics Commons

Recommended Citation

Harville, Kayla Davis, "On Binary And Regular Matroids Without Small Minors" (2013). *Electronic Theses and Dissertations*. 676.
https://egrove.olemiss.edu/etd/676

This Dissertation is brought to you for free and open access by the Graduate School at eGrove. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of eGrove. For more information, please contact egrove@olemiss.edu.

ON BINARY AND REGULAR MATROIDS WITHOUT SMALL MINORS

A Dissertation Presented for the

Doctor of Philosophy Degree

Department of Mathematics

The University of Mississippi

KAYLA DAVIS HARVILLE

May 2013

Copyright Kayla Davis Harville 2013

ALL RIGHTS RESERVED

Abstract

The results of this dissertation consist of excluded-minor results for Binary Matroids and excluded-minor results for Regular Matroids. Structural theorems on the relationship between minors and k-sums of matroids are developed here in order to provide some of these characterizations. Chapter 2 of the dissertation contains excluded-minor results for Binary Matroids. The first main result of this dissertation is a characterization of the internally 4-connected binary matroids with no minor that is isomorphic to the cycle matroid of the prism+e graph. This characterization generalizes results of Mayhew and Royle [**18**] for binary matroids and results of Dirac [**8**] and Lov*á*sz [**15**] for graphs. The results of this chapter are then extended from the class of internally 4-connected matroids to the class of 3-connected matroids. Chapter 3 of the dissertation contains the second main result, a decomposition theorem for regular matroids without certain minors. This decomposition theorem is used to obtain excluded-minor results for Regular Matroids. Wagner, Lov*á*sz, Oxley, Ding, Liu, and others have characterized many classes of graphs that are H-free for graphs H with at most twelve edges (see [**7**]). We extend several of these excluded-minor characterizations to regular matroids in Chapter 3. We also provide characterizations of regular matroids excluding several graphic matroids such as the octahedron, cube, and the Möbius Ladder on eight vertices. Both theoretical and computer-aided proofs of the results of Chapters 2 and 3 are provided in this dissertation.

Acknowledgments

In Philippians 4:13, the Apostle Paul wrote, "I can do all things through Christ who strengtheneth me." Though many people have encouraged, helped, and guided me along the way, none of this would have been possible without the love and guidance of my Saviour, Jesus Christ. He has been with me every step of the way, encouraging me each day.

This dissertation would not have been completed without the help and support of many wonderful people. Dr. Haidong Wu and Dr. James Reid showed me the mathematical beauty of matroids and taught me the art of research. They acted as mentors, guiding and encouraging me as I learned to navigate the world of research. Their support and guidance mean more to me than words can say and will never be forgotten. I would like to thank Dr. William Staton, Dr. Qingying Bu, and Dr. Dawn Wilkins for their support, patience, time, and willingness to serve on my committee. A special thanks goes to my computer expert, Yi Huang, for the time and patience she spent downloading Cygwin and MACEK and giving me the unix background I needed to use them. I would also like to thank Dr. Brian Hopkins and the Mississippi Center for Supercomputing Research for installing MACEK in the supercomputers and allowing me to use those supercomputers to run computations. I am so grateful for the funding and support provided by the GAANN team and the Department of Mathematics at Ole Miss over the last five years. A very special thanks goes to the faculty and staff at both The University of Mississippi and Mississippi State University for giving me the background in mathematics and support that I needed to succeed. It has been an honor to learn from them. Marlow Dorrough gave me the honor of coordinating a course while in graduate school. This act of confidence meant so much to me and I am grateful for the experience. The office staff and custodians Leslie Kendrick, Casandra Jenkins, Shelia Lewis, and Kay Phillips were always there to brighten my day and make sure that I was taken care of. A special thanks goes to my fellow graduate students from Ole Miss and Mississippi State. We rejoiced in each other successes and comforted each other when we hit rough patches. Thank you for the wonderful memories.

I have been blessed with a wonderful family who has stood by me every step of the way. My parents, Dennis and Sharon, have always encouraged me to follow my dreams and taught me to never back down from a challenge. When I felt like giving up and needed a kind and loving word, my parents, grandparents, and great-grandparents were there with hugs and words of support and love. My sister Miranda, brother Lee, and sister-in-law Glenda have encouraged me to do my best and never give up. The prayers, love, and support of all of my family members, friends, and my church family have seen me through good times and bad. Although there are too many of you to name here, know that I am thankful for each and every one of you. Through my family's unwavering faith in me, I found faith in myself.

Last, but not at all least, I would like to thank my husband Cody. He has loved and supported me on my worst days and my best days. Each day we spend together is a treasure, and I am so grateful to have him in my life.

Contents

List of Figures

CHAPTER 1

Introduction

The first section of this chaper contains an introduction to the excluded-minor results for graphs that motivate this research. The second section of this chapter gives the basic matroid concepts used here. The third section of this chapter contains the technical theory of matroids that underlies the research.

1. Area of Research

The concept of a matroid was introduced by Hassler Whitney in 1935 when he examined the basic properties of dependence found in both graphs and matrices [**34**]. The research problems considered in Matroid Theory are often motivated by research problems in Graph Theory and Projective Geometry. This research is broadly motivated by questions first considered in Graph Theory.

Suppose that G and H are graphs. Then G is said to be H-free if and only if no minor of G is isomorphic to H. A survey of results that characterize the H-free graphs for a particular graph H is given in [**7**]. A common theme among these results is that the graph H contains few edges. Here we generalize some of these results to the classes of regular and binary matroids. The complexity of the proofs of these H-free graph results increases as the graph H contains more edges. The class of graphic matroids is contained in the class of regular matroids which in turn is contained in the class of binary matroids. Hence the complexity of characterizing an N-free class of regular or binary matroids also increases as the number of elements of N increases. These matroid results require both the use of connectivity theory and computer-aided proofs.

We next give some background conjectures and theorems on classes of H-free graphs before exploring the concept of a matroid. Many important problems in combinatorics are related to characterizations of classes of H-free graphs. For example, the next two conjectures are among the most important open conjectures in Graph Theory. The complete graph on n-vertices is denoted by K_n (see [**33**] for graph terminology).

Conjecture 1.1 (Hadwiger [**10**]). *If a graph G is K_n-free, then G is $n-1$ colorable.*

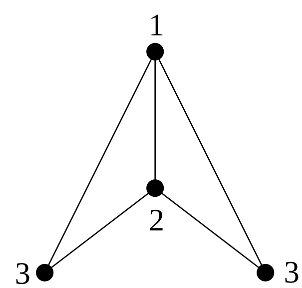


Figure 1.1. An example of a 3-colorable graph with no K_4-minor

Before stating Tutte's conjecture, we must first define a bridge and a 4-flow in a graph. A bridge of a graph is an edge whose deletion increases the number of components of the graph. A k-flow on a directed graph G, for $k \in \mathbb{Z}^+$, assigns a value in the set $\{0, 1, ..., k-1\}$ to each edge such that the sum of the flows into each vertex equals the sum of the flows out of each vertex. A nowhere-zero k-flow is a k-flow in which the value zero is not used on any edge (see figure 1.2).

CONJECTURE 1.2 (Tutte [**31**]). *If G is a bridgeless Petersen-free graph, then G admits a nowhere-zero 4-flow.*

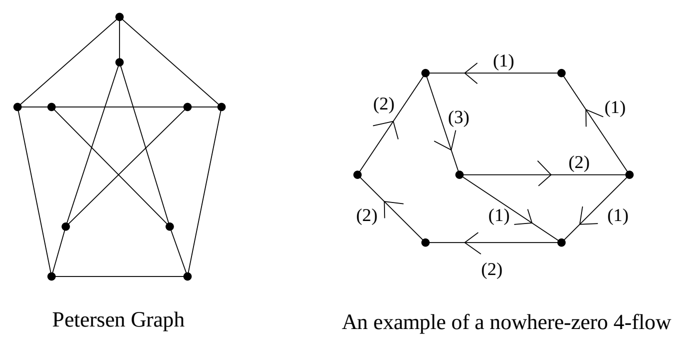


FIGURE 1.2. The Petersen graph and an example of a 4-flow

The Petersen graph has fifteen edges, so understanding the structure of the H-free graphs with fewer than fifteen edges may provide insight into the truth of Tutte's Conjecture. Understanding this structure may also provide insight into the truth of Hadwiger's Conjecture as suggested by the following result of Kawarabayashi, Norine, Thomas, and Wollan [**13**] since the graph K_6 has fifteen edges. An *apex graph* is one which contains a vertex whose deletion leaves a planar graph.

THEOREM 1.3 (Kawarabayashi et al., 2012). *There exists an absolute constant N such that every 6-connected graph on at least N vertices with no K_6-minor is apex.*

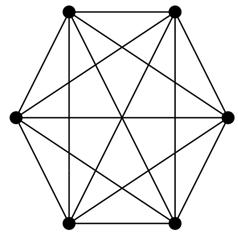


FIGURE 1.3. The complete graph on 6 vertices, K_6

Motivated by these results, Guoli Ding and Cheng Liu [**7**] announced a program of characterizing all classes of H-free graphs where H has with fewer than fifteen edges (see Chapter 3 Section 1 of the dissertation). It is this program that we will continue for regular and binary matroids. Accordingly, in Section 2 of this chapter, we discuss these classes of matroids after first introducing the concept of a matroid.

2. Matroid Concepts

In this section, we introduce the basic matroid concepts that will be used throughout the dissertation. The formal definition of a matroid as a set system is given below.

DEFINITION 1.4. *A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:*

(I1) $\emptyset \in \mathcal{I}$.

(I2) *If $I \in \mathcal{I}$ and $I' \subseteq I$, then $I' \in \mathcal{I}$.*

(I3) *If $I_1, I_2 \in \mathcal{I}$ and $|I_1| < |I_2|$, then there is an element e of $I_2 - I_1$ such that $I_1 \cup e \in \mathcal{I}$.*

The appeal and utility of matroids comes from the fact that they are associated with many important mathematical structures such as graphs. We next give some matroid terminology before discussing this association. Let $M = (E, \mathcal{I})$ be a matroid throughout this chapter. The members of $\mathcal{I}$ are called the *independent sets* of M. The subsets of E not contained in $\mathcal{I}$ are said to be *dependent*. The set E is called the ground set of M and can be denoted by $E(M)$. A *circuit* of a matroid is a minimal dependent set. The element of a circuit consisting of one element is called a *loop*. Two elements are said to be in *parallel* if they are members of a circuit of size two. The *simplification of* M, defined up to isomorphism, is the matroid obtained from M by deleting all loops of M and deleting all but one element in each non-trivial parallel class X of M. This new matroid is denoted by $si(M)$. The set of all circuits of M is denoted by $\mathcal{C}(M)$, or simply by $\mathcal{C}$. The *girth* of M, denoted by $g(M)$, is defined to be its minimum circuit cardinality if M contains a circuit; otherwise $g(M) = \infty$. Thus, a matroid M whose girth is three would contain a 3-element circuit but have no loops or parallel elements.

Let G be a graph on an edge set $E(G)$ of finite cardinality. Then G has an associated matroid $M(G)$, called the *cycle matroid of* G, defined as follows. The matroid $M(G)$ has $E(G)$ as its ground set and a subset of $E(G)$ is independent in $M(G)$ if and only if its induced subgraph does not contain a cycle. Thus $C \subseteq E(G)$ is circuit of $M(G)$ if and only if C is the edge set of a cycle of G. If $M = M(G)$ for some graph G, then M is said to be *graphic*. For example, consider the graph K_4 drawn in Figure 1.4. The associated cycle matroid $M(K_4)$ has $E = \{1, 2, 3, 4, 5, 6\}$ and circuits including $\{1, 2, 4\}$, $\{2, 3, 5\}$, $\{1, 3, 6\}$, and $\{4,5,6\}$ together with some 4-element circuits not listed above. A geometric representation for the matroid $M(K_4)$ is also given in Figure 1.4 (a), where the points of the diagram are labeled by the elements of E. The three-element circuits of this matroid are indicated by sets of three collinear points in the representation. Note that the sets $\{1,2,4\}$, $\{2,3,5\}$, $\{1,3,6\}$, and $\{4,5,6\}$ label three-point lines in the geometric representation given in Figure 1.4 (a). In general, circuits with three, four, or five points may be indicated in geometric representations of matroids with rank at most four as in Figure 1.4 (b). Note that the minimum circuit size of this matroid is three so $g(M(K_4)) = 3$.

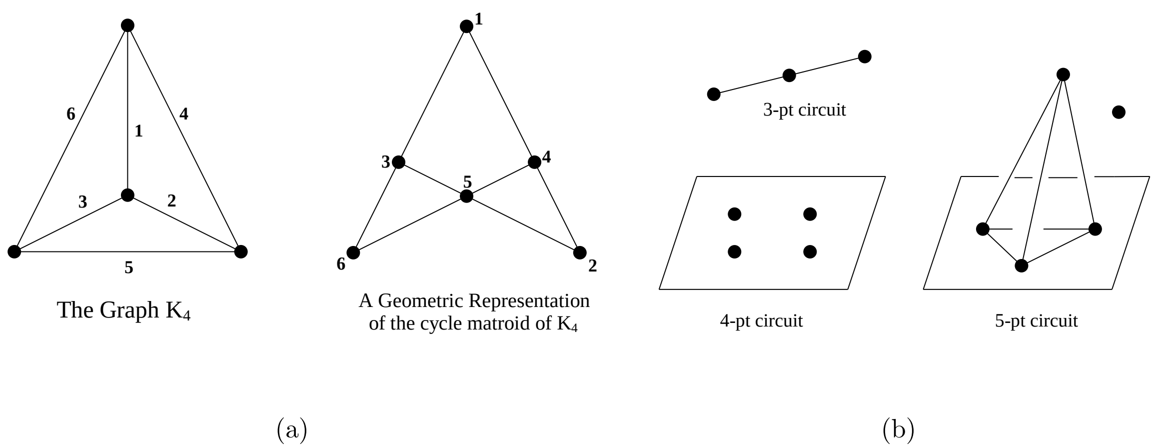


FIGURE 1.4. The complete graph K_4 and types of geometric circuits

Let E be the set of column labels of an $m \times n$ matrix A over a field F where $m, n \in \mathbb{Z}^+$. Suppose that $\mathcal{I}$ is the set of subsets X of E for which the multi-set of columns labeled by X is a linearly independent set in the vector space $V(m, F)$. Then $(E, \mathcal{I})$ is a matroid called the *vector matroid of A*. We denote this matroid by $M[A]$. A matroid is said to be *binary* if it is isomorphic to the vector matroid of a matroid that is representable over the finite field with two elements. A matroid is said to be *ternary* if it is isomorphic to the vector matroid of a matroid that is representable over the finite field with three elements. A matrix representation for the matroid $M(K_4)$ is given in Figure 1.5 where the entries of the matrix are taken from the field $GF(2)$. Here $E = \{1, 2, 3, 4, 5, 6\}$ is represented by the column labels. Note that the set $\{1, 2, 6\}$ corresponds to a set of three linearly dependent columns over $GF(2)$.

$$\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \end{array}$$

FIGURE 1.5. A binary matrix representation for the matroid $M(K_4)$

A maximal independent set in a matroid is called a *basis*. The set of bases of the matroid $M = (E, \mathcal{I})$ is denoted by $\mathcal{B}(M)$, or simply by $\mathcal{B}$. The bases of M all have a common cardinality. We call this cardinality the *rank* of M and denote it by $r(M)$. For $X \subseteq E$, the *restriction* of M to X, denoted by $M|X$, is the matroid on ground set X whose circuits are defined to as $\mathcal{C}(M|X) = \{C \subseteq X : C \in \mathcal{C}(M)\}$. We define the *deletion* of X from M, denoted by $M \backslash X$, to be the matroid $M|(E - X)$. The contraction of X from M, denoted by M/X, is the matroid on $E - X$ with circuits being the minimal nonempty members of the set $\{C - X : C \in \mathcal{C}(M)\}$. A *minor* N of M is of the form $N = M/X \backslash Y$ for disjoint subsets X and Y of E.

Let m and n be non-negative integers such that $m \leq n$. We say that M is the *uniform matroid* of rank m on an n-element set if $\mathcal{B}(M)$ is the collection of all m-element subsets of E. We denote this matroid by $U_{m,n}$. A geometric representation of the matroid $U_{2,4}$ is given in Figure 1.6.

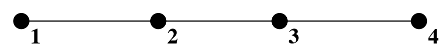


FIGURE 1.6. The matroid $U_{2,4}$

If N is a matroid, then we say that M is N-*free* if and only if M has no minor that is isomorphic to N. This generalizes the notion of a graph being H-free. Tutte [**28**] provided the following result on matroids that are representable over GF(2). This result further illustrates the fundamental importance of studying H-free matroids.

THEOREM 1.5 (Tutte, 1958). *A matroid is binary if and only if it is $U_{2,4}$-free.*

A *regular* matroid is one that can be represented as the vector matroid of a matrix over any field. Thus, it is a much stronger property that a matroid is regular than that it is binary. Here we focus on determining the structures of binary and regular matroids that are H-free for some specific matroid H. Certainly, one may consider classes of matroids that are free of several different minors. Accordingly, we make the following definition.

DEFINITION 1.6. *Let M and N be matroids. If $\mathcal{H}$ is a collection of matroids, then M is said to be $\mathcal{H}$-free if and only if M is N-free for all $N \in \mathcal{H}$.*

The *dual* of M, denoted by M^*, is a matroid on the set $E(M)$ whose set of bases is defined to be $\mathcal{B}^*(M) = \{E(M) - B : B \in \mathcal{B}(M)\}$. A circuit, basis, loop, and independent set of M^* is called a *cocircuit, cobasis, coloop*, and *coindependent set*, respectively, of M. The matroid M is called *cographic* if $M \cong M^*(G)$ for some graph G. The edge set of a minimal edge-cut of G corresponds to a cocircuit of $M(G)$. The dual of $si(M^*)$ is called the cosimplification of M. We denote this matroid by $co(M)$. The following theorem of Tutte illustrates the usefulness of an $\mathcal{H}$-free theorem by determining when a matroid is graphic or cographic in terms of excluding certain minors [**29**]. Note that the graph $K_{m,n}$ is the complete bipartite graph whose vertex set can be partitioned into two subsets X and Y such that $|X| = m$, $|Y| = n$, and each vertex in X is connected to each vertex in Y and vice versa. There are no edges between vertices of X, and no edges between vertices of Y (see Figure 1.7). Geometric representations of the Fano matroid, F_7, and its dual are also depicted in Figure 1.7.

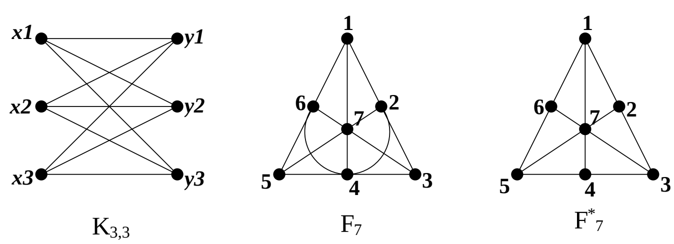


FIGURE 1.7. The complete bipartite graph $K_{3,3}$, the Fano matroid and its dual

THEOREM 1.7 (Tutte, 1959). *Let M be a matroid. Then the following statements are true.*

(i) M *is graphic if and only if* M *is* $\mathcal{X}$*-free where* $\mathcal{X} = \{U_{2,4}, F_7, F_7^*, M^*(K_5), M^*(K_{3,3})\}$.

(ii) M *is cographic if and only if* M *is* $\mathcal{Y}$*-free where* $\mathcal{Y} = \{U_{2,4}, F_7, F_7^*, M(K_5), M(K_{3,3})\}$.

The following proposition shows the relationship between minors of matroids and their duals and can be found in [**23**, Section 3.1].

Proposition 1.8. [**23**, Proposition 3.1.26] *A matroid* N *is a minor of a matroid* M *if and only if* N^* *is a minor of* M^*.

3. Technical Background

In this section, we give the technical background, concepts, and results used in this research. The terminology used here mostly follows [**23**]. Let $X \subseteq E$ throughout this section. The closure of X in M, denoted by $cl(X)$, is defined to be $cl(X) = \{x \in E : r(X \cup x) = r(X)\}$. The set X is a *flat* (sometimes called a *closed set*) of M if $cl(X) = X$. A flat of M of rank $r(M) - 1$ is called a *hyperplane*. In the following geometric representation of the matroid P_7, the flats of rank 2 are the members of the set $\{123, 345, 156, 147, 267\}$. As the rank of P_7 is three, each of these flats is also a hyperplane of P_7.

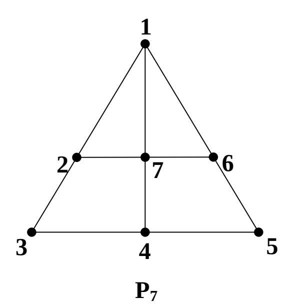


Figure 1.8. A geometric representation of P_7

Let X and Y be flats of a matroid M. Then (X, Y) is a *modular pair* of flats if and only if $r(X)+r(Y) = r(X \cup Y)+r(X \cap Y)$. If Z is a flat of M such that (Z, Y) is a modular pair for all flats Y, then Z is called a *modular flat* of M.

We next discuss the amalgam and generalized parallel connection of two matroids. Let M_i be a matroid with ground set E_i, rank function r_i, and closure operator cl_i for $i \in \{1, 2\}$. Let $E_1 \cap E_2 = T$ and $M_1|T = M_2|T = N$. If M is a rank-r matroid on ground set $E = E_1 \cup E_2$ such that $M|E_1 = M_1$ and $M|E_2 = M_2$, then M is said to be an *amalgam* of M_1 and M_2. A matroid M_0 is called a *free amalgam* of M_1 and M_2 if every independent set in M is also independent in M_0 for any other amalgam M. For any amalgam, M, of M_1 and M_2, the following holds for all $X \subseteq E$: $r_M(X) \leq r_1(X \cap E_1) + r_2(X \cap E_2) - r(X \cap T)$. Let $\zeta(X)$ be as defined in the equation below:

$$\zeta(X) = \min\{r_1(Y \cap E_1) + r_2(Y \cap E_2) - r(Y \cap T) \ : \ X \subseteq Y\}. \tag{3.1}$$

Then $\zeta(X) \geq r_M(X)$ for all $X \subseteq E$. Suppose that ζ is submodular, that is, $\zeta(X) + \zeta(Y) \geq \zeta(X \cup Y) + \zeta(X \cap Y)$ for all $X,\ Y \subseteq E$. Then the matroid M on ground set E with rank function ζ is known as the *proper amalgam* of M_1 and M_2. The following proposition reveals the relationship between the rank and ζ functions of flats in a proper amalgam (see [**23**, Proposition 11.4.3]).

Proposition 1.9. [**23**, Proposition 11.4.3] *A given matroid M is the proper amalgam of $M|E_1$ and $M|E_2$ if and only if, for every flat F of M,*

$$r(F) = \zeta(F) = r(F \cap E_1) + r(F \cap E_2) - r(F \cap T).$$

The generalized parallel connection of two matroids is an operation that allows one to combine two matroids across a common set of elements to produce another. An example of the generalized parallel connection of two matroids M_1 and M_2 across the set T is given in Figure 1.9. While generalized parallel connections can be defined in terms of amalgams, we choose an alternate definition to display here.

DEFINITION 1.10. *Let M_1 and M_2 be matroids with ground sets E_1 and E_2 such that $E_1 \cap E_2 = T$ and $M_1|T = M_2|T = N$. If $si(M_1|T)$ is a modular flat of $si(M_1)$, then $P_N(M_1, M_2)$ is the matroid on $E_1 \cup E_2$ whose flats are those subsets X of $E_1 \cup E_2$ such that $X \cap E_1$ and $X \cap E_2$ are flats of M_1 and M_2, respectively. The matroid $P_N(M_1, M_2)$ is called the generalized parallel connection of M_1 and M_2 across N.*

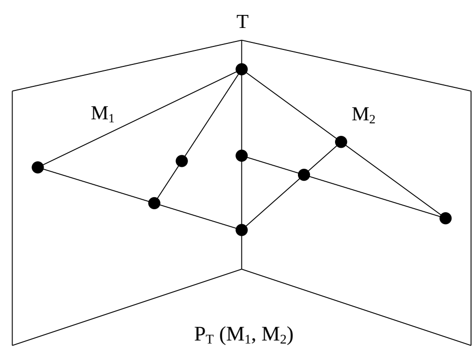


FIGURE 1.9. A geometric representation of a generalized paralled connection

The following propositions of Brylawski [**4**] give some useful properties of the generalized parallel connection $P_N(M_1, M_2)$. Many of these properties are used in the proof of Theorem 3.18 in Chapter 3 of this dissertation.

PROPOSITION 1.11 (Brylawski, 1975). *The generalized parallel connection $P_N(M_1, M_2)$ has the following properties:*

(i) $P_N(M_1, M_2)|E_1 = M_1$ *and* $P_N(M_1, M_2)|E_2 = M_2$.

(ii) *If* $si(T)$ *is a modular flat in* $si(M_2)$ *as well as in* $si(M_1)$, *then* $P_N(M_1, M_2) = P_N(M_2, M_1)$.

(iii) *The ground set of* $si(M_2)$ *is a modular flat of the simple matroid associated with* $P_N(M_1, M_2)$.

(iv) *If* $e \in E_1 - T$, *then* $P_N(M_1, M_2)\backslash e = P_N(M_1\backslash e, M_2)$.

(v) *If* $e \in E_1 - cl_1(T)$, *then* $P_N(M_1, M_2)/e = P_N(M_1/e, M_2)$.

(vi) *If* $e \in E_2 - T$, *then* $P_N(M_1, M_2)\backslash e = P_N(M_1, M_2\backslash e)$.

(vii) *If* $e \in E_2 - cl_2(T)$, *then* $P_N(M_1, M_2)/e = P_N(M_1, M_2/e)$.

(viii) *If* $e \in T$, *then* $P_N(M_1, M_2)/e = P_{N/e}(M_1/e, M_2/e)$.

(ix) $P_N(M_1, M_2)/T = (M_1/T) \oplus (M_2/T)$.

PROPOSITION 1.12 (Brylawski, 1975). *Let* $M = P_N(M_1, M_2)$ *where* $N = M_1|T = M_2|T$. *Let* cl_1, cl_2, *and* cl_M *denote the closure operators of* M_1, M_2, *and* M, *respectively. If* $X \subseteq E(M)$ *and* $X_i = cl_i(X \cap E_i) \cup X$, *then*

(i) $cl_M(X) = cl_1(X_2 \cap E_1) \cup cl_2(X_1 \cap E_2)$; *and*

(ii) $r(X) = r(X_2 \cap E_1) + r(X_1 \cap E_2) - r(T \cap [X_1 \cup X_2])$.

We next discuss the concepts of matroid connectivity that are essential to this research. The matroid M is said to be *connected* if and only if, for every pair of distinct elements in $E(M)$, there is a circuit containing both. A matroid is connected if and only if it is 2-connected. In order to define a matroid being n-connected for an integer n exceeding two, we need to introduce some additional terminology on matroid separations. We define the *connectivity function* λ_M of M as follows. For $X \subseteq E$ let

$$\lambda_M(X) = r_M(X) + r_M(E - X) - r(M). \tag{3.2}$$

One can show that λ is a submodular function, that is, $\lambda(X \cup Y) + \lambda(X \cap Y) \leq \lambda(X) + \lambda(Y)$ for $X, Y \in E(M)$. Another useful fact about the function λ_M can be found in [**23**, Section 8.1] and is provided next.

$$\lambda_M(X) = \lambda_M(E - X) = r_M(X) + r^*_M(X) - |X|. \tag{3.3}$$

Let $k \in \mathbb{Z}^+$. Then both X and $E - X$ are said to be *k-separating* if and only if $\lambda_M(X) = \lambda_M(E - X) < k$. If X and $E - X$ are k-separating and $\min\{|X|, |E - X|\} \geq k$, then $(X, E - X)$ is said to be a k-separation of M. Let $\tau(M)$ be $\min\{\ j : M$ has a j-separation$\}$ if M has a k-separation for some k; otherwise let $\tau(M) = \infty$. Let n be an integer exceeding one. Tutte defined M to be n-connected if and only if $\tau(M) \geq n$. Likewise, if $\lambda_M(X) = \lambda_M(E - X) < k$ and $\min\{r_M(X), r_M(E - X)\} \geq k$, then $(X, E - X)$ is said to be a *vertical k-separation* of M. Let $\kappa(M)$ be $\min\{\ j : M$ has a vertical j-separation$\}$ if M has a vertical k-separation for some k; otherwise let $\kappa(M) = r(M)$. Then M is *vertically n-connected* if and only if $\kappa(M) \geq n$. The concepts of graph connectivity and matroid connectivity do not generally coincide. However, the concept of vertical n-connectivity for matroids generalizes the concept of n-connectivity of graphs as we indicate in Theorem 1.13. Note that the connectivity of a graph G, denoted by $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or has only one vertex. A graph G is n-connected if its connectivity is at least k (see [**33**]).

THEOREM 1.13. [**23**, Theorem 8.6.1] *If G is a connected graph, then $\kappa(M(G)) = \kappa(G)$.*

Figure 1.10 shows an example of a graph whose cycle matroid is 3-connected.

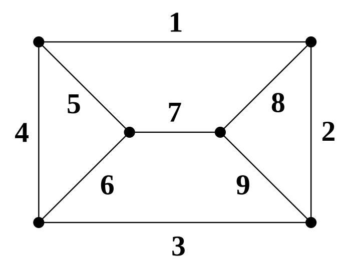


FIGURE 1.10. A graphic representation of the prism

Let $(X, E - X)$ be a k-separation of M. This separation is said to be a *minimal k-separation* if $\min\{|X|,\ |E - X|\} = k$. The matroid M is called *internally k-connected* if and only if M is $(k-1)$-connected and the only $(k-1)$-separations of M are minimal.

The following proposition reveals the connection between the connectivity of a matroid M and the connectivity of its dual.

PROPOSITION 1.14. [**23**, Corollary 8.1.5] *Let M be a matroid with ground set E. If $X \subseteq E$, then $\lambda_M(X) = \lambda_{M^*}(X)$. Moreover, M is n-connected if and only if M^* is n-connected.*

The next lemma, due to Seymour [**26**], provides the relationship between the connectivity function of a matroid M and that of a minor of M.

Lemma 1.15 (Seymour, 1980). *If N is a minor of M and $X \subset E(N)$, then $\lambda_N(X) \leq \lambda_M(X)$.*

The next proposition is due to Brylawski [**3**] and Seymour [**25**] independently.

Proposition 1.16. *If N is a connected minor of a connected matroid M and $e \in E(M) - E(N)$, then at least one of $M \backslash e$ and M/e is connected and has N as a minor.*

The next theorem connects the concepts of Tutte- and Vertical-connectivity. It is due, independently, to Oxley [**20**] and to Bixby and Cunningham [**2**].

Theorem 1.17. [**23**, Theorem 8.6.4] *Let M be a matroid and suppose that M is not isomorphic to any uniform matroid $U_{r,n}$ with $n \geq 2r-1$. Then $\tau(M) = \min\{\kappa(M), g(M)\}$.*

The operations of 1-sum (direct sum), 2-sum, and 3-sum are often used here and are described next.

Definition 1.18. *Let M_1 and M_2 be matroids on disjoint sets E_1 and E_2. The direct sum of M_1 and M_2, denoted by $M_1 \oplus M_2$, is the matroid $(E, \mathcal{I})$ where $E = E_1 \cup E_2$ and $\mathcal{I} = \{I_1 \cup I_2 : I_1 \in \mathcal{I}(M_1)$ and $I_2 \in \mathcal{I}(M_2)\}$ (see Figure 1.11).*

The definition of the 2-sum operation, as well as some properties of this operation, are discussed next.

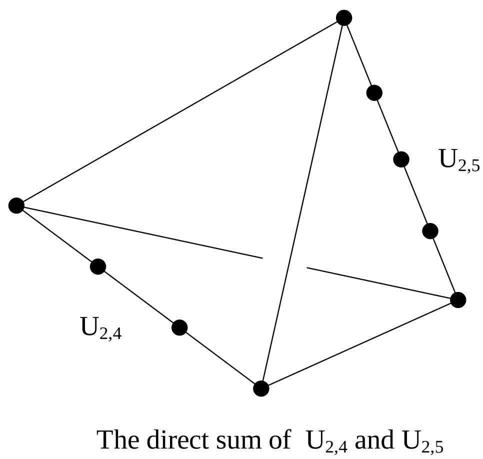


FIGURE 1.11. The direct sum of the uniform matroids $U_{2,4}$ and $U_{2,5}$

DEFINITION 1.19. *Let M and N be matroids, each with at least two elements, such that $E(M) \cap E(N) = \{p\}$ where p is neither a loop nor a coloop of M and N. Then the 2-sum of M and N, denoted by $M \oplus_2 N$, is $P(M, N) \backslash p$ (see Figure 1.12).*

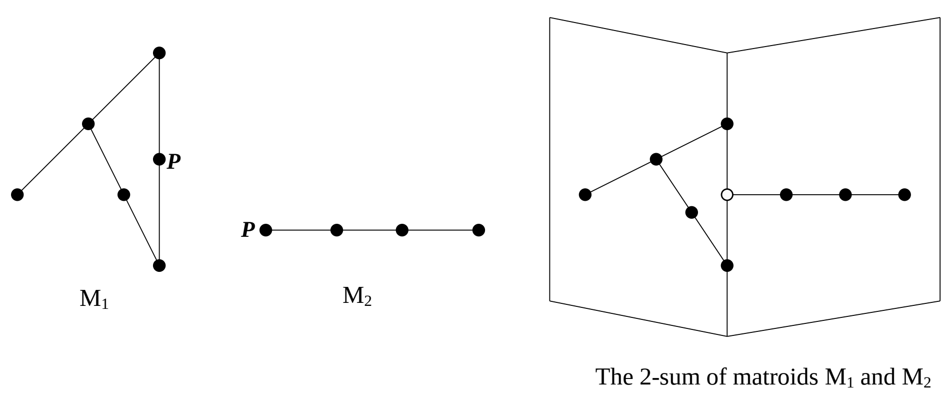


FIGURE 1.12. The 2-sum of matroids M_1 and M_2

The following proposition reveals important properties of the 2-sum operation and can be found in [**23**, Section 7.1].

Proposition 1.20. [**23**, Corollary 7.1.22] *Let M and N be matroids, each with at least two elements, such that $E(M) \cap E(N) = \{p\}$ where p is not a loop or coloop of M and N. Then the following statements are true.*

(i) $(M \oplus_2 N)^* = M^* \oplus_2 N^*$.

(ii) *Suppose that $|E(M)| \geq 2$ and $|E(N)| \geq 2$. Then $P(M, N)\backslash p$ is connected if and only if both M and N are connected. In particular, $M \oplus_2 N$ is connected if and only if both M and N are connected.*

The following result shows the relationship between minors and 2-sums of a matroid ([**23**, Proposition 8.3.5]).

Proposition 1.21. [**23**, Proposition 8.3.5] *Let M, N, M_1, and M_2 be matroids such that $M = M_1 \oplus_2 M_2$ and N is 3-connected. If M has an N-minor, then either M_1 or M_2 has an N-minor.*

The next proposition describes how a matroid that is not 3-connected can be constructed using the operations of direct sum and 2-sum [**23**, Corollary 8.3.4].

Proposition 1.22. [**23**, Corollary 8.3.4] *Every matroid that is not 3-connected can be constructed from 3-connected proper minors of itself by a sequence of the operations of direct sum and 2-sum.*

While the 2-sum operation involves the joining on and deletion of a single element p, the 3-sum operation is essentially the generalized parallel connection along a triangle followed by the deletion of said triangle.

Definition 1.23. *Let M_1 and M_2 be binary matroids such that $E(M_1) \cap E(M_2) = T$, where $|E(M_1)|, |E(M_2)| \geq 6$. Suppose that $M_1|T$ and $M_2|T$ are 3-circuits and that T does not contain a cocircuit of M_1 or M_2. Then the 3-sum $M_1 \oplus_3 M_2$ of M_1 and M_2 is $P_T(M_1, M_2)\backslash T$ (see Figure 1.13).*

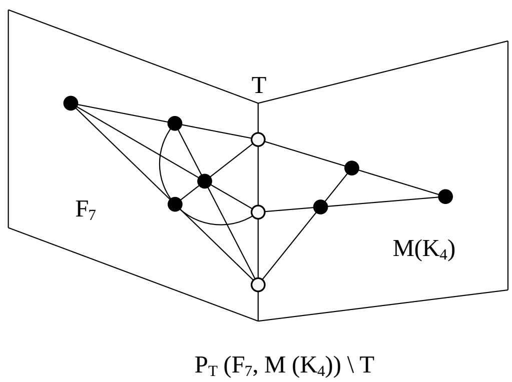


Figure 1.13. The 3-sum of matroids F_7 and $M(K_4)$

The following well-known result shows that the parts of a 3-sum $M = M_1 \oplus_3 M_2$ are isomorphic to minors of M provided that M is 3-connected [**26**].

Theorem 1.24 (Seymour, 1980). *If a 3-connected binary matroid M is the 3-sum of binary matroids M_1 and M_2, then M has minors that are isomorphic to each of M_1 and M_2 and $|E(M_i)| < |E(M)|$ for $i = 1,\ 2$.*

The definitions of the wheel graph W_n and the rank-r whirl matroid $\mathcal{W}^r$ are provided next [**9**]. These classes of graphs and matroids are of fundamental importance in Matroid Theory.

Definition 1.25. *For $n \geq 2$, the wheel W_n is the graph formed from an n-cycle C_n by adding a new vertex v and connecting v to each vertex on the rim C_n by a single edge called a spoke (see Figure 1.14).*

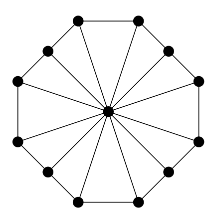


Figure 1.14. The wheel graph W_{12}

Definition 1.26. *For $r \geq 2$, the rank-r whirl $\mathcal{W}^r$ is the cycle matroid on the edge set of a wheel graph W_r whose set of circuits consists of all the cycles of W_r, except the rim, together with all sets of edges consisting of the rim plus a single spoke (see Figure 1.15).*

As you can see in Figure 1.15, the set $\{2, 4, 6\}$ is a circuit of $M(W_3)$ but not of $\mathcal{W}^3$.

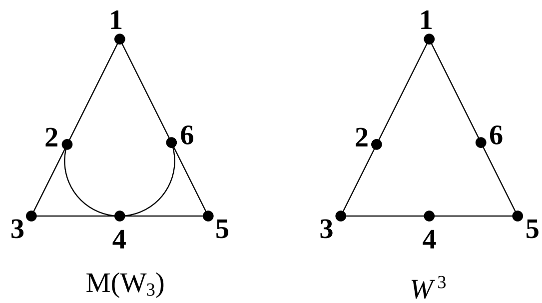


FIGURE 1.15. Geometric representations of the matroids $M(W_3)$ and $\mathcal{W}^3$

In 1966, Tutte [**30**] developed a theorem for determining when an element in a 3-connected matroid can be removed or contracted while still preserving the property of 3-connectedness.

THEOREM 1.27. *(Tutte's Wheels-and-Whirls Theorem) The following are equivalent for a* 3*-connected matroid M having at least one element.*

(i) *For every element e of M, neither $M\backslash e$ nor M/e is* 3*-connected.*

(ii) *M has rank at least three and is isomorphic to a wheel or a whirl.*

Seymour's Splitter Theorem [**26**] is a powerful inductive tool for determining classes of matroids by excluded-minors. It underlies all of the subsequent chain-type theorems that build from a small minor of a matroid up to the larger matroid. Seymour's Splitter Theorem (Theorem 1.29) considers when an element e of M can be removed or contracted without lowering the connectivity and maintaining the presence of an isomorphic copy of a particular minor of M. Before stating this theorem, we must first introduce the definition of a splitter.

Definition 1.28. *Let $\mathcal{N}$ be a class of matroids that is closed under minors and under isomorphism. A member N of $\mathcal{N}$ is called a splitter for $\mathcal{N}$ if and only if $\mathcal{N}$ has no 3-connected member having a proper N-minor.*

Theorem 1.29. *(Seymour's Splitter Theorem) Let $\mathcal{N}$ be a class of matroids that is closed under minors and under isomorphism. Let N be a 3-connected member of $\mathcal{N}$ having at least four elements such that if N is a wheel, it is the largest wheel in $\mathcal{N}$, while if N is a whirl. it is the largest whirl in $\mathcal{N}$. If there is no 3-connected member of $\mathcal{N}$ that has N as a minor and has one more element than N, then N is a splitter for $\mathcal{N}$* [**26**].

The next result of Seymour is the celebrated Decomposition Theorem for the class of regular matroids [**26**].

Theorem 1.30. *(Seymour's Decomposition Theorem) Every regular matroid M can be constructed by using direct sums, 2-sums, and 3-sums starting with matroids each of which is either graphic, cographic, or isomorphic to R_{10} (see Figure 1.16), and each of which is isomorphic to a minor of M.*

The following proposition is another result of Seymour [**26**]. It is useful in considering the relationship between a separation in a matroid and a minor of that matroid.

Theorem 1.31 (Seymour, 1980). *For disjoint subsets X and Y of the ground set of a matroid M, let $k_M(X,Y) = \min\{r(X')+r(Y')-r(M) : (X',Y')$ is a partition of E(M) with $X \subseteq X'$ and $Y \subseteq Y'\}$. Then the following statements are true.*

(i) $k_M(X,Y) = k_{M*}(X,Y)$.

(ii) *If N is a minor of M and $X, Y \subseteq E(N)$ with $X \cap Y = \emptyset$, then $k_N(X,Y) \leq k_M(X,Y)$.*

(iii) *If N is a j-connected minor of M and (X_1, Y_1) is an m-separation of M for some m with $1 \leq m < j$, then $\min\{|X_1 \cap E(N)|, |Y_1 \cap E(N)|\} \leq m - 1$.*

(iv) *If $e \in E(M) - (X \cup Y)$, then $k_M(X,Y)$ equals $k_{M\backslash e}(X,Y)$ or $k_{M/e}(X,Y)$.*

In 1980, Seymour also proved each of the next four theorems [**26**]. They are useful in describing classes of 3-connected and internally 4-connected regular and binary matroids. Standard representations of R_{10} and R_{12} are depicted over $\mathbb{R}$ in Figure 1.16. From this point on, all matrix representations of matroids will be presented in standard form, i.e. without the leading identity matrix.

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{bmatrix}$$

(a) R_{10}

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \end{bmatrix}$$

(b) R_{12}

FIGURE 1.16. Standard representations of R_{10} and R_{12} over $\mathbb{R}$

THEOREM 1.32 (Seymour, 1980). *Let M be a 3-connected regular matroid. Then either M is graphic, cographic, or M has a minor isomorphic to one of R_{10} and R_{12}.*

Theorem 1.33 (Seymour, 1980). *Let M be an internally 4-connected regular matroid. Then M is graphic, cographic, or isomorphic to R_{10}.*

Theorem 1.34 (Seymour, 1980). *If (X_1, X_2) is an exact 3-separation of a binary matroid M, with $|X_1|, |X_2| \geq 4$, then there are binary matroids M_1, M_2 on $X_1 \cup Z$, $X_2 \cup Z$, respectively (where Z contains three new elements), such that M is the 3-sum of M_1 and M_2. Conversely, if M is the 3-sum of M_1 and M_2, then $(E(M_1) - E(M_2), E(M_2) - E(M_1))$ is an exact 3-separation of M, and $|E(M_1) - E(M_2)|$, $|E(M_2) - E(M_1)| \geq 4$.*

Theorem 1.35 (Seymour, 1980). *Suppose that M is the 3-sum of binary matroids M_1 and M_2, and that M is 3-connected. If (Y_1, Y_2) is a 2-separation of M_1, then for some i, $Y_i = \{x, z\}$, where $x \in E(M_1) - E(M_2)$, $z \in E(M_2) - E(M_1)$, and x and z are parallel in M_1.*

CHAPTER 2

Binary Matroids Without a (Prism+e)-minor

The first section of this chapter gives some results from the literature on classes of graphic and binary matroids that are prism-free and (prism+e)-free. The second section of this chapter gives some lemmas that are used in the main result of the dissertation. This main result, a complete characterization of the internally 4-connected binary (prism+e)-free matroids is given in the third section of the chapter along with a classification of the 3-connected binary matroids with no (prism+e)-minor. Note that throughout the chapter, we will refer to the matroids M(prism) and M(prism+e) by simply prism and prism+e, respectively.

1. The Literature

A matroid M is said to be N-free for some matroid N if no minor of M is isomorphic to N. In this chapter, we will primarily consider T-free matroids where T represents the ten-element graphic matroid obtained from the prism by adding an edge. The graph T is the smallest twisted wheel. The twisted wheel graphs were first described in [**35**]. It is easy to check that $T \cong (prism + e)$ is self-dual and is a single-element extension of prism and a single-element coextension of $K_5\backslash e$.

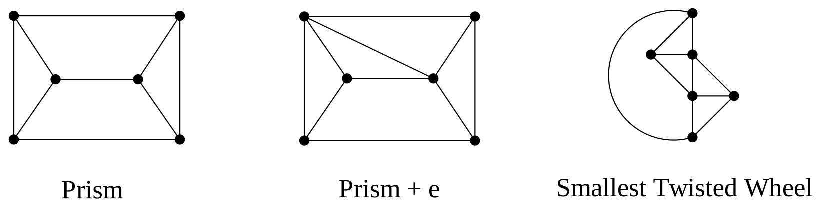


FIGURE 2.1. The graphs prism, prism+e, and the smallest twisted wheel

Dirac [**8**] and Lovász [**15**] independently characterized the class of 3-connected prism-free graphs. Let $\mathcal{K}$ be the class of 3-connected graphs G for which there exists a set X consisting of three vertices such that $G - X$ is edgeless. These graphs can be obtained from $K_{3,n}$ $(n \geq 1)$ by adding edges to its color class of size three. Let $\mathcal{W} = \{W_n : n \geq 3\}$. See Figure 2.2 for an example of a member of each class.

THEOREM 2.1 (Dirac, 1963; Lovász, 1965). *A simple* 3*-connected graph* G *is prism-free if and only if* $G \cong K_5$ *or* G *is a member of* $\mathcal{K}$ *or* $\mathcal{W}$.

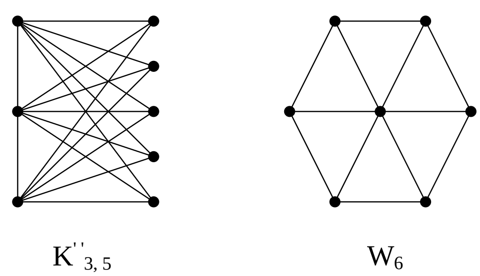


FIGURE 2.2. A member of each of the classes $\mathcal{K}$ and $\mathcal{W}$

Using Seymour's Splitter Theorem and Theorem 2.1, the class of 3-connected graphs with no (prism+e)-minor can be characterized as follows.

Theorem 2.2. *A simple* 3*-connected graph* G *is* $(prism + e)$*-free if and only if* G *is isomorphic to a graph in the set* $\{Prism, K_5\} \cup \mathcal{W} \cup \mathcal{K}$.

Mayhew and Royle [**18**] extended the result of Dirac and Lovász to characterize the class of internally 4-connected binary matroids with no prism-minor. The matroid P_{17} mentioned in the theorem below is derived from $AG(3, 2) \oplus U_{1,1}$ by completing the three-point line between every element in $AG(3, 2)$ and the single element of $U_{1,1}$. A standard representation (i.e. without the identity matrix) of the matroid P_{17} is provided in Figure 2.3.

Theorem 2.3 (Mayhew and Royle, 2012). *Let* M *be a* 3*-connected binary matroid with no prism-minor.*

(i) *If* M *is internally* 4*-connected, then* M *has rank at most five and* M *is a minor of* P_{17}.

(ii) *If* M *is* 3*-connected but not internally* 4*-connected, and* M *has an internally* 4*-connected minor with at least 6 elements that is not isomorphic to* $M(K_4)$, F_7, F_7^*, *or* $M(K_{3,3})$, *then* M *is isomorphic to one of five sporadic matroids.*

(iii) *If* M *is not internally* 4*-connected, then either* M *is isomorphic to one of five sporadic matroids or* M *can be constructed from copies of* $M(K_4)$ *and* F_7 *using parallel extensions and* 3*-sums.*

Kingan and Lemos also developed a theorem for the class of 3-connected binary non-regular matroids with no prism-minor [**14**]. Before giving this theorem, we first define the matroids that are presented there. The matroid P_9 is the generalized parallel connection,

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

FIGURE 2.3. The matroid P_{17} in standard representation form

$P_{\triangle}(F_7, W_3)$, of F_7 and W_3 across a triangle with the rim element of the triangle deleted. A matrix representation of P_9 in standard form is given in Figure 2.4.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

FIGURE 2.4. The matroid P_9 in standard representation form

Oxley characterized the 3-connected binary non-regular $\{P_9, P_9^*\}$-free matroids [**21**]. This class is made up of infinite families Z_r, Z_r^*, $Z_r \backslash y_r$, $Z_r \backslash t$ for $r \geq 3$, where Z_r is a rank r non-regular matroid with $2r + 1$ elements that can be represented by the binary matrix $[I_r|D]$ such that D has $r + 1$ columns labeled by $y_1, y_2, ..., y_r, t$. The first r columns in D have zeros along the diagonal and ones elsewhere. The last column, t, is all ones. The matroid Z_r is called the binary r-spike. All of the aforementioned infinite families are prism-free. As the prism has rank five, every binary non-regular 3-connected matroid with rank four is prism-free. Note that a matroid N is said to be a 3-decomposer of a matroid M if and only if every non-minimal exact 3-separation of M is induced by a non-minimal exact 3-separation of N. We now give Kingan and Lemos' theorem for prism-free matroids [**14**].

Theorem 2.4 (Kingan and Lemos, 2012). *Suppose M is a 3-connected binary non-regular matroid with no prism-minor. Then one of the following holds:*

(i) *M is isomorphic to Z_r, Z_r^*, $Z_r \backslash y_r$, $Z_r \backslash t$ for some $r \geq 4$,*

(ii) *P_9 is a 3-decomposer for M,*

(iii) *M is isomorphic to $(P_\triangle(F_7, F_7)\backslash z)^*$, or*

(iv) *M has rank at most five.*

Using this theorem, Kingan and Lemos [**14**] proved Mayhew and Royle's theorem [**18**].

Theorem 2.5 (Kingan and Lemos, 2012). *Let M be a binary matroid with no prism-minor.*

(i) *If M is internally 4-connected, then M has rank at most five, and is isomorphic to a minor of P_{17}.*

(ii) *If M is 3-connected but not internally 4-connected, and M has an internally 4-connected minor with at least six elements that is not isomorphic to $M(K_4)$, F_7, F_7^*, or $M(K_{3,3})$, then M is isomorphic to one of five matroids.*

2. Some Lemmas

The following theorem by Chun, Mayhew, and Oxley [**5**] is a chain theorem for internally 4-connected binary matroids that are not the cycle or dual matroids of certain classes of graphs: the terrahawk, the planar quartic ladders, and the Möbius quartic ladders. The *terrahawk* is obtained from the cube graph by adding a vertex adjacent to the four vertices in a face of the cube. The *planar quartic ladder* on $2n$ vertices for $n \geq 3$ consists of two disjoint cycles $\{u_0u_1, u_1u_2, ..., u_{n-1}u_0\} \cup \{v_0v_1, v_1v_2, ..., v_{n-1}v_0\}$ and two perfect matchings $\{u_0v_0, u_1v_1, ..., u_{n-1}v_{n-1}\} \cup \{u_0v_{n-1}, u_1v_0, ..., u_{n-1}v_{n-2}\}$. It is important to note that each planar quartic ladder contains all smaller planar quartic ladders as minors. The *octahedron* is the smallest planar quartic ladder. The *Möbius quartic ladder* on $2n-1$ vertices for $n \geq 3$ consists of a Hamilton cycle $\{v_ov_1, v_1v_2, ..., v_{2n-2}v_0\}$ and the edge set $\{v_iv_{i+n-1},\ v_iv_{i+n} | 0 \leq i \leq n-1\}$, where all subscripts are read modulo $2n-1$. As with planar quartic ladders, each Möbius quartic ladder contains all smaller ones as minors. The smallest graph in this particular class is K_5. Depicted below is the cube, terrahawk, octahedron, and the Möbius quartic ladder on seven vertices.

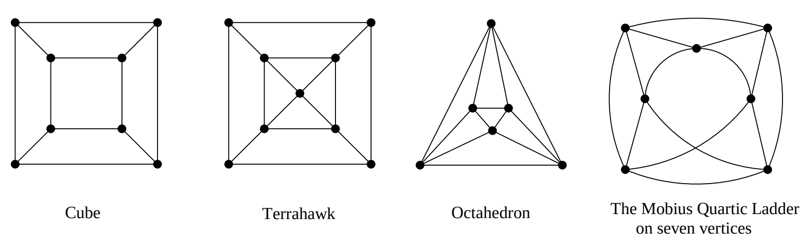


FIGURE 2.5. The graphs of the cube, terrahawk, octahedron, and the Möbius quartic ladder on 7 vertices

Theorem 2.6 (Chun, Mayhew, and Oxley, 2011). *Let M be an internally 4-connected binary matroid such that $|E(M)| \geq 7$. Then M has a proper internally 4-connected minor N with $|E(M)| - |E(N)| \leq 3$ unless M or its dual is the cycle matroid of a planar quartic ladder, Möbuis quartic ladder, or a terrahawk.*

The following two lemmas are analogues of results of Mayhew and Royle for the (prism+e)-matroid [**18**].

Lemma 2.7. *The cycle and dual matroids of the terrahawk, the planar quartic ladders, and the Möbius quartic ladders with at least 7 vertices all have prism+e as a minor. Note that the smallest Möbius quartic ladder, $M(K_5)$, and its dual do not have prism+e as a minor.*

Proof. We first note that the cube-matroid has prism+e as a minor. The cycle matorid of the terrahawk has a cube-minor which implies that terrahawk has a (prism+e)-minor as well. As the terrahawk is self-dual, the dual matroid of the terrahawk also has a (prism+e)-minor. The smallest planar quartic ladder is the octahedron which has a (prism+e)-minor. As all of the planar quartic ladders have an octahedron minor, they must also have prism+e as a minor. The dual of the octahedron, as well as the duals of all larger planar quartic ladders, have a cube-minor, which in turn has a (prism+e)-minor. The Möbius quartic ladder on seven vertices contains a (prism+e)-minor. Thus, all Möbius quartic ladders on at least seven vertices contain a (prism+e)-minor as well. The dual matroids of the Möbius quartic ladders on at least seven vertices also contain prism+e as a minor. □

The following lemma develops a sequence of (prism+e)-free 3-connected matroids. The proof is almost identical to Mayhew and Royle's proof for prism-free matroids [**18**]. The proof is included here for completeness.

LEMMA 2.8. *Let M be an internally 4-connected binary matroid such that $|E(M)| \geq 7$ and M is (prism+e)-free. If M is not isomorphic to $M(K_5)$, $M^*(K_5)$, $M(K_{3,3})$, or $M^*(K_{3,3})$, then there is a sequence $M_0, M_1, ..., M_t$ of 3-connected matroids such that:*

(i) *M_0 is internally 4-connected,*

(ii) *$M_t = M$,*

(iii) *$1 \leq t \leq 3$, and*

(iv) *M_{i+1} is a single-element extension or coextension of M_i for every $i \in \{0, 1, ..., t-1\}$.*

PROOF. By Theorem 2.6 and Lemma 2.7, M contains an internally 4-connected minor N such that $1 \leq |E(M)| - |E(N)| \leq 3$. Let $M_0 = N$. If N is not a wheel, then the result follows from Seymour's Splitter Theorem. Suppose that N is a wheel, say $N \cong M(W_n)$ for some n. If $n \geq 4$, then $M(W_n)$ is not internally 4-connected. Therefore, $N \cong M(W_3)$. If M has no larger wheel as a minor, the result follows from Seymour's Splitter Theorem. Assume that M has a W_4-minor. Note that $|E(M)| - |E(W_3)| \leq 3$ implies that $|E(M)| \leq |E(W_3)| + 3 = 9$. As $|M(W_4)| = 8$ and $M(W_4)$ is not internally 4-connected, we may conclude that M is a single-element extension or coextension of $M(W_4)$. Assume that M is an extension of $M(W_4)$. Consider the binary representation of $M(W_4)$:

$$\begin{array}{c}
\begin{array}{cccccccc} A & B & C & D & E & F & G & H \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 1
\end{pmatrix}
\end{array}$$

FIGURE 2.6. The matroid $M(W_4)$

As M is an extension of $M(W_4)$, M can be formed by adding a column I to the above matrix. If the first entry in the new column I is a zero, then $\{A, B, E\}$ is a triangle and $\{A, E, H\}$ is a triad of M; a contradiction as M is internally 4-connected. Thus, the first entry in column I must be a one. This argument can be repeated to show that each entry in I is a one. Hence $M \cong M^*(K_{3,3})$; a contradiction. Dually, if M is a coextension of $M(W_4)$, it can be shown that $M \cong M(K_{3,3})$; a contradiction. Hence the lemma holds. □

The next lemma follows directly from Proposition 1.14 and the definition of internal 4-connectivity.

LEMMA 2.9. *A matroid N is internally 4-connected if and only if N^* is internally 4-connected.*

The cycle matroid of the prism+e graph is self-dual. We note this useful fact in next lemma.

LEMMA 2.10. *Let T represent the cycle matroid of the prism graph plus an edge. Then a matroid N is T-free if and only if N^* is T-free.*

3. Results

Mayhew and Royle [**18**] classified all internally 4-connected binary prism-free matroids. They found that there are exactly forty-two such matroids in this class. In the main result of this chapter, this classification has been extended to find all internally 4-connected binary (prism+e)-free matroids. We determined that there are 90 such matroids: 42 of which have no prism-minor and were found by Mayhew and Royle [**18**], and 48 of which have a prism-minor but are (prism+e)-free. In order to name this class, we introduce the following notation: $EX_{nc}(M)$ where M is the matroid to be excluded from the set and nc represents the connectivity of the set. Henceforth, this set of 90 matroids will be denoted by $EX_{i4c}(prism+e)$, where $i4c$ represents internal 4-connectivity. The following theorem describes this class and reveals certain characteristics that each of these ninety matroids share. This theorem can be proven in one of two ways, both of which use the matroid computing software MACEK [**12**]. The five maximal matroids mentioned in the theorem are depicted in Figure 2.7 (see Appendix A for a complete list of the ninety matroids).

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) P_{17}

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(b) P_{17}^*

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(c) Q_{15}

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix}$$

(d) Q_{15}^*

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

(e) Q_{12}

FIGURE 2.7. The five maximal matroids in the class $EX_{i4c}(prism + e)$

Theorem 2.11. *An internally 4-connected binary matroid M has no (prism+e)-minor if and only if M is one of ninety matroids in the set $EX_{i4c}(prism + e)$. Each such matroid is an internally 4-connected minor of at least one of the matroids P_{17}, P_{17}^*, Q_{15}, Q_{15}^*, or Q_{12}; has at most 17 elements; and has rank or corank at most 5 with the exception of Q_{12}, where Q_{12} is a 12-element matroid in $EX_{i4c}(prism + e)$ having rank and corank 6.*

Proof. **Claim 1.** There are 90 internally 4-connected binary matroids that are (prism+e)-free and have at most 17 elements.

Proof of Claim 1. Let M be an internally 4-connected binary (prism+e)-free matroid. Then either M is prism-free or M has a prism-minor. If M is prism-free, then M is one of the forty-two matroids found by Mayhew and Royle [**18**]. Each of these forty-two matroids is a minor of P_{17} and has rank or corank at most 5. Suppose that M has a prism-minor. By Seymour's Splitter Theorem, there exists a chain of matroids beginning at prism and ending at M. Using MACEK [**12**], we extend and coextend prism eight times with the command

```
./macek '!extend bbbbbbbb;@ext-forbid prism+e;!print;!isconni4' prism
```

and obtain forty-eight internally 4-connected binary (prism+e)-free matroids with up to 17 elements. The four maximal (prism+e)-free matroids with a prism-minor are P_{17}^*, Q_{15}, Q_{15}^*, and Q_{12}. None of these matroids are minors of the other three and each of the other forty-four (prism+e)-free matroids with a prism-minor are minors of at least one of P_{17}^*, Q_{15}, Q_{15}^*, or Q_{12}. This can be verified through extensive computations using the MACEK command: ./**macek minor matroid1 matroid2** where **matroid1** and **matroid2** are distinct members of the set of forty-eight internally 4-connected binary (prism+e)-free matroids with a prism-minor and **matroid1** is the larger matroid. Each of the forty-eight matroids mentioned here have rank or corank at most 5 except for Q_{12}, which is a matroid with rank and corank 6. □

Claim 2. There does not exist an internally 4-connected binary (prism+e)-free matroid M such that $|E(M)| \geq 18$.

Proof of Claim 2. Suppose such a matroid M exists and $|E(M)| = 18$. By Theorem 2.6 and Lemma 2.7, M has a proper internally 4-connected minor L such that $|E(L)| \in \{15, 16, 17\}$. Suppose that $|E(L)| = 17$. If L has no prism-minor, then $L \cong P_{17}$ [**18**]. If L has a prism-minor, then $L \cong P_{17}^*$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) P_{17}

FIGURE 2.8. The matroid P_{17}

Using MACEK [**12**] to extend and coextend the matroids P_{17} and P_{17}^* with the commands

```
./macek '!extend b;@ext-forbid prism+e;!print;!isconni4' P17
./macek '!extend b;@ext-forbid prism+e;!print;!isconni4' 'P17;!dual'
```

yields no internally 4-connected (prism+e)-free matroids with 18 elements so $|E(L)| \neq 17$.

Suppose that $|E(L)| = 16$. If L is prism-free, then $L \cong M34$ or $M35$ [**18**]. If L has a prism-minor, then $L \cong (M34)^*$ or $(M35)^*$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) M34

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(b) M35

FIGURE 2.9. The matroids M34 and M35

Using MACEK [**12**] to extend and coextend all four of these possible matroids twice yields no internally 4-connected (prism+e)-free matroids with 18 elements. An example of the MACEK command used here is

```
./macek '!extend bb;@ext-forbid prism+e;!print;!isconni4' M34
```

Hence $|E(L)| = 15$. If L is prism-free, then L is isomorphic to $PG(3,2)$, $M31$, $M32$, or $M33$ by [**18**]. If L has an prism-minor, L is isomorphic to one of the following matroids: $(PG(3,2))^*$, $(M31)^*$, $(M32)^*$, $(M33)^*$, Q_{15} or Q_{15}^*.

$$\begin{bmatrix} 1&1&1&0&0&0&1&1&1&0&1\\ 1&0&0&1&1&0&1&1&0&1&1\\ 0&1&0&1&0&1&1&0&1&1&1\\ 0&0&1&0&1&1&0&1&1&1&1 \end{bmatrix}$$

(a) $PG(3,2)$

$$\begin{bmatrix} 0&1&1&1&1&0&0&0&0&1\\ 1&0&1&1&0&1&0&0&1&0\\ 1&1&0&1&0&0&1&0&1&1\\ 1&1&1&0&0&0&0&1&1&1\\ 0&0&0&0&1&1&1&1&1&1 \end{bmatrix}$$

(b) M31

$$\begin{bmatrix} 0&1&1&1&0&0&0&0&1&1\\ 1&0&1&0&1&0&0&1&0&1\\ 1&1&0&0&0&1&0&1&1&0\\ 1&1&1&0&0&0&1&1&1&1\\ 0&0&0&1&1&1&1&1&1&1 \end{bmatrix}$$

(c) M32

$$\begin{bmatrix} 0&1&1&1&0&0&0&0&1&1\\ 1&0&1&1&1&0&0&1&0&1\\ 1&1&0&1&0&1&0&1&1&0\\ 1&1&1&0&0&0&1&1&1&1\\ 0&0&0&0&1&1&1&1&1&1 \end{bmatrix}$$

(d) M33

$$\begin{bmatrix} 1&1&0&0&0&0&1&1&1&1\\ 1&0&0&1&1&1&0&0&1&1\\ 0&1&0&1&0&0&0&0&0&0\\ 0&1&1&0&1&1&0&1&0&1\\ 0&0&1&1&0&1&1&1&1&0 \end{bmatrix}$$

(e) Q_{15}

FIGURE 2.10. The matroids $PG(3,2)$, $M31$, $M32$, $M33$, and Q_{15}

Use MACEK [**12**] to extend and coextend each of these nine possible matroids three times. An example of the command used is

```
./macek '!extend bbb;@ext-forbid prism+e;!print;!isconni4' Q15.
```

This process yields no internally 4-connected matroids with eighteen elements. Hence $|E(M)| \neq 18$.

Suppose that $|E(M)| = 19$. By Theorem 2.6 and Lemma 2.7, M has a proper internally 4-connected minor L such that $|E(L)| \in \{16, 17, 18\}$. As there are no internally 4-connected (prism+e)-free matroids with 18 elements, L must have either 16 or 17 elements. Suppose $|E(L)| = 17$. Then, as before, L is either P_{17} or P_{17}^*. Using MACEK [**12**] to extend and coextend these matroids twice yields no internally 4-connected (prism+e)-free matroids with 19 elements. Hence $|E(L)| = 16$. Then L is isomorphic to $M34$, $M35$, $(M34)^*$, or $(M35)^*$. Extending and coextending these matroids three times yields no internally 4-connected (prism+e)-free matroids with 19 elements. Hence $|E(M)| \neq 19$.

Suppose that $|E(M)| = 20$. By Theorem 2.6 and Lemma 2.7, M has a proper internally 4-connected minor L such that $|E(L)| \in \{17, 18, 19\}$. As there are no internally 4-connected (prism+e)-free matroids with 18 or 19 elements, we conclude that L must have 17 elements and therefore is isomorphic to P_{17} or P_{17}^*. However, extending and coextending each of these matroids three times yields no internally 4-connected (prism+e)-free matroids with 20 elements.

Suppose that $|E(M)| \geq 21$. By continually using Theorem 2.6 and Lemma 2.7, it is implied that M has a proper internally 4-connected minor L such that $|E(L)| \in \{18, 19, 20\}$. However, there are no internally 4-connected (prism+e)-free matroids with 18, 19, or 20 elements. Therefore, no such matroid M exists by Theorem 2.6.Hence there does not exist any internally 4-connected (prism+e)-free matroids with more than 17 elements. This completes the proof of Theorem 2.11. □

Next we give another way to prove Theorem 2.11 using Lemma 2.7, Lemma 2.8, and Theorem 2.6 in addition to MACEK [**12**]. Let M be a minimum counterexample to the theorem. If M is an internally 4-connected binary (prism+e)-free matroid with no prism-minor, then M is one of the 42 matroids found by Mayhew and Royle [**18**] which are in the set $EX_{i4c}(prism + e)$. Now suppose that M is an internally 4-connected binary (prism+e)-free matroid that has a prism-minor. Then M is not isomorphic to $M(K_5)$, $M(K_{3,3})$, or $M^*(K_{3,3})$ as these are all prism-free. By Lemma 2.8, there is a sequence $M_0, M_1, ..., M_t$ such that the following statements (a) through (d) are true. Note that $M_0 \in EX_{i4c}(prism + e)$ by the minimality of M.

(a) M_0 has at least six elements,

(b) $M_t = M$,

(c) $1 \leq t \leq 3$, and

(d) M_{i+1} is a single-element extension or coextension of M_i for all $i \in \{0, 1, ..., t-1\}$.

(e) M does not contain a minor L where $|E(L)| = |E(M_0)| + 1$ and L is isomorphic to one of the 90 internally 4-connected binary (prism+e)-free matroids.

Assume that M and M_0 were chosen so that t is as small as possible. Then there does not exist a matroid $M' \in EX_{i4c}(prism + e)$ such that M' is a minor of M whose size is $|E(M_0)| + 1$, as that would contradict t being as small as possible. Hence (e) is true. Therefore, the matroid M will be found in the MACEK search satisfying (a) - (e).

For example, suppose that M_0 is isomorphic to the fifteen-element internally 4-connected matroid $PG(3,2)$. Use the MACEK command

```
./macek '!extend bbb;@ext-forbid prism+e M34
"M34;!dual" M35 "M35;!dual";!print;!isconni4' PG32
```

to find M. The five matroids we forbid (other than prism+e) are members of the set of 90 matroids that have sixteen elements. However, no new matroids satsifying (a) - (e) were found in the MACEK search starting from all matroids M_0 with at least six elements in the set $EX_{i4c}(prism+3)$. This contradiction completes the proof of Theorem 2.11. □

Next we determine all 3-connected binary (prism+e)-free matroids that are not internally 4-connected. This set is denoted by $EX_{3c-i4c}(prism+e)$. There are 42 sporadic matroids in $EX_{3c-i4c}(prism+e)$, each of which has between eleven and sixteen elements (see Appendix B for a list of the forty-two sporadic matroids as well as the description of how these matroids were found). The rest of the members of $EX_{3c-i4c}(prism+e)$ can be constructed from copies of $M(K_4)$, F_7, $M^*(K_{3,3})$, P_{10}, O_{10}, or P_{11} using parallel extensions and 3-sums. Standard representations of P_{10}, O_{10}, and P_{11} are provided in Figure 2.11.

THEOREM 2.12. *Let M be a 3-connected $(prism+e)$-free binary matroid that is not internally 4-connected. If M contains a minor isomorphic to a matroid in $EX_{i4c}(prism+e)$ having at least 6 elements and is not $M(K_4)$, F_7, F_7^*, $M(K_{3,3})$, $M^*(K_{3,3})$, P_{10}, P_{10}^*, O_{10}, R_{10}, P_{11}, or P_{11}^*, then M is one of 42 sporadic matroids contained in the set $EX_{3c-i4c}(prism+e)$. Each of these forty-two matroids has between 11 and 16 elements.*

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

(a) P_{10}

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(b) O_{10}

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(c) P_{11}

FIGURE 2.11. Standard representations of P_{10}, O_{10}, and P_{11}

PROOF. Using MACEK [**12**], it is possible to find all 3-connected binary (prism+e)-free matroids having the property that there is a sequence of matroids $M_0, M_1, ..., M_t$ such that the following statements are true.

(a) M_0 has at least six elements and is one of the 90 internally 4-connected (prism+e)-free matroids in $EX_{i4c}(prism + e)$,

(b) $M_t = M$,

(c) $1 \leq t \leq 5$,

(d) M_{i+1} is a single-element extension or coextension of M_i for all $i \in \{0, 1, ..., t-1\}$, and

(e) M does not contain a minor N where $|E(N)| = |E(M_0)| + 1$ and N is isomorphic to one of the 90 internally 4-connected binary (prism+e)-free matroids.

Assume that M is a minimal counterexample to the theorem. Then M has an internally 4-connected minor M_0 such that $|E(M_0)| \geq 6$ and M_0 is not isomorphic to a matroid in the set $D = \{M(K_4), F_7, F_7^*, M(K_{3,3}), M^*(K_{3,3}), P_{10}, P_{10}^*, O_{10}, R_{10}, P_{11}, \text{or } P_{11}^*\}$. By Theorem 2.11, M_0 is one of the internally 4-connected matroids described in the set $EX_{i4c}(prism + e) - D$.

Then there is a sequence of 3-connected matroids $M_0, M_1, ..., M_t$ such that $M_t = M$ and each M_k is a single-element extension or coextension of M_{k-1} for $1 \leq k \leq t$ by Seymour's Splitter Theorem. Assume that M and M_0 have been chosen so that $|E(M)| - |E(M_0)| = t$ is as small as possible. Suppose that M has an internally 4-connected minor N such that $|E(N)| = |E(M_0)| + 1$. As t is minimal, N must be isomorphic to a matroid in the set D. This implies that M_0 has at most ten elements and is therefore isomorphic to $M(K_4)$, F_7, F_7^*, $M(K_{3,3})$, $M^*(K_{3,3})$, P_{10}, P_{10}^*, O_{10}, R_{10}, $M(K_5)$, or $M^*(K_5)$. By the choice of M_0, $M \cong M(K_5)$ or $M^*(K_5)$. Thus N must be P_{11} or P_{11}^*. However, neither $M(K_5)$ nor $M^*(K_5)$ is a minor of either P_{11} or P_{11}^*. This contradiction shows that (e) holds.

If M_{t-1} is internally 4-connected, then $t = 1$. Assume that M_{t-1} is not internally 4-connected. By the minimality of t, M_{t-1} must be one of the 42 sporadic matroids in the set $EX_{3c-i4c}(prism + e)$. From Appendix B, we can see that each of these matroids M has an internally 4-connected minor M' such that $|E(M)| - |E(M')| \leq 4$ and $M \notin D$. It follows from this that $M_0 = M_{t-5}$, $M_0 = M_{t-4}$, $M_0 = M_{t-3}$, or $M_0 = M_{t-2}$ which implies that $t \leq 5$. Thus, the matroid M will be found by applying the MACEK procedure satsifying (a) - (e) to the matroids in $EX_{i4c}(prism + e) - D$. However, no new 3-connected matroids other than the 42 sporadic matroids are found by using Macek to extend and coextend the internally 4-connected matroids in the set $EX_{i4c}(prism + e) - D$ at most five times. This contradiction completes the proof of Theorem 2.12. $\square$

Next we determine all 3-connected binary $(prism + e)$-free matroids.

Theorem 2.13. *Let M be a 3-connected binary $(prism + e)$-free matroid. Then one of the following is true:*

(i) *M is one of the 90 matroids in $EX_{i4c}(prism + e)$,*

(ii) *M is one of 42 sporadic matroids in $EX_{3c-i4c}(prism + e)$, or*

(iii) *M can be constructed from copies of $M(K_4)$, F_7, $M^*(K_{3,3})$, P_{10}, O_{10}, or P_{11} using parallel extensions and 3-sums.*

Proof. Suppose that M is a counterexample chosen so that $|E(M)|$ is minimal. Then M is not internally 4-connected. Hence $M = M_1 \oplus_3 M_2$ where M_1 and M_2 are minors of M by Theorem 1.24 and $|E(M_i)| < |E(M)|$ for $i = 1, 2$. Note that $si(M_1)$ and $si(M_2)$ are 3-connected by Theorem 1.35. It follows from the facts that M is (prism+e)-free and M_1 and M_2 are minors of M that M_1 and M_2 are (prism+e)-free. By induction, the theorem holds for both $si(M_1)$ and $si(M_2)$. Assume that $si(M_1)$ is one of the forty-two sporadic matroids in $EX_{3c-i4c}(prism + e)$. Then $si(M_1)$ contains an internally 4-connected minor not isomorphic to a matroid in the set $D = \{M(K_4), F_7, F_7^*, M(K_{3,3}), M^*(K_{3,3}), P_{10}, P_{10}^*, O_{10}, R_{10}, P_{11}, P_{11}^*\}$ by Theorem 2.12. Hence M_1, and therefore M, contains an internally 4-connected minor not isomorphic to a matroid in D. By Theorem 2.12, M is isomorphic to one of the forty-two sporadic matroids in $EX_{3c-i4c}(prism + e)$; a contradiction.

Assume that $si(M_1)$ is internally 4-connected; that is, $si(M_1) \in EX_{i4c}(prism + e)$. Since M_1 is part of a 3-sum, M contains a triangle T that does not contain a cocircuit of M_1. This implies that $r(M_1) \geq 3$. It follows from the fact that $si(M_1)$ is 3-connected that $|si(M_1)| \geq 6$. Since $|E(si(M_1))| \geq 6$ and M is not one of the 42 sporadic matroids in $EX_{3c-i4c}(prism+e)$, it follows from Theorem 2.12 that $si(M_1) \in D$. As M_1 contains a triangle, $si(M_1)$ is isomorphic to $M(K_4)$, F_7, $M^*(K_{3,3})$, P_{10}, O_{10}, or P_{11}.

Suppose that $si(M_1)$ is not internally 4-connected. Then, by the induction hypothesis, M_1 can be constructed from copies of $M(K_4)$, F_7, $M^*(K_{3,3})$, P_{10}, O_{10}, or P_{11} using parallel extensions and 3-sums. Similarly, either $si(M_2)$ is internally 4-connected and is isomorphic to $M(K_4)$, F_7, $M^*(K_{3,3})$, P_{10}, O_{10}, or P_{11} or M_2 can be constructed from copies of $M(K_4)$, F_7, $M^*(K_{3,3})$, P_{10}, O_{10}, or P_{11} using parallel extensions and 3-sums. Hence $M = M_1 \oplus_3 M_2$ can be constructed in this manner as well. This completes the proof of the theorem. $\square$

CHAPTER 3

Some Excluded Minor Classes of Regular Matroids

Seymour's Decomposition Theorem states that every regular matroid M can be constructed by using direct sums, 2-sums, and 3-sums starting with matroids that are minors of M, each of which is either graphic, cographic, or is isomorphic to R_{10} [**26**]. In this chapter, we provide a decomposition theorem for regular matroids without certain minors. The first section of this chapter contains a number of results characterizing classes of H-free graphs for some graph H. The second section of this chapter contains technical lemmas needed to extend some of the results in Section 1 to the class of regular matroids. The third section of this chapter contains the proof of the aforementioned decomposition theorem as well as characterizations of regular matroids without certain minors such as $M(K_5)$, $M(K_5^{\perp})$, $M(V_8)$, $M^*(V_8)$, $M(cube)$, $M(octahedron)$, $\{M(W_5+e),\ M^*(W_5+e)\}$, $M(K_{3,3}')$, and $\{M(K_{3,3}''),\ M^*(K_{3,3}'')\}$.

1. The Literature

In the following result, Wagner [**32**] characterized all K_5-free graphs. Before stating this theorem, we must first define the k-clique-sum of a graph. If two graphs G and H each contain cliques of equal size, the *clique-sum* of G and H is formed from their disjoint union by identifying pairs of vertices in these two cliques and then possibly deleting some of the clique edges. A *k-clique-sum* is a clique-sum in which both cliques have k vertices.

THEOREM 3.1 (Wagner, 1937). *A simple 3-connected graph G is K_5-free if and only if $G \cong V_8$ or G can be constructed from 3-clique-sums of 3-connected planar graphs.*

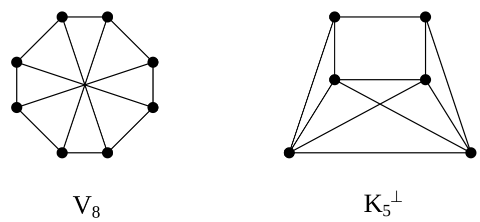


FIGURE 3.1. The graphs V_8 and $K_5^{\perp}$

The graph $K_5^{\perp}$, as shown in Figure 3.1, can be constructed from K_5 by splitting a vertex. The characterization of simple 3-connected $K_5^{\perp}$-graphs follows from Theorem 3.1 and is surely known.

THEOREM 3.2. *A simple 3-connected graph G is $K_5^{\perp}$-free if and only if $G \cong K_5$, $G \cong V_8$ or G can be constructed from 3-clique-sums of 3-connected planar graphs.*

The classes of graphs that are octahedron-free ($\mathcal{O}_8$-free), cube-free, and V_8-free are described next. In order to discuss the $\mathcal{O}_8$-free graphs, we must first define the square of odd cycles C_{2n+1}^2 for $n \geq 2$. The graph C_{2n+1}^2 can be obtained from the cycle C_{2n+1} by adding an edge between every pair of vertices of distance two in the cycle. Note that $C_5^2 = K_5$.

Maharry [**16**] proved that any 4-connected $\mathcal{O}_8$-free graph is isomorphic to C^2_{2n+1} for $n \geq 2$. Ding [**6**] characterized all $\mathcal{O}_8$-free graphs in the theorem below. Note that the 0-sum of two graphs G and H is the disjoint union of these two graphs.

THEOREM 3.3 (Ding, 2010). *A graph G is $\mathcal{O}_8$-free if and only if it is constructed by* 0-, 1-, 2-, *and* 3-*clique-sums starting from graphs in the set* $\{K_1, K_2, K_3, K_4\} \cup \{C^2_{2n+1} : n \geq 2\} \cup \{L_5, G_{0814}, G_{1015}, G_{1016}, G_{1117}\}$ *(see Figure 3.2).*

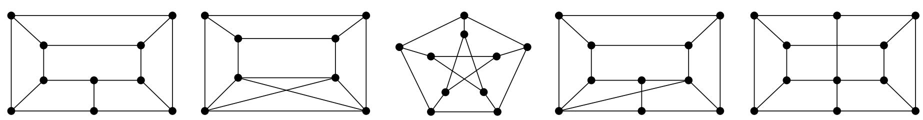


FIGURE 3.2. The graphs L_5, G_{0814}, G_{1015} (Petersen Graph), G_{1016}, G_{1117}

Maharry [**17**] proved that any 4-connected cube-free graph is a minor of the line graph of V_n for some $n \geq 6$ or a minor of one of five graphs. The graph V_n is sometimes referred to as the *Möbius Ladder on n vertices* and is depicted in Figure 3.3. In the paper *A Characterization of Graphs with No Cube Minor* [**17**], Maharry proves the existance of a unique 4-connected graph G and a unique 5-connected graph H that each have at least eight vertices and are each cube-free. He also shows that any cube-free graph can be constructed from 4-connected such graphs by $0-$, $1-$, and 2-summing, and 3-clique-summing over a specified triangle.

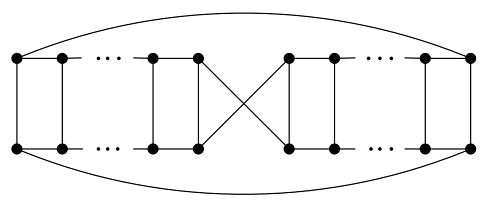


FIGURE 3.3. The graph V_n

The following theorem is an unpublished result by N. Robertson. This characterization of internally 4-connected V_8-free graphs can be found in [**27**]. Note that an internally 4-connected graph G is a simple 3-connected graph with at least five vertices such that at least one side of every 3-separation of G has at most three edges.

THEOREM 3.4. *(in [**27**]) Let G be an internally 4-connected graph. Then G has no V_8-minor if and only if one of the following holds:*

(a) *G is planar,*

(b) *G has two vertices u and v such that $G\backslash\{u, v\}$ is a circuit,*

(c) *there is a set $X \subseteq V(G)$ of cardinality four such that every edge of G has at least one end in X,*

(d) *G is isomorphic to the line graph of $K_{3,3}$, or*

(e) *G has at most seven vertices.*

Oxley characterized all 3-connected simple graphs having no W_5-minor [**22**]

THEOREM 3.5 (Oxley, 1989). *Let G be a graph. Then G is simple, 3-connected, and W_5-free if and only if $G \cong \{W_3, W_4\}$, $G \in \mathcal{K}$ or G is a 3-connected minor of the cube, octahedron, pyramid, or $K_5^{\perp}$.*

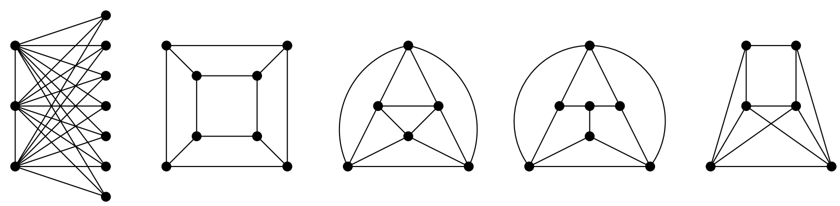


FIGURE 3.4. The graphs $K''_{3,7}$, cube, octahedron, pyramid, and $K_5^{\perp}$

Oxley also characterized all 3-connected regular matroids having no $M(W_5)$-minor [**22**].

THEOREM 3.6 (Oxley, 1989). *Let M be a regular matroid. Then M is 3-connected and $M(W_5)$-free if and only if M is*

(i) *a graphic matroid in Theorem 3.5,*

(ii) *the dual of a graphic matroid in (i), or*

(iii) R_{10}.

Ding and Liu found all 3-connected graphs having no $(W_5 + e)$-minor [**7**]. Recall that $\mathcal{K}$ is the class of 3-connected graphs G for which there exists a set X consisting of three vertices such that $G - X$ is edgeless. These graphs can be obtained from $K_{3,n} (n \geq 1)$ by adding edges to its color class of size three. Let $\mathcal{W} = \{W_n : n \geq 3\}$.

THEOREM 3.7 (Ding and Liu, 2013). *A simple* 3*-connected graph G is* $(W_5 + e)$*-free if and only if* $G \in \{\mathcal{K}, \mathcal{W}\}$ *or G is a* 3*-connected minor of* V_8*, cube, octahedron, pyramid,* A_1*,* A_2*, or* A_3 *(see Figure 3.5).*

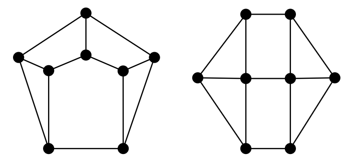
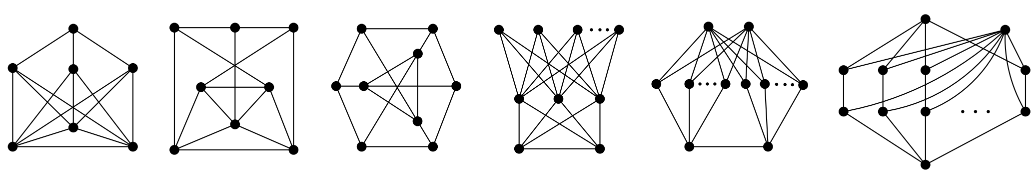


FIGURE 3.5. The graphs A_1, A_2, and A_3

Ding and Liu also characterized the class of 3-connected graphs that are $(W_5 + e)^*$-free [**7**].

THEOREM 3.8 (Ding and Liu, 2013). *A simple* 3*-connected graph G is* $(W_5 + e)^*$*-free if and only if* $G \in \mathcal{W}$ *or G is a* 3*-connected minor of the set* $\{K_6$*,* $K_{4,4}$*, Peterson graph, or graphs in the figure below*$\}$*.*



In 1943, Hall [**11**] characterized the set of $K_{3,3}$-free graphs.

THEOREM 3.9 (Hall, 1943). *A simple* 3*-connected graph G is* $K_{3,3}$*-free if and only if* $G \cong K_5$ *or G is a* 3*-connected planar graph.*

Let $K'_{3,3}$ represent the graph formed by adding an edge between any two vertices in $K_{3,3}$ that are in the same color class, and let $K''_{3,3}$ represent the simple graph formed by adding two edges between vertices of $K_{3,3}$ that are in the same color class. Examples of these two graphs are depicted in Figure 3.6.

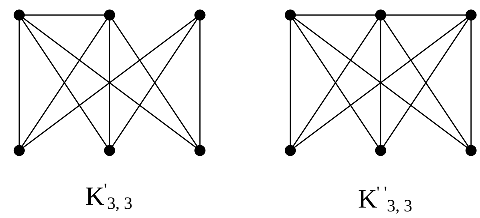


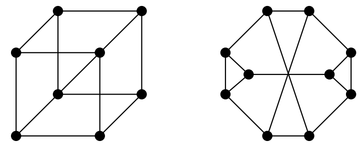
FIGURE 3.6. The graphs $K'_{3,3}$ and $K''_{3,3}$

The characterization of all $K'_{3,3}$-free graphs follows directly from Seymour's Splitter Theorem and Theorem 3.9.

THEOREM 3.10. *A simple 3-connected graph G is $K'_{3,3}$-free if and only if $G \cong \{K_{3,3}, K_5\}$ or G is a 3-connected planar graph.*

Ding and Liu [**7**] characterized the class of graphs that are $K''_{3,3}$-free.

THEOREM 3.11 (Ding and Liu, 2013). *A simple 3-connected graph G is $K''_{3,3}$-free if and only if G is a 3-connected planar graph or G is a 3-connected minor of V_8 or a 3-connected minor of one of the following graphs:*



2. Some Lemmas

This section contains technical results which are essential in proving the results found in the next section of this chapter. The first such result of Bixby [**1**] is among the most fundamental tools used in studying matroid structure.

Lemma 3.12 (Bixby, 1982). *Let e be an element of a 3-connected matroid M. Then either $M\backslash e$ or M/e has no non-minimal 2-separations. In the first case, $co(M\backslash e)$ is 3-connected, while, in the second case, $si(M/e)$ is 3-connected.*

The following lemma allows one to maintain the connectivity of a matroid under extensions and coextensions provided small circuits and cocircuits are not introduced (see [**23**, Proposition 8.2.7]).

Lemma 3.13. *Let e be an element of a matroid M. Suppose that $M\backslash e$ is n-connected but M is not. Then either e is a coloop of M, or M has a circuit that contains e and has fewer than n elements.*

The next result provides a relationship between vertically 4-connected and internally 4-connected matroids.

Lemma 3.14. *Let M be a simple binary matroid of rank at least four. Then M is vertically 4-connected if and only if M is triad-free and internally 4-connected. (If (X, Y) is a 3-separation of M, then either X or Y is a triangle.)*

Proof. Suppose that M is vertically 4-connected. It follows from M being simple and the definition of vertical 3-connectivity that M is 3-connected. Let T be a triad of M. Now $E(M) - T$ is a hyperplane so $r_M(E(M) - T) \geq 3$. Then $\lambda_M(T) = r_M(T) + r_M(E(M) - T) - r(M) = r_M(T) - 1$. As M is 3-connected, $r_M(T) \geq 3$; in fact, $r_M(T) = 3$ as T cannot be both a triangle and a triad of M. Hence $(T,\ E(M) - T)$ is a vertical 3-separation of M; a contradiction to the supposition that M is vertically 4-connected. Hence M is triad-free. Suppose that (X, Y) is a 3-separation of M. Then (X, Y) is not a vertical 3-separation of M. Hence $2 = r(X) < |X|$ or $2 = r(Y) < |Y|$. Without loss of generality, suppose the former holds. Then $M|X \cong U_{2,|X|}$. It follows from M being binary that $2 < |X| \leq 3$. Hence X is a triangle. Thus M is internally 4-connected.

Conversely, suppose that M is triad-free and internally 4-connected. Then M is 3-connected and hence vertically 3-connected. Suppose that (X, Y) is a vertical 3-separation of M. Then (X, Y) is a 3-separation of M. Since M is triad-free, either X or Y is a triangle. Hence $\min\{r(X), r(Y)\} = 2$; a contradiction. Thus, M is vertically 4-connected. $\square$

The following result is surely known. The proof is included here for completeness.

Lemma 3.15. *If N is a simple connected minor of a matroid M, then N is a minor of $si(M)$.*

PROOF. It follows from the definition of the direct-sum operation that N is a minor of a connected component M_1 of M. Let $e \in E(M_1) - E(N)$ be in a non-trivial parallel class of M. By Propostion 1.16 and the fact that M_1/e is disconnected, $M_1\backslash e$ is connected and has an N-minor. Continue this process for every element of $E(M_1) - E(N)$ that is in a non-trivial parallel class of M to obtain that $si(M_1)$ has an N-minor and hence $si(M)$ has an N-minor. □

LEMMA 3.16. *Let N be a 3-connected H-free binary matroid. Then N can be constructed from internally 4-connected H-free binary matroids by parallel extensions and 3-sums.*

PROOF. Suppose that N is not internally 4-connected. Then $N = N_1 \oplus_3 N_2$. By Theorem 1.24, both N_1 and N_2 are minors of N. Hence both N_1 and N_2 are H-free, $|E(N_i)| < |E(N)|$, and $si(N_1)$ and $si(N_2)$ are 3-connected by Theorems 1.34 and 1.35. By induction on the number of elements in N, $si(N_1)$ and $si(N_2)$ can be constructed from internally 4-connected H-free binary matroids by parallel extensions and 3-sums. Hence N can be constructed in the same way. □

Note the matroid $M_1 \oplus_3 M_2$ need not be 3-connected in the following lemma. To see this, consider the 3-sum of two prism graphs across a triangle. The resulting graph contains vertices of degree two. Hence the cycle matroid of this graph is not 3-connected (see Figure 3.7).

The proof of this lemma is due to James Oxley in a private communication [**24**].

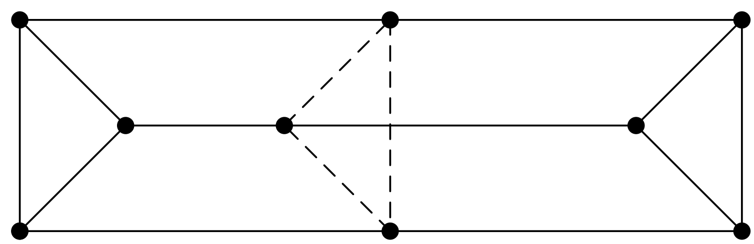


FIGURE 3.7. A 3-sum of two prisms

LEMMA 3.17. *If M_1 and M_2 are binary matroids with at least six elements, M_1 is 3-connected, M_2 is connected, $E(M_1) \cap E(M_2) = T$, the set T is a triangle of both, and neither M_1 nor M_2 has a cocircuit contained in T, then $M_1 \oplus_3 M_2$ is connected.*

PROOF. It follows from [**23**, Proposition 11.4.16] that $M_1 \oplus_3 M_2 = P_N(M_1, M_2)\backslash T$ where $N = M_1|T = M_2|T$. Let $M = P_N(M_1, M_2)$ so that $M\backslash T = M_1 \oplus_3 M_2$. The matroid M is connected (see [**23**, p. 447, ex. 9]). Let $X = E(M_1) - T$, $Y = E(M_2) - T$, and $E = E(M) = X \cup Y \cup T$. Suppose that $M\backslash T$ is not connected. Let (J, K) be a 1-separation of $M\backslash T$. We obtain a contradiction to complete the proof.

It follows from $\lambda_{M\backslash T}(J) = \lambda_{M\backslash T}(K) = 0$ and that the function λ is submodular that

(†) $\lambda_{M\backslash T}(X \cup J) + \lambda_{M\backslash T}(X \cap J) \leq \lambda_{M\backslash T}(X) + \lambda_{M\backslash T}(J) = \lambda_{M\backslash T}(X)$.

Now $\lambda_{M\backslash T}(X) = r_{M\backslash T}(X) + r_{M\backslash T}(Y) - r(M\backslash T) = r_M(X) + r_M(Y) - r(M) \leq r_M(X \cup T) + r_M(Y \cup T) - r(M) = r(M_1) + r(M_2) - r(M) = r_M(T) = 2$ (see Proposition 1.9). Hence (†) becomes

(†) $\lambda_{M\backslash T}(X \cup J) + \lambda_{M\backslash T}(X \cap J) \leq 2$.

It follows from T being a cocircuit of neither M_1 nor M_2 that X spans T in M_1 and Y spans T in M_2. Hence both X and Y span T in M. Thus $\lambda_{M\backslash T}(X \cup J) = \lambda_M(X \cup J \cup T)$. It follows from the complementary property of the function λ and that X and Y span T in M that $\lambda_{M\backslash T}(X \cap J) = \lambda_M(Y \cup K \cup T)$. From combining these observations we obtain that

(†) $\lambda_M(X \cup J \cup T) + \lambda_M(Y \cup K \cup T) \leq 2$.

Suppose that $X \subset J$. Then J spans T in M so that $\lambda_M(J \cup T) = r_M(J \cup T) + r_M(K) - r(M) = r_{M\backslash T}(J) + r_{M\backslash T}(K) - r(M\backslash T) = \lambda_{M\backslash T}(J) = 0$. Then $(J \cup T, K)$ is a 1-separation of M. This contradicts the fact that M is connected. Hence X meets K. Symmetric arguments yield that X meets J and that Y meets both J and K. Thus $(X \cup J \cup T, Y \cap K)$ and $(Y \cup K \cup T, X \cap J)$ are not 1-separations of the connected matroid M so that $\lambda_M(X \cup J \cup T)$ and $\lambda_M(Y \cup K \cup T)$ both exceed zero. Hence (†) implies that $\lambda_M(X \cup J \cup T) = \lambda_M(Y \cup K \cup T) = 1$.

The set X has at least three elements. We may assume that X meets J in at least two elements. Then M_1 is a minor of M. Hence Lemma 1.15 implies that $\lambda_{M_1}(X \cap J) \leq \lambda_M(X \cap J) = \lambda_M(Y \cup K \cup T) = 1$ so that $(X \cap J, X \cap K \cup T)$ is a 2-separation of the 3-connected matroid M_1; a contradiction. $\square$

One can modify the proof of Lemma 3.17 so that the result is still true when $si(M_1)$ is 3-connected and has at least six elements.

3. Results

The following theorem is useful in our characterizations of classes of regular H-free matroids. This result is an extension of Proposition 1.21 from the operation of 2-sum to the operation of 3-sum.

Theorem 3.18. *Let N be a simple binary vertically 4-connected matroid with rank at least four. If M is the 3-sum of binary N-free matroids M_1 and M_2, then M is N-free.*

Proof. Suppose that M has an N-minor and M_1 and M_2 are both N-free. Let $X = E(M_1) \cap E(N)$ and $Y = E(M_2) \cap E(N)$. Then N is minor of neither M_1 nor M_2 so that $X \neq \emptyset$ and $Y \neq \emptyset$. Fix $x \in X$. Suppose that $T \subseteq E(M_i)$ for $i \in \{1, 2\}$ so that $M = P_T(M_1, M_2) \backslash T$ where T is a triangle of M_1 and M_2 that contains no cocircuit of M_i for $i = 1, 2$. There exist disjoint subsets A and B of $E(P_T(M_1, M_2)) - (E(N) \cup E(T))$ such that

$$(P_T(M_1, M_2) \backslash A/B) \backslash T \cong N. \tag{3.1}$$

Choose A and B so that A is maximal with respect to condition 3.1. Let $A_i = A \cap (E(M_i) - (E(N) \cup T))$ and $B_i = B \cap (E(M_i) - (E(N) \cup T))$ for $i \in \{1, 2\}$. It follows from Proposition 1.11 (iv) and (vi) that $P_T(M_1, M_2) \backslash A = P_T(M_1 \backslash A_1,\ M_2 \backslash A_2)$. Let $B_i' \subseteq B_i$ for $i \in \{1, 2\}$ be maximal such that

$$P := P_T(M_1 \backslash A_1,\ M_2 \backslash A_2)/(B_1' \cup B_2') = P_T(M_1 \backslash A_1/B_1',\ M_2 \backslash A_2/B_2'). \tag{3.2}$$

Suppose that $x \in cl_P(T)$. Then x is not freely placed in $cl_P(T)$ as P is binary. Moreover, x is not a loop of N so it is not a loop of P. Hence

$$\text{if } x \in cl_P(T), \text{ then } x \text{ is in parallel in } P \text{ with some element of } T. \tag{3.3}$$

Suppose $B_1 - B_1' \neq \emptyset$. It follows from Proposition 1.11 (vii) that $B_1 - B_1' \subseteq cl_P(T)$. As the binary matroid P contains no four-point line restrictions, each element of $B_1 - B_1'$ is not freely placed in $cl_P(T)$. Each element of $B_1 - B_1'$ is not a loop of P because then it could be deleted instead of contracted to obtain the N-minor. This would contradict the maximality of the set A. Hence each element of $B_1 - B_1'$ is in parallel with some element of T in P. If distinct elements b_1 and b_2 of $B_1 - B_1'$ are in parallel with the same element of T, then $P/\{b_1, b_2\}$ has an N-minor and $P/b_1 \backslash b_2 \cong P/\{b_1, b_2\}$, contradicting the choice of A. So $B_1 - B_1'$ consists of at most three elements each of which is in parallel with a different element of T. Suppose $e \in B_1 - B_1'$ and $t \in T$ is in parallel with e in P. It follows from the fact that N is 3-connected, Theorem 1.31, and Proposition 1.11 (viii) that

$$\min\{|X|, |Y|, 2\} \leq \kappa_N(X, Y) \leq \kappa_{P/e}(X, Y) \leq$$

$$\kappa_{P/e}((E(M_1) - e) - (A_1 \cup B_1'), (E(M_2) - (A_2 \cup B_2')) =$$

$$\kappa_{P/t}((E(M_1) - t) - (A_1 \cup B_1'), ((E(M_2) - t) - (A_2 \cup B_2')) = 1.$$

Hence $\min\{|X|, |Y|\} = 1$. Suppose, without loss of generality, that x is the only element of X. If x is not in $cl_P(T)$, then x is a coloop of P. Hence x is a coloop of N; a contradiction. Hence x is in $cl_P(T)$. By Equation 3.3, x is in parallel in P with some element of T in P.

Let T' be the subset of T consisting of those elements that are in parallel with some element of $(B_1 - B_1') \cup x$. Then $P \backslash T' \cong P \backslash ((B_1 - B_1') \cup x) = P | E(M_2 \backslash A_2 / B_2')$ by Proposition 1.11 (i). The latter matroid is isomorphic to a minor of M_2. The former matroid has $si(P)$ as a minor. However $si(P)$ contains an N-minor by Lemma 3.15. Hence M_2 is not N-free; a contradiction. Thus $B_1 = B_1'$. Likewise, $B_2 = B_2'$. Thus $E(P) = X \cup Y \cup T$ and $P \backslash T = N$. If $X \subseteq cl_P(T)$, then, by Equation 3.3, each element of X is in parallel with an element of T. Then N is simple so that two distinct elements of X are in parallel with two different elements of T. Let T' be those elements of T that are in parallel with an element of X. Then $P \backslash T' \cong P \backslash X = P | (E(M_2 \backslash A_2 / B_2))$ again by Proposition 1.11 (i). So, as in the previous paragraph, M_2 is not N-free; a contradiction. This contradiction and symmetry imply that neither X nor Y is contained in $cl_P(T)$. Assume that $Y \cup T$ is spanning in P. Then $r_P(X \cup T) + r_P(Y \cup T) - r(P) = 2$ so that $2 = r_P(T) \leq r_P(X \cup T) = 2$. Hence $X \subset cl_P(T)$; a contradiction. This contradiction and symmetry imply that

$$\text{neither } X \cup T \text{ nor } Y \cup T \text{ is spanning in } P. \tag{3.4}$$

Suppose $\min\{|X|, |Y|\} = 1$. By Equation 3.4, either X or Y is the complement of a hyperplane of P. Hence either X or Y is a cocircuit of P. Thus X or Y is a cocircuit of N; contradiction. Hence $\min\{|X|, |Y|\} \geq 2$. It follows from Theorem 1.31 that

$$2 \leq \kappa_N(X, Y) \leq \kappa_P(X, Y) \leq \kappa_P(X \cup T_1, Y \cup T_2) \leq r_P(X \cup T) + r_P(Y \cup T) - r(P) = 2,$$

where (T_1, T_2) is any partition of T. The partition (X, Y) is not a vertical 3-separation of $P\backslash T = N$. Hence $\min\{r_N(X), r_N(Y)\} = 2$ as N is simple. Suppose that $r_N(X) = 2$, without loss of generality. Then $2 + r_N(Y) - r(N) = r_N(X) + r_N(Y) - r(N) = 2$. Hence Y is spanning in N. Thus $Y \cup T$ is spanning in P. This contradiction completes the proof of the theorem. □

The next result is a direct consequence of Theorem 3.18.

Corollary 3.19. *Let M_1 and M_2 be binary matroids with $M = M_1 \oplus_3 M_2$. If both M_1 and M_2 are N-free, then M is N-free for $N \in \{M(K_5), M(\mathcal{O}_8), M^*(V_8)\}$.*

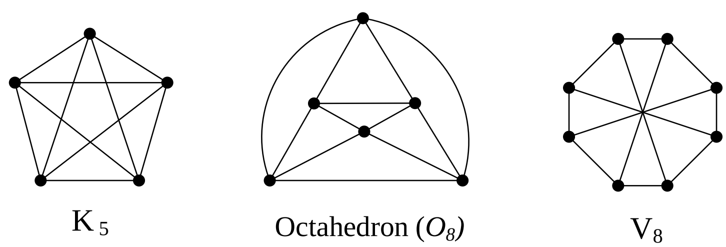


Figure 3.8. The graphs K_5, $\mathcal{O}_8$ (octahedron), and V_8

Proof. Suppose that both M_1 and M_2 are $M(K_5)$-free. Since the graph K_5 is 4-connected, $M(K_5)$ is vertically 4-connected by Theorem 1.13. The matroid $M(K_5)$ is also simple and has rank 4. Thus, M is $M(K_5)$-free by Theorem 3.18.

Similarly, suppose that M_1 and M_2 are both $M(\mathcal{O}_8)$-free. As $\mathcal{O}_8$ is a 4-connected graph, $M(\mathcal{O}_8)$ is vertically 4-connected by Theorem 1.13. The matroid $M(\mathcal{O}_8)$ is also simple and has rank 7. Hence M is $M(\mathcal{O}_8)$-free.

Now suppose that M_1 and M_2 are both $M^*(V_8)$-free. We wish to show that $M^*(V_8)$ is internally 4-connected and triad-free. It follows from the observation that $M(V_8)$ has no triangles that $M^*(V_8)$ is triad-free. Note that the graph V_8 is isomorphic to the 4-rung Möbius ladder which is internally 4-connected by Oporowski, Oxley, and Thomas [**19**]. Hence $M^*(V_8)$ is internally 4-connected by Proposition 1.14. Thus, M is $M^*(V_8)$-free by Theorem 3.18. $\square$

The forward direction of the following decomposition theorem follows directly from Seymour's Decomposition Theorem. However, the reverse direction does not. Note that the 3-sum of N-free matroids may no longer be N-free. For example, the $K_{3,3}$-free graphs $K_5\backslash e$ and K_4 can be 3-summed over a certain triangle to form the graph $K_{3,3}$.

THEOREM 3.20. *(First Decomposition Theorem) Let N be a simple, vertically 4-connected matroid with rank exceeding three. Then M is a regular N-free matroid if and only if M can be constructed by direct sums, 2-sums, or 3-sums starting with N-free matroids, each of which is isomorphic to a minor of M and each of which is graphic, cographic, or is isomorphic to R_{10} (if R_{10} is N-free).*

PROOF. Let M be an N-free regular matroid. It follows from Seymour's Decomposition Theorem that M can be constructed by using direct sums, 2-sums, and 3-sums starting from regular matroids, each of which is either graphic, cographic, or is isomorphic to R_{10}, and each of which is isomorphic to a minor of M. Therefore, each such matroid used in the construction of M is N-free.

Conversely, suppose that M is constructed by direct sums, 2-sums, or 3-sums starting with N-free matroids, each of which is isomorphic to a minor of M and each of which is graphic, cographic, or is isomorphic to R_{10} (if R_{10} is N-free). The given operations preserve the property of a matroid being regular so that M is regular. It follows from the definition of direct sum, Proposition 1.21, and Theorem 3.18 that M is N-free. $\square$

The Second Decomposition Theorem presented here is a decomposition theorem for a connected regular N-free matroid M.

THEOREM 3.21. *(Second Decomposition Theorem) Let N be a simple vertically 4-connected matroid with rank at least four. Then M is a connected regular N-free matroid if and only if M can be constructed by parallel extensions, 2-sums, and 3-sums starting from internally 4-connected N-free regular matroids, each of which is isomorphic to a minor of M and each of which is graphic, cographic, or is isomorphic to R_{10} (if R_{10} is N-free). During the construction, whenever the 3-sum operation is used, we require that the simplification of one side is an internally 4-connected N-free graphic or cographic matroid, or R_{10} (when it is N-free).*

PROOF. Suppose that M is a connected regular N-free matroid. If M is not 3-connected, then M can be constructed from 3-connected proper minors of itself by a sequence of 2-sum operations by Proposition 1.22. So M can be constructed using 2-sums with N-free 3-connected regular matroids. For each such 3-connected minor P during the construction, it follows from Lemma 3.16 that if P is not internally 4-connected and N-free, then P can be constructed from internally 4-connected N-free regular matroids by parallel extensions and 3-sums. During the recursive construction in this step, whenever the 3-sum is involved, the simplification of one side can be chosen to be an internally 4-connected N-free matroid. By Theorem 1.33, each such internally 4-connected matroid is either graphic, cographic, or is isomorphic to R_{10}.

Conversely, suppose that M is constructed as described above. As the operations of parallel extensions, 2-sums, and 3-sums of regular matroids produce a regular matroid, M is regular. It follows from Proposition 1.21 and Theorem 1.24 as well as the fact that each matroid in the construction is a minor of M that M is N- free. By Proposition 1.20 and Lemma 3.17, M is connected. This completes the proof of the theorem. □

The above decomposition theorems classify all N-free regular matroids where N is a simple vertically 4-connected matroid with rank at least four. Next, we apply these decomposition theorems to several classes of regular matroids.

Proposition 3.22. *Let M be a regular internally 4-connected matroid. Then M is $M(K_5)$-free if and only if M satisfies one of the following:*

(i) *M is cographic,*

(ii) *$M \cong R_{10}$ or $M(V_8)$, or*

(iii) *M is the cycle matroid of a graph G that is the 3-clique-sum of 3-connected planar graphs.*

Proof. Suppose that M is $M(K_5)$-free. Assume that M is neither cographic nor isomorphic to R_{10}. It follows from Theorem 1.33 that M is graphic. It follows from Theorem

3.1 that M is either $M(V_8)$ or is a graphic matroid that is a 3-clique-sum of 3-connected planar graphs.

Converserly, if M is isomorphic to R_{10} or $M(V_8)$, then M is regular and $M(K_5)$-free. If M is cographic, the result follows from Theorem 1.7. Suppose that M is the graphic matroid that is a 3-clique-sum of 3-connected planar graphs $M_1, M_2, ..., M_k$. First, assume that $k = 2$. Then $M = M_1' \oplus_3 M_2'$, where M_1' and M_2' can be obtained from M_1 and M_2 by possibly adding parallel edges to a triangle. As M_1 and M_2 are $M(K_5)$-free, so are M_1' and M_2'. Hence M, the clique-sum of M_1 and M_2, is also $M(K_5)$-free by Theorem 3.18. For general k, the result follows by an easy induction argument. □

The next theorem provides a characterization of the class of connected regular $M(K_5)$-free matroids.

Theorem 3.23. *A matroid M is connected, regular, and $M(K_5)$-free if and only if M can be constructed using parallel extensions, 2-sums and 3-sums of internally 4-connected regular matroids which are cographic, $M(V_8)$, R_{10}, or 3-clique-sums of 3-connected planar graphs. Whenever the 3-sum operation is used, assume that the simplification of one side is internally 4-connected.*

Proof. The proof of this theorem follows directly from the Second Decomposition Theorem and Proposition 3.22. □

Extending the result of Ding and Liu for $K_5^{\perp}$-free graphs to 3-connected regular matroids yields the following theorem.

Theorem 3.24. *Let M be a regular 3-connected matroid having no $M(K_5^{\perp})$-minor. Then either $M \cong M(K_5)$ or M has no $M(K_5)$-minor.*

Proof. Suppose that M has a $M(K_5)$-minor. By Seymour's Splitter Theorem, M has a 3-connected minor N such that some single-element extension or single-element coextension of $N \cong M(K_5)$. Using MACEK [**12**], we have determined that there are no such regular 3-connected extensions or coextensions of $M(K_5)$ having no $M(K_5^{\perp})$-minor. Hence $M \cong M(K_5)$ or M has no $M(K_5)$ minor. Thus, $M(K_5)$ is a splitter for the class of regular matroids having no $M(K_5^{\perp})$-minor. □

As a result of Theorem 3.23 and Theorem 3.24, we can completely characterize the class of regular matroids having no $M(K_5^{\perp})$-minor.

We now extend Ding's octahedron-free graph result and Maharry's cube-free graph result to determine the classes of regular matroids without minors isomorphic to $M(\mathcal{O}_8)$ or $M(cube)$. It is important to note the $M^*(\mathcal{O}_8) \cong M(cube)$.

Theorem 3.25. *Let M be a regular matroid. Then M is $M(\mathcal{O}_8)$-free if and only if M can be constructed using 1-, 2-, and 3-sums by the following matroids:*

(i) *graphic matroids that can be constructed by 0-, 1-, 2-, and 3-clique sums starting from graphs in the set* $\{K_1, K_2, K_3, K_4\} \cup \{C^2_{2n+1} : n \geq 2\} \cup \{L_5, G_{0814}, G_{1015}, G_{1016}, G_{1117}\}$ *(see Figure 3.2),*

(ii) *the dual of a graphic matroid described in* [**17**]*, or*

(iii) R_{10}.

Proof. This result follows immediately from the First Decomposition Theorem, Theorem 3.3, and Maharry's characterization of cube-free graphs [**17**]. □

As a consequence of the previous result, one can characterize the regular cube-free matroids by duality.

The theorem below provides a characterization of the class of regular $M^*(V_8)$-free matroids. By duality, we can also characterize the class of regular $M(V_8)$-free matroids.

Theorem 3.26. *Let M be a regular matroid. Then M is $M^*(V_8)$-free if and only if M can be constructed by direct sums, 2-sums, or 3-sums of matroids each of which is isomorphic to a minor of M and each of which is graphic, isomorphic to R_{10}, or is the dual of the cycle matroid of a graph H that is the 0-, 1-, 2-, or 3-clique-sums of internally 4-connected graphs G such that one of the following is true:*

(a) *G is planar,*

(b) *G has two vertices u and v such that $G\backslash\{u, v\}$ is a circuit,*

(c) *There is a set $X \subseteq V(G)$ of cardinality four such that every edge of G has at least one end in X,*

(d) *G is isomorphic to the line graph of $K_{3,3}$, or*

(e) *G has at most seven vertices.*

Proof. Theoreom 3.20 states that M can be constructed by direct sums, 2-sums, or 3-sums starting with $M^*(V_8)$-free matroids, each of which is isomorphic to a minor of M and each of which is graphic, cographic, or isomorphic to R_{10}. Let N be such a minor. If N is graphic, then N is $M^*(V_8)$-free. If N is cographic, then N^* is graphic and $M(V_8)$-free. Hence N^* is the cycle matroid of a graph that can be constructed from 0-, 1-, 2-, or 3-clique-sums of graphs G such that one of the conditions (a) - (e) hold. If $M \cong R_{10}$, then M is $M^*(V_8)$-free.

Conversely, suppose that M can be constructed as stated in the theorem. Then the result follows from Theorem 3.4, Proposition 1.21, Theorem 3.18 and the fact that $M^*(V_8)$ is vertically 4-connected. □

As W_5 is a minor of both $W_5 + e$ and $(W_5 + e)^*$, the following result is an extension of the characterization of $M(W_5)$-free matroids by Oxley [**22**]. First, it is proven that the matroid R_{12} is a splitter for the class of regular matroids that are $\{M(W_5+e),\ M^*(W_5+e)\}$-free. This proof uses the program MACEK [**12**]. Then a characterization of the 3-connected regular matroids that are $\{M(W_5 + e),\ M^*(W_5 + e)\}$-free is provided.

PROPOSITION 3.27. *The matroid R_{12} is a splitter for the class of regular matroids that are both $M(W_5 + e)$-free and $M^*(W_5 + e)$-free.*

PROOF. This proof follows directly from the MACEK command

```
./macek -pREG '!extend b;@ext-forbid W5+e "W5+e;!dual";!print' R12.
```

Theorem 3.27 can also be proven in a step-by-step fashion as follows. Suppose that M has a minor that is isomorphic to R_{12}. By Seymour's Splitter Theorem, M has a 3-connected minor N which is a single-element extension or single-element coextension of R_{12}. Finding the regular single-element extensions and coextensions of R_{12} with no $M(W_5 + e)$-minor yields the following two matroids in standard form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$$

Both of these matroids have the forbidden minor $M^*(W_5+e)$. Hence no results are obtained by finding the regular single-element extensions and coextensions of R_{12} with no $M(W_5+e)$-minor. Finding the regular single-element extensions and coextensions of R_{12} with no $M^*(W_5+e)$-minor yields the following two matroids in standard form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Each of these matroids has the forbidden minor $M(W_5 + e)$. Therefore, no results are obtained by finding the regular single-element extensions and coextensions of R_{12} with no $M^*(W_5+e)$-minor. Thus, either $M \cong R_{12}$ or M has no minor that is isomorphic to R_{12}. $\square$

A characterization of all 3-connected regular matroids that are $\{M(W_5{+}e),\ M^*(W_5{+}e)\}$-free may now be provided.

THEOREM 3.28. *A 3-connected regular matroid M is $\{M(W_5 + e),\ M^*(W_5 + e)\}$-free if and only if M is one of the following matroids:*

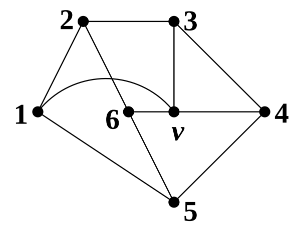
(i) *the cycle matroid of a graph G that is a member of $\mathcal{K}$, $\mathcal{W}$, or a 3-connected minor of V_8, cube, octahedron, pyramid, or $K_5^{\perp}$,*

(ii) *the dual matroid of a graph in (i),*

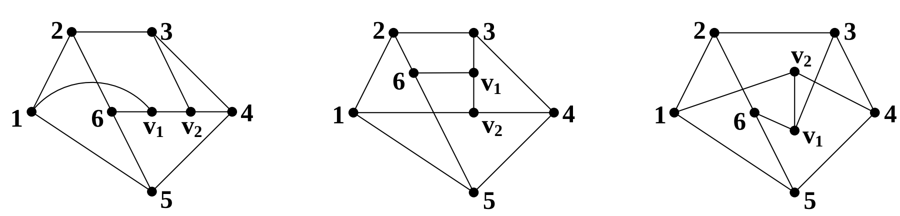
(iii) R_{10}, *or*

(iv) R_{12}.

PROOF. Suppose that M is a 3-connected regular matroid with no minor isomorphic to $M(W_5 + e)$ or $M^*(W_5 + e)$. By Theorem 1.32, M is graphic, cographic, or M has a minor isomorphic to R_{10} or R_{12}. As R_{10} is a splitter for the class of regular matroids, if M has an R_{10}-minor, then $M = R_{10}$. If M has an R_{12}-minor, then $M = R_{12}$ by Theorem 3.27. If M is cographic, we consider the dual of M. Assume that M is graphic, that is, $M = M(G)$ for some 3-connected graph G. If G has no W_5-minor, then G is either W_3, W_4, a member of $\mathcal{K}$, or a 3-connected minor of cube, octahedron, pyramid, or $K_5^{\perp}$ by Theorem 3.5. Now suppose that G has a W_5-minor. Suppose that G is not as given in (i) and is a minimal such graph. By Seymour's Splitter Theorem, either G is a wheel, or G has a 3-connected minor which is a single-edge extension or coextension of W_5. Each single-edge extension of W_5 is equal to $W_5 + e$, while W_5 has two non-isomorphic single-edge coextensions: one planar and one non-planar. The former is isomorphic to $(W_5 + e)^*$ and the latter graph H is depicted below:



However, the graph H is a minor of V_8. Hence $G \neq H$. By Seymour's Splitter Theorem, G has a 3-connected minor which is a single-edge extension or coextension of H. It is straightforward to check that each single-edge extension of H has either a $(W_5 + e)$-minor or a $(W_5 + e)^*$-minor. Thus G has a 3-connected minor which is a single-edge coextension of H. Since H has only vertex v with degree greater than three, one can only split this vertex. Splitting the vertex v yields three graphs:



(a) (b) (c)

Contracting the edge $6v_1$ of the graph in (a) yields $(W_5 + e)^*$. The graphs in (b) and (c) are both isomorphic to V_8 with cycle order $(1, 2, 3, 4, 5, 6, v_1, v_2)$. Hence G has a V_8-minor. However, each single-edge extension of V_8 contains a $(W_5 + e)$-minor. As each vertex of V_8 has degree three, there are no 3-connected single-edge coextensions of V_8. Hence $G \cong V_8$; a contradiction. Therefore, G is one of the graphs in (i).

Conversely, every graphic matroid in (i) is $\{M(W_5+e), M^*(W_5+e)\}$-free and so are their duals. The matroids R_{10} and R_{12} are also $\{M(W_5 + e), M^*(W_5 + e)\}$-free. This completes the proof of the theorem. □

The following results are extensions of the characterizations of $K_{3,3}$-free graphs by Hall [**11**] and the characterizations of $K'_{3,3}$- free graphs in Theorem 3.10. A characterization of the class of all 3-connected regular $M(K'_{3,3})$-free matroids is provided. It is then shown that R_{12} is a splitter for the class of regular matroids that are $\{M(K''_{3,3}),\ M^*(K''_{3,3})\}$-free. The class of regular 3-connected matroids without these two forbidden minors is also determined.

Theorem 3.29. *A 3-connected regular matroid is $M(K'_{3,3})$-free if and only if one of the following holds:*

(i) *M is isomorphic to the cycle matroid of $K_{3,3}$ or K_5 or M is a 3-connected planar graph,*

(ii) *M is cographic, or*

(iii) *$M = R_{10}$.*

PROOF. Suppose that M is a 3-connected regular $M(K'_{3,3})$-free matroid. By Theorem 1.32, either M is graphic, cographic, or M has a minor isomorphic to one of R_{10} and R_{12}. As R_{10} is a splitter for the class of regular matroids, $M \cong R_{10}$ if M has an R_{10}-minor. As R_{12} is not $M(K'_{3,3})$-free, M cannot have a minor isomorphic to R_{12}. Hence M is graphic, cographic, or R_{10}. If M is graphic, then $M \cong M(K_{3,3})$ or $M(K_5)$ or M is the cycle matroid of a 3-connected planar graph by Theorem 3.10.

Conversely, suppose that the matroid M satisfies one of the conditions (i) - (iii). Then the result follows from Theorem 3.10 and the fact that any cographic matroid is $M(K'_{3,3})$-free. $\square$

The result below shows that R_{12} is a splitter for the class of regular matroids that are $\{M(K''_{3,3}),\ M^*(K''_{3,3})\}$-free. The program MACEK [**12**] was used to prove this result.

PROPOSITION 3.30. *The matroid R_{12} is a splitter for the class of regular matroids that are both $M(K''_{3,3})$-free and $M^*(K''_{3,3})$-free.*

PROOF. This proof follows directly from the MACEK command

```
./macek -pREG '!extend b;@ext-forbid K33++ "K33++;!dual";!print' R12.
```

Theorem 3.30 can also be proven in a step-by-step fashion as follows. Suppose that M has a minor that is isomorphic to R_{12}. By Seymour's Splitter Theorem, M has a 3-connected minor N which is a single-element extension or single-element coextension of R_{12}. Finding the regular single-element extensions and coextensions of R_{12} with no $M(K''_{3,3})$-minor yields the following two matroids in standard form:

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 0 \end{bmatrix}$$

Both of these matroids have the forbidden minor $M^*(K''_{3,3})$. Hence no results are obtained by finding the regular single-element extensions and coextensions of R_{12} with no $M(K''_{3,3})$-minor. Finding the regular single-element extensions and coextensions of R_{12} with no $M^*(K''_{3,3})$-minor yields the following two matroids in standard form:

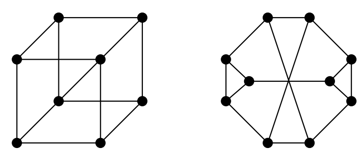
$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & -1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \end{bmatrix}$$

Each of these matroids has the forbidden minor $M(K''_{3,3})$. Therefore, no results are obtained by finding the regular single-element extensions and coextensions of R_{12} with no $M^*(K''_{3,3})$-minor. Thus, either $M \cong R_{12}$ or M has no minor that is isomorphic to R_{12}. $\square$

The class of 3-connected regular matroids that are $\{M(K''_{3,3}),\ M^*(K''_{3,3})\}$-free can now be determined.

THEOREM 3.31. *A* 3*-connected regular matroid* M *is* $\{M(K''_{3,3}),\ M^*(K''_{3,3})\}$*-free if and only if one of the following holds:*

(i) M *is isomorphic to the cycle matroid of either a* 3*-connected planar graph or a graph that is a* 3*-connected minor of* V_8 *or a* 3*-connected minor of one the following two graphs:*



(ii) M *is isomorphic to the dual matroid of one of the graphs in (i).*

(iii) $M = R_{10}$.

(iv) $M = R_{12}$.

PROOF. Let M be a regular 3-connected $\{M(K''_{3,3}),\ M^*(K''_{3,3})\}$-free matroid. It follows from Theorem 1.32 that M is either graphic, cographic, or M has a minor isomorphic to R_{10} or R_{12}. If M is graphic, the result follows from Theorem 3.11. If M is cographic, then (ii) holds. It follows from the fact that R_{10} is a splitter for the class of regular matroids that if

M has an R_{10}-minor, then $M = R_{10}$. It follows from Proposition 3.30 that if M has a minor isomorphic to R_{12}, then $M = R_{12}$.

Conversely, every graphic matroid in (i) has no $M(K''_{3,3})$-minor. As these matroids are graphic, they do not have $M^*(K''_{3,3})$ as a minor either. By duality, the dual of these graphic matroids are $\{M(K''_{3,3}),\ M^*(K''_{3,3})\}$-free. Moreover, both R_{10} and R_{12} are $\{M(K''_{3,3}),\ M^*(K''_{3,3})\}$-free. This completes the proof of the theorem. $\square$

Bibliography

Bibliography

1. Robert E. Bixby, *A simple theorem on 3-connectivity*, Linear Algebra Appl. **45** (1982), 123–126. MR 660982 (84j:05037)

2. Robert E. Bixby and William H. Cunningham, *Matroid optimization and algorithms*, Handbook of combinatorics, Vol. 1, 2, Elsevier, Amsterdam, 1995, pp. 551–609. MR 1373667 (97b:90089)

3. Thomas H. Brylawski, *A decomposition for combinatorial geometries*, Trans. Amer. Math. Soc. **171** (1972), 235–282. MR 0309764 (46 #8869)

4. ______, *Modular constructions for combinatorial geometries*, Trans. Amer. Math. Soc. **203** (1975), 1–44. MR 0357163 (50 #9631)

5. Carolyn Chun, Dillon Mayhew, and James Oxley, *A chain theorem for internally 4-connected binary matroids*, J. Combin. Theory Ser. B **101** (2011), no. 3, 141–189. MR 2771296

6. Guoli Ding, *A characterization of graphs with no octahedron minor*, preprint.

7. Guoli Ding and Cheng Liu, *Excluding a small minor*, Discrete Appl. Math. **161** (2013), no. 3, 355–368. MR 2998437

8. Gabriel A. Dirac, *Some results concerning the structure of graphs*, Canad. Math. Bull. **6** (1963), 183–210. MR 0157370 (28 #604)

9. Jonathan L. Gross and Jay Yellen (eds.), *Handbook of graph theory*, Discrete Mathematics and its Applications (Boca Raton), CRC Press, Boca Raton, FL, 2004. MR 2035186 (2004j:05001)

10. Hugo Hadwiger, *Über eine Klassifikation der Streckenkomplexe*, Vierteljschr. Naturforsch. Ges. Zürich **88** (1943), 133–142. MR 0012237 (6,281c)

11. Dick Wick Hall, *A note on primitive skew curves*, Bull. Amer. Math. Soc. **49** (1943), 935–936. MR 0009442 (5,151b)

12. Petr Hliněný, *The macek program: Version 1.2+*, http://www.mcs.vuw.ac.nz/research/macek/, 2002.

13. Ken-ichi Kawarabayashi, Serguei Norine, Robin Thomas, and Paul Wollan, *k_6-minors in large 6-connected graphs*, preprint.

14. Sandra Kingan and Manoel Lemos, *A decomposition theorem for binary matroids with no prism minor*, preprint.

15. László Lovász, *On graphs not containing independent circuits*, Mat. Lapok **16** (1965), 289–299. MR 0211902 (35 #2777)

16. John Maharry, *An excluded minor theorem for the octahedron*, J. Graph Theory **31** (1999), no. 2, 95–100. MR 1686280 (2000a:05170)

17. ———, *A characterization of graphs with no cube minor*, J. Combin. Theory Ser. B **80** (2000), 271–283. MR 1794690 (2001h:05087)

18. Dillon Mayhew and Gordon Royle, *The internally 4-connected binary matroids with no $m(k_5\backslash e)$-minor*, SIAM J. Discrete Math. **26** (2013), no. 2, 755–767.

19. Bogdan Oporowski, James Oxley, and Robin Thomas, *Typical subgraphs of 3- and 4-connected graphs*, J. Combin. Theory Ser. B **57** (1993), no. 2, 239–257. MR 1207490 (94a:05130)

20. James Oxley, *On a matroid generalization of graph connectivity*, Math. Proc. Cambridge Philos. Soc. **90** (1981), no. 2, 207–214. MR 620730 (82j:05045)

21. ———, *The binary matroids with no 4-wheel minor*, Trans. Amer. Math. Soc. **301** (1987), no. 1, 63–75. MR 879563 (88b:05043)

22. ———, *The regular matroids with no 5-wheel minor*, J. Combin. Theory Ser. B **46** (1989), no. 3, 292–305. MR 999699 (90b:05036)

23. ———, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. MR 2849819

24. ______, private communication, 2013.

25. Paul D. Seymour, *A note on the production of matroid minors*, J. Combinatorial Theory Ser. B **22** (1977), no. 3, 289–295. MR 0439663 (55 #12549)

26. ______, *Decomposition of regular matroids*, J. Combin. Theory Ser. B **28** (1980), no. 3, 305–359. MR 579077 (82j:05046)

27. Robin Thomas, *Recent excluded minor theorems for graphs*, Surveys in combinatorics, 1999 (Canterbury), London Math. Soc. Lecture Note Ser., vol. 267, Cambridge Univ. Press, Cambridge, 1999, pp. 201–222. MR 1725004 (2001e:05118)

28. William T. Tutte, *A homotopy theorem for matroids. I, II*, Trans. Amer. Math. Soc. **88** (1958), 144–174. MR 0101526 (21 #336)

29. ______, *Matroids and graphs*, Trans. Amer. Math. Soc. **90** (1959), 527–552. MR 0101527 (21 #337)

30. ______, *Connectivity in matroids*, Canad. J. Math. **18** (1966), 1301–1324. MR 0205880 (34 #5706)

31. ______, *On the algebraic theory of graph colorings*, J. Combinatorial Theory **1** (1966), 15–50. MR 0194363 (33 #2573)

32. Klaus Wagner, *Über eine Eigenschaft der ebenen Komplexe*, Math. Ann. **114** (1937), no. 1, 570–590. MR 1513158

33. Douglas B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996. MR 1367739 (96i:05001)

34. Hassler Whitney, *On the Abstract Properties of Linear Dependence*, Amer. J. Math. **57** (1935), no. 3, 509–533. MR 1507091

35. Haidong Wu, *Connectivity for matroids and graphs*, ProQuest LLC, Ann Arbor, MI, 1994, Thesis (Ph.D.)–Louisiana State University and Agricultural & Mechanical College. MR 2691691

List of Appendices

Appendix A: The 90 Matroids of the Set $EX_{i4c}(prism + e)$

The 90 Matroids of the Set $EX_{i4c}(prism + e)$

Forty-two of the ninety matroids in $EX_{i4c}(prism + e)$ were found by Mayhew and Royle [**18**]. Each of these matroids are prism-free and, hence, (prism+e)-free. The other forty-eight matroids found are (prism+e)-free but have prism as a minor. By Lemma 2.10, a matroid N is (prism+e)-free if and only if N^* is also (prism+e)-free. This is due to the fact that prism+e is self-dual. Therefore, it only makes sense that many of the forty-eight matroids with prism-minors are actually duals of those matroids without prism-minors. Some matroids, such as the ones beginning with N, have prism-minors and so do their duals. The matroids $M(K_4)$, $M15$, $M16 \cong R_{10}$, and Q_{12} are self-dual. A complete list of the 90 matroids is provided next. The 42 prism-free matroids found by Mayhew and Royle are listed below [**18**].

$U_{0,0}$, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,3}$, $M1 = M(K_4)$, $M2 = F_7$, $M3 = F_7^*$, $M4 = M^*(K_{3,3})$, $M5 = M(K_5)$, $M6 = P_{10}$, $M7$, $M8$, $M9$, $M10$, $M11$, $M12$, $M13 = PG(3,2)$, $M14 = M(K_{3,3})$, $M15 = O_{10}$, $M16 = R_{10}$, $M17$, $M18 = P_{11}$, $M19$, $M20$, $M21$, $M22$, $M23$, $M24$, $M25$, $M26$, $M27$, $M28$, $M29$, $M30$, $M31$, $M32$, $M33$, $M34$, $M35$, $M36 = P_{17}$.

The forty-eight matroids that have a prism-minor are listed next. As mentioned before, many of these matroids are duals of the prism-free members of $EX_{i4c}(prism + e)$.

$(M5)^* = M^*(K_5)$, $(M6)^* = P_{10}^*$, $(M7)^*$, $(M8)^*$, $(M9)^*$, $(M10)^*$, $(M11)^*$, $(M12)^*$, $(M13)^* = (PG(3,2))^*$, $(M17)^*$, $(M18)^* = P_{11}^*$, $(M19)^*$, $(M20)^*$, $(M21)^*$, $(M22)^*$, $(M23)^*$, $(M24)^*$, $(M25)^*$, $(M26)^*$, $(M27)^*$, $(M28)^*$, $(M29)^*$, $(M30)^*$, $(M31)^*$, $(M32)^*$, $(M33)^*$, $(M34)^*$,

$(M35)^*$, $(M36)^* = P_{17}^*$, $N1$, $(N1)^*$, $N2$, $(N2)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$, $N8$, $(N8)^*$, Q_{15}, Q_{15}^*, and Q_{12}.

Tables are provided to show these matroids in their standard reduced matrix form, i.e. without the leading identity matrix. Note that we do not include representations of $U_{0,0}$, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, $U_{2,3}$, or $M(K_4)$ in the tables. The matroids $M15$, $M16 \cong R_{10}$, and Q_{12} are self-dual. Of the remaining eighty matroids, forty matroids are duals of the other forty. For example, $M(K_5)$ and $M^*(K_5)$ are both members of the set $EX_{i4c}(prism + e)$. Therefore, we need only show the matrix representation of forty matroids plus the three that are self-dual. Note, however, that $M(K_{3,3})$ and $M^*(K_{3,3})$ are both included in the tables in keeping with the tables established by Mayhew and Royle [**18**]. Therefore, the matrix representations of forty-four matroids will be given in Figures 3.10, 3.11, 3.12, and 3.13. Figures 3.10 and 3.11 display the rank four and five members of $EX_{i4c}(prism + e)$ that were found by Mayhew and Royle [**18**]. Figures 3.12 and 3.13 display the members of $EX_{i4c}(prism + e)$ that have a prism-minor. In each table, the bullets indicate which columns are elements of the listed matroid. For example, a standard matrix representation for $M7$ is shown in Figure 3.9.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

FIGURE 3.9. Standard representation of $M7$

	1	1	1	0	0	0	1	1	1	0	1	
	1	0	0	1	1	0	1	1	0	1	1	
	0	1	0	1	0	1	1	0	1	1	1	
	0	0	1	0	1	1	0	1	1	1	1	
$M3$							$\bullet$	$\bullet$	$\bullet$			F_7^*
$M4$	$\bullet$				$\bullet$		$\bullet$			$\bullet$	$\bullet$	$M^*(K_{3,3})$
$M5$	$\bullet$			$\bullet$		$\bullet$	$\bullet$			$\bullet$	$\bullet$	$M(K_5)$
$M6$	$\bullet$		$\bullet$	$\bullet$	$\bullet$	$\bullet$			$\bullet$			P_{10}
$M7$		$\bullet$	$\bullet$	$\bullet$				$\bullet$	$\bullet$	$\bullet$	$\bullet$	
$M8$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\bullet$		$\bullet$			$\bullet$	
$M9$	$\bullet$	$\bullet$	$\bullet$				$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	
$M10$		$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$			$\bullet$	$\bullet$	$\bullet$	
$M11$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$			$\bullet$	$\bullet$	$\bullet$	
$M12$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$		$\bullet$	$\bullet$	$\bullet$	$\bullet$	
$M13$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$PG(3,2)$

FIGURE 3.10. Minors of $PG(3,2)$ in $EX_{i4c}(prism+e)$

	1	0	0	0	0	1	0	1	1	1	1	0	0	1	1	1	1	0	1	
	0	1	0	0	1	0	1	0	1	1	1	1	0	0	1	1	0	1	1	
	0	0	1	0	1	1	0	1	0	0	1	1	1	0	1	0	1	1	1	
	0	0	0	1	0	1	1	0	1	0	0	1	1	1	0	1	1	1	1	
	1	1	1	1	1	0	1	1	0	1	0	0	1	1	1	1	1	1	1	
$M14$											•	•	•						•	$M(K_{3,3})$
$M15$	•	•				•						•				•				O_{10}
$M16$										•	•	•	•	•						R_{10}
$M17$	•	•		•		•			•			•								
$M18$	•	•	•			•						•				•				P_{11}
$M19$						•		•	•	•		•		•						
$M20$	•	•		•		•			•			•				•				
$M21$	•	•		•		•			•			•			•					
$M22$	•	•	•	•		•			•			•								
$M23$						•		•	•	•	•	•		•						
$M24$	•	•		•		•			•			•			•	•				
$M25$	•	•	•	•		•			•		•	•								
$M26$	•	•	•	•		•			•			•						•		
$M27$						•		•	•	•	•	•		•					•	
$M28$	•	•	•	•		•			•		•	•						•		
$M29$	•	•	•	•		•			•			•					•	•		
$M30$						•		•	•	•	•	•	•	•					•	
$M31$	•	•	•	•		•			•		•	•					•	•		
$M32$	•	•	•	•		•			•			•				•	•	•		
$M33$						•	•	•	•	•	•	•	•	•					•	
$M34$	•	•	•	•		•			•		•	•				•	•	•		
$M35$					•	•	•	•	•	•	•	•	•	•					•	
$M36$	•	•	•	•		•			•			•			•	•	•	•		P_{17}

FIGURE 3.11. Minors of P_{17} in $EX_{i4c}(prism + e)$

	1	0	0	1	1	1	1	1	0	0	
	1	1	0	0	1	1	0	0	1	1	
	0	0	0	0	0	0	0	1	1	0	
	0	1	1	0	1	0	1	1	0	1	
	0	0	1	1	0	1	1	0	1	1	
$N1$	•	•	•	•				•	•		
$N2$	•	•	•				•	•	•		
$N3$	•	•	•	•				•	•	•	
$N4$	•	•	•				•	•	•	•	
$N5$	•	•	•			•	•	•	•		
$N6$	•	•	•	•			•	•	•	•	
$N7$	•	•	•	•		•		•	•	•	
$N8$	•	•	•	•		•	•	•	•	•	
$N9$	•	•	•	•	•	•	•	•	•	•	Q_{15}

FIGURE 3.12. Minors of Q_{15} in $EX_{i4c}(prism + e)$

$$
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
$$
$$
\begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0
\end{pmatrix}
$$

FIGURE 3.13. Standard representation of Q_{12}

Appendix B: The 42 Sporadic Matroids in the Set $EX_{3c-i4c}(prism+e)$

The 42 Sporadic Matroids in the Set $EX_{3c-i4c}(prism+e)$

In this section, we list the forty-two sporadic matroids in the set $EX_{3c-i4c}(prism+e)$ and discuss how these matroids were found. The eighty-four members of $EX_{i4c}(prism+e)$ that have at least six elements are extended and coextended in search of (prism+e)-free matroids that are 3-connected but not internally 4-connected. Recall that the only members of $EX_{i4c}(prism+e)$ that have fewer than six elements are the uniform matroids $U_{0,0}$, $U_{0,1}$, $U_{1,1}$, $U_{1,2}$, $U_{1,3}$, and $U_{2,3}$. In each of the following tables, we list the matroid M to be extended, the number of elements of M, the number of times M was extended and coextended, the number of internally 4-connected and not internally 4-connected matroids generated, the size of the largest extension or coextension found, and whether or not M will be excluded from the set of sporadic matroids of $EX_{3c-i4c}(prism+e)$. The letter 'b' in the command column represents the MACEK command to extend and coextend the matroid. The number of b's represents the number of times the matroid was extended and coextended using MACEK [**12**]. To simplify the computations, prism+e as well as members of $EX_{i4c}(prism+e)$ of size $|E(M)|+1$ are excluded when extending and coextending each matroid M. The list of excluded matroids can be found in the tables in the Command column. Note that the matroids with an asterisk symbol in the last column of the tables are the ones that are excluded from Theorem 2.12. This is because they continue to yield 3-connected extensions and/or coextensions that are not internally 4-connected and have no (prism+e)-minor. These matroids are deemed "out of control." As $M(K_4)$, F_7, F_7^*, and $M(K_{3,3})$ were considered "out of control" and excluded by Mayhew and Royle previously [**18**], the results for these four matroids are not included here.

The first two tables display the results of extending and coextending the prism-free members of $EX_{i4c}(prism + e)$. The results displayed in Table 3.14 are for matroids of size nine through eleven. The results displayed in Table 3.15 are for matroids of size twelve through seventeen. Note that twenty-two binary 3-connected (prism+e)-free matroids were found that are not internally 4-connected in Tables 3.14 and 3.15 while four matroids are to be excluded from the set of sporadic matroids in $EX_{3c-i4c}(prism + e)$.

Size	Matroid	Command	I4C	Not I4C	Largest	Excluded
9	$\mathbf{M^*(K_{3,3})}$	**bbbbbb forbid prism + e**, $\mathbf{O_{10}}$, $\mathbf{R_{10}}$, $\mathbf{M(K_5)}$, $\mathbf{M^*(K_5)}$, $\mathbf{P_{10}}$, $\mathbf{P_{10}^*}$	**0**	**108**	**15**	$*$
10	$M(K_5)$	bbbb forbid prism+e, $M7$, $(M7)^*$, $M8$, $(M8)^*$, $M17$, $(M17)^*$, P_{11}, P_{11}^*, $M19$, $(M19)^*$, $N1$, $(N1)^*$, $N2$, $(N2)^*$	0	2	12	
10	$\mathbf{P_{10}}$	**bbbbbbbbbb forbid prism + e**, $\mathbf{M7}$, $(\mathbf{M7})^*$, $\mathbf{M8}$, $(\mathbf{M8})^*$, $\mathbf{M17}$, $(\mathbf{M17})^*$, $\mathbf{P_{11}}$, $\mathbf{P_{11}^*}$, $\mathbf{M19}$, $(\mathbf{M19})^*$, $\mathbf{N1}$, $(\mathbf{N1})^*$, $\mathbf{N2}$, $(\mathbf{N2})^*$	**0**	**24**	**20**	$*$
10	$\mathbf{O_{10}}$	**bbbbbbbbbb forbid prism + e**, $\mathbf{M7}$, $(\mathbf{M7})^*$, $\mathbf{M8}$, $(\mathbf{M8})^*$, $\mathbf{M17}$, $(\mathbf{M17})^*$, $\mathbf{P_{11}}$, $\mathbf{P_{11}^*}$, $\mathbf{M19}$, $(\mathbf{M19})^*$, $\mathbf{N1}$, $(\mathbf{N1})^*$, $\mathbf{N2}$, $(\mathbf{N2})^*$	**0**	**41**	**20**	$*$
10	R_{10}	bbbb forbid prism+e, $M7$, $(M7)^*$, $M8$, $(M8)^*$, $M17$, $(M17)^*$, P_{11}, P_{11}^*, $M19$, $(M19)^*$, $N1$, $(N1)^*$, $N2$, $(N2)^*$	0	0		
11	$M7$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	4	13	
11	$M8$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	3	14	
11	$M17$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	1	12	
11	$\mathbf{P_{11}}$	**bbbbbbbbb forbid prism + e**, $\mathbf{M9}$, $(\mathbf{M9})^*$, $\mathbf{M10}$, $(\mathbf{M10})^*$, $\mathbf{M20}$, $(\mathbf{M20})^*$, $\mathbf{M21}$, $(\mathbf{M21})^*$, $\mathbf{M22}$, $(\mathbf{M22})^*$, $\mathbf{M23}$, $(\mathbf{M23})^*$, $\mathbf{N3}$, $(\mathbf{N3})^*$, $\mathbf{N4}$, $(\mathbf{N4})^*$, $\mathbf{N5}$, $(\mathbf{N5})^*$, $\mathbf{Q_{12}}$	**0**	**20**	**21**	$*$
11	$M19$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	0		

FIGURE 3.14. Extensions and coextensions of $EX_{i4c}(prism + e)$ Part 1

Size	Matroid	Command	I4C	Not I4C	Largest	Excluded
12	$M9$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	4	16	
12	$M10$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	3	13	
12	$M20$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	1	13	
12	$M21$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	0		
12	$M22$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	0		
12	$M23$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	0		
13	$M11$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	2	14	
13	$M24$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
13	$M25$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
13	$M26$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
13	$M27$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
14	$M12$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	1	15	
14	$M28$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	0		
14	$M29$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	0		
14	$M30$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	0		
15	$PG(3,2)$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	1	16	
15	$M31$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
15	$M32$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
15	$M33$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
16	$M34$	bbbb forbid prism+e, P_{17}, P_{17}^*	0	0		
16	$M35$	bbbb forbid prism+e, P_{17}, P_{17}^*	0	0		
17	$P17$	bbbb prism+e	0	0		

FIGURE 3.15. Extensions and coextensions of $EX_{i4c}(prism + e)$ Part 2

In Tables 3.16, 3.17, and 3.18 the forty-eight members of $EX_{i4c}(prism + e)$ that have a prism-minor are extended and coextended. Note that sixty-eight binary 3-connected (prism+e)-free matroids were found that are not internally 4-connected in Tables 3.16, 3.17, and 3.18 while two matroids are to be excluded from the set of sporadic matroids in $EX_{3c-i4c}(prism+e)$.

Size	Matroid	Command	I4C	Not I4C	Largest	Excluded
10	$M^*(K_5)$	bbbb forbid prism+e, $M7$, $(M7)^*$, $M8$, $(M8)^*$, $M17$, $(M17)^*$, P_{11}, P_{11}^*, $M19$, $(M19)^*$, $N1$, $(N1)^*$, $N2$, $(N2)^*$	0	2	12	
10	$\mathbf{P_{10}^*}$	**bbbbbbbbbb forbid prism + e**, $\mathbf{M7}$, $(\mathbf{M7})^*$, $\mathbf{M8}$, $(\mathbf{M8})^*$, $\mathbf{M17}$, $(\mathbf{M17})^*$, $\mathbf{P_{11}}$, $\mathbf{P_{11}^*}$, $\mathbf{M19}$, $(\mathbf{M19})^*$, $\mathbf{N1}$, $(\mathbf{N1})^*$, $\mathbf{N2}$, $(\mathbf{N2})^*$	**0**	**24**	**20**	*
11	$(M7)^*$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	4	13	
11	$(M8)^*$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	3	14	
11	$(M17)^*$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	1	12	
11	$\mathbf{P_{11}^*}$	**bbbbbbbbb forbid prism + e**, $\mathbf{M9}$, $(\mathbf{M9})^*$, $\mathbf{M10}$, $(\mathbf{M10})^*$, $\mathbf{M20}$, $(\mathbf{M20})^*$, $\mathbf{M21}$, $(\mathbf{M21})^*$, $\mathbf{M22}$, $(\mathbf{M22})^*$, $\mathbf{M23}$, $(\mathbf{M23})^*$, $\mathbf{N3}$, $(\mathbf{N3})^*$, $\mathbf{N4}$, $(\mathbf{N4})^*$, $\mathbf{N5}$, $(\mathbf{N5})^*$, $\mathbf{Q_{12}}$	**0**	**18**	**20**	*
11	$(M19)^*$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	0		
11	$N1$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	1	12	
11	$(N1)^*$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	1	12	
11	$N2$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	5	14	
11	$(N2)^*$	bbbb forbid prism+e, $M9$, $(M9)^*$, $M10$, $(M10)^*$, $M20$, $(M20)^*$, $M21$, $(M21)^*$, $M22$, $(M22)^*$, $M23$, $(M23)^*$, $N3$, $(N3)^*$, $N4$, $(N4)^*$, $N5$, $(N5)^*$, Q_{12}	0	5	14	

FIGURE 3.16. Extensions and coextensions of $EX_{i4c}(prism+e)$ Part 3

Size	Matroid	Command	I4C	Not I4C	Largest	Excluded
12	$(M9)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	4	16	
12	$(M10)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	3	13	
12	$(M20)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	1	13	
12	$(M21)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	0		
12	$(M22)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	0		
12	$(M23)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	0		
12	Q_{12}	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	8	16	
12	$N3$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	1	13	
12	$(N3)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	1	13	
12	$N4$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	1	13	
12	$(N4)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	1	13	
12	$N5$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	7	16	
12	$(N5)^*$	bbbb forbid prism+e, $M11$, $(M11)^*$, $M24$, $(M24)^*$, $M25$, $(M25)^*$, $M26$, $(M26)^*$, $M27$, $(M27)^*$, $N6$, $(N6)^*$, $N7$, $(N7)^*$	0	7	16	

FIGURE 3.17. Extensions and coextensions of $EX_{i4c}(prism + e)$ Part 4

Size	Matroid	Command	I4C	Not I4C	Largest	Excluded
13	$(M11)^*$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	2	14	
13	$(M24)^*$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
13	$(M25)^*$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
13	$(M26)^*$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
13	$(M27)^*$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	0		
13	$N6$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	1	14	
13	$(N6)^*$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	1	14	
13	$(N7)^*$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	1	14	
13	$N7$	bbbb forbid prism+e, $M12$, $(M12)^*$, $M28$, $(M28)^*$, $M29$, $(M29)^*$, $M30$, $(M30)^*$, $N8$, $(N8)^*$	0	1	14	
14	$(M12)^*$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	1	15	
14	$(M28)^*$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	0		
14	$(M29)^*$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	0		
14	$(M30)^*$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	0		
14	$N8$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	1	15	
14	$(N8)^*$	bbbb forbid prism+e, $PG(3,2)$, $(PG(3,2))^*$, $M31$, $(M31)^*$, $M32$, $(M32)^*$, $M33$, $(M33)^*$, Q_{15}, Q_{15}^*	0	1	15	
15	$(PG(3,2))^*$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	1	16	
15	$(M31)^*$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
15	$(M32)^*$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
15	$(M33)^*$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
15	Q_{15}	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	1	16	
15	Q_{15}^*	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	1	16	
16	$(M34)^*$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
16	$(M35)^*$	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		
17	P_{17}^*	bbbb forbid prism+e, $M34$, $(M34)^*$, $M35$, $(M35)^*$	0	0		

FIGURE 3.18. Extensions and coextensions of $EX_{i4c}(prism + e)$ Part 5

Between Tables 3.14, 3.15, 3.16, 3.17, and 3.18, ninety binary 3-connected (prism+e)-free matroids that are not internally 4-connected were found. Each of these matroids were then pairwise tested for isomorphisms. This test may be conducted by hand or using MACEK

[**12**]. Each matroid is denoted by the matroid from which it was extended/coextended and its number in the results. For example, $M(K_5)$ - 2 means that this is the second 3-connected (prism+e)-free matroid found when extending and coextending $M(K_5)$ that is not internally 4-connected. The next three tables provide a complete list of these matroids and their isomorphisms.

Size	Row x Column	Matroid	Isomorphic To
11	5 x 6	$M(K_5)$ - 1	$(M^*(K_5)$ - $1)^*$
11	6 x 5	$(M(K_5)$ - $1)^*$	$M^*(K_5)$ - 1
12	5 x 7	$M7$ - 1	$M17$ - 1
12	5 x 7	$M7$ - 2	$N1$ - 1
12	5 x 7	$M7$ - 3	
12	5 x 7	$M8$ - 1	$N2$ - 2
12	7 x 5	$(M7$ - $1)^*$	$(M17$ - $1)^*$
12	7 x 5	$(M7$ - $2)^*$	$(N1$ - $1)^*$
12	7 x 5	$(M7$ - $3)^*$	
12	7 x 5	$(M8$ - $1)^*$	$(N2$ - $2)^*$
12	6 x 6	$M(K_5)$ - 2	$(M^*(K_5)$ - $2)^*$
12	6 x 6	$(M(K_5)$ - $2)^*$	$M^*(K_5)$ - 2
12	6 x 6	$N2$ - 1	$(N2$ - $1)^*$

FIGURE 3.19. Isomorphism Table Part 1

Size	Row x Column	Matroid	Isomorphic To
13	5 x 8	$M9$ - 1	$N5$ - 2
13	5 x 8	$M10$ - 1	$N3$ - 1
13	5 x 8	$M10$ - 2	$N4$ - 1
13	5 x 8	$M10$ - 3	$M20$ - 1
13	8 x 5	$(M9$ - $1)^*$	$(N5$ - $2)^*$
13	8 x 5	$(M10$ - $1)^*$	$(N3$ - $1)^*$
13	8 x 5	$(M10$ - $2)^*$	$(N4$ - $1)^*$
13	8 x 5	$(M10$ - $3)^*$	$(M20$ - $1)^*$
13	6 x 7	$M7$ - 4	
13	6 x 7	$M8$ - 2	$N2$ - 4, $(N2-3)^*$
13	6 x 7	$N5$ - 1	Q_{12} - 2
13	7 x 6	$(M7$ - $4)^*$	
13	7 x 6	$(M8$ - $2)^*$	$N2$ - 3, $(N2-4)^*$
13	7 x 6	$(N5$ - $1)^*$	Q_{12} - 1
14	5 x 9	$M11$ - 1	$N7$ - 1
14	5 x 9	$M11$ - 2	$N6$ - 1
14	9 x 5	$(M11$ - $1)^*$	$(N7$ - $1)^*$
14	9 x 5	$(M11$ - $2)^*$	$(N6$ - $1)^*$
14	6 x 8	$M9$ - 2	Q_{12} - 5, $N5$ - 4
14	8 x 6	$(M9$ - $2)^*$	Q_{12} - 3, $(N5$ - $4)^*$
14	7 x 7	Q_{12} - 4	$N5$ - 3 , $(N5$ - $3)^*$
14	7 x 7	$M8$ - 3	$(M8$ - $3)^*$, $N2$ - 5, $(N2$ - $5)^*$

FIGURE 3.20. Isomorphism Table Part 2

Size	Row x Column	Matroid	Isomorphic To
15	5 x 10	$M12$ - 1	$N8$ - 1
15	10 x 5	($M12$ - 1)*	($N8$ - 1)*
15	7 x 8	$M9$ - 3	Q_{12} - 7, $N5$ - 6, $(N5-5)^*$
15	8 x 7	($M9$ - 3)*	Q_{12} - 6, $N5$ - 5, $(N5-6)^*$
16	5 x 11	$PG(3,2)$ - 1	Q_{15} - 1
16	11 x 5	($PG(3,2)$ - 1)*	(Q_{15} - 1)*
16	8 x 8	$M9$ - 4	($M9$ - 4)*, Q_{12} - 8, $N5$ - 7, ($N5$ - 7)*

FIGURE 3.21. Isomorphism Table Part 3

In the three tables above, we find a total of forty-two distinct binary 3-connected (prism+e)-free matroids that are not internally 4-connected. Note that 94 matroids were actually listed in the tables to emphasize that $M(K_5)$ - 1 $\cong$ ($M^*(K_5)$ - 1)*, ($M(K_5)$ - 1)* $\cong$ $M^*(K_5)$ - 1, $M(K_5)$ - 2 $\cong$ ($M^*(K_5)$ - 2)*, and ($M(K_5)$ - 2)* $\cong$ $M^*(K_5)$ - 2. These four isomorphisms allow us to use ($M(K_5)$ - 1)* and ($M(K_5)$ - 2)* in our set of 42 sporadic matroids to show that each non-self-dual matroid appears with their dual in the set. The complete list of the forty-two sporadic matroids in the set $EX_{3c-i4c}(prism+e)$ is as follows:

$M(K_5)$ - 1, ($M(K_5)$ - 1)*, $M(K_5)$ - 2, ($M(K_5)$ - 2)*, $M7$ - 1, ($M7$ - 1)*, $M7$ - 2, ($M7$ - 2)*, $M7$ - 3, ($M7$ - 3)*, $M7$ - 4, ($M7$ - 4)*, $M8$ - 1, ($M8$ - 1)*, $M8$ - 2, ($M8$ - 2)*, $M8$ - 3, $M9$ - 1, ($M9$ - 1)*, $M9$ - 2, ($M9$ - 2)*, $M9$ - 3, ($M9$ - 3)*, $M9$ - 4, $M10$ - 1, ($M10$ - 1)*, $M10$ - 2, ($M10$ - 2)*, $M10$ - 3, $(M10-3)^*$, $M11$ - 1, ($M11$ - 1)*, $M11$ - 2, ($M11$ - 2)*, $M12$ - 1, ($M12$ - 1)*, $PG(3,2)$ - 1, ($PG(3,2)$ - 1)*, Q_{12} - 4, $N2$ - 1, $N5$ - 1, and ($N5$ - 1)*.

The matroids $M8$ - 3, $M9$ - 4, Q_{12} - 4, and $N2$ - 1 are each self-dual. Nineteen of the remaining thirty-eight matroids are duals of the other nineteen. Therefore, we need only show matrix representations for twenty-three matroids rather than forty-two. These matroids are presented in standard form. In Tables 3.22 and 3.23, the rank five and rank six sporadic members of $EX_{3c-i4c}(prism + e)$ are presented. In each table, the bullets indicate which columns are elements of the listed matroid. Figures 3.24 and 3.25 display the standard representations of the rank seven and rank eight sporadic members of this set.

	1	0	0	1	1	0	1	0	0	1	1	0	1	0	0	1	1	1	1	1
	1	1	0	0	0	1	1	1	0	0	0	1	0	1	1	0	1	1	1	1
	0	1	1	0	1	0	0	1	1	0	1	0	1	1	1	1	0	1	1	1
	0	0	1	1	0	1	1	1	1	1	1	1	0	0	1	1	1	0	1	1
	0	0	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	1	0	1
$M(K_5)-1$	•	•							•						•			•	•	
$M7-1$				•	•			•						•		•	•		•	
$M7-2$				•	•									•	•	•	•		•	
$M7-3$					•			•		•				•		•	•			•
$M8-1$	•		•		•		•			•				•						•
$M9-1$	•				•		•			•	•				•			•	•	
$M10-1$				•	•			•	•		•	•		•					•	
$M10-2$			•	•	•							•		•	•	•			•	
$M10-3$								•	•	•	•	•	•	•					•	
$M11-1$	•			•	•			•	•		•	•		•					•	
$M11-2$	•		•		•	•		•		•	•			•						•
$M12-1$	•		•	•	•	•		•						•		•	•		•	
$PG(3,2)-1$	•	•		•	•	•		•	•		•						•	•	•	

FIGURE 3.22. Rank 5 sporadic members of $EX_{3c-i4c}(prism+e)$

	1	1	1	0	1	0	1	0	0	0	0	0	1	1	0	1	0	1	1	1	1	0	1	1	1
	1	0	1	1	0	0	0	1	1	0	0	0	0	1	1	0	1	0	1	1	1	1	1	1	1
	0	1	0	1	1	1	0	1	0	1	0	0	0	0	1	1	1	1	0	1	1	1	0	1	1
	0	0	0	0	1	1	1	0	1	1	1	1	1	0	0	1	1	1	1	0	1	1	1	0	1
	0	0	0	0	0	0	1	1	0	1	1	1	1	1	1	0	0	1	0	1	0	1	1	1	1
	0	1	1	1	0	1	0	0	1	0	0	1	1	1	1	1	1	0	1	0	0	1	1	1	0
$M(K_5)-2$			•	•						•										•	•	•			
$M7-4$		•											•		•		•	•					•		•
$M8-2$	•	•				•							•		•				•						•
$M9-2$	•	•					•									•			•		•	•		•	
$N2-1$	•						•		•			•			•	•									
$N5-1$	•				•			•	•		•		•	•											

FIGURE 3.23. Rank 6 sporadic members of $EX_{3c-i4c}(prism+e)$

$$\begin{bmatrix} 1&1&1&0&0&1&1\\ 1&0&0&1&0&1&1\\ 0&1&0&1&1&0&1\\ 0&0&1&0&1&1&1\\ 0&0&1&1&0&0&1\\ 0&1&1&1&1&1&0\\ 1&1&1&1&0&0&0 \end{bmatrix}$$

(a) $M8$ - 3

$$\begin{bmatrix} 1&1&1&1&1&1&0&1\\ 1&0&0&1&1&0&1&1\\ 0&1&0&1&0&1&1&1\\ 0&0&1&0&1&1&1&1\\ 0&0&1&1&0&0&1&0\\ 0&1&0&1&1&1&1&0\\ 0&1&1&0&1&0&1&1 \end{bmatrix}$$

(b) $M9$ - 3

$$\begin{bmatrix} 1&1&0&0&0&0&0\\ 1&0&0&1&0&1&1\\ 0&1&0&1&1&0&1\\ 0&1&1&0&0&1&1\\ 0&0&1&1&0&0&1\\ 1&0&1&1&1&0&0\\ 1&1&1&0&1&0&1 \end{bmatrix}$$

(c) Q_{12} - 4

FIGURE 3.24. The matroids $M8$ - 3, $M9$ - 3 and Q_{12} - 4

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

FIGURE 3.25. The matroid $M9$ - 4

Appendix C: MACEK Commands and Examples

MACEK Commands and Examples

The computer program MACEK [**12**] has been mentioned often throughout this dissertation. In this section, some background information as well as some sample commands and output used in this dissertation will be provided. The MACEK program was developed by Petr Hliněný in 2001 to assist in the research of Matroid Theory. Since that time, many upgrades have been made to make this software the powerful computational tool that it is today. This program has the capability to test for minors, isomorphisms, connectivity, etc. MACEK can be used to find the circuits, bases, and flats of specific matroids or to extend and coextend matroids while avoiding certain minors. For a complete guide to MACEK, see [**12**].

MACEK can be used to view matroids over different fields. For example, a binary and a regular representation of $M(W_3)$ can be found by using the command "-pGF2" or "-pREG" before the "print" command. When no field is specified, MACEK automatically defaults to the finite field over two elements, GF(2). Note that throughout this section, some of the non-essential output will be not be included to save space.

```
 ./macek -pGF2 print W3
    ~ Output of the command "!print (S)   [1]":
    ~ Matrix of the frame 0xee03a0 [W3] in GF(2): "the matroid W_3, wheel
of 3 spokes"
    ~   --------------------------------------------------------------
    ~   matrix 0xf16820 [W3], r=3, c=3, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')
    ~
    ~    '1')       1       o       1
    ~    '2')       1       1       o
    ~    '3')       o       1       1
    ~   --------------------------------------------------------------
```

```
 ./macek -pREG print W3
    ~ Output of the command "!print (S)   [1]":
~ Matrix of the frame 0xf6adc0 [W3] in regular: "the matroid W_3, wheel
of 3 spokes"
    ~   ------------------------------------------------------------
    ~   matrix 0xf75540 [W3], r=3, c=3, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')
    ~
    ~    '1')       1       o      -1
    ~    '2')      -1       1       o
    ~    '3')       o      -1       1
    ~   ------------------------------------------------------------
```

MACEK can also be used to check for the connectivity of a matroid in general or to see if it satisfies a particular connectivity. For example, the "connect" command reveals the connectivity of a matroid while "!isconni4" is designed to test for internal 4-connectivity. In the following example, MACEK is used to find the connectivity of the affine geometry $AG(3,2)$ and to show that this matroid is not internally 4-connected.

```
./macek connect AG32
~Matrix of the frame 0xfafb90 [AG32] in GF(2): "the matroid AG(3,2) binary aff cube"
~ Output of the command "!print (S)   [1]":
    ~   ------------------------------------------------------------
    ~   matrix 0xf9b388 [AG32], r=4, c=4, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')   '-4')
    ~
    ~    '1')       1       1       1       o
    ~    '2')       1       1       o       1
    ~    '3')       1       o       1       1
    ~    '4')       o       1       1       1
    ~   ------------------------------------------------------------
~ Output of the command "!connectivity (S)   [1]":
~ The #1 matroid [AG32] has connectivity exactly 3.


./macek '!isconni4' AG32
 ~ Output of the command "!isconni4 ((t)) 3 i [3]":
 ~ The #1 matroid [AG32] has -NOT- connectivity at least int-4.
```

The capability of this program to check for minors is displayed next. Multiple matroids can be tested for a specific minor using the "minor" command. The next example shows that $M(W_4)$ is a minor of R_{10} but not a minor of F_7.

```
./macek minor R10 F7 W4
   ~   matrix 0xcfb080 [R10], r=5, c=5, tr=0, ref=0x0
   ~           '-1')   '-2')   '-3')   '-4')   '-5')
   ~    '1')      1       1       o       o       1
   ~    '2')      1       1       1       o       o
   ~    '3')      o       1       1       1       o
   ~    '4')      o       o       1       1       1
   ~    '5')      1       o       o       1       1
   ~   matrix 0xd06f78 [F7], r=3, c=4, tr=0, ref=0x0
   ~           '-1')   '-2')   '-3')   '-4')
   ~    '1')      1       1       1       o
   ~    '2')      1       1       o       1
   ~    '3')      1       o       1       1
   ~   matrix 0xcfb080 [W4], r=4, c=4, tr=0, ref=0x0
   ~           '-1')   '-2')   '-3')   '-4')
   ~    '1')      1       o       o       1
   ~    '2')      1       1       o       o
   ~    '3')      o       1       1       o
   ~    '4')      o       o       1       1

~   Output of the command "!minor ((-1T)) ((-1)(T))  [2]":
~   The #1 matroid [R10] +HAS+ minor #1 [W4] in the list {W4 }.
~   The #2 matroid [F7] has -NO- minor #0 [] in the list {W4 }.
```

A particularly useful aspect of the MACEK package [**12**] is the capability to check matroids for isomorphisms using the "isomorph" command. The following example confirms that the cycle matroid of prism+e is isomorphic to its dual over the field GF(2). Note that there is no need to input $(prism+e)^*$ into MACEK. A simple "!dual" command can be used as seen in the command line. When printed, the dual of a matroid M is denoted by "M#".

```
./macek isomorph prism+e 'prism+e;!dual'
  ~   matrix 0xcfaee0 [prism+e], r=5, c=5, tr=0, ref=0x0
  ~           '-1')   '-2')   '-3')   '-4')   '-5')
  ~    '1')      1       1       o       o       o
  ~    '2')      1       o       o       1       o
  ~    '3')      o       1       o       1       1
  ~    '4')      o       1       1       o       1
  ~    '5')      o       o       1       1       o

  ~   matrix 0xcfaee0 [prism+e#], r=5, c=5, tr=1, ref=0x0
  ~           '1')    '2')    '3')    '4')    '5')
  ~    '-1')     1       1       o       o       o
  ~    '-2')     1       o       1       1       o
  ~    '-3')     o       o       o       1       1
  ~    '-4')     o       1       1       o       1
  ~    '-5')     o       o       1       1       o

~  Output of the command "!isomorph ((T)) (()(T))  [2]":
~  Matroid [prism+e] (over cur)  +IS+  isomorphic to matroid [prism+e#] (over GF(2)).
```

The capability of MACEK [**12**] to extend and coextend matroids while avoiding certain minors was instrumental in this dissertation. This function of MACEK enabled us to find all of the internally 4-connected binary (prism+e)-free matroids with at most 17 elements. By hand, this is quite a challenging computation. With MACEK, we were able to determine all of these matroids. An example of the command used to extend and coextend prism while avoiding prism+e and checking for internal 4-connectivity is given below. Note that the command "!extend b" simply means to both extend and coextend. Commands such as "!extend r" and "!extend c" can be used to add rows and columns, respectively.

```
./macek '!extend b;@ext-forbid prism+e;!print;!isconni4' prism
 [gener.c :gener_extframe_ex()133 ~505] Calling to get row co-extensions of the
seq 0xcd03c8[prism] (5x4) in GF(2)...
 [gener.c : gener_extensions()532 ~505] Gener - passed 3 out of 16 (co)extension
s of 5x4 matrix 0xcff870[prism].
        (seq3=6, struc3=6, canon3=4, struc2=4, canon2=4, struc1=4, canon1=4, seq
0=4, struc0=4, canon0=3)
~   Generated 3 non-equiv 3-conn row co-extens of the sequence [prism] (5x4|5x4).
 [gener.c :gener_extframe_ex()133 ~505] Calling to get column extensions of the
seq 0xcd03c8[prism] (5x4) in GF(2)...
```

```
 [gener.c : gener_extensions()532 ~505] Gener - passed 4 out of 32 (co)extension
s of 5x4 matrix 0xcff870[prism].
        (seq3=22, struc3=22, canon3=9, struc2=9, canon2=9, struc1=8, canon1=8, s
eq0=8, struc0=7, canon0=4)
~   Generated 4 non-equiv 3-conn column extens of the sequence [prism] (5x4|5x4).
~   In total 7 (co-)extensions of 1 matrix-sequences generated for "b" over GF(2).

    ~   matrix 0xcff870 [prism_r1], r=6, c=4, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')   '-4')
    ~    '1')       1       1       o       o
    ~    '2')       1       o       o       1
    ~    '3')       o       1       o       1
    ~    '4')       o       1       1       o
    ~    '5')       o       o       1       1
    ~    '6')       o       1       1       1

    ~   matrix 0xcff870 [prism_r2], r=6, c=4, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')   '-4')
    ~    '1')       1       1       o       o
    ~    '2')       1       o       o       1
    ~    '3')       o       1       o       1
    ~    '4')       o       1       1       o
    ~    '5')       o       o       1       1
    ~    '6')       1       o       1       o

    ~   matrix 0xcff870 [prism_r3], r=6, c=4, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')   '-4')
    ~    '1')       1       1       o       o
    ~    '2')       1       o       o       1
    ~    '3')       o       1       o       1
    ~    '4')       o       1       1       o
    ~    '5')       o       o       1       1
    ~    '6')       1       o       1       1

    ~   matrix 0xcff870 [prism_c1], r=5, c=5, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')   '-4')   '-5')
    ~    '1')       1       1       o       o       o
    ~    '2')       1       o       o       1       o
    ~    '3')       o       1       o       1       1
    ~    '4')       o       1       1       o       1
    ~    '5')       o       o       1       1       1

    ~   matrix 0xcff870 [prism_c2], r=5, c=5, tr=0, ref=0x0
    ~            '-1')   '-2')   '-3')   '-4')   '-5')
    ~    '1')       1       1       o       o       o
    ~    '2')       1       o       o       1       1
    ~    '3')       o       1       o       1       o
    ~    '4')       o       1       1       o       o
    ~    '5')       o       o       1       1       1
```

```
~   matrix 0xcff870 [prism_c3], r=5, c=5, tr=0, ref=0x0
~            '-1')   '-2')   '-3')   '-4')   '-5')
~     '1')       1       1       o       o       o
~     '2')       1       o       o       1       1
~     '3')       o       1       o       1       o
~     '4')       o       1       1       o       1
~     '5')       o       o       1       1       o

~   matrix 0xcff870 [prism_c4], r=5, c=5, tr=0, ref=0x0
~            '-1')   '-2')   '-3')   '-4')   '-5')
~     '1')       1       1       o       o       1
~     '2')       1       o       o       1       1
~     '3')       o       1       o       1       1
~     '4')       o       1       1       o       1
~     '5')       o       o       1       1       1

~   Output of the command "!isconni4 ((s)) 3 i [3]":
~   The #1 matroid [prism_r1] has -NOT- connectivity at least int-4.
~   The #2 matroid [prism_r2] has connectivity at least int-4.
~   The #3 matroid [prism_r3] has connectivity at least int-4.
~   The #4 matroid [prism_c1] has -NOT- connectivity at least int-4.
~   The #5 matroid [prism_c2] has -NOT- connectivity at least int-4.
~   The #6 matroid [prism_c3] has -NOT- connectivity at least int-4.
~   The #7 matroid [prism_c4] has -NOT- connectivity at least int-4.
~   Total 2 out of 7 matroids have connectivity at least int-4.
```

As can be seen in the examples provided, MACEK is a powerful tool for researchers in the field of Matroid Theory. A complete guide to MACEK as well as instructions on how to download this program can be found in [**12**].

Vita

The author, Kayla Davis Harville, was born in Lumberton, MS to Dennis and Sharon Davis. She was the valedictorian of her class at Lumberton High School in 2002. Kayla received an Associate's of the Arts degree from Pearl River Community College in 2004 before transferring to Mississippi State University to complete the degree requirements for both a Bachelor's of Science (2006) and a Master's of Science (2008) in Mathematics. She is currently a Ph.D. candidate in Mathematics at The University of Mississippi. She met and married her husband, Cody Harville, while attending Ole Miss.