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Zero Divisor Graphs and Poset Dimension

Bette Catherine Putnam

Zero Divisor Graphs and Poset Decomposition

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BETTE CATHERINE PUTNAM

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ABSTRACT

A graph is associated to any commutative ring R where the vertices are the non-zero zero divisors of R with two vertices adjacent if $x \cdot y = 0$. The zero-divisor graph has also been studied for various algebraic stuctures such as semigroups and partially ordered sets. In this paper, we will discuss some known results on zero-divisor graphs of posets as well as the concept of compactness as it relates to zero-divisor graphs. We will dicuss equivalence class graphs defined on the elements of various algebraic structures and also the reduced graph defined on the vertices of a compact graph. After introducing and discussing some known results on poset dimension, we will show that poset decomposition can be directly related to the equivalence classes represented in a reduced graph. Using this decomposition, we can build a poset of a compact graph with any dimension in a specified interval. Thus we have a device which gives us the ability to study the dimension of a poset of a zero-divisor graph.

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1 INTRODUCTION

In [3], Istvan Beck first introduced for any commutative ring R a simple graph G(R) whose vertices are labeled by the elements of R and any two vertices x and y of G(R) are adjacent if and only if $x \cdot y = 0$. By definition, G(R) is a simple graph, and therefore contains no loops. So the self-annihilating elements of R are not represented in G(R). Moreover, the zero vertex is adjacent to every ring element in the graph G(R).

Beck was interested in the coloring of the graph G(R). He conjectured that the chromatic number of G(R), the minimal number of colors that can be assigned to the vertices of G such that no two adjacent vertices share the same color, was equal to the clique number of G(R), the size of the largest complete subgraph of G(R). In 1993, D. D. Anderson and M. Neeser presented a counterexample of a commutative local ring for which the chromatic number was strictly greater than the clique number [2]. Later, D. F. Anderson and P. Livingston [1] simplified Beck's zero-divisor graph by restricting the graph to the nonzero zero-divisors.

Recently, the zero-divisor graphs of various algebraic structures have been studied in [1], [4], [7], and [10]. We will discuss some of these results; however, the main focus of this paper will be on a wider class of relational structures, that being the class of partially ordered sets. In 2009, Halas and Jakl introduced the zero-divisor graph for a partially ordered set, or a poset. In [7], D. Wu and T. Lu charaterized the zero-divisor graphs of posets and provided a description for building a poset of a compact graph, which we will discuss in Section 2. In their characterization, Wu and Lu showed that a simple graph G is the zero-divisor graph of a poset if and only if G is compact. In Section 3, we will see that for various algebraic structures, a simple graph G is the zero-divisor graph of that algebraic structure if and only if G statisfies conditions similar to compactness.

In Section 4, we will dicuss equivalence class graphs defined on the elements of various algebraic structures as well as the reduced graph defined on the vertices of a compact graph. Then after introducing and discussing some known results on poset dimension in Section 5, we will show that poset decomposition can be directly related to the equivalence classes represented in a reduced graph. In the main theorem of this paper, using poset decomposition, we can build a poset of a compact graph with any dimension in a specified interval. Thus we have a device which gives us the ability to study the dimension of a poset of a zero-divisor graph.

2 ZERO-DIVISOR GRAPHS OF POSETS

We begin by defining the simplified zero-divisor graph given by D. F. Anderson and P. Livingston in [1].

Definition 1 *[1] Given a ring R, let $Z^*(R)$ denote the set of nonzero zero-divisors of R. Let $\Gamma(R)$ denote the graph whose vertex set is $Z^*(R)$, such that any two distinct vertices x and y of $Z^*(R)$ are adjacent if $x \cdot y = 0$.*

Figure 1 below shows the zero-divisor graph of $\Gamma(\mathbb{Z}_{12})$ as defined in the previous definition. Note that $Z^*(\mathbb{Z}_{12}) = \{2, 3, 4, 6, 8, 9, 10\}$.

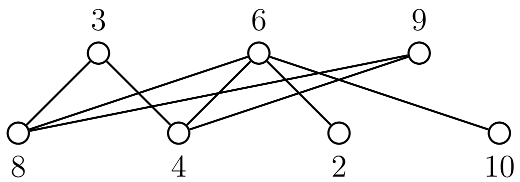


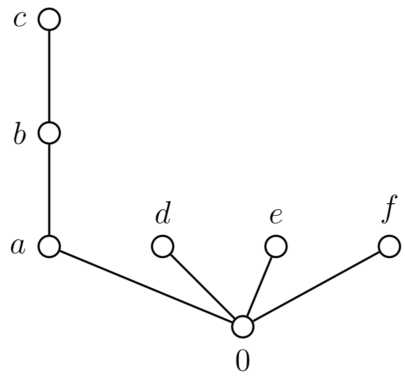
Figure 1: The Zero-Divisor Graph $\Gamma(\mathbb{Z}_{12})$

In this paper, we will discuss the zero-divisor graph for a partially ordered set, or a poset for short, which was introduced by Halas and Jakl in [5]. A poset with a zero-divisor graph G will be referred to from now on as a poset of graph G. We define a poset **P** to be a pair (X,P) where X is a set and P is a reflexive, antisymmetric, and transitive binary relation on X. We call X the ground set and P the partial order on X [11]. Given a partially ordered set, and a finite set $A \subseteq X$ such that $A = \{a_1, a_2, ..., a_k\}$, define $L(A) = L(a_1, a_2, ...a_k) := \{y \in X : \forall a_i \in A, y \leq a_i\}$ as the *lower cone* of A. Also, for any poset, we say the x is the least element of the poset if $x \leq a$ for all elements $a \in X$.

Definition 2 *Let* $\boldsymbol{P}$= $(X, \leq)$ *be a poset with a least element* 0. *An element* $x \in X$ *is called a zero-divisor if* $L(x,y) = 0$ *for some nonzero element* $y \in X$. *Then the zero-divisor graph* $\Gamma(\boldsymbol{P})$ *of a poset* $\boldsymbol{P}$ *is the graph where the vertex set* $V(\Gamma(\boldsymbol{P}))$ *consists of all nonzero zero-divisors of* $\boldsymbol{P}$ *where two distinct vertices* a *and* b *are adjacent if and only if* $L(a,b) = \{0\}$.

In a poset $\mathbf{P} = (X, P)$, we say that two elements $x, y \in X$ are called *comparable* in P if either $x < y$ or $y < x$ in P. Two elements are called *incomparable* in P if neither $x < y$ nor $y < x$ in P. An element x is said to be *covered* by another element y in P when $x < y$ in P and there is no element $z \in X$ such that $x < z$ and $z < y$ in P. We say that (X, P) is a *chain* if every distinct pair of elements in X are comparable and (X, P) is an *antichain* if every distinct pair of elements in X are incomparable.

A poset can be graphically represented by a Hasse diagram where each element of the poset is represented by a vertex in the diagram. For any two elements $x, y \in X$, if $x < y$ in P, then x appears lower than y in the Hasse diagram. Also, x and y are adjacent if and only if x covers y or y covers x in P. The chromatic number $\chi(P)$ is the chromatic number of $\Gamma(P)$, and for any subset C of P, C is a clique if for any two elements x and y of C, we have that $L(x,y) = 0$. Then the clique number of the poset P, denoted *clique*(P), is the size of the largest clique of P. The figure below shows the graphical representation of a poset. The elements $\{a, b, c\}$ of the poset form a chain, and the elements $\{d, e, f\}$ of the poset form an antichain.



$\mathbf{P} = (X, P)$

Figure 2: Hasse Diagram of a Poset

We now introduce the concept of compactness.

Definition 3 *[7] A simple graph G is called compact if G has no isolated vertices and for any two distinct nonadjacent vertices x and y of G, there is a vertex z of G such that $N(x) \cup N(y) \subseteq N(z)$.*

In 2009, D. Lu and T. Wu proved the following theorem:

Theorem 4 *[7] A simple graph G is the zero-divisor graph of a poset if and only if G is a compact graph.*

Note that if G is a compact graph, then for any pair of vertices x and y of G, we can show that either $N(x) \cap N(y) \neq \emptyset$ or each vertex in $N(x)$ is adjacent to all of the vertices of $N(y)$. Therefore G is connected with diameter at most 3.

We now introduce some basic terminology. A simple graph G is called an *n-compact graph* when G is compact and the clique number $\omega(G) = n$. For n a cardinal number, an *n-partite graph* is a graph whose vertex set can be partitioned into n subsets such that no edge of the graph has both ends in the same partite set. Furthermore, a *complete n-partite graph* is an n-partite graph where each vertex is adjacent to every vertex not in the same partite set, and a graph G is a *proper n-partite graph* if G is not an r-partite graph for any $r < n$. It can be shown that if G is an n-compact graph, then G is a proper n-partite graph with clique size n.

For convenience, denote $\{1, 2, ..., n\}$ by $[1, n]$, and let $\Delta_n(G)$ be the set of all vertices which lie in some clique of size n in G. For an n-compact graph G with parts A_i for $i \in [1, n]$, if $a_i \in \Delta_n(G) \cap A_i$ for each $i \in [1, n]$, then the set $\{a_i : i \in [1, n]\}$ is a clique in G. Then we can see that for G an n-compact graph, the induced subgraph on $\Delta_n(G)$ is a complete n-partite graph. Then we can see that for each $i \in [1, n]$, the elements of $\Delta_n(G) \cap A_i$ have the same neighborhoods. We can define an equivalence relation $\sim$ such that for any two elements x and y of G, $x \sim y$ if and only if $N(x) = N(y)$. Then for each $i \in [1, n]$, we have that $[a_i] = \Delta_n(G) \cap A_i$ is an equivalence class.

For any n-compact graph G with parts $A_i, i \in [1, n]$ with a vertex $a \in V(G)$, we denote $W(a) = \{i \in [1, n] : N(a) \cap \Delta_n(G) \cap A_i \neq \emptyset\}$. Now we define a generalized complete n-partite graph.

Definition 5 *[7] A simple connected graph G is called a generalized complete n-partite graph if V(G) is a disjoint union of A and H satisfying the following conditions:*

1. *$A = \Delta_n(G)$ and the induced graph on A is a complete n-partite graph with parts $A_i, i \in [1, n]$.*
2. *For any $h \in H$ and $i \in [1, n]$, h is adjacent to some vertex of A_i if and only if h is adjacent to every vertex of A_i.*
3. *For any $h_1, h_2 \in H$, h_1 is adjacent to h_2 if and only if $W(h_1) \cup W(h_2) = [1, n]$.*

Now we can give a characterization of the zero-divisor graph of a poset.

Theorem 6 *[7] Let G be a graph with $\omega(G) = n < \infty$. Then the following statements are equivalent:*

1. *G is the zero-divisor graph of a poset.*
2. *G is an n-compact graph.*
3. *G is a generalized complete n-partite graph.*

In the proof of Theorem 4, Lu and Wu showed how to build a poset given a n-compact graph G. We begin by denoting $X = V(G) \cup \{0\}$ where $0 \notin V(G)$. Now we define an ordering P on X where for any element x of G, $0 \leq x$ and $x \leq x$. Now for two elements x and y in G, we define an equivalence relation $\sim$ such that $x \sim y$ if and only if $N(x) = N(y)$

Then we can choose, without loss of generality, an element b_i from the equivalence class as a representative of that set. Then let $\{b_i : i \in [1, n]\}$ be the set of representatives chosen from each equivalence class in G. Then for $x, y \in V(G)$, we say that $x \leq y$ if and only if $N(y) \subsetneq N(x)$ or $N(y) = N(x)$ and $x \in \{b_i : i \in [1, n]\}$. Then we can make all of the elements of $[b_i] \setminus \{b_i\}$ comparable to b_i but incomparable to each other. With the given ordering, we have the poset $\mathbf{P} = (X, P)$ with $\Gamma(P) = G$.

Notice that for a given zero-divisor graph, we can sometimes build more than one poset. We can do this by the way in which we define the ordering on the sets $\Delta_n(G) \cap A_i$, for $i \in [1, n]$. Consider the compact graph G in Figure 3 and the posets of G in Figure 4.

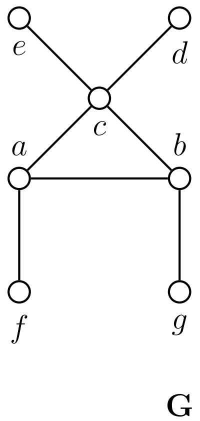


Figure 3: A Compact Graph G

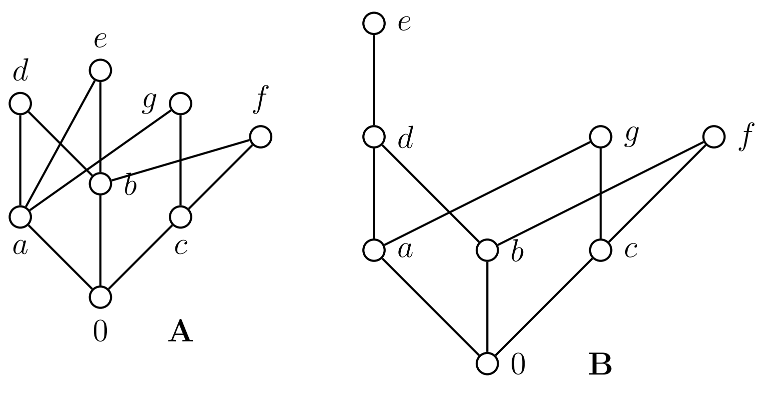


Figure 4: Examples of Posets of a Compact Graph G

The minimal non-zero elements of the poset will always be the in the set $\Delta_n(G)$. In the case of Figure 3, $\{a, b, c\} \in \Delta_n(G)$. Notice also in Figure 3 that $N(e) = N(d)$, and thus e and d is also in the set A_i for some $i \in [1, n]$. Thus, for example, we have that

$a \leq e$, $b \leq e$, $a \leq d$ and $b \leq d$, but there is no restriction on the relationship between the elements e and d in the poset. On the other hand, we also have that $f \leq c$ and $f \leq b$ and also $g \leq a$ and $g \leq c$. In Figure 4 above, the pair of elements d and e were drawn as a chain in B and as an antichain in A. For any collection of elements such that each element in the collection has the same neighborhood, we can express that collection of elements without restriction and still preserve the required ordering on X.

3 ZERO-DIVISOR GRAPHS OF SOME ALGEBRAIC SRUCTURES

Recall from Section 2 that I. Beck conjectured that for an arbitrary ring R, the clique number of G(R) is equal to the chromatic number of G(R). While it has been shown that his conjecture is not true in general, it does hold for both commutative semigroups and partially ordered sets.

A semigroup is a set of elements with an associative binary operation. A semigroup is called *reduced* if for any element $x \in S$ we have that if $x^n = 0$ for any positive integer n then $x = 0$, and idempotent if $x^2 = x$ for each $x \in S$. A semigroup that is commutative and idempotent is a semilattice. In this paper, the semigroup S will be a commutative semigroup with a zero element. In 2007, it was shown in [10] by Nimbhorkar, Wasadikhar and Demeyer that Beck's conjecture, that is the chromatic number of the graph is equal to the clique number of the graph, holds for commutative semigroups with a zero element.

As with commutative rings, we can consider the zero-divisor graph $\Gamma(S)$ of a commutative semigroup with a zero element, which was defined by DeMeyer et al. in [4]. We can easily see that the zero-divisor graph $\Gamma(S)$ of a semigroup S is a compact graph [7].

D. Lu and T. Wu also proved a characterization of the zero-divisor graphs of reduced semigroups with 0.

Theorem 7 *[7] Let G be a simple graph with $\omega(G) = n < \infty$. Then the following statements are equivalent:*

1. *G is the zero-divisor graph of a reduced semigroup with* 0.

2. *G is a generalized complete n-partite graph such that for any nonadjacent vertices $x, y \in V(G)$, there is a vertex $z \in V(G)$ with $W(z) = W(x) \cup W(y)$.*

3. *G is the zero-divisor graph of a semilattice with* 0.

Notice that this theorem is similar to the characterization of the zero-divisor graph of a poset given in Section 2.

In [5], Beck's conjecture was proven by Halas and Jukl for posets with 0. First recall that a prime ideal is an ideal I such that if $ab \in I$ then either $a \in I$ or $b \in I$. A prime ideal is said to be a minimal prime ideal if it does not properly contain any other prime ideal. Also, the annihilator of an element x of a poset, denoted $\mathrm{ann}(x)$, is the set of all elements y in the poset such that $L(x, y) = 0$. Then by [5], every minimal prime ideal of a poset $\mathbf{P} = (X, P)$ is of the form $\mathrm{ann}(x)$ for some $x \in X$, and if the clique number of the poset is finite, then the number of minimal prime ideals, say n, is finite and $\chi(P) = clique(P) = n + 1$.

4 EQUIVALENCE GRAPHS

We first note that the annihilator of an element x of a ring R, denoted ann(x) is the set of all elements r in R such that $x \cdot r = 0$. Then we define equivalence classes on the nonzero zero-divisor elements such that for any $x, y \in Z^*(R)$, we have that $x \sim y$ if and only if ann(x) =ann(y). In 2002, S. Mulay [8] introduced the graph of equivalence classes of a ring R, denoted EG(R), whose vertices correspond to the equivalence classes of the elements of $Z^*(R)$.

In 2006, T. Lucas showed that, like the zero-divisor graphs, the diameter of an equivalence class graph is at most three [6]. However, unlike the zero-divisor graphs, there are no complete equivalence class graphs with more than two vertices.

In the equivalence class graph above, we defined our equivalence relation on the elements of the ring. But we can also define an equivalence relation on the vertices of a compact graph G. Define the equivalence relation $x \sim y$ if and only if $N(x) = N(y)$, for $x, y \in V(G)$. Then the reduced graph EG is the graph whose vertices correspond to the equivalence classes of the elements of G. Then any two elements, say $[a]$ and $[b]$ of EG, are adjacent if and only if $[a] \cdot [b] = 0$, or equivalently, for $a_1 \in [a]$ and $b_1 \in [b]$, we have that $a_1 \cdot b_2 = 0$. Figure 5 below shows the reduced graph of $\mathbb{Z}_{12}$.

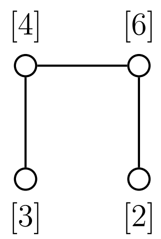


Figure 5: The Reduced Graph of $\mathbb{Z}_{12}$

Lu and Wu showed that a graph G is compact if and only if its reduced graph

EG is compact. Furthermore, G is the zero-divsor graph of a poset if and only if EG is the zero-divisor graph of a poset [7]. Therefore, given a reduced graph EG, we can build a poset with a least element 0 whose zero-divisor graph is EG.

To do so, we consider a compact graph G and let EG be the graph of equivalence classes of G. Then we can build the poset of the reduced graph EG using the construction defined by Lu and Wu in [7] and outlined in Section 2. Figure 6 gives the poset for the equivalence class graph $\Gamma(\mathbb{Z}_{12})$ shown in Figure 5.

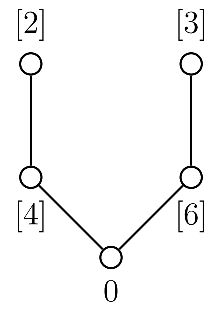


Figure 6: The Poset of the Equivalence Class Graph of $\mathbb{Z}_{12}$

Recall from Figure 4 that with the poset of a zero-divisor graph we could interpret elements with the same neighborhoods as any collection of chains and antichains. In the poset of the equivalence class graph, those elements are captured by the equivalence classes, giving a single poset.

The equivalence class graph of a ring cannot be complete; however, the reduced graph of a compact graph G could be complete as the graph is defined on the neighborhoods of the elements of G and not on the annihilators of the ring. Using the construction of the poset of a reduced graph outlined above, we can see that a complete reduced graph EG with n vertices produces a poset which, excluding the least element 0, is an antichain with n elements.

5 POSET DIMENSION

The concept of poset dimension was first introduced by Dushnik and Miller in 1941. The dimension of a poset depends upon the the linear extensions of the poset. Let P and Q be two partial orders on a set X. If $x \leq y$ in P implies that $x \leq y$ in Q for $x, y \in X$, or equivalently $P \subseteq Q$ then we call Q an extension of P. The set of all extensions of P is also a partially ordered set by inclusion. Then P is the unique minimal element, and the maximal elements are the linear orders on X, called linear extensions of P. Then we have the following definition:

Definition 8 *[11] Let $\boldsymbol{P} = (X, P)$ be a finite poset. Then the dimension of (X, P), denoted* $\dim(X, P)$*, is the least positive integer n for which there is a family $\mathcal{R} = \{L_1, L_2, ..., L_n\}$ of linear extensions of P such that $\boldsymbol{P} = \cap\mathcal{R} = \bigcap_{i=1}^{n} L_i$.*

A family $\mathcal{R}$ of linear extensions of P is called a realizer of P on X if $P = \cap\mathcal{R}$. As repetitions of linear orders are allowed in a realizer of P, a poset of dimension n has a realizer $\mathcal{R} = \{L_1, L_2, ..., L_m\}$ for every $m \geq n$. Figure 7 illustrates a poset with five linear extensions. However, $\mathcal{R} = \{L_4, L_5\}$ realizes P. So the dimension of the poset is at most 2.

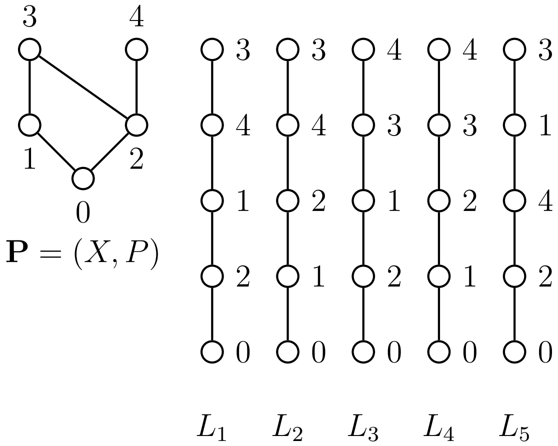


Figure 7: Linear Extensions of a Poset $\mathbf{P} = (X, P)$

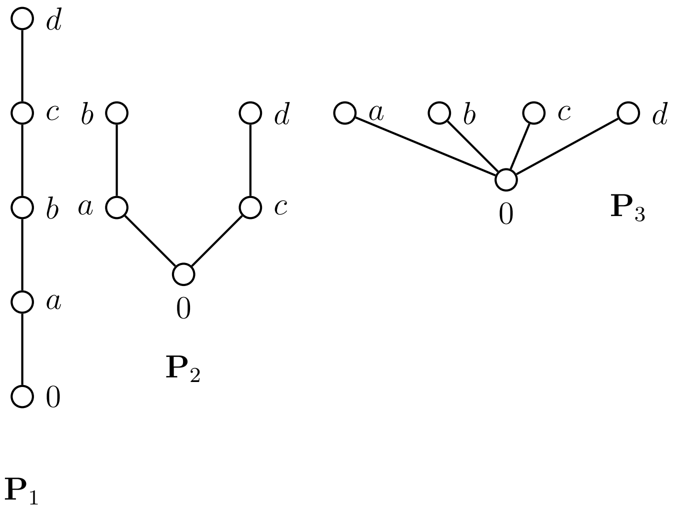


Figure 8: Posets $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$

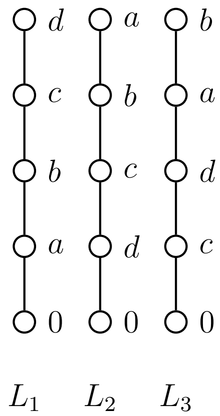


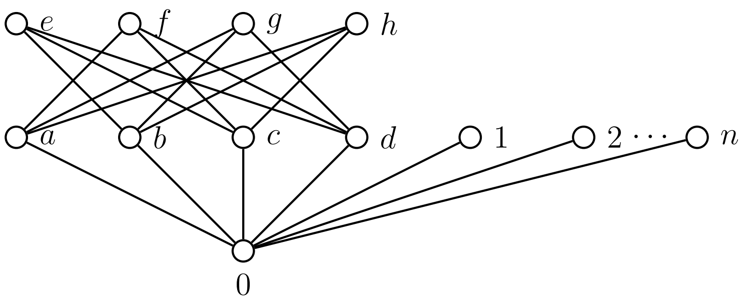
Figure 9: Linear Extensions of Posets $\mathbf{P}_1$, $\mathbf{P}_2$, and $\mathbf{P}_3$

In Figure 8, we have a chain $\mathbf{P}_1 = (X, P_1)$, and the posets $\mathbf{P}_2 = (X, P_2)$ and $\mathbf{P}_3 = (X, P_3)$ where, excluding the least element 0, the poset $\mathbf{P}_2$ is a collection of chains and the poset $\mathbf{P}_3$ is an antichain. From Figure 9, the family $\mathcal{R}_1 = \{L_1\}$ realizes $\mathbf{P}_1$, and the family $\mathcal{R}_2 = \{L_1, L_3\}$ realizes $\mathbf{P}_2$. Also, note that the family $\mathcal{R}_3 = \{L_1, L_2\}$ realizes $\mathbf{P}_3$. In fact, a poset has dimension one if and only if it is a single chain. Therefore, the poset given in Figure 7 has dimension 2.

A crown on $2n$ vertices is a poset whose Hasse diagram is two sets of nonzero elements $A = \{u_i : i \in [1, n]\}$ and $B = \{v_i : i \in [1, n]\}$ where $u_i \leq v_j$ whenever $i \neq j$.

Proposition 9 *A poset whose non-zero elements are the union of a crown with $2n$ elements and an antichain of any size has dimension n. Similarly, the poset whose non-zero elements are the union of a crown with $2n$ elements and one or more chains of any length has dimension n.*

It is known that the poset dimension of a crown with $2n$ elements is n. Therefore we know that a crown of size $2n$ has a realizer of size n. If we consider the union of a crown and an antichain, then we need the elements of the antichain to be incomparable with each other and with the elements of the crown. Then we can list the elements of the anitchain $1, 2, ..., n$ at the bottom of one linear extension of the crown, and list the elements of the anitchain in the opposite order, that is $n - 1, n, ..., 2, 1$ at the top of the remaining $n - 1$ linear extensions of the crown. Thus the dimension of the poset is n. Similarly, we can show that the dimension of a poset whose non-zero elements is the union of a crown and one or more chains of any length is also n. Figure 10 illustrates the union of a crown and an antichain below.



$$L_1 = 1 < 2 < \cdots < n < b < c < d < h < e < a < f < g < h$$
$$L_2 = a < c < d < f < b < e < g < h < n < n - 1 < \cdots < 1$$
$$L_3 = a < b < d < g < c < e < f < h < n < n - 1 < \cdots < 1$$
$$L_4 = a < b < c < h < d < e < f < g < n < n - 1 < \cdots < 1$$

Figure 10: An Example of a Poset with Dimension 4

Now let a subposet be a restriction of the ordering P to a nonempty subset Y of X, denoted $(Y, P(Y))$. We can show that poset dimension is continuous in that small removals or additions to the poset cannot greatly affect the dimension. In fact, for a chain $C \subseteq X$ where $X - C \neq \emptyset$, the removal of C changes the dimension by at most 2.

Theorem 10 *[11] Let $\boldsymbol{P} = (X, P)$ be a poset and let $C \subseteq X$ be a chain with $X - C \neq \emptyset$. Then* $\dim(X, P) \leq 2 + \dim(X - C, P(X - C))$.

Note that a single point is also a chain. Then there is a better result on the removal of a single point.

Theorem 11 *[11] Let $\boldsymbol{P} = (X, P)$ be a poset and with $|X| \geq 2$ and let $x \in X$. Then* $\dim(X, P) \leq 1 + \dim(X - \{x\}, P(X - \{x\})$.

Given a poset $\mathbf{P} = (X, P)$, we can also consider the dimension of a subposet of $\mathbf{P}$. With the following property, we see that dimension is monotonic.

Monotonic Property of Dimension *[11] Let $\boldsymbol{P} = (X, P)$ be a poset, and let $\emptyset \neq Y \subseteq X$. Then* $\dim(Y, P(Y)) \leq \dim(X, P)$.

Thus we see that the dimension of a poset is both continuous and monotonic.

Now we can relate the dimension of a poset to the width of that poset, or the number of elements in a maximum antichain.

Theorem 12 *[11] Let $\boldsymbol{P} = (X, P)$ be a poset. Then* $\dim(X, P) \leq \mathrm{width}(X, P)$.

Therefore, given a poset (X, P) where $|X| = n$, we know that $\dim(X, P) \leq \mathrm{width}(X, P) \leq n$. We can easily show that for $|X| \leq 5$, $\dim(X, P) \leq 2$, and similarly for $|X| \leq 7$, we have that $\dim(X, P) \leq 3$. We can find a tighter bound for the dimension of small posets.

Theorem 13 *[11] Let $\boldsymbol{P} = (X, P)$ be a poset with $|X| \geq 4$. Then* $\dim(X, P) \leq \frac{|X|}{2}$.

Now for a poset (X, P) let $\mathcal{F} = \{(Y_x, Q_x) : x \in X\}$ be a family of posets. We define the lexicographical sum of $\mathcal{F}$ over (X, P), denoted $\sum\limits_{x \in (X,P)} (Y_x, Q_x)$, as the poset (Z, R) where $Z = \{(x, y) : x \in X, y \in Y_x\}$ and $(x_1, y_1) \leq (x_2, y_2)$ in R if and only if $x_1 < x_2$ in P or $x_1 = x_2$ and $y_1 \leq y_2$ in Q_x. Then we call $\{(Y_x, Q_x) : x \in X\}$ the poset decomposition of (X, P). Now we can state the following result.

Lemma 14 *[11] Let* $\boldsymbol{P} = (X, P)$ *be a poset and let* $\mathcal{F} = \{(Y_x, Q_x) : x \in X\}$ *be a family of posets. Then*

$$\dim(\sum_{x \in (X,P)} (Y_x, Q_x)) = \max\{\dim(X, P), \max\{\dim(Y_x, Q_x) : x \in X\}\}.$$

We are now able to state and prove the main theorem of this paper. In the following theorem, we will consider the reduced graph (built from the equivalence classes defined on the neighborhoods) of a compact graph G. We can then extend the ordering of the poset of the reduced graph to the elements of the compact graph G, allowing us to define the ordering on each equivalence class in the poset of G without restriction. We then generate a poset decomposition of the poset of G by defining an ordering on each equivalence class. We do this by building the subposet on each set in the decomposition using chains, antichains and crowns. Using Lemma 14, we can then describe the dimension of the poset of a compact graph.

Theorem 15 *Let G be a compact graph. Then there are posets* $\boldsymbol{Q}= (Y, Q)$ *such that* $\Gamma(Q)$ *=EG,* $\boldsymbol{P}= (X, P)$ *such that* $\Gamma(P)$ *=G, and a family* (Y_v, Q_v) *such that* $(X, P) = \sum_{v \in (Y,Q)} (Y_v, Q_v)$ *and*

$$2 \leq \dim(X, P) \leq \max\{\dim{(Y, Q)}, \lfloor \tfrac{k}{2} \rfloor\}$$

where $k = \max\{|Y_v| : v \in Y\}$.

Moreover, for each n in the interval $[\dim{(Y, Q)}, \lfloor \frac{k}{2} \rfloor]$*, the theorem above can be satisfied with* $\dim(X, P) = n$.

Proof Let G be a compact graph and let $\mathbf{P}= (X, P)$. Now suppose that $\dim(X, P) = 1$. Then $\mathbf{P}$ is a single chain with $|X|$ elements. Then for any $x, y \in X$ where $x, y \neq 0$, $L(x, y) \neq 0$. Thus $\{0\}$ is the only zero-divisor element. So for a nonempty zero-divisor graph, the dimension of the poset is at least 2.

Let G be a compact graph. Then from each equivalence class of G, we can choose a representative b_i for $i \in [1, n]$. Then let the set $\{b_i : i \in [1, n]\}$ denote the set of vertices in the reduced graph EG. Then we can build a poset on EG, say $\mathbf{Q} = (Y, Q)$.

Now consider $X = V(G) \cup \{0\}$. Then we say that $Y_v = [v]$ for any $v \in \{b_i : [1, n]\}$. Thus $X = \bigcup_{v \in Y} Y_v$. We extend the ordering Q to the set X such that for any $a \in Y_v$ and $b \in Y_w$, we have that $a \leq b$ if and only if $v \leq w$ in Q. Then we can define any ordering on Y_v and call it Q_v. Thus we have an ordering P on X such that $(X, P) = \sum_{v \in (Y,Q)} (Y_v, Q_v)$.

Now we can define each Q_v without restriction. Consider $|Y_v| = k$, and build Q_v as an antichain with k elements. Then $\dim(Y_v, Q_v) = 2$. Now let $k \geq 2n$ for some $3 \leq n \leq \left\lfloor \frac{k}{2} \right\rfloor$. Now build Q_v as the union of a crown with $2n$ elements and an antichain with $k - 2n$ elements. Note that by Proposition 9, $\dim(Y_v, Q_v) = n$.

Thus $\mathbf{P} = (X, P)$ is a poset of the graph G, and

$$2 \leq \dim(Y, Q) \leq \dim(X, P) \leq \max\{\dim(Y, Q), \left\lfloor \tfrac{k}{2} \right\rfloor\}$$

where $k = \max\{|Y_v| : v \in Y\}$. Moreover, for each n in the interval $[\dim{(Y, Q)}, \left\lfloor \frac{k}{2} \right\rfloor]$, the theorem above can be satisfied with $\dim(X, P) = n$.

$\square$

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Appendix

A APPENDIX

1. A graph is *r-partite* if the vertices can be partitioned into r disjoint subsets so that every edge joins vertices in distinct subsets.

2. A *clique* in a graph is a subset of vertices of the graph that are all pairwise adjacent; i.e. a vertex set which induces a complete subgraph.

3. If a graph G contains a clique of size n and no clique has more than n elements, then the *clique number* of the graph is said to be n; if the clique size is unbounded, then the clique number is infinite. It is denoted by $\omega(G)$.

4. The *chromatic number* of a graph G, denoted $\chi(G)$, is the minimal number of colors which can be assigned to the vertices of G such that no pair of adjacent vertices has the same color.

5. A graph is *compact* if it is a simple connected graph satisfying the property that for every pair of non-adjacent vertices x and y, there is vertex z adjacent to every vertex adjacent to x and/or y.

6. A graph is said to be *complete* if every vertex in the graph is adjacent to every other vertex in the graph. The notation for a complete graph on n vertices is K_n.

7. A *complete bipartite graph* is a bipartite graph such that every vertex in one partitioning subset is adjacent to every vertex in the other partitioning subset. If the subsets have cardinality m and n, then this graph is denoted by $K_{m,n}$.

8. A *complete r-partite graph* is an r-partite graph such that every vertex in any partitioning subset is adjacent to every vertex in all of the other partitioning subsets.

9. A graph is said to be *connected* if there is a path between every pair of vertices of the graph.

10. The *diameter* of a connected graph is the greatest distance between any two vertices.

11. A *graph* consists of a set of vertices, a set of edges, and an incident relation, describing which vertices are adjacent (i.e., joined by an edge) to which.

12. The *neighborhood* of a vertex v in a graph is the set of all vertices adjacent to v. It is denoted by $N(v)$.

13. A *simple graph* is one with no loops on a vertex and no multiple edges between a pair of vertices.

VITA

Education

- Bachelor of Arts in Mathematics, Magna Cum Laude, May 2009
  Minor: Philosophy
  University of Mississippi, Oxford, MS

Work Experience

- Gear Up Mississippi, Mathematics Instructor, June 2011
- Graduate Assistant, Instructor, University of Mississippi, 2010-2011
  Courses Taught: Elementary Statistics
- Graduate Assistant, Tutor, University of Mississippi, 2009-2010
- Tutor, Kinard Math Lab, University of Mississippi, Fall 2008

Scholarly Presentations

- *"On Generalized Mersenne and Fermat Primes."* Nebraska Conference for Undergraduate Women in Mathematics. The University of Nebraska-Lincoln, 2009.
- *"On Generalized Mersenne and Fermat Primes."* Pi Mu Epsilon National Meeting 2009. Portland, Oregon, 2009

Academic Honors and Awards

- Sally McDonnell Barksdale Honors College Graduate, 2009
- Participant in 2009 Student Paper Competition for Louisiana/Mississippi Section of MAA, second place
- Participant in 2009 Student Team Competition for Louisiana/Mississippi Section of MAA, third place
- Phi Kappa Phi Honor Society, University of Mississippi, 2008
- Chancellor's Honor Roll, University of Mississippi, 2005-2006, Spring 2007, Spring 2008, Fall 2008
- Pi Mu Epsilon Honor Society, University of Mississippi, 2008-present Vice President, 2008-2009 Secretary, 2008
- Alpha Lambda Delta Honor Society, 2006
- Mississippi Eminent Scholar, 2005-2009