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Independent Domination Of Subcubic Graphs

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INDEPENDENT DOMINATION OF SUBCUBIC GRAPHS

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ABSTRACT

The first topic of this dissertation is independent domination. Let $G = (V, E)$ be a simple graph. The independent domination number $i(G)$ is the minimum cardinality among all maximal independent sets of G. A graph is called subcubic whenever the maximum degree is at most three. In this paper, we will show that a connected subcubic graph G of order n having minimum degree at least two satisfies $i(G) \leq 3(n + 1)/7$, providing a sharp upper bound for subcubic connected graphs with minimum degree at least two.

The second topic is independent bondage. The independent bondage number of a nonempty graph G, denoted $b_i(G)$, is the minimum cardinality among all sets of edges $B \subseteq E$ for which $i(G - B) > i(B)$. We will calculate b_i for paths, cycles, complete graphs, and complete t-partite graphs in terms of the order of the graph. We will provide an upper bound for any graph in terms of the degree sum of two adjacent vertices and give an upper bound for trees.

DEDICATION

¡Para Lorena! Mi amiga y mi amor.

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I would like to thank Dr. Wei for his patience, guidance, cleverness, tenacity, dedication, and much more. Dr. Wei made all that follows possible and without him I could not have proved such a nice result. Secondly, I want to thank Dr. Reid, Dr. Wilkins, and Dr. Wu for agreeing to be my committee members. I would like to thank Dr. Miña, Dr. Reid, Dr. Staton, and Dr. Wei for the seemingly endless requests for letters of recommendation. A special thanks to Dr. Miña and Dr. Staton for the many conversations about mathematics, religion, politics, and other topics which made life more interesting. Also, I would like to thank Dr. Song for all those late afternoons when she waited while Dr. Wei and I discussed graph theory well past the time to leave for the day.

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CHAPTER 1

INTRODUCTION

The topic of this dissertation involves graphs and their properties. Informally, a graph is the mathematical tool we employ to analyze or model objects when there is some type of binary relationship between the objects in question. Graph theory is a modern branch of combinatorics which has many different applications. In recent decades, graph theory has been established as an important area of research in the field of mathematics. It has applications in many other areas including: computer science, logic, physics, biology, and engineering.

1.1 Definitions

A graph $G = (V, E)$ is an ordered pair where V is a finite set of points called **vertices** and E is a collection of two element subsets of V called **edges**. To avoid absurdity we also require that $V \neq \emptyset$. If $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\{v_i, v_j\} \in E(G)$, then we usually denote this by letting $e = v_i v_j = v_j v_i \in E(G)$ instead of the more cumbersome notation $e = \{v_i, v_j\} \in E(G)$ and in this case we say that v_i and v_j are **adjacent**. Here, we say that e and v_i (or v_j) are **incident**. A graph is called simple provided it contains no loops, $vv \in E(G)$ for some $v \in V(G)$, or repeated edges (i.e. if $uv \in E(G)$ then for any other edge xy with $x = u$, $y = v$ then $xy = uv$). Our focus is on simple undirected graph so let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. The **order** of G is given by $|V(G)| = n(G) = n$ and the **size** is defined as $|E(G)| = m(G) = m$ where $| * |$ denotes the number of elements in the set (i.e. the cardinality). It is common in literature to use the notation $|G|$ for the order of a graph and $||G||$ for the size so we may sometimes use either of these notations.

The **neighborhood** of a vertex $v \in V(G)$ is the set of vertices adjacent to v, denoted $N_G(v)$ or just $N(v)$, and the **closed neighborhood** of v is given by $N[v] = N(v) \cup \{v\}$. Thus, $N(v) = \{u \in V(G) : uv \in E(G)\}$ and $N(v)$ is sometimes referred to as the **open neighborhood** of v. The degree of a vertex $v \in V$ is defined as $d(v) = |N(v)|$. The **minimum degree** of a graph G is given by $\delta(G) = \min\{d(v) : v \in V\}$ and the **maximum degree** of a graph G is given by $\Delta(G) = \max\{d(v) : v \in V\}$. A graph is **r-regular** provided every vertex has degree r or when $\delta(G) = \Delta(G) = r$. A 3-regular graph is usually referred to as **cubic** graph and a **subcubic** graph is any graph satisfying $\Delta(G) \leq 3$. For vertices of degree k we define $S_k = \{v \in V : d(v) = k\}$ and we also let $n_k = |S_k|$.

For any natural number $n \in \mathbb{N}$ we let $[n] = \{1, 2, \ldots, n\}$. For any real number $x \in \mathbb{R}$ we define the **ceiling function** as the smallest integer greater than or equal to x or

$$\lceil x \rceil = \min\{z \in \mathbb{Z} : z \geq x\}$$

and similarly we define the **floor function** as the largest integer smaller than or equal to x or

$$\lfloor x \rfloor = \max\{z \in \mathbb{Z} : z \leq x\}.$$

A graph is G is **complete** provided any two distinct vertices of G are adjacent. A graph G is **bipartite** if $V(G)$ can be partitioned into two subsets X and Y called **partite sets**, such that every edge of G joins a vertex of X and a vertex of Y. Similarly, A graph G is t**-partite** provided $V(G)$ can be partitioned into t subsets such that if $xy \in E(G)$ then x and y are in different partite sets. Additionally, when every two vertices in different partite sets are joined by an edge then G is called a **complete t-partite**. If $|V_i| = n_i$ for $1 \leq i \leq t$, then the complete t-partite graph can be denoted $K(n_1, n_2, \ldots, n_t)$.

Let $G = (V, E)$ and $G' = (V', E')$ be graphs. We say that G' is a **subgraph** of G, denoted $G' \subseteq G$, whenever $V' \subseteq V$ and $E' \subseteq E$. Thus,

$$G' \subseteq G \Longleftrightarrow V' \subseteq V \text{ and } E' \subseteq E.$$

We say that a subgraph $H \subseteq G$ is a **spanning subgraph** if in addition $V' = V$. Two graphs G and H are **equal** provided $V(G) = V(H)$ and $E(G) = E(H)$ and in this case we write $G = H$. Two graphs G and H are **isomorphic** if there is a bijection $\phi : V(G) \to V(H)$ such that

$$uv \in E(G) \Longleftrightarrow \phi(u)\phi(v) \in E(H).$$

and in this case we write $G \cong H$. If G and H are not isomorphic we say they are **non-isomorphic** and denote this by $G \ncong H$. For a subset $S \subseteq V(G)$, the subgraph **induced by** S, denoted by $G[S]$, has vertex set S and edge set $E(G[S]) = \{uv \in E(G) : u, v \in S\}$. This is sometimes denoted as $\langle S \rangle_G$ or just $\langle S \rangle$.

Let G be a graph and let e be an edge of G. Then $H = G \setminus e = G - e$ is the graph obtained from G by removing the edge e. Thus, $H = G - e$ is the spanning subgraph of G whose edge set is $E(H) = E(G - e) = E(G) - e$ and if G has order n and size m, then $|H| = n$ and $||H|| = m - 1$. For any subset $F \subseteq E$, $G - F = (V, E \setminus F)$ and $G + F = (V, E \cup F)$.

Likewise, for $v \in V(G)$, the graph $H = G - v$ is the graph obtained from

G by deleting v and removing all edges incident with v. Thus, $|H| = n - 1$ and $||H|| = m - |N(v)|$. For a subset $X \subseteq V(G)$ the graph $G - X$ is obtained by removing the vertices of X and removing all the edges incident with any vertex from X. Thus, the graph $G - X$ is a subgraph of G induced by $V(G) \setminus X$ or simply $G[V(G) \setminus X]$.

Finally, by $G' = G - X + uv = G \setminus X + uv$ we mean that we obtain G' by removing the vertex set $X \subseteq V(G)$ and all vertices incident with each vertex in X then adding the edge uv provided that $uv \notin E(G)$. Note that we may add or subtract any finite number of edges or vertices when constructing new graphs from old ones. So we could have a finite sequence of operations such as $G' = G - (X + u) + e_1 + e_2$ and so forth where $X + u_1 = X \cup \{u_1\}$.

For a subset $S \subseteq V(G)$ we define the **complement** of S by $S^c = V(G) \setminus S = \{x : x \in V \text{ and } x \notin S\}$ and similarly when $F \subseteq E(G)$. Both of these are purely set theoretical concepts and we only mention this as a reminder in an effort to create a self-contained manuscript. Thus, we use the minus symbol for two different types of operations. Namely, for set minus and as an operation on graphs (i.e. $S^c = S \setminus V$ and $G \setminus X$). However, this is not ambiguous since the context is usually clear.

A maximal connected subgraph of G is called a **component** and we define $\omega(G)$ as the number of components in G. Let $U \subseteq V(G)$ and I, J be a partition of U. Then, we define $e(I, J)$ to be the number of edges between I and J.

A **path** P is a graph having vertex set $V = \{v_1, v_2, \ldots, v_k\}$ and having edge set $E = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k\}$ where all the v_i are distinct. The vertices v_1 and v_k are called the **ends** or **end vertices**. A path on n vertices, denoted P_n, has **length** $m = n - 1$ where the length is assumed to be the number of edges between the end vertices of P_n. Thus, for any path P_n, on n vertices, we have $|P_n| = n$ and

$||P_n|| = n - 1$. We say the vertices v_1 and v_k are **linked**, **joined**, or **connected** in G. A **cycle** is a graph with vertex set $V = \{v_1, v_2, \ldots, v_k\}$ and edge set $E = \{v_1v_2, v_2v_3, \ldots, v_{k-1}v_k, v_kv_1\}$ where each v_i is distinct. A cycle on n vertices, denoted C_n, is a connected 2-regular graph satisfying $m = n$. When there is a path joining two vertices u and v we may refer to this as a (u, v) - path where the ordered pair notation indicates that property of joining and is, in fact, an equivalence relation on the vertex set $V(G)$ [5]. Additionally, we say there is a (u, X)-path in $H \subseteq G$ provided there is a (u, x)-path in H for some $x \in X$ where $u \in V(H) \subseteq V(G)$ and $X \subseteq V(H)$.

A graph is said to be **connected** if any two vertices are connected by a path. A graph is **k-connected** if $|V| > k$ and $G - S$ is connected for every $S \subseteq V$ with $|S| < k$ and the **connectivity**, denoted $\kappa(G)$, is the greatest integer k for which G is k-connected. A vertex $v \in V(G)$ is a **cut-vertex** of a connected graph G provided $G - v$ is no longer connected. Similarly, a **cut-edge** is an edge whose removal increases the number of connected components of the graph. A **block** of a graph G is a maximal connected subgraph without a cut vertex. An **endblock** of G is a block containing exactly one cut vertex of G and we define $\omega_B(G)$ as the number of endblocks of G. For vertices $u, v \in V$, we define the **distance** between u and v, denoted $d(u, v)$, to be the shortest path connecting them. The **diameter** of a graph G is the greatest distance between any two vertices in G. The **girth** of G, denoted $g(G)$, is the length of the shortest cycle.

For $u, v \in V(G)$ we say that u and v are **non-adjacent** provided $uv \notin E(G)$. An **independent** set of vertices in G is a set of pairwise non-adjacent vertices. A independent set of vertices S of G is a **maximal** independent set provided $S \cup \{u\}$ is not independent for any $u \in V \setminus S$ or equivalently if for any $u \in V \setminus S$ there is some

$v \in S$ such that $uv \in E(G)$. For $e, e' \in E(G)$ with $e \neq e'$ we say that e and e' are **non-adjacent** provided they are not incident with a common vertex. That is, there is no $v \in V(G)$ such that $e = uv \in E(G)$ and $e' = u'v \in E(G)$. A **matching** in G is a set of pairwise nonadjacent (or independent) edges and a **perfect matching** $M \subseteq E(G)$ is a matching such that every vertex of G is incident with an edge of M.

A **dominating set** of G is a set of vertices $D \subseteq V$ such that every vertex not in D is adjacent to a vertex in D. Formally, this can be stated as $(\forall y \in V \setminus D)(\exists x \in D)[xy \in E(G)]$. There are several equivalent definitions of a dominating set and we give one of these below.

$$(\forall y \in V \setminus D)(\exists x \in D)[xy \in E(G)] \iff N[D] = V.$$

An **independent dominating set** of G is a set that is both independent and dominating. The **domination number** of a graph G is the minimum cardinality of a dominating set of vertices of the graph G and is usually denoted $\gamma(G)$. Similarly, the **independent domination number** of a graph G is the minimum cardinality of an independent dominating set and is usually denoted $i(G)$. We will sometimes refer to a minimum independent dominating set by use of the acronym MIDS. A set of vertices $D \subseteq V$ is a **non-dominating set** of G provided there is some vertex not in D that is non-adjacent to every vertex in D (i.e. the negation of dominating). Sometimes we may also say not dominating when a set fails to dominate G.

The **bondage number** of a nonempty graph G, denoted $b(G)$, is the minimum cardinality among all sets of edges $B \subseteq E(G)$ for which $\gamma(G - B) > \gamma(G)$. Stated differently, the bondage number of a graph G is the minimum number of edges whose removal renders every minimum dominating set of G a non-dominating set in the resulting spanning subgraph $G - B$. The **independent bondage number** of a nonempty graph G, denoted $b_i(G)$, is the minimum cardinality among all sets of edges $B \subseteq E$ for which $i(G-B) > i(G)$. Here, when we say a graph is nonempty we mean that $E(G) \neq \emptyset$ which implies that $n \geq 2$. There is a large class of variants for the concept of domination and for many of these concepts removing edges has been explored. The bibliography contains a list of just some of these results [1]. Further notation and terminology may be found in [15].

1.2 Motivation and Historical perspective

In this brief section, we give a historical overview of domination and bondage of graphs. Both areas of research, independent domination and independent bondage, are derived from the original ideas of domination and bondage of graphs. Domination is a modern branch of graph theory and there is essentially no published research before 1962.

1.2.1 Domination

The general topic of domination has been studied quite extensively over the years and more recently, the topic of independent domination became a topic of interest to some researchers in the field of graph theory. Independent domination has been studied quite thoroughly and a nice survey of some of these results may be found in [12].

Much of the early development of graph theory originated from questions about games of chance or recreational mathematics problems and is similar to development in early probability [21]. In 1848, Max Bezzel a German chess composer proposed the problem of how to place eight queens on a chessboard so that every square on the board was reachable in one move by one of the queens and every queen's position was also reachable by another queen (protected) [27]. This problem, known as the eight queens puzzle, can be formulated in terms of a graph [3]. This idea developed into a problem now known as the N Queens problem in modern combinatorics which takes place on an $N \times N$chessboard. More generally, the (m, n) queens graph is a graph with mn vertices where each vertex represents a square of an $m \times n$ chessboard and the edges represent one legal move by the queen in accordance with the standard rules of chess. Later, in 1862, de Jaenisch [19] proposed a problem concerning how to place the minimum number queens on a chessboard in such a way that each square of the chessboard can be attacked (reached in one legal move) by one of the queens and no queen may attack another queen (not protected). This problem is equivalent to finding a minimum independent dominating set for the queens graph.

Although the study of independent domination began from a question about the minimum number of non-attacking queens required to cover a standard chessboard (8 x 8) and domination began in a similar manner as the minimum number of queens required to cover a standard chessboard, in more modern times, the idea has become a useful tool in such areas as radar design, scheduling problems, networks, artificial intelligence [25], surveillance algorithms, and search algorithms (e.g. DNA, Amino acid, hueristics [14]). Domination and other similar parameters often provide certain information about the graph which might be used in variety of applications.

In general, the problem of finding the independent domination number of a graph can be very difficult. Independent domination is an extremal graph theory problem where the graph invariant called the independent domination number of a graph G, denoted $i(G)$, is compared to other graph invariants such as order n (number of vertices), size m (number of edges), minimum degree $\delta(G)$, maximum degree $\Delta(G)$, and girth $g(G)$ (length of the shortest cycle). The independent domination number of a graph may be defined as the minimum cardinality of maximal independent set of the graph G. Alternatively, it may be defined as the minimum cardinality of an independent dominating set of G. Most of the results that have been published have focused on establishing the best upper bound for $i(G)$ in terms of the parameters listed above. Several results have been published which determine a bound for $i(G)$ in terms of order and minimum or maximum degree where the degree is not fixed but allowed to vary. Thus, the bound is a relationship between several parameters, e. g. $i(G) \leq f(n, m, \delta, \Delta, g)$, as in the results given below.

1.2.2 Bondage

The background of bondage is somewhat convoluted since two groups of researchers published the same result about trees independently. Bauer et. al. published a result on trees and in this paper referred to vertices as points, edges as lines, and bondage as domination line stability. Later Fink et al [9] published the same theorem for trees as well as results for bondage of paths, cycles, trees, complete graphs, and complete r-partite graphs. See [1] and [9] for more details and [16] for an earlier survey of results of bondage and reinforcement.

If we know that some vertex v is in every dominating set, then we can remove all the edges incident with v and the bondage number is easily found. However, most graphs do not satisfy this property and the challenge becomes how to show that a certain edge set renders every dominating set a non-dominating set.

Considering the effect of the addition or removal of one or more edges from a graph is sometimes referred to as an alteration of the original graph [16]. Furthermore, a new problem is easily constructed by asking what happens to certain parameters such as the domination number when we add or remove an edge. This will often provide additional information about a graph that might be helpful in certain applications. The concept of bondage was introduced in 1990 Fink et. al. in [9]. The motivation for their idea was as follows (see [9] and [16]). Consider a communication network of existing communication links between a fixed set of sites. The problem is to find the smallest set of sites at which to place transmitters so that every site without a transmitter is connected directly to a site that does. This smallest set is a minimum dominating set of the graph where each site represents a vertex and the relationship for the edges is two vertices share an edge if and only if the two sites are connected by a direct communication link. Now, we can introduce the concept of bondage in this application by examining the vulnerability of the network under a link failure. Suppose that a malicious entity (saboteur) does not know at which sites the transmitters are located but does know that such a site corresponds to a minimum dominating set of the associated graph. What is the fewest number of links that must be severed so that an additional transmitter is now required for the network to function properly.

In the following section, we will introduce the concept of bondage and then extend this idea to independent bondage. In this way, we can formulate new questions about independent bondage that mirror the development of the original bondage concept. This provides a variety of interesting problems to examine. Independent bondage is a new concept and to the best of our knowledge no previous work has been done on this topic.

CHAPTER 2

INDEPENDENT DOMINATION

The concept of independent domination has been studied quite thoroughly and a nice survey of these results may be found in [12]. Lam et. al. [20] established an upper bound for the independent domination number of cubic graphs.

Theorem 1. *If G is a connected cubic graph of order n, where $n \geq 8$, then*

$$i(G) \leq \frac{2n}{5}.$$

In addition, Henning et al. [17] established a bound for cubic bipartite graphs having girth at least six (2014).

Theorem 2. *If G is a cubic bipartite graph of order n and of girth at least six, then*

$$i(G) \leq \frac{4n}{11}.$$

More recently, Henning et al. [18] established a new bound for connected cubic graphs containing no subgraph isomorphic to $K_{2,3}$.

Theorem 3. *If G is a connected cubic graph of order $n > 10$ and G does not contain a subgraph isomorphic to $K_{2,3}$, then*

$$i(G) \leq \frac{3n}{8}.$$

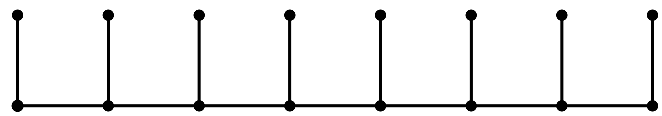

Figure 2.1: The Extremal Graph G when $\delta(G) = 1$

In this chapter of the dissertation, we will establish an upper bound for a subclass of subcubic graphs. Namely, we shall investigate connected graphs that are subcubic with the added condition that $\delta(G) \geq 2$. This assumption eliminates pendant vertices since there is a graph with $\delta(G) = 1$ having independent domination number $i(G) = n/2$ (see Figure 2.1).

Theorem 4. *If G is a connected subcubic graph of order n with $\delta(G) \geq 2$, then*

$$i(G) \leq \frac{3(n+1)}{7}.$$

Note that our bound is necessary since $i(K_{3,3}) = 3$ and for $n = 6$ we have $3(n+1)/7 = 3$. Additionally, the graph of Figure 2.2, G^*, has order $n = 21$ and satisfies $i(G^*) = 9 < 3(n+1)/7$. Clearly, the pattern in graph of Figure 2.2 repeats giving an infinite family of graphs that satisfy the bound $3n/7$. However, the graph of Figure 2.2 has girth 4 and so we might ask whether adding a girth condition like $g \geq 5$ might allow one to find a new upper bound. To this end, the following open conjecture for cubic graphs was proposed by Jacques Verstraete [26].

Conjecture 1. *If G is a cubic graph of order n having girth at least six, then*

$$i(G) \leq \frac{n}{3}.$$

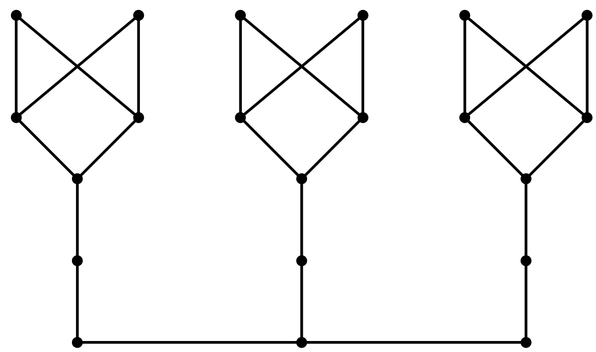

Figure 2.2: The Extremal Graph G^*

2.1 Proof of Theorem 4

2.1.1 Preliminaries

Before we can proceed, we must introduce several more results and some terminology. Afterward, we will begin by proving a series of lemmas which will allow a more efficient proof of our main theorem. Let G be a connected subcubic graph satisfying $\delta \geq 2$. An edge $e = uv \in E$ is called a *critical edge* of G provided any of the following statements hold:

1. e does not lie on a cycle,

2. $\beta(u,v) = \min\{d(u), d(v)\} = 2$,

3. $i(G-e) < i(G)$.

If every edge of G is a critical edge then G is called a *critical graph* or sometimes just critical. As an example consider $K_{3,3}$. In this case, we can remove any edge $e = uv$ and then choose $I = \{u, v\}$ so that $i(K_{3,3} - e) = 2$ and since $i(K_{3,3}) = 3$ we see that every edge is critical. Thus, $K_{3,3}$ is a critical graph. Although many graphs may have a critical edge, they will not all be critical graphs. We shall distinguish between these two classes of graphs in the following manner. First, we will prove the following useful result.

Fact 1. *Suppose G is a connected subcubic graph having $\delta(G) \geq 2$. Then there is a connected subcubic spanning critical subgraph G' with $\delta(G') \geq 2$ such that $i(G) \leq i(G')$.*

Proof. If G is a critical graph, then we let $G' = G$ and we are done. Otherwise, there exists a non-critical edge $e = uv$ so we let $G' = G - e$. Thus, G' is connected since e is not a cut edge (negation of property 1) and since $u, v \in S_3$ we have $\delta(G') \geq 2$ (negation of property 2). Finally, from property 3 we must have $i(G) \leq i(G')$. If necessary, we can repeat the process to produce the desired graph G'. □

Since any spanning critical subgraph satisfies the inequality from Fact 1 we can restrict the argument and consider only critical graphs. That is, we will use the fact that the critical graphs serve as an upper bound for the noncritical graphs and if we show the bound holds for all critical graphs we have also shown the result holds for every graph. In light of this observation, for the remainder of the proof we will consider only critical graphs.

Remark 1. *Henceforth, we will assume that all graphs are critical graphs.*

Now, we introduce several more useful facts which are easily shown to be true and which will be needed quite often in the remainder of our proof. First, we define the following structure or subgraph which will occur often in the arguments that follow. Let $Y = \{u_1, u_2, u_3, u_4, u_5\}$ be a set of vertices and $G_Y = (W, E_W)$ be a graph where $W = Y \cup \{u, v, w\}$ and $E_W = \{u_1u_2, u_1u_4, u_1u_5, u_2u_3, u_3u, u_4v, u_5w\}$.

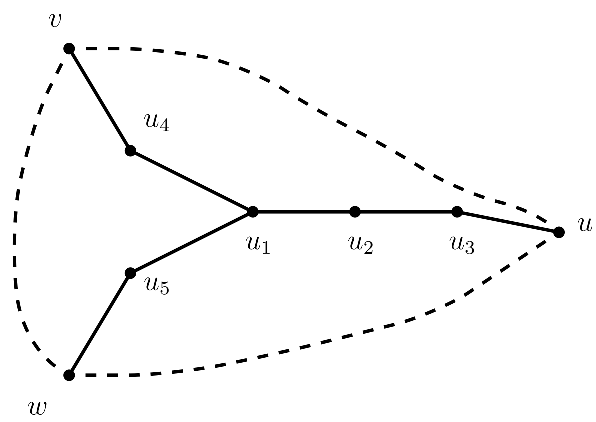

Figure 2.3: The Y-Graph G_Y

The graph G_Y, given in Figure 2.3, will be called the Y-graph or sometimes just Y. We have stated each fact below in most general term and so we should note that even though some vertices are shown as degree two vertices they may be degree three vertices in the larger graph since each will be a subgraph of a subcubic graph G having minimum degree $\delta(G) \geq 2$. Thus, when $d(u_3) = 3$ we let $u, u' \in N(u_3)$. However, note that in the second case below (when adding vw) we require that $d(u_2) = 2$.

Remark 2. *Additionally, for the remainder of the proof we declare that $G' = G$ if the edge we need to add already exists in the original graph G and in this case we point out that the surgery works exactly the same although we may need to show connectivity in some cases. Otherwise, set $G' = G \setminus R + uv$ where $uv \notin E(G)$ and R is the set of vertices we need to remove when we argue by induction on the number of vertices (removal set).*

Fact 2. *Let G be a subcubic graph which satisfies $\delta(G) \geq 2$ and suppose that $G_Y \subseteq G$ where G_Y is given above. If $G' = G \setminus Y + uv$ where $uv \notin E(G)$ or $G' = G \setminus Y + uw$*

where $uw \notin E(G)$*, then* $i(G) \leq i(G') + 2$*. Furthermore, if* $d(u_2) = 2$ *and* $G' = G \setminus Y + vw$ *where* $vw \notin E(G)$*, then* $i(G) \leq i(G') + 2$*. Additionally, if* $G' = G \setminus Y$*, then* $i(G) \leq i(G') + 2$*.*

Proof. Suppose as stated above and consider when $G' = G \setminus Y + uv$ and $uv \notin E(G)$. If we remove the vertices in Y and add the edge uv, the resulting graph $G' = G \setminus Y + uv$ has a MIDS I'. We need to use I' to construct I and we must add no more than two vertices to I' when we do this. Furthermore, when we construct I from I' we must ensure that the vertices we remove are dominated in G and that they are not adjacent to each other since I must be an independent dominating set of G. Now, we must show that $i(G) \leq i(G') + 2$. If $u, v \notin I'$, we let $I = I' \cup \{u_1, u_3\}$ provided $N(u_3) \cap I' = \emptyset$ and $I = I' \cup \{u_1\}$ when $N(u_3) \cap I' \neq \emptyset$. Observe that $N(u_3) \cap I' \neq \emptyset$ exactly when $d(u_3) = 3$ and $u' \in I'$ where $u' \in N(u_3) \setminus \{u, u_2\}$ since $u_2 \notin V(G')$ and we have assumed that $u \notin I'$. Additionally, since $u_2 \in V(G) \setminus V(G')$, $N(u_3) \cap I' = (N_G(u_3) \setminus \{u_2\}) \cap I'$ and in this case we always use the notation $N(u_3) \cap I'$. Also, we use the notation $u, v \notin I'$ instead of $\{u, v\} \cap I' = \emptyset$ for convenience and point out these two statements are logically equivalent and we will always write $N(u_3)$ instead of $N_G(u_3)$. Now, consider when $\{u, v\} \cap I' \neq \emptyset$. Then, either $u \in I'$ or $v \in I'$ but never both since $uv \in E(G')$ and I' is an independent set of G'. If $u \in I'$, we can let $I = I' \cup \{u_1, v\}$ when $(N(v) \setminus \{u_4\}) \cap I' = \emptyset$ and $I = I' \cup \{u_1\}$ otherwise. If $v \in I'$, let $I = I' \cup \{u_1, u\}$ when $(N(u) \setminus \{u_3\}) \cap I' = \emptyset$. Now, assume that $(N(u) \setminus \{u_3\}) \cap I' \neq \emptyset$. Then, let $I = I' \cup \{u_1, u_3\}$ when $N(u_3) \cap I' = \emptyset$ and $I = I' \cup \{u_1\}$ otherwise. Therefore, we have shown that $i(G) \leq i(G') + 2$ as required.

For the second case, consider when $G' = G \setminus Y + uw$ provided $uw \notin E(G)$ we repeat the argument from above letting w replace v. Furthermore, when $\{u_3, u_4, u_5\} \cap S_3 \neq \emptyset$ and $u' \in N(u_3) \setminus \{u, u_2\}$, $v' \in N(u_4) \setminus \{v, u_1\}$, and $w' \in N(u_5) \setminus \{w, u_1\}$

this surgery works if we remove Y and add any one of $\{uv', u'v, u'v', uw', u'w, u'w'\}$. Finally, the surgery also works for $d(u_2) = 3$ since we never need to choose u_2 when constructing I from I'.

For third case, consider when $G' = G \setminus Y + vw$ with $vw \notin E(G)$. Here, we require that $d(u_2) = 2$ since the surgery fails when $d(u_2) = 3$ and $(N(u_2) \setminus \{u_1, u_3\}) \cap I' \neq \emptyset$, $(N(w) \setminus \{u_1\}) \cap I' = \emptyset$, with $v, w', z \in I'$, $u, u' \notin I'$ where $z \in N(u_2) \setminus \{u_1, u_3\}$. If $v, w \notin I'$, let $I = I' \cup \{u_1, u_3\}$ when $N(u_3) \cap I' = \emptyset$ and $I = I' \cup \{u_1\}$ otherwise. If $v \in I'$, let $I = I' \cup \{u_2, u_5\}$ when $(N(u_5) \setminus \{u_1\}) \cap I' = \emptyset$ and $I = I' \cup \{u_2\}$ otherwise. If $w \in I'$, let $I = I' \cup \{u_2, u_4\}$ when $(N(u_4) \setminus \{u_1\}) \cap I' = \emptyset$ and $I = I' \cup \{u_2\}$ otherwise.

For the final case, when $G' = G \setminus Y$, let $I = I' \cup \{u_1, u_3\}$ when $N(u_3) \cap I' = \emptyset$ and $I = I' \cup \{u\}$ otherwise. Thus, in each case we have shown that $i(G) \leq i(G') + 2$.

$\square$

For the next result, we will construct several graphs using a path of order five as the base structure. Next, let Q_3 be the graph shown in Figure 2.4 and let $Q_{2,4}$ be the graph shown in Figure 2.5. Graphs of this type, i.e. like Q_3 and $Q_{2,4}$, are sometimes called caterpillars. A *caterpillar* is a tree in which a path is incident to

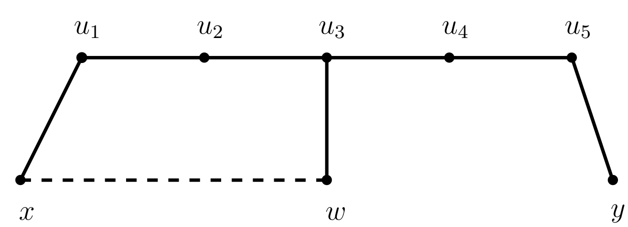

Figure 2.4: The Graph Q_3

every edge. Alternatively, a caterpillar is a tree such that removal all end vertices of the tree leaves a path. The remaining path is called the *spine* of the caterpillar and in our case is a P_5. For the fact below, we let $Q = \{u_1, u_2, u_3, u_4, u_5\}$. We can refer to the following fact as our *path* or *caterpillar lemma.* The graphs Q_3 and $Q_{2,4}$ are shown in Figure 2.4 and Figure 2.5 where the edges with dotted lines are the edges we need to add when we make the induction argument in Fact 3.

Fact 3. *Let G be a subcubic graph with $\delta(G) \geq 2$ and suppose that $Q_3 \subseteq G$ (see Figure 2.4). If $G' = G \setminus Q + xw$ with $u_1, u_2, u_4 \in S_2$ or $G' = G \setminus Q + wy$ with $u_2, u_4, u_5 \in S_2$, then $i(G) \leq i(G')+2$. Likewise, for any subcubic graph with $\delta(G) \geq 2$ containing $Q_{2,4}$ (Figure 2.5), if $G' = G \setminus Q + uv$, then $i(G) \leq i(G') + 2$.*

Proof. Suppose G contains the subgraph Q_3 and suppose that $d(u_1) = d(u_2) = d(u_4) = 2$ as above. If we remove Q and add xw the graph $G' = G \setminus Q + xw$ has a MIDS I' which we use to construct I. If $x, w \notin I'$, we let $I = I' \cup \{u_2, u_4\}$. If $x \in I'$, we can let $I = I' \cup \{u_3, u_5\}$ when $N(u_5) \cap I' = \emptyset$ and $I = I' \cup \{u_3\}$ otherwise. If $w \in I'$, we can let $I = I' \cup \{u_1, u_4\}$. Next, the argument for $G' = G \setminus Q + yw$ is the same if we replace x with y in the previous case and we assume that $d(u_2) = d(u_4) = d(u_5) = 2$. Finally, note that this surgery fails if $d(u_5) = 3$ and similarly for the first case when

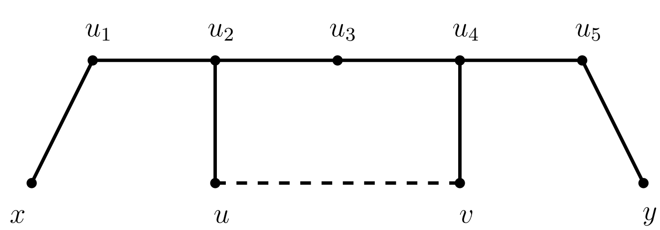

Figure 2.5: The Graph $Q_{2,4}$

$d(u_1) = 3$.

For the second graph, $Q_{2,4}$, if we remove Q and add uv, the resulting graph $G' = G \setminus Q + uv$ has a MIDS I'. If $u, v \notin I'$, we let $I = I' \cup \{u_2, u_4\}$. If $u \in I'$, we can let $I = I' \cup \{u_1, u_4\}$ when $N(u_1) \cap I' = \emptyset$ and $I = I' \cup \{u_4\}$ otherwise. Similarly, if $v \in I'$, we can let $I = I' \cup \{u_2, u_5\}$ when $N(u_5) \cap I' = \emptyset$ and $I = I' \cup \{u_2\}$ otherwise. □

Next, we let G be a graph and suppose that $G[S_2]$ contains one or more independent edges. Let $e = uv$ be such an edge and suppose that $x \in N(u) \setminus \{v\}$ and $y \in N(v) \setminus \{u\}$ with $x, y \in S_3$. The next result is useful if we encounter one or more independent edges adjacent to the vertex set S_3 and at least one of those vertices is not a cut vertex. Although we will not prove this in general, we might think of the result as a restricted version of an *independent edge lemma* where the edges lie in $G[S_2]$ (see Figure 2.6).

Fact 4. *For any subcubic graph G containing $e = uv$ as defined above, if $G' = G \setminus \{x, u, v\}$ or $G' = G \setminus \{u, v, y\}$, then $i(G) \leq i(G') + 1$. Furthermore, if $u' \in S_3$ and $u'_1, u'_2, u'_3 \in N(u')$, then $G' = G - N[u']$ satisfies $i(G) \leq i(G') + 1$. Additionally, if $v' \in S_2$ and $v'', v''' \in N(v')$, then $G' = G - N[v']$ satisfies $i(G) \leq i(G') + 1$.*

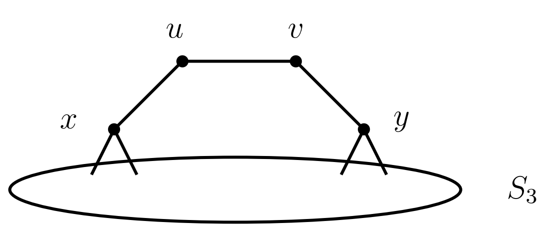

Figure 2.6: The independent edge lemma or Fact 4

Proof. Suppose as stated and let $G' = G \setminus \{x, u, v\}$. Then G' has a MIDS I' so we let $I = I' \cup \{u\}$. If $G' = G \setminus \{u, v, y\}$. Then, let $I = I' \cup \{v\}$. Next, for the second statement if $u' \in S_3$ with $u'_1, u'_2, u'_3 \in N(u')$ and $G' = G - N[u']$, then we can let $I = I' \cup \{u'\}$. Finally, for the third case let $v' \in S_2$ with $v'', v''' \in N(v')$ and $G' = G - N[v']$, then we can let $I = I' \cup \{v'\}$. Thus, in each case we have shown that $i(G) \leq i(G') + 1$.

□

Next, we let G_S be the graph of Figure 2.7 where $S = \{u, v, u_1, u_2, v_1, v_2\}$. Furthermore, we let $S' = \{u, v, u_1, u_2, v_1\}$ where $u, v \in S_3$ with $uv \in G_S$ and $u_1, u_2 \in N(u)$ and $v_1, v_2 \in N(v)$. The graph G_S contains a *double-star*. The dotted lines show the edges to be added. The following fact allows us to eliminate the case of a double star. As a note, we can allow $v_1, v_2 \in S_2 \cup S_3$ however when we add two edges we will require that $u_1, u_2 \in S_2$ since the surgery fails otherwise.

Fact 5. *Let G be a subcubic graph with $\delta(G) \geq 2$ containing $G_S \subseteq G$. If $G' = G \setminus S' + v_2v_3$, then $i(G) \leq i(G') + 2$ and if $G' = G \setminus S' + u_3u_4$ and $u_1, u_2 \in S_2$, then $i(G) \leq i(G') + 2$. If $G' = G \setminus S' + v_2v_3 + u_3u_4$ and $u_1, u_2 \in S_2$, then $i(G) \leq i(G') + 2$. Additionally, if $G' = G \setminus S'$, then $i(G) \leq i(G') + 2$.*

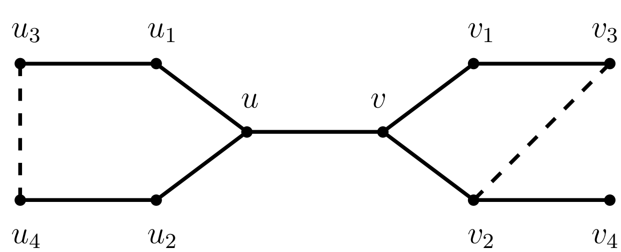

Figure 2.7: The Graph G_S

Proof. Suppose as stated above. For the first case, let $G' = G - S' + v_2v_3$. Then, if $I' \cap \{v_2, v_3\} = \emptyset$ or $v_2 \in I'$, let $I = I' \cup \{u, v_1\}$ when $N(v_1) \cap I' = \emptyset$ and $N(v_1) \cap I' \neq \emptyset$ let $I = I' \cup \{u, v_3\}$ if $N(v_3) \cap I' = \emptyset$ and $I = I' \cup \{u\}$ otherwise. If $v_3 \in I$, then $I = I' \cup \{u, v_2\}$ if $N(v_2) \cap I' = \emptyset$ and $I = I' \cup \{u\}$ otherwise.

For the second case, if $G' = G \setminus S' + u_3u_4$ and $u_1, u_2 \in S_2$, then if $\{u_3, u_4\} \cap I' = \emptyset$ or when $u_3, u_4 \notin I'$, then let $I = I' \cup \{u, v_1\}$ when $N(v_1) \cap I' = \emptyset$ and $I = I' \cup \{u\}$ otherwise. If $u_3 \in I'$, let $I = I' \cup \{u_2, v\}$ when $v_2 \notin I'$ and if $v_2 \in I'$ let $I = I' \cup \{u_2, v_1\}$ when $N(v_1) \cap I' = \emptyset$ and $I = I' \cup \{u_2\}$ otherwise. If $u_4 \in I'$, let $I = I' \cup \{u_1, v\}$ when $v_2 \notin I'$ and if $v_2 \in I'$ let $I = I' \cup \{u_1, v_1\}$ when $N(v_1) \cap I' = \emptyset$ and $I = I' \cup \{u_1\}$ otherwise.

For the third case, if we remove S' and add both v_2v_3 and u_3u_4, the resulting graph $G' = G \setminus S' + v_2v_3 + u_3u_4$ has a MIDS I' and we can use this set to construct I. There are three cases to consider since $0 \leq |I' \cap \{u_3, u_4, v_2, v_3\}| \leq 2$. For the first case, suppose that $|I' \cap \{u_3, u_4, v_2, v_3\}| = 0$. Then, $u_3, u_4, v_2, v_3 \notin I'$, and we let $I = I' \cup \{u, v_1\}$. For the second case, suppose that $|I' \cap \{u_3, u_4, v_2, v_3\}| = 1$. Then, if $u_3 \in I'$, let $I = I' \cup \{u_2, v_1\}$. If $u_4 \in I'$, let $I = I' \cup \{u_1, v_1\}$. If $v_2 \in I'$, let $I = I' \cup \{u, v_1\}$. If $v_3 \in I'$, let $I = I' \cup \{u, v_2\}$ when $N(v_2) \cap I' = \emptyset$ and $I = I' \cup \{u\}$ otherwise. Finally suppose that $|I' \cap \{u_3, u_4, v_2, v_3\}| = 2$. If $u_3, v_2 \in I'$, let $I = I' \cup \{u_2, v_1\}$. If $u_3, v_3 \in I'$, let $I = I' \cup \{v, u_2\}$. If $u_4, v_2 \in I'$, let $I = I' \cup \{u_1, v_1\}$. If $u_4, v_3 \in I'$, let $I = I' \cup \{u_1, v\}$.

For the final case, when $G' = G \setminus S'$, let $I = I' \cup \{u, v_1\}$ when $N(v_1) \cap I' = \emptyset$ and $I = I' \cup \{u\}$ otherwise. □

Next, consider any subcubic graph G of minimum degree $\delta(G) \geq 2$ which contains a path of length two in $G[S_2]$ (i.e. $P_3 \subseteq G[S_2] \subseteq G$) where we label the path as follows (show in Figure 2.8). Let the path have vertices $\{w_1, w_2, w_3, w_4, w_5\}$ where

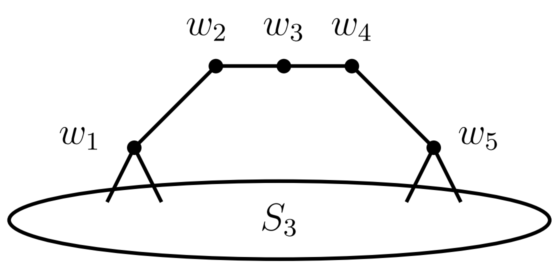

Figure 2.8: A path of length two lying in $G[S_2]$

$w_2, w_3, w_4 \in S_2$ and having edges $w_1w_2, w_2w_3, w_3w_4, w_4w_5 \in E(G)$ with $w_1, w_5 \in S_3$. If we require that $w_1 \neq w_5$, then we can show that $i(G) \leq i(G') + 1$ where $G' = G - \{w_2, w_3, w_4\} + w_1w_5$ provided $w_1w_5 \notin E(G)$.

Fact 6. *Let G be a subcubic graph with $\delta(G) \geq 2$ containing a path of length two in $G[S_2]$ with vertices $\{w_1, w_2, w_3, w_4, w_5\}$ where $w_2, w_3, w_4 \in S_2$ and having edges $w_1w_2, w_2w_3, w_3w_4, w_4w_5 \in E(G)$ with $w_1, w_5 \in S_3$. If $G' = G - \{w_2, w_3, w_4\} + w_1w_5$ provided $w_1w_5 \notin E(G)$, then $i(G) \leq i(G') + 1$.*

Proof. Suppose as stated above. Then, there are three cases to consider. If $w_1, w_5 \notin I'$, then let $I = I' + \{w_3\}$. If $w_1 \in I'$, let $I = I' + \{w_4\}$ and if $w_5 \in I'$, let $I = I' + \{w_2\}$. ☐

2.1.2 Lemmas

Next, the following series of lemmas allow us to exclude certain structures from consideration by means of the definition of a critical edge (or graph) along with the application of Fact 1 above. The first lemma shows that a critical edge cannot lie on a triangle having two or more degree three vertices. Once we restrict our focus to critical graphs, the only possibility for a critical graph having girth three is when the triangle has only one degree three vertex.

Lemma 1. *Suppose G is a connected subcubic critical graph which satisfies $\delta(G) \geq 2$. If $e = uv$ is a critical edge of G and $u, v \in S_3$, then e must not lie on a triangle.*

Proof. Suppose $e = uv$ is a critical edge of G where $u, v \in S_3$ and that e lies on a triangle. We label the triangle as $\{u, v, w\}$. Furthermore, let $x \in N(u) \setminus \{v, w\}$ and $y \in N(v) \setminus \{u, w\}$. Next, we let $H = G - e$ and then suppose that I_H is a MIDS of H. First, observe that e must satisfy part three of the definition of a critical edge since $u, v \in S_3$ and e lies on a triangle. If $u \in I_H$ but $v \notin I_H$, then $|I| \leq |I_H|$ for any IDS I of G which contradicts our definition of a critical edge. Thus, we must have $u, v \in I_H$. Next, if $(N(x) \setminus \{u\}) \cap I_H \neq \emptyset$, then there is some $z \in (N(x) \setminus \{u\}) \cap I_H$ which dominates x so then letting $I = I_H \setminus \{u\}$ gives $|I| < |I_H|$ which is a contradiction. If $(N(x) \setminus \{u\}) \cap I_H = \emptyset$, then we let $I = (I_H \setminus \{u\}) \cup \{x\}$ which again gives a contradiction.

□

We have a second result for the case of girth 4 which eliminates certain edges in the neighborhood of a vertex which is incident with a critical non-cut edge.

Lemma 2. *Suppose G is a connected subcubic critical graph of order $n \geq 7$ having $\delta(G) \geq 2$. If $e = uv$ is not a cut edge of G with $u, v \in S_3$ and $H = G - e$, then there is no perfect matching in $G[N_H(\{u, v\})]$.*

Proof. By Lemma 1, $N(u) \cap N(v) = \emptyset$. Suppose there is a perfect matching in $G[N_H(\{u, v\})]$ which we denote by M and let $Z = N_H(\{u, v\})$ with $z_1, z_2 \in N(u) \setminus \{v\}$ and $z_3, z_4 \in N(v) \setminus \{u\}$ (see Figure 2.9). Additionally, let $F_Z =$

$\{z_1z_2, z_1z_3, z_1z_4, z_2z_3, z_2z_4, z_3z_4\}$ whenever $z_i \in S_3$ we let $z_i' \in N(z_i) \setminus (Z \cup \{u, v\})$. First, note that e satisfies part three of the definition of a critical edge since it is not a cut edge and $\beta(u, v) = 3$. Now, let $H = G - e$ and suppose I_H is a MIDS of H. Next, since G is subcubic and every member of Z has neighbor in $\{u, v\}$ then $E(G[Z]) \neq F_Z$ which is only possible when $G[Z] \cong K_4$. Furthermore, by Lemma 1, $z_1z_2 \notin E(G)$ whenever $z_1 \in S_3$ or $z_2 \in S_3$ (or both). This is a also true for $z_3z_4 \notin E(G)$ whenever $z_3 \in S_3$ or $z_4 \in S_3$. Also, we must not have $z_i = z_j$ for any $i, j \in [4]$ when $i \neq j$ since this gives either a triangle with more than one degree three vertex or a multi-edge.

<u>**Case 1:**</u> $M = \{z_1z_2, z_3z_4\}$.

Suppose that $z_1z_2, z_3z_4 \in E$ (see Figure 2.9). Since e is a not a cut edge there must be a path in H joining some z_i and some z_j. Since $n \geq 7$ this path must include an additional vertex $z_i' \in N(z_i) \setminus (Z \cup \{u, v\})$. But this would imply that G contains a triangle with two or more degree three vertices and since G is a critical graph this contradicts Lemma 1. Thus, Case 1 isn't possible.

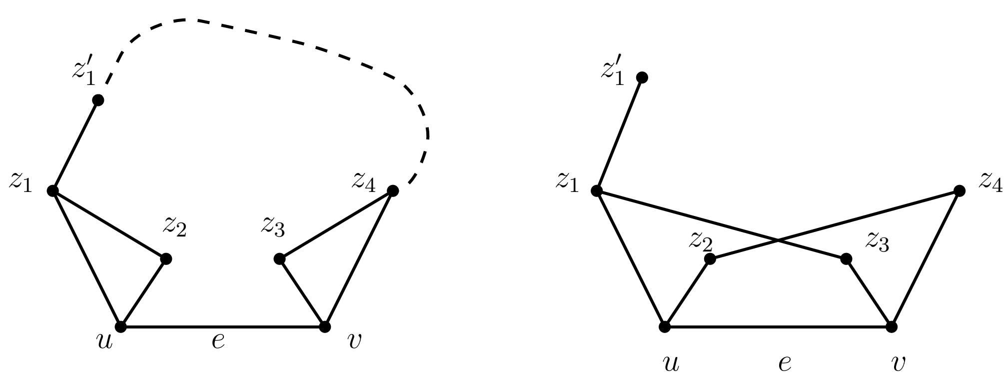

Figure 2.9: The Graph of Lemma 2

Case 2: $M = \{z_1z_3, z_2z_4\}$ (or $M = \{z_1z_4, z_2z_3\}$).

Suppose $z_1z_3, z_2z_4 \in E$ (see Figure 2.9). If $G[Z] \cong C_4 \cong K_{2,2}$, then $n = 6$ and we have assumed that $n \geq 7$ so there must be an edge missing from the set of available edges in $E(G[Z]) \subsetneq F_Z$. Since $z_1z_2, z_3z_4 \notin E$ and $z_1z_3, z_2z_4, z_1u, z_2u, z_3v, z_4v \in E$ this leaves only three possibilities. Thus, either $z_1z_4 \notin E(G)$ or $z_2z_3 \notin E(G)$ or $z_1z_4, z_2z_3 \notin E(G)$. Thus, we will assume that $z_1z_4 \notin E$. This implies that $M = \{z_1z_3, z_2z_4\}$. Since $n \geq 7$ there is some $z_i' \in N(z_i) \setminus (Z \cup \{u, v\})$. Because e is critical $i(H) < i(G)$, we must have $u, v \in I_H$ and consequently $Z \cap I_H = \emptyset$. There are two subcases to consider.

Case 2.1: Suppose for some $i \in [4]$ that $N(z_i) \cap I_H - \{u, v\} \neq \emptyset$ (or when some $z_i' \in I_H$). Next, choose j such that $z_ix, z_jx \in E(G)$ where $x \in \{u, v\}$. Then, let $I = I_H \setminus \{x\} \cup \{z_j\}$ when $(N(z_j) \setminus \{x\}) \cap I_H = \emptyset$ and $I = I_H \setminus \{x\}$ when $(N(z_j) \setminus \{x\}) \cap I_H \neq \emptyset$. Hence, $|I| \leq |I_H|$ a contradiction.

Case 2.2: Suppose for each $i \in [4]$ that $N(z_i) \cap I_H - \{u, v\} = \emptyset$. Then, since $z_1z_4 \notin E$, letting $I = (I_H \setminus \{u, v\}) \cup \{z_1, z_4\}$ gives $|I| \leq |I_H|$. Again a contradiction. Thus, in both cases there is no perfect matching in $G[Z]$. □

Lemma 3. *Suppose G is a connected subcubic critical graph which satisfies $\delta(G) \geq 2$. Suppose that $X = \{x_1, x_2, x_3, x_4\}$ forms a 4-cycle where $x_1, x_2 \in S_3$ and $x_3, x_4 \in S_2$ with $u_1 \in N(x_1) \setminus \{x_2, x_4\}$ and $u_2 \in N(x_2) \setminus \{x_1, x_3\}$. Now, suppose that B is an endblock of G containing X such that $|V(B)| \geq 6$. If $u_1 \in S_3$ and $e = u_1x_1$, then e is not a critical edge of G and a similar result holds for u_2x_2 when $u_2 \in S_3$.*

Proof. Suppose as stated above and see Figure 2.10 for the labeling of X. Furthermore, assume that G is a critical graph containing $e = u_1x_1$. Again, let $H = G - e$. Since B is an endblock of G, it has at exactly one cut vertex. Since $e = x_1u_1$ is not a cut edge of G it must lie on a cycle and since $\beta = 3$ we must have $i(H) < i(G)$. We know that $u_1, x_1 \in I_H$ (see Lemma 1 or Lemma 2). Then, $x_3 \in I_H$ since I_H must be an independent dominating set. Then, letting $I = I_H \setminus \{x_1\}$ gives $|I| < i(H)$. This is a contradiction. Therefore, $e = u_1x_1$, as defined above, can only occur as a non-critical edge. This arguments also works for $e = u_2x_2$.

□

Lemma 4. *Suppose G is a connected subcubic critical graph which satisfies $\delta(G) \geq 2$. Suppose that $X = \{x_1, x_2, x_3, x_4\}$ forms a 4-cycle where $x_1, x_2, x_3 \in S_3$ and $x_4 \in S_2$ with $u_1 \in N(x_1)\setminus\{x_2, x_4\}$, $v_1 \in N(x_2)\setminus\{x_1, x_3\}$, $z_1 \in N(x_3)\setminus\{x_1, x_2\}$. Now, suppose that B is an endblock of G containing X. If $e = v_1x_2$ is not a cut edge and $v_1 \in S_3$, then e is not a critical edge of G.*

Proof. Suppose as stated above and see Figure 2.11) for the labeling of X. Furthermore, assume that G is a critical graph containing $e = v_1x_2$. Again, let $H = G - e$.

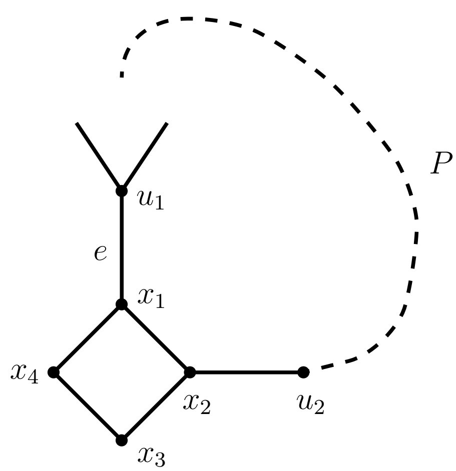

Figure 2.10: The endblock of Lemma 3

Since e is not a cut edge and $\beta = 3$ we must have $i(H) < i(G)$. We know that $v_1, x_2 \in I_H$ (see Lemma 1 or Lemma 2). Then, $x_4 \in I_H$ since I_H must be an independent dominating set. Then, letting $I = I_H \setminus \{x_1\}$ gives $|I| < i(H)$. This is a contradiction. Therefore, $e = v_1x_2$, as given above, can only occur as a non-critical edge. □

Before we proceed, we need to show that the base step of our induction argument is valid and we do so by using the following lemma. We require the following result from [4] which originally appeared in a text by O. Ore. This result will be used in the following lemma. The next result shows that any graph having order at most 7 satisfies our bound and will be used as the base step for the induction arguments to follow.

Theorem 5. *(Ore 1962) If G is a connected graph with no isolated vertices, then*

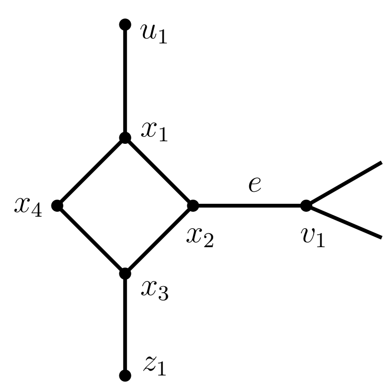

Figure 2.11: The endblock of Lemma 4

the complement of every minimal dominating set is also a dominating set.

Lemma 5. *Let G be a connected subcubic graph with $\delta(G) \geq 2$ and having order $n \leq 7$. Then, G satisfies the bound*

$$i(G) \leq \frac{3(n+1)}{7}.$$

Proof. We proceed by contradiction and for each case we let $J = V \setminus I$. When $n = 3$ the result is trivial. If $4 \leq n \leq 5$, then we require $|I| \geq 3$ for a contradiction so suppose that I is an MIDS such that $|I| \geq 3$. First, if $n = 4$ with $|I| \geq 3$, then $|J| \leq 1$ with $\Delta(G) \leq 3$ and would be a smaller independent dominating set so this can't occur. If $n = 5$ with $|I| \geq 3$, then $|J| \leq 2$. Furthermore, J must contain at least one edge since otherwise it would be a smaller independent dominating set by Ore's theorem above. Then, since $\delta(G) \geq 2$, counting edges leaving I we have $e(I, J) \geq 3 * 2 = 6$. Similarly, since $\Delta(G) \leq 3$ and J contains an edge, we have $e(I, J) \leq 6 - 2 = 4$ which is impossible.

Now suppose that $6 \leq n \leq 7$. Here we require $|I| \geq 4$ for a contradiction so suppose that I is an MIDS such that $|I| \geq 4$. If $n = 6$, with $|I| \geq 4$, then $|J| \leq 2$, so $e(I, J) \geq 4 * 2 = 8$ and $e(I, J) \leq 6 - 2 = 4$ which is impossible. Finally, when $n = 7$, with $|I| \geq 4$, then $|J| = 3$ so $e(I, J) \geq 4 * 2 = 8$ and $e(I, J) \leq 6 - 2 + 3 = 7$ which is again a contradiction. This completes the lemma and we can use this result as our base step for the induction arguments below.

□

Before we begin the next result, we need to introduce some terminology. We will be using induction in each result below and we will employ a few specialized terms for convenience. The process of removing vertices and adding edges to maintain the degree condition and connectivity will become somewhat involved and we may refer to this process as surgery. In the following lemma, we show that any graph with an induced path in $G[S_2]$ of length at least two satisfies the bound of our main theorem.

Lemma 6. *Let G be a connected subcubic critical graph with minimum degree $\delta(G) \geq 2$ and let B be an endblock of G such that $|V(B)| \geq 6$. If B contains an induced path in $G[S_2]$ of length at least two, then*

$$i(G) \leq \frac{3(n+1)}{7}.$$

Proof. We proceed by induction on $|V| = n$. We have just shown that the result holds for every graph with $n \leq 7$. Suppose that every graph with fewer than n vertices satisfies our claim and consider a graph with n vertices that satisfies the hypothesis.

Next, suppose that $G[S_2]$ contains a path of length at least three and label this path $P = w_1 \dots w_6$ where $w_2, w_3, w_4, w_5 \in S_2$. Next, let $G' = G - \{w_3, w_4, w_5\} + w_2w_6$. Then, if $w_2 \in I'$, we let $I = I' \cup \{w_5\}$. If $w_6 \in I'$, we let $I = I' \cup \{w_3\}$ and if neither occur let $I = I' \cup \{w_4\}$.

Now, suppose that $G[S_2]$ contains a path of length two and label this path $P = w_1 \dots w_5$ where $w_2, w_3, w_4 \in S_2$. There are three cases to consider.

Case 1: $w_1 \neq w_5$ and $w_1w_5 \notin E$.

Here, we let $H = G[\{w_2, w_3, w_4\}]$. Then, removing H and adding the edge w_1w_5 gives a new graph G', with fewer vertices, that satisfies our hypothesis and so there is an MIDS I' corresponding to G'. Then, if $w_1 \in I'$, we let $I = I' \cup \{w_4\}$. If $w_5 \in I'$, we let $I = I' \cup \{w_2\}$ and if neither occur let $I = I' \cup \{w_3\}$. Thus, we have shown that in each case we only need one additional vertex for I. Hence, by the Induction

Hypothesis (IH), we have

$$i(G) = |I| = |I'| + 1 = \frac{3(n-3+1)}{7} + 1 = \frac{3n+1}{7} \leq \frac{3(n+1)}{7}$$

Case 2: $w_1 \neq w_5$ and $w_1 w_5 \in E$.

There are several subcases to be considered. First, if both $w_1, w_5 \in S_2$, then $G = C_5$ and we have already considered this case. Now, suppose that both $w_1, w_5 \in S_3$ and again let $H = G[\{w_2, w_3, w_4\}]$. Then, we can remove H to obtain the graph $G' = G - H$ which satisfies the IH, so there is an MIDS I' for G'. Thus, we let $I = I' \cup \{w_3\}$. If $w_1 \in S_2$ and $w_5 \in S_3$ or $w_1 \in S_3$ and $w_5 \in S_2$, then $|V(B)| \leq 5$ and we assumed that $|V(B)| \geq 6$.

Case 3: $w_1 = w_5$.

In this case, we have a four cycle with only one degree three vertex on the cycle (the others being degree two) which is not possible in B since $|V(B)| \geq 6$.

□

Although we checked the case above, whenever we let $G' = G - R$ where $|R| = n_R$ we can always choose k vertices to dominate the vertices of R and create I from I' since if

$$\frac{3(n-n_R+1)}{7} + k = \frac{3n - 3n_R + 3 + 7k}{7} \leq \frac{3(n+1)}{7} = \frac{3n+3}{7}$$

then,

$$\frac{-3n_R + 7k}{7} \leq 0$$

or equivalently

$$\frac{3n_R}{7} \geq k.$$

Thus, if we let $k = \lfloor 3n_R/7 \rfloor$ we can always satisfy the above relationship. Thus, we don't need to check each case but rather just use the relationship above.

Lemma 7. *Let G be a connected subcubic critical graph of order n satisfying $\delta(G) \geq 2$ and let B be an end-block of G which satisfies $g(B) \geq 5$. If $u \in V(B) \cap S_3$ is not a cut vertex of G and satisfies $|N(u) \cap S_2| \geq 2$, then*

$$i(G) \leq \frac{3(n+1)}{7} .$$

Proof. We proceed by induction on $|V| = n$. By Lemma 5, we may assume that $n \geq 8$. Suppose that every graph with fewer than n vertices satisfies our claim and consider a graph with n vertices that satisfies the hypothesis. Suppose that $u \in V(B) \cap S_3$ is not a cut vertex of G where B is an end-block of G. Let $u_1, u_2, u_3 \in N(u)$.

__**Case 1:**__ $|N(u) \cap S_2| = 3$.

Let $u_4 \in N(u_2) \backslash \{u\}$, $u_5 \in N(u_3) \backslash \{u\}$, and let $u_1' \in N(u_1) \backslash \{u\}$. If $|\{u_1', u_4, u_5\} \cap S_2| \geq 2$, then let $u_4, u_5 \in S_2$. Now, suppose that $u_6 \in N(u_4) \setminus \{u_2\}$ and $u_7 \in N(u_5) \setminus \{u_3\}$. Then, by Lemma 6, $u_6, u_7 \in S_3$. Next, let $Q = \{u, u_2, u_3, u_4, u_5\}$. Then, since u is not a cut vertex, $G - Q$ is connected. Thus, at least one of $G_1' = G - Q + u_1u_6$ or $G_2' = G - Q + u_1u_7$ is also connected and both graphs satisfy $\delta(G_i') \geq 2$ since we added an edge. Finally, applying Fact 3 shows the result holds in this case. Now, we may assume that $|\{u_1', u_4, u_5\} \cap S_2| \leq 1$. There are two subcases to consider.

Case 1.1: $|\{u_1', u_4, u_5\} \cap S_2| = 1$.

Suppose that $u_4 \in S_2$. First, if $u_5 = u_6$, then let $G' = G - Y + u_1'u_5$ where $Y = N[u] \cup \{u_4\}$. Next, since u is not a cut vertex the graph G' is connected. Then, applying Fact 2 shows the result holds in this case. Next, if $u_5 \neq u_6$, then we can let $G' = G - Y$. Since u is not a cut vertex, the graph G' is connected. Then, applying Fact 2 show that the result holds in this case. Finally, note that the same argument works if $u_1' = u_6$ and $u_1' \neq u_6$ by symmetry, so this completes Case 1.1.

Case 1.2: $|\{u_1', u_4, u_5\} \cap S_2| = 0$.

Next, since we have assumed that $g(B) \geq 5$, then $u_4 \neq u_5$. Then, $G' = G - N[u]$ is connected since u is not a cut vertex so we can apply Fact 4. This completes the case.

Case 2: $|N(u) \cap S_2| = 2$.

Suppose that $u_2, u_3 \in S_2$. By Case 1, we can show that $|\{u_4, u_5\} \cap S_2| \leq 1$. There are two cases to consider.

Case 2.1: $|\{u_4, u_5\} \cap S_2| = 1$.

Suppose that $u_4 \in S_2$ and $u_5 \in S_3$ and let $u_1', u_1'' \in N(u_1) \setminus \{u\}$ where $d(u_1'') \leq d(u_1')$. First, we can show the result for $\{u_1', u_1''\} \cap S_2 \neq \emptyset$. Next, let $u_1''' \in N(u_1'') \setminus \{u_1\}$. Let $G' = G - \{u, u_1, u_2, u_3, u_1''\} + u_4u_5 + u_1'u_1'''$. Then, since u is not a cut vertex G' is connected. Finally, applying Fact 5 (The Double Star Lemma) shows the result holds in this case.

Now, suppose that $\{u_1', u_1''\} \subseteq S_3$. First, if $u_5 \neq u_6$, then let $Y = \{u, u_1, u_2, u_3, u_4\}$. Then, at least one of $G_1' = G - Y + u_5u_6$, $G_2' = G - Y + u_1'u_6$, or $G_3' = G - Y + u_1''u_6$ is connected since u is not a cut vertex. Then, applying Fact 2 shows the result holds in this case.

If $u_5 = u_6$, then let $X = \{u, u_2, u_3, u_4, u_5\}$ and $G' = G - X + u_1u_5'$ where $u_5' \in N(u_5) \setminus \{u_3\}$. If $u_1, u_5' \notin I'$, let $I = I' \cup \{u, u_5\}$. If $u_1' \in I'$, let $I = I' \cup \{u_2, u_5\}$ and if $u_5 \in I'$, let $I = I' \cup \{u, u_4\}$ Then, since G' is connected the result holds.

<u>Case 2.2:</u> $|\{u_4, u_5\} \cap S_2| = 0$.

First, since we have assumed that $g(B) \geq 5$, then $u_4 \neq u_5$. By Case 1, we can assume that $\{u_1', u_1''\} \subseteq S_3$. Suppose that $\{u_1', u_1''\} \subseteq S_3$. Next, at least one of $\{u_4, u_5\}$ is not a cut vertex, so we may assume that u_4 is not a cut vertex. Next, let $u_4', u_4'' \in N(u_4) \setminus \{u_2\}$ with $d(u_4'') \leq d(u_4')$. By Case 1, we have $|\{u_4', u_4''\} \cap S_3| \leq 1$. Thus, $d(u_4') = 3$. Now, set $Y = N[u] \cup \{u_4\}$. Then, at least one of $G_1' = G - Y + u_4''u_1'$, $G_2' = G - Y + u_4''u_1''$, or $G_3' = G - Y + u_4''u_5$ is connected and each graphs satisfies $\delta(G_i') \geq 2$. Applying, Fact 2 completes the Lemma.

□

Now, we may assume that any vertex $x \in V(B) \cap S_3$ which is not cut vertex satisfies $|N(x) \cap S_2| \leq 1$. We will proceed with the main theorem in three stages. The following lemma allows us to eliminate degree two vertices that are adjacent to any vertex on the shortest cycle of the endblock that isn't itself a cut vertex. Additionally, we will address the cases of girth 3 and 4 in the main theorem.

Lemma 8. *Let G be a connected subcubic critical graph of order n satisfying $\delta(G) \geq 2$ and having girth at least 5. Let B be an end-block of G with $|V(B)| \geq 6$ and containing a shortest cycle X. If $x \in V(X) \cap S_3$ is not a cut vertex of G and x has a degree two neighbor not lying on the cycle, then*

$$i(G) \leq \frac{3(n+1)}{7}.$$

Proof. We proceed by induction on the number of vertices. By Lemma 5, we may assume that the bound holds for every graph with order $n \le 7$. Suppose that every graph with fewer than n vertices satisfies our lemma and consider a graph with n vertices that satisfies the hypothesis. Let X be a shortest cycle of an end-block B and suppose that $x = x_1 \in V(X) \cap S_3$ is not a cut vertex of G. For the remainder of our proof we label the cycle X as follows. In the clockwise direction, we let vertices have odd subscripts, that is x_1, x_3, x_5, x_7, and in the counter clockwise direction we label with even subscripts. Although our labeling might exceed five, we never violate the girth five condition. Let $u_1 \in N(x_1) \setminus X$ with $u_1 \in S_2$. Then, since x_1 is not a cut vertex and $u_1 \in S_2$, $x_2, x_3 \in S_3$ by Lemma 7. Next, suppose that $u_2 \in N(u_1) \setminus \{x_1\}$. By Fact 4 and Lemma 6, $u_2 \in S_3$. There are two possibilities to consider. If u_2 is not a cut vertex of G, then $N(u_2) \setminus \{u_1\} \subseteq S_3$. Then, $G' = G - \{x_1, u_1, u_2\}$ is connected. Applying Fact 4 show the result holds.

Now, we may assume that u_2 is a cut vertex of G. In this case, since $u_2 \in V(B)$ it is the only such vertex in B. Then, x_3 is not a cut vertex so we must have $|N(x_3) \cap S_2| \le 1$ (By Lemma 7). This might occur in three different ways so we must consider either $u_3 \in S_2$ and $x_5 \in S_3$ or $u_3 \in S_3$ and $x_5 \in S_2$ or $u_3, u_5 \in S_3$ where $u_3 \in N(x_3) \setminus \{x_1, x_2\}$ and x_5 is the neighbor of x_3 lying on the cycle.

<u>Case 1:</u> $u_3 \in S_2$ and $x_5 \in S_3$.

Then, let $u_4 \in N(u_3) \setminus \{x_3\}$. Since u_4 is not a cut vertex and $u_3 \in S_2$, $N(u_4) \setminus \{u_3\} \subseteq S_3$ by Lemma 7. Then, $G' = G - \{x_3, u_3, u_4\}$ is connected. Applying Fact 4 show the result holds in this case.

<u>Case 2:</u> $u_3 \in S_3$ and $x_5 \in S_2$.

Next, $x_9 \in S_3$ by Lemma 6. By Fact 4, $x_7 \in S_3$. Let $u_7 \in N(x_7) \setminus X$ and $u_3', u_3'' \in N(u_3) \setminus X$ with $d(u_3'') \le d(u_3')$. Since u_2 is a cut vertex, u_3 is not a cut vertex. Thus,

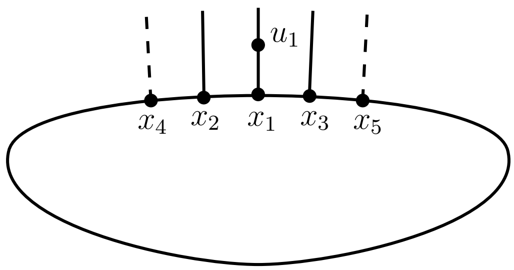

Figure 2.12: The Cycle of Lemma 8

$u_3' \in S_3$ by Lemma 7. Let $P = \{x_3, x_5, x_7\}$, $P' = \{x_1, u_1, u_2\}$, and $Y = N[x_3] \cup \{u_1\}$. If $G - P$ is connected, we can apply Fact 4. Suppose that $\omega(G - P) = 2$ with components H and H' where $u_3, u_7 \in V(H)$. Since $u_3 \in V(H)$, $u_3', u_3'' \in V(H)$. Next, $u_2 \in V(B)$ is a cut vertex but only one of it's neighbors is not in $V(B)$. Then, since x_1 is not a cut vertex, there is a (u_2, X)-path in $G - Y$ and one of the neighbors of u_2 must be in $V(H')$. Therefore, $G' = G - Y + u_2u_3''$ is connected. Applying Fact 2 show the result holds in this case.

<u>Case 3:</u> $u_3, x_5 \in S_3$.

Let $u_5 \in N(x_5) \setminus X$, $R = N[x_1]$, $P = \{x_1, x_3, x_5\}$, $Y = R \cup \{x_5\}$, $Y' = R \cup \{x_4\}$, and $w \in N(x_2) \setminus X$. Since u_2 is a cut vertex, x_2 is not a cut vertex so $w \in S_3$ by Case 1 of this Lemma (i.e. repeat the argument we used to show $u_3 \in S_3$) and $x_4 \in S_3$ by Case 2 of this Lemma (symmetry where we replace x_5 with x_4 and u_3 with w). If $G - R$ is connected, we can apply Fact 4. Thus, we may assume that $\omega(G - R) = 2$ with components H and H'. There are three subcases.

<u>Case 3.1:</u> $u_2, u_3, w \in V(H)$.

Then, $G' = G - Y + u_2x_7$ is connected since $u_5 \in V(H' - \{x_5\}) \subseteq V(H')$. Applying

Fact 2 shows the result holds.

Case 3.2: $u_2, w \in V(H)$ and $u_3 \in V(H')$.

In this case, $2 \leq \omega(G-Y) \leq 3$ and $1 \leq \omega(G-P) \leq 2$ so there are two further subcases to consider.

Case 3.2.1: $\omega(G-Y) = 2$.

Then, $\omega(G-P) = 1$. In this case, u_2 and u_5 lie in different components of $G-P$. Therefore, $G' = G - Y + u_2x_7$ is connected. Applying Fact 2 shows the result holds.

Case 3.2.2: $\omega(G-Y) = 3$.

Then, $\omega(G-P) = 2$. In this case, u_2 and u_5 lie in the same component of $G-P$. Therefore, $G' = G - Y' + u_2x_6$ is connected. Applying Fact 2 shows the result holds.

Case 3.3: $u_3, w \in V(H)$ and $u_2 \in V(H')$.

Again, $2 \leq \omega(G-Y) \leq 3$ and $1 \leq \omega(G-P) \leq 2$ so there are two further subcases to consider.

Case 3.3.1: $\omega(G-Y) = 2$.

Then, $\omega(G-P) = 1$. In this case, u_2 and u_5 lie in different components of $G-P$. Therefore, $G' = G - Y + u_2x_7$ is connected. Applying Fact 2 shows the result holds.

Case 3.3.2: $\omega(G-Y) = 3$.

Then, $\omega(G-P) = 2$. In this case, u_2 and u_5 lie in the same component of $G-P$. Therefore, $G' = G - Y' + u_2x_6$ is connected. Applying Fact 2 shows the result holds. This completes Lemma 8.

□

Using the previous lemma, we can show the following result effectively eliminating degree two vertices from the cycle altogether provided $g(G) \geq 5$.

Lemma 9. *Let G be a connected subcubic critical graph of order n satisfying $\delta(G) \geq 2$*

and having girth at least 5. Let B be an end-block of G with $|V(B)| \geq 6$ containing a shortest cycle X. If $x \in V(X) \cap S_3$ is not a cut vertex of G and x has a degree two neighbor lying on the cycle, then

$$i(G) \leq \frac{3(n+1)}{7}.$$

Proof. We proceed by induction on the number of vertices. By Lemma 5, we may assume that $n \geq 8$. Suppose that every graph with fewer than n vertices satisfies our lemma and consider a graph with n vertices that satisfies the hypothesis. Let X be a shortest cycle of the end-block B and suppose that $x_1 \in V(X) \cap S_3$ is not a cut vertex of G. Label X as in the previous Lemma (see Figure 2.13) and suppose that $u_1 \in N(x_1) \setminus X$ where $u_1 \in S_3$ by Lemma 8. Since x_1 is not a cut vertex and $u_1 \in S_3$, either $x_2 \in S_3$ and $x_3 \in S_2$ or $x_2 \in S_2$ and $x_3 \in S_3$ (by Lemma 7). Suppose that $x_2 \in S_3$ and $x_3 \in S_2$. Next, let $u_2, u_3 \in N(u_1) \setminus \{x_1\}$ with $d(u_2) \leq d(u_3)$. We may assume that u_1 is not a cut vertex since otherwise we could move to x_5 and repeat the following argument. Then, since u_1 is not a cut vertex, we must have $u_3 \in S_3$ (By Lemma 7). Thus, either $u_2 \in S_2$ or $u_2 \in S_3$.

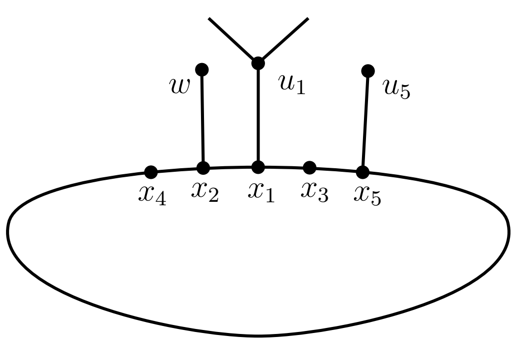

Figure 2.13: The Cycle of Lemma 9

Case 1: $u_2 \in S_2$.

Let $u_2' \in N(u_2) \setminus \{u_1\}$. There are two subcases.

Case 1.1: u_2' is not a cut vertex.

Then, $N(u_2') \setminus \{u_2\} \subseteq S_3$ by Lemma 7. Let $u_2'', u_2''' \in N(u_2') \setminus \{u_2\}$. Next, if $G' = G - \{u_1, u_2, u_2'\}$ is connected, we are done since we can apply Fact 4. Suppose that $\omega(G') = 2$ with components H and H'. In this case, either $u_2'', u_3 \in V(H)$ and $u_2''' \in V(H')$ where $V(X) \subseteq V(H')$ or $u_2''', u_3 \in V(H)$ and $u_2'' \in V(H')$ where $V(X) \subseteq V(H')$. Suppose that $u_2'', u_3 \in V(H)$ and $u_2''' \in V(H')$. Then, let $G' = G - \{x_3, x_1, u_1, u_2, u_2'\} + u_3x_5$. If $u_3, x_5 \notin I'$ or $u_3 \in I'$, let $I = I' \cup \{x_3, u_2\}$. When $x_5 \in I'$, let $I = I' \cup \{u_1, u_2'\}$ provided $N(u_2') \cap I' = \emptyset$ and $I = I' \cup \{u_1\}$ otherwise. This completes Case 1.1

Case 1.2: u_2' is a cut vertex.

Then, u_3 is not a cut vertex. Suppose that $u_3', u_3'' \in N(u_3) \setminus \{u_1\}$ with $d(u_3'') \leq d(u_3')$ where $u_3' \in S_3$ by Lemma 7.

Case 1.2.1: $u_3'' \in S_3$.

Let $Y = N[u_1] \cup \{x_3\}$. Then, at least one of $G' = G - Y + x_5u_3'$ or $G' = G - Y + x_5u_3''$ is connected. Applying Fact 2 shows the result holds.

Case 1.2.2: $u_3'' \in S_2$.

Let $u_3''' \in N(u_3'') \setminus \{u_3\}$. Then, $1 \leq \omega(G - Y) \leq 2$ and there are several cases. If $\omega(G - Y) = 1$, then let $G' = G - Y + u_3''x_5$ and apply Fact 2. Suppose that $\omega(G - Y) = 2$ with components H and H'. Now, suppose that $u_3''', u_2' \in V(H)$ and $u_3' \in V(H')$ and there is a (u_3', X)-path in $G - Y$. In this case, let $G' = G - Y + x_5u_3''$. Then, G' is connected so applying Fact 2 shows the result holds. Now, suppose that $u_3', u_2' \in V(H)$ and $u_3''' \in V(H')$ where there is a (u_3''', X)-path in $G - Y$. Let $P' = \{u_3, u_3'', u_3'''\}$. Since u_3''' is not a cut vertex and $u_3'' \in S_2$ we know that

$N(u_3''') \setminus \{u_3''\} \subseteq S_3$. Thus, $G' = G - P'$ is connected and satisfies $\delta(G') \geq 2$. Applying Fact 4 shows the result holds. This completes Case 1.

Case 2: $u_2 \in S_3$.

Suppose that $u_2, u_3 \in S_3$ with $u_2', u_2'' \in N(u_2) \setminus \{u_1\}$ and $u_3', u_3'' \in N(u_3) \setminus \{u_1\}$.

Case 2.1: x_5 is not a cut vertex.

By our assumptions, u_1, x_1, x_3, x_5 are not cut vertices. Since x_5 is not a cut vertex and $x_3 \in S_2$, $x_7 \in S_3$ by Lemma 7. Let $P'' = \{x_1, x_3, x_5\}$. If $G' = G - P''$ is connected, apply Fact 4. Suppose $\omega(G - P'') = 2$. Let $G - P''$ have components H and H' with $u_1, u_2, u_3, u_5 \in V(H)$ where $u_5 \in N(x_5) \setminus X$ and $x_2, x_7 \in V(H')$. There are two major subcases to consider.

Case 2.1.1: x_2 is not a cut vertex.

Let $w \in N(x_2) \setminus X$ and $Y = N[x_1] \cup \{x_5\}$. Since x_2 is not a cut vertex, $w \in S_3$ by Lemma 8. Since u_1 and x_2 are not a cut vertices, $\omega(G - Y) = 2$ with components $H_1 = H - \{u_1\}$ and $H_2 = H' - \{x_2\}$ so $G' = G - Y + x_4u_5$ is connected. Applying Fact 2 shows the result holds in this case.

Case 2.1.2: x_2 is a cut vertex.

By Lemma 7, $u_5, x_7 \in S_3$ since x_5 is not cut vertex and $x_3 \in S_2$. Let $u_5', u_5'' \in N(u_5) \setminus \{x_5\}$ with $d(u_5'') \leq d(u_5') = 3$ and let $u_7 \in N(x_7) \setminus X$. Since x_7 is not a cut vertex, $u_7 \in S_3$ by Lemma 8. There are two subcases.

Case 2.1.2.1: $u_5'' \in S_3$.

Let $Y = N[x_5] \cup \{x_1\}$. Then, $\omega(G - Y) = 2$ with components $F_1 = H - \{u_5\}$ and $F_2 = H' - \{x_7\}$ and since $u_5 \in V(H)$ we know that $u_5', u_5'' \in V(F_1)$. Therefore, $G' = G - Y + x_9u_1$ is connected. Applying Fact 2 shows the result holds in this case.

Case 2.1.2.2: $u_5'' \in S_2$.

Let $u_5''' \in N(u_5'') \setminus \{u_5\}$ and $Y = N[x_5] \cup u_5''$. Then, $G' = G - Y + x_9u_5'''$ is connected.

Applying Fact 2 shows the result holds in this case.

Case 2.2: x_5 is a cut vertex.

Since x_5 is a cut vertex, x_2 is not a cut vertex. Let $w \in N(x_2) \setminus X$. Then, $w \in S_3$ by Lemma 8.

Case 2.2.1: $x_4 \in S_2$.

Let $Y = N[x_1] \cup \{x_4\}$. Then, at least one of $G' = G - Y + u_2x_6$ or $G' = G - Y + u_3x_6$ is connected. Applying Fact 2 shows the result holds.

Case 2.2.2: $x_4 \in S_3$.

Let $w', w'' \in N(w) \setminus X$ with $d(w'') \leq d(w')$. Also, let $R = N[x_1]$ and $Y = R \cup \{w\}$. Since x_5 is a cut vertex, w is not a cut vertex so $w' \in S_3$ by Lemma 7. If $\omega(G-R) = 1$, we can apply Fact 4. Now, suppose that $\omega(G - R) = 2$ with components F and F'. Since u_1 and x_2 are not cut vertices u_2 and u_3 are in different components of $G - R$ and w and x_4 are in different components of $G - R$. Thus, we may assume (without loss of generality) that $u_2, w \in V(F)$ and $u_3, x_4 \in V(F')$. Since w is not a cut vertex, at least one of $G_1' = G - Y + w''x_5$ or $G_2' = G - Y + w''u_2$ is connected. Applying Fact 2 shows the result holds. This completes Lemma 9.

$\square$

Finally, we show that any neighbor of a vertex x which is not a cut vertex and which lies on a shortest cycle must have degree three neighbors. At this stage, we have a critical graph which is almost locally cubic.

Lemma 10. *Let G be a connected subcubic critical graph of order n satisfying $\delta(G) \geq 2$ and having girth at least 5. Let B be an end-block of G with $|V(B)| \geq 6$ containing a shortest cycle X. Suppose that $x \in V(X)$ is not a cut vertex of G with $N[x] \subseteq S_3$. Let $u \in N(x) \setminus X$ and suppose that u is not a cut vertex of G. If $N(u) \cap S_2 \neq \emptyset$,*

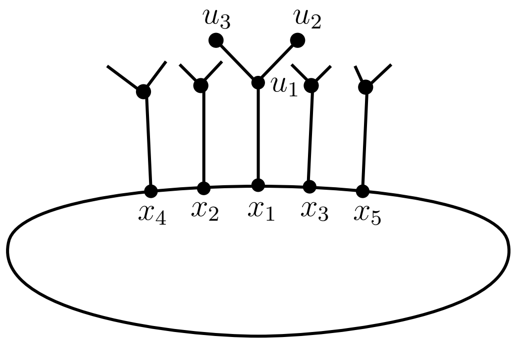

Figure 2.14: The Cycle of Lemma 10

$$i(G) \leq \frac{3(n+1)}{7}.$$

Proof. We proceed by induction on the number of vertices. By Lemma 5, we may assume that the bound holds for every graph with order $n \leq 7$. Suppose that every graph with fewer than n vertices satisfies our lemma and consider a graph with n vertices that satisfies the hypothesis. Let X be a shortest cycle of the end-block B and suppose that $x = x_1 \in V(X) \cap S_3$ is not a cut vertex of G.

From our previous work, we may now assume that for each $x \in V(X)$ that $N[x] \subseteq S_3$ provided x is not a cut vertex. If x' is a cut vertex and $u' \in N(x') \setminus X$, then $2 \leq d(u') \leq 3$. Additionally, if x' is not a cut vertex but u' is a cut vertex, then $u' \in S_3$ since otherwise x' would also be a cut vertex. Thus, there can be at most one degree two neighbor in $U = N(X) \setminus X$. Thus, we may assume that $X \cap S_2 = \emptyset$ and $|U \cap S_2| \leq 1$ provided $|V(B)| \geq 6$.

Next, we can label X as in Lemma 9 (see Figure 2.14). Let $u = u_1 \in N(x_1) \setminus X$ and let $u_2, u_3 \in N(u_1) \setminus \{x_1\}$ with $d(u_2) \leq d(u_3)$. Then, since u_1 is not a cut vertex,

$u_3 \in S_3$ by Lemma 7. Then, $u_2 \in S_2$ so let $u_2' \in N(u_2) \setminus \{u_1\}$. Since u_1 is not a cut vertex, $u_2' \in S_3$ by Fact 4 so let $u_2'', u_2''' \in N(u_2') \setminus \{u_2\}$. Let $P = N[u_2]$ and $R = N[u_1]$.

<u>**Case 1:**</u> u_3 is not a cut vertex.

Let $u_3', u_3'' \in N(u_3) \setminus \{u_1\}$ with $d(u_3'') \leq d(u_3')$ where $u_3' \in S_3$ by Lemma 7. There are two subcases.

<u>**Case 1.1:**</u> $u_3'' \in S_2$.

Let $u_3''' \in N(u_3'') \setminus \{u_3\}$ where $u_3''' \in S_3$ by Fact 4. Let $Y = R \cup \{u_3''\}$. Then, at least one of $G_1' = G - Y + u_3'''x_3$ or $G_2' = G - Y + u_3'''u_2'$ is connected. Applying Fact 2 shows the result holds.

<u>**Case 1.2:**</u> $u_3'' \in S_3$.

<u>**Case 1.2.1:**</u> u_2' is not a cut vertex.

If $\omega(G - P) = 1$, apply Fact 4. Suppose that $\omega(G - P) = 2$ with components H and H'. Since at most one of x_2, x_3 can be a cut vertex we can assume that x_3 is not a cut vertex. Then u_2'' and u_2''' are in different components of $G - P$. Let $Y = R \cup \{x_3\}$. Then, $\omega(G - Y) \leq 2$ since u_3 is not a cut vertex. Therefore, at least one of $G_1' = G - Y + u_2'x_5$, $G_2' = G - Y + u_3'x_5$, or $G_3' = G - Y + u_3''x_5$ is connected. Applying Fact 2 shows the result holds.

<u>**Case 1.2.2:**</u> u_2' is a cut vertex.

Then, x_2 and x_3 are not cut vertices. If $\omega(G - R) = 1$, then apply Fact 4. Suppose that $\omega(G - R) = 2$ with components F and F' and without loss of generality we may assume that $u_2', u_3' \in V(F)$ and $u_3'' \in V(F')$. Then, u_3' and u_3'' are in different components of $G - R$. Let $Y = R \cup \{x_3\}$ and $Y' = R + \{x_2\}$. Then, either $G_1 = G - Y$ or $G_2 = G - Y'$ has at most two components so assume that $\omega(G_1) \leq 2$. Then, at least one of $G_1 + u_3'x_5$ or $G_1 + u_3''x_5$ is connected. Applying Fact 2 shows the result holds.

Case 2: u_3 is a cut vertex.

Then, u_2' is not a cut vertex. If $\omega(G-P) = 1$, apply Fact 4. Suppose that $\omega(G-P) = 2$ with components H and H'. Then, u_2'' and u_2''' are in different components of $G-P$ so we may assume that $u_2'', u_3 \in V(H)$ and $u_2''' \in V(H')$. Let $Y = N[x_1] \cup \{u_2\}$ and $w \in N(x_2) \setminus X$, $w' \in N(x_3) \setminus X$ with $w, w' \in S_3$ by Lemma 9. Then, since x_2 and x_3 are not a cut vertices, $\omega(G - Y) = 2$. Therefore, at least one of $G_1' = G - Y + u_2'x_5$ or $G_2' = G - Y + u_2'w$ is connected. Applying Fact 2 shows the result holds. This completes Lemma 10.

$\square$

Using the lemmas just proven we can now assume that any vertex on the cycle that isn't a cut vertex has degree three neighbors. This will allow us to perform the necessary induction surgery and eliminate many cases that use the same surgery repeatedly.

Before we can begin the proof of Theorem 3, we need to introduce several theorems involving the properties of blocks and the idea of the block decomposition of a connected graph.

Since the graph of Theorem 3 is connected and has minimum degree two we know it contains a cycle and that each cycle of G lies in some block of G. Thus, we let B be an endblock of G provided G is 1-connected or if $\kappa(G) \geq 2$ then we can let $B = G$. In either case, we can consider the smallest cycle in B and when G is 1-connected we can exploit the fact that the endblock contains at most one cut vertex of G when we perform the required surgery.

2.1.3 Proof of Main Result

We are ready to prove the main theorem. Using the lemmas above we can now complete the proof of Theorem 4.

Theorem 4 If G is a connected subcubic graph of order n and minimum degree $\delta(G) \geq 2$, then

$$i(G) \leq \frac{3(n+1)}{7}.$$

Proof. We proceed by induction on the number of vertices. By Lemma 5, the bound holds when $n \leq 7$. Suppose that every graph with fewer than n vertices satisfies the bound and consider a graph with n vertices that satisfies the hypothesis. Before proceeding, recall that we only need to show that the result holds for critical graphs since non-critical graphs satisfy $i(G) \leq i(G')$ where G' was constructed in Fact 1.

To finish the proof, we can choose an endblock B with the maximum number of vertices. Then, since $\delta(G) \geq 2$ and the cycles are contained in the blocks of G, we can choose the shortest cycle in B and proceed with cases. When $|V(B)| \geq 6$ we can apply the Lemmas above (8,9, & 10) and this will be Case A below but for $|V(B)| \leq 5$ we must address several different cases (Case B below). We can further subdivide the two cases by considering the girth of G (i.e. $g(B) = 3, 4, ...$). Additionally, recall that when $\kappa(G) \geq 2$ we let $B = G$. Also, observe by Lemma 1 we can eliminate the case of $g(B) = 3$ when the triangle has more than one degree three vertex because we only need to consider critical graphs.

Case A: $|V(B)| \geq 6$.

Let B be an endblock of G which satisfies $|V(B)| \geq 6$ and let X be a shortest cycle in B. Since $|V(B)| \geq 6$ we may assume that $g(B) \geq 4$. We can proceed with increasing girth in each of the cases below.

Case 1: $g(B) = 4$.

Let $X = \{x_1, x_2, x_3, x_4\}$ form a cycle where the index increases in the clockwise direction so that $x_1x_2x_3x_4x_1$ is a 4-cycle. For each of the subcases below, we consider the number of degree three vertices on the cycle in increasing order (i.e. $|X \cap S_3| \geq 2$).

Case 1.1: $|X \cap S_3| = 2$.

There are two further subcases that must be considered.

Case 1.1.1: $x_1, x_2 \in S_3$ and $x_3, x_4, \in S_2$.

Let $u_1 \in N(x_1) \setminus \{x_2, x_4\}$ and $u_2 \in N(x_2) \setminus \{x_1, x_3\}$. Since B is an endblock x_1 and x_2 cannot be cut vertices. This is true because if x_1 were a cut vertex there would be no path P in $G - X$ joining u_1 and u_2 (shown in Figure 2.10). Thus, x_2 would also be a cut vertex and since B is an endblock it has exactly one cut vertex. So, x_1 and x_2 are not cut vertices. Therefore, by Lemma 7, x_1 satisfies $|N(x_1) \cap S_2| \leq 1$ and since $x_4 \in N(x_1) \cap S_2$ we must have $u_1 \in S_3$. Similarly, x_2 satisfies $|N(x_2) \cap S_2| \leq 1$ and since $x_3 \in N(x_2) \cap S_2$ we know that $u_2 \in S_3$. By Lemma 3, $e_1 = x_1u_1$ and $e_2 = x_2u_2$ are non-critical edges and we have assumed that G is a critical graph. Therefore, this case can't occur.

Now, that we have completed Case 1.1.1, we may assume that $|X \cap S_3| \geq 2$ and $x_1, x_3 \in S_3$. Therefore, we can show the following result and apply it in each of the subcases below (for the remainder of Case 1). Suppose that $|X \cap S_3| \geq 2$ with $x_1, x_3 \in S_3$ and $u_1 \in S_2$ where $u_1' \in N(u_1) \setminus X$. If u_1' is not a cut vertex and $u_1' = u_3 \in S_3$, then $e = u_1'x_3$ is not a critical edge of G. Suppose as stated and let $H = G - e$. Then, let I_H be an MIDS of H. Suppose that e is a critical edge of G. Next, since u_1 is not a cut vertex, e is not a cut edge and $\beta(x_1, u_1') = 3$, so $i(H) < i(G)$. Then, as in Lemmas 1-4, $x_1, u_1' \in I_H$. Therefore, $I = I_H \setminus \{x_3\} \cup \{x_1\}$, which gives a contradiction. Hence, $u_1' \neq u_3$.

<u>**Case 1.1.2:**</u> $x_1, x_3 \in S_3$ and $x_2, x_4, \in S_2$.

Let $u_1 \in N(x_1) \setminus \{x_2, x_4\}$ and $u_3 \in N(x_3) \setminus \{x_2, x_4\}$. Since B is an endblock and satisfies $|V(B)| \geq 6$, $u_1 \neq u_3$. Next, using the same argument as in Case 1.1.1, since B is an endblock x_1 and x_3 cannot be cut vertices. Then, by Lemma 7, we know that $u_1, u_3 \in S_3$. Suppose that $u_1u_3 \notin E$. If u_1 is not a cut vertex, then by Lemma 7 $|N(u_1) \cap S_2| \leq 1$ so we may assume that $d(u_1'') \leq d(u_1') = 3$ where $u_1', u_1'' \in N(u_1) \setminus \{x_1\}$ (shown in Figure 2.15). Next, if $u_1''u_3 \notin E$, then $G' = G - (X + u_1) + u_1''u_3$ is connected and satisfies $\delta(G') \geq 2$. By induction, G' has a MIDS I' so we can use I' to construct I. Therefore, we must show that $i(G) \leq i(G') + 2$ since we removed 5 vertices. If $u_1'', u_3 \notin I'$ or $u_1'' \in I'$, let $I = I' \cup \{x_1, x_3\}$. If $u_3 \in I'$, let $I = I' \cup \{x_1, u_1''\}$ when $N(u_1'') \cap I' = \emptyset$ and $I = I' \cup \{x_1\}$ otherwise. Hence, $i(G) \leq i(G') + 2$ as required. If $u_1''u_3 \in E$, then let $G' = G - (X + u_1)$ and $I = I' \cup \{x_1, x_3\}$ when $u_3 \notin I'$ and $I = I' \cup \{x_1\}$ when $u_3 \in I'$.

If u_1 is a cut vertex, then u_3 is not a cut vertex so let $d(u_3'') \leq d(u_3') = 3$ where $u_3', u_3'' \in N(u_3) \setminus \{x_3\}$. Then, let $G' = G - (X + u_3) + u_3''u_1$ and repeat the argument from above (i.e. exchanging the roles of u_1 and x_1 with u_3 and x_3).

Now, consider when $u_1u_3 \in E$. Since B is an endblock, u_1 and u_2 are not cut vertices so $|\{u_1, u_3\} \cap S_3| \geq 1$. If $u_1, u_3 \in S_3$, then let $u_1' \in N(u_1) \setminus \{x_1\}$ and $u_3' \in N(u_3) \setminus \{x_3\}$. If $u_1', u_3' \in S_3$, we can let $G' = G - (X + u_1 + u_3)$ and let $I = I' \cup \{x_1, x_3\}$. If $u_1', u_3' \in S_2$ or $u_1' \in S_2$ and $u_3' \in S_3$, let $G' = G - (X + u_1 + u_3 + u_1') + u_1''u_3'$ where $u_1'' \in N(u_1') \setminus \{u_1\}$. Then, if $u_1'', u_3' \notin I'$, we can let $I = I' \cup \{u_1, x_3\}$. If $u_1'' \in I'$, we can let $I = I' \cup \{u_3, x_1\}$ and if $u_3' \in I'$, we can let $I = I' \cup \{x_1, x_3, u_1'\}$.

Now, suppose that $u_1u_3 \in E$ where $u_3 \in S_2$ and $u_1 \in S_3$. Again, we let $u_1' \in N(u_1) \setminus \{x_1\}$. If $u_1' \in S_3$, let $G' = G - (X + u_1 + u_3)$. Then, $|V(B)| = 6$

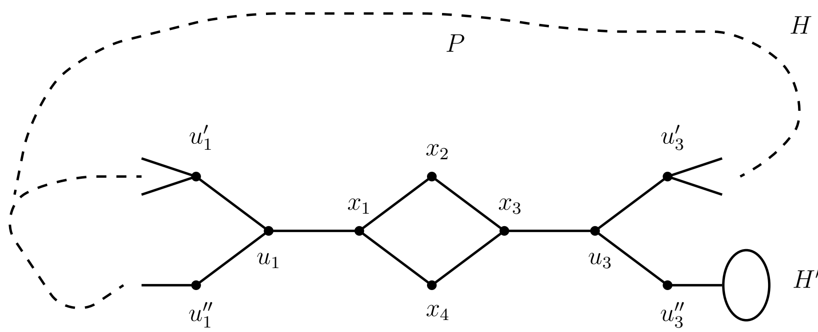

Figure 2.15: u_1 is not a cut vertex and $\omega(G-(X+u_3))=2$

and so we let $I = I' \cup \{x_1, x_3\}$. If $u_1' \in S_2$, let $u_1'' \in N(u_1') \setminus \{u_1\}$. If $u_1'' \in S_3$, let $G' = G - (X + u_1 + u_3 + u_1')$ and $I = I' \cup \{u_1, x_3\}$. If $u_1'' \in S_2$, let $u_1''' \in N(u_1'') \setminus \{u_1'\}$. Then, $u_1''' \in S_3$ by Lemma 6, so we can let $G' = G - (X + u_1 + u_3 + u_1' + u_1'')$ and $I = I' \cup \{u_1', x_1, x_3\}$. This completes Case 1.1.

Case 1.2: $|X \cap S_3| = 3$.

Suppose that $x_1, x_2, x_3 \in S_3$ and $x_4, \in S_2$. As before, let $U = N(X) \setminus X$ and $u_i \in N(x_i) \setminus X$ so that $u_1 \in N(x_1) \setminus \{x_2, x_4\}$, $u_2 \in N(x_2) \setminus \{x_1, x_3\}$, and $u_3 \in N(x_3) \setminus \{x_2, x_4\}$. By Lemma 2, we know that $u_1u_2, u_2u_3 \notin E(G)$. There are several subcases to consider: x_1 is a cut vertex, x_2 is a cut vertex, and x_3 is a cut vertex. However, the case x_1 is a cut vertex and x_3 is a cut vertex are the same (symmetry). Additionally, we need to consider when u_1 is cut vertex but x_1 is not a cut vertex. In this case, the argument is the same as when x_1 is a cut vertex. Next, if u_2 is a cut vertex but x_2 is not a cut vertex, this would imply that $u_2 \in S_3$ but then the graph would not be critical (see Lemma 4) so this case isn't possible.

Observe that when x_3 is a cut vertex, then $u_1 \in V(B)$ but $u_3 \notin V(B)$. Therefore, $u_1 \neq u_3$ and a similar result holds when x_1 is a cut vertex. When x_2 is

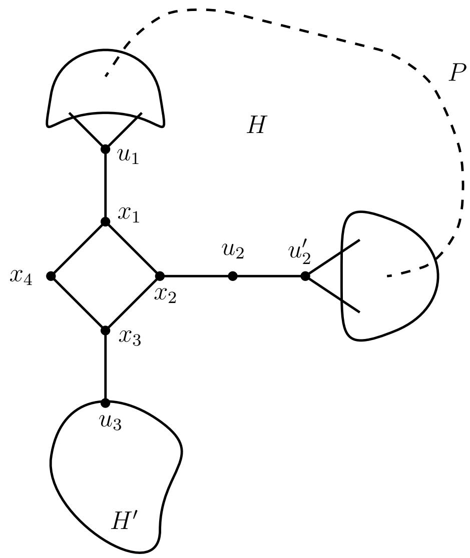

Figure 2.16: x_3 is a cut vertex and $G - (X + u_2)$ has two components H and H'

a cut vertex, x_1 and x_3 are not cut vertices with $u_1, u_3 \in S_3$ by Lemma 7. Then $u_1 \neq u_3$ because B is an endblock and $u_1, u_3 \in V(B)$ are not cut vertices. Thus, $u_1 \neq u_3$ for Cases 1.2.1-1.2.3. We can also assume that $u_1u_3 \notin E(G)$ when x_3 is a cut vertex (or x_1 is cut) since $u_1 \in V(B)$ but $u_3 \notin V(B)$.

Case 1.2.1: Suppose that x_3 is a cut vertex, $u_1 \neq u_3$, and $u_1u_3 \notin E(G)$. Since B is an endblock and x_3 is a cut vertex, x_1 is not a cut vertex so $u_1 \in S_3$ by Lemma 7. Since x_2 is not a cut vertex and G is a critical graph, $u_2 \in S_2$ by Lemma 4. Let $u_2' \in N(u_2) \setminus \{x_2\}$. Next, since x_2 is not a cut vertex, there is a (u_1, u_2')-path P in $G - (X + u_2)$ (see Figure 2.16). Then, $G - (X + u_2) + u_2'u_3$ is connected and $\delta(G') \geq 2$. Next, we need to show that $i(G) \leq i(G') + 2$.

Surgery A $\quad G' = G - (X + u_2) + u_2'u_3$.

If $u_2', u_3 \notin I'$, we let $I = I' \cup \{x_2, x_4\}$. If $u_3 \in I'$, let $I = I' \cup \{u_2, x_4\}$. If $u_2' \in I'$, let

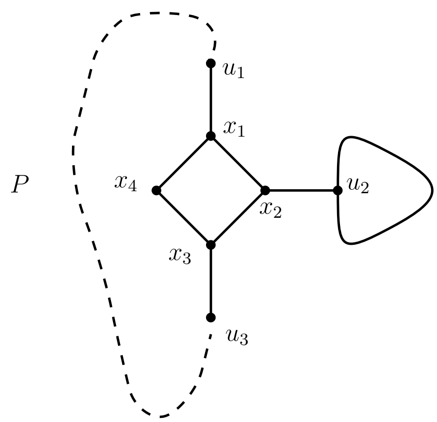

Figure 2.17: Surgery B where x_2 is a cut vertex

$I = I' \cup \{x_1, x_3\}$ provided $u_1 \notin I'$ and $I = I' \cup \{x_3\}$ when $u_1 \in I'$. Finally, observe that this surgery also works when u_3 is a cut vertex but x_3 is not a cut vertex. We will call this Surgery A since we need it again below.

Case 1.2.2: Suppose that x_1 is a cut vertex. Then, by symmetry, we can repeat the argument in Case 1.2.1.

Case 1.2.3: Suppose that x_2 is a cut vertex (see Figure 2.17) and $u_1 \neq u_3$. Then, x_1 is not a cut vertex so it must satisfy $|N(x_1) \cap S_2| \leq 1$ (again Lemma 7). Since $x_4 \in N(x_1) \cap S_2$, then $u_1 \in S_3$. Similarly, x_3 is not a cut vertex so $u_3 \in S_3$ by Lemma 7. Additionally, since B is an endblock and x_2 is a cut vertex, u_1 is not a cut vertex and satisfies $|N(u_1) \cap S_2| \leq 1$ again by applying lemma 7. Let $u_1', u_1'' \in N(u_1) \setminus \{x_1\}$ with $d(u_1'') \leq d(u_1') = 3$. Suppose that $u_1u_3 \notin E(G)$. Then, $\omega(G - (X + u_1)) = 2$ so $G' = G - (X + u_1) + u_1''u_2$ is connected and satisfies $\delta(G') \geq 2$. We call this Surgery B.

Surgery B $G' = G - (X + u_1) + u_1''u_2$.

Then, if $u_1'', u_2 \notin I'$, let $I = I' \cup \{x_1, x_3\}$ when $u_3 \notin I'$ and $I = I' \cup \{x_1\}$ when $u_3 \in I'$. If $u_1'' \in I'$, then let $I = I' \cup \{x_2, x_4\}$. If $u_2 \in I'$, then let $I = I' \cup \{u_1'', x_4\}$ when $N(u_1'') \cap I' = \emptyset$. If $N(u_1'') \cap I' \neq \emptyset$, let $I = I' \cup \{x_1, x_3\}$ if $u_3 \notin I'$ and $I = I' \cup \{x_1\}$ when $u_3 \in I'$. Finally, observe that the surgery works for $d(u_1'') = 2$ and $d(u_1'') = 3$.

Now, suppose that $u_1u_3 \in E(G)$. If $u_2 \in S_2$, then we can let $G' = G - (X + u_2) + u_2'u_3$ and use Surgery A of Case 1.2.1. Suppose that $u_2 \in S_3$ and let $u_2', u_2'' \in N(u_2) \setminus \{x_2\}$. Then, $G' = G - (X + u_2) + u_1u_2'' + u_3u_2'$ is connected and satisfies the minimum degree condition but we must show that $i(G) \leq i(G') + 2$ since we removed 5 vertices. First, if $u_1, u_3, u_2', u_2'' \notin I'$, then let $I = I' \cup \{x_2, x_4\}$. Next, if $u_1 \in I'$ but $u_3, u_2', u_2'' \notin I'$, let $I = I' \cup \{x_3, u_2\}$. If $u_3 \in I'$ but $u_1, u_2', u_2'' \notin I'$, let $I = I' \cup \{x_1, u_2\}$. If $u_2' \in I'$ but $u_1, u_3, u_2'' \notin I'$ or $u_2'' \in I'$ but $u_1, u_3, u_2' \notin I'$ let $I = I' \cup \{x_1, x_3\}$. If $u_1, u_2' \in I'$, then let $I = I' \cup \{u_2'', x_3\}$ and if $u_3, u_2'' \in I'$, then let $I = I' \cup \{u_2', x_3\}$. If $u_2', u_2'' \in I'$, then $I = I' \cup \{x_1, x_3\}$. This completes Case 1.2.3.

Case 1.2.4: No cut vertex in $X \cup U$. Now, suppose that $u_1 \neq u_3$. Then, from above we know that $u_1, u_3 \in S_3$ by Lemma 7 and $u_2 \in S_2$ by Lemma 4. Let $u_2' \in N(u_2) \setminus \{x_2\}$. Then, we can let $G' = G - (X + u_2) + u_2'u_3$ and use Surgery A above. Next, if $u_1 = u_3 \in S_3$, then let $G' = G - (X + u_1 + u_2) + u_1u_2'$. If $u_1, u_2' \notin I'$, let $I = I' \cup \{x_2, x_4\}$. If $u_1 \in I'$, let $I = I' \cup \{u_2, x_4\}$. If $u_2' \in I'$, let $I = I' \cup \{x_1, x_3\}$. This completes Case 1.2.

Case 1.3: $|X \cap S_3| = 4$.

Suppose that $x_i \in S_3$ and let $u_i \in N(x_i) \setminus X$ for $1 \leq i \leq 4$. Additionally, let $u_i' \in N(u_i) \setminus \{x_i\}$ and when $u_i \in S_3$ let $u_i'' \in N(u_i) \setminus \{x_i\}$. By Lemma 2, we know that $u_1u_2, u_2u_3, u_3u_4, u_4u_1 \notin E(G)$. Since $g(B) > 3$, $u_i \neq u_{i+1}$ for $1 \leq i \leq 4$. Next, X can have only one cut vertex. Then, if x_1 is a cut vertex, x_3 is not a cut vertex so $u_1 \neq u_3$. Similarly, when x_3 is a cut vertex, x_1 is not a cut vertex so $u_1 \neq_3$. By the same argument, $u_2 \neq u_4$ whenever x_2 isn't a cut vertex (or x_4). If X has no cut vertex but U has a cut vertex, then since U has at most one cut vertex $u_1 \neq u_3$ and $u_2 \neq u_4$ (by the same argument). If $u_1 = u_3 \in S_2$, let $G' = G - (X + u_1) + u_2u_4$. If $u_2, u_4 \notin I'$, let $I = I' \cup \{x_1, x_3\}$. If $u_2 \in I'$, let $I = I' \cup \{u_1, x_4\}$. If $u_4 \in I'$, let $I = I' \cup \{u_1, x_2\}$.

There are two cases to consider. Either X has a cut vertex or X has no cut vertex but U has a cut vertex. Additionally, we can suppose (without loss of generality) that x_3 is the cut vertex of G lying on the cycle whenever there is a cut vertex on the cycle (use symmetry). Before we complete the proof, we need the following claim.

Claim 1. *If x_i is not a cut vertex, then $u_i \in S_3$.*

Proof. Suppose that x_i is not a cut vertex and $u_i \in S_2$. Let $u_i' \in N(u_i) \setminus X$ and let $i = 1$.

Case C.1: $u_1' \in S_2$.

Let $u_1'' \in N(u_1') \setminus \{u_1\}$. Then, $u_1'' \in S_3$ by Lemma 6. Then, $G' = G - \{x_1, u_1, u_1'\}$ is connected. Applying Fact 4 shows the result holds in this case.

Case C.2: $u_1' \in S_3$.

Again, since x_1 is not a cut vertex, neither is u_1 but we must consider whether or not u_1' is a cut vertex.

Case C.2.1: u_1' is not a cut vertex.

Then $N(u_1') \setminus \{u_1\} \subseteq S_3$ by Lemma 7. Then, $G' = G - \{x_1, u_1, u_1'\}$ is connected. Applying Fact 4 shows the result holds.

Case C.2.2: u_1' is a cut vertex.

First, since u_1' is a cut vertex, we know that $u_1' \neq u_2, u_4$. Also, we have shown above that $u_1' \neq u_3$ (see the discussion between Case 1.1.1 and Case 1.1.2). Then u_1' is the only cut vertex of B, so $\omega(G - u_1') = 2$. Therefore, $\omega(G - (X + u_1)) = 2$ with components H and H'. Next, since we can't use Lemmas 7-10 we cannot assume $U \setminus \{u_1\} \subseteq S_3$. Thus, we must add two edges in order to reduce the number of cases. Then at least one of the following is connected: $G' = G - (X + u_1) + u_1'u_3 + u_2u_4$, $G' = G - (X + u_1) + u_1'u_4 + u_2u_3$, or $G' = G - (X + u_1) + u_1'u_2 + u_3u_4$. We use Surgery C.2 when $u_2, u_4 \in H'$ and $u_1', u_3 \in H$ and use Surgery C.1 otherwise. The graph G' is always connected and $\delta(G') \geq 2$ in each case.

Surgery C.1 $G' = G - (X + u_1) + u_1'u_3 + u_2u_4$.

Next, if $u_1' \in I'$ and $u_2, u_3, u_4 \notin I'$, let $I = I' \cup \{x_1, x_3\}$. If $u_2 \in I'$ and $u_1', u_3, u_4 \notin I'$, let $I = I' \cup \{u_1, x_4\}$. If $u_3 \in I'$ and $u_1', u_2, u_4 \notin I'$, let $I = I' \cup \{x_1, u_1'\}$ when $N(u_1') \cap I' \neq \emptyset$ and $I = I' \cup \{x_1\}$ otherwise. If $u_4 \in I'$ and $u_1', u_2, u_3 \notin I'$, let $I = I' \cup \{u_1, x_2\}$. If $u_2, u_3 \in I'$, let $I = I' \cup \{u_1, x_4\}$ and $u_3, u_4 \in I'$, let $I = I' \cup \{u_1, x_2\}$. If $u_1', u_4 \in I'$, let $I = I' \cup \{x_2, u_3\}$ when $N(u_3) \cap I' \neq \emptyset$ and $I = I' \cup \{x_2\}$ otherwise. If $u_1', u_2 \in I'$, let $I = I' \cup \{x_4, u_3\}$ when $N(u_3) \cap I' \neq \emptyset$ and $I = I' \cup \{x_4\}$ otherwise.

Surgery C.2 $G' = G - (X + u_1) + u_1'u_4 + u_2u_3$ or $G' = G - (X + u_1) + u_1'u_2 + u_3u_4$.

First, note that both surgery are the same by symmetry so we will show the first case when $G' = G - (X + u_1) + u_1'u_4 + u_2u_3$. Next, if $u_1' \in I'$ and $u_2, u_3, u_4 \notin I'$, let $I = I' \cup \{x_2, x_4\}$. If $u_2 \in I'$ and $u_1', u_3, u_4 \notin I'$, let $I = I' \cup \{x_1, x_3\}$. If $u_3 \in I'$ and $u_1', u_2, u_4 \notin I'$, let $I = I' \cup \{x_1, u_2\}$ when $N(u_2) \cap I' \neq \emptyset$ and $I = I' \cup \{x_1\}$ otherwise. If $u_4 \in I'$ and $u_1', u_2, u_3 \notin I'$, let $I = I' \cup \{u_1, x_2\}$. If $u_2, u_4 \in I'$, let $I = I' \cup \{u_1, x_3\}$ and $u_3, u_4 \in I'$, let $I = I' \cup \{u_1, x_2\}$. If $u_1', u_3 \in I'$, let $I = I' \cup \{x_2, x_4\}$. If $u_1', u_2 \in I'$, let $I = I' \cup \{x_4, u_3\}$ when $N(u_3) \cap I' \neq \emptyset$ and $I = I' \cup \{x_4\}$ otherwise.

□

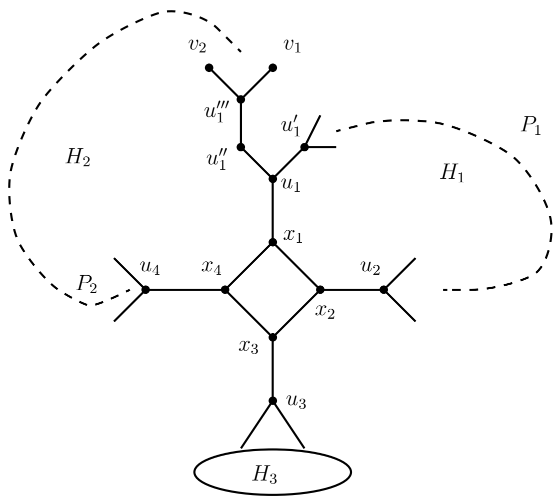

Figure 2.18: $\omega(G-(X+u_1))=3$

Now, we may assume that $u_i \in S_3$ whenever x_i is not a cut vertex. Thus, for Cases 1.3.1 and 1.3.2 we may assume that $u_1, u_2, u_4 \in S_3$ whenever x_3 is a cut vertex or u_3 is a cut vertex.

<u>Case 1.3.1:</u> X has a cut vertex.

Suppose (without loss of generality) that x_3 is the cut vertex. Then, as stated above, $u_1, u_2, u_4 \in S_3$. Thus, $N(X) \setminus \{u_3\} \subseteq S_3$. Next, let $u_1', u_1'' \in N(u_1) \setminus \{x_1\}$ with $d(u_1'') \leq d(u_1') = 3$. Then, $u_1' \in S_3$ by Lemma 7.

<u>Case 1.3.1.1:</u> $u_1'' \in S_2$.

Let $u_1''' \in N(u_1'') \setminus \{u_1\}$. Next, let $v_1, v_2 \in N(u_1''') \setminus \{u_1''\}$ and since u_1''' is not a cut vertex, $v_1, v_2 \in S_3$ (by Lemma 7). If $\omega(G-P)=1$, apply Fact 4. Suppose that $\omega(G-P)=2$ with components H and H'. Then v_1 and v_2 are in different components of $G-P$. Therefore, $G = G' - (X+u_1) + u_1''u_3$ is connected and satisfies $\delta(G') \geq 2$ but we need to show that $i(G) \leq i(G')+2$. If $u_1'', u_3 \notin I'$ or $u_1'' \in I'$, let

$I = I' \cup \{x_1, x_3\}$. If $u_3 \in I'$, let $I = I' \cup \{x_1, u_1''\}$ when $u_1''' \notin I'$ and $I = I' \cup \{x_1\}$ otherwise.

Case 1.3.1.2: $u_1'' \in S_3$.

Then, if we let $G' = G - (X + u_1) + u_2u_4 + u_1''u_3$, the graph remains connected because x_3 is not a cut vertex. Then, applying Surgery C below shows the result holds.

Surgery C: $G' = G - (X + u_1) + u_1''u_3 + u_2u_4 \implies i(G) \leq i(G') + 2$

If $\{u_1'', u_2, u_3, u_4\} \cap I' = \emptyset$, then let $I = I' \cup \{x_1, x_3\}$. Now, suppose that $|\{u_1'', u_2, u_3, u_4\} \cap I'| = 1$. If $u_1'' \in I'$, let $I = I' \cup \{x_1, x_3\}$. Next, suppose $u_2 \in I'$. Then if $u_1' \notin I'$, let $I = I' \cup \{x_4, u_1\}$. If $u_1' \in I'$, let $I = I' \cup \{x_4, u_1''\}$ provided $N(u_1'') \cap I' = \emptyset$ and $I = I' \cup \{x_4\}$ otherwise. If $u_3 \in I'$, let $I = I' \cup \{x_1, u_1''\}$ provided $N(u_1'') \cap I' = \emptyset$ and $I = I' \cup \{x_1\}$ otherwise. Now, suppose $u_4 \in I'$. Then if $u_1' \notin I'$, let $I = I' \cup \{x_2, u_1\}$. If $u_1' \in I'$, let $I = I' \cup \{x_2, u_1''\}$ provided $N(u_1'') \cap I' = \emptyset$ and $I = I' \cup \{x_2\}$ otherwise. Finally, suppose that $|\{u_1'', u_2, u_3, u_4\} \cap I'| = 2$. If $u_1'', u_2 \in I'$, let $I = I' \cup \{x_4, u_3\}$ provided $N(u_3) \cap I' = \emptyset$ and $I = I' \cup \{x_4\}$ otherwise. If $u_1'', u_4 \in I'$, let $I = I' \cup \{x_2, u_3\}$ provided $N(u_3) \cap I' = \emptyset$ and $I = I' \cup \{x_2\}$ otherwise. Suppose that $u_2, u_3 \in I'$. If $u_1' \notin I'$, let $I = I' \cup \{x_4, u_1\}$. If $u_1'' \in I'$ and $N(u_1'') \cap I' = \emptyset$, let $I = I' \cup \{x_4, u_1''\}$. If $u_1'' \in I'$ and $N(u_1'') \cap I' \neq \emptyset$, let $I = I' \cup \{x_4\}$. Suppose that $u_3, u_4 \in I'$. If $u_1' \notin I'$, let $I = I' \cup \{x_2, u_1\}$. If $u_1'' \in I'$ and $N(u_1'') \cap I' = \emptyset$, let $I = I' \cup \{x_2, u_1''\}$. If $u_1'' \in I'$ and $N(u_1'') \cap I' \neq \emptyset$, let $I = I' \cup \{x_2\}$.

Case 1.3.2: X has no cut vertex but U has a cut vertex.

Let $u_i', u_i'' \in N(u_i) \setminus \{x_i\}$ with $d(u_i'') \leq d(u_i')$ where $u_i' \in S_3$ by Lemma 7 and $1 \leq i \leq 4$. Then, $\omega(G - X) \leq 2$. Since U has at most one cut vertex we may assume that u_1 is not the cut vertex. If $\omega(G - X) = 1$, then let $G' = G - (X + u_1) + u_1''u_3$ and use the surgery of Case 1.3.1.1. If $\omega(G - X) = 2$ with u_1'' and u_3 in different components of $G - X$, then let $G' = G - (X + u_1) + u_1''u_3$ and use the surgery of Case

1.3.1.1. If $\omega(G-X)=2$ with u_1'' and u_3 in the same component of $G-X$, then let $G_1' = G-(X+u_1)+u_1'u_2+u_1''u_4$, and use surgery D (below).

Surgery D $\quad G_i' = G-(X+u_i)+u_i'u_{i+1}+u_i''u_{i+3} \implies i(G) \leq i(G_i')+2$

Let $i=1$. If $\{u_1', u_1'', u_2, u_4\} \cap I' = \emptyset$, then let $I = I' \cup \{x_1, x_3\}$ when $u_3 \notin I'$ and $I = I' \cup \{x_1\}$ otherwise. Next, suppose that $|\{u_1', u_1'', u_2, u_4\} \cap I'| = 1$. If $u_1' \in I'$ or $u_1'' \in I'$, let $I = I' \cup \{x_2, x_4\}$. If $u_2 \in I'$, let $I = I' \cup \{u_1, x_4\}$. If $u_4 \in I'$, let $I = I' \cup \{u_1, x_2\}$. Now, suppose that $|\{u_1', u_1'', u_2, u_4\} \cap I'| = 2$. If $u_1', u_1'' \in I'$, let $I = I' \cup \{x_2, x_4\}$. If $u_1', u_4 \in I'$, let $I = I' \cup \{x_2, u_1''\}$ when $N(u_1'') \cap I' \neq \emptyset$ and $I = I' \cup \{x_2\}$ otherwise. If $u_1'', u_2 \in I'$, let $I = I' \cup \{x_4, u_1'\}$ when $N(u_1') \cap I' \neq \emptyset$ and $I = I' \cup \{x_4\}$ otherwise. If $u_2, u_4 \in I'$, let $I = I' \cup \{u_1, x_3\}$ when $u_3 \notin I'$ and $I = I' \cup \{u_1\}$ otherwise.

Case 2: $\quad g(B) = 5$.

Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ form a 5-cycle with $x_1x_2x_3x_4x_5x_1$ and $|X \cap S_3| = 5$ by Lemma 9. Let $U = N(X) \setminus X$ where $u_i \in N(x_i) \setminus X$. Additionally, note that $u_i \neq u_{i+1}$, $u_i \neq u_{i+2}$ and $u_iu_{i+1} \notin E(G)$ for all $i \in [5]$ since $g(B) = 5$.

Case 2.1: $\quad X$ has no cut vertex.

Then, $G-X$ satisfies $\omega(G-X) \leq 2$ and $|U \cap S_3| = 5$. If $\omega(G-X) = 1$, then let $G' = G-X+u_2u_4$. Then, if $u_2, u_4 \notin I'$, let $I = I' \cup \{x_2, x_4\}$. If $u_2 \in I'$, let $I = I' \cup \{x_1, x_4\}$ if $u_1 \notin I'$ and $I = I' \cup \{x_4\}$ otherwise. If $u_4 \in I'$, let $I = I' \cup \{x_2, x_5\}$ if $u_5 \notin I'$ and $I = I' \cup \{x_2\}$ otherwise. If $\omega(G-X) = 2$, then $u_i, u_j, u_k \in V(H)$ and $u_r, u_s \in V(H')$ where H and H' are the two components of $G-X$ and $i, j, k, r, s \in [5]$. Here, if we remove X, we need to add an edge which joins a vertex in $\{u_i, u_j, u_k\} \subseteq V(H)$ to a vertex in $\{u_r, u_s\} \subseteq V(H')$ to maintain connectivity. Thus, if $G' = G-X+u_iu_r$, then we can show that whenever $\{i, r\} \in \{\{1,3\}, \{1,4\}, \{2,4\}, \{2,5\}, \{3,5\}\} = \mathcal{A}$ that G' is connected, $\delta(G') \geq 2$, and $i(G) \leq i(G')+2$. Thus, when $i=1$ and $r=3$

with $u_1 \in V(H)$ and $u_3 \in V(H')$ we let $G' = G - X + u_1u_3$. Then, if $u_1, u_3 \notin I'$, let $I = I' \cup \{x_1, x_3\}$. If $u_1 \in I'$, let $I = I' \cup \{x_3, x_5\}$ if $u_5 \notin I'$ and $I = I' \cup \{x_3\}$ otherwise. If $u_3 \in I'$, let $I = I' \cup \{x_1, x_4\}$ if $u_4 \notin I'$ and $I = I' \cup \{x_1\}$ otherwise. The same argument works for all the other members of $\mathcal{A}$ by shifting the index. Since $G - X$ has only two components the graph $G' = G - X + u_iu_r$ for $\{i, r\} \in \mathcal{A}$ is always connected and satisfies $\delta(G') \geq 2$.

Case 2.2: X has a cut vertex.

Now, suppose that x_1 is the cut vertex. Then, $G - X$ satisfies $2 \leq \omega(G - X) \leq 3$. Next, $u_i \in S_3$ for $i \geq 2$ by Lemma 8. Next, let $u_i', u_i'' \in N(u_i) \setminus \{x_i\}$. Then, for $i \geq 2$, $u_i', u_i'' \in S_3$ by Lemma 10.

Next, if $\omega(G-X) = 2$, then suppose that $G-X$ has two components H and H' with $u_1 \in V(H)$ and $u_i \in V(H')$ for $i \geq 2$. In this case, we can let $G' = G - X + u_1u_3$ and use the surgery of Case 2.1. Now, suppose that $\omega(G - X) = 3$ and let H, H', and H'' be the three components of $G - X$ where $u_1 \in V(H'')$. Then, one of the following must occur. Either $u_2, u_5 \in V(H)$ and $u_3, u_4 \in V(H')$ or $u_2, u_3 \in V(H)$ and $u_4, u_5 \in V(H')$ or $u_2, u_4 \in V(H)$ and $u_3, u_5 \in V(H')$. If $u_2, u_5 \in V(H)$ and $u_3, u_4 \in V(H')$ or $u_2, u_3 \in V(H)$ and $u_4, u_5 \in V(H')$, then $G' = G - X + u_2u_4 + u_1u_5$ is connected. Then, we can apply Surgery E (below).

Surgery E: $G' = G - X + u_2u_4 + u_1u_5 \implies i(G) \leq i(G') + 2$

If $|\{u_1, u_2, u_4, u_5\} \cap I'| = 0$, let $I = I' \cup \{x_2, x_4\}$. Suppose that $|\{u_1, u_2, u_4, u_5\} \cap I'| = 1$. If $u_1 \in I'$ or $u_4 \in I'$, let $I = I' \cup \{x_2, x_5\}$ and if $u_2 \in I'$ or $u_5 \in I'$, let $I = I' \cup \{x_1, x_4\}$. Suppose that $|\{u_1, u_2, u_4, u_5\} \cap I'| = 2$. If $u_1, u_2 \in I'$, let $I = I' \cup \{x_4, u_5\}$ provided $N(u_5) \cap I' = \emptyset$ and $I = I' \cup \{x_4\}$ otherwise. If $u_4, u_5 \in I'$, let $I = I' \cup \{x_2, u_1\}$ provided $N(u_1) \cap I' = \emptyset$ and $I = I' \cup \{x_2\}$ otherwise. If $u_1, u_4 \in I'$, let $I = I' \cup \{x_2, x_5\}$ and if $u_2, u_5 \in I'$, let $I = I' \cup \{x_1, x_4\}$.

Now, suppose that $u_2, u_4 \in V(H)$ and $u_3, u_5 \in V(H')$. Let $u_3', u_3'' \in N(u_3) \setminus \{x_3\}$ and $Y' = \{u_3, u_4, x_3, x_4, x_5\}$. Then, $G' = G - Y' + x_2u_3'$ is connected. Now, we must show that $i(g) \leq i(G') + 2$. Then, if $x_2, u_3' \notin I'$, let $I = I' \cup \{u_3, x_4\}$ when $u_3'' \notin I'$ and $I = I' \cup \{x_4\}$ otherwise. Now, suppose that $x_2 \in I'$. If $u_3'' \notin I'$, let $I = I' \cup \{u_3, x_4\}$. If $u_3'' \in I'$ we can let $I = I' \cup \{u_3', x_4\}$ when $N(u_3') \cap I' = \emptyset$ and $I = I' \cup \{x_4\}$ otherwise. If $u_3' \in I'$, let $I = I' \cup \{x_2, x_4\}$ when $N(x_2) \cap I' = \emptyset$ and $I = I' \cup \{x_4\}$ otherwise.

Case 3: $g(B) \geq 6$.

Let $U = N(X) \setminus X$ where $u_i \in N(x_i) \setminus X$ and $u_i', u_i'' \in N(u_i) \setminus \{x_i\}$. Then, from our previous arguments, we know that $u_i, u_i', u_i'' \in S_3$. Suppose that $g(B) \geq k \geq 6$ and let X be the shortest cycle of B with $X = \{x_i : i \leq k\}$ and $x_ix_{i+1} \in E(B)$ for $i \in [k]$. Also note that $u_i \neq u_{i+1}$, $u_i \neq u_{i+2}$, $u_i \neq u_{i+3}$ and $u_iu_{i+1}, u_iu_{i+2} \notin E(G)$ since $g(B) \geq 6$. Additionally, since B can have at most one cut vertex, we may assume that x_i and u_i are not cut vertices for $1 \leq i \leq 5$ (we can always relabel the index to guarantee this condition). Next, let $R = N[x_3]$.

Case 3.1: $\omega(G - R) = 1$.

Then, apply Fact 4.

Case 3.2: $\omega(G - R) = 2$.

Let $G - R$ have components H and H' with $Y = R \cup x_1$ and $Y' = R \cup x_5$. Then, we may assume (without loss of generality) that $u_2, u_3' \in V(H)$ and $u_3'', u_4 \in V(H')$ where there is either a (u_3'', X)-path or a (u_4, X)-path in $G - R$. If $\omega(G - Y) = 1$, apply Fact 2. Suppose that $\omega(G - Y) = 2$. If u_1 is not in the same component of $G - Y$ as u_3'' and u_4, then $G' = G - Y + u_3''u_1$ is connected. If u_1 is in the same component of $G - Y$ as u_3'' and u_4, then at least one of $G' = G - Y' + u_3''u_5$ or $G' = G - Y' + u_3''x_6$ is connected. Applying Fact 2 to shows the result holds in both cases.

Case 3.3: $\omega(G-R)=3$.

Let $G-R$ have components H, H', and H'' where $x_1, x_5 \in V(H'')$. We may assume (again without loss of generality) that $u_2, u_3' \in V(H)$ and $u_3'', u_4 \in V(H')$. Next, let $R' = N[x_4]$. Then, since $u_4 \in V(H')$, we know that $u_4', u_4'' \in V(H')$. Hence $\omega(G-R')=1$. Applying Fact 4 shows the result holds. This completes Case A.

Case B: $|V(B)| \leq 5$.

First, define the endblock B_1 to have the minimum number of vertices. Then, $3 \leq |V(B_1)| \leq 5$ and we know that G has at least two endblocks since B has a cut vertex. Then, $3 \leq g(B) \leq 5$ for every endblock of G since $|V(B)| \leq 5$. Additionally, note that even though an endblock of order $|V(B)|=4$ might contain a path of length two we cannot use Lemma 6 for this case because Lemma 6 requires $|V(B)| \geq 6$. This condition was necessary because the surgery $G' = G - V(B_1)$ might fail in certain cases (when there is a cut vertex and when we try to add edges).

Case 1: $|V(B_1)| = g(B_1) = 3$.

Let $V(B_1) = \{v_1, v_2, v_3\}$ where $v_3 \in S_3$ is a cut vertex and $d(v_1) = d(v_2) = 2$ (By Lemma 1). Suppose that $v_3v_4 \in E(G)$ and $v_4 \in S_3$. Then, let $G' = G - V(B_1)$ and $I = I' \cup \{v_2\}$. Now, suppose that $v_4 \in S_2$ and $v_3v_4, v_4v_5 \in E(G)$ where $v_5 \in S_3$. Then, let $G' = G - (V(B_1) \cup \{v_4\})$ and $I = I' \cup \{v_3\}$. Next, suppose that $v_4, v_5 \in S_2$ and $v_3v_4, v_4v_5, v_5v_6 \in E(G)$ where $v_6 \in S_3$. Then, let $G' = G - (V(B_1) \cup \{v_4, v_5\})$ and $I = I' \cup \{v_4, v_2\}$. Finally, suppose that $v_4, v_5, v_6 \in S_2$ and $v_3v_4, v_4v_5, v_5v_6, v_6v_7 \in E(G)$ where $v_7 \in S_3$. Then, let $G' = G - \{v_4, v_5, v_6\} + v_3v_7$ and apply Fact 6. This completes Case 1.

Case 2: $|V(B_1)| = 4$ and $g(B) = 4$ for each endblock B.

Let $V(B_1) = \{v_1, v_2, v_3, v_4\}$ where $v_4 \in S_3$ is a cut vertex. By Lemma 1, we know that $v_1v_3 \notin E(G)$ and this holds for each of the cases below so we may assume that $V(B_1) \setminus \{v_4\} \subseteq S_2$. Next, let $v_4v_5 \in E(G)$. Then, v_4 is a cut vertex of G so $\omega_B(G) \geq 2$ (i.e. there are at least two endblocks). There are two major cases to consider. Either $v_5 \in S_2$ or $v_5 \in S_3$. Whenever $v_5 \in S_2$ we just remove the endblock and any neighboring degree two vertices just as in Case 1. However, if $v_5 \in S_3$ we must consider several cases.

Case 2.1: $v_5 \in S_2$.

Suppose that $v_5 \in S_2$ and $v_4v_5, v_5v_6 \in E(G)$ where $v_6 \in S_3$. Then, let $G' = G - (V(B_1) \cup \{v_5\})$ and $I = I' \cup \{v_2, v_4\}$. Next, suppose that $v_5, v_6 \in S_2$ and $v_4v_5, v_5v_6, v_6v_7 \in E(G)$ where $v_7 \in S_3$. Then, let $G' = G - (V(B_1) \cup \{v_5, v_6\})$ and $I = I' \cup \{v_2, v_5\}$. Finally, suppose that $v_5, v_6, v_7 \in S_2$ and $v_4v_5, v_5v_6, v_6v_7, v_7v_8 \in E(G)$ where $v_8 \in S_3$. Then, let $G' = G - \{v_5, v_6, v_7\} + v_4v_8$ and apply Fact 6. This completes Case 2.1.

Case 2.2: $v_5 \in S_3$.

Recall, that there are at least two endblocks. There are two possibilities. Either every endblock has order 4 or every endblock satisfies $4 \leq |V(B)| \leq 5$.

Case 2.2.1: There is an endblock $B_2 \neq B_1$ such that $|V(B_2)| = 4$.

Let $V(B_1) = \{v_1, v_2, v_3, v_4\}$ where v_4 is a cut vertex and $V(B_2) = \{u_1, u_2, u_3, u_4\}$ where u_4 is a cut vertex. Next, let $v_1, v_2, v_3, u_1, u_2, u_3 \in S_2$ and suppose that $v_4v_5, u_4u_5 \in E(G)$. We may assume that both $u_5, v_5 \in S_3$ since otherwise we just repeat the argument of Case 2.1. Then, let $G' = G - (V(B_2) \cup \{v_1, v_2, v_3\}) + v_4u_5$. Then, if $v_4, u_5 \notin I'$ or $v_4 \in I'$, let $I = I' \cup \{v_2, u_2, u_4\}$. If $u_5 \in I'$, let $I = I' \cup \{u_2, v_1, v_3\}$. Now, we may assume that there is only one endblock of order 4 (i.e. B_1) and all the other endblocks have order 5. However, every endblock satisfies $g(B) = 4$.

Case 2.2.2: For every endblock $B_2 \neq B_1$, $|V(B_2)| = 5$.

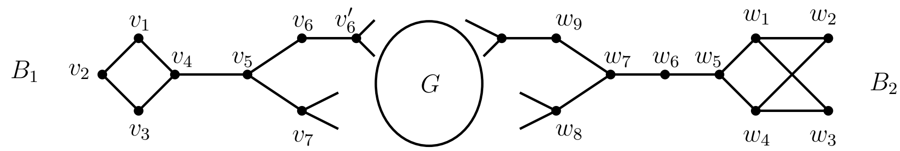

Figure 2.19: G has at least two endblocks B_1 and B_2

Let $V(B_2) = \{w_1, w_2, w_3, w_4, w_5\}$ where $w_5 \in S_3$ is the cut vertex. Suppose that $w_5w_6 \in E(G)$. Next, because $g(B) = 4$ for every endblock of G, we must have $w_1w_2, w_1w_3, w_1w_5, w_2w_4, w_3w_4, w_4w_5 \in E(B_2)$ and $w_1w_4, w_2w_3 \notin E(B_2)$ (see Figure 2.19).

Now, suppose that $w_5w_6 \in E(G)$ and $w_6 \in S_3$. Then, let $G' = G - V(B_2)$ and $I = I' \cup \{w_1, w_4\}$. We postpone the case $w_5w_6, w_6w_7 \in E(G)$ with $w_6 \in S_2$ and $w_7 \in S_3$ until last (because w_7 might be a cut vertex). Now, suppose that $w_5w_6, w_6w_7, w_7w_8 \in E(G)$ with $w_6, w_7 \in S_2$ and $w_8 \in S_3$. Then let $G' = G - (V(B_2) \cup \{w_6, w_7\})$ and $I = I' \cup \{w_1, w_4, w_6\}$. Next, suppose that $w_5w_6, w_6w_7, w_7w_8, w_8w_9 \in E(G)$ with $w_6, w_7, w_8 \in S_2$ and $w_9 \in S_3$. Then, let $G' = G - \{w_6, w_7, w_8\} + w_5w_9$ and apply Fact 6.

Next, v_5 might be a cut vertex and w_7 might be a cut vertex so we must consider two separate cases. Now, suppose that $w_5w_6, w_6w_7 \in E(G)$ with $w_7 \in S_3$ (Figure 2.19). Next, let $w_8, w_9 \in N(w_7) \setminus \{w_6\}$ and $l = \omega_B(G) - 2$. Then, we label the endblocks as follows. For each endblock $B \cong B_2$ we label the endblock B_2^i and let $w_k^i \in (V(B_2^i) \cup \{w_6^i, w_7^i, w_8^i, w_9^i\}) \leftrightarrow w_k \in (V(B_2) \cup \{w_6, w_7, w_8, w_9\})$ and $w_k^i w_j^i \in E(B_2^i) \cup \{w_5^i w_6^i, w_6^i w_7^i, w_7^i w_8^i, w_7^i w_9^i\} \leftrightarrow w_k w_j \in E(B_2) \cup \{w_5w_6, w_6w_7, w_7w_8, w_7w_9\}$ where $k, j \le 9$ and $i \le l$. Then, we can let $U = \{v_5, w_7\} \cup \{w_7^i : i \le l\}$.

<u>Case 2.2.2.1:</u> For any $u \in U$, $\omega(G - u) = 2$.

Let $v_6, v_7 \in N(v_5) \setminus \{v_4\}$. If $v_6, v_7 \in S_3$, let $G' = G - (V(B_1) \cup \{v_5\})$ and $I = I' \cup \{v_2, v_4\}$. Since $\omega(G - v_5) = 2$, G' is connected.

Now, suppose that $v_6 \in S_2$ and $v_7 \in S_3$. First, suppose that $v_6'', v_7 \in S_3$ and $v_6, v_6', v_7 \in S_2$. Then, let $G = G' - (V(B_1) \cup \{v_5, v_6, v_6'\})$ and $I = I' \cup \{v_2, v_4, v_6\}$. Now, suppose that $v_6' \in S_3$ and let $v_7', v_7'' \in N(v_7) \setminus \{v_5\}$. Since $\omega(G - v_5) = 2$ there must be a (v_6, v_7)-path in $G - v_5$. Therefore, $\omega(G - v_7) \leq 2$. Let $G - v_7$ have components H and H'. Next, we may assume (without loss of generality) that $v_6', v_7' \in H$ and $v_7'' \in H'$. Thus, $\omega(G - (V(B_1) \cup \{v_5, v_6, v_7\}) = 2$ with $v_6', v_7' \in H$ and $v_7'' \in H'$. There are three further subcases to consider. Either $v_7', v_7'' \in S_3$, $v_7', v_7'' \in S_2$, or $|\{v_7', v_7''\} \cap S_2| = 1$. Next, we need the following claim.

Claim 2. *If $v_7' \in S_3$, $e = v_7 v_7'$, and there is a (v_6, v_7')-path in $G - v_5$, then e is not a critical edge of G. A similar result holds for v_7''.*

Proof. We only need to show one case. Suppose that $v_7' \in S_3$, $e = v_7 v_7'$, and there is a (v_6, v_7')-path in $G - v_5$. Now, by contradiction, suppose that e is a critical edge of G. Let $H = G - e$ and I_H be a MIDS of H. Then, since $\beta = 3$ and e lies on a cycle, $i(H) < i(G)$. If $v_7 \in I_H$ but $v_7' \notin I_H$, then $|I| \leq i(H)$ which is a contradiction. Thus, we know that $v_7, v_7' \in I_H$. Next, let $Z = N(v_7'') \setminus \{v_7\}$. There are two cases. First, suppose that $Z \cap I_H \neq \emptyset$. If $v_4 \in I_H$, let $I = I_H \setminus \{v_7\}$. If $v_4 \notin I_H$, then $v_1, v_3 \in I_H$. Thus, let $I = I_H \setminus \{v_1, v_3, v_7\} \cup \{v_2, v_4\}$. In each case, we get a contradiction.

Now, suppose that $Z \cap I_H = \emptyset$. If $v_4 \in I_H$, let $I = I_H \setminus \{v_7\} \cup \{v_7''\}$. If $v_4 \notin I_H$, then $v_1, v_3 \in I_H$. Then, let $I = I_H \setminus \{v_1, v_3, v_7\} \cup \{v_2, v_4, v_7''\}$. Again, we get a contradiction in each case. □

Now, by our claim, we may assume that $|\{v_7', v_7''\} \cap S_2| \geq 1$. Next, we may assume (without loss of generality) $d(v_7'') \leq d(v_7')$ where $v_7'' \in S_2$, $2 \leq d(v_7') \leq 3$, and where v_6' and v_7'' are in the same component of $G - v_7$ (since $\omega(G - v_7) = 2$). Then, let $G = G' - (V(B_1) \cup \{v_5, v_6, v_7, v_7''\}) + v_7'v_7'''$ where $v_7''' \in N(v_7'') \setminus \{v_7\}$. If $v_7', v_7''' \notin I'$ or $v_7' \in I'$, let $I = I' \cup \{v_2, v_5, v_7''\}$. If $v_7''' \in I'$, let $I = I' \cup \{v_2, v_5, v_7'\}$ when $N(v_7') \cap I' = \emptyset$ and $I = I' \cup \{v_2, v_5\}$ otherwise.

Suppose that $v_6, v_7 \in S_2$. Let $v_6' \in N(v_6) \setminus \{v_5\}$ and $v_7' \in N(v_7) \setminus \{v_5\}$. If $v_6', v_7' \in S_2$, then $N(\{v_6', v_7'\}) \setminus \{v_6, v_7\} \subseteq S_3$) by Fact 6. Let $v_6'' \in N(v_6') \setminus \{v_6\}$ and $v_7'' \in N(v_7') \setminus \{v_7\}$. In this case, we can let $G = G' - (V(B_1) \cup \{v_5, v_6, v_6', v_7, v_7'\}) + v_4v_6''$ and apply Fact 3. Now, suppose that $v_6'', v_7' \in S_3$ and $v_6, v_6', v_7 \in S_2$. Then, let $G = G' - (V(B_1) \cup \{v_5, v_6, v_6', v_7\})$ and $I = I' \cup \{v_2, v_5, v_6'\}$ when $v_6'' \notin I'$ and $I = I' \cup \{v_2, v_5\}$ otherwise. Finally, suppose that $v_6, v_7 \in S_2$ and $v_6', v_7' \in S_3$. Then, let $G = G' - (V(B_1) \cup \{v_5, v_6\})$ and $I = I' \cup \{v_2, v_5\}$. This completes Case 2.2.2.1.

Case 2.2.2.2: For any $u \in U$, $\omega(G - u) = 3$.

For this case, we can choose a longest path P (in G) joining two different vertices of U. Since there can be at most one B_1 at the end of P, there are two subcases that must be considered. Either some vertex in U is adjacent to B_1 ($v_4v_5 \in E(G)$ and $v_5 \in U$) or every endblock is isomorphic to B_2.

First, suppose that for some B, $B \cong B_1$. Then all the other endblocks are isomorphic to B_2. For simplicity, let us assume that $v_5 = w_7 \in S_3$. That is, B_1 and B_2 lie at the same end of P and all the other endblocks are isomorphic to B_2^i for some $i \leq l$. Then, we must have $w_7^i = w_7^j \in S_3$ for some $i, j \in [l]$. Thus, we may assume that $i = 1$ and $j = 2$. Next, let $y \in N(w_7) \setminus \{v_4, w_6\}$ and $y' \in N(w_7^1) \setminus \{w_6^1, w_6^2\}$. Then, if $yy'' \in E(G)$, let $G' = G - (V(B_1) \cup V(B_2) \cup V(B_2^1) \cup V(B_2^2)) \cup \{w_6, w_6^1, w_6^2, w_7, w_7^1\}$. If $y' \in I'$ or $y, y' \notin I'$, let $I = I' \cup \{v_2, w_1, w_4, w_7, w_1^1, w_4^1, w_6^1, w_1^2, w_4^2, w_6^2\}$. If $y \in I'$, let $I = I' \cup \{v_2, v_4, w_1, w_4, w_6, w_1^1, w_4^1, w_7^1, w_1^2, w_4^2\}$. Now, suppose that $yy' \notin E(G)$.

Then, let $G' = G - (V(B_1) \cup V(B_2) \cup V(B_2^1) \cup V(B_2^2) \cup \{w_6, w_6^1, w_6^2, w_7, w_7^1\}) + yy'$ and repeat the surgery above.

Now, suppose that every endblock is isomorphic to B_2. Then, label as above but replace B_1 with $B_2^3 + w_6$ where $w_7 = w_7^3$. If $yy' \in E(G)$, let $G' = G - (V(B_2) \cup V(B_2^1) \cup V(B_2^2) \cup V(B_2^3)) \cup \{w_6, w_6^1, w_6^2, w_6^3, w_7, w_7^1\}$. If $y, y' \notin I'$ or $y' \in I'$, let $I = I' \cup \{w_1, w_4, w_7, w_1^1, w_4^1, w_6^1, w_1^2, w_4^2, w_6^2, w_1^3, w_4^3\}$. If $y \in I'$, we can let $I = I' \cup \{w_1, w_4, w_6, w_1^1, w_4^1, w_7^1, w_1^2, w_4^2, w_1^3, w_4^3, w_6^3\}$. If $yy' \notin E(G)$, let $G' = G - (V(B_2) \cup V(B_2^1) \cup V(B_2^2) \cup V(B_2^3) \cup \{w_6, w_6^1, w_6^2, w_6^3, w_7, w_7^1\}) + yy'$ and repeat the surgery above. This completes Case 2.

<u>Case 3:</u> $|V(B_1)| = g(B_1) = 5$.

We proceed as we did in Case 1 by removing the endblock B_1 and the neighboring degree two vertices for each of the cases below. Let $V(B_1) = \{v_1, v_2, v_3, v_4, v_5\}$. Since B_1 is an endblock and $n \geq 8$, we may assume that $v_5 \in S_3$, $v_1, v_2, v_3, v_4 \in S_2$, $v_1v_2, v_2v_3, v_3v_4, v_4v_5 \in E(B_1)$, and $v_1v_3, v_1v_4, v_2v_4 \notin E(B_1)$. Suppose that $v_5v_6 \in E(G)$ and $v_6 \in S_3$. Then, let $G' = G - V(B_1)$ and $I = I' \cup \{v_1, v_4\}$. Now, suppose that $v_6 \in S_2$ and $v_5v_6, v_6v_7 \in E(G)$ where $v_7 \in S_3$. Then, let $G' = G - (V(B_1) \cup \{v_6\})$ and $I = I' \cup \{v_2, v_5\}$. Next, suppose that $v_6, v_7 \in S_2$ and $v_5v_6, v_6v_7, v_7v_8 \in E(G)$ where $v_8 \in S_3$. Then, let $G' = G - (V(B_1) \cup \{v_6, v_7\})$ and $I = I' \cup \{v_1, v_4, v_6\}$. Finally, suppose that $v_6, v_7, v_8 \in S_2$ and $v_5v_6, v_6v_7, v_7v_8, v_8v_9 \in E(G)$ where $v_9 \in S_3$. Then, let $G' = G - \{v_6, v_7, v_8\} + v_5v_9$ and apply Fact 6. This completes Theorem 4.

$\square$

CHAPTER 3

INDEPENDENT BONDAGE

3.1 Overview

The second topic focuses on a new idea called the independent bondage number which is a variation of the bondage number of a graph. In the following section, we will introduce the concept of bondage and then extend this idea to independent bondage. In this way, we can formulate new questions about independent bondage that mirror the development of the original bondage concept. This provides a variety of interesting problems to examine. Independent bondage is a relatively new concept and not much work has been done on this topic.

3.2 Bondage

The concept of bondage is somewhat convoluted since two groups of researchers published the same result about bondage in trees independently. Bauer et al. [1] published a result on trees and in this paper referred to vertices as points, edges as lines, and bondage as domination line stability. Later Fink et al. [9] published the same theorem for trees as well as results for bondage of paths, cycles, trees, complete graphs, and complete r-partite graphs. See [1] and [9] for more details and [16] for an earlier survey of results of bondage and reinforcement.

Recall the following definitions. The **bondage number** of a nonempty graph G, denoted $b(G)$, is the minimum cardinality among all sets of edges $B \subseteq E(G)$ for which $\gamma(G-B) > \gamma(G)$. Stated differently, the bondage number of a graph G is the minimum number of edges whose removal renders every minimum dominating set of G a non-dominating set in the resulting spanning subgraph $G-B$. The **independent bondage number** of a nonempty graph G, denoted $b_i(G)$, is the minimum cardinality among all sets of edges $B \subseteq E$ for which $i(G-B) > i(G)$.

If we know that some vertex v is in every dominating set, then we can remove all the edges incident with v and the bondage number is easily found. However, most graphs do not satisfy this property and the challenge becomes how to show that a certain edge set renders every dominating set a non-dominating set.

The first published result on bondage is due to Bauer et al. [1] and also independently by Fink et al. [9].

Theorem 6. *If T is a tree with at least two vertices, then*

$$b(G) \leq 2.$$

The following fact is due to Fink et. al [9]. However, the result is not true for every minimum independent dominating set of a graph.

Observation 1. *If $v \in V(G)$ is adjacent to more than one end-vertex, then v must be in every minimum dominating set of G.*

As mentioned above, Fink et al. [9] published the first results (other than for a tree) on bondage which included paths, cycles, trees, complete graphs, and complete k-partite graphs.

Theorem 7. *The bondage number of the complete graph K_n $(n \geq 2)$ is*

$$b(K_n) = \lceil n/2 \rceil.$$

Theorem 8. *The bondage number of a cycle C_n is*

$$b(C_n) = \begin{cases} 3 & \textit{if } n \equiv 1 \pmod 3, \\ 2 & \textit{otherwise.} \end{cases}$$

Theorem 9. *The bondage number of a path P_n ($n \geq 2$) is given by*

$$b(P_n) = \begin{cases} 2 & \textit{if } n \equiv 1 \pmod 3, \\ 1 & \textit{otherwise.} \end{cases}$$

The following result provides an upper bound for the bondage number in terms of the degree of a graph and can be used to give a relationship between the bondage number and the minimum and maximum degree of a graph [1], [9].

Theorem 10. *If G is a nonempty graph, then*

$$b(G) \leq \min\{d(u) + d(v) - 1 : uv \in E(G)\}.$$

As a corollary to this result, if we choose an edge incident with a minimum degree vertex we have the following.

Corollary 1. *If G is a nonempty connected graph, then*

$$b(G) \leq \Delta(G) + \delta(G) - 1.$$

Fink et. al also gave a result for multi-partite graphs.

Theorem 11. *The bondage number of a t-partite graph $G = K(n_1, n_2, \ldots, n_t)$ where $n_1 \leq n_2 \leq \ldots \leq n_k$ is given by*

$$b(G) = \begin{cases} \lceil m/2 \rceil & \textit{if } n_m = 1 \textit{ and } n_{m+1} \geq 2, \textit{ for some } m, 1 \leq m \leq t, \\ 2t-1 & \textit{if } n_1 = n_2 = \cdots = n_t = 2, \\ \sum_{i=1}^{t-1} n_i & \textit{otherwise.} \end{cases}$$

3.3 Independent Bondage

Before we present our results, we would like to point out that unlike bondage it is possible for the independent domination number to decrease after removing an edge. Consider the complete bipartite graph $K_{r,r}$ on $n = 2r$ vertices. Then, $i(K_{r,r}) = n/2 = r$ but remove any edge and $i(K_{r,r} - e) = 2$. This is not the case with domination since the removal of edges never decreases the domination number. Of course the independent domination number can increase as well. Consider the case when G is a star $K_{1,r}$. Here, $i(G) = 1$ but for any edge e, we have $i(G - e) = 2$. In fact, since removal of each edge isolates a vertex that is not in the MIDS of the original graph, i increases each time we remove one or more edges.

The following result is in terms of degree and provides many useful relationships. In particular, it gives a bound for the independent bondage number in terms of the minimum and maximum degree of a graph.

Theorem 12. *If G is a nonempty graph and $uv \in E(G)$, then*

$$b_i(G) \leq \min\{d(u) + d(v) - 1 - |N(u) \cap N(v)| : uv \in E(G)\}.$$

Proof. For any two adjacent vertices u and v, let $E_u = \{ux : x \in N(u)\}$, $E_v = \{vy : y \in N(v)\}$ and $E_{S_u} = \{ux : x \in N(u) \backslash N[v]\}$. Consider a spanning subgraph $G' = G - E_{S_u} - E_v$, which is obtained by removing edges of $E_{S_u} \cup E_v$ from G, where

$V(G') = V(G)$. We claim that $i(G') > i(G)$. Suppose not. Then, $i(G') \leq i(G)$. Let I' and I be minimum independent dominating sets of G' and G respectively. By the structure of the spanning subgraph G', $v \in I'$ because v is an isolated vertex in G'. Next, there is at least one common vertex between I' and $[N(u) \cap N(v)] \cup \{u\} = Z$, that is, $I' \cap Z \neq \emptyset$. Thus, $I' - \{v\}$ is a MIDS of G and then $|I' - \{v\}| < |I'|$, which contradicts with $i(G') \leq i(G)$. Thus, we have found a spanning subgraph $G' = G - E_{S_u} - E_v$ obtained by removing s edges from G such that $i(G') > i(G)$, where $s = |E_{S_u} \cup E_v| = d(u) + d(u) - |N(u) \cap N(v)| - 1$ and this holds for any edge $uv \in E(G)$. Therefore, by the definition of the independent bondage number, $b_i(G) \leq s = d(u) + d(u) - |N(u) \cap N(v)| - 1$ for any edge $uv \in E(G)$. This completes the proof. □

Using the Theorem 12, we can establish several useful results. For the following result, we need only choose a vertex of maximum degree.

Corollary 2. *If G is a nonempty graph and $uv \in E(G)$, then*

$$b_i(G) \leq 2\Delta(G) - |N(u) \cap N(v)| - 1.$$

If $N(u) \cap N(v) = \emptyset$ we get the following result.

Corollary 3. *If G is a nonempty graph and $uv \in E(G)$, then*

$$b_i(G) \leq d(u) + d(v) - 1.$$

By the above Corollary, if we choose a vertex of minimum degree, the result follows immediately.

Corollary 4. *If G is a nonempty graph, then*

$$b_i(G) \leq \Delta(G) + \delta(G) - 1.$$

If we choose a vertex of minimum degree and apply the corollary above we get the following upper bound for any path of order n in terms of minimum and maximum degree.

Corollary 5. *The independent bondage number of a path P_n ($n \geq 2$) satisfies*

$$b_i(P_n) \leq 2.$$

Similarly we have an upper bound for a cycle of order n.

Corollary 6. *The independent bondage number of a cycle C_n ($n \geq 3$) satisfies*

$$b_i(C_n) \leq 3.$$

Additionally, since $i(P_n) = i(C_n) = \lceil n/3 \rceil$, we must remove more than one edge from every independent dominating set in order to increase the independent domination number of a cycle and so have the following.

Corollary 7. *The independent bondage number of a cycle C_n ($n \geq 3$) satisfies*

$$2 \leq b_i(C_n) \leq 3.$$

Now, we are ready to find the exact value of b_i for a cycle.

Theorem 13. *The independent bondage number of a cycle C_n of order $n \geq 3$ is* *given by*

$$b_i(C_n) = \begin{cases} 3 & \textit{if } n \equiv 1 \pmod 3, \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. Since $i(C_n) = i(P_n) = \lceil \frac{n}{3} \rceil$ for $n \geq 3$, we must remove at least two edges from C_n to increase the independent domination number. Hence, $b_i(C_n) \geq 2$. If $n \equiv 1 (\text{mod } 3)$, the removal of two edges from C_n leaves a graph H consisting of two paths P and Q. If P has order n_1 and Q has order n_2, then either $n_1 \equiv n_2 \equiv 2 \pmod 3$, or, without loss of generality, $n_1 \equiv 0 \pmod 3$ and $n_2 \equiv 1 \pmod 3$. In the former case, $i(H) = i(P) + i(Q) = \lceil \frac{n_1}{3} \rceil + \lceil \frac{n_2}{3} \rceil = \frac{n_1+1}{3} + \frac{n_2+1}{3} = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil = i(C_n)$. In the latter case, $i(H) = \frac{n_1}{3} + \frac{n_2+2}{3} = \frac{n+2}{3} = \lceil \frac{n}{3} \rceil = i(C_n)$. In either case, when $n \equiv 1 \pmod 3$, we have $b_i(C_n) \geq 3$. To obtain the upper bounds that, by trichotomy, will yield the desired equalities of our theorem's statement, we must consider two cases as follows.

Case 1. Suppose that $n \equiv 0, 2 \pmod 3$. The graph H obtained by removing two adjacent edges from C_n consists of an isolated vertex and a path of order $n - 1$. Thus, $i(H) = 1 + i(P_{n-1}) = 1 + \lceil \frac{n-1}{3} \rceil = 1 + \lceil \frac{n}{3} \rceil = 1 + i(C_n)$ and so $b_i(C_n) \geq 2$ in this case. Combining this with the upper bound obtained earlier, we have shown that $b_i(C_n) = 2$ whenever $n \equiv 0, 2 \pmod 3$.

Case 2. Suppose that $n \equiv 1 \pmod 3$. The graph H resulting from the deletion of three consecutive edges of C_n consists of two isolated vertices and a path of order $n - 2$. Thus, $i(H) = 2 + \lceil \frac{n-2}{3} \rceil = 2 + \lceil \frac{n-1}{3} \rceil = 2 + (\lceil \frac{n}{3} - 1) = 1 + i(C_n)$. So that $b_i(C_n) \leq 3$ in this case. With the earlier inequality we conclude that $b_i(C_n) = 3$ whenever $n \equiv 1 \pmod 3$. □

The exact value of b_i for a path follows from above.

Corollary 8. *The independent bondage number of a path P_n $(n \geq 2)$ is given by*

$$b_i(P_n) = \begin{cases} 2 & \textit{if } n \equiv 1 \pmod 3, \\ 1 & \textit{otherwise.} \end{cases}$$

For the complete graph we need to remove $n/2$ independent edges (a perfect matching) when n is even to increase the independent domination number and $(n-1)/2$ edges plus one additional edge (from the last vertex having degree $n-1$) when n is odd (a maximum matching). This is required because any graph with at least one vertex of degree $n-1$ has independent domination number $i(G) = 1$. This observation leads to the following result.

Theorem 14. *The independent bondage number of the complete graph K_n having order n is given by*

$$b_i(K_n) = \lceil n/2 \rceil.$$

Proof. If H is a spanning subgraph of K_n that is obtained by removing fewer than $\lceil \frac{n}{2} \rceil$ edges (possibly none) from K_n, then H contains a vertex of degree $n-1$. Hence $i(H) = 1$. Thus, $b_i(K_n) \geq \lceil \frac{n}{2} \rceil$.

If n is even, the removal of $n/2$ independent edges from K_n reduces the degree of each vertex to $n-2$ and therefore yields a graph H with domination number $i(H) = 2$, but $i(K_n) = 1$.

If n is odd, the removal of $(n-1)/2$ independent edges from K_n leaves a graph having exactly one vertex of degree $n-1$. If we remove one edge incident with this vertex, we obtain a graph H with $i(H) = 2$.

In both cases (n is even or n is odd), the graph H is obtained by the removal of $\lceil \frac{n}{2} \rceil$ edges from K_n. Thus, $b_i(K_n) = \lceil \frac{n}{2} \rceil$. □

Let T be a tree. A vertex $v \in V(T)$ is a **leaf** or end-vertex provided $d_T(v) = 1$

and the edge adjacent with v is called a **leaf edge**. A vertex which is adjacent to a leaf is called a **support vertex** and we let $S_p = \{v : N(v) \cap S_1 \neq \emptyset\}$ and recall that $S_1 = \{v : d(v) = 1\}$.

Theorem 15. *If T is a nontrivial tree, then*

$$b_i(T) \leq 2.$$

Proof. If $|V(T)| = 2$ then $b_i(T) = 1$. Assume $|V(T)| \geq 3$ and let $x_0x_1 \cdots x_k$ be a longest path in T. Let $e_1 = x_0x_1$. Then, $d_T(x_0) = d_T(x_k) = 1$ and $k \geq 2$.

Case 1. $d_T(x_1) = 2$. Then $N(x_0) \cap N(x_1) = \emptyset$. Then, choosing x_0x_1 and applying Theorem 12 gives $b_i(T) \leq d(x_0) + d(x_1) - |N(x_0) \cap N(x_0)| - 1 = 1 + 2 - 0 - 1 = 2$. This completes Case 1.

Now, we may assume that $S_p \cap S_2 = \emptyset$.

Case 2. $d_T(x_1) > 2$. Then we consider the following two cases.

Case 2.1. $d_T(x_2) = 1$. Then, since $d_T(x_1) > 2$ and $x_0x_1x_2$ is a longest path, every longest path has length 2. Thus, T is a star. If we let $B = x_0x_1$, then $i(T - B) = 2 > i(T) = 1$. Thus, $b_i(T) = 1 \leq 2$.

Case 2.2. $d_T(x_2) = 2$. Let $e_2 = x_1x_2$ and $e_3 = x_2x_3$. Now, consider the spanning subgraph T' obtained by removing e_2 and e_3 from T. Let I and I' be MIDS of T and T' respectively. By the structure of the spanning subgraph T', $x_2 \in I'$ because $d_{T'}(x_2) = 0$.

First, we claim that $i(T') > i(T)$ or equivalently that $|I'| > |I|$. Suppose not. Then $|I'| \leq |I|$. Next, we can show that $x_1 \notin I'$. If not, then $I' - \{x_0\}$ is a MIDS of T. But this implies that $|I| = |I' - \{x_0\}| < |I'|$ which is a contradiction. Thus,

$x_1 \notin I'$ and $N(x_1) \subset I'$. Next, $[I' \setminus (N(x_1) \cup \{x_2\})] \cup \{x_1\}$ must be an independent dominating set of T. Then, $|I| \leq |[I' \setminus (N(x_1) \cup \{x_2\})] \cup \{x_1\}| < |I'|$ which contradicts the assumption $|I'| \leq |I|$. Thus, $i(T') > i(T)$ and $B = \{e_1, e_2\}$. Therefore, $b_i(T) \leq 2$.

Case 2.3. $d_T(x_2) \geq 3$.

Case 2.3.1. $x_2 \in S_p$. Then there is a leaf, denoted x_2', such that $e_2 = x_2 x_2' \in E(T)$. Next, consider a spanning subgraph T' which is obtained by removing e_1 and e_2 from T. Let I and I' be MIDS of T and T' respectively. By the structure of the spanning subgraph T', $x_0, x_2' \in I'$ because they are isolated vertices in T'.

Again, we claim that $i(T') > i(T)$ or equivalently that $|I'| > |I|$. Suppose not. Then $|I'| \leq |I|$. Next, we can show that $x_1 \notin I'$. If not, then $I' - \{x_0\}$ is a MIDS T. But this implies that $|I| = |I' - \{x_0\}| < |I'|$ which is a contradiction. Similarly, $x_2 \notin I'$. Then $N(x_1) \subset I'$ because $x_1 \notin I'$. Next, the subset $[I' - N(x_1)] \cup \{x_1\}$ must be an independent dominating set of T. Then, $|I| \leq |[I' - N(x_1)] \cup \{x_1\}| < |I'| - 1 < |I'|$ which contradicts the assumption $|I'| \leq |I|$. Thus, $i(T') > i(T)$ and $B = \{e_1, e_2\}$. Therefore, $b_i(T) \leq 2$.

Case 2.3.2. $x_2 \notin S_p$. Let $A = N(x_2) \setminus \{x_1, x_3\} = \{v_2, v_3, \ldots v_r\}$ where $r = d(x_2) - 2$. Then, $A \cap S_p \neq \emptyset$ since otherwise there would be a longer path in T contrary to our choice of P. Then, $|A \cap S_p| \geq r \geq 1$. Next, consider v_i for $2 \leq i \leq r$. If $d_T(v_i) = 2$, we can use Theorem 12 to show that $b_i(T) \leq 2$ so we may assume that $d_T(v_i) \geq 3$ and there are at least two leaves v_i' and v_i'' adjacent to v_i for each i $(2 \leq i \leq r)$. Let $e_2' = v_2 v_2'$ and consider a spanning subgraph T' which is obtained by removing e_1 and e_2' from T. Again, let I and I' be MIDS of T and T'. As in the previous cases, we claim that $i(T') > i(T)$ or equivalently that $|I'| > |I|$. Suppose not. Then $|I'| \leq |I|$. From the previous two cases, we know that $x_0, v_2' \in I'$ and $x_1, v_2 \notin I'$. Now, let $Z_i = N(v_i) \setminus \{x_2\}$ for $2 \leq i \leq r$. Then $Z_1 = N(x_1) \setminus \{x_2\} \subseteq I'$ and

$Z_2 \subseteq I'$ (see the previous case). Additionally, we have already shown that $|Z_i| \geq 2$ for $2 \leq i \leq r$ and since $d(x_1) \geq 3$ we know that $|Z_1| \geq 2$. Then, either $x_2 \notin I'$ or $x_2 \in I'$. If $x_2 \notin I'$, then we can let $I = [I' \setminus (Z_1 \cup Z_2)] \cup \{x_1, v_2\}$. Again, I is an independent dominating set and $|I'| > |I|$ which is a contradiction. Now, suppose that $x_2 \in I'$. Then, there are two further cases. If $[N(x_3) \setminus \{x_2\}] \cap I' \neq \emptyset$, then we can let $I = [I' \setminus (\bigcup_{i=1}^{r} Z_i \cup \{x_2\})] \cup \{x_1\} \cup A$. Then, I is an independent dominating set and $|I'| > |I|$ which is a contradiction. If $[N(x_3) \setminus \{x_2\}] \cap I' = \emptyset$, then we can let $I = [I' \setminus (\bigcup_{i=1}^{r} Z_i \cup \{x_2\})] \cup \{x_1, x_3\} \cup A$. Then, I is an independent dominating set and $|I'| > |I|$ which is a contradiction. This completes Theorem 15.

□

Next, we need the following result involving equivalent definitions of dominating sets of a graph.

Observation 2. *The following are equivalent definitions of a dominating set $X \subseteq V$ of any graph G.*

$$(\forall y \in V \setminus X)(\exists x \in X)[xy \in E(G)] \iff N[X] = V$$

Proof. Suppose that $N[X] \neq V$. Then, there is some $y \in V$ such that $y \notin N[X]$. Hence, there is some $y \in V \setminus X$ such that for every $x \in X$ we have $xy \notin E$. In each step, we have equivalent statements. Finally, since the statement is equivalent to it's contrapositive we are done. □

Lemma 11. *Let G be a complete t-partite graph where $V = \bigcup_{i=1}^{t} A_i$ and suppose that $|A_1| = \min\{|A_i| : 1 \leq i \leq t\}$. Then $E' \subseteq E(G)$ is a subset with the minimum size required to render A_1 a non-dominating set in $G' = G - E'$ if and only if there is some $v \in V \setminus A_1$ such that $E' = E[v, A_1] = \{vu : u \in A_1\}$.*

Proof. Suppose that $E' \subseteq E(G)$ is an edge set having the minimum number of edges required to render A_1 non-dominating in G'. Then, there is some $v \in V \setminus A_1$ such that $v \notin N_{G'}[A_1]$ so $E' \supseteq E[v, A_1]$. Since E' is the minimum set of edges for which this can occur we must have $E' = E[v, A_1]$. Now suppose that there is some $v \in V \setminus A_1$ such that $E' = E[v, A_1]$. Then, $v \notin N_{G'}[A_1]$ so A_1 is not dominating in G'. □

Lemma 12. *Let G be a complete t-partite graph where $V = \bigcup_{i=1}^{t} A_i$ and suppose that $|A_i| = |A_j| = n_1$ where A_i and A_j are MIDS. Then $E' \subseteq E(G)$ is a subset with the minimum size required to render A_i and A_j non-dominating sets in $G' = G - E'$ if and only if $E' = E[v_i, A_j] \cup E[v_j, A_i]$ for some $v_i \in A_i$ and $v_j \in A_j$.*

Proof. ($\Longrightarrow$) Since A_i is not dominating in G' by Lemma 11 there is some $v \in V \setminus A_i$ such that $E'_i = E[v, A_i]$. Similarly, there is some $u \in V \setminus A_j$ such that $E'_j = E[u, A_j]$. Let $E' = E'_i \cup E'_j$. If $v \in A_i$ and $u \in A_j$, then $|E'_i \cap E'_j| = 1$. Since $|E'|$ is minimum size required to render A_i and A_j non-dominating in G' we must have $|E'| = 2n_1 - 1$. If $v \notin A_i$ or $u \notin A_j$, then $|E'_i \cap E'_j| = 0$ and $|E'| = |E'_i| + |E'_j| = 2n_1 - 1$.
($\Longleftarrow$) For the other direction if $E' = E[v_i, A_j] \cup E[v_j, A_i]$ where $v_i \in A_i$ and $v_j \in A_j$, then $N_{G'}[A_i] = V \setminus \{v_j\} \neq V$ and $N_{G'}[A_j] = V \setminus \{v_i\} \neq V$. Thus, A_i and A_j are not dominating in G'. □

We need to define the following function for the next theorem.

$$f(k) = \begin{cases} kn_1 - \frac{k}{2} & \text{if } k \text{ } \textit{even}, \\ kn_1 - \frac{(k-1)}{2} & \text{if } k \text{ } \textit{odd}. \end{cases}$$

Theorem 16. *Let $G = K(n_1, n_2, \ldots n_t)$ be a complete t-partite graph where $V = \bigcup_{i=1}^{t} A_i$ and $|A_i| = n_i$ with $n_1 \leq n_2 \leq \ldots \leq n_t$. Let k be the largest number such that*

$n_1 = n_2 = \cdots = n_k$ and $n_{k+1} > n_k$ if $k \neq t$. Then, the independent bondage number of G is given by

$$b_i(G) = f(k).$$

Proof. Let G be complete t-partite graph where $V = \bigcup_{i=1}^{t} A_i$ and $|A_i| = n_i$ with $n_1 \leq n_2 \leq \ldots \leq n_t$. Let k be the largest number such that $n_1 = n_2 = \cdots = n_k$ and $n_{k+1} > n_k$ if $k \neq t$.

Case 1. $k = 1$. First, we know that A_1 is the only MIDS. By Lemma 11, $|E'| \geq n_1$. Thus $b_1(G) = n_1$.

Case 2. $k \geq 2$. There are two cases to consider. Either k is even or k is odd.

Case 2.1. k is even.

We can pair the partitions as follows. Let $(A_1, A_2), (A_3, A_4), \ldots, (A_{k-1}, A_k)$. Next, let $B_i = E[v_i, A_{i+1}] \cup E[v_{i+1}, A_i]$ for $v_i \in A_i$ and $v_{i+1} \in A_{i+1}$ where $i \geq 1, 3, 5, \ldots k-1$. If we only remove the edges between the pairs $(A_1, A_2), (A_3, A_4), \ldots, (A_{k-1}, A_k)$, we have removed $2n_1 - 1$ edges from each pair where there are a total of $r = \frac{k}{2}$ pairs. Then, letting $E' = \bigcup_{i=1}^{r} B_i$ gives

$$|E'| = r(2n_1 - 1) = \frac{1}{2}k(2n_1 - 1) = kn_1 - \frac{k}{2}.$$

Then, by Lemma 12, each pair satisfies $N_{G'}[A_i] \neq V$ and $N_{G'}[A_{i+1}] \neq V$. By the structure of $G' = G - E'$, any vertex set $A \neq A_i$ with $|A| \leq n_1$ $(1 \leq i \leq k)$ cannot form an IDS. Hence, $b_i(G) \leq f(k)$.

Now, we need to show that $b_i(G) \geq f(k)$. If any edge set E' satisfies $|E'| \leq f(k) - 1$, then there are A_i and A_{i+1} for $i \geq 1, 3, 5, \ldots k - 1$ such that $|E' \cap E[A_i, A_{i+1}]| \leq 2n_1 - 2$. By Lemma 12, either $N[A_i] = V$ or $N[A_{i+1}] = V$.

Thus, $b_i(G) \geq f(k)$.

Case 2.2. k is odd.

In this case, we have an extra dominating set A_k that we must "render" non-dominating in G' so we pair the partitions as follows. Let $(A_1, A_2), (A_3, A_4), \ldots,$ (A_{k-2}, A_{k-1}) and then we still have another dominating set A_k. The argument for each pair of partitions is the same as in the previous case. But for A_k we still need to remove another n_1 edges and in this case there are only $((k-1)/2)$ pairs. Thus, we obtain

$$\frac{(k-1)}{2} \times (2n_1 - 1) + n_1 = kn_1 - \frac{(k-1)}{2}.$$

Now, we can repeat the argument of Case 1 for the $k-1$ pairs and apply Lemma 11 to A_k (for both directions) to get the needed inequalities $b_i(G) \leq f(k)$ and $b_i(G) \geq f(k)$. Which implies $b_i(G) = f(k)$ when k is odd. Therefore, in both cases (k even and k odd), we have shown that $b_i(G) = f(k)$.

□

CHAPTER 4

CONCLUSION

4.1 Future Research

4.1.1 Independent Domination

In the process of proving Theorem 4, we observed that requiring that our graph have girth at least six would eliminate troublesome counterexamples and allow us to possibly improve the upper bound. Recently, Michael Henning et. al. [17] improved the bound of Lam et. al. [20] by considering a cubic bipartite graph. In the same paper, Henning communicates a conjecture for cubic bipartite graphs of girth at least six proposed by Verstraete. The best possible bound for a cubic graph of girth six has been conjectured by Jacques Verstraete [26] where the girth condition which eliminates several problematic counterexamples all of which have smaller girth. Furthermore, Duckworth and Wormald [7] have shown that any random cubic graph almost surely satisfies $i(G) \leq 0.2794n$. If it is possible to show that Verstraete's conjecture holds then it is likely to be very difficult [17].

Conjecture 2. *(Verstraete) If G is a cubic graph of order n having girth at least six, then*

$$i(G) \leq \frac{n}{3}.$$

Theorem 17. *(Henning et al.) If G is a cubic bipartite graph of order n and of* *girth at least six, then*

$$i(G) \leq \frac{4n}{11}.$$

In light of these facts, we propose the same question of subcubic graphs having girth at least six with the added condition of minimum degree at least two but including all graphs and not just restricting our interest to bipartite graphs.

Problem 1. *If G is a connected subcubic graph of order n with $\delta(G) \geq 2$ and having girth at least 6, then*

$$i(G) \leq \frac{4n}{11}.$$

We might also ask the same question about graph that are k-regular when $k \geq 4$. That is, can we find a bound $f(n, k)$ such that the following holds.

Problem 2. *If G is a k-regular connected graph of order n where $k \geq 4$,*

$$i(G) \leq f(n, k).$$

4.1.2 Independent Bondage

As we discussed earlier, we can explore many of the same graphs that were examined with the bondage number and formulate similar questions about the independent bondage number. We have obtained results for the independent bondage number of paths, cycles, the complete graph, complete bipartite graphs, mutli-partite graphs, and trees. Additionally, we might explore relationships involving minimum degree or maximum degree as well as diameter. Some other graphs of interest are cubic or subcubic graphs, claw-free graphs, or planar graphs.

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