May 20th, 2:30 PM - 3:30 PM

Landau-Siegel zeros and their illusory consequences

Kyle Pratt
University of Illinois

Landau-Siegel zeros and their illusory consequences

Kyle Pratt

May 20, 2019

Prelude: effective and ineffective constants

- Since the terms *effective* and *ineffective* will arise often, let us briefly define them
- A constant, implied or otherwise, is *effective* if, given enough time and patience, one could go through a proof and write down an actual value for the constant like $\pi\sqrt{17}/1234$ or $10^{-10^{10}}$ etc
- A constant is *ineffective* if it is not possible (!) to make the constant effective. That is, the very mechanism of the proof does not allow one to compute the constant. Ineffective constants usually arise from invoking the *law of the excluded middle*

Landau-Siegel zeros and class numbers

- Dirichlet (1837) introduced his eponymous characters $\chi \pmod{D}$ to prove there are infinitely many primes $p \equiv a \pmod{D}$, $(a, D) = 1$. Key step in proof is to show that $L(1, \chi) \neq 0$ for each non-principal character $\chi \pmod{D}$
- Fairly easy to show that $L(1, \chi) \neq 0$ if χ is complex (so that $\overline{\chi} \neq \chi$), but the nonvanishing of $L(1, \chi)$ for real characters χ is more subtle
- To this end, Dirichlet developed his class number formula

$$L(1, \chi_D) = \frac{\pi h(-D)}{\sqrt{D}}, \qquad D > 4$$

Landau-Siegel zeros and class numbers

$$L(1,\chi_D) = \frac{\pi h(-D)}{\sqrt{D}}$$

- Class number formula relates special value of L-function to class number of $\mathbf{Q}(\sqrt{-D})$
- Class number $h(-D)$ is order of a finite group, hence is a positive integer, so

  $$L(1,\chi_D) \gg D^{-1/2}$$

  with effective constant
- Henceforth, we always write χ to mean a real primitive Dirichlet character modulo D

Landau-Siegel zeros and class numbers

- For most applications, lower bound $L(1,\chi) \gg D^{-1/2}$ is not strong enough
- Strongest possible bounds come from GRH:

$$\log\log D \gg L(1,\chi) \gg \frac{1}{\log\log D}$$

- Unconditionally, can show $L(1,\chi) \ll \log D$, but lower bounds are more difficult and more important
- Not able to rule out a real zero β of $L(s,\chi)$ with β close to $s=1$
- Such a real zero β is a *Landau-Siegel zero*

Landau-Siegel zeros and class numbers

- Classical zero-free region shows $L(\sigma + it, \chi)$ has at most one real zero β in region

$$\sigma \geq 1 - \frac{c}{\log(q(2 + |t|))}$$

- We say χ is an *exceptional character*, or that χ has a Landau-Siegel zero, if $L(\beta, \chi) = 0$ for some $\beta \geq 1 - c/\log q$
- We do not make constant $c > 0$ explicit, but it is fixed and effective
- Landau showed that exceptional characters, if they exist, appear only rarely

Landau-Siegel zeros and class numbers

- Hecke showed that no real zero in classical zero-free region implies

$$L(1,\chi) \gg \frac{1}{\log D}$$

with effective implied constant

- In such a situation, this yields respectable bound

$$h(-D) \gg \frac{\sqrt{D}}{\log D}$$

- Taking contrapositive of Hecke's result gives

$$L(1,\chi) = o((\log D)^{-1}) \implies L(s,\chi) \text{ has Landau-Siegel zero}$$

- We will soon discuss many other consequences of small values of $L(1,\chi)$

Landau-Siegel zeros and class numbers

- One can prove stronger lower bounds on $L(1,\chi)$, but the constants are ineffective
- For instance, Landau (1935) showed

$$L(1,\chi) \gg_\varepsilon \frac{1}{D^{3/8+\varepsilon}},$$

  and Siegel (1935) improved this to

$$L(1,\chi) \gg_\varepsilon \frac{1}{D^{\varepsilon}}$$

- Landau's result gives $h(-D) \gg_\varepsilon D^{1/8-\varepsilon}$, but ineffective constant means one cannot solve Gauss class number problem along these lines

Landau-Siegel zeros and class numbers

- Strongest *effective* lower bound for class number comes from Goldfeld-Gross-Zagier, who showed

$$h(-D) \gg (\log D) \prod_{p|D} \left(1 - \frac{2\sqrt{p}}{p+1}\right)$$

- Allows one, in principle, to solve $h(-D) = h$ for any fixed h, and has been carried out in practice for all $h \leq 100$ (Watkins, 2004)
- This lower bound uses ideas similar to those who shall discuss shortly

Landau-Siegel zeros: both blessing and curse?

- Linnik's theorem on primes in arithmetic progressions is good place to showcase several principles about Landau-Siegel zeros and exceptional characters
- Recall statement of Linnik's theorem: $\exists$ absolute $L > 0$ such that for any $(a, D) = 1$ there exists $p \equiv a \pmod{D}$ with $p \ll D^L$
- Naive application of Siegel-Walfisz theorem only gives $p \ll \exp(D^\varepsilon)$, so Linnik's theorem is substantial improvement
- Current record is $L = 5$, due to Xylouris (2011)

Landau-Siegel zeros: both blessing and curse?

- At first glance, Landau-Siegel zero for some $\chi \pmod{D}$ makes Linnik's theorem harder to prove
- Recall from Davenport that

$$\sum_{\substack{p \le x \\ p \equiv a \pmod{D}}} \log p = \frac{x}{\varphi(D)} \left(1 - \chi(a) \frac{x^{\beta - 1}}{\beta}\right) + O(E),$$

  where β is the Landau-Siegel zero of $L(\beta, \chi)$
- Since β close to 1, main term badly affected: $\chi(a) = -1$ implies main term twice as large as expected, and if $\chi(a) = 1$ then main term much smaller than expected
- So, seems like it is hard to find $p \equiv a \pmod{D}$ when $\chi(a) = 1$. This is the distorting influence of the Landau-Siegel zero manifesting itself

Landau-Siegel zeros: both blessing and curse?

- Actually, can get better value of L when Landau-Siegel zero exists!
- The error term $O(E)$ involves a sum over zeros of $L(s, \psi)$, where ψ runs over characters mod D.
- One can show that a Landau-Siegel zero forces the zeros of other L-functions farther away from 1-line (this is the *Deuring-Heilbronn phenomenon*), leading to improved error term that more than compensates losses in main term
- Can go even further. Assuming very strong Landau-Siegel zero (i.e. $L(1, \chi)$ very small), Friedlander-Iwaniec (2003) have shown in certain ranges that $L < 2 - \frac{1}{59}$
- GRH only gives $L < 2 + \varepsilon$

Some illusory results

- This is not an isolated phenomenon. One can prove many amazing theorems assuming the existence of a Landau-Siegel zero
- However, since we do not believe Landau-Siegel zeros exist, we think of these results as being "illusory"
- They look impressive, but they will lose content once such zeros are finally eliminated

Some illusory results

- Why prove illusory results?
- Many results, like Linnik's theorem, require bifurcation: case where Landau-Siegel zero exists, and case where it does not
- More philosophically, illusory results test strength of the hypothesis "Landau-Siegel zeros exist"
- Gives us a way to measure strength of our tools and power of our technology, and how close we are to eliminating Landau-Siegel zeros (not close!)

Some illusory results

- Class number $h(-D)$ is smaller than expected (classical)
- Half the arithmetic progressions contain twice as many primes as expected (classical)
- There are infinitely many prime pairs $p, p+h$ for any fixed nonzero h (Heath-Brown, 1983; expected main term)
- Primes in the short interval $(x-y, x]$ for any $y > x^{1/2-1/58+\varepsilon}$ (Friedlander-Iwaniec, 2004; expected main term)
  - RH only gives $y > x^{1/2+\varepsilon}$
- infinitely many primes of the form $p = a^6 + b^2$ (Friedlander-Iwaniec, 2005; expected main term)
  - Can also get prime values of discriminant $-4a^6 - 27b^2$, giving infinitely many elliptic curves with only one place of bad reduction

Some illusory results

- If $\chi \pmod{D}$ is exceptional, then D is sum of a prime and a square (Friedlander-Iwaniec, 2013; main term is conjecturally incorrect)
  - Hardy-Littlewood (1923) conjectured every large nonsquare integer is sum of a prime and a square
- Montgomery's pair correlation function $F(\alpha, T)$ is essentially periodic with period 2, for T in certain ranges depending on D (Montgomery and Heath-Brown–independently; see also Baluyot, 2016; this behavior of $F(\alpha, T)$ is conjecturally incorrect)
  - Quasi-equivalent way to think of this: almost always, distance between zeros of zeta is at least half of the average spacing (Conrey-Iwaniec, 2002)
- There are others...

Technical consequences of exceptional characters

- What is the mechanism underlying all these illusory results?
- The key point is that small value of $L(1, \chi)$ forces $\chi(p) = -1$ for "most" primes p
- Heuristic for why this is true:

$$L(1, \chi) = \prod_p \left(1 - \frac{\chi(p)}{p}\right)^{-1},$$

  so if LHS is small must have $\chi(p) = -1$ for many p on RHS
- More algebraic way to think about it: if class number is small then many primes are inert

Technical consequences of exceptional characters

- More rigorously, it is elementary to prove

$$\sum_{n\le x}\frac{(1\star\chi)(n)}{n} = L(1,\chi)(\log x+\gamma)+L'(1,\chi) + O\left(\frac{D^{1/4}(\log Dx)^2}{x^{1/2}}\right) \qquad (1)$$

- By subtraction, we find

$$\sum_{D^2<n\le D^A}\frac{(1\star\chi)(n)}{n}\le L(1,\chi)\log(D^A) \qquad (2)$$

  provided D large enough (this uses the effective bound $L(1,\chi)\gg D^{-1/2}$)
- Can take $A>0$ to be large constant, or even a function going slowly to infinity

Technical consequences of exceptional characters

$$\sum_{D^2 < n \leq D^A} \frac{(1 \star \chi)(n)}{n} \leq L(1, \chi) \log(D^A)$$

- The inequality (2) is key. Not interesting under GRH because in that case $L(1, \chi) \log D \to \infty$. But under assumption of Landau-Siegel zero, $L(1, \chi) \log D = o(1)$
- Thus, the sequence $(1 \star \chi)(n)$ is *lacunary*: very often we have $(1 \star \chi)(n) = 0$
- Since $(1 \star \chi)(p) = 1 + \chi(p)$ this is more quantitative way of expressing earlier heuristic
- Among other things, this implies $\mu(n) \approx \chi(n)$ for squarefree n. This is a powerful piece of information

Illusory nonvanishing of Dirichlet L-functions

- Now focus on a specific illusory situation: nonvanishing of $L(s, \psi)$ at central point $s = \frac{1}{2}$
- Work with family of $L(s, \psi)$ where ψ ranges over primitive characters modulo q
- For technical convenience take q to be large prime, but this assumption could be weakened
- It is believed that $L(\frac{1}{2}, \psi) \neq 0$ for almost all $\psi \pmod{q}$
  - More ambitiously, one might conjecture that $L(\frac{1}{2}, \psi) \neq 0$ for all ψ

Illusory nonvanishing of Dirichlet L-functions

- Balasubramanian and Murty (1992) were first to show that a positive proportion of $L(\frac{1}{2}, \psi) \neq 0$, but they obtained only a small percentage
- Iwaniec and Sarnak (1999) developed simpler, stronger method and showed at least 33% are nonzero
- Bui (2012) introduced refinements and obtained 34.11%
- Khan and Ngo (2016) obtained 37.5% when q is prime (previous results hold for general q)
- On GRH one can show 50% of central values $L(\frac{1}{2}, \psi) \neq 0$
- Difficulty with obtaining large proportions of nonvanishing is related to family having "unitary symmetry'

Illusory nonvanishing of Dirichlet L-functions

- Can get larger percentages by working with derivatives $L^{(k)}(\frac{1}{2}, \psi)$
- Bui and Milinovich (2011) obtained a lower bound on proportion of ψ with $L^{(k)}(\frac{1}{2}, \psi) \neq 0$ of the form $1 - O(k^{-2})$
- In particular, proportion goes to 1 as $k \to \infty$
- Again, we expect proportion is equal to 1 for every k
- They obtain that more than 75.44% of $L(s, \psi)$ have at most a simple zero at $s = \frac{1}{2}$. This is roughly on pare with what one can obtain from GRH

Illusory nonvanishing of Dirichlet L-functions

- We can improve upon these percentages under the assumption that Landau-Siegel zeros exist
- The results are unconditional and effective, but only useful if $L(1, \chi)$ is small
- We assume that χ is even, but the arguments work identically if χ is odd

Illusory nonvanishing of Dirichlet L-functions

Theorem (Bui, P., Zaharescu, 2019+)

Let $C > 300$ be a fixed real number. For any $\varepsilon > 0$ and any prime q satisfying $D^{300} < q \leq D^C$

$$\frac{1}{\varphi(q)} \sideset{}{^*}\sum_{\substack{\psi \pmod q \\ L(1/2,\psi)\neq 0}} 1 \geq \frac{1}{2} + O_{\varepsilon,C}\left((\log q)^{-1/2+\varepsilon} + L(1,\chi)^{1/2}(\log q)^{25/2+\varepsilon}\right)$$

- Thus, on assumption of exceptional character can match GRH result
- One might be able to increase percentage by using more complicated mollifier
- Could work harder and decrease power of $\log q$ in error term
- $\log q$ must be comparable to $\log D$. If q is much larger then distorting influence of Landau-Siegel zero disappears

Illusory nonvanishing of Dirichlet L-functions

Theorem (Bui, P., Zaharescu, 2019+)

Assume the same hypotheses as the previous theorem. Then the number of characters $\psi \pmod q$ *such that* $L(s,\psi)$ *has a multiple zero at* $s = \frac{1}{2}$ *is*

$$\ll_{\varepsilon,C} (\log q)^{-1/2+\varepsilon} + L(1,\chi)^{1/2}(\log q)^{25/2+\varepsilon}.$$

That is, almost every $L(s,\psi)$ *has at most a simple zero at* $s = \frac{1}{2}$.

Mollification and nonvanishing

- We discuss the proof of the 50% result. By working with linear combinations of functions and their derivatives one can similarly prove the result about almost all having at most a simple zero
- Like almost all modern works on nonvanishing, we use the mollification method. Key observation is the following: for any complex numbers b_ψ, we have

$$\left| \sideset{}{^*}\sum_{\psi \pmod q} b_\psi \right|^2 \leq \left(\sideset{}{^*}\sum_{\psi \pmod q} |b_\psi|^2 \right) \left(\sideset{}{^*}\sum_{\substack{\psi \pmod q \\ b_\psi \neq 0}} 1 \right) \tag{3}$$

Mollification and nonvanishing

- If we apply (3) directly with $b_\psi = L(\frac{1}{2}, \psi)$ then we do not obtain a positive proportion result, because the first moment has size $\approx q$, while the second moment has size $\approx q \log q$
- To take advantage of lacunarity, we actually study nonvanishing of $L(\frac{1}{2}, \psi)L(\frac{1}{2}, \chi\psi)$. Observe that

$$L(s, \psi)L(s, \chi\psi) = \sum_{n \geq 1} \frac{\psi(n)(1 \star \chi)(n)}{n^s}$$

  has lacunary coefficients, and this will be very useful
- We therefore apply (3) with $b_\psi = L(\frac{1}{2}, \psi)L(\frac{1}{2}, \chi\psi)M(\psi)$, where $M(\psi)$ is a short Dirichlet polynomial designed to dampen large values of $L(\frac{1}{2}, \psi)L(\frac{1}{2}, \chi\psi)$

Mollification and nonvanishing

- Concretely, we take

$$M(\psi) = \sum_{a \leq D^{20}} \frac{\rho(a)\psi(a)}{\sqrt{a}},$$

  where $\rho(a) = (\mu \star \mu\chi)(a)$
- The function ρ is supported on cubefree integers, with $\rho(p) = \mu(p)(1 \star \chi)(p)$ and $\rho(p^2) = \chi(p)$
- ρ arises from the Dirichlet series coefficients of $\zeta(s)^{-1}L(s,\chi)^{-1}$ for $\mathrm{Re}(s) > 1$
- Recall that $q \geq D^{300}$, so $M(\psi)$ will be short compared to the sums coming from the approximate functional equations for the L-values

Mollification and nonvanishing

- The inequality (3) reduces the problem of nonvanishing to that of computing mollified first and second moments.
- Actually, because we study nonvanishing of product of L-values, it is better conceptually to think of a second moment and a fourth moment
  - We will discuss this fourth moment notion a bit more at the end

Mollification and nonvanishing

- We give a heuristic argument before discussing some of the technical details
- By the approximate functional equation, we have

$$L(\tfrac{1}{2}, \psi) \approx \sum_{n \leq \sqrt{q}} \frac{\psi(n)}{\sqrt{n}} + \epsilon(\psi) \sum_{n \leq \sqrt{q}} \frac{\overline{\psi}(n)}{\sqrt{n}},$$

where $\epsilon(\psi)$ is a root number (normalized Gauss sum)

Mollification and nonvanishing

- Clearly the character values $\psi(1) = \overline{\psi}(1) = 1$ are not oscillatory, so we pull these out, and get

$$L(\tfrac{1}{2}, \psi) = 1 + \epsilon(\psi) + \left(\sum_{n>1} \right)$$

- One might hope that, for $n > 1$, the character values $\psi(n)$ will oscillate and cancel out on average. Thus, one might expect that, on average, we have

$$L(\tfrac{1}{2}, \psi) \approx 1 + \epsilon(\psi).$$

- Furthermore, the sum of $\epsilon(\psi)$ over ψ experiences cancellation, so we have

$$\sum_{\psi (\text{mod q})} L(\tfrac{1}{2}, \psi) \approx \sum_{\psi (\text{mod } q)} [1 + \epsilon(\psi)] \approx \sum_{\psi (\text{mod } q)} 1.$$

Mollification and nonvanishing

- Similarly, we have

$$|L(\tfrac{1}{2},\psi)|^2 \approx 2 \sum_{mn\leq q} \frac{\psi(m)\overline{\psi}(n)}{\sqrt{mn}} = 2 + \left(\sum_{mn>1}\right).$$

- Again, one might hope that the oscillatory values $\psi(m)\overline{\psi}(n)$ might cancel out on average when $mn > 1$, so that

$$|L(\tfrac{1}{2},\psi)|^2 \approx 2$$

on average over ψ

- The inequality (3) would then give

$$\sum_{\substack{\psi (\mathrm{mod}\ q)\\ L(\frac{1}{2},\psi)\neq 0}} 1 \geq \frac{\left|\sum_\psi L(\frac{1}{2},\psi)\right|^2}{\sum_\psi |L(\frac{1}{2},\psi)|^2} \gtrapprox \frac{\varphi(q)^2}{2\varphi(q)} = \frac{1}{2}\varphi(q),$$

and this would yield that 50% of the central values are nonzero

Mollification and nonvanishing

- This heuristic works for the first moment, but does not work for the second moment, even with mollification
- However, in the exceptional circumstances under which we work this heuristic actually can be turned into a rigorous argument
- Thus, we achieve almost perfect mollification, from the point of view of the approximate functional equation: the non-oscillatory $n = 1$ terms are all that contribute to the percentage of nonvanishing
- From this point of view 50% is therefore a natural barrier in the problem

Some technical points

- The hard part is the second moment (which, we recall, is more like a fourth moment)
- Use the approximate functional equation to write our L-values, open up the definition of $M(\psi)$, and use character orthogonality
- The second moment is then something like

$$\varphi(q) \sum_{\substack{a,b \le D^{20} \\ m,n \le q\sqrt{D} \\ am \equiv bn(q)}} \frac{\rho(a)\rho(b)(1 \star \chi)(m)(1 \star \chi)(n)}{\sqrt{abmn}}$$

- The diagonal $am = bn$ gives the main term
- The off-diagonal $am \neq bn$, $am \equiv bn \pmod{q}$ contributes only to error term

Some technical points

- Contribution of main term is comparatively straightforward
- Make repeated use of lacunarity (2) to separate variables, and turn sums into Euler products (!)
- The fact that all the sums eventually turn into products is heavily responsible for strong mollification

Some technical points

- For instance, (1) gives

$$L'(1,\chi) = \sum_{n \le D^4} \frac{(1 \star \chi)(n)}{n} + O(L(1,\chi)\log D)$$

- Terms involving $L'(1,\chi)$ arise when looking at contour integrals with $L(1+s,\chi)$ and a double pole at $s=0$
- Up to error of $O(L(1,\chi)(\log D)^{O(1)})$ we have

$$\sum_{n \le D^4} \frac{(1 \star \chi)(n)}{n} = \prod_{p \le D^4} \left(1 + \frac{(1 \star \chi)(p)}{p} + \cdots\right)$$

- Since $1 \star \chi$ is lacunary, this is analogous to

$$\sum_{n \le x} \frac{1}{n^2} \sim \prod_{p \le x} \left(1 + \frac{1}{p^2} + \cdots\right)$$

Some technical points

- The off-diagonal

$$\varphi(q) \sum_{\substack{a,b \le D^{20} \\ m,n \le q\sqrt{D} \\ am \equiv bn (q) \\ am \ne bn}} \frac{\rho(a)\rho(b)(1 \star \chi)(m)(1 \star \chi)(n)}{\sqrt{abmn}}$$

  contributes only to the error term
- Difficult to see rigorously, but we can give a heuristic
- Write $am - bn = qr$, where r is nonzero integer; note that $|r|$ is small
- for each fixed a, b, r, we have a sort of shifted convolution problem for $1 \star \chi$

Some technical points

- Consider for illustration the simpler shifted convolution

$$\sum_n (1 \star \chi)(n)(1 \star \chi)(n+h),$$

  where the shift $h \neq 0$ is fixed
- We expect multiplicative structure of $n, n+h$ to be essentially independent
- Therefore, expect lacunarity of $1 \star \chi$ to strike twice, giving us something like $L(1,\chi)^2$ in the main term of the off-diagonal

Some technical points

- We mentioned that the second moment is really more like a fourth moment
- Recent work on the fourth moment of Dirichlet L-functions has required deep inputs from spectral theory of automorphic forms to control off-diagonal contributions
- In our exceptional setup, do not need any spectral theory; suffices to use simpler delta method of Duke, Friedlander, Iwaniec, and the Weil bound for Kloosterman sums

Partial bibliography

- S. Baluyot. *On the pair correlation conjecture and the alternative hypothesis*. J. Number Theory 169 (2016), 183-226
- H. M. Bui. *Non-vanishing of Dirichlet L-functions at the central point*. Int. J. Number Theory 8 (2012), 1855-1881
- H. M. Bui, M. B. Milinovich. *Central values of derivatives of Dirichlet L-functions*. Int. J. Number Theory 7 (2011), 371-388
- H. M. Bui, K. Pratt, A. Zaharescu. *Exceptional characters and nonvanishing of Dirichlet L-functions*. Preprint, 2019
- B. Conrey, H. Iwaniec. *Spacing of zeros of Hecke L-functions and the class number problem*. Acta Arith. 103 (2002), no. 3, 259-312
- B. Conrey, H. Iwaniec. *Critical zeros of lacunary L-functions*. https://arxiv.org/abs/1607.03288

Partial bibliography

- W. Duke, J. B. Friedlander, H. Iwaniec. *A quadratic divisor problem*. Invent. Math. 115 (1994), no. 2, 209-217
- J. B. Friedlander, H. Iwaniec. *Exceptional characters and prime numbers in arithmetic progressions*. Int. Math. Res. Not. 2003, no. 37, 2033-2050
- J. B. Friedlander, H. Iwaniec. *Exceptional characters and prime numbers in short intervals*. Selecta Math. (N.S.) 10 (2004), no. 1, 61-69
- J. B. Friedlander, H. Iwaniec. *The illusory sieve*. Int. J. Number Theory 1 (2005), no. 4, 459-494
- J. B. Friedlander, H. Iwaniec. *Exceptional discriminants are the sum of a square and a prime*. Q. J. Math. 64 (2013), no. 4, 1099-1107
- J. B. Friedlander, H. Iwaniec. *The temptation of exceptional characters*. Exploring the Riemann zeta function, 67-81, Springer, Cham, 2017

Partial bibliography

- J. B. Friedlander, H. Iwaniec. *A note on Dirichlet L-functions*. Expo. Math. 36 (2018), no. 3-4, 343-350
- D. R. Heath-Brown. *Prime twins and Siegel zeros*. Proc. London Math. Soc. (3) 47 (1983), no. 2, 193-224
- H. Iwaniec. *Conversations on the exceptional character*. Analytic number theory, 97-132, Lecture Notes in Math., 1891, Springer, Berlin, 2006
- H. Iwaniec, E. Kowalski. *Analytic number theory*. American Mathematical Society Colloquium Publications, 53. American Mathematical Society, Providence, RI, 2004
  - especially chapters 22 and 23
- R. Khan, H. Ngo. *Nonvanishing of Dirichlet L-functions*. Algebra Number Theory 10 (2016), 2081-2091
- M. Watkins. *Class numbers of imaginary quadratic fields*. Math. Comp. 73 (2004), no. 246, 907-938

Partial bibliography

- T. Xylouris. *Über die Nullstellen der Dirichletschen L-Funktionen und die kleinste Primzahl in einer arithmetischen Progression*. Universität Bonn, Mathematisches Institut, Bonn, 2011. 110 pp.
- M. P. Young. *The fourth moment of Dirichlet L-functions*. Ann. of Math. 173 (2011), 1-50