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An effective Chebotarev density theorem for families of fields, with an application to class groups

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(Joint work with Lillian Pierce and Melanie Matchett Wood)

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Gauss

- classified the binary quadratic forms with a given discriminant $D := b^2 - 4ac$;
- formed the *class group*, the group of equivalence classes of binary quadratic forms of a given D with group action Gauss composition;
- showed that, for any given discriminant D, there exist *only finitely* many equivalence classes of binary quadratic forms.

Quadratic forms and quadratic number fields

Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic number field. To each form

$$(a, b, c) := ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$, we may associate an ideal I of $\mathcal{O}_K$, where

$$I = \left\langle a, \frac{-b + \sqrt{D}}{2} \right\rangle.$$

Binary quadratic forms	$\longleftrightarrow$	**Nonzero ideals of** $\mathcal{O}_{\mathbb{Q}[\sqrt{D}]}$
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$$\mathrm{Cl}_K := \text{ the ideal class group of } K = \mathbb{Q}(\sqrt{D})$$

$$h(K) = |\mathrm{Cl}_K| := \text{ the class number of } K = \mathbb{Q}(\sqrt{D})$$

Note: $h(K)$ is finite via the correspondence.

Class group of K, $[K : \mathbb{Q}] \geq 2$

The ideal class group of K is defined by

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Question: How big is $|\mathrm{Cl}_K|$ in general?

Landau observed that if $[K : \mathbb{Q}] = n$, then

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For any integer $\ell > 1$, the ℓ-**torsion subgroup** of Cl_K is given by

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Natural Question: What is the size of $\mathrm{Cl}_K[\ell]$ as K varies within a family of fields of fixed degree?

How big is $|\mathrm{Cl}_K[\ell]|$?

Trivial Bound – For $[K:\mathbb{Q}] = n$, any integer $\ell \geq 1$, and $\varepsilon > 0$

$$|\mathrm{Cl}_K[\ell]| \leq |\mathrm{Cl}_K| \ll_{n,\varepsilon} D_K^{1/2+\varepsilon}$$

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Conjecture – For $[K:\mathbb{Q}] = n$, any integer $\ell \geq 1$, and $\varepsilon > 0$

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Recorded by

- Brumer-Silverman, ’96
- Duke, ’98
- Zhang, ’05
- Ellenberg-Venkatesh, ’07

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Implied by

- Cohen-Lenstra-Martinet heuristics on the distribution of class groups and ℓ-torsion subgroups within families

What do we know is true?

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Theorem (Gauss)

For all quadratic fields K, we have $|\mathrm{Cl}_K[2]| \ll_{\varepsilon} D_K^{\varepsilon}$.

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- **Question:** Are there cases for which nontrivial bounds known?

Nontrivial bounds on $|\mathrm{Cl}_K[\ell]|$

Theorem (Ellenberg & Venkatesh, 2007)

Let $K/\mathbb{Q}$ be a number field of degree 2 or 3. We have

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Let $K/\mathbb{Q}$ be a non-D_4 number field of degree 4. We have

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Theorem (Bhargava, Shankar, Taniguchi, Thorne, Tsimerman & Zhao, 2017)

Let $K/\mathbb{Q}$ be a number field of degree $n > 2$. For some $\delta_n > 0$ we have

$$|\mathrm{Cl}_K[2]| \ll_{n,\varepsilon} D_K^{\frac{1}{2}-\delta_n+\varepsilon}.$$

NONTRIVIAL BOUNDS ON $|\mathrm{Cl}_K[\ell]|$ ... UNDER GRH

Theorem (Ellenberg & Venkatesh, 2007)

Let $K/\mathbb{Q}$ be a number field of degree n and ℓ a positive integer. Assuming GRH, we have

$$|\mathrm{Cl}_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\frac{1}{2}-\frac{1}{2\ell(n-1)}+\varepsilon}.$$

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- **Question:** What can we say unconditionally for all but a possible exceptional set of fields K within a family?

Nontrivial bounds on $|\mathrm{Cl}_K[\ell]|$ ... in families

Theorem (Soundararajan, 2000)

Let ℓ be prime. For all but a possible zero-density exceptional family of imaginary quadratic fields $K/\mathbb{Q}$, we have

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Theorem (Heath-Brown & Pierce, 2014)

Let $\ell \geq 5$ be prime. For all but a possible zero-density exceptional family of imaginary quadratic fields $K/\mathbb{Q}$, we have

$$|\mathrm{Cl}_K[\ell]| \ll_{\ell,\varepsilon} D_K^{\frac{1}{2}-\frac{3}{2\ell+2}+\varepsilon}.$$

NONTRIVIAL BOUNDS ON $|\mathrm{Cl}_K[\ell]|$ ... IN FAMILIES

Theorem (Ellenberg, Pierce, & Wood, 2016)

Let $\ell \geq 1$, and let $[K : Q] = 2, 3$ or 5. For all but a possible zero-density exceptional family of fields $K/\mathbb{Q}$, we have

$$|\mathrm{Cl}_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\frac{1}{2} - \frac{1}{2\ell(n-1)} + \varepsilon}.$$

If $[K : \mathbb{Q}] = 4$, then the same bound applies for K non-D_4.

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- Note that the bound is as strong as on GRH.

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Pierce, T., and Wood, (2017 preprint)
Under certain conditions (but never under GRH), we extend this result to different families in which $[K : \mathbb{Q}] \geq 2$.

Starting point

Theorem (Ellenberg & Venkatesh, 2007)

Suppose that there are M rational primes

$$p_1, p_2, \ldots, p_M$$

that split completely in K, where $p_j \leq D_K^{\delta}$ *and* $\delta < \frac{1}{2\ell(n-1)}$. *Then for any* $\varepsilon > 0$,

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Answer: via a Chebotarev Density Theorem

An Effective Chebotarev Density Theorem

$$K \;\Big|\; n \quad \mathrm{Gal}(K/\mathbb{Q}) \cong G \;\Big|\; \mathbb{Q}$$

Theorem (Lagarias-Odlyzko*, 1975)

If GRH holds for $\zeta_K(s)$, then

$$\left| \#\{p \le x \text{ that split completely in } K\} - \frac{\mathrm{Li}(x)}{|G|} \right| \le \frac{C_0}{|G|} x^{1/2} \log(D_K x^{n_K})$$

for every $x \geq 2$ and C_0 is effectively computable.

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- We may take $x = D_K^{\delta - \epsilon_0}$, with $\delta = \dfrac{1}{2\ell(n-1)}$.
- Obtain at least $M \gg D_K^{1/(2\ell(n-1)) - \varepsilon_0}$ sufficiently small primes that split completely in K.

BOUNDING ℓ-TORSION ASSUMING GRH

Ellenberg-Venkatesh (2007)

$$|\mathrm{Cl}_K[\ell]| \quad \ll_{\ell,n,\varepsilon} \quad D_K^{\frac{1}{2}+\varepsilon} M^{-1}$$

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Goal: Remove GRH and obtain the same ℓ-torsion bound.

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– We can do this at the cost of proving the result for all but a possible zero-density family of fields.

Same starting point as before

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- has a low threshold on x.

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Let us first recall how to count primes.

Counting Primes

Motivating Question

Given a large number x, how many primes are there less than or equal to x?

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how does $\pi(x)$ behave as $x \to \infty$?

Prime Number Theorem (Hadamard, de la Vallée Poussin 1896)

$$\pi(x) \sim \mathrm{Li}(x), \qquad x \to \infty$$

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Count primes with a weight:

$$\psi(x) := \sum_{n \leq x} \Lambda(n),$$

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We have

$$\psi(x) = x - \sum_{|\gamma| \leq x} \frac{x^\rho}{\rho} + O\left(\log^2 x\right)$$

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$\rho = \beta + i\gamma$ is a nontrivial zero of $\zeta(s)$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}, \quad \Re(s) > 1$$

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- Since $|x^\rho| = x^\beta$, if $\beta < 1$, then the contribution from the nontrivial zeros is not too big.
- Key to proof of the Prime Number Theorem:

$$\zeta(s) \neq 0 \text{ for } \Re(s) = 1$$

Counting primes in arithmetic progressions

Siegel-Walfisz Theorem (1935)

If $n \geq 2$ and a is coprime to q then as $x \to \infty$,

$$\pi(x; a, q) := \sum_{\substack{p \leq x \\ p \equiv a \pmod{q}}} 1 = \frac{1}{\varphi(q)} \mathrm{Li}(x) + \text{"error term"}.$$

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$$L(s, \chi_q) := \sum_{n=1}^{\infty} \frac{\chi_q(n)}{n^s}, \quad \Re(s) > 1$$

Counting prime ideals in number fields

$$\begin{array}{ccc} \mathcal{O}_k & \quad & k \\ | & & | \\ \mathbb{Z} & & \mathbb{Q} \end{array}$$

Prime Ideal Theorem (Landau 1918)
As $x \to \infty$,

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Example 2: When $k = \mathbb{Q}(\sqrt{q})$, one can show $\zeta_k(s) = \zeta(s)L(s, \chi_q)$.

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Generalized Riemann Hypothesis: Nontrivial zeros of $\zeta_K(s)$ have real part equal to $1/2$.

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Let L/k be a normal extension with Galois group $G = \mathrm{Gal}(L/k)$.

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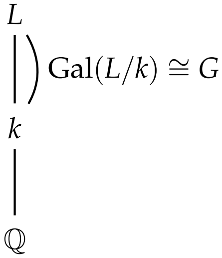

Counting prime ideals in conjugacy classes

Let L/k be a normal extension with Galois group $G = \mathrm{Gal}(L/k)$.

$$\pi_{\mathcal{C}}(x, L/k) := \#\left\{\mathfrak{p} \subset \mathcal{O}_k : \mathfrak{p} \text{ unramified in } L, \left[\frac{L/k}{\mathfrak{p}}\right] = \mathcal{C}, \mathrm{Nm}_{k/\mathbb{Q}}\mathfrak{p} \leq x\right\}$$

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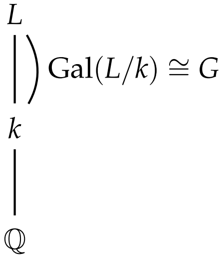

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- $\mathfrak{p}$ is a prime ideal in $\mathcal{O}_k$ which is unramified in L.
- $\left[\frac{L/k}{\mathfrak{p}}\right]$ is the Artin symbol, which denotes the fixed, targeted conjugacy class $\mathcal{C}$ within G.

Counting prime ideals in conjugacy classes

Chebotarev Density Theorem (1922)

$$\pi_{\mathcal{C}}(x; L/k) \sim \frac{|\mathcal{C}|}{|G|}\mathrm{Li}(x), \quad x \to \infty$$

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Counting prime ideals in conjugacy classes

Effective **Chebotarev Density Theorem**

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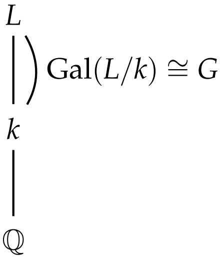

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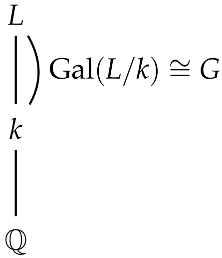

COUNTING PRIME IDEALS IN CONJUGACY CLASSES

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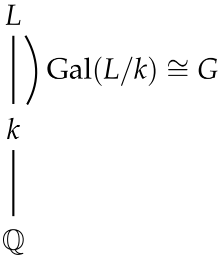
$$\zeta_L(s) := \zeta_k(s) \prod_{\substack{\rho \in \hat{G} \\ \rho \neq \rho_0}} L(s, \rho, L/k)^{\dim \rho}$$



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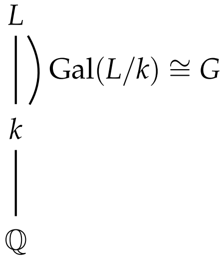

- Each $L(s, \rho, L/k)$ is an Artin L-function.

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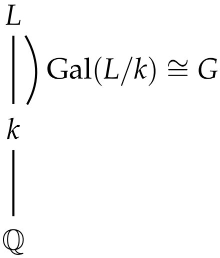

- Each $L(s, \rho, L/k)$ is an Artin L-function.
- The product is over the nontrivial irreducible representations of G.

Example of a Dedekind zeta-function $\zeta_L(s)$

Let $k = \mathbb{Q}$ and $G = \mathrm{Gal}(L/\mathbb{Q}) \cong S_3$.

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- ρ_1 – sign representation, 1-dimensional
- ρ_2 – standard representation, 2-dimensional

$$\zeta_L(s) = \zeta(s)\, L(s, \rho_1)\, L(s, \rho_2)^2$$

An Effective Chebotarev Density Theorem

Let L/k be a normal extension with Galois group $G = \mathrm{Gal}(L/k)$, $D_L = |\mathrm{Disc}\, L/\mathbb{Q}|$, and $n_L = [L : \mathbb{Q}]$.

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Theorem (Lagarias-Odlyzko, 1975)

For any fixed conjugacy class $\mathcal{C} \subset G$,

$$\left| \pi_{\mathcal{C}}(x, L/k) - \frac{|\mathcal{C}|}{|G|}\mathrm{Li}(x) \right| \leq \underbrace{\frac{|\mathcal{C}|}{|G|}\mathrm{Li}(x^{\beta_0}) + c_1 x \exp\left(-c_2 n_L^{1/2} (\log x)^{1/2}\right)}_{\text{Error term depends on zero-free region of } \zeta_L(s).}$$

for $x \geq \exp(10 n_L (\log D_L)^2)$, where

- *β_0 is a real, simple exceptional zero of $\zeta_L(s)$;*
- *c_1, c_2 are effectively computable constants.*

A *Conditional* Effective Chebotarev Density Theorem

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Theorem (Lagarias-Odlyzko, 1975)

If the **generalized Riemann hypothesis** *holds for the Dedekind zeta-function $\zeta_L(s)$, then for any fixed conjugacy class $\mathcal{C} \subset G$*

$$\left| \pi_{\mathcal{C}}(x, L/k) - \frac{|\mathcal{C}|}{|G|}\mathrm{Li}(x) \right| \leq \underbrace{C_0 \frac{|\mathcal{C}|}{|G|} x^{1/2} \log(D_L x^{n_L})}_{\text{Error term relies on GRH for } \zeta_L(s).}$$

for every $x \geq 2$, where

- *C_0 is an effectively computable constant.*

Comparing the theorems (Lagarias-Odlyzko, 1975)

Theorem (Unconditional)

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for every $x \geq 2$.

Want: An *unconditional* effective CDT with a low threshold on x, no β_0 term, and an acceptable error term.

Skeleton of Theorem (Pierce, T., Wood)

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We prove *most* Dedekind zeta-functions in the family satisfy a certain zero-free region.

Application to bounding ℓ-torsion

Skeleton of Corollary (Pierce, T., Wood)

Let $\mathcal{F}(X)$ be a family of fields for which the previous Chebotarev Density Theorem holds. For the nonexceptional fields $K \in \mathcal{F}(X)$, we have

$$|\mathrm{Cl}_K[\ell]| \ll_{n,\ell,\varepsilon} D_K^{\frac{1}{2}-\frac{1}{2\ell(n-1)}+\varepsilon}.$$

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Question:
To which families does our Chebotarev Density Theorem apply?

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Overview of argument

Ellenberg-Venkatesh

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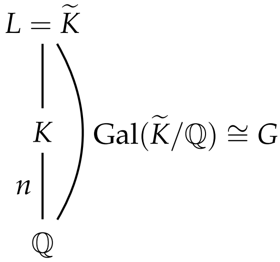

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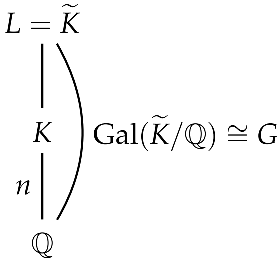

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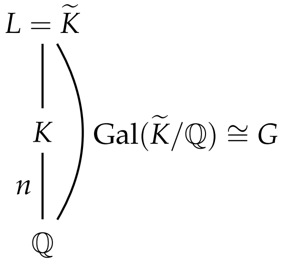

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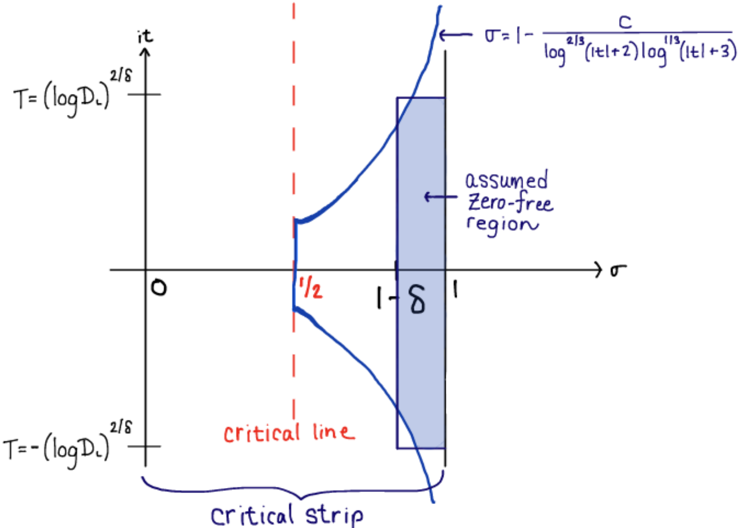

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- We work delicately to provide both an acceptable effective error term, and a sufficiently small threshold for x depending on D_L.

Theorem (Pierce, T., Wood)

Let $0 < \delta \leq 1/4$ be a fixed positive constant. For any normal extension of number fields $L/\mathbb{Q}$ with $[L:\mathbb{Q}] = n_L$ such that D_L is sufficiently large and $\zeta_L(s)$ obeys the assumed zero-free region, we have that for any $A \geq 2$ and any conjugacy class $\mathcal{C} \subset G = \mathrm{Gal}(L/\mathbb{Q})$

$$\left|\pi_{\mathcal{C}}(x, L/\mathbb{Q}) - \frac{|\mathcal{C}|}{|G|}\mathrm{Li}(x)\right| \leq \underbrace{\frac{|\mathcal{C}|}{|G|}\frac{x}{(\log x)^A}}_{\text{error term depends on assumed zero-free region}}$$

for all

$$x \geq \underbrace{c_1 \exp\left\{c_2(\log\log(D_L^{c_3})^{3/2} \log\log\log(D_L^{c_4}))^{1/2}\right\}}_{\geq (\log D_L)^{\text{small power}}},$$

where all the constants can be written explicitly.

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Kowalski and Michel have given a bound for $N(\pi; \alpha, T)$ that holds on average for an appropriately defined family of cuspidal automorphic representations.

Theorem (Kowalski & Michel, 2002)

Let $S(q)$, $q \geq 1$ be a family of cuspidal automorphic representations satisfying a prescribed set of conditions. Let $\alpha \geq 3/4$ and $T \geq 2$. Then there exists $c_0 > 0$, depending on the family, such that

$$\sum_{\pi \in S(q)} N(\pi; \alpha, T) \ll T^B q^{c_0 \frac{1-\alpha}{2\alpha-1}}$$

for all $q \geq 1$ and some $B \geq 0$ that depends on the family. The implied constant only depends on the choice of c_0.

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Applied to $L(s, \pi)$ for $\pi \in S(q)$ $\implies$ a zero-free region of the desired shape that holds for all but a possible zero-density sub-family of L-functions

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A couple of issues:

1. We are working with Artin L-functions, which in general are not known to be automorphic.

2. Kowalski & Michel's result applies to family of cuspidal automorphic representations. We would like to apply it to a family of *isobaric* automorphic representations.

$$\frac{\zeta_{\widetilde{K}}(s)}{\zeta(s)} = \prod_{\substack{\rho \in \hat{G} \\ \rho \neq \rho_0 \text{ irreducible}}} L(s, \rho, \widetilde{K}/\mathbb{Q})^{d_j}, \quad d_j = \deg(\rho_j).$$

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Assuming the strong Artin conjecture, we have that each $L(s, \rho, \widetilde{K}/\mathbb{Q})$ is automorphic, i.e. we can write

$$L(s, \rho, \widetilde{K}/\mathbb{Q}) = L(s, \pi)$$

for each $L(s, \rho, \widetilde{K}/\mathbb{Q})$ in our product.

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- We decompose each Dedekind zeta function into a product of cuspidal automorphic L-functions.
- We apply the Kowalski-Michel result to the sub-family generated by each factor.

A new obstacle:

In generalizing Kowalski-Michel, we uncover a technical barrier:

– *a priori*, each sub-family could lead to many bad fields for which our Chebotarev Density Theorem does not apply.

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In generalizing Kowalski-Michel, we uncover a technical barrier:

– *a priori*, each sub-family could lead to many bad fields for which our Chebotarev Density Theorem does not apply.

Must define our families of fields to avoid this situation – where potential "bad" elements in each sub-family propagate to create a "large" family of "bad" Dedekind zeta-functions $\zeta_{\widetilde{K}}(s)$.

Bounding ℓ-torsion without assuming GRH

Ellenberg-Venkatesh

$$|\mathrm{Cl}_K[\ell]| \ll_{\ell,n,\varepsilon} D_K^{\frac{1}{2}+\varepsilon} M^{-1}$$

↓

Prove an effective Chebotarev Density Theorem assuming non-GRH zero-free region

↓

Show that within an appropriate family of fields K, most $\zeta_{\widetilde{K}}(s)$ obey the zero-free region

↓

Control the propagation of "bad" fields within the family

↓

Without assuming GRH, conclude

$|\mathrm{Cl}_K[\ell]| \ll_{\ell,n,\varepsilon} D_K^{\frac{1}{2}-\frac{1}{2\ell(n-1)}+\varepsilon}$ for non-exceptional K.

Controlling propagation of bad fields

Sketch of new idea

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- Then we quantify how many K can have a particular discriminant.

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Thanks for y’all’s attention!