HARMONIC MAPPINGS AND CONFORMAL MINIMAL IMMERSIONS OF RIEMANN SURFACES INTO $\mathbb{R}^N$

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ABSTRACT. We prove that for any open Riemann surface $\mathcal{N}$, natural number $N \geq 3$, non-constant harmonic map $h : \mathcal{N} \to \mathbb{R}^{N-2}$ and holomorphic 2-form $\mathfrak{H}$ on $\mathcal{N}$, there exists a weakly complete harmonic map $X = (X_j)_{j=1,\ldots,N} : \mathcal{N} \to \mathbb{R}^N$ with Hopf differential $\mathfrak{H}$ and $(X_j)_{j=3,\ldots,N} = h$. In particular, there exists a complete conformal minimal immersion $Y = (Y_j)_{j=1,\ldots,N} : \mathcal{N} \to \mathbb{R}^N$ such that $(Y_j)_{j=3,\ldots,N} = h$.

As some consequences of these results:

- There exists complete full non-decomposable minimal surfaces with arbitrary conformal structure and whose generalized Gauss map is non-degenerate and fails to intersect N hyperplanes of $\mathbb{CP}^{N-1}$ in general position.
- There exists complete non-proper embedded minimal surfaces in $\mathbb{R}^N$, $\forall N > 3$.

1. INTRODUCTION

In this paper we use methods coming from the study of minimal surfaces to construct harmonic mappings of Riemann surfaces into $\mathbb{R}^N$ with prescribed geometry. A basic reference for this topic is, for instance, Klotz's work [**K**].

Our main result states (see Corollary 4.5):

> **Theorem A.** *For any open Riemann surface $\mathcal{N}$, natural number $N \geq 3$, non-constant harmonic map $h : \mathcal{N} \to \mathbb{R}^{N-2}$ and holomorphic 2-form $\mathfrak{H}$ on $\mathcal{N}$, there exists a weakly complete harmonic map $X = (X_j)_{j=1,\ldots,N} : \mathcal{N} \to \mathbb{R}^N$ with Hopf differential $\mathfrak{H}$ and $(X_j)_{j=3,\ldots,N} = h$.*

Recall that the Hopf differential Q_X of a harmonic map $X : \mathcal{N} \to \mathbb{R}^N$ is the holomorphic 2-form given by $Q_X := \langle \partial_z X, \partial_z X \rangle$, where ∂_z means complex differential. By definition, X is said to be weakly complete if $\Gamma_X := |\partial_z X|^2$ is a complete conformal Riemannian metric in $\mathcal{N}$ (see [**K**]).

The fact that conformal minimal immersions are harmonic maps strongly influences the global theory of this kind of surfaces. It is well known that a harmonic immersion $X : \mathcal{N} \to \mathbb{R}^N$ is minimal if and only if it is conformal, or equivalently, $Q_X = 0$. Weakly completeness is equivalent to Riemannian completeness under minimality assumptions. The geometry of complete minimal surfaces in $\mathbb{R}^N$, specially those properties regarding the Gauss map, has been the object of extensive study over the last past decades (see for instance [**O1**, **CO**, **C**, **F3**, **R**]).

In the recent paper [**AFL**], the authors constructed complete minimal surfaces in $\mathbb{R}^3$ with arbitrarily prescribed conformal structure and non-constant third coordinate function (see also [**AF**]). As a consequence, any open Riemann surface admits a complete conformal minimal immersion in

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$\mathbb{R}^3$ whose Gauss map omits two antipodal points of the unit sphere. Theorem A let us generalize these results to arbitrary higher dimensions (see Corollary 4.6):

> **Theorem B.** *For any open Riemann surface $\mathcal{N}$, natural number $\mathrm{N} \geq 3$ and non-constant harmonic map $h : \mathcal{N} \to \mathbb{R}^{\mathrm{N}-2}$, there exists a complete conformal minimal immersion $X = (X_j)_{j=1,\ldots,\mathrm{N}} : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ with $(X_j)_{j=3,\ldots,\mathrm{N}} = h$.*

Under some compatibility conditions depending on the map h, the flux map of the immersion X can be also prescribed. Recall that the flux map of a conformal minimal immersion $X : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ is given by $\mathrm{p}_X(\gamma) = \mathrm{Im} \int_\gamma \partial_z X$ for all $\gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$. In particular, if h is the real part of a holomorphic map $H : \mathcal{N} \to \mathbb{C}^{\mathrm{N}-2}$, Theorem B provides a complete null holomorphic curve $F = (F_j)_{j=1,\ldots,\mathrm{N}} : \mathcal{N} \to \mathbb{C}^{\mathrm{N}}$ such that $(F_j)_{j=3,\ldots,\mathrm{N}} = H$. Likewise, $Y = (F_j)_{j=2,\ldots,\mathrm{N}} : \mathcal{N} \to \mathbb{C}^{\mathrm{N}-1}$ is a complete holomorphic immersion whose last $\mathrm{N} - 2$ coordinates coincide with H.

Theorem B also includes some information about the Gauss map of minimal surfaces in $\mathbb{R}^{\mathrm{N}}$. Given a conformal minimal immersion $X : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$, its generalized Gauss map $G_X : \mathcal{N} \to \mathbb{CP}^{\mathrm{N}-1}$, $G_X(P) = \partial_z X(P)$, is holomorphic and takes values on the complex hyperquadric $\{\sum_{j=1}^{\mathrm{N}} w_j^2 = 0\}$. Chern and Osserman [**C**, **CO**] showed that if X is complete then either $X(\mathcal{N})$ is a plane or $G_X(\mathcal{N})$ intersects a dense set of complex hyperplanes. Even more, Ru [**R**] proved that if X is complete and non-flat then G_X cannot omit more than $\mathrm{N}(\mathrm{N}+1)/2$ hyperplanes in $\mathbb{CP}^{\mathrm{N}-1}$ located in general position (see also the works of Fujimoto [**F2**, **F3**] for a good setting). Under the non-degeneracy assumption on G_X, this upper bound is sharp for some values of N, see [**F4**]. However, the number of exceptional hyperplanes strongly depends on the underlying conformal structure of the surface. Indeed, Ahlfors [**A**] proved that any holomorphic map $G : \mathbb{C} \to \mathbb{CP}^{\mathrm{N}-1}$, $\mathrm{N} \geq 3$, avoiding $\mathrm{N}+1$ hyperplanes of $\mathbb{CP}^{\mathrm{N}-1}$ in general position is degenerate, that is to say, $G(\mathbb{C})$ lies in a proper projective subspace of $\mathbb{CP}^{\mathrm{N}-1}$ (see [**W**, Chapter 5, §5] and [**F1**] for further generalizations). So, it is natural to wonder whether any open Riemann surface admits a complete conformal minimal immersion in $\mathbb{R}^{\mathrm{N}}$ whose generalized Gauss map is non-degenerate and omits N hyperplanes in general position. An affirmative answer to this question can be found in the following (see Corollary 4.8)

> **Theorem C.** *Let $\mathcal{N}$ be an open Riemann surface, and let $\mathrm{p} : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^{\mathrm{N}}$ be a group morphism, $\mathrm{N} \geq 3$.*
>
> *Then there exists a complete conformal full non-decomposable minimal immersion $X : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ with $\mathrm{p}_X = \mathrm{p}$ and whose generalized Gauss map is non-degenerate and omits N hyperplanes in general position.*

On the other hand, Theorem B leads to some interesting consequences regarding Calabi-Yau conjectures. The embedded Calabi-Yau problem for minimal surfaces asks for the existence of complete bounded embedded minimal surfaces in $\mathbb{R}^3$. Complete embedded minimal surfaces in $\mathbb{R}^3$ with finite genus and countably many ends are proper in space [**MPR**, **CM**]. However, this result fails to be true for arbitrary higher dimensions. For instance, taking $\mathcal{N}$ the unit disc $\mathbb{D}$ in $\mathbb{C}$ and $h : \mathbb{D} \to \mathbb{R}^2$ the map $h(z) = (\mathrm{Re}(z), \mathrm{Im}(z))$, Theorem B generates complete non-proper embedded minimal discs in $\mathbb{R}^4$ (so in $\mathbb{R}^{\mathrm{N}}$ for all $\mathrm{N} \geq 4$), see Corollary 4.7 for more details.

The paper is laid out as follows. In Section 2 we introduce the necessary background and notations. In Section 3 we prove a basic approximation result by holomorphic 1-forms in open Riemann surfaces (Lemma 3.3), which is the key tool for proving our main results. Finally, in Section 4 we state and prove Theorems A, B and C. It is worth mentioning that all these theorems actually follows from the more general result Theorem 4.4 in Section 4.

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2. PRELIMINARIES

Given a topological surface M, ∂M will denote the one dimensional topological manifold determined by the boundary points of M. Given $S \subset M$, call by S° and $\overline{S}$ the interior and the closure of S in M, respectively. Open connected subsets of $M - \partial M$ will be called *domains*, and those proper topological subspaces of M being surfaces with boundary are said to be *regions*. The surface M is said to be *open* if it is non-compact and $\partial M = \emptyset$.

If M is a Riemann surface, ∂_z will denote the global complex operator given by $\partial_z|_U = \frac{\partial}{\partial w} dw$ for any conformal chart (U, w) on M.

Remark 2.1. Throughout this paper $\mathcal{N}$ and N will denote a fixed but arbitrary open Riemann surface and natural number greater than or equal to three, respectively.

Let S denote a subset of $\mathcal{N}$, $S \neq \mathcal{N}$. We denote by $\mathcal{F}_0(S)$ as the space of continuous functions $f : S \to \mathbb{C}$ which are holomorphic on an open neighborhood of S in $\mathcal{N}$. Likewise, $\mathcal{F}_0^*(S)$ will denote the space of continuous functions $f : S \to \mathbb{C}$ being holomorphic on S°.

As usual, a 1-form θ on S is said to be of type $(1,0)$ if for any conformal chart (U, z) in $\mathcal{N}$, $\theta|_{U\cap S} = h(z)dz$ for some function $h : U \cap S \to \mathbb{C}$. We denote by $\Omega_0(S)$ as the space of holomorphic 1-forms on an open neighborhood of S in $\mathcal{N}$. We call $\Omega_0^*(S)$ as the space of 1-forms θ of type $(1,0)$ on S such that $(\theta|_U)/dz \in \mathcal{F}_0^*(S \cap U)$ for any conformal chart (U, z) on $\mathcal{N}$.

We denote by $\mho_0(S)$ as the space of holomorphic 2-forms on an open neigborhood of S in $\mathcal{N}$.

Let $\mathfrak{Div}(S)$ denote the free commutative group of divisors of S with multiplicative notation. A divisor $D \in \mathfrak{Div}(S)$ is said to be *integral* if $D = \prod_{i=1}^n Q_i^{n_i}$ and $n_i \geq 0$ for all i. Given D_1, $D_2 \in \mathfrak{Div}(S)$, we write $D_1 \geq D_2$ if and only if $D_1 D_2^{-1}$ is integral. For any $f \in \mathcal{F}_0(S)$ we denote by (f) its associated integral divisor of zeros in S. Likewise we define (θ) for any $\theta \in \Omega_0(S)$.

In the sequel we will assume that S is a *compact subset* of $\mathcal{N}$.

A compact Jordan arc in $\mathcal{N}$ is said to be analytical if it is contained in an open analytical Jordan arc in $\mathcal{N}$. By definition, a connected component V of $\mathcal{N} - S$ is said to be *bounded* if $\overline{V}$ is compact, where $\overline{V}$ is the closure of V in $\mathcal{N}$. Moreover, a subset $K \subset \mathcal{N}$ is said to be *Runge* (in $\mathcal{N}$) if $\mathcal{N} - K$ has no bounded components.

Definition 2.2. A compact subset $S \subset \mathcal{N}$ is said to be *admissible* if and only if (see Figure 2.1):

- S is Runge,
- $M_S := \overline{S^\circ}$ consists of a finite collection of pairwise disjoint compact regions in $\mathcal{N}$ with $\mathcal{C}^0$ boundary,
- $C_S := \overline{S - M_S}$ consists of a finite collection of pairwise disjoint analytical Jordan arcs, and
- any component α of C_S with an endpoint $P \in M_S$ admits an analytical extension β in $\mathcal{N}$ such that the unique component of $\beta - \alpha$ with endpoint P lies in M_S.

Let W be a domain in $\mathcal{N}$, and let S be either a compact region or an admissible subset in $\mathcal{N}$. W is said to be a *tubular neighborhood* of S if $S \subset W$ and $W - S$ consists of a finite collection of pairwise disjoint open annuli. In addition, if $\overline{W}$ is a compact region isotopic to W then $\overline{W}$ is said to be a *compact tubular neighborhood* of S. Here *isotopic* means that $\jmath_* : \mathcal{H}_1(W, \mathbb{Z}) \to \mathcal{H}_1(\overline{W}, \mathbb{Z})$ is an isomorphism, where $\jmath : W \to \overline{W}$ is the inclusion map.

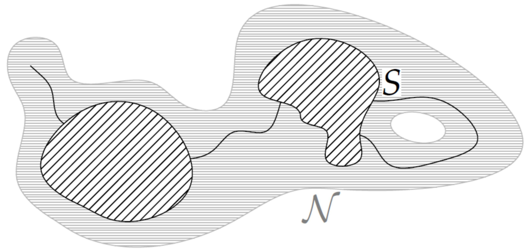

FIGURE 2.1. An admissible subset.

Let $W \subset \mathcal{N}$ be a domain with $S \subset W$. We shall say that a function $f \in \mathcal{F}_0^*(S)$ can be *uniformly approximated* on S by functions in $\mathcal{F}_0(W)$ if there exists $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(W)$ such that $\{|f_n - f|\}_{n\in\mathbb{N}} \to 0$ uniformly on S. A 1-form $\theta \in \Omega_0^*(S)$ can be uniformly approximated on S by 1-forms in $\Omega_0(W)$ if there exists $\{\theta_n\}_{n\in\mathbb{N}} \subset \Omega_0(W)$ such that $\{\frac{\theta_n-\theta}{dz}\}_{n\in\mathbb{N}} \to 0$ uniformly on $S \cap U$, for any conformal closed disc (U, dz) on W.

Given an admissible compact set $S \subset W$, a function $f : S \to \mathbb{C}^n$, $n \in \mathbb{N}$, is said to be smooth if $f|_{M_S}$ admits a smooth extension f_0 to an open domain V in W containing M_S, and for any component α of C_S and any open analytical Jordan arc β in W containing α, f admits an smooth extension f_β to β satisfying that $f_\beta|_{V\cap\beta} = f_0|_{V\cap\beta}$. Likewise, an 1-form θ of type $(1,0)$ on S is said to be smooth if for any closed conformal disc (U,z) on $\mathcal{N}$ such that $S \cap U$ is admissible, the function $\frac{\theta}{dz}$ is smooth on $S \cap U$. Given a smooth $f \in \mathcal{F}_0^*(S)$, we set $df \in \Omega_0^*(S)$ as the smooth 1-form given by $df|_{M_S} = d(f|_{M_S})$ and $df|_{\alpha\cap U} = (f\circ\alpha)'(x)dz|_{\alpha\cap U}$, where $(U, z = x+iy)$ is a conformal chart on W such that $\alpha \cap U = z^{-1}(\mathbb{R} \cap z(U))$. Obviously, $df|_\alpha(t) = (f\circ\alpha)'(t)dt$ for any component α of C_S, where t is any smooth parameter along α. This definition makes sense also for smooth functions with poles in S°.

A smooth 1-form $\theta \in \Omega_0^*(S)$ is said to be *exact* if $\theta = df$ for some smooth $f \in \mathcal{F}_0^*(S)$, or equivalently if $\int_\gamma \theta = 0$ for all $\gamma \in \mathcal{H}_1(S,\mathbb{Z})$.

2.1. **Harmonic maps and minimal surfaces in $\mathbb{R}^{\mathrm{N}}$.** Given a non-constant harmonic map $X = (X_j)_{j=1,\ldots,\mathrm{N}} : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$, the holomorphic quadratic differential

$$Q_X := \langle \partial_z X, \partial_z X\rangle = \sum_{j=1}^{\mathrm{N}} (\partial_z X_j)^2$$

is said to be the Hopf differential of X. We also consider the conformal metric, possibly with isolated singularities,

$$\Gamma_X := \frac{1}{2}\sum_{j=1}^{\mathrm{N}} |\partial_z X_j|^2.$$

It is clear that $2\Gamma_X$ is greater than or equal to the Riemannian metric on $\mathcal{N}$ (possibly with singularities) induced by X. When X is an immersion then Γ_X is a Riemannian metric, and if in addition X is complete then Γ_X is complete as well **[K]**. However, the reciprocal does not hold in general.

Definition 2.3. We say that a harmonic map $X : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ is *weakly complete* (or complete in the sense of **[K]**) if Γ_X is a complete metric in $\mathcal{N}$.

We also associate to X the group morphism

$$\mathrm{p}_X : \mathcal{H}_1(\mathcal{N},\mathbb{Z}) \to \mathbb{R}^{\mathrm{N}}, \quad \mathrm{p}_X(\gamma) = \mathrm{Im}\int_\gamma \partial_z X.$$

Remark 2.4. If $Q_X = 0$ and Γ_X never vanishes, then X is a conformal minimal immersion, Γ_X is the metric induced on $\mathcal{N}$ by X, and p_X is the flux map of X.

If in addition X is a conformal minimal immersion and we write $\partial_z X_j = f_j d\zeta$ in terms of a local parameter ζ on $\mathcal{N}$, $j = 1, \ldots, \mathrm{N}$, then the *(generalized) Gauss map* of X is given by

$$G_X : \mathcal{N} \to \mathbb{CP}^{\mathrm{N}-1}, \quad G_X(\zeta) = [(f_j(\zeta))_{j=1,\ldots,\mathrm{N}}],$$

where $[w]$ is the class of w in $\mathbb{CP}^{\mathrm{N}-1}$, $\forall w \in \mathbb{C}^{\mathrm{N}}$. It is well known that G_X is a holomorphic map taking values in the complex quadric $\{[(w_j)_{j=1,\ldots,\mathrm{N}}] \in \mathbb{CP}^{\mathrm{N}-1} \mid \sum_{j=1}^{\mathrm{N}} w_j^2 = 0\}$.

A set of hyperplanes in $\mathbb{CP}^{\mathrm{N}-1}$ is said to be in general position if each subset of k hyperplanes, with $k \leq \mathrm{N} - 1$, has an $(\mathrm{N} - 1 - k)$-dimensional intersection.

Definition 2.5 ([**O2**])**.** Let $X : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ be a conformal minimal immersion.

- X is said to be *decomposable* if, with respect to suitable rectangular coordinates in $\mathbb{R}^{\mathrm{N}}$, one has $\sum_{k=1}^{m} (\partial_z X_k)^2 = 0$ for some $m < \mathrm{N}$.
- X is said to be *full* if $X(\mathcal{N})$ is contained in no hyperplane of $\mathbb{R}^{\mathrm{N}}$.
- The Gauss map G_X is said to be *degenerate* if $G_X(\mathcal{N})$ lies in a hyperplane of $\mathbb{CP}^{\mathrm{N}-1}$.

When $\mathrm{N} = 3$, decomposable, non-full and degenerate are equivalent. However, if one passes to higher dimensions then no two of these conditions are equivalent (see [**O2**]).

3. The Approximation Lemma

The next two lemmas are the key tools in the proof of the main result of this section (Lemma 3.3). They represent a slight generalization of Lemmas 2.4 and 2.5 in [**AL**].

From now on, $\imath$ denotes the imaginary unit and the symbol $\not\equiv 0$ means *non-identically zero*.

Lemma 3.1. *Let $W \subset \mathcal{N}$ be a domain with finite topology and $S \subset \mathcal{N}$ an admissible subset with $S \subset W$. Consider $f \in \mathcal{F}_0^*(S) \cap \mathcal{F}_0(M_S)$ with $f|_{M_S} \not\equiv 0$.*

Then f can be uniformly approximated on S by functions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}_0(W)$ satisfying that $(f_n) = (f|_{M_S})$ on W. In particular, f_n never vanishes on $W - M_S$ for all n.

Proof. Let $\{M_k\}_{k\in\mathbb{N}}$ be a sequence of compact tubular neighborhoods of M_S in W such that $M_k \subset M_{k-1}^\circ$ for any k, $\cap_{k\in\mathbb{N}} M_k = M_S$ and f holomorphically extends (with the same name) to M_1 and has no zeros on $M_1 - M_S$ (take into account that $f|_{M_S} \not\equiv 0$). Choose M_k so that, in addition, the compact set $S_k := M_k \cup C_S \subset W$ is admissible and $\alpha - M_k^\circ$ is a (non-empty) Jordan arc for any component α of C_S. In particular, $M_{S_k} = M_k$ and $C_{S_k} = C_S - M_k^\circ$, $k \in \mathbb{N}$.

For any $k \in \mathbb{N}$ take $g_k \in \mathcal{F}_0^*(S_k) \cap \mathcal{F}_0(M_{S_k})$ satisfying

- $g_k|_{M_{S_k}} = f|_{M_{S_k}}$,
- g_k never vanishes on $S_k - S_k^\circ$ (recall that f has no zeros on $M_1 - M_S$), and
- the sequence $\{g_k|_S\}_{k\in\mathbb{N}}$ uniformly converges to f on S.

The construction of such functions is standard, we omit the details. Since g_k satisfies the hypotheses of Lemma 2.4 in [**AL**], it can be uniformly approximated on S_k by a sequence $\{g_{k,n}\}_{n\in\mathbb{N}} \subset \mathcal{F}_0(W)$ with $(g_{k,n}) = (g_k|_{M_{S_k}}) = (f|_{M_S})$ on W, for any k. A standard diagonal argument concludes the proof. □

Lemma 3.2. *Let $W \subset \mathcal{N}$ be a domain with finite topology and $S \subset \mathcal{N}$ an admissible subset with $S \subset W$. Consider $\theta \in \Omega_0^*(S) \cap \Omega_0(M_S)$ with $\theta|_{M_S} \not\equiv 0$.*

Then θ can be uniformly approximated on S by 1-forms $\{\theta_n\}_{n\in\mathbb{N}}$ in $\Omega_0(W)$ satisfying that $(\theta_n) = (\theta|_{M_S})$ on W. In particular, θ_n never vanishes on $W - M_S$ for all n.

Proof. Let ϑ be a never vanishing 1-form in $\Omega_0(W)$. Define $f := \theta/\vartheta \in \mathcal{F}_0^*(S) \cap \mathcal{F}_0(M_S)$. By Lemma 3.1, f can be uniformly approximated on S by a sequence $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{F}_0(W)$ satisfying that $(f_n) = (f|_{M_S})$ on W for all n. It suffices to set $\theta_n := f_n\vartheta$, $n \in \mathbb{N}$. □

Lemma 3.3. *Let $W \subset \mathcal{N}$ be a domain with finite topology and $S \subset \mathcal{N}$ an admissible subset with $S \subset W$. Let $\Theta \in \mho_0(W)$ and $\Phi = (\phi_1, \phi_2)$ be a smooth pair in $\Omega_0^*(S)^2 \cap \Omega_0(M_S)^2$ satisfying $\phi_1^2 + \phi_2^2 = \Theta|_S$ and either of the following conditions:*

(A) *$\phi_1|_{M_S}$ and $\phi_2|_{M_S}$ are linearly independent in $\Omega_0(M_S)$ and Θ has no zeros on C_S.*
(B) *$\Theta = 0$ and $\phi_1|_{M_S} \not\equiv 0$.*

Then Φ can be uniformly approximated on S by a sequence $\{\Phi_n = (\phi_{1,n}, \phi_{2,n})\}_{n\in\mathbb{N}} \subset \Omega_0(W)^2$ satisfying

(a) *$\phi_{1,n}^2 + \phi_{2,n}^2 = \Theta$,*
(b) *$\Phi_n - \Phi$ is exact on S, and*
(c) *the zeros of Φ_n on W are those of Φ on M_S (in particular, Φ_n never vanishes on $W - M_S$).*

Proof. Assume (A) holds.

Claim 3.4. *Without loss of generality it can be assumed that ϕ_1, ϕ_2 and $d\xi$ never vanish on C_S, where $\xi := \phi_1/\phi_2$.*

Proof. Assume for a moment that the conclusion of the lemma holds when ϕ_1, ϕ_2 and $d\xi$ never vanish on C_S.

Take a sequence $\{M_k\}_{k\in\mathbb{N}}$ as in the proof of Lemma 3.1 such that Φ holomorphically extends (with the same name) to M_1, and ϕ_1, ϕ_2 and $d\xi$ never vanish on $M_1 - M_S$, for all n (take into account (A)). Recall that $S_k := M_k \cup C_S \subset W^\circ$ is an admissible set and $C_{S_k} = C_S - M_k^\circ$, $k \in \mathbb{N}$.

Since Θ never vanishes on C_S, which consists of a finite collection of pairwise disjoint analytical Jordan arcs, then we can find $\theta \in \Omega_0(C_S)$ with $\theta^2 = \Theta|_{C_S}$. Consider $f_j : C_S \to \mathbb{C}$, $f_j = \phi_j/\theta$, $j = 1, 2$, and notice that $f_1^2 + f_2^2 = 1$ and $\xi|_{C_S} = f_1/f_2$. Consider a sequence $\{(f_{1,k}, f_{2,k})\}_{k\in\mathbb{N}}$ of pairs of smooth functions on C_{S_k} satisfying:

i) $f_{1,k}$, $f_{2,k}$ and $df_{1,k}$ never vanish on C_{S_k},
ii) $f_{1,k}^2 + f_{2,k}^2 = 1$,
iii) the function $g_{j,k}$ given by $g_{j,k}|_{M_{S_k}} = f_j$, $g_{j,k}|_{C_{S_k}} = f_{j,k}$, lies in $\mathcal{F}_0^*(S_k)$ and is smooth, $j = 1, 2$,
iv) $\{f_{j,k}\}_{k\in\mathbb{N}}$ uniformly converges to f_j on C_S, $j = 1, 2$, and
v) $\Psi_k|_S - \Phi$ is exact on S, where $\Psi_k := (g_{j,k}\theta)_{j=1,2} \in \Omega_0^*(S_k)^2 \cap \Omega_0(M_{S_k})^2$.

The construction of this data is standard, we omit the details. Write $\Psi_k = (\psi_{j,k})_{j=1,2}$ and $\xi_k = \psi_{1,k}/\psi_{2,k}$. From *i*), *ii*) and the definition of θ follow that $\psi_{1,k}^2 + \psi_{2,k}^2 = \Theta$ and $d\xi_k$ never vanishes on C_{S_k}. Moreover, *iv*) gives that $\{\Psi_k|_S\}_{k\in\mathbb{N}}$ uniformly converges to Φ on S.

By hypothesis, Lemma 3.3 holds for any Ψ_k, then there exists a sequence $\{\Psi_{k,n}\}_{n\in\mathbb{N}}$ uniformly converging to Ψ_k on S_k and satisfying (a), (b) and (c) of Lemma 3.3 for $\Phi = \Psi_k$ and $S = S_k$. Using that $\{\Psi_k|_S\}_{k\in\mathbb{N}}$ converges to Φ, the zeros of Ψ_k in M_{S_k} are those of Φ in M_S, *v*), and a standard diagonal argument, we can obtain a sequence satisfying the conclusion of the lemma, proving the claim. □

In the sequel we will assume that ϕ_1, ϕ_2 and $d\xi$ never vanish on C_S.

Label $\eta = \phi_1 - \imath\phi_2 \in \Omega_0^*(S) \cap \Omega_0(M_S)$ and observe that $\Theta/\eta = \phi_1 + \imath\phi_2 \in \Omega_0^*(S) \cap \Omega_0(M_S)$. Notice that $(\Theta/\eta)|_{M_S}, \eta|_{M_S} \not\equiv 0$,

$$\phi_1 = \frac{1}{2}\left(\eta + \frac{\Theta}{\eta}\right) \quad \text{and} \quad \phi_2 = \frac{\imath}{2}\left(\eta - \frac{\Theta}{\eta}\right).$$

Let $\mathcal{B}_S$ be a homology basis of $\mathcal{H}_1(S,\mathbb{Z})$ and label ν as its cardinal number. Consider in $\mathcal{F}_0^*(S)$ the maximum norm and the Fréchet differentiable map

$$\mathcal{P} : \mathcal{F}_0^*(S) \to \mathbb{C}^{2\nu}, \quad \mathcal{P}(f) = \left(\int_c \left(e^f\eta + e^{-f}\frac{\Theta}{\eta} - 2\phi_1 \, , \, e^f\eta - e^{-f}\frac{\Theta}{\eta} + 2\imath\phi_2\right)\right)_{c\in\mathcal{B}_S}.$$

Label $\mathcal{A} : \mathcal{F}_0^*(S) \to \mathbb{C}^{2\nu}$ as the Fréchet derivative of $\mathcal{P}$ at 0.

Claim 3.5. *$\mathcal{A}|_{\mathcal{F}_0(W)}$ is surjective.*

Proof. Reason by contradiction and assume that $\mathcal{A}(\mathcal{F}_0(W))$ lies in a complex subspace $\mathcal{U} = \{\left((x_c, y_c)\right)_{c\in\mathcal{B}_S} \in \mathbb{C}^{2\nu} \mid \sum_{c\in\mathcal{B}_S}\left(A_c x_c + B_c y_c\right) = 0\}$, where $A_c, B_c \in \mathbb{C}$, $\forall c \in \mathcal{B}_S$, and

$$\sum_{c\in\mathcal{B}_S}\left(|A_c| + |B_c|\right) \neq 0. \tag{3.1}$$

Then, writing $\Gamma_1 = \sum_{c\in\mathcal{B}_S} A_c\, c$ and $\Gamma_2 = \sum_{c\in\mathcal{B}_S} B_c\, c$, we have

$$\int_{\Gamma_1} f\phi_2 + \imath\int_{\Gamma_2} f\phi_1 = 0, \quad \forall f \in \mathcal{F}_0(W). \tag{3.2}$$

Denote by $\Sigma = \{f \in \mathcal{F}_0(W) \mid (f) \geq (\phi_2|_{M_S})^2\}$ (recall that ϕ_2 never vanishes on C_S). Then for any $f \in \Sigma$ the function $df/\phi_2 \in \mathcal{F}_0^*(S) \cap \mathcal{F}_0(M_S)$, so it can be uniformly approximated on S by functions in $\mathcal{F}_0(W)$. This fact is trivial when f is constant, otherwise use Lemma 3.1. Hence equation (3.2) applies and gives

$$0 = \int_{\Gamma_2} \xi df = \int_{\Gamma_2} f\, d\xi, \quad \forall f \in \Sigma, \tag{3.3}$$

where we have used integration by parts (notice that $f\xi$, ξdf and $fd\xi$ are smooth).

Suppose $\Gamma_2 \neq 0$ and take $[\tau] \in \mathcal{H}^1_{\text{hol}}(W)$ (the first holomorphic De Rham cohomology group of W) and $g \in \mathcal{F}_0(W)$ so that $\int_{\Gamma_2}\tau \neq 0$, the function $f := (\tau + dg)/d\xi$ lies in $\mathcal{F}_0^*(S) \cap \mathcal{F}_0(M_S)$ and $(f|_{M_S}) \geq (\phi_2|_{M_S})^2$. The existence of such 1-form and function follows from well known arguments on Riemann surfaces theory (take into account that (A) implies $d\xi|_{M_S} \not\equiv 0$). By Lemma 3.1, f can be uniformly approximated on S by functions in Σ, so equation (3.3) applies and shows that $0 = \int_{\Gamma_2} fd\xi = \int_{\Gamma_2}(\tau + dg) = \int_{\Gamma_2}\tau \neq 0$, a contradiction. Therefore $\Gamma_2 = 0$.

Replacing $(\xi, \phi_1, \phi_2, \Gamma_1, \Gamma_2)$ by $(1/\xi, \phi_2, \phi_1, \Gamma_2, \Gamma_1)$ and using a symmetric argument, we can prove that $\Gamma_1 = 0$. This contradicts (3.1) and concludes the proof. □

Let $\{e_1, \ldots, e_{2\nu}\}$ be a basis of $\mathbb{C}^{2\nu}$, fix $f_i \in \mathcal{A}^{-1}(e_i) \cap \mathcal{F}_0(W)$ for all i, and set $\mathcal{Q} : \mathbb{C}^{2\nu} \to \mathbb{C}^{2\nu}$ as the analytical map given by

$$\mathcal{Q}((z_i)_{i=1,\ldots,2\nu}) = \mathcal{P}(\sum_{i=1,\ldots,2\nu} z_i f_i).$$

By Claim 3.5 the differential $d\mathcal{Q}_0$ of $\mathcal{Q}$ at 0 is an isomorphism, then there exists a closed Euclidean ball $U \subset \mathbb{C}^{2\nu}$ centered at the origin such that $\mathcal{Q} : U \to \mathcal{Q}(U)$ is an analytical diffeomorphism. Furthermore, notice that $0 = \mathcal{Q}(0) \in \mathcal{Q}(U)$ is an interior point of $\mathcal{Q}(U)$.

Consider a sequence $\{\theta_n\}_{n\in\mathbb{N}} \subset \Omega_0(W)$ uniformly approximating η on S and with $(\theta_n) = (\eta|_{M_S})$ for all n (recall that $\eta|_{M_S} \not\equiv 0$ and see Lemma 3.2).

Label $\mathcal{P}_n : \mathcal{F}_0^*(S) \to \mathbb{C}^{2\nu}$ as the Fréchet differentiable map given by

$$\mathcal{P}_n(f) = \left(\int_c \left(e^f \theta_n + e^{-f} \frac{\Theta}{\theta_n} - 2\phi_1 \, , \, e^f \theta_n - e^{-f} \frac{\Theta}{\theta_n} + 2\imath\phi_2 \right) \right)_{c \in \mathcal{B}_S}, \quad \forall n \in \mathbb{N}.$$

Call $\mathcal{Q}_n : \mathbb{C}^{2\nu} \to \mathbb{C}^{2\nu}$ as the analytical map $\mathcal{Q}_n((z_i)_{i=1,\ldots,2\nu}) = \mathcal{P}_n(\sum_{i=1,\ldots,2\nu} z_i f_i)$ for all $n \in \mathbb{N}$. Since $\{\mathcal{Q}_n\}_{n\in\mathbb{N}} \to \mathcal{Q}$ uniformly on compacts subsets of $\mathbb{C}^{2\nu}$, without loss of generality we can suppose that $\mathcal{Q}_n : U \to \mathcal{Q}_n(U)$ is an analytical diffeomorphism and $0 \in \mathcal{Q}_n(U)$ for all n. Label $\alpha_n = (\alpha_{1,n}, \ldots, \alpha_{2\nu,n})$ as the unique point in U such that $\mathcal{Q}_n(\alpha_n) = 0$ and note that $\{\alpha_n\}_{n\in\mathbb{N}} \to 0$. Set

$$\eta_n := e^{\sum_{i=1}^{2\nu} \alpha_{i,n} f_i} \theta_n, \quad \phi_{1,n} := \frac{1}{2}\left(\eta_n + \frac{\Theta}{\eta_n}\right) \quad \text{and} \quad \phi_{2,n} := \frac{\imath}{2}\left(\eta_n - \frac{\Theta}{\eta_n}\right), \quad \forall n \in \mathbb{N}$$

and let us check that the sequence $\{\Phi_n = (\phi_{1,n}, \phi_{2,n})\}_{n\in\mathbb{N}}$ satisfies the conclusion of the lemma. Indeed, since $(\eta_n) = (\theta_n) = (\eta|_{M_S})$ one has $\Theta/\eta_n \in \Omega_0(W)$ and so $\Phi_n \in \Omega_0(W)^2$. The convergence of $\{\Phi_n\}_{n\in\mathbb{N}}$ to Φ on S follows from the ones of $\{\theta_n\}_{n\in\mathbb{N}}$ to η and of $\{\alpha_n\}_{n\in\mathbb{N}}$ to 0. A straightforward computation gives (a). The fact that $\mathcal{Q}_n(\alpha_n) = 0$, $n \in \mathbb{N}$, implies (b). Finally, $(\eta_n) = (\eta|_{M_S})$ for all n implies (c).

The proof of the lemma in case (B) goes as follows.

Notice that $\Theta = 0$ is nothing but $\phi_2 = \beta\phi_1$, where $\beta \in \{\imath, -\imath\}$.

As above, we can assume without loss of generality that ϕ_1 never vanishes on C_S (we omit the details). Reasoning as in case (A), we can prove that $\hat{\mathcal{A}}|_{\mathcal{F}_0(W)} : \mathcal{F}_0(W) \to \mathbb{C}^\nu$ is surjective, where $\hat{\mathcal{A}}$ is the Fréchet derivative of $\hat{\mathcal{P}} : \mathcal{F}_0^*(S) \to \mathbb{C}^\nu$, $\hat{\mathcal{P}}(f) = \left(\int_c (e^f - 1)\phi_1\right)_{c\in\hat{\mathcal{B}}_S}$, at 0. Take $\hat{f}_i \in \hat{\mathcal{A}}^{-1}(\hat{e}_i) \cap \mathcal{F}_0(W)$ for all i, where $\hat{\mathcal{B}}_S = \{\hat{e}_1, \ldots, \hat{e}_\nu\}$ is a basis of $\mathbb{C}^\nu$, and define $\hat{\mathcal{Q}} : \mathbb{C}^\nu \to \mathbb{C}^\nu$ by $\hat{\mathcal{Q}}((z_i)_{i=1,\ldots,\nu}) = \hat{\mathcal{P}}(\sum_{i=1,\ldots,\nu} z_i \hat{f}_i)$. Now, consider a sequence $\{\hat{\theta}_n\}_{n\in\mathbb{N}} \subset \Omega_0(W)$ that uniformly approximates ϕ_1 on S and $(\hat{\theta}_n) = (\phi_1|_{M_S})$ for all n (as above, recall that $\phi_1|_{M_S} \not\equiv 0$ and see Lemma 3.2). Set $\hat{\mathcal{P}}_n : \mathcal{F}_0^*(S) \to \mathbb{C}^\nu$ by $\hat{\mathcal{P}}_n(f) = \left(\int_c (e^f \hat{\theta}_n - \phi_1)\right)_{c\in\hat{\mathcal{B}}_S}$, and call $\hat{\mathcal{Q}}_n : \mathbb{C}^\nu \to \mathbb{C}^\nu$ as the analytical map $\hat{\mathcal{Q}}_n((z_i)_{i=1,\ldots,\nu}) = \hat{\mathcal{P}}_n(\sum_{i=1,\ldots,\nu} z_i \hat{f}_i)$ for all $n \in \mathbb{N}$. To finish, reason as in case (A) but setting $\phi_{1,n} := e^{\sum_{i=1}^{\nu} \hat{\alpha}_{i,n} \hat{f}_i} \hat{\theta}_n$ and $\phi_{2,n} := \beta\phi_{1,n}$, where $\hat{\alpha}_n = (\hat{\alpha}_{1,n}, \ldots, \hat{\alpha}_{\nu,n})$ is chosen so that $\hat{\mathcal{Q}}_n(\hat{\alpha}_n) = 0$ and $\{\hat{\alpha}_n\}_{n\in\mathbb{N}} \to 0$. □

4. MAIN RESULTS

The main results of this paper follow as consequence of Lemma 4.1 below. Although the proof of this lemma is inspired by the technique developed in [**AFL**, Lemma 3.1], it represents a wide generalization of that result.

We need the following notations and definitions.

Fix a nowhere zero $\tau_0 \in \Omega_0(\mathcal{N})$ (the existence of such a τ_0 is well known, anyway see [**AFL**] for a proof). Then for any compact subset $K \subset \mathcal{N}$ and any $\theta \in \Omega_0^*(K)$ we set $\|\theta\| := \max_K\{|\theta/\tau_0|\}$. This norm induces the topology of the uniform convergence on $\Omega_0^*(K)$.

Let $K \subset \mathcal{N}$ be a connected compact region and σ^2 a Riemannian metric on K possibly with singularities. Given $P, Q \in K$ we denote by $\text{dist}_{(K,\sigma)}(P,Q) = \min\{\text{length}_\sigma(\alpha) \mid \alpha \text{ curve in } K \text{ joining } P \text{ and } Q\}$. If K_1 and K_2 are two compact sets in K we set $\text{dist}_{(K,\sigma)}(K_1, K_2) = \min\{\text{dist}_{(K,\sigma)}(P,Q) \mid P \in K_1,\ Q \in K_2\}$.

Lemma 4.1. *Let M, K be two compact regions in $\mathcal{N}$ with $M \subset K^\circ$. Assume that M is Runge, K is connected and consider $P_0 \in M^\circ$. Let $\mathcal{I}$ be a conformal Riemannian metric on K possibly with isolated singularities. Let $\mathtt{f} = (\mathtt{f}_1, \mathtt{f}_2) : \mathcal{H}_1(K, \mathbb{Z}) \to \mathbb{C}^2$ be a group homomorphism, $\Theta \in \mho_0(K)$ and*

$\Phi = (\phi_1, \phi_2) \in \Omega_0(M)^2$ *satisfying*

$$\phi_1^2 + \phi_2^2 = \Theta|_M, \quad \mathtt{f}(\gamma) = \int_\gamma \Phi, \forall \gamma \in \mathcal{H}_1(M, \mathbb{Z}),$$

and either of the following conditions:

(A) ϕ_1 *and* ϕ_2 *are linearly independent in* $\Omega_0(M)$.
(B) $\Theta = 0$, $\phi_1 \not\equiv 0$ *and there is* $\beta \in \{\imath, -\imath\}$ *such that* $\mathtt{f}_2 = \beta \mathtt{f}_1$ *and* $\phi_2 = \beta\phi_1$.

Then, for any $\epsilon > 0$ *there exists* $\Psi = (\psi_1, \psi_2) \in \Omega_0(K)^2$ *so that*

(L1) $\|\Psi - \Phi\| < \epsilon$ *on* M,
(L2) $\psi_1^2 + \psi_2^2 = \Theta$,
(L3) $\mathtt{f}(\gamma) = \int_\gamma \Psi, \forall \gamma \in \mathcal{H}_1(K, \mathbb{Z})$,
(L4) $\mathrm{dist}_{(K, \sigma_{(\Psi, \mathcal{I})})}(P_0, \partial K) > 1/\epsilon$, *where* $\sigma^2_{(\Psi,\mathcal{I})} := |\psi_1|^2 + |\psi_2|^2 + \mathcal{I}$, *and*
(L5) *the zeros of* Ψ *on* K *are those of* Φ *on* M *(in particular,* Ψ *never vanishes on* $K - M$*).*

Proof. The proof goes by induction on minus the Euler characteristic of $W - M^\circ$. Since M is Runge then no component of $K - M^\circ$ is a closed disc, and so $-\chi(K - M^\circ) \geq 0$. The basis of the induction is proved in the following

Claim 4.2. *Lemma 4.1 holds if* $\chi(K - M^\circ) = 0$.

Proof. In this case $K^\circ - M = \cup_{j=1}^k A_j$, where A_j are pairwise disjoint open annuli, $k \in \mathbb{N}$. On each A_j we construct a Jorge-Xavier's type labyrinth of compact sets as follows (see **[JX]**). Let $z_j : A_j \to \mathbb{C}$ be a conformal parametrization, and let $C_j \subset A_j$ be a compact region such that C_j contains no singularities of $\mathcal{I}$, $z_j(C_j)$ is a compact annulus of radii r_j and R_j, where $r_j < R_j$, and $z_j(C_j)$ contains the homology of $z_j(A_j)$. This choice is possible since the singularities of $\mathcal{I}$ are isolated. Since $\mathcal{I}|_{C_j}$ has no singularities, we can find a positive constant μ with

$$\mathcal{I} > \mu^2 |dz_j|^2 \quad \text{on } C_j,\ j = 1, \ldots, k. \tag{4.1}$$

Consider a large $m \in \mathbb{N}$ (to be specified later) such that $2/m < \min\{R_j - r_j \mid j = 1, \ldots, k\}$. For any $j \in \{1, \ldots, k\}$ label $s_{j,0} := R_j$ and for any $n \in \{1, \ldots, 2m^2\}$ set $s_{j,n} := R_j - n/m^3$ and consider the compact set in C_j (see Figure 4.1):

$$\mathcal{K}_{j,n} = \left\{ P \in A_j \;\middle|\; s_{j,n} + \frac{1}{4m^3} \leq |z_j(P)| \leq s_{j,n-1} - \frac{1}{4m^3},\ \frac{1}{m^2} \leq \arg((-1)^n z_j(P)) \leq 2\pi - \frac{1}{m^2} \right\}.$$

Then, define

$$\mathcal{K}_j = \bigcup_{n=1}^{2m^2} \mathcal{K}_{j,n} \qquad \text{and} \qquad \mathcal{K} = \bigcup_{j=1}^{k} \mathcal{K}_j.$$

Consider the pair $\Xi = (\varphi_1, \varphi_2) \in \Omega_0(M \cup \mathcal{K})^2$ given by

$$\Xi|_M = \Phi, \qquad \Xi|_{\mathcal{K}_j} = \begin{cases} \left(\frac{1}{2}(\lambda dz_j + \frac{\Theta}{\lambda dz_j}),\ \frac{\imath}{2}(\lambda dz_j - \frac{\Theta}{\lambda dz_j})\right) & \text{if (A) holds} \\ (\lambda dz_j,\ \beta\lambda dz_j) & \text{if (B) holds,} \end{cases} \qquad j = 1, \ldots, k,$$

where $\lambda > \sqrt{2}\,\mu m^4$ is a constant. Notice that $\varphi_1^2 + \varphi_2^2 = \Theta|_{M \cup \mathcal{K}}$.

Let $W \subset \mathcal{N}$ be a domain with finite topology containing K. Applying Lemma 3.3 to the data

$$\hat{W} = W, \quad \hat{S} = M \cup \mathcal{K}, \quad \hat{\Theta} = \Theta, \quad \text{and} \quad \hat{\Phi} = \Xi,$$

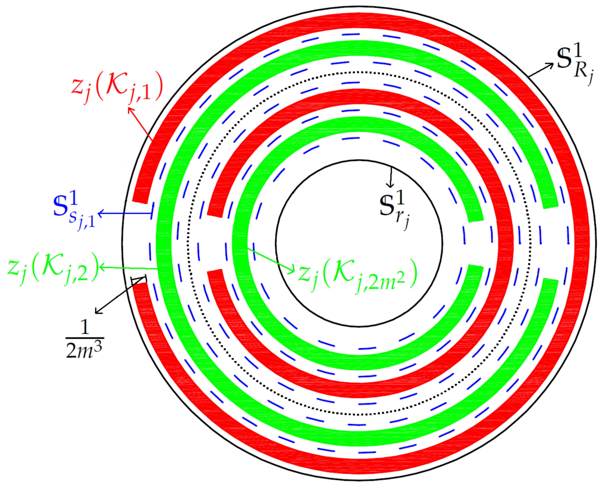

FIGURE 4.1. The labyrinth of compact sets on the annulus $z_j(C_j)$.

we obtain a pair $\Psi = (\psi_1, \psi_2) \in \Omega_0(K)^2$ satisfying (L1), (L2), (L3), (L5) and

$$|\psi_1|^2 + |\psi_2|^2 > \mu^2 m^8 |dz_j|^2 \quad \text{on } \mathcal{K}_j,\ j = 1, \ldots, k. \tag{4.2}$$

Then, taking into account (4.1), (4.2) and the definition of $\mathcal{K}_j$, it is straightforward to check the existence of a positive constant ρ_j depending neither on μ nor m such that

$$\text{length}_{\sigma_{(\Psi,\mathcal{I})}}(\alpha) > \rho_j \cdot \mu \cdot m$$

for any α curve in C_j joining the two components of ∂C_j. Thus, we can choose m large enough so that $\rho_j \cdot \mu \cdot m > 1/\epsilon$ for any $j = 1, \ldots, k$. This choice gives (L4) and we are done. □

The inductive step and so Lemma 4.1 are proved in the following

Claim 4.3. *Consider $n > 0$ and assume that Lemma 4.1 holds if $-\chi(K - M^\circ) < n$. Then it also holds if $-\chi(K - M^\circ) = n$.*

Proof. Since M is Runge, $\jmath_* : \mathcal{H}_1(M, \mathbb{Z}) \to \mathcal{H}_1(K, \mathbb{Z})$ is a monomorphism, where $\jmath : M \to K$ is the inclusion map. Up to this natural identification we will consider $\mathcal{H}_1(M, \mathbb{Z}) \subset \mathcal{H}_1(K, \mathbb{Z})$. Since $-\chi(K - M^\circ) = n > 0$, there exists $\hat{\gamma} \in \mathcal{H}_1(K, \mathbb{Z}) - \mathcal{H}_1(M, \mathbb{Z})$ intersecting $K - M^\circ$ in a compact Jordan arc γ with endpoints $P_1, P_2 \in \partial M$ and otherwise disjoint from $\partial M \cup \partial K$, and such that $S := M \cup \gamma$ is admissible. Notice that in this case $\gamma = C_S$ and $M = M_S$.

Assume (A) holds, and in addition choose $\hat{\gamma}$ so that Θ never vanishes on γ. Consider a pair $\hat{\Phi} = (\hat{\phi}_1, \hat{\phi}_2) \in \Omega_0^*(S)^2 \cap \Omega_0(M_S)^2$ satisfying $\hat{\Phi}|_M = \Phi$, $\hat{\phi}_1^2 + \hat{\phi}_2^2 = \Theta|_S$ and $\int_{\hat{\gamma}} \hat{\Phi} = \mathtt{f}(\hat{\gamma})$ (we leave the details to the reader). By Lemma 3.3, case (A), applied to $\hat{\Phi}$, S, Θ and K°, we can find a compact tubular neighborhood U of S in K° and $\Xi = (\varphi_1, \varphi_2) \in \Omega_0(U)^2$ such that φ_1 and φ_2 are linearly independent in $\Omega_0(U)^2$, $\|\Xi - \Phi\| < \epsilon/2$ on M, $\varphi_1^2 + \varphi_2^2 = \Theta|_U$, the zeros of Ξ on U are those of Φ on M, and $\Xi - \hat{\Phi}$ is exact on S. Since $-\chi(K - U^\circ) < n$, the induction hypothesis applied to Ξ and $\epsilon/2$ gives the existence of a pair $\Psi \in \Omega_0(K)^2$ satisfying the conclusion of the lemma.

Assume now that (B) holds, and take a function $\hat{\phi}_1 \in \Omega_0^*(S) \cap \Omega_0(M_S)$ such that $\hat{\phi}_1|_M = \phi_1$ and $\int_{\hat{\gamma}} \hat{\phi}_1 = \mathtt{f}_1(\hat{\gamma})$. Apply Lemma 3.3, case (B), to the data K°, S and $(\hat{\phi}_1, \beta\hat{\phi}_1)$, and obtain a compact tubular neighborhood U of S in K° and a 1-form $\varphi_1 \in \Omega_0(U)$ such that $\|\varphi_1 - \phi_1\| < \epsilon/4$ on

M, the zeros of φ_1 on U are those of ϕ_1 on M, and $\varphi_1 - \hat{\phi}_1$ is exact on S. As above, the induction hypothesis applied to $(\varphi_1, \beta\varphi_1)$ and $\epsilon/2$ gives a pair $\Psi \in \Omega_0(K)^2$ proving the claim. $\square$

This finishes the proof of the lemma. $\square$

Now we can state and prove the main theorem of this paper.

Theorem 4.4. *Let $M \subset \mathcal{N}$ be a Runge compact region. Let $\mathcal{I}$ be a conformal Riemannian metric on $\mathcal{N}$ possibly with isolated singularities. Consider $\mathtt{f} = (\mathtt{f}_1, \mathtt{f}_2) : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{C}^2$ be a group homomorphism, $\Theta \in \mho_0(\mathcal{N})$ and $\Phi = (\phi_1, \phi_2) \in \Omega_0(M)^2$ satisfying*

$$\phi_1^2 + \phi_2^2 = \Theta|_M, \quad \mathtt{f}(\gamma) = \int_\gamma \Phi, \forall \gamma \in \mathcal{H}_1(M, \mathbb{Z}),$$

and either of the following conditions:

(A) *ϕ_1 and ϕ_2 are linearly independent in $\Omega_0(M)$.*
(B) *$\Theta = 0$, $\phi_1 \not\equiv 0$ and there is $\beta \in \{\imath, -\imath\}$ such that $\mathtt{f}_2 = \beta \mathtt{f}_1$ and $\phi_2 = \beta\phi_1$.*

Then, for any $\epsilon > 0$ there exists $\Psi = (\psi_1, \psi_2) \in \Omega_0(\mathcal{N})^2$ so that

(T1) *$\|\Psi - \Phi\| < \epsilon$ on M,*
(T2) *$\psi_1^2 + \psi_2^2 = \Theta$,*
(T3) *$\mathtt{f}(\gamma) = \int_\gamma \Psi, \forall \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$,*
(T4) *$|\psi_1|^2 + |\psi_2|^2 + \mathcal{I}$ is a complete conformal Riemannian metric on $\mathcal{N}$ with singularities at the zeros of $|\phi_1|^2 + |\phi_2|^2 + \mathcal{I}$ on M, and*
(T5) *the zeros of Ψ on $\mathcal{N}$ are those of Φ on M (in particular, Ψ never vanishes on $\mathcal{N} - M$).*

Proof. Label $M_1 = M$ and let $\{M_n \mid n \geq 2\}$ be an exhaustion of $\mathcal{N}$ by Runge connected compact regions with $M_n \subset M_{n+1}^\circ$ for all $n \in \mathbb{N}$. Fix a base point $P_0 \in M^\circ$ and a positive $\varepsilon < \min\{\epsilon, 1\}$ which will be specified later.

Label $\Phi_1 = \Phi$, and by Lemma 4.1 and an inductive process, construct a sequence of pairs $\{\Phi_n = (\phi_{j,n})_{j=1,2}\}_{n \in \mathbb{N}}$ satisfying that

(a) $\Phi_n \in \Omega_0(M_n)^2, \forall n \in \mathbb{N}$,
(b) $\|\Phi_n - \Phi_{n-1}\| < \varepsilon/2^n$ on $M_{n-1}, \forall n \geq 2$,
(c) $\phi_{1,n}^2 + \phi_{2,n}^2 = \Theta|_{M_n}, \forall n \in \mathbb{N}$,
(d) $\mathtt{f}(\gamma) = \int_\gamma \Phi_n, \forall \gamma \in \mathcal{H}_1(M_n, \mathbb{Z}), \forall n \in \mathbb{N}$,
(e) $\mathrm{dist}_{(M_n, \sigma_{(\Phi_n, \mathcal{I})})}(P_0, \partial M_n) > 2^n$, where $\sigma^2_{(\Phi_n, \mathcal{I})} = |\phi_{1,n}|^2 + |\phi_{2,n}|^2 + \mathcal{I}, \forall n \geq 2$, and
(f) the zeros of Φ_n on M_n are those of Φ on $M, \forall n \in \mathbb{N}$.

Since $\cup_{n \in \mathbb{N}} M_n = \mathcal{N}$, items (a) and (b) and Harnack's theorem, then the sequence $\{\Phi_n\}_{n \in \mathbb{N}}$ uniformly converges on compact subsets of $\mathcal{N}$ to a pair $\Psi = (\psi_j)_{j=1,2} \in \Omega_0(\mathcal{N})$ satisfying (T1). Items (c) and (d) directly give (T2) and (T3), respectively. Since $\{\Phi_n\}_{n \in \mathbb{N}}$ uniformly converges to Ψ and (f), Hurwitz's theorem gives that either the zeros of Ψ on $\mathcal{N}$ are those of Φ on M or $\psi_1 = 0$ or $\psi_2 = 0$. However, (b) gives $\|\Psi - \Phi\| \leq \varepsilon$ on M and so $\psi_j|_M \not\equiv 0$, $j = 1, 2$, provided that ε is small enough. This proves (T5). Finally (T5) and (e) imply (T4) and we are done. $\square$

Corollary 4.5. *Let $\mathfrak{H}$, $X = (X_i)_{i=3,\ldots,\mathrm{N}} : \mathcal{N} \to \mathbb{R}^{\mathrm{N}-2}$ and $\mathtt{p} = (\mathtt{p}_j)_{j=1,\ldots,\mathrm{N}} : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^{\mathrm{N}}$ be a 2-form in $\mho_0(\mathcal{N})$, a non-constant harmonic map and a group homomorphism, respectively, satisfying that*

- *$\mathtt{p}_i(\gamma) = \mathrm{Im} \int_\gamma \partial_z X_i, \forall \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z}), \forall i = 3, \ldots, \mathrm{N}$, and*
- *$\mathtt{p}_1 = \mathtt{p}_2 = 0$ when $\mathfrak{H} = \sum_{i=3}^{\mathrm{N}} (\partial_z X_i)^2$.*

Then there exists a weakly complete harmonic map $Y = (Y_j)_{j=1,\ldots,\mathrm{N}} : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ with

(I) $(Y_i)_{i=3,\ldots,\mathrm{N}} = X$,
(II) $\mathrm{p}_Y = \mathrm{p}$, *and*
(III) $Q_Y = \mathfrak{H}$.

Furthermore, if X is full then Y can be chosen to be full, and if X is an immersion then Y is.

Proof. Label $\Theta := \mathfrak{H} - \sum_{i=3}^{\mathrm{N}} (\partial_z X_i)^2$, and assume for a moment that $\Theta \not\equiv 0$. Consider a compact disc $K \subset \mathcal{N}$ and $\eta \in \Omega_0(K)$ such that both η and ϕ_1 never vanish on K, and ϕ_1 and ϕ_2 are linearly independent in $\Omega_0(K)$, where $\phi_1 := \frac{1}{2}(\eta + \Theta/\eta)$ and $\phi_2 := \frac{\imath}{2}(\eta - \Theta/\eta)$. Consider a pair $\Psi = (\psi_1, \psi_2)$ obtained from Theorem 4.4, case (A), applied to the data

$$\mathcal{N}, \quad M = K, \quad \mathcal{I} = \sum_{i=3}^{\mathrm{N}} |\partial_z X_i|^2, \quad \Theta, \quad \Phi = (\phi_1, \phi_2), \quad \mathtt{f} = \imath(\mathrm{p}_1, \mathrm{p}_2)$$

and $\epsilon > 0$ to be specified later. Fix a point $P_0 \in \mathcal{N}$ and define $Y_k(P) = \operatorname{Re} \int_{P_0}^P \psi_k$, $\forall P \in \mathcal{N}$, $k = 1, 2$, and $Y_k = X_k$, $\forall\, k = 3, \ldots, \mathrm{N}$.

Statements (I), (II) and (III) trivially follow from the definition of Θ and $\mathtt{f}$, and properties (T2) and (T3). Moreover, (T4) and the fact that ϕ_1 never vanishes on K give that $\sum_{j=1}^{\mathrm{N}} |\partial_z Y_j|^2$ is a complete conformal metric on $\mathcal{N}$, and so Y is weakly complete. Finally, if X is full then we can choose η so that the map

$$K \to \mathbb{R}^{\mathrm{N}}, \quad P \mapsto \left(\int_{P_0}^P \phi_1 \,,\, \int_{P_0}^P \phi_2 \,,\, X(P) \right)$$

is full as well. Then (T1) gives the fullness of Y provided that ϵ is chosen small enough.

Assume now that $\Theta = 0$. Take an exact $\phi_1 \in \Omega_0(M)$, $\phi_1 \not\equiv 0$, and consider a pair Ψ obtained by applying Theorem 4.4, case (B), to the data

$$\mathcal{N}, \quad M = K, \quad \mathcal{I} = \sum_{i=3}^{\mathrm{N}} |\partial_z X_i|^2, \quad \Theta = 0, \quad \Phi = (\phi_1, \imath\phi_1), \quad \mathtt{f} = 0$$

and $\epsilon > 0$. To finish argue as above. □

Corollary 4.6. *Let* $X = (X_i)_{i=3,\ldots,n} : \mathcal{N} \to \mathbb{R}^{\mathrm{N}-2}$ *and* $\mathrm{p} = (\mathrm{p}_j)_{j=1,\ldots,\mathrm{N}} : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^{\mathrm{N}}$ *be a non-constant harmonic map and a group homomorphism, respectively, satisfying that*

- $\mathrm{p}_i(\gamma) = \operatorname{Im} \int_\gamma \partial_z X_i$, $\forall \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$, $\forall i = 3, \ldots, \mathrm{N}$, *and*
- $\mathrm{p}_1 = \mathrm{p}_2 = 0$ *when* $\sum_{i=3}^{\mathrm{N}} (\partial_z X_i)^2 = 0$.

Then there exists a complete conformal minimal immersion $Y = (Y_j)_{j=1,\ldots,\mathrm{N}} : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ *with* $(Y_i)_{i=3,\ldots,\mathrm{N}} = X$ *and* $\mathrm{p}_Y = \mathrm{p}$. *Furthermore, Y can be chosen full provided that X is.*

Proof. Apply Corollary 4.5 for $\mathfrak{H} = 0$ and see Remark 2.4. □

Corollary 4.7. *Let* $\mathcal{N}$ *be a bounded planar domain. Then there exists a complete non-proper holomorphic embedding of* $\mathcal{N}$ *in* $\mathbb{C}^2$.

Proof. Consider $X = (X_3, X_4) : \mathcal{N} \to \mathbb{R}^2 \equiv \mathbb{C}$ given by $X(z) = z$. Let $Y = (Y_j)_{j=1,\ldots,4} : \mathcal{N} \to \mathbb{R}^4$ be an immersion obtained from Corollary 4.6 applied to the data $\mathcal{N}$, X and $\mathrm{p} = 0$. Since X is injective, Y is an embedding. Finally, observe that Y is non-proper. Indeed, otherwise the holomorphic function $Y_1 + \imath Y_2$ would be proper on $\mathcal{N}$, contradicting that $\mathcal{N}$ is hyperbolic. □

Corollary 4.8. *Let* $\mathrm{p} : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}^{\mathrm{N}}$ *be a group homomorphism.*

Then there exists a conformal complete minimal immersion $Y : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}$ *satisfying*

- $\mathrm{p}_Y = \mathrm{p}$,

- *Y is non-decomposable and full,*
- *G_Y is non-degenerate, and*
- *G_Y fails to intersect* N *hyperplanes of* $\mathbb{CP}^{\mathrm{N}-1}$ *in general position.*

Proof. We need the following

Claim 4.9 ([**AFL**, Theorem 4.2])**.** *For any group homomorphism $\hat{\mathrm{p}} : \mathcal{H}_1(\mathcal{N}, \mathbb{Z}) \to \mathbb{R}$ there exists a never vanishing $\phi \in \Omega_0(\mathcal{N})$ with $\int_\gamma \phi = \imath\hat{\mathrm{p}}(\gamma)$, $\forall \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$.*

Assume first that N is even.

Consider a nowhere zero $\phi \in \Omega_0(\mathcal{N})$ (see Claim 4.9) and a compact disc $M \subset \mathcal{N}$. Fix $P_0 \in M^\circ$ and take $\lambda_j \in \mathbb{C} - \{0\}$ and $\Phi_j = (\phi_{j,1}, \phi_{j,2}) \in \Omega_0(M)^2$, $j = 1, \ldots, \mathrm{N}/2$, so that

- $\sum_{j=1}^{\mathrm{N}/2} \lambda_j^2 = 0$,
- $\phi_{j,1}$ and $\phi_{j,2}$ are linearly independent in $\Omega_0(M)$ and $\phi_{j,1}^2 + \phi_{j,2}^2 = \lambda_j^2 \phi^2|_M$, $\forall j = 1, \ldots, \mathrm{N}/2$,
- the minimal immersion $X : M \to \mathbb{R}^{\mathrm{N}}$, $X(P) = \mathrm{Re}(\int_{P_0}^P (\Phi_j)_{j=1,\ldots,\mathrm{N}/2})$ is non-decomposable and full, and
- G_X is non-degenerate.

Write $\mathrm{p} = (\mathrm{p}_k)_{k=1,\ldots,\mathrm{N}}$, and for any $j = 1, \ldots, \mathrm{N}/2$ consider $\Psi_j = (\psi_{j,1}, \psi_{j,2}) \in \Omega_0(\mathcal{N})^2$ given by Theorem 4.4, case (A), applied to the data

$$\mathcal{N}, \quad M, \quad \mathcal{I} = |\phi|^2, \quad \mathtt{f} = \imath(\mathrm{p}_{2j-1}, \mathrm{p}_{2j}), \quad \Theta = \lambda_j^2 \phi^2, \quad \Phi = \Phi_j,$$

and $\epsilon > 0$ which will be specified later. Define

$$Y : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}, \quad Y(P) = \mathrm{Re}\left(\int_{P_0}^P (\Psi_j)_{j=1,\ldots,\mathrm{N}/2}\right).$$

Statement (T3) in Theorem 4.4 gives that Y is well defined. From (T2) follows that $\sum_{j=1}^{\mathrm{N}/2}(\psi_{j,1}^2 + \psi_{j,2}^2) = 0$, and so Y is conformal. Moreover, $\sum_{j=1}^{\mathrm{N}/2}(|\psi_{j,1}|^2 + |\psi_{j,2}|^2) \geq |\psi_{1,1}|^2 + |\psi_{1,2}|^2 \geq \frac{1}{|\lambda_1|^2+1}(|\psi_{1,1}|^2 + |\psi_{1,2}|^2 + |\phi|^2)$ that is a complete Riemannian metric on $\mathcal{N}$ (take into account (T4)). Therefore, Y is a complete conformal minimal immersion. Item (T3) implies that $\mathrm{p}_Y = \mathrm{p}$. Since X is non-decomposable and full and G_X is non-degenerate, then Y and G_Y are, provided that ϵ is chosen small enough (see (T1)). Finally, observe that $\psi_{j,1}^2 + \psi_{j,2}^2$ never vanishes on $\mathcal{N}$ for all $j = 1, \ldots, \mathrm{N}/2$, hence G_Y fails to intersect the hyperplanes

$$\Pi_{j,\delta} := \left\{[(w_k)_{k=1,\ldots,\mathrm{N}}] \in \mathbb{CP}^{\mathrm{N}-1} \mid w_{2j-1} + (-1)^\delta \imath w_{2j} = 0\right\}, \quad \forall (j, \delta) \in \{1, \ldots, \mathrm{N}/2\} \times \{0, 1\},$$

which are located in general position.

Assume now that N is odd.

Write $\mathrm{p} = (\mathrm{p}_k)_{k=1,\ldots,\mathrm{N}}$ and consider a nowhere zero $\phi \in \Omega_0(\mathcal{N})$ with $\int_\gamma \phi = \imath \mathrm{p}_{\mathrm{N}}(\gamma)$, $\forall \gamma \in \mathcal{H}_1(\mathcal{N}, \mathbb{Z})$ (see Claim 4.9). Fix a compact disc $M \subset \mathcal{N}$ and a point $P_0 \in M^\circ$. Take $\lambda_j \in \mathbb{C} - \{0\}$ and $\Phi_j = (\phi_{j,1}, \phi_{j,2}) \in \Omega_0(M)^2$, $j = 1, \ldots, (\mathrm{N}-1)/2$ so that:

- $\sum_{j=1}^{(\mathrm{N}-1)/2} \lambda_j^2 = -1$,
- $\phi_{j,1}$ and $\phi_{j,2}$ are linearly independent in $\Omega_0(M)$ and $\phi_{j,1}^2 + \phi_{j,2}^2 = \lambda_j^2 \phi^2|_M$, $\forall j = 1, \ldots, (\mathrm{N} - 1)/2$,
- the minimal immersion $X : M \to \mathbb{R}^{\mathrm{N}}$, $X(P) = \mathrm{Re}(\int_{P_0}^P ((\Phi_j)_{j=1,\ldots,(\mathrm{N}-1)/2}, \phi)$ is non-decomposable and full, and
- G_X is non-degenerate.

For any $j = 1, \ldots, (\mathrm{N}-1)/2$ consider $\Psi_j = (\psi_{j,1}, \psi_{j,2}) \in \Omega_0(\mathcal{N})^2$ given by Theorem 4.4, case (A), applied to the data

$$\mathcal{N}, \quad M, \quad \mathcal{I} = |\phi|^2, \quad \mathtt{f} = \imath(\mathrm{p}_{2j-1}, \mathrm{p}_{2j}), \quad \Theta = \lambda_j^2 \phi^2, \quad \Phi = \Phi_j,$$

and $\epsilon > 0$ which will be specified later.

As above

$$Y : \mathcal{N} \to \mathbb{R}^{\mathrm{N}}, \quad Y(P) = \operatorname{Re}\left(\int_{P_0}^{P} ((\Psi_j)_{j=1,\ldots,(\mathrm{N}-1)/2}, \phi) \right)$$

is the immersion we are looking for, provided that ϵ is small enough. In this case G_Y fails to intersect the following hyperplanes of $\mathbb{CP}^{\mathrm{N}-1}$ located in general position:

$$\Pi_{j,\delta} := \left\{ [(w_k)_{k=1,\ldots,\mathrm{N}}] \in \mathbb{CP}^{\mathrm{N}-1} \mid w_{2j-1} + (-1)^{\delta} \imath w_{2j} = 0 \right\},$$

$\forall (j, \delta) \in \{1, \ldots, (\mathrm{N}-1)/2\} \times \{0, 1\}$, and

$$\Pi := \left\{ [(w_k)_{k=1,\ldots,\mathrm{N}}] \in \mathbb{CP}^{\mathrm{N}-1} \mid w_{\mathrm{N}} = 0 \right\}.$$

The proof is done. □

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