A vanishing theorem for a class of logarithmic $\mathcal{D}$-modules

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To our beloved friend and colleague Pilar Pisón, in memoriam.

Abstract

Let $\mathcal{O}_X$ (resp. $\mathcal{D}_X$) be the sheaf of holomorphic functions (resp. the sheaf of linear differential operators with holomorphic coefficients) on $X = \mathbb{C}^n$. Let $D \subset X$ be a locally weakly quasi-homogeneous free divisor defined by a polynomial f. In this paper we prove that, locally, the annihilating ideal of $1/f^k$ over $\mathcal{D}_X$ is generated by linear differential operators of order 1 (for k big enough). For this purpose we prove a vanishing theorem for the extension groups of a certain logarithmic $\mathcal{D}_X$–module with $\mathcal{O}_X$. The logarithmic $\mathcal{D}_X$–module is naturally associated with D (see Notation 1.1). This result is related to the so called Logarithmic Comparison Theorem.

1 Introduction

Let us denote by $\mathcal{O}_X$ the sheaf of holomorphic functions on $X := \mathbb{C}^n$ and by $\mathcal{D}_X$ the sheaf of linear differential operators with holomorphic coefficients on X. A local section P of $\mathcal{D}_X$ is a finite sum $P = \sum_{\alpha\in\mathbb{N}^n} a_\alpha(x)\partial^\alpha$ where $x = (x_1, \ldots, x_n)$, $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $a_\alpha(x)$ is a local section of the sheaf $\mathcal{O}_X$ and ∂^α stands for $\partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$ each ∂_i being the partial derivative with respect to the variable x_i. The order of such an element P is by definition the non negative integer $\mathrm{ord}(P) := \max\{|\alpha| := \sum_i \alpha_i \,|\, a_\alpha(x) \neq 0\}$. For each point $p \in X$ we will write $\mathcal{O}_p := \mathcal{O}_{X,p}$ and $\mathcal{D}_p := \mathcal{D}_{X,p}$.

Let us fix a point $p \in X$. Denote by $Der(\mathcal{O}_p)$ the $\mathcal{O}_p$-module of $\mathbb{C}$-derivations of $\mathcal{O}_p$. The elements in $Der(\mathcal{O}_p)$ are called (germs of) *vector fields* at the point p. This yields to the sheaf $Der(\mathcal{O}_X)$ of vector fields on X. Vector fields are linear differential operators of order 1.

Let D be a divisor (i.e. a hypersurface) on X. Following K. Saito [Saito 1980], a (germ of) vector field $\delta \in Der(\mathcal{O}_p)$ is said to be *logarithmic* with respect to D if $\delta(f) = af$ for some $a \in \mathcal{O}_p$, where f is a local (reduced) equation of the germ $(D, p) \subset (X, p)$. The $\mathcal{O}_p$-module of logarithmic vector fields (or logarithmic derivations) with respect to D is denoted by $Der(-\log D)_p$ and it is closed under the bracket product $[-,-]$. This yields a coherent $\mathcal{O}$-module sheaf denoted by $Der(-\log D)$, which is a sub-module sheaf of $Der(\mathcal{O}_X)$.

Definition 1.1 [Saito 1980] *The germ of divisor $(D,p) \subset (\mathbb{C}^n, p)$ is said to be free if the $\mathcal{O}_p$-module $Der(-\log D)_p$ of germs of logarithmic vector field with respect to D is free (and in this case it is necessarily of rank n). If (D,p) is free we also say that D is free at p. A divisor $D \subset \mathbb{C}^n$ is said to be free if the germ (D,p) is free for any $p \in D$.*

Saito's criterion [Saito 1980] says that a divisor $D \equiv (f=0)$ is free at a point $p \in D$ if and only if there exists a basis $\{\delta_1, \ldots, \delta_n\}$ of $Der(-\log D)_p$, say $\delta_i = \sum_j a_{ij}\partial_j$, whose determinant $\det((a_{ij}))$ is equal to $u \cdot f$, for some invertible power series $u \in \mathcal{O}_p$ (i.e. such that $u(p) \neq 0$). Smooth divisors and normal crossing divisors are free. By [Saito 1980], any plane curve $D \subset \mathbb{C}^2$ is a free divisor.

Notation 1.1 *The quotient*

$$M^{\log D} := \frac{\mathcal{D}_X}{\mathcal{D}_X Der(-\log D)}$$

plays a fundamental role in what follows. It is a coherent left $\mathcal{D}_X$–module and has been introduced in this context in [Calderón 1999]. We are going to consider later some others related $\mathcal{D}_X$-modules.

Attached to each germ (D,p) we will also consider the following two left ideals in $\mathcal{D}_p$. First of all, the annihilating ideal

$$Ann_{\mathcal{D}_p}(1/f) = \{P \in \mathcal{D}_p \text{ such that } P(1/f) = 0\}$$

where f is a local (reduced) equation of the germ (D,p), and the ideal

$$Ann^{(1)}_{\mathcal{D}_p}(1/f) = \mathcal{D}_p\{P \in \mathcal{D}_p \text{ such that } P(1/f) = 0 \text{ and } \operatorname{ord}(P) = 1\}.$$

An order 1 operator $P \in \mathcal{D}_p$ such that $P(1/f) = 0$ must have the form $\delta + \frac{\delta(f)}{f}$ where δ is a logarithmic derivation in $Der(-\log D)_p$.

Notation 1.2 *Let us denote by $\widetilde{M}^{\log D}$ the coherent $\mathcal{D}_X$-module with stalks*

$$(\widetilde{M}^{\log D})_p := \frac{\mathcal{D}_p}{Ann^{(1)}_{\mathcal{D}_p}(1/f)}$$

for f a local reduced equation of (D,p). Although the previous quotient module depends on the reduced equation f of the germ (D,p) they are all isomorphic for different reduced equations.

The module $\widetilde{M}^{\log D}$ admits in the free Spencer case (see [Castro and Ucha 2002]; see also [Calderón and Narváez 2005] for an intrinsic treatment of these objects) a free resolution (called the *logarithmic Spencer resolution*) analogous to the one of $M^{\log D}$ (see [Calderón 1999]):

$$\mathcal{D}_X \otimes_{\mathcal{O}_X} \wedge^{\bullet} \widetilde{Der}(-\log D) \to \widetilde{M}^{\log D} \to 0$$

where $\widetilde{Der}(-\log D)$ denotes the free $\mathcal{O}_X$-module whose stalks are

$$\widetilde{Der}(-\log D)_p := \{\delta + \frac{\delta(f)}{f} \mid \delta \in Der(-\log D)_p\}.$$

2 Weak quasi-homogeneity

In this paper we will consider a weight vector as an element $(w_1, \ldots, w_n) \in \mathbb{Q}^n$ with non negative coordinates and such that at least one w_i is strictly positive.

A weight vector $w = (w_1, \ldots, w_n)$ defines a filtration on the ring $\mathcal{O} = \mathbb{C}\{x\} = \mathbb{C}\{x_1, \ldots, x_n\}$ of convergent power series with complex coefficients.

If $g(x) = \sum_\alpha g_\alpha x^\alpha$ is a non zero element in $\mathcal{O}$ we define its weight or its w-order as $\mathrm{ord}_w(g) := \min\{\alpha \cdot w = \sum_i \alpha_i w_i \mid g_\alpha \neq 0\}$. By definition the w-order of 0 is $+\infty$.

The so called w-filtration on $\mathcal{O}$, which is a decreasing filtration, is defined by

$$F_\nu = F_\nu(\mathcal{O}) := \{g \in \mathcal{O} \mid \mathrm{ord}_w(g) \geq \nu\}$$

for all $\nu \in \mathbb{Q}$. We have $F_\nu = \mathcal{O}$ for $\nu < 0$.

The associated graded ring is by definition

$$\mathrm{gr}^w(\mathcal{O}) := \oplus_{\nu \geq 0} \frac{F_\nu}{F_{\nu+1}}.$$

Let us denote by $r = r(w)$ the number of non zero coordinates of w. By assumption $1 \leq r \leq n$. The graded ring $\mathrm{gr}^w(\mathcal{O})$ is isomorphic to a polynomial ring in r variables with coefficients in a convergent power series ring in $n - r$ variables. To this end, assume (applying if necessary a permutation of the components $(x_1, \ldots, x_n)$) that $w_i > 0$ for $i = 1, \ldots, r$ and $w_j = 0$ for $j = r+1, \ldots, n$. Let us write $x' = (x_1, \ldots, x_r)$, $x'' = (x_{r+1}, \ldots, x_n)$ and define

$$\mathbb{C}\{x''\}[x']_\nu = \left\{ \sum_{\beta \in \mathbb{N}^r} g_\beta(x'')(x')^\beta \in \mathbb{C}\{x''\}[x'] \mid \sum_i \beta_i w_i = \nu \text{ if } g_\beta \neq 0 \right\}.$$

Then the vector space $F_\nu / F_{\nu+1}$ is isomorphic to $\mathbb{C}\{x''\}[x']_\nu$. In this way, the weight vector w induces a graded structure on the ring $\mathbb{C}\{x''\}[x'] = \oplus_{\nu \geq 0} \mathbb{C}\{x''\}[x']_\nu$. The graded rings $\mathrm{gr}^w(\mathcal{O})$ and $\mathbb{C}\{x''\}[x']$ are then isomorphic as graded rings.

If no confusion arises elements in $\mathbb{C}\{x''\}[x']_\nu$ are called weakly quasi-homogeneous (or WQH) power series of weight ν (with respect to the weight vector w). Any non zero element g in $\mathcal{O}$ can be written in an unique way as a sum

$$g = \sum_{\nu \geq 0} g_\nu$$

where each g_ν is a WQH power series of weight ν. If the weight w has no zero coordinates (i.e. if $r = r(w) = n$) then $F_\nu / F_{\nu+1}$ is isomorphic to $\mathbb{C}[x]_\nu$ the vector space of quasi-homogeneous (QH) polynomials of weight ν.

If a WQH power series $f(x)$ has strictly positive weight $\nu = \mathrm{ord}_w(f)$ then it is WQH of weight 1 with respect to the weight vector $(w_1/\nu, \ldots, w_n/\nu)$.

Definition 2.1 *Let p be a point in $\mathbb{C}^n$. A germ of divisor $(D, p) \subset (\mathbb{C}^n, p)$ is said to be weakly quasi-homogeneous (WQH) if it can be defined by a WQH germ of convergent power series around p. If $U \subset \mathbb{C}^n$ is a non empty open set and $D \subset U$ is a divisor, we say that D is locally weakly quasi-homogeneous (LWQH) if for any point $p \in D$ the germ (D, p) is WQH.*

If $r(w) = n$ then weakly quasi-homogeneity is nothing but classical quasi-homogeneity and locally weakly quasi-homogeneity coincides with the notion of locally quasi-homogeneity (see [Castro et al. 1996]), i.e. every locally quasi-homogeneous (LQH) divisor is LWQH. The reciprocal does not hold. For example, the surface defined in $\mathbb{C}^3$ by the polynomial $xy(x+y)(xz+y)$ is LWQH but it is not LQH (see [Calderón et al. 2002]).

The $\mathcal{O}$–module of germs of holomorphic vector fields $\mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ is also filtered with respect to the given weight vector w, just by giving to each variable x_i the weight w_i and the weight $-w_i$ to the partial derivative ∂_i. The w–order of a non zero element $\delta = \sum_i a_i \partial_i \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ is then the (possibly negative) rational number

$$\mathrm{ord}_w(\delta) = \min\{\mathrm{ord}_w(a_i) - w_i \mid i = 1, \ldots, n\}.$$

A vector field $\delta = \sum_i a_i \delta_i = \sum_{i,\alpha} a_{i,\alpha} x^\alpha \partial_i \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ is said to be WQH of weight (or w–order) $\mu \in \mathbb{Q}$ with respect to the weight vector w if all monomials $a_{i,\alpha} x^\alpha \partial_i$ in δ have the same weight μ, i.e. if $a_{i,\alpha} \neq 0$ then $\alpha \cdot w - w_i = \mu$. Any non zero vector field $\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$ can be written in a unique way as a sum $\delta = \sum_{\mu \in \mathbb{Q}} \delta_\mu$ where δ_μ is the WQH part of δ of w-order μ.

We denote by $\chi = \sum w_i x_i \partial_i$ the Euler vector field with respect to w. It is WQH of weight 0. If $g \in \mathcal{O}$ is WQH of weight ν then $\chi(g) = \nu g$.

Remark 2.2 *For any WQH vector field δ of weight ν, a straightforward calculation proves that $[\chi, \delta] = \chi\delta - \delta\chi = \nu\delta$.*

3 Two basic lemmata

Let $(D, 0) \subset (\mathbb{C}^n, 0)$ be a germ of a WQH free divisor, defined by some WQH power series $f \in \mathcal{O}$ with respect to a weight vector $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n$ and assume that the weight of f is 1. We recall that $Der(-\log D)_0$ stands for the $\mathcal{O}_0$–module of germs of logarithmic derivations with respect to $(D, 0)$.

Lemma 3.1 *There exists a basis* $\{\delta_1, \ldots, \delta_n\}$ *of* $Der(-\log D)_0$ *such that:*

1. $\delta_1 = \chi$.
2. *Every* δ_i *is WQH with respect to the weight vector* w *and* $\delta_i(f) = 0$ *for* $i \geq 2$.
3. *If we write* $\delta_i = \sum a_{ij}\partial_j$ *for some* a_{ij} *in* $\mathcal{O}$, *then* $\det(a_{ij}) = f$.

Proof. The result being well known, we include a complete proof for the sake of completeness. First of all, we have

$$Der(-\log D)_0 = \Theta_f \oplus \mathcal{O} \cdot \chi,$$

where Θ_f is the $\mathcal{O}$–module of vector fields annihilating f. The above decomposition follows from the equality $\delta = (\delta - \frac{\delta(f)}{f}\chi) + \frac{\delta(f)}{f}\chi$ and the fact that $(\delta - \frac{\delta(f)}{f}\chi)(f) = 0$, which holds for any $\delta \in Der(-\log D)_0$.

As D is free, that is, $Der(-\log D)_0$ is free of rank n, then Θ_f is free of rank $n-1$.

As f is WQH of weight or w-order 1, so $f_i := \frac{\partial f}{\partial x_i}$ is WQH of weight or w–order $1 - w_i$ (we consider, as usual, the power series 0 to be WQH of order ν for any $\nu \in \mathbb{Q}$).

Let us denote by $\mathrm{Syz}_{\mathcal{O}}(f_1, \ldots, f_n)$ the $\mathcal{O}$–module of syzygies among $(f_1, \ldots, f_n)$. The $\mathcal{O}$-modules Θ_f and $\mathrm{Syz}_{\mathcal{O}}(f_1, \ldots, f_n)$ are naturally isomorphic.

Let us write A for the graded ring $\mathbb{C}\{x''\}[x']$ and $A_\nu = \mathbb{C}\{x''\}[x']_\nu$ for $\nu \in \mathbb{Q}$ (see the introduction for the notations). Let us consider the A–module $\Theta_{A,f}$ of vector fields with coefficients in A annihilating f. The A-modules $\Theta_{A,f}$ and $\mathrm{Syz}_A(f_1, \ldots, f_n)$ are naturally isomorphic. By assumption $f, f_1, \ldots, f_n$ are homogeneous elements in A (more precisely, they are WQH power series of w-order $1, 1 - w_1, \ldots, 1 - w_n$ respectively). Then the syzygy A-module $\mathrm{Syz}_A(f_1, \ldots, f_n)$ (resp. $\Theta_{A,f}$) is finitely generated and it is also graded with respect to the weight vector w. An element $(a_1, \ldots, a_n) \in \mathrm{Syz}_A(f_1, \ldots, f_n)$ (resp. $\sum_i a_i\partial_i \in \Theta_{A,f}$) is WQH of w-order $\mu \in \mathbb{Q}$ if and only if it satisfies the condition $a_i \in A_{\mu+w_i}$ for $i = 1, \ldots, n$. Then the A–modules $\Theta_{A,f}$ and $\mathrm{Syz}_A(f_1, \ldots, f_n)$ are naturally isomorphic as graded A–modules. In fact this graded structure is induced on $\mathrm{Syz}_A(f_1, \ldots, f_n) \subset A^n[-w]$ by the shifted graded structure on $A^n[-w]$ where $(A^n[-w])_\mu = \sum_i A_{\mu+w_i}$ for any $\mu \in \mathbb{Q}$. As the inclusion $A \subset \mathcal{O}$ is flat, the $\mathcal{O}$–module $\mathrm{Syz}_{\mathcal{O}}(f_1, \ldots, f_n)$ (and so Θ_f) has a finite system $\{\eta_1, \ldots, \eta_m\}$ of WQH generators.

Applying Saito's criterion (see [Saito 1980]) to $\{\chi, \eta_1, \ldots, \eta_m\}$ we can choose among these vectors fields n elements generating $Der(-\log D)_0$ and the determinant of its coefficients being equal to uf, with u inversible in $\mathcal{O}$. Moreover, χ must be in this generating system, so we write the system $\{\delta_1, \delta_2, \ldots, \delta_n\}$ with $\delta_1 = \chi$ and $\mathrm{weight}(\delta_i) = \nu_i$ for some $\nu_i \in \mathbb{Q}$, $i = 2, \ldots, n$. If we write $\delta_i = \sum a_{ij}\partial_j$ then $\mathrm{weight}(a_{ij}) = w_j + \nu_i$, so the determinant $\det((a_{ij})) = uf$ is WQH of weight $\nu = \sum w_i + \sum \nu_i$.

If $u = \sum u_\mu$ then $u_\mu = 0$ for all $\mu \neq \nu - 1$, that is, we have $u = u_{\nu-1}$, so $\nu - 1 = 0$. Changing δ_2 by $\frac{1}{u}\delta_2$ we obtain the desired basis.

Lemma 3.2 *Let $(D,0) \subset (\mathbb{C}^n,0)$ be a germ of free divisor as before and $\{\chi, \delta_2, \ldots, \delta_n\}$ a basis of $Der(-\log D)_0$ as in Lemma 3.1, with* weight$(\delta_i) = \nu_i$. *Then for all subset $J \subset \{2, \ldots, n\}$*

$$1 - \sum_{j \in J} \nu_j > 0.$$

Proof. Each $\delta_i = \sum_{j=1}^{n} a_{ij}\partial_j$ is a WQH vector field of weight ν_i and because weight$(\partial_j) = -\omega_j$ it follows that $\nu_i + \omega_j = \text{weight}(a_{ij}) \geq 0$, if $a_{ij} \neq 0$. Let Δ be the matrix whose rows are the weights of the a_{ij}:

$$\Delta = \begin{pmatrix} \omega_1 & \ldots & \omega_r & 0 & \ldots & 0 \\ \nu_2 + \omega_1 & \ldots & \nu_2 + \omega_r & \nu_2 & \ldots & \nu_2 \\ \vdots & & & & & \\ \nu_n + \omega_1 & \ldots & \nu_n + \omega_r & \nu_n & \ldots & \nu_n \end{pmatrix}.$$

Each summand in the determinant of the matrix (a_{ij}) is WQH and since weight$(f) = 1$, at least one summand is non zero and has weight 1. So there exists some $i \in \{1, \ldots, r\}$ such that

$$1 = \omega_i + \sum_{j=2}^{n} (\nu_j + \omega_{i(j)}), \text{ with } i(j) \neq i.$$

If $J \subset \{2, \ldots, n\}$ then

$$1 - \sum_{j \in J} \nu_j = \omega_i + \sum_{j \notin J} (\nu_j + \omega_{i(j)}) + \sum_{j \in J, i(j) \neq i} \omega_{i(j)} > 0.$$

Remark 3.3 *In Theorem 3.5 we will compute some Ext groups of the $\mathcal{D}_p$–module $(\widetilde{M}^{\log D})_p$ for a class of free divisors D and $p \in D$. For this purpose we will use the logarithmic Spencer resolution of $(\widetilde{M}^{\log D})_p$ (see [Calderón 1999]; see also [Castro and Ucha 2002]):*

$$\mathcal{D}_p \bigotimes_{\mathcal{O}_p} \bigwedge^{\bullet} \widetilde{Der}(-\log D)_p \to (\widetilde{M}^{\log D})_p \to 0$$

whose differential is defined as

$$\phi_\ell(P \otimes \widetilde{\delta}_1 \wedge \cdots \wedge \widetilde{\delta}_\ell) = \sum_{i=1}^{\ell} (-1)^{i-1} P\widetilde{\delta}_i \otimes \widetilde{\delta}_1 \wedge \cdots \widehat{()}_i \cdots \wedge \widetilde{\delta}_\ell$$

$$+ \sum_{1 \leq i < j \leq \ell} (-1)^{i+j} P \otimes [\widetilde{\delta}_i, \widetilde{\delta}_j] \wedge \widetilde{\delta}_1 \wedge \cdots \widehat{()}_i \cdots \widehat{()}_j \cdots \wedge \widetilde{\delta}_\ell$$

where $\widehat{()}_i, \widehat{()}_j$ means that corresponding elements are missing and $\widetilde{\delta}$ is nothing but $\delta + \frac{\delta(f)}{f}$ for any $\delta \in Der(-\log D)_p$, once a local reduced equation f of (D,p) has been chosen.

We will take a good basis $\{\delta_1, \ldots, \delta_n\}$ of $Der(-\log D)_p$ as in Lemma 3.1 which gives a corresponding basis $\{\widetilde{\delta}_1, \ldots, \widetilde{\delta}_n\}$ of $\widetilde{Der}(-\log D)_p$.

In addition, it will be useful for handling bases in $\bigwedge^\ell Der(-\log D)_p$ (and in $\bigwedge^\ell \widetilde{Der}(-\log D)_p$) to consider a lexicographical ordering with respect to the indexes of the elements δ_i: $\delta_{i_1} \wedge \cdots \wedge \delta_{i_\ell}$ precedes $\delta_{j_1} \wedge \cdots \wedge \delta_{j_\ell}$ if $i_1 = j_1, \ldots, i_s = j_s$ and $i_{s+1} < j_{s+1}$ for some $s < \ell$.

With respect to these bases we will identify

$$\mathcal{D}_p \bigotimes_{\mathcal{O}_p} \bigwedge^\ell Der(-\log D)_p \text{ and } \mathcal{D}_p \bigotimes_{\mathcal{O}_p} \bigwedge^\ell \widetilde{Der}(-\log D)_p$$

with $\mathcal{D}_p^{\binom{n}{\ell}}$. We will also write $\mathcal{D}_p^{\binom{n}{\ell}}$ as a direct sum $R_\ell \oplus S_\ell$ where

$$R_\ell = \bigoplus_{2 \le j_2 < \cdots < j_\ell \le n} \mathcal{D}_p \cdot \widetilde{\delta}_1 \wedge \widetilde{\delta}_{j_2} \wedge \cdots \wedge \widetilde{\delta}_{j_\ell} \text{ and } S_\ell = \bigoplus_{2 \le i_1 < \cdots < i_\ell \le n} \mathcal{D}_p \cdot \widetilde{\delta}_{i_1} \wedge \cdots \wedge \widetilde{\delta}_{i_\ell}.$$

With this choice of bases the matrices of the morphisms $\phi_\ell : R_\ell \oplus S_\ell \to R_{\ell-1} \oplus S_{\ell-1}$ have a special form:

- *The coordinate of $\phi_\ell(\widetilde{\delta}_1 \wedge \widetilde{\delta}_{i_2} \wedge \cdots \wedge \widetilde{\delta}_{i_\ell})$ corresponding to its "tail" $\widetilde{\delta}_{i_2} \wedge \cdots \wedge \widetilde{\delta}_{i_\ell}$ is*
$$\widetilde{\delta}_1 - \sum_{j \in \{i_2, \ldots, i_\ell\}} \nu_j.$$
- *The coordinate of $\phi_\ell(\widetilde{\delta}_1 \wedge \widetilde{\delta}_{i_2} \wedge \cdots \wedge \widetilde{\delta}_{i_\ell})$ corresponding to the element $\widetilde{\delta}_{j_2} \wedge \cdots \wedge \widetilde{\delta}_{j_\ell}$ is zero if $1 \notin \{j_2, \ldots, j_\ell\}$ and $(j_2, \ldots, j_\ell) \neq (i_2, \ldots, i_\ell)$.*

So the matrices written by rows of the morphisms have the form

$$\begin{pmatrix} A_\ell & X_\ell \\ B_\ell & C_\ell \end{pmatrix},$$

where X_ℓ is a diagonal $\binom{n-1}{\ell-1} \times \binom{n-1}{\ell-1}$ matrix with elements of the form $\widetilde{\delta}_1 - \sum_{j \in \{i_2, \ldots, i_\ell\}} \nu_j$ in its principal diagonal for all $2 \le i_2 < \ldots < i_\ell = n$. For our purposes we do not require to know the particular shape of matrices A_ℓ, B_ℓ, C_ℓ.

Let us remark that, as proven in Lemma 3.1, we have $\widetilde{\delta}_1 = \delta_1 + 1 = \chi + 1$ and $\widetilde{\delta}_i = \delta_i$ for $i = 2, \ldots, n$.

Remark 3.4 *Given an Euler vector field $\chi = \sum_{i=1}^n \omega_i x_i \partial_i$ with $\omega_i > 0, i = 1, \ldots, r$, $\omega_i = 0, i = r+1, \ldots, n$, and $\psi \in \mathcal{O} = \mathbb{C}\{x_1, \ldots, x_n\}$, it is clear that the equation*

$$(\chi + c)(h) = \psi$$

has a convergent solution h for any $c > 0$. To prove that, we decompose the given power series ψ as the sum of its WQH parts (with respect to the weight vector *w) $\psi = \sum_{\nu \in \mathbf{Q}^+} \psi_\nu$. We write the unknown power series h as $h = \sum_{\nu \in \mathbf{Q}^+} h_\nu$. From the equation*

$$(\chi + c)(h) = \sum_{\nu \in \mathbf{Q}^+} (\nu + c) h_\nu = \sum_{\nu \in \mathbf{Q}^+} \psi_\nu$$

we get $h_\nu = \frac{1}{c+\nu}\psi_\nu$ and so h is the unique convergent solution. Moreover, $(\chi + c)(h) = 0$ implies $h = 0$ and then the solution h of the non-homogeneous equation $(\chi + c)(h) = \psi$ is unique once ψ is fixed. More generally, if $c_1, \ldots, c_r$ are strictly positive real numbers, the $\mathbb{C}$–linear morphism from $\mathcal{O}^r$ to $\mathcal{O}^r$ defined by the diagonal matrix

$$\begin{pmatrix} \chi + c_1 & & \\ & \ddots & \\ & & \chi + c_r \end{pmatrix}$$

is an isomorphism. This fact will plays a crucial role in what follows (see the proof of Theorem 3.5).

Notation 3.1 *For any integer $k \geq 0$ we also consider, as in the Introduction, the coherent $\mathcal{D}_X$-module $\widetilde{M}^{(k)\log D}$ with stalks*

$$(\widetilde{M}^{(k)\log D})_p := \frac{\mathcal{D}_p}{Ann^{(1)}_{\mathcal{D}_p}(1/f^k)}$$

where f is a local reduced equation of the germ (D, p). For $k = 1$ one has $\widetilde{M}^{(1)\log D} = \widetilde{M}^{\log D}$ (see Introduction).

We have a natural $\mathcal{D}_X$–module morphism $\varphi^k_D : \widetilde{M}^{(k)\log D} \longrightarrow \mathcal{O}_X[*D]$ verifying $\varphi^k_{D,p}(\overline{P}) = P(\frac{1}{f^k})$ for any $P \in \mathcal{D}_{X,p}$. Once a local reduced equation $f \in \mathcal{O}_p$ of (D, p) has been chosen, we will write $\widetilde{M}^{(k)\log f} = \widetilde{M}^{(k)\log D}$.

Theorem 3.5 *Given a Spencer free divisor (D, p) defined at $p \in X = \mathbb{C}^n$ by a WQH power series f, then*

$$\mathrm{Ext}^i_{\mathcal{D}_p}((\widetilde{M}^{\log f})_p, \mathcal{O}_p) = 0$$

for $i = 0, \ldots, n$.

Proof. We can assume $p = 0 \in X$. The preceding Lemmata have prepared the computations:

- We choose an adapted basis $\{\delta_1 = \chi, \delta_2, \ldots, \delta_n\}$ of $Der(-\log D)_0$ as in Lemma 3.1 with $[\chi + 1, \delta_j] = \nu_j \delta_j$ and $[\delta_i, \delta_j] = \sum_{l=2}^n \nu^{ij}_l \delta_l$.
- We use the logarithmic Spencer resolution of $(\widetilde{M}^{\log D})_0$ with respect to this basis, so the matrices of the morphisms in this resolution are like in Remark 3.3.

- The elements $\chi + 1 - \sum_{j\in\{i_2,\ldots,i_\ell\}} \nu_j$ in the main diagonal of the upper-right blocks of the matrices X_ℓ (see Remark 3.3) verify by Lemma 3.2 that $1 - \sum_{j\in\{i_2,\ldots,i_p\}} \nu_j > 0$.

To compute the Ext groups, we apply the functor $\mathrm{Hom}_{\mathcal{D}_0}(-, \mathcal{O}_0)$ to the corresponding logarithmic Spencer complex (obtained by truncation of the logarithmic Spencer resolution of $\widetilde{M}^{\log f}$)

$$0 \longrightarrow \mathcal{D}_0 \xrightarrow{\phi_n} \mathcal{D}_0^{\binom{n}{n-1}} \xrightarrow{\phi_{n-1}} \cdots \xrightarrow{\phi_2} \mathcal{D}_0^{\binom{n}{1}} \xrightarrow{\phi_1} \mathcal{D}_0 \longrightarrow 0,$$

obtaining the complex

$$0 \longrightarrow \mathcal{O}_0 \xrightarrow{\phi_1^*} \mathcal{O}_0^{\binom{n}{1}} \xrightarrow{\phi_2^*} \cdots \xrightarrow{\phi_{n-1}^*} \mathcal{O}_0^{\binom{n}{n-1}} \xrightarrow{\phi_n^*} \mathcal{O}_0 \longrightarrow 0,$$

using the isomorphism $\mathrm{Hom}_{\mathcal{D}_0}(\mathcal{D}_0^p, \mathcal{O}_0) \simeq \mathcal{O}_0^p$. The morphisms ϕ_i^* come from the ϕ_i just by applying the functor $\mathrm{Hom}_{\mathcal{D}_0}(-, \mathcal{O}_0)$.

Using the notation of Remark 3.3 we will write the $\mathcal{O}_0$–module $\bigwedge^\ell \widetilde{\mathrm{Der}}(-\log f) \simeq \mathcal{O}_0^{\binom{n}{\ell}}$ as a direct sum $G_\ell \oplus H_\ell$ where

$$G_\ell = \bigoplus_{2\leq j_2<\cdots<j_\ell\leq n} \mathcal{O}_0 \cdot \widetilde{\delta}_1 \wedge \widetilde{\delta}_{j_2} \wedge \cdots \wedge \widetilde{\delta}_{j_\ell} \text{ and } H_\ell = \bigoplus_{2\leq i_1<\cdots<i_\ell\leq n} \mathcal{O}_0 \cdot \widetilde{\delta}_{i_1} \wedge \cdots \wedge \widetilde{\delta}_{i_\ell}.$$

We will identify the isomophic $\mathcal{O}$–modules $H_{\ell-1}$ and G_ℓ. The matrix of the $\mathbb{C}$–linear map $\phi_\ell^* : G_{\ell-1} \oplus H_{\ell-1} \to G_\ell \oplus H_\ell$ is nothing but

$$\begin{pmatrix} A_\ell & X_\ell \\ B_\ell & C_\ell \end{pmatrix}.$$

We have

$$\mathrm{Ext}^\ell_{\mathcal{D}_0}((\widetilde{M}^{\log D})_0, \mathcal{O}_0) = \frac{\ker(\phi_{\ell+1}^*)}{\mathrm{Im}\,(\phi_\ell^*)}.$$

For $(g, h) \in \ker(\phi_{\ell+1}^*) \subset G_\ell \oplus H_\ell$ we have $A_{\ell+1}g + X_{\ell+1}h = B_{\ell+1}g + C_{\ell+1}h = 0$. By Remark 3.4 there is a unique $h' \in H_{\ell-1}$ such that $X_\ell h' = g$. So, $\phi_\ell^*(0, h') = (g, h'')$ where $h'' = C_\ell(h') \in H_\ell$. So $(g, h) - (g, h'') = (0, h - h'')$ belongs to $\ker(\phi_{\ell+1}^*)$. In particular $X_{\ell+1}(h - h'') = 0$ and then, again by Remark 3.4, $h = h''$. We have proven $(g, h) \in \mathrm{Im}\,(\phi_\ell^*)$ and then $\mathrm{Ext}^\ell_{\mathcal{D}_0}((\widetilde{M}^{\log D})_0, \mathcal{O}_0) = 0$.

Remark 3.6 *The main idea of the proof of Theorem 3.5 has its origin in [Ucha 1999] (see also [Castro and Ucha 2001]). Given an integer $k \geq 1$, if $\widetilde{M}^{(k)\log f}$ admits a logarithmic Spencer resolution (analogous to the one of Remark 3.3) then Theorem 3.5 holds for $\widetilde{M}^{(k)\log f}$.*

Theorem 3.7 *Let $D \subset X$ be a Spencer free divisor. We also assume that D is LWQH on X (i.e. the germ (D, p) can be defined by a WQH germ of* *holomorphic function in $\mathcal{O}_p$ for any $p \in D$). Then, as long as $(\widetilde{M}^{(k)\log f})_p$ admits a logarithmic Spencer resolution, we have*

$$Ann_{\mathcal{D}_p}\left(\frac{1}{f^k}\right) = Ann^{(1)}_{\mathcal{D}_p}\left(\frac{1}{f^k}\right)$$

for any $p \in D$, any reduced equation f of (D, p) and $k \gg 0$.

Proof. Let us consider the $\mathcal{D}_X$–module $\mathcal{O}_X[*D]$ of meromorphic functions on X with poles along D. If $p \in X$ and f is a local equation of (D, p) we have $(\mathcal{O}_X[^*D])_p = \mathcal{O}_p[\frac{1}{f}]$.

Let $p \in D$ and $-k_0$ be the least integer root of the local b-function $b_{f,p}$ where $f \in \mathcal{O}_p$ is a local reduced equation of the germ (D, p). We know that $-n \leq -k_0 \leq -1$. Let k be an integer $k \geq k_0$. We have an exact sequence

$$0 \to L_{k,p} \to (\widetilde{M}^{(k)\log f})_p \to \frac{\mathcal{D}_p}{Ann_{\mathcal{D}_p}(\frac{1}{f^k})} = \mathcal{O}_p[\frac{1}{f}] \to 0 \qquad (1)$$

where $L_{k,p}$ is the kernel of the morphism $\varphi^k_{D,p}$ which is surjective because $k \geq k_0$.

By considering the long exact sequence associated to the exact sequence (1) we get

$$\ldots \to \mathrm{Ext}^i(\mathcal{O}_p[\frac{1}{f}], \mathcal{O}_p) \to \mathrm{Ext}^i((\widetilde{M}^{(k)\log f})_p, \mathcal{O}_p) \to \mathrm{Ext}^i(L_{k,p}, \mathcal{O}_p) \to$$

$$\to \mathrm{Ext}^{i+1}(\mathcal{O}_p[\frac{1}{f}], \mathcal{O}_p) \to \ldots$$

where $i \geq 0$ and the Ext groups have been considered with respect to the ring $\mathcal{D}_p$.

Since $p \in D$ the vector space $\mathrm{Ext}^i_{\mathcal{D}_p}(\mathcal{O}_p[\frac{1}{f}], \mathcal{O}_p)$ is equal to 0 for $i \geq 0$ (see e.g. [Mebkhout 1989, Chap. II, Th. 2.2.4]).

So, from the equality $\mathrm{Ext}^i_{\mathcal{D}_p}((\widetilde{M}^{(k)\log f})_p, \mathcal{O}_p) = 0$ (see Theorem 3.5 and Remark 3.6) we get $\mathrm{Ext}^i_{\mathcal{D}_p}(L_{k,p}, \mathcal{O}_p) = 0$ for $i \geq 0$. If $p \notin D$ then $(\widetilde{M}^{(k)\log f})_p \simeq \mathcal{O}_p \simeq \mathcal{O}_p[\frac{1}{f}]$ and $L_{k,p} = 0$ (see the exact sequence (1)).

So, we have proved that $\mathrm{Ext}^i_{\mathcal{D}_p}(L_{k,p}, \mathcal{O}_p) = 0$ for $p \in X$ and $k \geq k_0$.

Since $L_{k,p} = \ker(\varphi^k_D)_p$ and

$$(\mathcal{E}xt^i_{\mathcal{D}_X}(\ker(\varphi^k_D), \mathcal{O}_X))_p \simeq \mathrm{Ext}^i_{\mathcal{D}_p}(L_{k,p}, \mathcal{O}_p) = 0$$

then the following $\mathcal{E}xt$ sheaf vanishes

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\ker(\varphi^k_D), \mathcal{O}_X) = 0.$$

By [Mebkhout 2004, Corollary 11.4.-1] this implies that $\ker(\varphi^k_D) = 0$ for $k \geq k_0$. This proves the theorem.

Remark 3.8 *We do not know if the hypothesis about the LWQH condition on f is necessary in Theorem 3.7. We notice that by using [Mebkhout 2004, Corollary 11.4.-1] in the last part of the proof we do not need to assume the holonomy of $\widetilde{M}^{(k)\log D}$. Let us also notice that the proof of Theorem 3.7 uses very deep results in $\mathcal{D}$–module theory: The Grothendieck Comparison Theorem (as presented in [Mebkhout 1989, Chap. II, Th. 2.2.4]) and the biduality Theorem for $\mathcal{D}^\infty$–modules (as presented in [Mebkhout 2004, 11.4.]).*

Corollary 3.9 *Under the hypotheses of Theorem 3.7, if -1 is the least integer root of the local Bernstein polynomial $b_{f,p}(s)$ for all $p \in D$, then the Logarithmic Comparison Theorem (LCT) holds for D.*

Proof. Let us recall that the divisor $D \subset X$ satisfies the Logarithmic Comparison Theorem (see [Castro et al. 1996]) if the inclusion of the logarithmic de Rham complex $\Omega^\bullet(\log D)$ in the meromorphic de Rham complex $\Omega^\bullet(*D)$ is a quasi-isomorphism. Under the hypothesis of the Corollary we have

$$Ann_{\mathcal{D}_p}(\frac{1}{f}) = Ann^{(1)}_{\mathcal{D}_p}(\frac{1}{f}),$$

for any $p \in D$ and then by [Castro and Ucha 2004, Criterion 3.1] LCT holds for the divisor D. L. Narváez-Macarro pointed out that [Castro and Ucha 2004, Criterion 3.1] uses [Calderón and Narváez 2005, Cor.4.2].

Remark 3.10 *If D is a locally quasi-homogenous (LQH) free divisor we do not need to assume that -1 is the least integer root of the Bernstein-Sato polynomial $b_{f,p}(s)$ for $p \in D$. To this end, any LQH free divisor is of Spencer type (see [Calderón and Narváez 2002a, Theorem 3.2]) and from the proof of [Castro and Ucha 2002, Theorem 5.2] we deduce, for each $p \in D$, the equality $Ann_{\mathcal{D}_p}(\frac{1}{f}) = Ann^{(1)}_{\mathcal{D}_p}(\frac{1}{f})$ where f is a local reduced equation of the germ (D, p). This implies (see e.g. [Torrelli 2004, Proposition 1.3]) that -1 is the least integer root of $b_{f,p}(s)$ for any $p \in D$. Then by [Castro and Ucha 2004, Criterion 3.1] LCT holds for D.*

Remark 3.11 *After reading the first version of this paper L. Narváez- Macarro told us that any LWQH free divisor of Spencer type actually satisfies LCT. This is more general than Corollary 3.9 because it is not necessary to assume the condition about the roots of the b-function of f. The sketch of his proof is as follows. By [Calderón and Narváez 2006, Th. 2.1.1] any Koszul free divisor $D \subset \mathbb{C}^n$ satisfies LCT if and only if the canonical morphism*

$$j_!\mathbb{C}_U \longrightarrow \Omega^\bullet_X(\log D)(\mathcal{O}_X(-D)) \tag{2}$$

is an isomorphism in the derived category of complexes of sheaves of complex vector spaces. Here $X = \mathbb{C}^n$, $j : U = X \setminus D \hookrightarrow X$ is the inclusion and $\Omega^\bullet_X(\log D)(\mathcal{O}_X(-D))$ is the tensor product of $\Omega^\bullet_X(\log D)$ with the invertible $\mathcal{O}_X$–module $\mathcal{O}_X(-D)$. In fact, the argument in the proof of [Calderón and Narváez 2006, *Th. 2.1.1] also shows that the previous equivalence is also true for free divisors of Spencer type, because of the isomorphism* $\Omega^\bullet_X(\log D) \overset{\simeq}{\to} \Omega^\bullet_X(\log D)(O_X(-D))^\vee$, *where* $^\vee$ *denotes the Verdier's dual, which is nothing but the intrinsic version of the duality theorem in [Castro and Ucha 2002, Th. 4.3]. Moreover, the quasi-isomorphism (2) holds if and only if the complex* $\Omega^\bullet_X(\log D)(\mathcal{O}_X(-D))_p$ *is exact at any point* $p \in D$. *The last complex is nothing but*

$$f\mathcal{O}_{X,p} \to f\Omega^1_X(\log D)_p \to \cdots \to f\Omega^n_X(\log D)_p$$

(for f *a reduced equation of the germ* (D,p)*) which is a filtered complex using the weight* w. *By an argument of [Mond 2000, Lemma 3.3,6] (also used in [Castro et al. 1996, Section 2]) this complex is quasi-isomorphic to its subcomplex of weight 0 which is in fact 0 because the weight of* f *is 1 and any logarithmic differential form has a non negative weight.*

This proves that any LWQH free divisor of Spencer type satisfies the Logarithmic Comparison Theorem and we do not need to assume the condition of Corollary 3.9 about the b-function of f.

The free divisor D *defined in the space* $M_{n,n+1}(\mathbb{C})$ *of* $n \times (n+1)$ *matrices by the vanishing of the product of the maximal minors [Granger et al. 2006] is LWQH and as shown in* loc. cit. *it is not LQH. Nevertheless,* D *satisfies the so called Global Logarithmic Comparison Theorem [Granger et al. 2006]. For* $n = 3$ *the divisor is also of Spencer type and then it satisfies LCT. It seems that for* $n \geq 4$ *the divisor* D *is also of Spencer type.*

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