Mathematical and Computational Applications

MDPI

Article

The Invariant Two-Parameter Function of Algebras $\bar{\psi}$

José María Escobar [1], Juan Núñez-Valdés [1,*] and Pedro Pérez-Fernández [2,3]

1 Dpto. de Geometría y Topología, Facultad de Matemáticas, Universidad de Sevilla, Calle Tarfia s/n, 41012 Sevilla, Spain; pinchamate@gmail.com

2 Dpto. de Física Aplicada III, Escuela Técnica Superior de Ingeniería, Universidad de Sevilla, Camino de los Descubrimientos, 41092 Sevilla, Spain; pedropf@us.es

3 Instituto Carlos I de Física Teórica y Computacional, Universidad de Granada, Fuentenueva s/n, 18071 Granada, Spain

* Correspondence: jnvaldes@us.es

Received: 12 September 2019; Accepted: 11 October 2019; Published: 14 October 2019

Abstract: At present, the research on invariant functions for algebras is very extended since Hrivnák and Novotný defined in 2007 the invariant functions ψ and φ as a tool to study the Inönü–Wigner contractions (IW-contractions), previously introduced by those authors in 1953. In this paper, we introduce a new invariant two-parameter function of algebras, which we call $\bar{\psi}$, as a tool which makes easier the computations and allows researchers to deal with contractions of algebras. Our study of this new function is mainly focused in Malcev algebras of the type Lie, although it can also be used with any other types of algebras. The main goal of the paper is to prove, by means of this function, that the five-dimensional classical-mechanical model built upon certain types of five-dimensional Lie algebras cannot be obtained as a limit process of a quantum-mechanical model based on a fifth Heisenberg algebra. As an example of other applications of the new function obtained, its computation in the case of the Lie algebra induced by the Lorentz group $SO(3,1)$ is shown and some open physical problems related to contractions are also formulated.

Keywords: invariant functions; contractions of algebras; Lie algebras; Malcev algebras; Heisenberg algebras

1. Introduction

Regarding the concept of *limit process* between physical theories in terms of contractions of their associated symmetry groups, formulated by Erdal Inönü and Eugene Wigner [1,2], these authors introduced the so-called *Inönü–Wigner contractions (IW-contractions)* in 1953. Later, other extensions of these IW-contractions have also been addressed, for instance the *generalized Inönü–Wigner contractions*, introduced by Melsheiner [3], the *parametric degenerations* [4–6], widely used in the Algebraic Invariants Theory, and the *singular contractions* [2]. To study these contractions, Hrivnák and Novotný introduced the invariant functions ψ and φ as a tool in 2007 [7]. These invariant functions depend on one parameter.

Continuing with this topic, the main goal of this paper is to introduce a new invariant function, in this case depending on two parameters, which we call *the two-parameter invariant function* $\bar{\psi}$, to get some advances on this research. Indeed, the objective is to prove, by means of this function, that the five-dimensional classical-mechanical model built upon certain types of five-dimensional Lie algebras cannot be obtained as a limit process of a quantum-mechanical model based on a fifth Heisenberg algebra.

Indeed, the study of this function is mainly focused in the frame of the Malcev algebras of the type Lie. Thus, this paper can be considered as the natural continuation of a previous one dealing with Lie algebras [8]. We try to generalize the properties obtained on that to the case of Malcev algebras.

The structure of the paper is as follows. In Section 2, we recall some preliminaries on the mathematical objects dealt with in this paper, Lie algebras and Malcev algebras. Section 3 is devoted to introducing and proving the main properties of the two-parameter invariant function $\bar{\psi}$. For computations, we used the SAGE symbolic computation package and in this section we prove that this new function is different from others previously defined, which are used as a tool to study contractions of algebras. We also prove the main result of the paper: no proper contraction between a fifth Heisenberg algebra and a filiform Lie algebra of dimension 5 exists. It implies that the five-dimensional classical-mechanical model built upon a five-dimensional filiform Lie algebra cannot be obtained as a limit process of a quantum-mechanical model based on a fifth Heisenberg algebra. In this way, the new function allows us to step forward in the research on contractions. In Section 4, we show some of our discussion and conclusions regarding the research done. Finally, in Section 5, we give some comments on the materials and methods used in such a research.

2. Preliminaries

We show in this section some preliminaries on Lie algebras, Malcev algebras and on Heisenberg algebras, which are the main mathematical objects used in the paper.

2.1. Preliminaries on Lie Algebras

In this subsection, we show some preliminaries on Lie algebras. For a further review on this topic, the reader can consult [9].

An n-dimensional *Lie algebra* $\mathfrak{g}$ over a field K is an n-dimensional vector space over K endowed with a second inner law, named *bracket product*, which is bilinear and anti-commutative and satisfies the *Jacobi identity*

$$J(u,v,w) = [u,[v,w]] + [v,[w,u]] + [w,[u,v]] = 0, \text{ for all } u,v,w \in \mathfrak{g}. \tag{1}$$

The law of the n-dimensional Lie algebra $\mathfrak{g}$ is determined by the products

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k, \quad \text{for} \quad 1 \le i < j \le n,$$

where $c_{i,j}^k \in K$ are called *structure constants* of $\mathfrak{g}$. If all these constants are zero, then the Lie algebra is called *abelian*.

Two Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are *isomorphic* if there exists a vector space isomorphism f between them such that $f([u,v]) = [f(u), f(v)]$, for all $u, v \in \mathfrak{g}$.

A mapping $d : \mathfrak{g} \longrightarrow \mathfrak{g}$ is a *derivation* of $\mathfrak{g}$ if $d([u,v]) = [d(u),v] + [u,d(v)]$, for all $u,v \in \mathfrak{g}$. The set of derivations of $\mathfrak{g}$ is denoted by $Der\mathfrak{g}$.

The *lower central series* of a Lie algebra $\mathfrak{g}$ is defined as $\mathfrak{g}^1 = \mathfrak{g}$, $\mathfrak{g}^2 = [\mathfrak{g}^1, \mathfrak{g}]$, $\ldots$, $\mathfrak{g}^k = [\mathfrak{g}^{k-1}, \mathfrak{g}]$, $\ldots$

If there exists $m \in \mathbb{N}$ such that $\mathfrak{g}^m \equiv 0$, then $\mathfrak{g}$ is called *nilpotent*. The *nilpotency class* of $\mathfrak{g}$ is the smallest natural c such that $\mathfrak{g}^{c+1} \equiv 0$.

An n-dimensional nilpotent Lie algebra $\mathfrak{g}$ is said to be *filiform* if it is verified that $\dim \mathfrak{g}^k = n - k$, for all $k \in \{2, \ldots, n\}$. Filiform Lie algebras were introduced by Vergne in her Ph.D. Thesis, in 1966, later published in [10] in 1970.

The only n-dimensional filiform Lie algebra for $n < 3$ is the abelian. For $n \geq 3$, it is always possible to find an *adapted basis* $\{e_1, \ldots, e_n\}$ of $\mathfrak{g}$ such that $[e_1, e_2] = 0,\quad [e_1, e_j] = e_{j-1}$, for all $j \in \{3, \ldots, n\}$ and $[e_2, e_j] = [e_3, e_j] = 0$, for all $j \in \{3, \ldots, n\}$.

From the condition of filiformity and the Jacobi identity in Equation (1), the bracket product of $\mathfrak{g}$ is determined by

$$[e_i, e_j] = \sum_{k=2}^{min\{i-1,n-2\}} c_{ij}^k e_k, \quad \text{for} \quad 4 \leq i < j \leq n,$$

where $c_{i,j}^k \in K$ are called *structure constants* of $\mathfrak{g}$. If all these constants are zero, then the filiform Lie algebra $\mathfrak{g}$ is called *model*. The model algebra is not isomorphic to any other algebra of the same dimension and every n-dimensional filiform Lie algebra $\mathfrak{g}$ having an adapted basis $\{e_1, \ldots, e_n\}$ verifies that $\mathfrak{g}^2 = \langle e_2, \ldots, e_{n-1} \rangle$, $\mathfrak{g}^3 = \langle e_2, \ldots, e_{n-2} \rangle, \ldots, \mathfrak{g}^{n-1} = \langle e_2 \rangle$, $\mathfrak{g}^n = 0$.

2.2. Preliminaries on Malcev Algebras

Now, we recall some preliminary concepts on Malcev algebras, taking into account that a general overview can be consulted in [11]. From here on, we only consider finite-dimensional Malcev algebras over the complex number field $\mathbb{C}$.

A *Malcev algebra* $\mathcal{M}$ is a vector space with a second bilinear inner composition law ($[\cdot, \cdot]$) called the *bracket product* or *commutator*, which satisfies: (a) $[u, v] = -[v, u]$, $\forall u, v \in \mathcal{M}$; and (b) $[[u, v], [u, w]] = [[[u, v], w], u] + [[[v, w], u], u] + [[[w, u], u], v]$, $\forall u, v, w \in \mathcal{M}$. Condition (b) is named *Malcev identity* and we use the notation $M(u, v, w) = [[u, v], [u, w]] - [[[u, v], w], u] - [[[v, w], u], u] - [[[w, u], u], v]$.

Given a basis $\{e_i\}_{i=1}^n$ of a n-dimensional Malcev algebra $\mathcal{M}$, the *structure constants* $c_{i,j}^h$ are defined as $[e_i, e_j] = \sum_{h=1}^n c_{i,j}^h e_h$, for $1 \leq i, j \leq n$.

It is immediate to see that Malcev algebras and Lie algebras are not disjoint sets. Indeed, every Lie algebra is a Malcev algebra, but the converse is not true. Therefore, we can distinguish between Malcev algebras of the type Lie and Malcev algebras of the type non-Lie. Obviously, those Malcev algebras which are of the type Lie verify both identities: Jacobi and Malcev.

If the Jacobi identity does not hold, then the Malcev algebra is said to have a *Jacobi anomaly*. In quantum mechanics, the existence of Jacobi anomalies in the underlying non-associative algebraic structure related to the coordinates and momenta of a quantum non-Hamiltonian dissipative system was already claimed by Dirac [12] in the process of taking Poisson brackets. In string theory, for instance, one such anomaly is involved by the non-associative algebraic structure that is defined by coordinates ($\vec{x}$) and velocities or momenta ($\vec{v}$) of an electron moving in the field of a constant magnetic charge distribution, at the position of the location of the magnetic monopole [13]. In particular, $J(v_1, v_2, v_3) = -\vec{\nabla} \circ \vec{B}(\vec{x})$, where $\vec{\nabla} \circ \vec{B}(\vec{x})$ denotes the divergence of the magnetic field $\vec{B}(\vec{x})$. The underlying algebraic structure constitutes a non-Lie Malcev algebra [14], with the commutation relations $[x_a, x_b] = 0$, $[x_a, v_b] = i\,\delta_{ab}$ and $[v_a, v_b] = i\,\varepsilon_{abc}\,B_c(\vec{x})$, where $a, b, c \in \{1, 2, 3\}$, δ_{ab} denotes the Kronecker delta and ε_{abc} denotes the Levi–Civita symbol. If the magnetic field is proportional to the coordinates, the latter can be normalized and $B_c(\vec{x})$ can then be supposed to coincide with x_c. The resulting algebra is then called magnetic [15]. A generalization to electric charges has recently been considered [15] by defining the products $[x_a, x_b] = -i\,\varepsilon_{abc}\,\vec{E}_c(\vec{x}, \vec{v})$, where the electric field $\vec{E}$ as well as the magnetic field $\vec{B}$ must depend not only on coordinates but also on velocities. It is worth remarking that both magnetic and electric algebras constitute magma algebras (see [16] for this last concept).

If $\mathfrak{g}$ is a Malcev algebra of the type Lie and $D \in Der\mathfrak{g}$ a derivation of $\mathfrak{g}$, then, according to the anti-commutative property of $\mathfrak{g}$ and the Jacobi identity in Equation (1) of Lie algebras, we get that

$$[d[x,y],[x,z]] + [[x,y],d[x,z]] = d[[[x,z],y],x] + d[[[z,x],x],y] \quad \forall x,y,z \in \mathfrak{g}$$

Starting from here and due to reasons of length, only Malcev algebras of type Lie, that is to say, actually Lie algebras, are used in this paper. Malcev algebras of type non-Lie will be dealt with in future work.

2.3. Preliminaries on Heisenberg Algebras

Let n be a non-negative integer or infinity. The nth Heisenberg algebra (so-called after Werner Karl Heisenberg) is the Lie algebra with basis $\mathcal{B} = \{p_1, \ldots, p_n, q_1, \ldots, q_n, z\}$ with the following relations, known as *canonical commutation relations*

1. $[p_i, q_j] = c_{ij}\, z, \quad 1 \leq i, j \leq n.$
2. $[p_i, z] = [q_i, z] = [p_i, p_j] = [q_i, q_j] = 0, \quad 1 \leq i, j \leq n.$

Note that the dimension of an nth Heisenberg algebra is not n, but $2n+1$. In fact, the n in the above definition is called the *rank* of the Heisenberg algebra, although it is not, however, a rank in any of the usual meanings that this word has in the theory of Lie algebras. Thus, this Lie algebra is also known as the *Heisenberg algebra of rank n*.

In any case, from here on and to avoid confusions we designate under the notation fifth Heisenberg algebras to those Heisenberg algebras generated by five generators.

3. Results

In this section, which is divided by subheadings, we provide a concise and precise description of our experimental results. They are the following.

3.1. Introducing a New Invariant Function

Let $\mathfrak{g} = (V, [\,,\,])$ be a Lie algebra. $End\,\mathfrak{g}$ denotes the vector space of all linear operators of $\mathfrak{g}$ over V.

Definition 1. *Let $\mathfrak{g}$ be a Lie algebra. The set*

$$Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = \{d \in End\,\mathfrak{g} : \alpha[d[x,y],[x,z]] + \beta[[x,y],d[x,z]] = \gamma d[[[x,z],y]x] + \tau d[[[z,x],x],y]\}$$

$\forall(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$, *is called* the set of the $(\alpha,\beta,\gamma,\tau)$-derivations *of the algebra* $\mathfrak{g}$. *It is denoted by* $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$.

It is obvious that $\dim\big(Der_{(1,1,1,1)}\mathfrak{g}\big) = \dim\big(Der\mathfrak{g}\big)$. Then, as $\dim\big(Der\mathfrak{g}\big)$ is an invariant of $\mathfrak{g}$, it follows that $\dim\big(Der_{(1,1,1,1)}\mathfrak{g}\big)$ is an invariant of $\mathfrak{g}$. This leads the following result.

Proposition 1. *If $\mathfrak{g}$ is a Lie algebra, then $dim_{(1,1,1,1)}\mathfrak{g}$ is an algebraic invariant of $\mathfrak{g}$.* □

Theorem 1. *Let $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ be two Malcev algebras of the type Lie and let $f : \mathfrak{g} \to \bar{\mathfrak{g}}$ be an isomorphism. Then, the mapping $\rho : End\ \ \mathfrak{g} \to End\ \ \bar{\mathfrak{g}}$, defined by $D \longrightarrow fDf^{-1}$, is an isomorphism between the vector spaces $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$ and $Der_{(\alpha,\beta,\gamma,\tau)}\bar{\mathfrak{g}}$, $\forall(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$.,*

Proof. Let $\mathfrak{g} = (V, \cdot)$ and $\bar{\mathfrak{g}} = (\bar{V}, *)$ be two Malcev algebras of the type Lie and let us consider $D \in Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$, for any $(\alpha,\beta,\gamma,\tau) \in \mathbb{C}^4$ and for all $x, y, z \in \bar{\mathfrak{g}}$. Then,

$$\alpha D\Big(f^{-1}(x)\cdot f^{-1}(y)\Big)\cdot\Big(f^{-1}(x)\cdot f^{-1}(z)\Big)+\beta\Big(f^{-1}(x)\cdot f^{-1}(y)\Big)\cdot D\Big(f^{-1}(x)\cdot f^{-1}(z)\Big)=$$
$$\gamma D\Big(\Big(\Big(f^{-1}(x)\cdot f^{-1}(z)\Big)\cdot f^{-1}(y)\Big)\cdot f^{-1}(x)\Big)+\tau D\Big(\Big(\Big(f^{-1}(z)\cdot f^{-1}(x)\Big)\cdot f^{-1}(x)\Big)\cdot f^{-1}(y)\Big).$$

It is deduced that

$$\gamma D\Big(\Big(\Big(f^{-1}(x)\cdot f^{-1}(z)\Big)\cdot f^{-1}(y)\Big)\cdot f^{-1}(x)\Big)=\gamma D\Big(\Big(f^{-1}(x*z)\cdot f^{-1}(y)\Big)\cdot f^{-1}(x)\Big)=$$
$$\gamma D f^{-1}((x*z)*y)\cdot f^{-1}(x))=\gamma D f^{-1}(((x*z)*y)*x),$$

and, similarly,

$$\tau D\Big(\Big(\Big(f^{-1}(z)\cdot f^{-1}(x)\Big)\cdot f^{-1}(x)\Big)\cdot f^{-1}(y)\Big)=\tau D f^{-1}(((z*x)*x)*y)$$

$$\alpha D\Big(f^{-1}(x)\cdot f^{-1}(y)\Big)\cdot\Big(f^{-1}(x)\cdot f^{-1}(z)\Big)=\alpha D f^{-1}(x*y)\cdot f^{-1}(x*z)$$

$$\beta\Big(f^{-1}(x)\cdot f^{-1}(y)\Big)\cdot D\Big(f^{-1}(x)\cdot f^{-1}(z)\Big)=\beta f^{-1}(x*y)\cdot D f^{-1}(x*z).$$

Thus,

$$\alpha D f^{-1}(x*y)\cdot f^{-1}(x*z)+\beta f^{-1}(x*y)\cdot D f^{-1}(x*z)=\gamma D f^{-1}(((x*z)*y)*x)+\tau D f^{-1}(((z*x)*x)*y).$$

Now, the result of applying f to the previous expression is

$$\alpha\Big(fDf^{-1}\Big)(x*y)*(x*z)+\beta(x*y)*\Big(fDf^{-1}\Big)(x*z)=\gamma\Big(fDf^{-1}\Big)(((x*z)*y)*x)+\tau\Big(fDf^{-1}\Big)(((z*x)*x)*y).$$

Thus, $fDf^{-1}\in Der_{(\alpha,\beta,\gamma,\tau)}\bar{\mathfrak{g}}$, which concludes the proof. □

An immediate consequence of this result is the following.

Corollary 1. *Let $\mathfrak{g}$ be a Lie algebra. The dimension of the vector space $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$ is an invariant of the algebra, for all $(\alpha,\beta,\gamma,\tau)\in\mathbb{C}^4$.*

Lemma 1. *(Technical Lemma) Let d be a derivation of a Lie algebra $\mathfrak{g}$. The following expressions are verified*

1. *d[[[z,x],x],y]=d[[x,y],[x,z]]-d[[[y,z],x],x]*
2. *d[[[y,x],x],z]=d[[x,z],[x,y]]-d[[[z,y],x],x]*
3. *d[[[x,z],y],x]=d[[x,y],[x,z]]-d[[[z,x],x],y]*
4. *d[[[x,y],z],x]=d[[x,z],[x,y]]-d[[[y,x],x],z].*

Proof. All expressions are immediate consequences of the properties of the derivations (see Section 2). □

Lemma 2. *Let* $\mathfrak{g} = (V, [,])$ *be a Lie algebra. Then,*

$$Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$$

Proof. Suppose $D \in Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g}$. Then, for all $(x, y, z) \in \mathfrak{g}$, we have

$$\alpha[d[x,y],[x,z]] + \beta[[x,y],d[x,z]] = \gamma d[[[x,z],y],x] + \tau d[[[z,x],x],y].$$

Charging now y and z between themselves, we have

$$\alpha[d[x,z],[x,y]] + \beta[[x,z],d[x,y]] = \gamma d[[[x,y],z],x] + \tau d[[[y,x],x],z]$$

and by adding the two first expressions of Lemma 1 and taking the anti-skew property of the Lie bracket into consideration, we have

$$\begin{aligned}&(\alpha-\beta)[d[x,y],[x,z]] + (\beta-\alpha)[[x,y],d[x,z]] = \\ &\quad \gamma(d[[[x,z],y]x] + d[[[x,y],z]x]) + \tau(d[[[z,x],x],y] + d[[[y,x],x],z]).\end{aligned}$$

Similarly, starting from the two last expressions of Lemma 1, we obtain

$$d[[[z,x],x],y] + d[[[y,x],x],z] = 0$$

and by repeating the same procedure we obtain

$$d[[[x,z],y],x] + d[[[x,y],z],x] = 0.$$

Now, starting from both expressions, we have

$$(\alpha-\beta)[d[x,y],[x,z]] + (\beta-\alpha)[[x,y],d[x,z]] = 0.$$

Therefore, $D \in Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$.

Now, by subtracting the two first expressions of the proof and taking into account the anti-skew property, we have $(\alpha+\beta)[d[x,y],[x,z]] + (\beta+\alpha)[[x,y],d[x,z]] = \gamma(d[[[x,z],y],x] - d[[[x,y],z],x]) + \tau(d[[[z,x],x],y] - d[[[y,x],x],z])$.

We use now in the previous equality the two expressions $d[[[y,x],x],z] = -d[[[z,x],x],y]$ and $d[[[x,y],z],x] = -d[[[x,z],y],x]$, respectively, obtained from previous expressions.

We have that $(\alpha+\beta)[d[x,y],[x,z]] + (\alpha+\beta)[[x,y],d[x,z]] = 2\gamma d[[[x,z],y],x] + 2\tau d[[[z,x],x],y]$. It involves that $D \in Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g}$. Therefore, it is verified that $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} \subset Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$.

If $D \in Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$, then D verifies both equations $(\alpha+\beta)[d[x,y],[x,z]] + (\alpha+\beta)[[x,y],d[x,z]] = 2\gamma d[[[x,z],y],x] + 2\tau d[[[z,x],x],y]$ and $(\alpha-\beta)[d[x,y],[x,z]] + (\beta-\alpha)[[x,y],d[x,z]] = 0$.

Then, by adding these last equations and simplifying, we observe that D verifies

$$\alpha[d[x,y],[x,z]] + \beta[[x,y],d[x,z]] = \gamma d[[[x,z],y],x] + \tau d[[[z,x],x],y].$$

Thus, $D \in Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$.

Therefore, $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(\alpha+\beta,\alpha+\beta,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(\alpha-\beta,\beta-\alpha,0,0)}\mathfrak{g}$, which completes the proof. □

Theorem 2. *Let $\mathfrak{g}$ be a Lie algebra. Then, for all $(\alpha, \beta, \gamma, \tau) \in \mathbb{C}^4$, it exists $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ such that $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} \subset \mathbb{C}^2$ is one of the following four sets: $Der_{(0,0,\lambda_1,\lambda_2)}\mathfrak{g}$; $Der_{(1,-1,\lambda_1,\lambda_2)}\mathfrak{g}$; $Der_{(1,0,\lambda_1,\lambda_2)}\mathfrak{g}$; or $Der_{(1,1,\lambda_1,\lambda_2)}\mathfrak{g}$.*

Proof. Consider $(\alpha, \beta, \gamma, \tau) \in \mathbb{C}^4$. We distinguish the following cases

Case 1: $\alpha + \beta = 0$. We distinguish now the following two subcases:

1.1 $\alpha = \beta = 0$. Then, $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(0,0,\gamma,\tau)}\mathfrak{g}$. Therefore, $\gamma = \lambda_1$ and $\lambda_2 = \tau$.

1.2 $\alpha = -\beta$. In this subcase, by Lemma 2, we have that

$$Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(0,0,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(-2\beta,2\beta,0,0)}\mathfrak{g} = Der_{(0,0,\gamma,\tau)}\mathfrak{g} \cap Der_{(-1,1,0,0)}\mathfrak{g}.$$

Apart from that, it is also verified that

$$Der_{(-1,1,\gamma,\tau)}\mathfrak{g} = Der_{(0,0,2\gamma,2\tau)}\mathfrak{g} \cap Der_{(-2,2,0,0)}\mathfrak{g} = Der_{(0,0,\gamma,\tau)}\mathfrak{g} \cap Der_{(-1,1,0,0)}\mathfrak{g}.$$

Therefore, $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(-1,1,\gamma,\tau)}\mathfrak{g}$. It involves that $\lambda_1 = \gamma$ and $\lambda_2 = \tau$.

Case 2: $\alpha + \beta \neq 0$. Two subcases are also considered:

2.1 $\alpha \neq \beta$.

By Lemma 2, we have $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(1,1,\frac{2\gamma}{\alpha+\beta},\frac{2\tau}{\alpha+\beta})}\mathfrak{g} \cap Der_{(1,-1,0,0)}\mathfrak{g}$.

Since $Der_{(1,0,\frac{\gamma}{\alpha+\beta},\frac{\tau}{\alpha+\beta})}\mathfrak{g} = Der_{(1,1,\frac{2\gamma}{\alpha+\beta},\frac{2\tau}{\alpha+\beta})}\mathfrak{g} \cap Der_{(1,-1,0,0)}\mathfrak{g}$, it is deduced that $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(1,0,\frac{\gamma}{\alpha+\beta},\frac{\tau}{\alpha+\beta})}\mathfrak{g}$. It involves that $\lambda_1 = \frac{\gamma}{\alpha+\beta}$ and $\lambda_2 = \frac{\tau}{\alpha+\beta}$.

2.2 $\alpha = \beta$.
In this subcase, $Der_{(\alpha,\beta,\gamma,\tau)}\mathfrak{g} = Der_{(1,1,\frac{\gamma}{\alpha},\frac{\tau}{\alpha})}\mathfrak{g}$. Therefore, $\lambda_1 = \frac{\gamma}{\alpha}$ and $\lambda_2 = \frac{\tau}{\alpha}$.

□

These two two-parameter sets $Der_{(1,0,\lambda_1,\lambda_2)}\mathfrak{g}$ and $Der_{(1,1,\lambda_1,\lambda_2)}\mathfrak{g}$ previously defined allow us to define the following invariant two-parameter functions of Lie algebras.

Definition 2. *The functions $\bar{\psi}_{\mathfrak{g}}, \bar{\psi}^0_{\mathfrak{g}} : \mathbb{C}^2 \mapsto \mathbb{N}$ defined, respectively, as $(\bar{\psi}_{\mathfrak{g}})(\alpha, \beta) = \dim Der_{(1,1,\alpha,\beta)}\mathfrak{g}$ and $(\bar{\psi}^0_{\mathfrak{g}})(\alpha, \beta) = \dim Der_{(1,0,\alpha,\beta)}\mathfrak{g}$ are called $\bar{\psi}_{\mathfrak{g}}$ and $\bar{\psi}^0_{\mathfrak{g}}$* invariant functions *corresponding to the $(\alpha, \beta, \gamma, \tau)$-derivations of $\mathfrak{g}$.*

Corollary 2. *If two Malcev algebras of the type Lie $\mathfrak{g}$ and $\mathfrak{f}$ are isormorphic, then $\bar{\psi}_{\mathfrak{g}} = \bar{\psi}_{\mathfrak{f}}$ and $\bar{\psi}^0_{\mathfrak{g}} = \bar{\psi}^0_{\mathfrak{f}}$.*

Note that the function $\bar{\psi}$ is a two-parameter function, whereas the function ψ by Novotný and Hrivnák [7] is one-parameter. It implies that both functions are structurally different. However, it can be thought that ψ could be obtained as a particular case of $\bar{\psi}$ by simply taking one of the parameters as a constant. The following counter-example shows that it is not possible.

Indeed, we now compare the function $\bar{\psi}$ with the invariant function ψ and prove that both functions are totally different. To do this, we compute both functions for a same Lie algebra, in the particular case of being $\alpha = 1$. Concretely, we use the Lie algebra induced by the Lorentz group $SO(3, 1)$, which we denote by $\mathfrak{g}_6$.

Computing $\psi_{\mathfrak{g}_6}$, for $\alpha = 1$

Let us recall that Minkowski defined the spacetime as a four-dimensional manifold with the metric $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$. We introduce the metric tensor

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

If we rename $(ct, x, y, z) \to (x^0, x^1, x^2, x^3)$, then the expression ds^2 can be written as $ds^2 = \eta_{\mu\gamma}dx^\mu dx^\gamma$ (summed over μ and γ). Recall that this distance is invariant under the following type of transformations $x^\mu \to \lambda^\mu_\gamma x^\gamma$ such that the coefficients λ^μ_γ are the elements of a matrix Λ (which is called *Lorentz transformations*) that satisfies $\Lambda^t\eta\Lambda = \eta$. Since the metric in the three-dimensional Euclidean space corresponds to the identity matrix, if R is the matrix of a rotation, then $R^t 1R = 1$ and comparing this expression with $\Lambda^t\eta\Lambda = \eta$ it is possible to say that the Lorentz transformations are rotations in the Minkowski space. These transformations form a group called the *Lorentz group* $SO(3,1)$.

Now, we focus our study on the *infinitesimal* Lorentz transformations. A Lorentz transformation matrix can be written as $\Lambda^\mu_\gamma = \delta^\mu_\gamma + \lambda^\mu_\gamma$, where the parameters λ^μ_γ are infinitesimal and verify that $\lambda^\mu_\gamma = -\lambda^\gamma_\mu$ so that the Lorentz transformation is valid. The action of this transformation on the coordinates x^μ in the Minkowski space can be written as $\delta x^\mu = \Lambda^\mu_\gamma x^\gamma$.

If we define $A_{\rho\sigma}$ such that $\Lambda^\mu_\gamma = \dfrac{1}{2}\lambda^{\rho\sigma}(A_{\rho\sigma})^\mu_\gamma$, we can write the above action as $\delta x^\mu = \dfrac{1}{2}\lambda^{\rho\sigma}(A_{\rho\sigma})^\mu_\gamma x^\gamma$. Then, it is easily proved that $(A_{\rho\sigma})^\mu_\gamma = \delta^\mu_\rho \eta_{\sigma\gamma} - \delta^\mu_\sigma \eta_{\rho\gamma}$.

Explicitly,

$$A_{10} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_{20} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_{30} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad A_{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad A_{31} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Now, by defining the Lie product as the usual commutator $[A_{ij}, A_{hk}] = A_{ij} \cdot A_{hk} - A_{hk} \cdot A_{ij}$, $A_{10}, A_{20}, A_{30}, A_{12}, A_{23}$ and A_{31} generate a Lie algebra, which we denote by $\mathfrak{g}_6$.

Let us consider $d \in Der_{(1,1,1,1)}\mathfrak{g}_6$ and let $A = (a_{ij})$, $1 \le i, j \le 6$ be the 6×6 square matrix associated with the endomorphism d.

To obtain the elements of this matrix, for the pair of generators (e_i, e_j), with $i < j$, the derivation d satisfies $d\big([e_i, e_j]\big) = [d(e_i), e_j] + [e_i, d(e_j)]$ and $d(e_i) = \sum_{h=1}^6 a_{ih}\, e_h$. In this way, the following conditions are obtained. This can be seen in the following Table 1.

Table 1. Condition obtained.

From Pair (e_i, e_j)	Conditions
(e_1, e_2)	$a_{41} = a_{14},\ a_{42} = a_{24},\ a_{43} = -a_{15} - a_{26},$ $a_{44} = a_{11} + a_{22},\ a_{45} = -a_{13},\ a_{46} = -a_{23}.$
(e_1, e_3)	$a_{61} = a_{16},\ a_{62} = -a_{15} - a_{34},\ a_{63} = a_{36},$ $a_{64} = -a_{32},\ a_{65} = -a_{12},\ a_{66} = a_{33} + a_{11}.$
(e_1, e_4)	$a_{21} = -a_{12},\ a_{22} = a_{11} + a_{44},\ a_{23} = -a_{46},$ $a_{24} = a_{42},\ a_{25} = -a_{16},\ a_{26} = a_{15} - a_{43}.$
(e_1, e_5)	$a_{13} = 0,\ a_{54} = 0,\ a_{12} - a_{56} = 0,$ $a_{16} + a_{52} = 0,\ a_{14} + a_{53} = 0.$
(e_1, e_6)	$a_{31} = -a_{13},\ a_{32} = -a_{64},\ a_{33} = a_{11} + a_{66},$ $a_{34} = a_{15} - a_{62},\ a_{35} = -a_{14},\ a_{36} = a_{63}.$
(e_2, e_3)	$a_{51} = -a_{26} - a_{34},\ a_{52} = a_{25},\ a_{53} = a_{35},$ $a_{54} = -a_{31},\ a_{55} = a_{22} + a_{33},\ a_{56} = -a_{21}.$
(e_2, e_4)	$a_{11} = a_{22} + a_{44},\ a_{12} = -a_{21},\ a_{13} = -a_{45},$ $a_{14} = a_{41},\ a_{15} = a_{26} - a_{43},\ a_{16} = -a_{25}.$
(e_2, e_5)	$a_{31} = -a_{54},\ a_{32} = -a_{23},\ a_{33} = a_{22} + a_{55},$ $a_{34} = a_{26} - a_{51},\ a_{35} = a_{53},\ a_{36} = -a_{24}.$
(e_2, e_6)	$a_{23} - a_{64} = 0,\ -a_{21} + a_{65} = 0,\ a_{25} + a_{61} = 0,$ $a_{24} + a_{63} = 0.$
(e_3, e_5)	$a_{21} = -a_{56},\ a_{22} = a_{33} + a_{55},\ a_{23} = -a_{32},$ $a_{24} = -a_{36},\ a_{25} = a_{52},\ a_{26} = a_{34} - a_{51}.$
(e_3, e_6)	$a_{11} = a_{33} + a_{66},\ a_{12} = -a_{65},\ a_{13} = -a_{31},$ $a_{14} = -a_{35} + a_{34},\ a_{15} = -a_{62},\ a_{16} = a_{61}.$
(e_4, e_5)	$a_{61} = -a_{52},\ a_{62} = a_{43} + a_{51},\ a_{63} = -a_{42},$ $a_{64} = -a_{46},\ a_{65} = -a_{56},\ a_{66} = a_{44} + a_{55}.$
(e_4, e_6)	$a_{51} = a_{43} + a_{62},\ a_{52} = -a_{61},\ a_{53} = -a_{41},$ $a_{54} = -a_{45},\ a_{55} = a_{44} + a_{66},\ a_{56} = -a_{65}.$
(e_5, e_6)	$a_{41} = -a_{53},\ a_{42} = -a_{63},\ a_{43} = a_{51} + a_{62},$ $a_{44} = a_{55} + a_{66},\ a_{45} = -a_{54},\ a_{46} = -a_{64}.$
(e_3, e_4)	$-a_{32} + a_{46} = 0,\ a_{31} - a_{45} = 0,\ a_{36} + a_{42} = 0,$ $a_{35} + a_{41} = 0.$

From these conditions on a_{ij} and $\forall\, a_{41}, a_{42}, a_{44}, a_{46}, a_{55}, a_{61}, a_{65}, a_{66} \in \mathbb{C}$, we have the following conditions shown in Table 2.

Table 2. Conditions obtained.

$a_{11} = a_{55},$	$a_{12} = -a_{65},$	$a_{13} = 0,$	$a_{14} = a_{41},$	$a_{15} = 0,$	$a_{16} = a_{61}.$
$a_{21} = a_{65},$	$a_{22} = a_{66},$	$a_{23} = -a_{46},$	$a_{24} = a_{42},$	$a_{25} = -a_{61},$	$a_{26} = 0.$
$a_{31} = 0,$	$a_{32} = a_{46},$	$a_{33} = a_{44},$	$a_{34} = 0,$	$a_{35} = -a_{41},$	$a_{36} = -a_{42},$
		$a_{43} = 0,$		$a_{45} = 0.$	
$a_{51} = 0,$	$a_{52} = -a_{61},$	$a_{53} = -a_{41},$	$a_{54} = 0,$		$a_{56} = -a_{65}.$
	$a_{62} = 0,$	$a_{63} = -a_{42},$	$a_{64} = -a_{46}.$		

This implies that $\psi_{\mathfrak{g}_6}(1) = dim\big(Der_{(1,1,1,1)}\mathfrak{g}_6\big) = 8.$

Computing $\bar{\psi}_{\mathfrak{g}_6}$, for $\alpha = 1$

Let us consider $d \in Der_{(1,1,1,1)}\mathfrak{g}_6$. Then, $[d[u,v],[u,w]] + [[u,v],d[u,w]] = d[[[u,w],v],u] + d[[[w,u],u],v], \quad \forall u,v,w \in \mathfrak{g}_6.$

To obtain the elements a_{ij} of the corresponding 6×6 square matrix associated with d, we see that for each triplets of generators (e_i, e_j, e_k) of the algebra, the previous expression is written as

$$[d[e_i, e_j], [e_i, e_k]] + [[e_i, e_j], d[e_i, e_k]] = d[[[e_i, e_k], e_j], e_i] + d[[[e_k, e_i], e_i], e_j].$$

Starting from it, we obtain the following conditions shown in Table 3.

Table 3. Conditions obtained.

From Triplet (e_i, e_j, e_k)	**Conditions**
(e_1, e_2, e_3)	$a_{51} = a_{43} + a_{62}, \quad a_{52} = -a_{61}, \quad a_{53} = -a_{41},$ $a_{54} = -a_{45}, \quad a_{55} = a_{66} + a_{44}, \quad a_{56} = -a_{65}.$
(e_1, e_2, e_4)	$a_{11} = a_{22} + a_{44}, \quad a_{12} = -a_{21}, \quad a_{13} = -a_{45},$ $a_{14} = -a_{41}, \quad a_{15} = a_{26} - a_{43}, \quad a_{16} = -a_{25}.$
(e_1, e_2, e_5)	$0 = 0$
(e_1, e_2, e_6)	$a_{32} + a_{46} = 0, \quad e_{31} + a_{45} = 0, \quad a_{36} + a_{42} = 0,$ $a_{35} + a_{41} = 0.$
(e_1, e_3, e_4)	$-a_{23} + a_{64} = 0, \quad a_{21} - a_{65} = 0, \quad a_{25} + a_{61} = 0$ $a_{24} + a_{63} = 0.$
(e_1, e_3, e_5)	$0 = 0$
(e_1, e_3, e_6)	$a_{33} + a_{66} = a_{11}, \quad a_{65} = -a_{12}, \quad a_{31} = -a_{13},$ $a_{35} = -a_{14}, \quad a_{34} - a_{62} = a_{15}, \quad a_{61} = a_{16}.$
(e_1, e_4, e_5)	$0 = 0$
(e_1, e_4, e_6)	$a_{51} = -a_{26} - a_{34}, \quad a_{52} = a_{25}, \quad a_{53} = a_{35},$ $a_{54} = -a_{31}, \quad a_{55} = a_{22} + a_{33}, \quad a_{56} = -a_{21}.$
(e_1, e_5, e_6)	$0 = 0$
(e_2, e_3, e_4)	$-a_{13} + a_{54} = 0, \quad a_{12} - a_{56} = 0, \quad a_{16} + a_{52} = 0,$ $a_{14} + a_{53} = 0.$
(e_2, e_3, e_5)	$a_{21} = -a_{56}, \quad a_{22} = a_{33} + a_{55}, \quad a_{23} = -a_{32},$ $a_{24} = -a_{36}, \quad a_{25} = a_{52}, \quad a_{26} = a_{34} - a_{51}.$
(e_2, e_3, e_6)	$0 = 0$
(e_2, e_4, e_5)	$a_{61} = -a_{16}, \quad a_{62} = a_{15} + a_{34}, \quad a_{63} = -a_{36},$ $a_{64} = a_{32}, \quad a_{65} = a_{12}, \quad a_{66} = -a_{11} - a_{33}.$
(e_2, e_4, e_6)	$0 = 0$
(e_2, e_5, e_6)	$0 = 0$
(e_3, e_4, e_5)	$0 = 0$
(e_3, e_4, e_6)	$0 = 0$
(e_4, e_5, e_6)	$a_{41} = -a_{53}, \quad a_{42} = -a_{63}, \quad a_{43} = a_{51} + a_{62},$ $a_{44} = a_{55} + a_{66}, \quad a_{45} = -a_{54}, \quad a_{46} = -a_{64}.$
(e_3, e_5, e_6)	$a_{41} = a_{14}, \quad a_{42} = a_{24}, \quad a_{43} = -a_{15} - a_{26},$ $a_{44} = a_{11} + a_{22}, \quad a_{45} = -a_{13}, \quad a_{46} = -a_{23}.$

It follows from these conditions for a_{ij} that $a_{ij} = 0, \quad \forall\, i, j \in \{1, 2, 3, 4, 5, 6\}$. This implies that $\bar{\psi}_{\mathfrak{g}_6}(1,1) = dim\big(Der_{(1,1,1,1)}\mathfrak{g}_6\big) = 0$, which proves that $\psi \neq \bar{\psi}$ in general.

3.2. The Quantum-Mechanical Model Based on a 5th Heisenberg Algebra

In this section, and by using the invariant function previously introduced $\bar{\psi}$, we prove the following result.

Theorem 3. Main Theorem

The five-dimensional classical-mechanical model built upon certain types of five-dimensional Lie algebras cannot be obtained as a limit process of a quantum-mechanical model based on a fifth Heisenberg algebra.

Proof. Let $\mathbb{H}_5$ be the fifth Heisenberg algebra generated by $\{e_1, \ldots, e_5\}$ and defined by the brackets $[e_1, e_3] = e_5$ and $[e_2, e_4] = e_5$.

Let us consider $d \in Der_{(1,1,1,1)}\mathbb{H}_5$. Then, $[d[u,v],[u,w]] + [[u,v],d[u,w]] = d[[[u,w],v],u] + d[[[w,u],u],v]$, $\forall u,v,w \in \mathbb{H}_5$.

To obtain the elements a_{ij} of the corresponding 5×5 square matrix associated with d, we see that for each triplet of generators (e_i, e_j, e_k) of the algebra, the previous expression is written as

$$[d[e_i,e_j],[e_i,e_k]] + [[e_i,e_j],d[e_i,e_k]] = d[[[e_i,e_k],e_j],e_i] + d[[[e_k,e_i],e_i],e_j].$$

Note that, in this case, there is no restriction on the elements of the matrix associated with d and, thus, $\bar{\psi}_{\mathbb{H}_5}(1,1) = dim\big(Der_{(1,1,1,1)}\mathbb{H}_5\big) = 25$.

For another part, let $\mathfrak{f}_5$ be the five-dimensional filiform Lie algebra, defined by $[e_1, e_3] = e_2$, $[e_1, e_4] = e_3$ and $[e_1, e_5] = e_4$.

Let us consider $d \in Der_{(1,1,1,1)}\mathfrak{f}_5$. Then, it is verified that $[d[u,v],[u,w]] + [[u,v],d[u,w]] = d[[[u,w],v],u] + d[[[w,u],u],v]$, $\forall u,v,w \in \mathfrak{f}_5$.

Similar to the previous case, to obtain the elements a_{ij} of the corresponding 5×5 square matrix associated with d, we see that, for each triplet of generators (e_i, e_j, e_k) of the algebra, the previous expression is written as

$$[d[e_i,e_j],[e_i,e_k]] + [[e_i,e_j],d[e_i,e_k]] = d[[[e_i,e_k],e_j],e_i] + d[[[e_k,e_i],e_i],e_j].$$

In this case, the restrictions of the matrix associated with d are $a_{21} = 0$, obtained from the bracket (e_1, e_3, e_5) and $a_{31} = a_{41}$ from (e_1, e_4, e_5), therefore $\bar{\psi}(1,1) = 23$.

Next, we use the highly non-trivial result, which was originally proved by Borel [17]: *If* $\mathfrak{g}_0$ *is a proper contraction of a complex Lie algebra* $\mathfrak{g}$*, then it holds:* $\dim\big(Der\mathfrak{g}\big) < \dim\big(Der\mathfrak{g}_0\big)$.

Indeed, according to Proposition 1 we obtain that

$$\bar{\psi}_{\mathbb{H}_5}(1,1) = \dim\big(Der_{(1,1,1,1)}\mathbb{H}_5\big) = \dim\big(Der(\mathbb{H}_5)\big) = 25$$

and

$$\bar{\psi}_{\mathfrak{f}_5}(1,1) = \dim\big(Der_{(1,1,1,1)}\mathfrak{f}_5\big) = \dim\big(Der(\mathfrak{f}_5)\big) = 23.$$

It implies that no proper contraction transforming the Heisemberg algebra $\mathbb{H}_5$ into the filiform Lie algebra $\mathfrak{f}_5$ exists. Thus, since both algebras are not isomorphic, the five-dimensional classical-mechanical model built upon a five-dimensional filiform Lie algebra cannot be obtained as a limit process of a quantum-mechanical model based on a fifth Heisenberg algebra. □

4. Discussion and Conclusions

In this paper, we introduce an invariant two-parameter function of algebras, $\bar{\psi}$, and we have used it as a tool to study contractions of certain particular types of algebras.

Indeed, by means of this function, we have proved that there is no proper contraction between a fifth Heisenberg algebra and a filiform Lie algebra of dimension 5. It implies, as a main result, that the five-dimensional classical-mechanical model built upon a five-dimensional filiform Lie algebra cannot be obtained as a limit process of a quantum-mechanical model based on a fifth Heisenberg algebra.

We have also computed this function in the case of other types of algebras, for instance, Malcev algebras of the type Lie and the Lie algebra induced by the Lorentz group $SO(3,1)$.

Apart from continuing this study with with higher-dimensional algebras, we indicate next some open problems to be dealt with in future work, most of them with the objective of trying to find some possible interesting physical applications for the filiform Lie algebras. They are the following

1. As mentioned above, in 2007, Hrivnák and Novotný introduced the invariant functions ψ and φ as a tool to study contractions of Lie algebras [7]. Those are one-parameter functions. We have now defined the two-parameter invariant function $\bar{\psi}$. It would be good to search new invariant functions to continue with this research, for instance, some related with twisted cocycles of Lie algebras.
2. It would also be good to find necessary and sufficient conditions which characterize contractions of Lie algebras.
3. One of the possible physical applications of the present topic is given by the possibility of describing a many-body system based on interacting spinless boson particles located in a lattice of n sites by means of a filiform Lie algebra. This system could be a kind of Bose–Hubbard model, which is well known in the condensed matter community and widely studied. The Hamiltonian corresponding to that system can be described in terms of semi-simple Lie algebras and is a quadratic model since it contains up to two-body operators. Therefore, we wonder if we could describe the same system employing filiform Lie algebras and if we could obtain new information using the tools developed in this manuscript.

   To perform this task, it is necessary to write the boson operators involved in the Hamiltonian in term of new ones that fulfill the commutation relations for a given filiform Lie algebra. However, at that point, we find the difficulty that we should employ a tensorial product of two filiform Lie algebras in order to describe the system properly. That means that an isomorphism between the semi-simple Lie algebra of the original hamiltonian and the filiform Lie algebra proposed to describe the physical system should exist. Fortunately, it seems that we have obtained a theorem that can confirm that kind of isomorphism.

   Now, the advantage that we gain employing a filiform Lie algebra instead of a semi-simple Lie algebra is that we could map a non-linear problem such as the problem described by a system with up to two-body interactions onto a linear problem with just one-body interactions. On the other hand, once we have described the system in terms of the filiform Lie algebra, it is necessary to define the branching rules, that is to find the irreducible representations of an algebra $\mathfrak{g}'$ contained in a given representation of $\mathfrak{g}$. Since the representations are interpreted as quantum mechanical states, it is necessary to provide a complete set of quantum numbers (labels) to characterize uniquely the basis of the system. This is a non-trivial task that it may even lead to a further research.
4. Another possible physical applications of the present topic is to study phase spaces by using filiform Lie algebras as a tool.

   In this respect, Arzano and Nettel [18] in 2016 introduced a general framework for describing deformed phase spaces with group valued momenta. Using techniques from the theory of Poisson–Lie groups and Lie bialgebras, they developed tools for constructing Poisson structures on the deformed phase space starting from the minimal input of the algebraic structure of the generators of the momentum Lie group. These tools developed are used to derive Poisson structures on examples of group momentum space much studied in the literature such as the n-dimensional generalization of the κ-deformed momentum space and the $SL(2,R)$ momentum space in three space-time dimensions. They also discussed classical momentum observables associated to multiparticle systems and argued that these combined according the usual four-vector addition despite the non-Abelian group structure of momentum space (see [18] for further information).

In that paper, the authors work with a phase space $\Gamma = T \times G$, given by the Cartesian product of a n-dimensional Lie group configuration space T and a n-dimensional Lie group momentum space G. Since T and G are Lie groups, we can consider their associated Lie algebras $\mathfrak{t}$ and $\mathfrak{g}$ so that we can define a Lie–Poisson algebra, which can endow a mathematical structure to the phase space Γ. Indeed, Arzano and Nettel considered a phase space Γ in which the component related to momentum is an n-dimensional Lie sub-group of the $(n+2)$-dimensional Lorentz group $SO(n+1,1)$, denoted as $AN(n)$.

Taking into consideration this paper, we have tried to construct a phase space similar to the one by those authors, although we have taken the $(n+2)$-dimensional Lorentz group $SO(n+1,2)$ as the Lie group related to momentum.

We began our research on this subject considering the Lie group $SO(2,2)$ and using the same procedure as Arzano and Nettel did. However, we realized that that attempt was going to be very complicated because of the great dimensions of the matrices involved (in the computations, a 49×49 $r-$matrix appeared).

Therefore, the fact of finding a Poisson structure that allows us to endow the phase space $\Gamma = T \times SO(n+1,2)$ with a mathematical structure is another problem, which we consider open.

5. Finally, semi-invariant functions of algebras could also be considered to study contractions of Lie Algebras (see [19], for instance).

We will dedicate our efforts to these objectives in future work.

5. Materials and Methods

Since this is a work on pure and applied mathematics, no type of materials different from the usual ones in a theoretical investigation was needed. Indeed, on the one hand, only the existing bibliography on the subject and, on the other hand, a suitable symbolic computation package were used. In the same way, with regard to the methodology used for the writing of the manuscript, it was also the usual one in research work of this nature, namely, based on already established hypotheses and known results.

We used the SAGE symbolic computation package for computations. SageMath, which is a free open-source mathematics software system licensed under the GPL, builds on top of many existing open-source packages, such as matplotlib, Sympy, Maxima, GAP, R and many more (see [20], for instance).

Author Contributions: All of the authors wrote the paper and J.N.-V. and P.P.-F. checked the proofs.

Funding: This research was funded by the Spanish Ministerio de Ciencia e Innovación and Junta de Andalucía via grants No. MTM2013-40455-P and No. FQM-326 (J.N.-V.) and No. FQM-160 (P.P.-F.).

Acknowledgments: The authors gratefully acknowledge the financial support above mentioned.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Inönü, E.; Wigner, E. On the contraction of groups and their representations. *Proc. Nat. Acad. Sci. USA* **1953**, *39*, 510–524. [CrossRef] [PubMed]

2. Inönü, E.; Wigner, E. On a particular type of convergence to a singular matrix. *Proc. Nat. Acad. Sci. USA* **1954**, *40*, 119–121. [CrossRef] [PubMed]
3. Doebner, H.D.; Melsheimer O. On a class of generalized group contractions. *Nuovo Cimento A* **1967**, *49*, 306–311. [CrossRef]
4. Burde, D. Degenerations of nilpotent Lie algebras. *J. Lie Theory* **1999**, *9*, 193–202.
5. Burde, D. Degenerations of 7-dimensional nilpotent Lie algebras. *Commun. Algebra* **2005**, *33*, 1259–1277. [CrossRef]
6. Steinhoff, C. Klassifikation und Degeneration von Lie Algebren. Ph.D. Thesis, Universität Düsseldorf, Düsseldorf, Germany, 1997.
7. Novotný, P.; Hrivnák, J. On (α, β, γ)-derivations of Lie algebras and corresponding invariant functions. *J. Geometry Phys.* **2008**, *58*, 208–217. [CrossRef]
8. Escobar, J.M.; Núñez, J.; Pérez-Fernández, P. On contractions of Lie algebras. *Math. Comput. Sci.* **2016**. [CrossRef]
9. Humphreys, J.E. *Introduction to Lie Algebras and Representation Theory*; Springer: New York, NY, USA, 1972.
10. Vergne, M. Cohomologie des algèbres de Lie nilpotentes. Application à l'étude de la variété des algebres de Lie nilpotentes. *Bull. Soc. Math. France* **1970**, *98*, 81–116. [CrossRef]
11. Sagle, A.A. Malcev Algebras. *Trans. Am. Math. Soc.* **1961**, *101*, 426–458. [CrossRef]
12. Dirac, P.A.M. *Lectures on Quantum Mechanics*; Yeshiva University: New York, NY, USA, 1964.
13. Lipkin, H.J.; Weisberger, W.I.; Peshkin, M. Magnetic Charge Quantization and Angular Momentum. *Ann. Phys.* **1969**, *53*, 203–214. [CrossRef]
14. Günaydin, M. Exceptionality, supersymmetry and non-associativity in Physics. In Proceedings of the Bruno Zumino Memorial Meeting, Geneva, Switzerland, 27–28 April 2015.
15. Günaydin, M.; Minic, D. Nonassociativity, Malcev algebras and string theory. *Fortschritte der Physik* **2013**, *61*, 873–892.
16. Falcón, O.J.; Falcón, R.M.; Núñez, J. A computational algebraic geometry approach to enumerate Malcev magma algebras over finite fields. *Math. Methods Appl. Sci.* **2016**, *39*, 4901–4913. [CrossRef]
17. Borel, A. *Linear Algebraic Groups*; Benjamin, Inc.: New York, NY, USA, 1969.
18. Arzano, M.; Nettel, F. Deformed phase spaces with group valued momenta. *Phys. Rev. D* **2016**, *94*, 085004. [CrossRef]
19. Nesterenko, M.; Popovych, R. Contractions of Low-Dimensional Lie Algebras. *J. Math. Phys.* **2006**, *47*, 123515. [CrossRef]
20. On SAGE symbolic computation package. Available online: http://www.sagemath.org/ (accessed on 11 October 2019).