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SYMMETRIES AND EXACT SOLUTIONS OF CONFORMABLE FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper Lie group analysis is used to investigate invariance properties of nonlinear fractional partial differential equations with conformable fractional time derivative. The analysis is applied to Korteweg-de Vries, modified Korteweg-de Vries, Burgers, and modified Burgers equations. For each equation, all of the vector fields and the Lie symmetries are obtained. Moreover, exact solutions are given to these equations.

1 Introduction

In the last decades the interest in studding fractional calculus was rapidly growing due to its applications in many fields of applied sciences such as: Mathematics, Physics, Chemistry, Engineering, Finance, and Social sciences. These applications show the importance of fractional calculus. As a result several definitions for fractional derivatives appear in the literature that are utilized to present more accurate models for real life phenomena. Some of known fractional derivatives are Riemann-Liouville, modified Riemann-Liouville, Caputo, Hadmard, Erdélyi-Kober, Riesz, Grünwald-Letnikov, Marchaud and others (see [1]–[10]). It should be noted that all definitions satisfy the linear property, which is the only property inherited from the usual derivative. However, properties such as the derivative of a constant is zero, the product rule, the quotient rule, and the chain rule does not hold or too complicated in many fractional derivatives.

Recently, a new definition of fractional derivative that extends the familiar limit definition of the derivatives of a function has been introduced by Khalil et. al. [11]. The new definition is called the conformable fractional derivative. Unlike other definitions, this new definition prominently compatible with the classical derivative and it seems to satisfy all the requirements of the standard derivative. The importance of it lies in satisfying the product formula, the quotient formula, and it has a simpler formula for the chain rule. After Khalil et. al. [11] many studies related to this new fractional derivative were published [12]–[22]. A generalization of this definition can be found in [23].

The Lie symmetry theory plays a significant role in the analysis of differential equations. The Norwegian mathematician Sophus Lie proposed the first work devoted exclusively to the subject of Lie symmetry in the 19th century. The Lie symmetry analysis is regarded as the most important approach for constructing analytical solutions of nonlinear differential equations. A huge number of papers and many excellent textbooks have been devoted to the theory of Lie symmetry groups and their applications to differential equations; some of these are [24]–[29]. In recent years, Lie group analysis of fractional differential equations was investigated [30]–[48]. The Lie symmetry analysis of time-fractional Burgers and Korteweg-de Vries (KdV) equations with Riemann-Liouville time derivative was studied in [32]. The Lie symmetry analysis of the KdV equation with modified Riemann-Liouville time-fractional derivative was studied in [35]. It was shown that each of these equations can be reduced to a nonlinear ordinary differential equation of fractional order with a new independent variable. The fractional derivative in the reduced equation is the Erdelyi-Kober fractional derivative. The new equations are not solvable as in the classical derivative. It is well known that the KdV and Burgers equations with classical derivatives can be transformed to equations related to Painlevé and Riccati equations respectively.

In this paper we propose prolongation formulas for conformable fractional derivatives and apply the method of Lie group to conformable fractional partial differential equations ($CFPDEs$). We study the Lie analysis of Korteweg-de Vries, modified Korteweg-de Vries, Burgers, and modified Burgers equations with conformable fractional time partial derivative. For each equation, all of the vector fields and the Lie symmetries are obtained. We show that the equations under consideration can be reduced to ordinary differential equations with classical derivative.

2 Conformable Fractional Calculus

Definition 2.1. [11] Given a function $f : [0, \infty) \to \mathbb{R}$, the conformable fractional derivative of f of order α is defined by

$$D^\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}, \tag{2.1}$$

for all $t > 0$, $\alpha \in (0, 1]$. If $D^\alpha(f)(t)$ exists for t in some interval $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} D^\alpha(f)(t)$ exists, then $D^\alpha(f)(0) = \lim_{t \to 0^+} D^\alpha(f)(t)$.

If $D^\alpha(f)(t)$ exists for $t \in [0, \infty)$, then f is said to be α-differentiable at t.

Theorem 2.2. *[11] Let $\alpha \in (0, 1]$ and $f,\ g$ be α-differentiable at a point $t > 0$. Then*

(i) $D^\alpha(af + bg)(t) = a(D^\alpha f)(t) + b(D^\alpha g)(t)$, *for all* $a, b \in \mathbb{R}$.

(ii) $D^\alpha(t^p) = pt^{p-\alpha}$, *for all* $p \in \mathbb{R}$.

(iii) $D^\alpha(c) = 0$, *for any constant c.*

(iv) $D^\alpha(fg)(t) = f(t)D^\alpha(g)(t) + g(t)D^\alpha(f)(t)$.

(v) $D^\alpha\left(\frac{f}{g}\right)(t) = \frac{g(t)D^\alpha(f)(t) - f(t)D^\alpha(g)(t)}{g^2(t)}$.

(vi) If, in addition, f is differentiable, then $D^\alpha(f)(t) = t^{1-\alpha}\frac{d}{dt}f(t)$.

Definition 2.3. [11] $I^\alpha(f)(t) = I(t^{\alpha-1}f)(t) = \int_0^t \frac{f(x)}{x^{1-\alpha}}dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1]$.

Theorem 2.4. *[11] $D^\alpha I^\alpha(f)(t) = f(t)$, for $t \geq 0$, where f is any continuous function in the domain of I^α.*

Lemma 2.5. *[13] Let $f : [0, b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > 0$ we have $I^\alpha D^\alpha(f)(t) = f(t) - f(0)$.*

Lemma 2.6. *[21] Let $0 < \alpha \leq 1$, f be differentiable at $g(t)$, and g be α-differentiable at $t > 0$, then $D^\alpha(fog)(t) = f'(g(t))D^\alpha(g)(t)$.*

3 Lie Symmetry Analysis of Conformable Fractional Partial Differential Equations

Consider the following time-fractional partial differential equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = F[u], \quad 0 < \alpha \leq 1, \tag{3.1}$$

where $u = u(x, t)$, $F[u]$ is a nonlinear differential operator and $\frac{\partial^\alpha}{\partial t^\alpha}$ is the conformable fractional derivative. Our aim is to study the symmetry transformations of equation (3.1).

The invertible point transformations

$$\hat{x} = X(t, x, u, \varepsilon), \quad \hat{t} = T(t, x, u, \varepsilon), \quad \hat{u} = U(t, x, u, \varepsilon), \tag{3.2}$$

depending on a continuous parameter ε, are said to be symmetry transformations of equation (3.1), if equation (3.1) has the same form in the new variables $\hat{x}, \hat{t}, \hat{u}$. The set G of all such transformations forms a continuous group called the symmetry group. The symmetry group G is also known as the Lie group admitted by equation (3.1).

The key step to find a Lie group of symmetry transformations is to find the infinitesimal generator of the group. In order to provide a basis of group generators one has to create and then to solve the so called determining system of equations.

The infinitesimal transformations of (3.2) read

$$\begin{aligned}\hat{x} &= x + \varepsilon\xi(t,x,u) + o(\varepsilon^2),\\ \hat{t} &= t + \varepsilon\tau(t,x,u) + o(\varepsilon^2),\\ \hat{u} &= u + \varepsilon\eta(t,x,u) + o(\varepsilon^2).\end{aligned} \tag{3.3}$$

It is convenient to introduce the operator

$$V = \xi(t,x,u)\frac{\partial}{\partial x} + \tau(t,x,u)\frac{\partial}{\partial t} + \eta(t,x,u)\frac{\partial}{\partial u}, \tag{3.4}$$

which is known as the infinitesimal operator (or generator of the group G). The group transformations (3.2) corresponding to operator (3.4) can be obtained by solving the Lie equations

$$\frac{d\hat{x}}{d\varepsilon} = \xi(\hat{t},\hat{x},\hat{u}), \qquad \frac{d\hat{t}}{d\varepsilon} = \tau(\hat{t},\hat{x},\hat{u}), \qquad \frac{d\hat{u}}{d\varepsilon} = \eta(\hat{t},\hat{x},\hat{u}), \tag{3.5}$$

subject to the initial conditions

$$\hat{x}|_{\varepsilon=0} = x, \qquad \hat{t}|_{\varepsilon=0} = t, \qquad \hat{u}|_{\varepsilon=0} = u.$$

A surface $u = u(t,x)$ is mapped to it self by the group of transformations generated by V if

$$V(u - u(t,x)) = 0 \qquad \text{when} \qquad u = u(t,x). \tag{3.6}$$

By definition, the transformations (3.2) form a symmetry group G of equation (3.1) if the function $\hat{u}(\hat{t},\hat{x})$ satisfies the equation

$$\frac{\partial^\alpha \hat{u}}{\partial \hat{t}^\alpha} = F[\hat{u}], \qquad 0 < \alpha \le 1, \tag{3.7}$$

whenever the function $u = u(t,x)$ satisfies equation (3.1). Extending transformation (3.3) to the operator of fractional differentiation $\frac{\partial^\alpha u}{\partial t^\alpha}$ and to the operator of x differentiation of various orders $\frac{\partial^r u}{\partial x^r}$, $r = 1, 2, 3, ...$, one can obtain

$$\begin{aligned}\frac{\partial^\alpha \hat{u}}{\partial \hat{t}^\alpha} &= \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon\eta_\alpha^t(t,x,u) + o(\varepsilon^2),\\ \frac{\partial \hat{u}}{\partial \hat{x}} &= \frac{\partial u}{\partial x} + \varepsilon\eta^x(t,x,u) + o(\varepsilon^2),\\ \frac{\partial^2 \hat{u}}{\partial \hat{x}^2} &= \frac{\partial^2 u}{\partial x^2} + \varepsilon\eta^{xx}(t,x,u) + o(\varepsilon^2),\\ \frac{\partial^3 \hat{u}}{\partial \hat{x}^3} &= \frac{\partial^3 u}{\partial x^3} + \varepsilon\eta^{xxx}(t,x,u) + o(\varepsilon^2),\\ &\ \ \vdots\end{aligned} \tag{3.8}$$

where

$$\begin{aligned}\eta^x &= D_x(\eta) - u_t D_x(\tau) - u_x D_x(\xi),\\ \eta^{xx} &= D_x(\eta^x) - u_{xt} D_x(\tau) - u_{xx} D_x(\xi),\\ \eta^{xxx} &= D_x(\eta^{xx}) - u_{xxt} D_x(\tau) - u_{xxx} D_x(\xi).\\ &\ \ \vdots\end{aligned} \tag{3.9}$$

Here D_x denotes the total derivative operator and is defined as

$$D_x = \frac{\partial}{\partial x} + u_x \frac{\partial}{\partial u} + u_{xx} \frac{\partial}{\partial u_x} + u_{tx} \frac{\partial}{\partial u_t} + \cdots$$

Proposition 3.1. *The prolongation of the point transformation (3.2) to the αth derivative for some $\alpha \in (0,1]$ (with the conformable fractional operator (2.1)) is given by $\frac{\partial^\alpha \hat{u}}{\partial \hat{t}^\alpha} = \frac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon \eta_\alpha^t + o(\varepsilon^2)$, where η_α^t is the αth extended infinitesimal related to conformable fractional time derivative with u* and *$\hat{u}$ are differentiable functions.*

Proof. Assume that $\hat{u}$ is a differentiable function, then

$$\frac{\partial^\alpha \hat{u}}{\partial \hat{t}^\alpha} = \hat{t}^{1-\alpha} \frac{\partial \hat{u}}{\partial \hat{t}}. \tag{3.10}$$

But

$$\hat{t} = t + \varepsilon \tau(t,x,u) + o(\varepsilon^2), \qquad \frac{\partial \hat{u}}{\partial \hat{t}} = \frac{\partial u}{\partial t} + \varepsilon \eta^t + o(\varepsilon^2), \tag{3.11}$$

where $\eta^t = D_t \eta - u_x D_t \xi - u_t D_t \tau$. Substituting equation (3.11) into equation (3.10), gives

$$\begin{aligned} \frac{\partial^\alpha \hat{u}}{\partial \hat{t}^\alpha} &= [t + \varepsilon \tau(t,x,u) + o(\varepsilon^2)]^{1-\alpha} [\tfrac{\partial u}{\partial t} + \varepsilon \eta^t + o(\varepsilon^2)] \\ &= [t^{1-\alpha} + \varepsilon(1-\alpha)\tau t^{-\alpha} + o(\varepsilon^2)][\tfrac{\partial u}{\partial t} + \varepsilon \eta^t + o(\varepsilon^2)] \\ &= t^{1-\alpha} u_t + \varepsilon[t^{1-\alpha}\eta^t + (1-\alpha)\tau t^{-\alpha} u_t] + o(\varepsilon^2) \\ &= \tfrac{\partial^\alpha u}{\partial t^\alpha} + \varepsilon \eta_\alpha^t + o(\varepsilon^2). \end{aligned}$$

The αth extended infinitesimal related to conformable fractional time derivative reads

$$\eta_\alpha^t = D_t^\alpha \eta - u_x D_t^\alpha \xi - u_t D_t^\alpha \tau + (1-\alpha)\tau t^{-\alpha} u_t, \tag{3.12}$$

where the operator D_t^α express the total fractional derivative operator, whenever the function is differentiable it is given by $D_t^\alpha = t^{1-\alpha} D_t$, and D_t is the total derivative operator given by

$$D_t = \frac{\partial}{\partial t} + u_t \frac{\partial}{\partial u} + u_{xt} \frac{\partial}{\partial u_x} + u_{tt} \frac{\partial}{\partial u_t} + \cdots$$

□

If the vector field (3.4) generates a symmetry of (3.1), then V must satisfy Lie symmetry condition

$$pr^{(n)} V(\Delta_1)\Big|_{\Delta_1 = 0} = 0, \tag{3.13}$$

where $\Delta_1 = \frac{\partial^\alpha u}{\partial t^\alpha} - F[u]$.

4 The fractional Korteweg-de Vries equation

In this section we consider the following time-fractional Korteweg-de Vries (KdV) equation of the form

$$\frac{\partial^\alpha u}{\partial t^\alpha} + a u u_x + b u_{xxx} = 0, \tag{4.1}$$

where $0 < \alpha \leq 1$, a, b are constants, and α is a parameter describing the order of the conformable fractional time derivative. According to the Lie theory, applying the third prolongation $pr^{(3)} V$ to (4.1), one can find the infinitesimal criterion (3.13) to be

$$a \eta u_x + \eta_\alpha^t + a u \eta^x + b \eta^{xxx} = 0, \tag{4.2}$$

which must be satisfied whenever $\frac{\partial^\alpha u}{\partial t^\alpha} + a u u_x + b u_{xxx} = 0$. Substituting the general formulas for η^x, η^{xxx} and η_α^t from (3.9) and (3.12) into (4.2), replacing u_{xxx} by $-\frac{1}{b}\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{a}{b} u u_x$ whenever it occurs, and equating the coefficients of the various monomials in partial derivatives of u, one can get the full determining equations for the symmetry group of (4.1). Solving these equations, one can obtain

$$\tau = \frac{-3c_1}{2\alpha}t + c_4 t^{1-\alpha}, \qquad \xi = \frac{-c_1}{2}x + \frac{ac_2}{\alpha}t^{\alpha} + c_3, \qquad \eta = c_1 u + c_2, \tag{4.3}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants. Therefore, the symmetry group of (4.1) is spanned by the four vector fields

$$V_1 = t^{1-\alpha}\frac{\partial}{\partial t}, \;\; V_2 = \frac{\partial}{\partial x}, \;\; V_3 = \frac{at^{\alpha}}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \;\; V_4 = \frac{-3t}{2\alpha}\frac{\partial}{\partial t} - \frac{x}{2}\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}. \tag{4.4}$$

It is easily checked that the set of these vector fields is closed under the Lie bracket ($[\rho,\sigma] = \rho\sigma - \sigma\rho$). In fact we have the following commutation relations between these vector fields

$$[V_1, V_2] = 0, \qquad [V_1, V_3] = aV_2, \qquad [V_1, V_4] = \frac{-3}{2}V_1,$$

$$[V_2, V_3] = 0, \qquad [V_2, V_4] = \frac{-1}{2}V_2, \qquad [V_3, V_4] = V_3.$$

It is remarkable that the Lie algebra for fractional and classical KdV equation have the same dimension. Moreover, when $\alpha = 1$ the Lie algebra for the fractional KdV equation reduces to that of classical KdV equation [25].

As a next step, we can fined the invariant solution of equation (4.1). For example, the similarity variables for the infinitesimal generator V_4 can be found by solving the corresponding characteristic equation

$$\frac{-2dx}{x} = \frac{2\alpha dt}{-3t} = \frac{du}{u}. \tag{4.5}$$

The corresponding invariants are

$$\zeta = x(3t)^{\frac{-\alpha}{3}}, \qquad u = (3t)^{\frac{-2\alpha}{3}}\Psi(\zeta). \tag{4.6}$$

Substituting transformation (4.6) into equation (4.1), one can find that (4.1) can be reduced to a nonlinear ODE with the classical derivative. Consequently, we have

$$b\Psi'''(\zeta) + a\Psi(\zeta)\Psi'(\zeta) - \alpha\zeta\Psi'(\zeta) - 2\alpha\Psi(\zeta) = 0, \tag{4.7}$$

where $\Psi'(\zeta) := \frac{d\Psi(\zeta)}{d\zeta}$.

For the special case $a = 6,\ b = 1$, the time fractional KdV (4.1) becomes

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} + 6uu_x + u_{xxx} = 0. \tag{4.8}$$

By using the scale in (4.6), then (4.7) reduces to

$$\Psi'''(\zeta) + 6\Psi(\zeta)\Psi'(\zeta) - \alpha\zeta\Psi'(\zeta) - 2\alpha\Psi(\zeta) = 0. \tag{4.9}$$

The change of variables $z = \alpha^{\frac{1}{3}}\zeta,\ \ \Psi(\zeta) = \alpha^{\frac{2}{3}}W(z)$ transforms (4.9) to the form

$$\dddot{W}(z) + 6W(z)\dot{W}(z) - z\dot{W}(z) - 2W(z) = 0, \tag{4.10}$$

where $\dot{W} = \frac{dW}{dz}$, whose solutions are also expressible in terms of solutions of second Painlevé equation P_{II}. Indeed there exists a one-to-one correspondence between solutions of (4.10) and those of P_{II}, given by

$$W = -\dot{\Phi} - \Phi^2, \qquad \Phi = \frac{\dot{W} + \gamma}{2W - z}, \tag{4.11}$$

where Φ satisfies the P_{II} equation

$$\ddot{\Phi}(z) = 2\Phi^3(z) + z\Phi(z) + \gamma, \tag{4.12}$$

(for further details see [49], [50]).

Another special solution of the time fractional KdV equation (4.1) can be obtained from the linear combination $V_3 - \frac{1}{\mu}V_1 = \frac{at^\alpha}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{1}{\mu}t^{1-\alpha}\frac{\partial}{\partial t}$, ($\mu$ is a constant). Solving the corresponding characteristic equation

$$\frac{\alpha dx}{at^\alpha} = \frac{-\mu dt}{t^{1-\alpha}} = \frac{du}{1}, \tag{4.13}$$

we obtain the invariants

$$\zeta = x + \frac{a\mu}{2\alpha^2}t^{2\alpha}, \qquad u = \frac{-\mu}{\alpha}t^\alpha + \Psi(\zeta). \tag{4.14}$$

Substituting transformation (4.14) into equation (4.1), one can find that (4.1) can be reduced to a nonlinear ODE with the classical derivative

$$b\Psi'''(\zeta) + a\Psi(\zeta)\Psi'(\zeta) - \mu = 0, \tag{4.15}$$

where $\Psi'(\zeta) := \frac{d\Psi(\zeta)}{d\zeta}$. Integrating equation (4.15) gives

$$b\Psi''(\zeta) + \frac{a}{2}\Psi^2(\zeta) - \mu\zeta = \gamma, \tag{4.16}$$

where γ is a constant of integration.

When $a = 6,\ b = 1$, the time fractional KdV (4.1) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} + 6uu_x + u_{xxx} = 0. \tag{4.17}$$

Using the scale in (4.14), and integrating the result then (4.17) can be reduced to

$$\Psi''(\zeta) + 3\Psi^2(\zeta) - \mu\zeta = \gamma. \tag{4.18}$$

Equation (4.18) can be converted by the scale $z = \left(\frac{-\mu}{2}\right)^{\frac{1}{5}}\left(\zeta + \frac{\gamma}{\mu}\right)$ and $\Psi(\zeta) = -2\left(\frac{-\mu}{2}\right)^{\frac{2}{5}}\Phi(z)$ to the first Painlevé equation (P_I)

$$\ddot{\Phi}(z) = 6\Phi^2(z) + z, \tag{4.19}$$

with $\ddot{\Phi}(z) := \frac{d^2\Phi(z)}{dz^2}$.

5 The fractional modified Korteweg-de Vries equation

In this part, the same methodology as in Section 4 will be used to investigate the Lie symmetry analysis of the time fractional modified Korteweg-de Vries (mKdV) equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au^2u_x + bu_{xxx} = 0, \tag{5.1}$$

where $0 < \alpha \leq 1,\ a,\ b$ are constants, and α is a parameter describing the order of the conformable fractional time derivative. According to the Lie theory, applying the third prolongation $pr^{(3)}V$ to (5.1), one can find the infinitesimal criterion (3.13) to be

$$2a\eta uu_x + \eta_\alpha^t + a\eta^x u^2 + b\eta^{xxx} = 0, \tag{5.2}$$

which must be satisfied whenever $\frac{\partial^\alpha u}{\partial t^\alpha} + au^2u_x + bu_{xxx} = 0$. Direct substitution of $\eta^x,\ \eta^{xxx}$ and η_α^t from (3.9), (3.12) into (5.2), replacing u_{xxx} by $-\frac{1}{b}\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{a}{b}u^2u_x$ whenever it occurs, and equating the coefficients of the various monomials in partial derivatives of u, one can get the full determining equations for the symmetry group of (5.1). Solving these equations, one can obtain

$$\tau = \frac{-3c_2}{\alpha}t + c_3t^{1-\alpha}, \qquad \xi = -c_2x + c_1, \qquad \eta = c_2u, \tag{5.3}$$

where c_1, c_2 and c_3 are arbitrary constants. Therefore, the symmetry group of (5.1) is spanned by the three vector fields

$$V_1 = t^{1-\alpha}\frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x}, \quad V_3 = \frac{3t}{\alpha}\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} - u\frac{\partial}{\partial u}. \tag{5.4}$$

These vector fields satisfy Lie bracket relations

$$[V_1, V_2] = 0, \quad [V_1, V_3] = 3V_1, \quad [V_2, V_3] = V_2.$$

Note that when $\alpha = 1$, the vector fields of the fractional mKdV equation reduces to the vector fields of the classical mKdV equation [24].

The similarity variables for the infinitesimal generator V_3 can be found by solving the corresponding characteristic equations

$$\frac{dx}{x} = \frac{\alpha dt}{3t} = \frac{du}{-u}. \tag{5.5}$$

The corresponding invariants are

$$\zeta = x(3t)^{\frac{-\alpha}{3}}, \qquad u = (3t)^{\frac{-\alpha}{3}}\Psi(\zeta). \tag{5.6}$$

Substituting transformation (5.6) into equation (5.1), one can find that (5.1) can be reduced to a nonlinear ODE with the classical derivative

$$b\Psi'''(\zeta) + a\Psi^2(\zeta)\Psi'(\zeta) - \alpha[\zeta\Psi'(\zeta) + \Psi(\zeta)] = 0, \tag{5.7}$$

where $\Psi'(\zeta) := \frac{d\Psi(\zeta)}{d\zeta}$. As a result, we have

$$b\Psi''(\zeta) + \frac{a}{3}\Psi^3(\zeta) - \alpha\zeta\Psi(\zeta) = \gamma, \tag{5.8}$$

where γ is a constant of integration.

For the special case $a = -6,\ b = 1$, the time fractional mKdV (5.1) becomes

$$\frac{\partial^\alpha u}{\partial t^\alpha} - 6u^2u_x + u_{xxx} = 0. \tag{5.9}$$

By using the scale in (5.6) and integrating the result, then (5.9) reduces to

$$\Psi''(\zeta) = 2\Psi^3(\zeta) + \alpha\zeta\Psi(\zeta) + \gamma. \tag{5.10}$$

Equation (5.10) can be converted by the scale $\omega = \alpha^{\frac{1}{3}}\zeta,\ \Psi(\zeta) = \alpha^{\frac{1}{3}}\Phi(\omega)$ to the second Painlevé equation P_{II}

$$\ddot{\Phi}(\omega) = 2\Phi^3(\omega) + \omega\Phi(\omega) + \mu, \tag{5.11}$$

where $\ddot{\Phi}(\omega) := \frac{d^2\Phi(\omega)}{d\omega^2}$, and $\mu = \frac{\gamma}{\alpha}$.

6 The fractional Burgers equation

In this section we consider the following time-fractional Burgers equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + auu_x + bu_{xx} = 0, \tag{6.1}$$

where $0 < \alpha \leq 1,\ a,\ b$ are constants, and α is a parameter describing the order of the conformable fractional time derivative. According to the Lie theory, applying the second prolongation $pr^{(2)}V$ to (6.1), one can find the infinitesimal criterion (3.13) to be

$$a\eta u_x + \eta^t_\alpha + au\eta^x + b\eta^{xx} = 0, \tag{6.2}$$

which must be satisfied whenever $\frac{\partial^\alpha u}{\partial t^\alpha} + auu_x + bu_{xx} = 0$. Using (3.9), (3.12) into (6.2) to substitute $\eta^x,\ \eta^{xx}$ and η^t_α, replacing u_{xx} by $-\frac{1}{b}\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{a}{b}uu_x$ whenever it occurs, and equating the coefficients of the various monomials in partial derivatives of u, one can get the full determining equations for the symmetry group of (6.1). Solving these equations, one can obtain

$$\tau = \frac{c_1}{\alpha^2}t^{1+\alpha} + \frac{c_4}{\alpha}t + c_5 t^{1-\alpha}, \qquad \xi = \left(\frac{c_1}{\alpha}t^{\alpha} + \frac{c_4}{2}\right)x + \frac{ac_2}{\alpha}t^{\alpha} + c_3,$$
$$\eta = \left(\frac{-c_1}{\alpha}t^{\alpha} - \frac{c_4}{2}\right)u + \frac{c_1}{a}x + c_2, \tag{6.3}$$

where $c_1,\ c_2,\ c_3,\ c_4$ and c_5 are arbitrary constants. Therefore, the symmetry group of (6.1) is spanned by the five vector fields

$$V_1 = t^{1-\alpha}\frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = \frac{at^{\alpha}}{\alpha}\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$
$$V_4 = \frac{x}{2}\frac{\partial}{\partial x} + \frac{t}{\alpha}\frac{\partial}{\partial t} - \frac{u}{2}\frac{\partial}{\partial u}, \quad V_5 = \frac{xt^{\alpha}}{\alpha}\frac{\partial}{\partial x} + \frac{t^{1+\alpha}}{\alpha^2}\frac{\partial}{\partial t} + \left(\frac{x}{a} - \frac{t^{\alpha}u}{\alpha}\right)\frac{\partial}{\partial u}. \tag{6.4}$$

It is easily checked that these five vector fields satisfy

$$[V_1, V_2] = [V_2, V_3] = [V_3, V_5] = 0, \quad [V_2, V_4] = \frac{1}{2}V_2, \quad [V_2, V_5] = \frac{1}{a}V_3,$$
$$[V_1, V_3] = aV_2, \quad [V_1, V_4] = V_1, \quad [V_1, V_5] = 2V_4, \quad [V_3, V_4] = \frac{-1}{2}V_3, \quad [V_4, V_5] = V_5.$$

Thus the Lie algebra of infinitesimal symmetries of equation (6.1) is spanned by these five vector fields. The number of the vector fields coincides with that of the classical Burgers equation and when $\alpha = 1$ these vector fields reduces to that of the classical Burgers equation [29].

The similarity variables for the infinitesimal generator V_4 can be found by solving the corresponding characteristic equations

$$\frac{2dx}{x} = \frac{\alpha dt}{t} = \frac{-2du}{u}, \tag{6.5}$$

and the corresponding invariants are

$$\zeta = xt^{\frac{-\alpha}{2}}, \qquad u = t^{\frac{-\alpha}{2}}\Psi(\zeta). \tag{6.6}$$

Substituting transformation (6.6) into equation (6.1), one can find that (6.1) can be reduced to a nonlinear ODE with a new independent variable

$$b\Psi''(\zeta) + a\Psi(\zeta)\Psi'(\zeta) - \frac{\alpha}{2}[\zeta\Psi'(\zeta) + \Psi(\zeta)] = 0, \tag{6.7}$$

$\Psi'(\zeta) := \frac{d\Psi(\zeta)}{d\zeta}$. Consequently, we have

$$b\Psi'(\zeta) + \frac{a}{2}\Psi^2(\zeta) - \frac{\alpha}{2}\zeta\Psi(\zeta) = \gamma, \tag{6.8}$$

where γ is a constant of integration. The produced equation is a first order nonlinear differential equation with classical derivative (Riccati equation).

As another similarity variable we consider the linear combination $V_3 - \frac{1}{\mu}V_1$, (μ is a constant). As a result a similarity reduction can be found by solving the corresponding characteristic equations

$$\frac{\alpha dx}{at^{\alpha}} = \frac{-\mu dt}{t^{1-\alpha}} = \frac{du}{1}, \tag{6.9}$$

and the corresponding invariants are

$$\zeta = x + \frac{a\mu}{2\alpha^2}t^{2\alpha}, \qquad u = \frac{-\mu}{\alpha}t^{\alpha} + \Psi(\zeta). \tag{6.10}$$

Substituting transformation (6.10) into equation (6.1), one can find that (6.1) can be reduced to a nonlinear ODE with the classical derivative

$$b\Psi''(\zeta) + a\Psi(\zeta)\Psi'(\zeta) - \mu = 0, \tag{6.11}$$

where $\Psi'(\zeta) := \frac{d\Psi(\zeta)}{d\zeta}$. From which we obtain

$$b\Psi'(\zeta) + \frac{a}{2}\Psi^2(\zeta) - \mu\zeta = \gamma, \tag{6.12}$$

where γ is a constant of integration. The produced equation is a Riccati equation with classical derivative.

7 The fractional modified Burgers equation

In this section we will study the Lie theory of the following nonlinear time-fractional modified Burgers equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} + au^2u_x + bu_{xx} = 0, \tag{7.1}$$

where $0 < \alpha \le 1$, a, b are constants, and α is a parameter describing the order of the conformable fractional time derivative. According to the Lie theory, applying the second prolongation $pr^{(2)}V$ to (7.1), one can find the infinitesimal criterion (3.13) to be

$$2a\eta u u_x + \eta^t_\alpha + a\eta^x u^2 + b\eta^{xx} = 0, \tag{7.2}$$

which must be satisfied whenever $\frac{\partial^\alpha u}{\partial t^\alpha} + au^2u_x + bu_{xx} = 0$. Using (3.9), (3.12) to substitute η^x, η^{xx} and η^t_α into (7.2), replacing u_{xx} by $-\frac{1}{b}\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{a}{b}u^2u_x$ whenever it occurs, and equating the coefficients of the various monomials in partial derivatives of u, one can get the full determining equations for the symmetry group of (7.1). Solving these equations, one can obtain

$$\tau = \frac{c_2}{\alpha}t + c_3t^{1-\alpha}, \qquad \xi = \frac{c_2}{2}x + c_1, \qquad \eta = \frac{-c_2}{4}u, \tag{7.3}$$

where c_1, c_2 and c_3 are arbitrary constants. Therefore, the symmetry group of (7.1) is spanned by the three vector fields

$$V_1 = t^{1-\alpha}\frac{\partial}{\partial t}, \qquad V_2 = \frac{\partial}{\partial x}, \qquad V_3 = \frac{t}{\alpha}\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x} - \frac{u}{4}\frac{\partial}{\partial u}. \tag{7.4}$$

The commutation relations between these vector fields are given by

$$[V_1, V_2] = 0, \qquad [V_1, V_3] = V_1, \qquad [V_2, V_3] = \frac{1}{2}V_2.$$

Once again the vector fields of the fractional modified Burgers equation reduces to those of the classical equations as α reduces to 1 [41].

The one-parameter group generated by V_3 can be found by solving the corresponding characteristic equations

$$\frac{dx}{x} = \frac{\alpha dt}{2t} = \frac{-2du}{u}, \tag{7.5}$$

and the corresponding invariants are

$$\zeta = xt^{\frac{-\alpha}{2}}, \qquad u = t^{\frac{-\alpha}{4}}\Psi(\zeta). \tag{7.6}$$

Substituting transformation (7.6) into equation (7.1), one can find that (7.1) can be reduced to a nonlinear ODE with a new independent variable. Consequently, we have

$$b\Psi''(\zeta) + a\Psi^2(\zeta)\Psi'(\zeta) - \frac{\alpha}{2}\zeta\Psi'(\zeta) - \frac{\alpha}{4}\Psi(\zeta) = 0, \tag{7.7}$$

$\Psi'(\zeta) := \frac{d\Psi(\zeta)}{d\zeta}$. The produced equation is a second order nonlinear differential equation with classical derivative.
The scale $z = \left(\frac{\alpha}{4}\right)^{\frac{1}{2}}\zeta$, $\Psi(\zeta) = \left(\frac{\alpha}{4}\right)^{\frac{1}{4}}\Phi(z)$ transform (7.7) to an equivalent form

$$b\ddot{\Phi}(z) + a\Phi^2(z)\dot{\Phi}(z) - 2z\dot{\Phi}(z) - \Phi(z) = 0, \tag{7.8}$$

where $\ddot{\Phi}(z) := \frac{d^2\Phi(z)}{dz^2}$.

8 Conclusion

We have applied the Lie group analysis to the time fractional Korteweg-de Vries, modified Korteweg-de Vries, Burgers, and modified Burgers equations, where the time derivative is the conformable fractional derivative. All the generating vector fields for each equation have been calculated. Thus it is evident that the Lie group analysis can be used successfully to study conformal fractional partial differential equations. It worth to note that the number of the generating vector fields for each of the four time-fractional equations is the same as that of the classical equation and the generating vector fields of each of these equation reduce to that of the corresponding classical equation when $\alpha = 1$.

Using the obtained Lie symmetries, we have shown that the equations under consideration can be transformed to ordinary differential equations with classical derivative. More precisely, we have shown that the time fractional KdV equation can be transformed into the first and second Painlevé equations. For the time fractional modified KdV equation, we obtained a solution in terms of the second Painlevé equation. In the case of Burgers equation, we derived solutions in terms of Riccati equations.

It should be noted that the similarity reduction method convert a fractional partial differential equation with conformable fractional derivative to an ordinary differential equation with classical derivative. However, fractional partial differential equation with Riemann-Liouville fractional derivative is transformed to an ordinary fractional differential equation with an Erdélyi-Kober derivative depending on a parameter α.

It is interesting to apply the Lie group analysis to other partial differential equations with time and space fractional derivatives.

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